Elasticities

With a one level nested logit, elasticities calculated from quantities are given by:

$$e_{jj} = \alpha \left(\frac{1}{1-\sigma} - \frac{\sigma}{(1-\sigma)} \frac{q_j}{q_{g_j}} - \frac{q_j}{m_1} \right) p_j$$

$$e_{jk} = -\alpha \left(\frac{\sigma}{(1-\sigma)} \frac{q_j}{q_{g_j}} + \frac{q_j}{m_1} \right) p_j$$

Unweighted average elasticities are then

$$\bar{e}_{jj} = \alpha \left(\frac{1}{1-\sigma} \frac{1}{J} \sum_j p_j - \frac{\sigma}{(1-\sigma)} \frac{1}{J} \sum_j \frac{p_j q_j}{q_{g_j}} - \frac{1}{m_1} \frac{1}{J} \sum_j p_j q_j \right)$$

$$\bar{e}_{jk} = -\alpha \left(\frac{\sigma}{(1-\sigma)} \frac{1}{J} \sum_j p_j \frac{q_j}{q_{\text{tot}}} + \frac{1}{m_1} \frac{1}{J} \sum_j p_j q_j \right)$$

Defining $\bar{p}$, A and B as the three sums above

$$C_1 = \frac{\alpha}{1-\sigma} = \frac{\bar{e}_{jj} - \bar{e}_{jk}}{\bar{p}}$$

and thus

$$\bar{e}_{jk} = -\sigma C_1 A - (1-\sigma) C_1 B$$
$$\bar{e}_{jk} + C_1 B = \sigma C_1 (B - A)$$

The one level nested logit parameters can thus be expressed in terms of shares

$$\sigma = \frac{\bar{e}_{jk} + C_1 B}{C_1 (B - A)}$$

$$\alpha = (1-\sigma) C_1 = \frac{C_1 A - \bar{e}_{jk}}{B - A}$$

Two level nests

Basing calculations on shares in the two level unit demand logit, we have:

$$\bar{e}_{jj} = \alpha \left(\frac{1}{1-\sigma_1} \bar{p} - \left(\frac{1}{1-\sigma_1} - \frac{1}{1-\sigma_2} \right) p \cdot s_{hg}/J - \frac{\sigma_2}{1-\sigma_2} p \cdot s_g/J - p \cdot s/J \right)$$

$$\bar{e}_{jk} = -\alpha \left(\left(\frac{1}{1-\sigma_1} - \frac{1}{1-\sigma_2} \right) p \cdot s_{hg}/J + \frac{\sigma_2}{1-\sigma_2} p \cdot s_g/J + p \cdot s/J \right)$$

$$\bar{e}_{jl} = -\alpha \left(\frac{\sigma_2}{1-\sigma_2} p \cdot s_{hg}/J + p \cdot s/J \right)$$

Let $Z_g = p \cdot s_g/J$, $Z_{hg} = p \cdot s_{hg}/J$, and $Z_j = p \cdot s/J$.

$$\bar{e}_{jj} = \alpha \left(\frac{1}{1-\sigma_1}\bar{p} - \left(\frac{1}{1-\sigma_1} - \frac{1}{1-\sigma_2} \right) Z_{hg} - \frac{\sigma_2}{1-\sigma_2} Z_g - Z_j \right)$$

We now have

$$C_1 = \frac{\alpha}{1-\sigma_1} = \frac{\bar{e}_{jj} - \bar{e}_{jk}}{\bar{p}}$$

and

$$\bar{e}_{jl} - \bar{e}_{jk} = \alpha \left(\frac{1}{1-\sigma_1} - \frac{1}{1-\sigma_2} \right) Z_{hg}$$

$$\bar{e}_{jl} - \bar{e}_{jk} = \left(C_1 - \frac{\alpha}{1-\sigma_2} \right) Z_{hg}$$

$$C_2 = \frac{\alpha}{1-\sigma_2} = C_1 - \frac{\bar{e}_{jl} - \bar{e}_{jk}}{Z_{hg}}$$

$$\bar{e}_{jl} = -\sigma_2 \frac{\alpha}{1-\sigma_2} Z_g - \frac{\alpha}{1-\sigma_2} (1-\sigma_2) Z_j$$

$$\bar{e}_{jl} = -\sigma_2 C_2 Z_g - C_2 (1-\sigma_2) Z_j$$

$$\bar{e}_{jl}/C_2 + Z_j = -\sigma_2 Z_{gg} + \sigma_2 Z_j$$

Thus the solution is

$$\sigma_2 = \frac{\bar{e}_{jl}/C_2 - Z_j}{Z_j - Z_g}$$

$$\alpha = (1-\sigma_2) C_2$$

$$\alpha = (1-\sigma_1) C_1 = (1-\sigma_2) C_2$$

$$\sigma_1 = 1 - \frac{(1-\sigma_2) C_2}{C_1}$$

CES

For CES demand, the C_1 and Z_* parameters above have to be modified as follows

$$\bar{e}_{jj} = \alpha \left(\frac{1}{1-\sigma_1} - \left(\frac{1}{1-\sigma_1} - \frac{1}{1-\sigma_2} \right) s_{hg}/J - \frac{\sigma_2}{1-\sigma_2} s_g/J - s/J \right) - 1$$

In this case

$$C_1 = \bar{e}_{jj} - \bar{e}_{jk} + 1 = \frac{\alpha}{1-\sigma}$$

Also $Z_{jh} = \sum_j s_{jh}/J$, $Z_{hg} = \sum_j s_{hg}/J$, and $Z_j = \sum_j s_j/J$. In the formulas above $\bar{p}$ is replaced with 1.

Market size and outer nest

Here we show that a one level nested logit with an outer nest that includes all J products corresponds to an unnested logit with a different α and market size m.

Shares are given by $s_j = q_j/m$ and group shares $s_{j|g} = q_j/q_{tot}$. Own and cross price elasticities in the one level nested logit are given by

$$e_{jj} = -\alpha_1 \left(\frac{1}{1-\sigma} - \frac{\sigma}{(1-\sigma)} \frac{q_j}{q_{\text{tot}}} - \frac{q_j}{m_1} \right) p_j$$

and

$$e_{jk} = \alpha_1 \left(\frac{\sigma q_j}{(1-\sigma) q_{\text{tot}}} + \frac{q_j}{m_1} \right) p_j$$

For the unnested logit elasticities depend on α_2 and m_2:

$$e_{jj} = -\alpha_2 \left(1 - \frac{q_j}{m_2} \right) p_j$$

and

$$e_{jk} = \alpha_2 \frac{q_j}{m_2} p_j$$

Elasticities are equal for all products if

$$\alpha_2 = \frac{\alpha_1}{1-\sigma}$$

and

$$m_2 = \frac{m_1}{1 - \sigma + \frac{m_1}{q_{\text{tot}}} \sigma}$$

We have $m_2 > q_{tot}$ if $m_1 > q_{tot}$ and $\sigma < 1$. Thus for any set of nested logit parameters (α_1, σ, m_1) there exist a unique set(α_2, m_2) such that all elasiticities are equal. The relationship only depends on total quantities q_{tot}, not on individual prices or quantites.