methods and neural networks and shows various applications of nonlinear models in finance. Chapter 5 is concerned with analysis of high-frequency financial data, the effects of market microstructure, and some applications of high-frequency finance. It shows that nonsynchronous trading and bid–ask bounce can introduce serial correlations in a stock return. It also studies the dynamic of time duration between trades and some econometric models for analyzing transactions data. In Chapter 6, we introduce continuous-time diffusion models and Ito's lemma. Black–Scholes option pricing formulas are derived, and a simple jump diffusion model is used to capture some characteristics commonly observed in options markets. Chapter 7 discusses extreme value theory, heavy-tailed distributions, and their application to financial risk management. In particular, it discusses various methods for calculating value at risk and expected shortfall of a financial position. Chapter 8 focuses on multivariate time series analysis and simple multivariate models with emphasis on the lead–lag relationship between time series. The chapter also introduces cointegration, some cointegration tests, and threshold cointegration and applies the concept of cointegration to investigate arbitrage opportunity in financial markets, including pairs trading. Chapter 9 discusses ways to simplify the dynamic structure of a multivariate series and methods to reduce the dimension. It introduces and demonstrates three types of factor model to analyze returns of multiple assets. In Chapter 10, we introduce multivariate volatility models, including those with time-varying correlations, and discuss methods that can be used to reparameterize a conditional covariance matrix to satisfy the positiveness constraint and reduce the complexity in volatility modeling. Chapter 11 introduces state-space models and the Kalman filter and discusses the relationship between state-space models and other econometric models discussed in the book. It also gives several examples of financial applications. Finally, in Chapter 12, we introduce some Markov chain Monte Carlo (MCMC) methods developed in the statistical literature and apply these methods to various financial research problems, such as the estimation of stochastic volatility and Markov switching models.

The book places great emphasis on application and empirical data analysis. Every chapter contains real examples and, in many occasions, empirical characteristics of financial time series are used to motivate the development of econometric models. Computer programs and commands used in data analysis are provided when needed. In some cases, the programs are given in an appendix. Many real data sets are also used in the exercises of each chapter.

1.1 ASSET RETURNS

Most financial studies involve returns, instead of prices, of assets. Campbell, Lo, and MacKinlay (1997) give two main reasons for using returns. First, for average investors, return of an asset is a complete and scale-free summary of the investment opportunity. Second, return series are easier to handle than price series because the former have more attractive statistical properties. There are, however, several definitions of an asset return.

Let P_t be the price of an asset at time index t. We discuss some definitions of returns that are used throughout the book. Assume for the moment that the asset pays no dividends.

One-Period Simple Return

Holding the asset for one period from date $t-1$ to date t would result in a *simple gross return*:

$$1 + R_t = \frac{P_t}{P_{t-1}} \qquad \text{or} \qquad P_t = P_{t-1}(1 + R_t). \tag{1.1}$$

The corresponding one-period *simple net return* or *simple return* is

$$R_t = \frac{P_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}}. \tag{1.2}$$

Multiperiod Simple Return

Holding the asset for k periods between dates $t-k$ and t gives a k-period simple gross return:

$$\begin{aligned} 1 + R_t[k] = \frac{P_t}{P_{t-k}} &= \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \cdots \times \frac{P_{t-k+1}}{P_{t-k}} \\ &= (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}) \\ &= \prod_{j=0}^{k-1} (1 + R_{t-j}). \end{aligned}$$

Thus, the k-period simple gross return is just the product of the k one-period simple gross returns involved. This is called a compound return. The k-period simple net return is $R_t[k] = (P_t - P_{t-k})/P_{t-k}$.

In practice, the actual time interval is important in discussing and comparing returns (e.g., monthly return or annual return). If the time interval is not given, then it is implicitly assumed to be one year. If the asset was held for k years, then the annualized (average) return is defined as

$$\text{Annualized}\{R_t[k]\} = \left[\prod_{j=0}^{k-1} (1 + R_{t-j})\right]^{1/k} - 1.$$

This is a geometric mean of the k one-period simple gross returns involved and can be computed by

$$\text{Annualized}\{R_t[k]\} = \exp\left[\frac{1}{k}\sum_{j=0}^{k-1} \ln(1 + R_{t-j})\right] - 1,$$

where $\exp(x)$ denotes the exponential function and $\ln(x)$ is the natural logarithm of the positive number x. Because it is easier to compute arithmetic average than geometric mean and the one-period returns tend to be small, one can use a first-order Taylor expansion to approximate the annualized return and obtain

$$\text{Annualized}\,\{R_t[k]\} \approx \frac{1}{k}\sum_{j=0}^{k-1} R_{t-j}. \tag{1.3}$$

Accuracy of the approximation in Eq. (1.3) may not be sufficient in some applications, however.

Continuous Compounding

Before introducing continuously compounded return, we discuss the effect of compounding. Assume that the interest rate of a bank deposit is 10% per annum and the initial deposit is \$1.00. If the bank pays interest once a year, then the net value of the deposit becomes $\$1(1+0.1) = \$1.1$ one year later. If the bank pays interest semiannually, the 6-month interest rate is $10\%/2 = 5\%$ and the net value is $\$1(1+0.1/2)^2 = \$1.1025$ after the first year. In general, if the bank pays interest m times a year, then the interest rate for each payment is $10\%/m$ and the net value of the deposit becomes $\$1(1+0.1/m)^m$ one year later. Table 1.1 gives the results for some commonly used time intervals on a deposit of \$1.00 with interest rate of 10% per annum. In particular, the net value approaches \$1.1052, which is obtained by exp(0.1) and referred to as the result of continuous compounding. The effect of compounding is clearly seen.

In general, the net asset value A of continuous compounding is

$$A = C\,\exp(r \times n), \tag{1.4}$$

where r is the interest rate per annum, C is the initial capital, and n is the number of years. From Eq. (1.4), we have

$$C = A\,\exp(-r \times n), \tag{1.5}$$

TABLE 1.1 Illustration of Effects of Compounding: Time Interval Is 1 Year and Interest Rate Is 10% per Annum

Type	Number of Payments	Interest Rate per Period	Net Value
Annual	1	0.1	\$1.10000
Semiannual	2	0.05	\$1.10250
Quarterly	4	0.025	\$1.10381
Monthly	12	0.0083	\$1.10471
Weekly	52	0.1/52	\$1.10506
Daily	365	0.1/365	\$1.10516
Continuously	∞		\$1.10517

which is referred to as the *present value* of an asset that is worth A dollars n years from now, assuming that the continuously compounded interest rate is r per annum.

Continuously Compounded Return

The natural logarithm of the simple gross return of an asset is called the continuously compounded return or *log return*:

$$r_t = \ln(1 + R_t) = \ln \frac{P_t}{P_{t-1}} = p_t - p_{t-1}, \tag{1.6}$$

where $p_t = \ln(P_t)$. Continuously compounded returns r_t enjoy some advantages over the simple net returns R_t. First, consider multiperiod returns. We have

$$\begin{aligned} r_t[k] &= \ln(1 + R_t[k]) = \ln[(1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1})] \\ &= \ln(1 + R_t) + \ln(1 + R_{t-1}) + \cdots + \ln(1 + R_{t-k+1}) \\ &= r_t + r_{t-1} + \cdots + r_{t-k+1}. \end{aligned}$$

Thus, the continuously compounded multiperiod return is simply the sum of continuously compounded one-period returns involved. Second, statistical properties of log returns are more tractable.

Portfolio Return

The simple net return of a portfolio consisting of N assets is a weighted average of the simple net returns of the assets involved, where the weight on each asset is the percentage of the portfolio's value invested in that asset. Let p be a portfolio that places weight w_i on asset i. Then the simple return of p at time t is $R_{p,t} = \sum_{i=1}^{N} w_i R_{it}$, where R_{it} is the simple return of asset i.

The continuously compounded returns of a portfolio, however, do not have the above convenient property. If the simple returns R_{it} are all small in magnitude, then we have $r_{p,t} \approx \sum_{i=1}^{N} w_i r_{it}$, where $r_{p,t}$ is the continuously compounded return of the portfolio at time t. This approximation is often used to study portfolio returns.

Dividend Payment

If an asset pays dividends periodically, we must modify the definitions of asset returns. Let D_t be the dividend payment of an asset between dates $t - 1$ and t and P_t be the price of the asset at the end of period t. Thus, dividend is not included in P_t. Then the simple net return and continuously compounded return at time t become

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1, \qquad r_t = \ln(P_t + D_t) - \ln(P_{t-1}).$$

Excess Return

Excess return of an asset at time t is the difference between the asset's return and the return on some reference asset. The reference asset is often taken to be riskless such as a short-term U.S. Treasury bill return. The simple excess return and log excess return of an asset are then defined as

$$Z_t = R_t - R_{0t}, \qquad z_t = r_t - r_{0t}, \tag{1.7}$$

where R_{0t} and r_{0t} are the simple and log returns of the reference asset, respectively. In the finance literature, the excess return is thought of as the payoff on an arbitrage portfolio that goes long in an asset and short in the reference asset with no net initial investment.

Remark. A long financial position means owning the asset. A short position involves selling an asset one does not own. This is accomplished by borrowing the asset from an investor who has purchased it. At some subsequent date, the short seller is obligated to buy exactly the same number of shares borrowed to pay back the lender. Because the repayment requires equal shares rather than equal dollars, the short seller benefits from a decline in the price of the asset. If cash dividends are paid on the asset while a short position is maintained, these are paid to the buyer of the short sale. The short seller must also compensate the lender by matching the cash dividends from his own resources. In other words, the short seller is also obligated to pay cash dividends on the borrowed asset to the lender. □

Summary of Relationship

The relationships between simple return R_t and continuously compounded (or log) return r_t are

$$r_t = \ln(1 + R_t), \qquad R_t = e^{r_t} - 1.$$

If the returns R_t and r_t are in percentages, then

$$r_t = 100 \ln\left(1 + \frac{R_t}{100}\right), \qquad R_t = 100\left(e^{r_t/100} - 1\right).$$

Temporal aggregation of the returns produces

$$1 + R_t[k] = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}),$$
$$r_t[k] = r_t + r_{t-1} + \cdots + r_{t-k+1}.$$

If the continuously compounded interest rate is r per annum, then the relationship between present and future values of an asset is

$$A = C \exp(r \times n), \qquad C = A \exp(-r \times n).$$

Example 1.1. If the monthly log return of an asset is 4.46%, then the corresponding monthly simple return is $100[\exp(4.46/100) - 1] = 4.56\%$. Also, if the monthly log returns of the asset within a quarter are 4.46%, −7.34%, and 10.77%, respectively, then the quarterly log return of the asset is $(4.46 - 7.34 + 10.77)\% = 7.89\%$.

1.2 DISTRIBUTIONAL PROPERTIES OF RETURNS

To study asset returns, it is best to begin with their distributional properties. The objective here is to understand the behavior of the returns across assets and over time. Consider a collection of N assets held for T time periods, say, $t = 1, \ldots, T$. For each asset i, let r_{it} be its log return at time t. The log returns under study are $\{r_{it}; i = 1, \ldots, N; t = 1, \ldots, T\}$. One can also consider the simple returns $\{R_{it}; i = 1, \ldots, N; t = 1, \ldots, T\}$ and the log excess returns $\{z_{it}; i = 1, \ldots, N; t = 1, \ldots, T\}$.

1.2.1 Review of Statistical Distributions and Their Moments

We briefly review some basic properties of statistical distributions and the moment equations of a random variable. Let R^k be the k-dimensional Euclidean space. A point in R^k is denoted by $\boldsymbol{x} \in R^k$. Consider two random vectors $\boldsymbol{X} = (X_1, \ldots, X_k)'$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_q)'$. Let $P(\boldsymbol{X} \in A, \boldsymbol{Y} \in B)$ be the probability that $\boldsymbol{X}$ is in the subspace $A \subset R^k$ and $\boldsymbol{Y}$ is in the subspace $B \subset R^q$. For most of the cases considered in this book, both random vectors are assumed to be continuous.

Joint Distribution

The function

$$F_{X,Y}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\theta}) = P(\boldsymbol{X} \le \boldsymbol{x}, \boldsymbol{Y} \le \boldsymbol{y}; \boldsymbol{\theta}),$$

where $\boldsymbol{x} \in R^p$, $\boldsymbol{y} \in R^q$, and the inequality $\le$ is a component-by-component operation, is a joint distribution function of $\boldsymbol{X}$ and $\boldsymbol{Y}$ with parameter $\boldsymbol{\theta}$. Behavior of $\boldsymbol{X}$ and $\boldsymbol{Y}$ is characterized by $F_{X,Y}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\theta})$. If the joint probability density function $f_{x,y}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\theta})$ of $\boldsymbol{X}$ and $\boldsymbol{Y}$ exists, then

$$F_{X,Y}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\theta}) = \int_{-\infty}^{\boldsymbol{x}} \int_{-\infty}^{\boldsymbol{y}} f_{x,y}(\boldsymbol{w}, \boldsymbol{z}; \boldsymbol{\theta})\, d\boldsymbol{z}\, d\boldsymbol{w}.$$

In this case, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are continuous random vectors.

Marginal Distribution

The marginal distribution of $\boldsymbol{X}$ is given by

$$F_X(\boldsymbol{x}; \boldsymbol{\theta}) = F_{X,Y}(\boldsymbol{x}, \infty, \cdots, \infty; \boldsymbol{\theta}).$$

Thus, the marginal distribution of $\boldsymbol{X}$ is obtained by integrating out $\boldsymbol{Y}$. A similar definition applies to the marginal distribution of $\boldsymbol{Y}$.

If $k = 1$, X is a scalar random variable and the distribution function becomes

$$F_X(x) = P(X \le x; \boldsymbol{\theta}),$$