OLS Asymptotics

AECN 396/896-002

Before we start

Learning objectives

Understand the consequences of the violation of the homoskedasticity assumption and how to deal with the problem

Table of contents

OLS Asymptotics

Large Sample Properties of OLS

- Properties of OLS that hold only when the sample size is infinite very large
- (loosely put) How OLS estimators behave when the number of observations goes infinite (really large)

Small Sample Properties of OLS

Under certain conditions:

- OLS estimators are unbiased
- OLS estimators are efficient

These hold whatever the sample size is.

Consistency

Consistency

Verbally (and very loosely)

An estimator is consistent if the probability that the estimator produces the true parameter is 1 when sample size is infinite.

MC simulation: consistency of OLS estimators

OLS estimator of the coefficient on x in the following model with all $MLR.1$ through $MLR.4$ satisfied:

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

with all the conditions necessary for the unbiasedness property of OLS satisfied.

Conceptual steps of the MC simulations

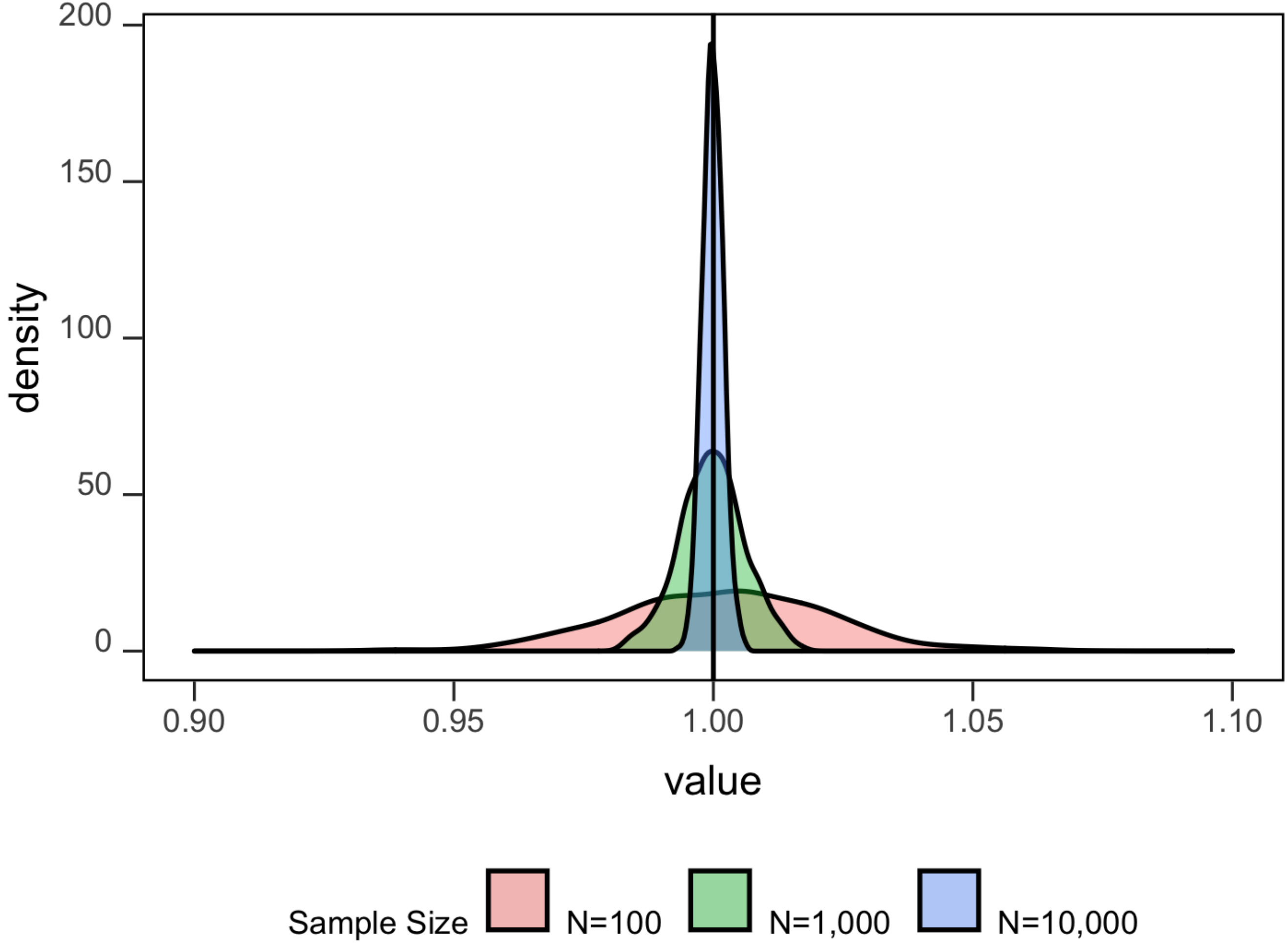
- Generate data according to $y_i = \beta_0 + \beta_1 x_i + u_i$
- Estimate the coefficients and store them
- Repeat the above experiment 1000 times
- Examine how the coefficient estimates are distributed

What you should see is

As N gets larger (more observations), the distribution of $\hat{\beta}_1$ get more tightly centered around its true value (here, 1)

```r
#--- Preparation ---#
B <- 1000 # the number of iterations
N_list <- c(100, 1000, 10000) # sample size
N_len <- length(N_list)
estimate_storage <- matrix(0, B, 3) # estimates storage

for (j in 1:N_len) {
  temp_N <- N_list[j]
  for (i in 1:B) {
    #--- generate data ---#
    x <- rnorm(temp_N) # indep var 1
    u <- rnorm(temp_N) * 0.2 # error
    y <- 1 + x + u # dependent variable 1
    data <- data.frame(y = y, x = x)

    #--- OLS ---#
    reg <- lm(y ~ x, data = data) # OLS

    #--- store coef estimates ---#
    estimate_storage[i, j] <- reg$coef[2]
  }
}
```

```r
plot_data <- melt(data.frame(estimate_storage))

#--- create a figure ---#
g_co_ex <- ggplot(data = plot_data) +
  geom_density(aes(x = value, fill = variable), alpha = 0.4) +
  geom_vline(xintercept = 1) +
  xlim(0.9, 1.1) +
  scale_fill_discrete(
    name = "Sample Size",
    labels = c("N=100 ", "N=1,000 ", "N=10,000")
  ) +
  theme(
    legend.position = "bottom"
  )
```


200
150
100
50
0
density
0.90
0.95
1.00
1.05
1.10
value
Sample Size
N=100
N=1,000
N=10,000

Consistency of OLS estimators

Under $MLR.1$ through $MLR.4$, OLS estimators are consistent

MC simulations: Inconsistency of OLS estimators

Conceptual steps of MC simulations

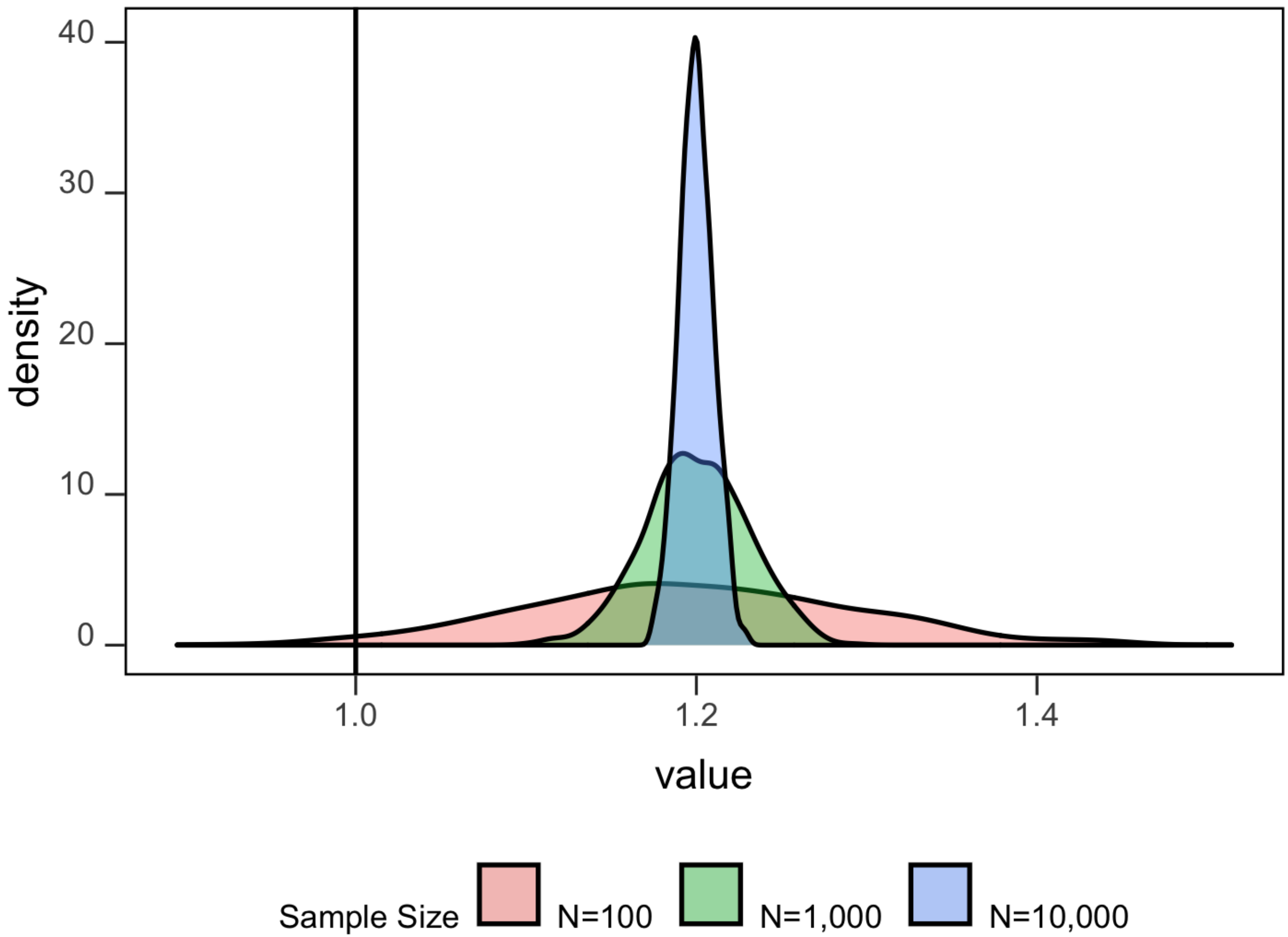
- generate data (N observations) according to $y_i = \beta_0 + \beta_1 x_i + u_i$ with $E[u_i|x_i] \neq 0$
- Estimate the coefficients and store them
- repeat the above experiment 1000 times
- examine how the coefficient estimates are distributed

Question

Would the bias disappear as N gets larger?

```r
#--- Preparation ---#
N_list <- c(100, 1000, 10000) # sample size
N_len <- length(N_list)
estimate_storage <- matrix(0, B, 3) # estimates storage

for (j in 1:N_len) {
  temp_N <- N_list[j]
  for (i in 1:B) {
    #--- generate data ---#
    mu <- rnorm(temp_N) # shared term between x and u
    x <- rnorm(temp_N) + 0.5 * mu # <<
    u <- rnorm(temp_N) + 0.5 * mu # <<
    y <- 1 + x + u # dependent variable
    data <- data.frame(y = y, x = x)

    #--- OLS ---#
    reg <- lm(y ~ x, data = data) # OLS

    #--- store coef estimates ---#
    estimate_storage[i, j] <- reg$coef[2]
  }
}
```

```r
#--- wide to long format ---#
plot_data <- melt(data.frame(estimate_storage))

#--- create a figure ---#
g_inco_ex <- ggplot(data = plot_data) +
  geom_density(aes(x = value, fill = variable), alpha = 0.4) +
  geom_vline(xintercept = 1) +
  scale_fill_discrete(
    name = "Sample Size",
    labels = c("N=100 ", "N=1,000 ", "N=10,000")
  ) +
  theme(
    legend.position = "bottom"
  )
```


40
30
20
10
0
density
1.0
1.2
1.4
value
Sample Size
N=100
N=1,000
N=10,000

Asymptotic Normality

Testing

When we talked about hypothesis testing, we made the following assumption:

Normality assumption

The population error u is independent of the explanatory variables $x_1, \ldots, x_k$ and is normally distributed with zero mean and variance σ^2:

$$u \sim Normal(0, \sigma^2)$$

Remember

- If the normality assumption is violated, t-statistic and F-statistic we constructed before are no longer distributed as t-distribution and F-distribution, respectively
- So, whenever $MLR.6$ is violated, our t- and F-tests are invalid

Fortunately

You can continue to use t- and F-tests because (slightly transformed) OLS estimators are approximately normally distributed when the sample size is large enough.

Central Limit Theorem (CLT)

Suppose $\{x_1, x_2, ...\}$ is a sequence of idetically independently distributed} random variables with $E[x_i] = \mu$ and $Var[x_i] = \sigma^2 < \infty$. Then, as n approaches infinity,

$$\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n} x_i - \mu) \xrightarrow{d} N(0, \sigma^2)$$

Verbally

Sample mean less its expected value multiplied by $\sqrt{n}$ (square root of the sample size) is going to be distributed as Normal distribution as n goes infinity where its expected value is 0 and variance is the variance of x.

Example

$x_i \sim Bern[p = 0.3]$

1 with probability p and 0 with probability $1 - p$.

- $E[x_i] = p = 0.3$
- $Var[x_i](\sigma^2) = p(1-p) = 0.21$

According to CLT

$\left(\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n} x_i - \mu) \xrightarrow{d} N(0, \sigma^2)\right)$

$\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n} x_i - 0.3) \xrightarrow{d} N(0, 0.21)$

MC simulations: CLT

Conceptual steps of the MC simulation

- draw n observations from $x_i \sim Bern(0.3)$
- find its mean, subtract the expected value (here, $E[x_i] = 0.3$), multiply by $\sqrt{n}$ ($\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n} x_i - \mu)$
- store the calculated value
- repeat the above experiment 1000 times
- examine how the calculated values are distributed

What you should see is

As N gets larger (more observations), the distribution of $\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n} x_i - 0.3)$ looks more and more like $N(0, 0.21)$

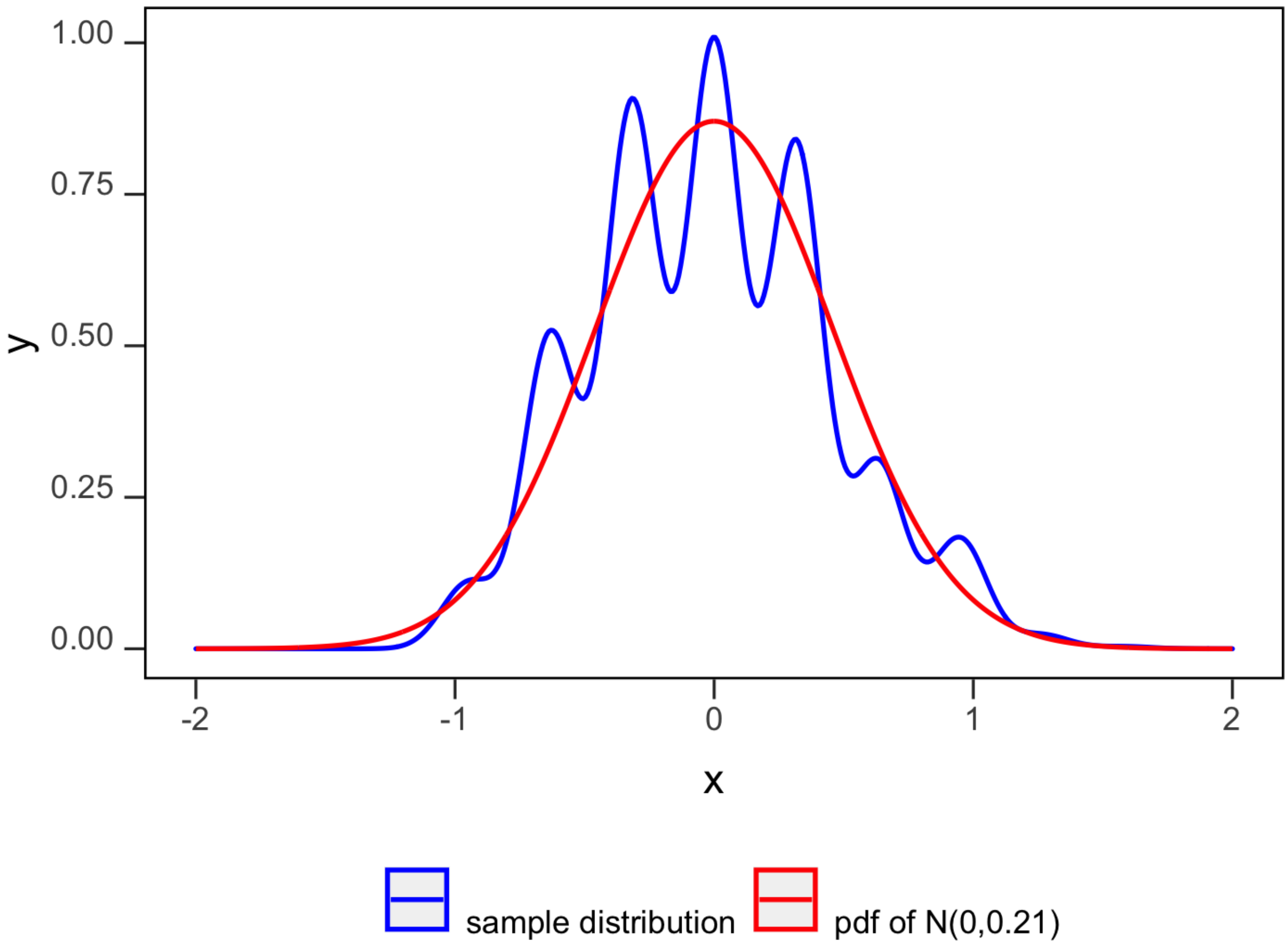
MC simulations (N = 10)

```r
set.seed(893269)
#--- the number of observations ---#
# this is what we change
N <- 10 # number of observations
B <- 1000 # number of iterations
p <- 0.3 # mean of the Bernoulli distribution
storage <- rep(0, B)

for (i in 1:B) {
  #--- draw from Bern[0.3] (x distributed as Bern[0.3]) ---#
  x_seq <- runif(N) <= p

  #--- sample mean ---#
  x_mean <- mean(x_seq)

  #--- normalize ---#
  lhs <- sqrt(N) * (x_mean - p)

  #--- save lhs to storage ---#
  storage[i] <- lhs
}
```

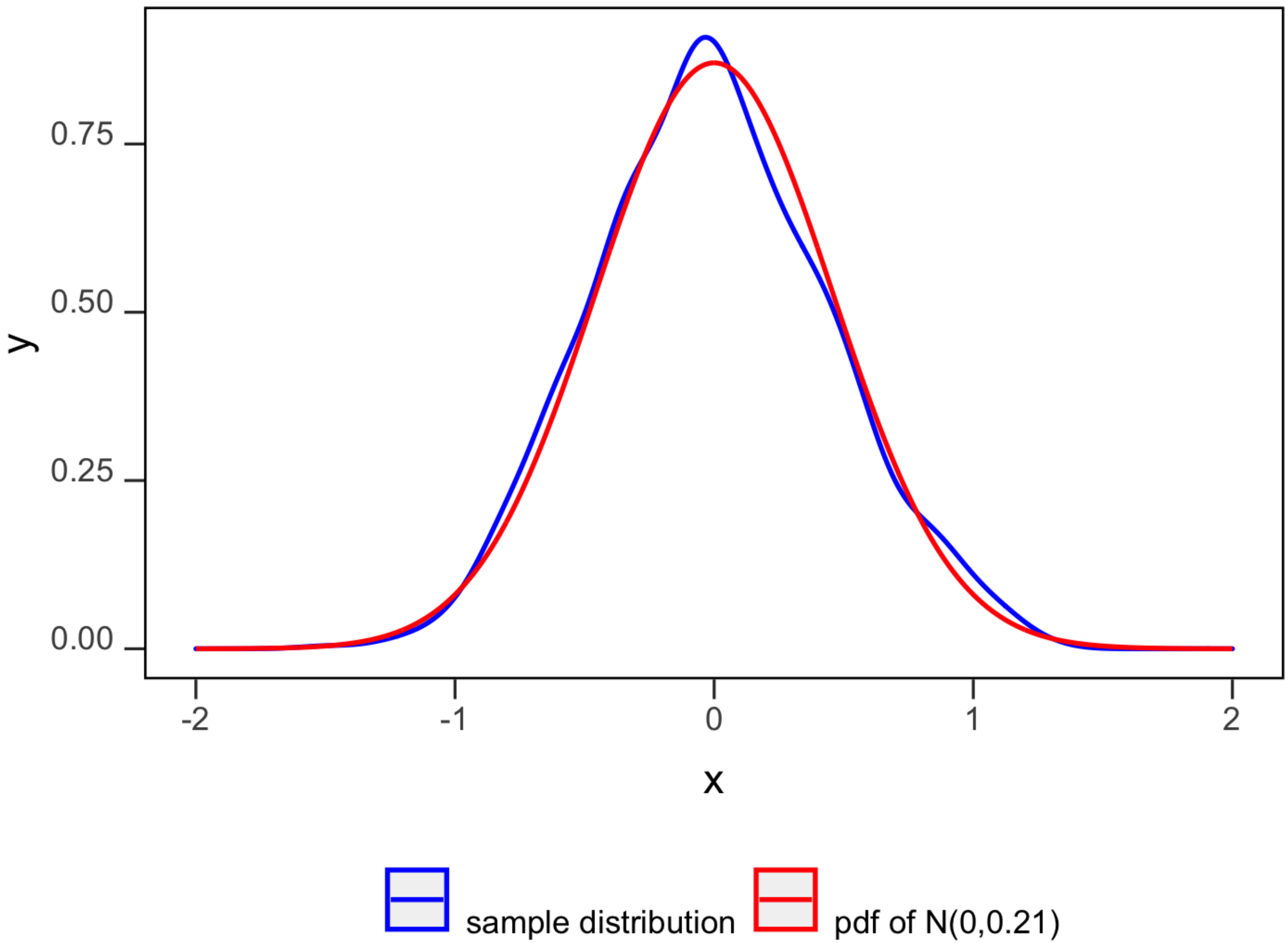
Visualization

```
data_pdf <- data.frame(
  x = seq(-2, 2, length = 1000),
  y = dnorm(seq(-2, 2, length = 1000), sd = sqrt(p * (1 - p)))
)

g_N_10 <-
  ggplot() +
  geom_density(
    data = data.frame(x = storage),
    aes(x = x, color = "sample distribution")
  ) +
  geom_line(
    data = data_pdf,
    aes(y = y, x = x, color = "pdf of N(0,0.21)")
  ) +
  scale_color_manual(
    values = c("sample distribution" = "blue", "pdf of N(0,0.21)" = "red"),
    name = ""
  ) +
  theme(
    legend.position = "bottom"
  )
```

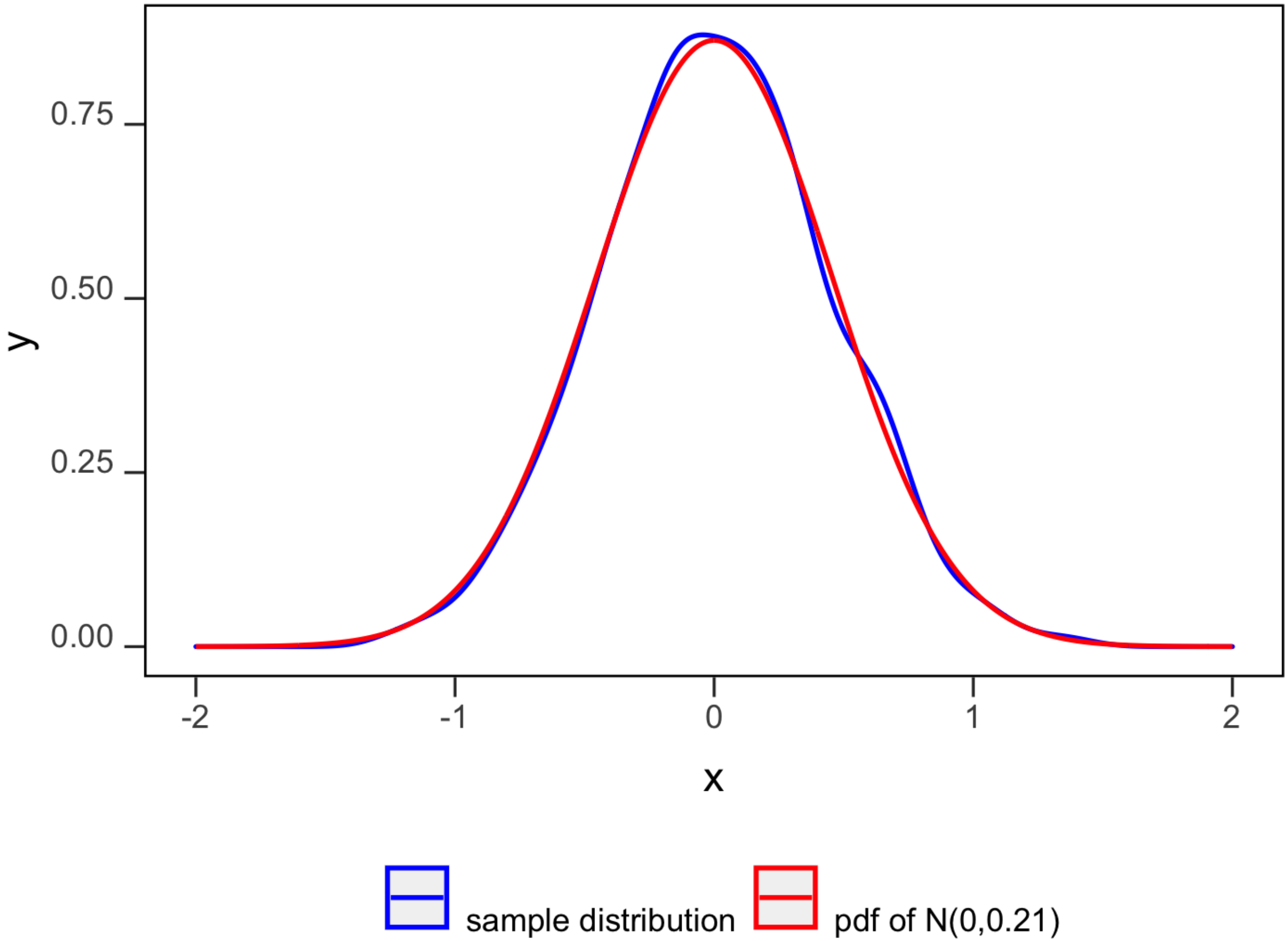
MC simulations (N = 10)



MC simulations (N = 100)



MC simulations (N = 10000)



Important

CLT holds for any distribution of x_i as long as it has a finite expected value and variance.

Under assumptions $MLR.1$ through $MLR.5$ (MLR.6 not necessary!!),

Asymptotic Normality of OLS

$\sqrt{n}(\hat{\beta}_j - \beta_j) \overset{a}{\rightarrow} N(0, \sigma^2/\alpha_j^2)$

where $\alpha_j^2 = plim(\frac{1}{n}\sum_{i=1}^{n} r_{i,j}^2)$, where $r_{i,j}^2$ are the residuals from regressing x_j on the other independent variables.

Consistency

$\hat{\sigma}^2 \equiv \frac{1}{n-k-1}\sum_{i=1}^{n} \hat{u}_i^2$ is a consistent estimator of of σ^2 $(Var(u))$

Further

- $(\hat{\beta}_j - \beta_j)/se(\hat{\beta}_j) \xrightarrow{a} N(0,1)$
- $(\hat{\beta}_j - \beta_j)/\widehat{se(\hat{\beta}_j)} \xrightarrow{a} N(0,1)$, where $\widehat{se(\hat{\beta}_j)} = \sqrt{\frac{\hat{\sigma}^2}{SST_j(1-R_j^2)}}$

Small sample (any sample size)

Under $MLR.1$ through $MLR.5$ and $MLR.6$ $(u_i \sim N(0, \sigma^2))$,

- $(\hat{\beta}_j - \beta_j)/sd(\hat{\beta}_j) \sim N(0,1)$
- $(\hat{\beta}_j - \beta_j)/se(\hat{\beta}_j) \sim t_{n-k-1}$

Large sample (when（n）goes infinity)

Under $MLR.1$ through $MLR.5$ without $MLR.6$,

- $(\hat{\beta}_j - \beta_j)/sd(\hat{\beta}_j) \xrightarrow{a} N(0,1)$
- $(\hat{\beta}_j - \beta_j)/se(\hat{\beta}_j) \xrightarrow{a} N(0,1)$

Testing under large sample

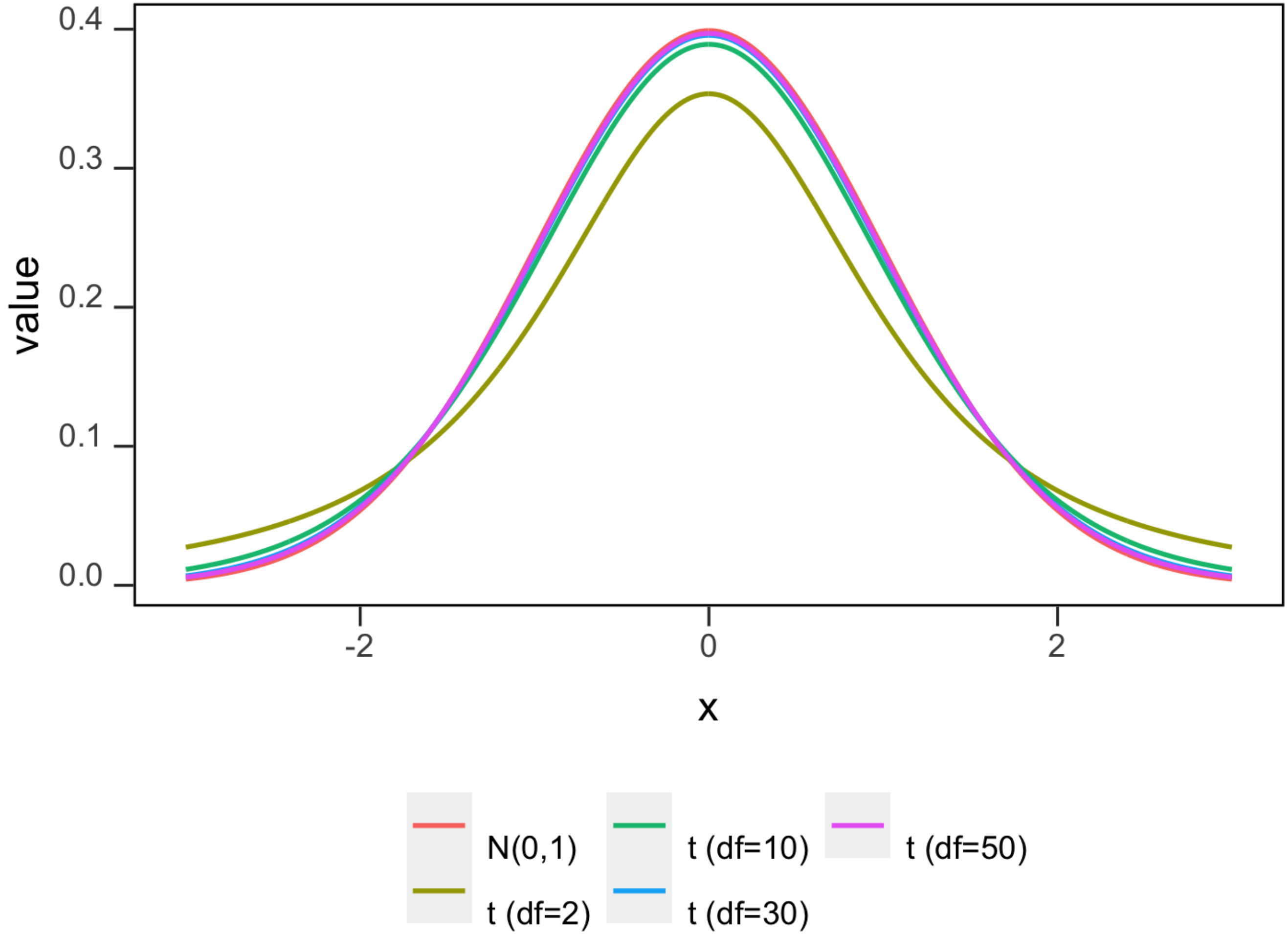
It turns out,

You can proceed exactly the same way as you did before (practically speaking)!!

- calculate $(\hat{\beta}_j - \beta_j)/\widehat{se(\hat{\beta}_j)}$
- check if the obtained value is greater than (in magnitude) the critical value for the specified significance level under t_{n-k-1}

But,

Shouldn't we use $N(0,1)$ when you find the critical value?


0.4
0.3
0.2
0.1
0.0
value
-2
0
2
x
N(0,1)
t (df=2)
t (df=10)
t (df=30)
t (df=50)

Testing under large sample

Since t_{n-k-1} and $N(0,1)$ are almost identical when n is large, there is very little error in using t_{n-k-1} instead of $N(0,1)$ to find the critical value.

When the homoskedasticity is violated (as almost always the case)

Important

The consistency of the estimation of $\widehat{Var(\hat{\beta})}$ DOES require the homoskedasticity assumption (MLR.5)!!

- the usual t-statistics and confidence intervals are invalid no matter how large the sample size is if error is heteroskedastic
- so, we should use heteroskedasticity-robust or cluster-robust standard error estimators even when the sample size is large