Econometric Modeling

AECN 396/896-002

Before we start

Learning objectives

1. Enhance the understanding of the interpretation of various models
2. Post-estimation simulation

Table of contents

1.
2.
3.

More on functional forms

Various econometric models

log-linear

$$log(y_i) = \beta_0 + \beta_1 x_i + u_i$$

linear-log

$$y_i = \beta_0 + \beta_1 log(x_i) + u_i$$

log-log

$$log(y_i) = \beta_0 + \beta_1 log(x_i) + u_i$$

quadratic

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + u_i$$

Quadratic

Model

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + u_i$$

Calculus

Differentiating the both sides wrt x_i,

$$\frac{\partial y_i}{\partial x_i} = \beta_1 + 2 * \beta_2 x_i \Rightarrow \Delta y_i = (\beta_1 + 2 * \beta_2 x_i) \Delta x_i$$

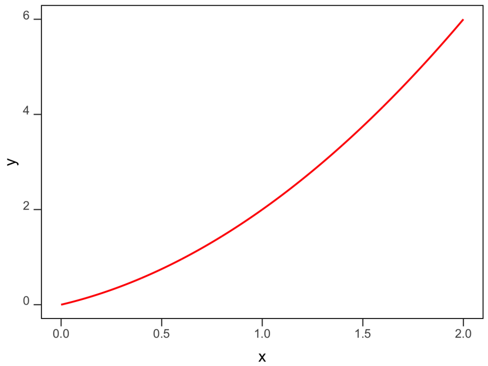
Interpretation

When x increases by 1 unit $(\Delta x_i = 1)$, y increases by $\beta_1 + 2 * \beta_2 x_i$

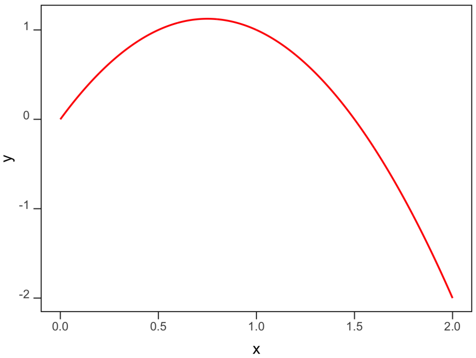
Visualizarion

Quadratic functional form is quite flexible.

$y = x + x^2 \; (\beta_1 = 1, \beta_2 = 1)$



$y = 3x - 2x^2 \; (\beta_1 = 3, \beta_2 = -2)$



Example

Education impacts of income

The marginal impact of education (the impact of a small change in education on income) may differ what level of education you have had:

- How much does it help to have two more years of education when you have had education until elementary school?
- How much does it help to have two more years of education when you have have graduated a college?
- How much does it help to spend two more years as a Ph.D student if you have already spent six years in a Ph.D program

Observation

The marginal impact of education does not seem to be linear.

Implementation in R

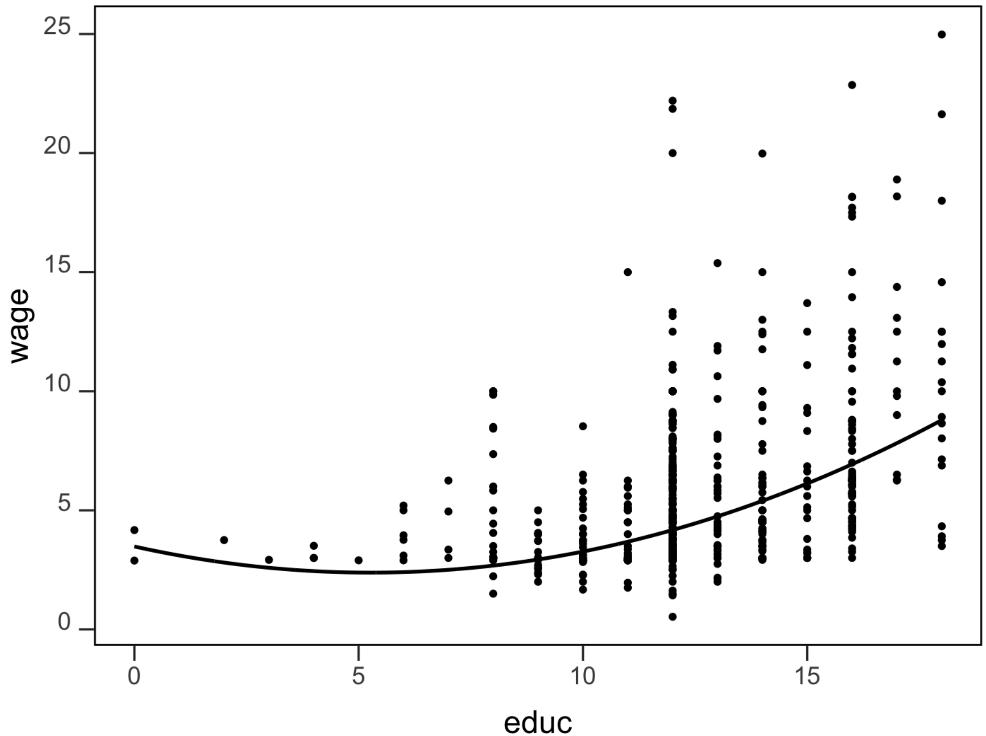
When you include a variable that is a transformation of an existing variable, use `I()` function in which you write the mathematical expression of the desired transformation.

```
#--- prepare a dataset ---#
wage <- readRDS('wage1.rds')

#--- run a regression ---#
quad_reg <- feols(wage ~ female + educ + I(educ^2), data = wage)

#--- look at the results ---#
tidy(quad_reg)
```

```
## # A tibble: 4 × 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)   5.61     1.38         4.05 5.91e- 5
## 2 female       -2.13     0.277       -7.67 8.50e-14
## 3 educ         -0.416    0.231       -1.81 7.14e- 2
## 4 I(educ^2)     0.0395   0.00964      4.10 4.80e- 5
```

Estimated Model

$$wage = 5.60 - 2.12 \times female - 0.416 \times educ + 0.039 \times educ^2$$



Estimated Model

$$wage = 5.60 - 2.12 \times female - 0.416 \times educ + 0.039 \times educ^2$$

Problem

What is the marginal impact of $educ$?

$$\frac{\partial wage}{\partial educ} = ?$$

Answer

$$\frac{\partial wage}{\partial educ} = -0.416 + 0.039 \times 2 \times educ$$

- When $educ = 4$, additional year of education is going to increase hourly wage by -0.104 on average
- When $educ = 10$, additional year of education is going to increase hourly wage by 0.364 on average

Statistical significance of the marginal impact

The marginal impact of $educ$ is:

$$\frac{\partial wage}{\partial educ} = -0.416 + 0.039 \times 2 \times educ$$

- $educ$: -0.416 (t-stat $= -1.80$)
- $educ^2$: 0.039 (t-stat $= 4.10$)

Question

So, is the marginal impact of $educ$ statistically significantly different from 0?

In the linear case

```
linear_reg <- feols(wage ~ female + educ, data = wage)
tidy(linear_reg)
```

```
## # A tibble: 3 × 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)    0.623    0.673      0.926 3.55e- 1
## 2 female        -2.27     0.279     -8.15  2.76e-15
## 3 educ           0.506    0.0504    10.1   7.56e-22
```

Estimated model

$$wage = 0.62 + 0.51 \times educ$$

Estimated model

$$wage = 0.62 + 0.51 \times educ$$

Question

- What is the marginal impact of $educ$?

0.51

- Does the marginal impact of education vary depending on the level of education?

No, the model we estimated assumed that the marginal impact of education is constant.

Testing

You can just test if $\hat{\beta}_{educ}$ (the marginal impact of education) is statistically significantly different from 0, which is just a t-test.

Going back to the quadratic case

With the quadratic specification

- The marginal impact of education varies depending on your education level
- There is no single test that tells you whether the marginal impact of education is statistically significant universally
- Indeed, you need different tests for different values education levels

Example 1

Marginal impact of education

$\hat{\beta}_{educ} + \hat{\beta}_{educ^2} \times 2 \times educ$

Hypothesis testing

Does additional year of education has a statistically significant impact (positive or negative) if your current education level is 4?

- H_0: $\hat{\beta}_{educ} + \hat{\beta}_{educ^2} \times 2 \times 4 = 0$
- H_1: $\hat{\beta}_{educ} + \hat{\beta}_{educ^2} \times 2 \times 4 \neq 0$

Question

Is this

- test of a single coefficient? (t-test)
- test of a single equation with multiple coefficients? (t-test)
- test of multiples equations with multiple coefficients? (F-test)

t-statistic

$$t = \frac{\hat{\beta}_{educ} + \hat{\beta}_{educ^2} \times 2 \times 4}{se(\hat{\beta}_{educ} + \hat{\beta}_{educ^2} \times 2 \times 4)} = \frac{\hat{\beta}_{educ} + \hat{\beta}_{educ^2} \times 8}{se(\hat{\beta}_{educ} + \hat{\beta}_{educ^2} \times 8)}$$

R implementation

Remember, a trick to do this test using R is take advantage of the fact that $F_{1,n-k-1} \sim t_{n-k-1}$.

```
linearHypothesis(quad_reg, "educ + 8*I(educ^2)=0")
```

```
## Linear hypothesis test
##
## Hypothesis:
## educ  + 8 I(educ^2) = 0
##
## Model 1: restricted model
## Model 2: wage ~ female + educ + I(educ^2)
##
##   Df  Chisq Pr(>Chisq)
## 1
## 2  1 0.4126     0.5207
```

Since the p-value is 0.529, we do not reject the null.

t-statistic

$$t = \frac{\hat{\beta}_{educ} + \hat{\beta}_{educ^2} \times 2 \times 10}{se(\hat{\beta}_{educ} + \hat{\beta}_{educ^2} \times 2 \times 10)} = \frac{\hat{\beta}_{educ} + \hat{\beta}_{educ^2} \times 20}{se(\hat{\beta}_{educ} + \hat{\beta}_{educ^2} \times 20)}$$

R implementation

```
linearHypothesis(quad_reg, "educ + 20*I(educ^2)=0")
```

```
## Linear hypothesis test
##
## Hypothesis:
## educ  + 20 I(educ^2) = 0
##
## Model 1: restricted model
## Model 2: wage ~ female + educ + I(educ^2)
##
##   Df  Chisq Pr(>Chisq)
## 1
## 2  1 39.831  2.769e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Since the much lower than is 0.01, we can reject the null at the 1% level.

Interaction terms

An interaction term

A variable that is a multiplication of two variables

Example

$educ \times exper$

A model with an interaction term

$$wage = \beta_0 + \beta_1 exper + \beta_2 educ \times exper + u$$

Marginal impact of experience

$$\frac{\partial wage}{\partial exper} = \beta_1 + \beta_2 \times educ$$

Implications

The marginal impact of experience depends on education

- β_1: the marginal impact of experience when $educ = ?$
- if $\beta_2 > 0$: additional year of experience is worth more when you have more years of education

Regression with interaction terms

Just like the quadratic case with $educ^2$, you can use `I()`.

```
reg_int <- feols(wage ~ female + exper + I(exper*educ), data = wage)
```

	Model 1
(Intercept)	6.121***
	(0.267)
exper	-0.188***
	(0.024)
female	-2.418***
	(0.277)
I(exper * educ)	0.020***
	(0.002)
Std. errors	IID

* p < 0.1, ** p < 0.05, *** p < 0.01

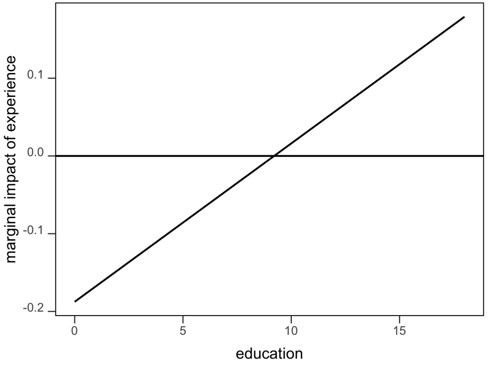
Estimated Model

$$wage = 6.121 - 2.418 \times female - 0.188 \times exper + 0.020 \times educ \times exper$$

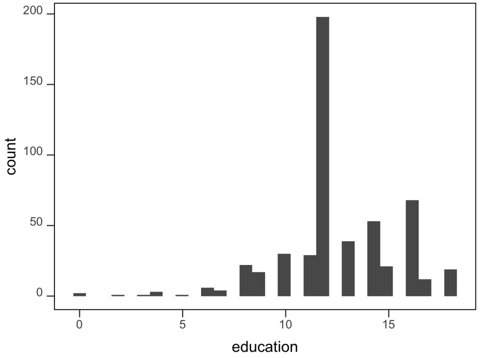
Marginal impact of experience

$$\frac{\partial wage}{\partial exper} = -0.188 + 0.020 \times educ$$

Marginal impact of $exper$:



Histogram of education:



Testing of the marginal impact

- Just like the case of the quadratic specification of education, marginal impact of experience is not constant

- We can test if the marginal impact of experience is statistically significant for a given level of education
  - When $educ = 10$, $\frac{\partial wage}{\partial exper} = -0.188 + 0.020 \times 10 = 0.012$
  - When $educ = 15$, $\frac{\partial wage}{\partial exper} = -0.188 + 0.020 \times 15 = 0.112$

Question

Does additional year of experience has a statistically significant impact (positive or negative) if your current education level is 10?

Hypothesis

- H_0: $\hat{\beta}_{exper} + \hat{\beta}_{exper\_educ} \times 10 = 0$
- H_1: $\hat{\beta}_{exper} + \hat{\beta}_{exper\_educ} \times 10 = 0$

R implementation

```
linearHypothesis(reg_int,'exper+10*I(exper * educ)=0')
```

```
## Linear hypothesis test
##
## Hypothesis:
## exper  + 10 I(exper * educ) = 0
##
## Model 1: restricted model
## Model 2: wage ~ female + exper + I(exper * educ)
##
##   Df  Chisq Pr(>Chisq)
## 1
## 2  1 2.4627     0.1166
```

Including qualitative information

Qualitative information

Issue

How do we include qualitative information as an independent variable?

Examples

- male or female (binary)
- married or single (binary)
- high-school, college, masters, or Ph.D (more than two states)

Binary variables

Dummy variable

- Relevant information in binary variables can be captured by a zero-one variable that takes the value of 1 for one state and 0 for the other state

- We use "dummy variable" to refer to a binary (zero-one) variable

Example

```r
wage <- readRDS('wage1.rds')

dplyr::select(wage, wage, educ, exper, female, married) %>%
    head()
```

```
##   wage educ exper female married
## 1 3.10   11     2      1       0
## 2 3.24   12    22      1       1
## 3 3.00   11     2      0       0
## 4 6.00    8    44      0       1
## 5 5.30   12     7      0       1
## 6 8.75   16     9      0       1
```

Model with dummy a variable

$$wage = \beta_0 + \sigma_f female + \beta_2 educ + u$$

Interpretation

- `female`: $E[wage|female = 1, educ] = \beta_0 + \sigma_f + \beta_2 educ$
- `male`: $E[wage|female = 0, educ] = \beta_0 + \beta_2 educ$

This means that

$$\sigma_f = E[wage|female = 1, educ] - E[wage|female = 0, educ]$$

$$\sigma_f = E[wage|female = 1, educ] - E[wage|female = 0, educ]$$

Verbally,

- σ_f is the difference in the expected wage conditional on education between female and male
- σ_f measures how much more (less) female workers make compared to male workers (baseline) if they were to have the same education level

Regression with a dummy variable

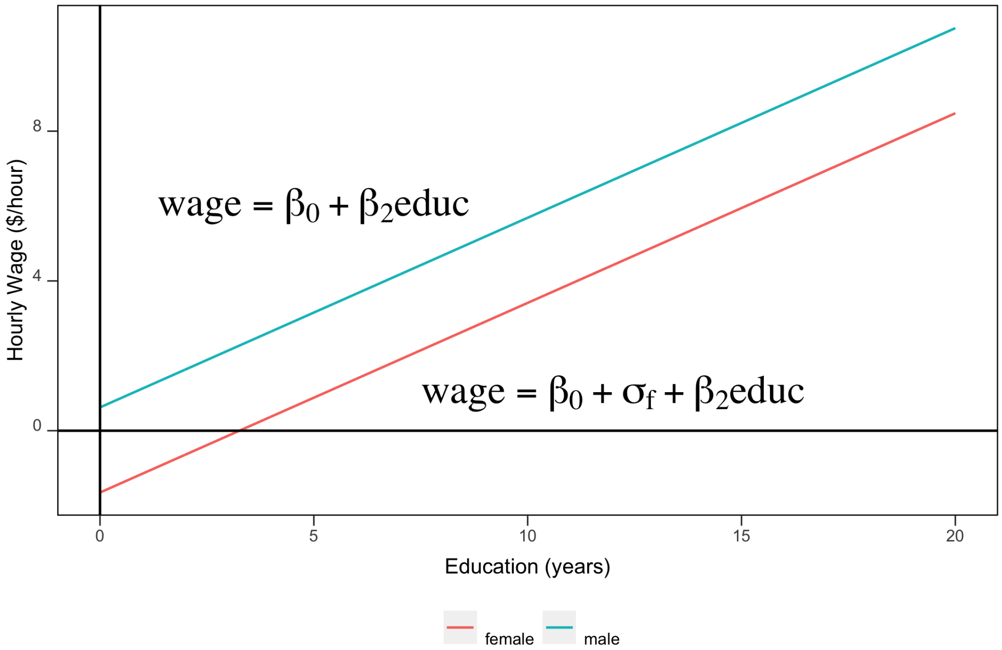
```
reg_df <- feols(wage ~ female + educ, data = wage)

reg_df
```

```
## OLS estimation, Dep. Var.: wage
## Observations: 526
## Standard-errors: IID
##              Estimate Std. Error   t value   Pr(>|t|)
## (Intercept)  0.622817   0.672533  0.926076 3.5483e-01
## female      -2.273362   0.279044 -8.146954 2.7642e-15 ***
## educ         0.506452   0.050391 10.050520  < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## RMSE: 3.17642   Adj. R2: 0.255985
```

Interpretation

Female workers make -2.2733619 (\$/hour) less than male workers on average even though they have the same education level.

Visualization of the estimated model



Model with dummy a variable

$$wage = \beta_0 + \sigma_m male + \beta_2 educ + u$$

Interpretation

- `male`: $E[wage|male = 1, educ] = \beta_0 + \sigma_m + \beta_2 educ$
- `female`: $E[wage|male = 0, educ] = \beta_0 + \beta_2 educ$

This means that

$$\sigma_m = E[wage|male = 1, educ] - E[wage|male = 0, educ]$$

$$\sigma_m = E[wage|male = 1, educ] - E[wage|male = 0, educ]$$

Verbally,

- σ_m is the difference in the expected wage conditional on education between female and male
- σ_m measures how much more (less) male workers make compared to female workers (baseline) if they were to have the same education level

Important: whichever status that is given the value of 0 becomes the baseline

Regression with a dummy variable

```
wage <- mutate(wage, male = 1- female)

reg_df <- feols(wage ~ male + educ, data = wage)

reg_df
```

```
## OLS estimation, Dep. Var.: wage
## Observations: 526
## Standard-errors: IID
##              Estimate Std. Error  t value   Pr(>|t|)
## (Intercept) -1.650545   0.652317 -2.53028 1.1689e-02 *
## male         2.273362   0.279044  8.14695 2.7642e-15 ***
## educ         0.506452   0.050391 10.05052  < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## RMSE: 3.17642   Adj. R2: 0.255985
```

Interpretation

Female workers make 2.2733619 ($/hour) more than female workers on average even though they have the same education level.

Question

Why do you think will happen if we include both male and female dummy variables?

Answer

- They contain redundant information
- Indeed, including both of them along with the intercept would cause perfect collinearity problem
- So, you need to drop either one of them

Perfect Collinearity

intercept = male + female

Here is what happens if you include both:

```r
reg_dmf <- feols(wage ~ male + female + educ, data = wage)

reg_dmf
```

```
## OLS estimation, Dep. Var.: wage
## Observations: 526
## Standard-errors: IID
##              Estimate Std. Error  t value   Pr(>|t|)
## (Intercept) -1.650545   0.652317 -2.53028 1.1689e-02 *
## male         2.273362   0.279044  8.14695 2.7642e-15 ***
## educ         0.506452   0.050391 10.05052  < 2.2e-16 ***
## ... 1 variable was removed because of collinearity (female)
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## RMSE: 3.17642   Adj. R2: 0.255985
```

Interactions with a dummy variable

Issue

- In the previous example, the impact of education on wage was modeled to be exactly the same
- Can we build a more flexible model that allows us to estimate the differential impacts of education on wage between male and female?

A more flexible model

$$wage = \beta_0 + \sigma_f female + \beta_2 educ + \gamma female \times educ + u$$

- [female]: $E[wage|female = 1, educ] = \beta_0 + \sigma_f + (\beta_2 + \gamma)educ$
- [male]: $E[wage|female = 0, educ] = \beta_0 + \beta_2 educ$

Interpretation

For female, education is more effective by γ than it is for male.

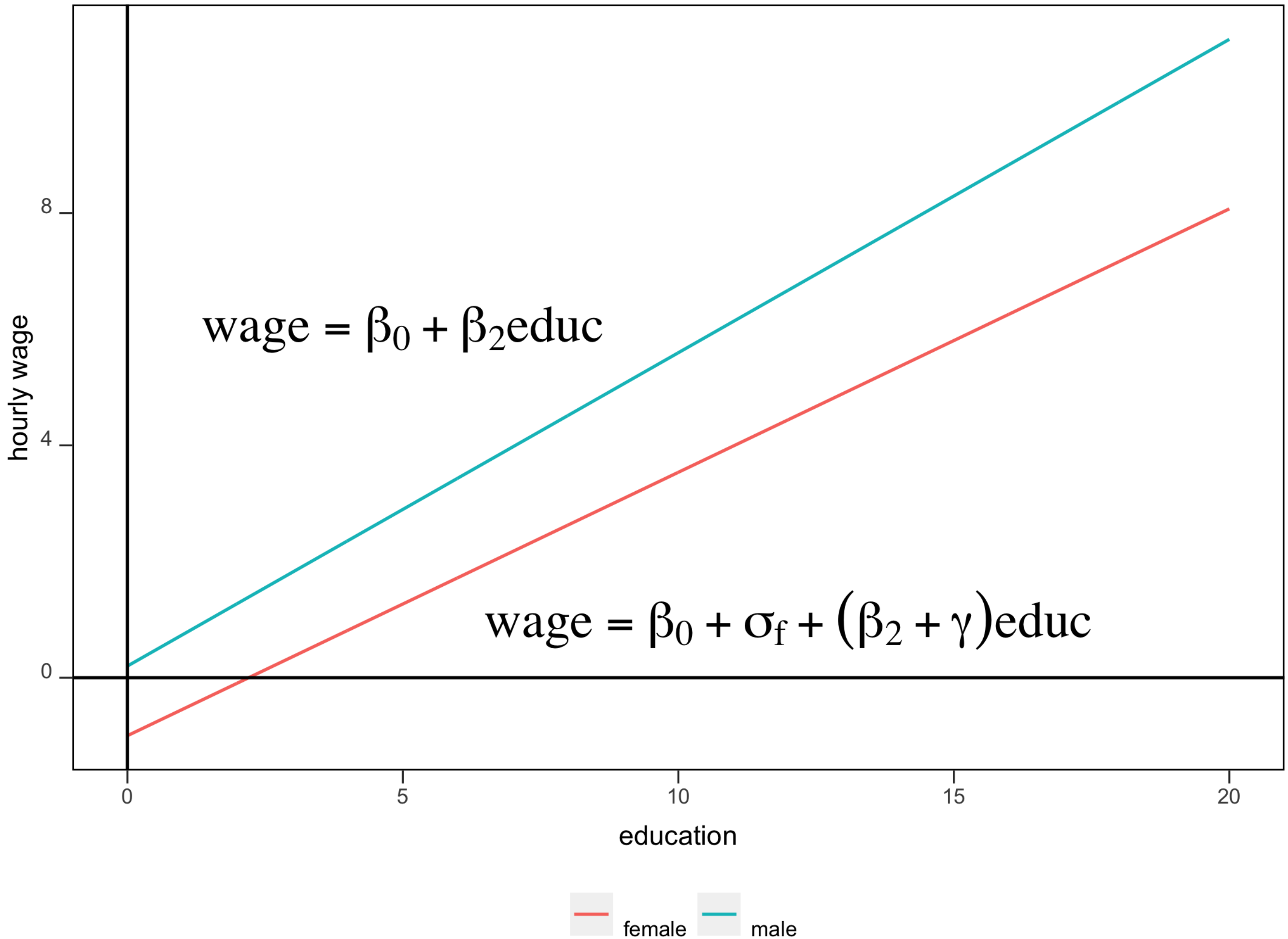
Example using R

```
reg_di <- lm(wage ~ female + educ + I(female * educ), data = wage)
reg_di
```

```
##
## Call:
## lm(formula = wage ~ female + educ + I(female * educ), data = wage)
##
## Coefficients:
##      (Intercept)            female              educ  I(female * educ)
##           0.2005           -1.1985            0.5395           -0.0860
```

Interpretation

The marginal benefit of education is 0.086 ($/hour) less for females workers than for male workers on average.



hourly wage
8
4
0
0
5
10
15
20
education
female
male
$wage = \beta_0 + \beta_2 educ$
$wage = \beta_0 + \sigma_f + (\beta_2 + \gamma) educ$

Categorical variable: more than two states

Issue

- Consider a variable called $degree$ which has three status values: college, master, and doctor.
- Unlike a binary variable, there are three status values.
- How do we include a categorical variable like this in a model?

What do we do about this?

You can create three dummy variables likes below:

- `college`: 1 if the highest degree is college, 0 otherwise
- `master`: 1 if the highest degree is Master's, 0 otherwise
- `doctor`: 1 if the highest degree is Ph.D., 0 otherwise

You then include two (the number of status values - 1) of the three dummy variables:

Model

$$wage = \beta_0 + \sigma_m master + \sigma_d doctor + \beta_1 educ + u$$

- [college]: $E[wage|master = 0, doctor = 0, educ] = \beta_0 + \beta_1 educ$
- [master]: $E[wage|master = 1, doctor = 0, educ] = \beta_0 + \sigma_m + \beta_1 educ$
- [doctor]: $E[wage|master = 0, doctor = 1, educ] = \beta_0 + \sigma_d + \beta_1 educ$

Interpretation

σ_m: the impact of having a MS degree relative to having a college degree

σ_d: the impact of having a Ph.D. degree relative to having a college degree

Important

The omitted category (here, `college`) becomes the baseline.

Structural differences across groups

Definition

Structural difference refers to the fundamental differences in the model of a phenomenon in the population:

Example

Male: $cumgpa = \alpha_0 + \alpha_1 sat + \alpha_2 hsperc + \alpha_3 tothrs + u$

Female: $cumgpa = \beta_0 + \beta_1 sat + \beta_2 hsperc + \beta_3 tothrs + u$

- $cumgpa$: college grade points averages for male and female college athletes
- sat: SAT score
- $hsperc$: high school rank percentile
- $tothrs$: total hours of college courses

In this example,

$cumgpa$ are determined in a fundamentally different manner between female and male students.

You do not want to run a single regression that fits a single model for both female and male students.

What to do?

If you suspect that the underlying process of how the dependent variable is determined vary across groups, then you should test that hypothesis!

To do so,

You estimate the model that allows to estimate separate models across groups within a single regression analysis.

$$cumgpa = \beta_0 + \sigma_0 female + \beta_1 sat + \sigma_1 (sat \times female)$$

$$+\beta_2 hsperc + \sigma_2 (hsperc \times female)$$

$$+\beta_3 tothrs + \sigma_3 (tothrs \times female) + u$$

The flexible model

$$cumgpa = \beta_0 + \sigma_0 female + \beta_1 sat + \sigma_1 (sat \times female)$$
$$+\beta_2 hsperc + \sigma_2 (hsperc \times female)$$
$$+\beta_3 tothrs + \sigma_3 (tothrs \times female) + u$$

Male

$$E[cumgpa] = \beta_0 + \beta_1 sat + \beta_2 hsperc + \beta_3 tothrs$$

Feale

$$E[cumgpa] = (\beta_0 + \sigma_0) + (\beta_1 + \sigma_1) sat + (\beta_2 + \sigma_2) hsperc + (\beta_3 + \sigma_3) tothrs$$

Interpretation

- βs are commonly shared by female and male students
- σs capture the differences between female and male students

Hypothesis (verbal) The model of GPA for male and female students are not structurally different.

Null Hypothesis

$$H_0 : \quad \sigma_0 = 0, \;\; \sigma_1 = 0, \;\; \sigma_2 = 0, \;\; \text{and} \;\; \sigma_3 = 0$$

Question

What test do we do? t-test or F-test?

R code

Run the unrestricted model with all the interaction terms:

```
gpa <- read.dta13("GPA3.dta") %>%
    filter(!is.na(ctothrs)) %>%
    #--- create interaction terms ---#
    mutate(
      female_sat:=female*sat,
      female_hsperc:=female*hsperc,
      female_tothrs:=female*tothrs
      )

#--- regression with female dummy ---#
reg_full <- feols(cumgpa~female
  +sat+female_sat
  +hsperc+female_hsperc
  +tothrs+female_tothrs,
  data=gpa
)
```

What do you see?

- None of the variables that involve $female$ are statistically significant at the 5% level individually.
- Does this mean that $male$ and $female$ students have the same regression function?
- No, we are testing the joint significance of the coefficients. We need to do an F-test!

	Model 1
(Intercept)	1.481***
	(0.207)
female	-0.353
	(0.411)
female_hsperc	-0.001
	(0.003)
female_sat	0.001*
	(0.000)
female_tothrs	-0.000
	(0.002)
hsperc	-0.008***
	(0.001)
sat	0.001***
	(0.000)
tothrs	0.002***
	(0.001)
* p < 0.1, ** p < 0.05, *** p < 0.01	

```r
linearHypothesis(reg_full,
  c(
    "female = 0",
    "female_hsperc = 0",
    "female_sat = 0",
    "female_tothrs = 0"
  )
)
```

```
## Linear hypothesis test
##
## Hypothesis:
## female = 0
## female_hsperc = 0
## female_sat = 0
## female_tothrs = 0
##
## Model 1: restricted model
## Model 2: cumgpa ~ female + sat + female_sat + hsperc + female_hsperc +
##     tothrs + female_tothrs
##
##   Df  Chisq Pr(>Chisq)
## 1
## 2  4 32.716  1.365e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

More on R^2

Goodness of fit: R^2

Important

Small value of R^2 does not mean the end of the world (In fact, we could not care less about it.)

Example

$$ecolabs = \beta_0 + \beta_1 regprc + \beta_2 ecoprc$$

- $ecolabs$: the (hypothetical) pounds of ecologically friendly (ecolabled) apples a family would demand
- $regprc$: prices of regular apples
- $ecoprc$: prices of the hypothetical ecolabled apples

Key

- The data was obtained via survey and $ecoprc$ was set randomly (So, we know $E[u|x] = 0$).
- The objective of the study is to understand the impact of the price of ecolabled apple on the demand for ecolabled apples.

	Dependent variable:
	ecolbs
regprc	3.029***
	(0.711)
ecoprc	-2.926***
	(0.588)
Constant	1.965***
	(0.380)
Observations	660
R^2	0.036

Suppose you are challenged by somebody who claim that your regression is not good because the R^2 is tiny. How would your respond to his/her attack?

Scaling

Questions

What happens if you scale up/down variables used in regression?

- coefficients
- standard errors
- t-statistics
- R^2

```r
#--- regression with female dummy ---#
reg_no_scale <- lm(wage ~ female + educ, data = wage)
reg_scale <- lm(wage ~ female + I(educ * 12), data = wage)
```

```r
tidy(reg_no_scale)
```

```
## # A tibble: 3 × 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)    0.623    0.673      0.926 3.55e- 1
## 2 female        -2.27     0.279     -8.15  2.76e-15
## 3 educ           0.506    0.0504    10.1   7.56e-22
```

```r
tidy(reg_scale)
```

```
## # A tibble: 3 × 5
##   term          estimate std.error statistic  p.value
##   <chr>            <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)     0.623    0.673       0.926 3.55e- 1
## 2 female         -2.27     0.279      -8.15  2.76e-15
## 3 I(educ * 12)    0.0422   0.00420    10.1   7.56e-22
```

So,

- coefficient: 1/12
- standard error: 1/12
- t-stat: the same

Interpretation

- Regression without scaling

hourly wage increases by 0.506 if education increases by a year

- Regression with scaling (e.g., 48 means 4 years)

hourly wage increases by 0.0422 if education increases by a month

Note

According to the scaled model, hourly wage increases by $0.0422 * 12$ if education increases by a year (12 months).

That is, the estimated marginal impact of education on wage from the scaled model is the same as that from the unscaled model.

Summary

When an independent variable is scaled,

- its coefficient estimate and standard error are going to be scaled up/back to the exact degree the variable is scaled up/back
- t-statistics stays the same (as it should be)
- R^2 stays the same (the model does not improve by simply scaling independent variables)