Minimax Adaptive Control for a Finite Set of Linear Systems

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Abstract

An adaptive controller with bounded l_2-gain from disturbances to errors is derived for linear time-invariant systems with uncertain parameters restricted to a finite set. The gain bound refers to the closed loop system, including the non-linear learning procedure. As a result, robustness to unmodelled dynamics (possibly nonlinear and infinite-dimensional) follows from the small gain theorem. The approach is based on a new zero-sum dynamic game formulation, which optimizes the trade-off between exploration and exploitation. An explicit upper bound on the optimal value function is stated in terms of semi-definite programming and a corresponding simple formula for an adaptive controller achieving the upper bound is given. Once the uncertain parameters have been sufficiently estimated, the controller behaves like standard H_∞ optimal control.

Keywords: adaptive control, real-time learning

1. Introduction

The history of adaptive control dates back at least to aircraft autopilot development in the 1950s. Following the landmark paper Åström and Wittenmark (1973), a surge of research activity during the 1970s derived conditions for convergence, stability, robustness and performance under various assumptions. For example, Ljung (1977) analysed adaptive algorithms using averaging, Goodwin et al. (1981) derived an algorithm that gives mean square stability with probability one, while Guo (1995) gave conditions for the optimal asymptotic rate of convergence. On the other hand, conditions that may cause instability were studied in Egardt (1979), Ioannou and Kokotovic (1984) and Rohrs et al. (1985). Altogether, the subject has a rich history documented in numerous textbooks, such as Åström and Wittenmark (2013), Goodwin and Sin (2014), Narendra and Annaswamy (2012), Sastry and Bodson (2011) and Astolfi et al. (2007). In this paper, the focus is on worst-case models for disturbances and uncertain parameters, as discussed in Cusumano and Poolla (1988); Sun and Ioannou (1987); Vinnicombe (2004) and Megretski (2004). The "minimax adaptive" paradigm illustrated in Figure 1 was introduced for linear systems in Didinsky and Basar (1994) and nonlinear systems in Pan and Basar (1998).

More recently, there has been an explosion of research interest in the boundary between machine learning, system identification and adaptive control. For a review, see for example Matni et al. (2019). Most of the studies are carried out in a stochastic setting, but recently works connecting to H_∞ control have started to appear Dean et al. (2018). There is also recent work connecting adversarial learning and minimax regret analysis to robust control Zhang et al. (2020). However, it
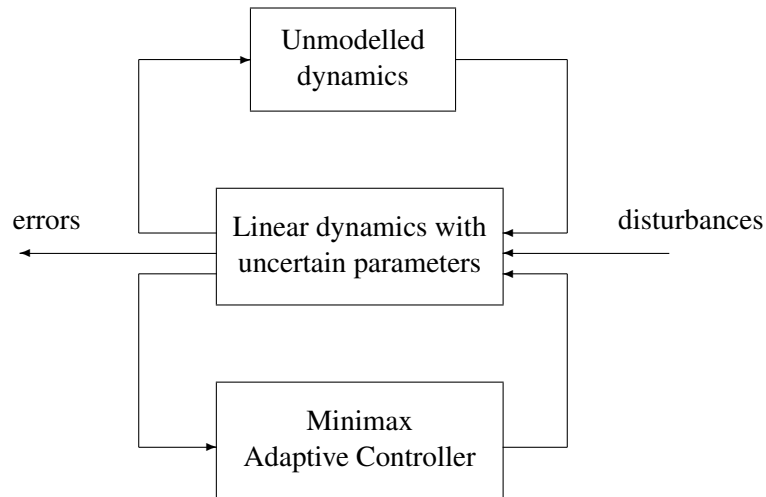
Figure 1: Given a finite set of stabilizable linear models, the objective of this paper is to construct an adaptive controller that minimizes the l_2-gain from disturbances to errors under worst case values of the uncertain parameters. The gain bound gives a guaranteed robustness to unmodelled (possibly nonlinear and infinite-dimensional) dynamics. Optimal trade-off between exploration and exploitation is needed for minimization of the gain.

is worth noting that unlike most recent contributions the minimax approach of this paper will not assume that a stabilizing controller is known in advance.

This paper is an extension and generalization of the approach presented in Rantzer (2020). The outline is as follows: After some notation in sections 2, we give the problem formulation in section 3, followed by preliminary results on minimax dynamic programming in section 4. The main result is first presented in section 5 for the special case of input sign uncertainty with a double integrator as an illustrating example, then proved for the general case in section 6. Some calculations are deferred to an appendix.

2. Notation

The set of $n \times m$ matrices with real coefficients is denoted $\mathbb{R}^{n \times m}$. The transpose of a matrix A is denoted $A^\top$. For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, we write $A \succ 0$ to say that A is positive definite, while $A \succeq 0$ means positive semi-definite. For $A, B \in \mathbb{R}^{n \times m}$, the expression $\langle A, B \rangle$ denotes the trace of $A^\top B$. Given $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$, the notation $|x|_A^2$ means $x^\top A x$. Similarly, given $B \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{n \times n}$, the trace of $B^\top A B$ is denoted $\|B\|_A^2$.

3. Minimax Adaptive Control

Let $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ be positive definite. Given a compact set $\mathcal{M} \subset \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ and a number $\gamma > 0$, we are interested to compute

$$J_*(x_0) := \inf_{\mu} \underbrace{\sup_{w,A,B,N} \sum_{t=0}^{N} \left(|x_t|_Q^2 + |u_t|_R^2 - \gamma^2 |w_t|^2 \right)}_{J_\mu(x_0)}, \tag{1}$$

where $(A, B) \in \mathcal{M}$, $w_t \in \mathbb{R}^n$, $N \geq 0$ and the sequences x and u are generated according to

$$x_{t+1} = Ax_t + Bu_t + w_t \qquad\qquad t \geq 0 \tag{2}$$

$$u_t = \mu_t(x_0, \ldots, x_t, u_0, \ldots, u_{t-1}). \tag{3}$$

The problem above can be viewed as a dynamic game, where the μ-player tries to minimize the cost, while the (w, A, B)-player tries to maximize it. If it wasn't for the uncertainty in (A, B), this would be the standard game formulation of H_∞ optimal control Basar and Bernhard (1995). In our formulation, the maximizing player can choose not only w, but also the matrices A, B. The matrices are unknown but constant, so an optimal feedback law tends to "learn" A and B early on, in order to exploit this knowledge later. Such nonlinear adaptive controllers can stabilize and optimize the behavior also when no linear controller can simultaneously stabilize (2) for all $(A, B) \in \mathcal{M}$.

To accommodate the uncertainty in (A, B) when deciding u_t, it is natural for the controller to consider historical data collected in the matrix

$$Z_t = \sum_{\tau=0}^{t-1} \begin{bmatrix} -x_{\tau+1} \\ x_\tau \\ u_\tau \end{bmatrix} \begin{bmatrix} -x_{\tau+1} \\ x_\tau \\ u_\tau \end{bmatrix}^\top \tag{4}$$

since this gives $\left\| \begin{bmatrix} I & A & B \end{bmatrix}^\top \right\|_{Z_t}^2 = \sum_{\tau=0}^{t-1} |w_\tau|^2$. In fact, we will prove in Theorem 1 that knowledge of x_t and Z_t is sufficient for optimal control. The result will be based on the following reformulated problem: Given $Q \succ 0, R \succ 0, \gamma > 0$, a set $\mathcal{M}$ and the system

$$\begin{cases} x_{t+1} = v_t \\ Z_{t+1} = Z_t + \begin{bmatrix} -v_t \\ x_t \\ u_t \end{bmatrix} \begin{bmatrix} -v_t \\ x_t \\ u_t \end{bmatrix}^\top, \qquad Z_0 = 0, \end{cases} \tag{5}$$

find a control law

$$u_t = \eta(x_t, Z_t) \tag{6}$$

to minimize

$$\inf_{\eta} \sup_{v,N} \left[\sum_{t=0}^{N} \left(|x_t|_Q^2 + |u_t|_R^2 \right) - \gamma^2 \min_{(A,B) \in \mathcal{M}} \left\| \begin{bmatrix} I & A & B \end{bmatrix}^\top \right\|_{Z_{N+1}}^2 \right] \tag{7}$$

when x, u, Z are generated from v according to (5)-(6).

In this formulation, the unknown (A, B) does not appear in the dynamics, only in the penalty of the final state. As a consequence, no past states are needed in the control law (6), only the current state (x_t, Z_t). In fact, the problem is a standard zero-sum dynamic game Basar and Olsder (1999), which will next be addressed by dynamic programming.

4. Minimax Dynamic Programming

Define the operators $\mathcal{F}$ and $\mathcal{F}_u$ by

$$\mathcal{F}V(x,Z) := \min_u \underbrace{\max_v \left\{ |x|_Q^2 + |u|_R^2 + V\left(v, Z + \begin{bmatrix} -v \\ x \\ u \end{bmatrix} \begin{bmatrix} -v \\ x \\ u \end{bmatrix}^\top \right) \right\}}_{\mathcal{F}_u V(x,Z)}.$$

Then the following result holds:

Theorem 1 *Given $Q, R, \mathcal{M}$, define the operator $\mathcal{F}$ as above and $V_0, V_1, V_2 \ldots$ according to the iteration*

$$V_0(x,Z) = -\gamma^2 \min_{(A,B) \in \mathcal{M}} \left\| \begin{bmatrix} I & A & B \end{bmatrix}^\top \right\|_Z^2 \tag{8}$$

$$V_{k+1}(x,Z) = \mathcal{F}V_k(x,Z). \tag{9}$$

The expressions (1) and (7) have the same value. The value is finite if and only if the sequence $\{V_k(x,0)\}_{k=0}^\infty$ is upper bounded. If so, the limit $V_* := \lim_{k \to \infty} V_k$ exists and $J_*(x_0) = V_*(x_0, 0)$. Defining $\eta(x,Z)$ as the minimizing value of u in the expression for $\mathcal{F}V_*(x,Z)$ gives an optimal η^* for (7), while the control law μ^* defined by

$$\mu_t^*(x_0, \ldots, x_t, u_0, \ldots, u_{t-1}) := \eta^* \left(x_t, \sum_{\tau=0}^{t-1} \begin{bmatrix} -x_{\tau+1} \\ x_\tau \\ u_\tau \end{bmatrix} \begin{bmatrix} -x_{\tau+1} \\ x_\tau \\ u_\tau \end{bmatrix}^\top \right) \tag{10}$$

is optimal for (1). Moreover, if there exists a function $\bar{V}$ satisfying $\bar{V} \geq V_0$ and $\mathcal{F}_{\bar{\eta}} \bar{V} \leq \bar{V}$, then the control law $\bar{\mu}$ defined by $u_t = \bar{\eta}(x_t, Z_t)$ satisfies $J_{\bar{\mu}}(x_0) \leq \bar{V}(x_0, 0)$.

Proof.

$$V_1(x,Z) \geq \min_u \max_v V_0\left(v, Z + \begin{bmatrix} -v \\ x \\ u \end{bmatrix} \begin{bmatrix} -v \\ x \\ u \end{bmatrix}^\top \right)$$

$$= -\gamma^2 \max_u \min_{A,B,v} \left(|Ax + Bu - v|^2 + \left\| \begin{bmatrix} I & A & B \end{bmatrix}^\top \right\|_Z^2 \right)$$

$$= V_0(x,Z)$$

so $V_1 \geq V_0$ and the sequence $V_0, V_1, V_2, \ldots$ is monotonically non-decreasing.

For any fixed $N \geq 0$, the value of (1) is bounded below by the expression

$$\inf_\mu \sup_{w,A,B} \sum_{t=0}^N \left(|x_t|_Q^2 + |u_t|_R^2 - \gamma^2 |w_t|^2 \right), \tag{11}$$

4

where $(A, B) \in \mathcal{M}$, $w_t \in \mathbb{R}^n$ and the sequences x and u are generated according to (2)-(3). The value of (11) grows monotonically with N and (1) is obtained in the limit. A change of variables with $v_t := x_{t+1}$ and Z_t given by (4) shows that (11) is equal to

$$\inf_{\mu} \sup_{v} \left[\sum_{t=0}^{N} \left(|x_t|_Q^2 + |u_t|_R^2 \right) - \gamma^2 \min_{A,B} \| \begin{bmatrix} I & A & B \end{bmatrix}^\top \|_{Z_{N+1}}^2 \right] \tag{12}$$

where x, Z, u are generated by (5) combined with (3). Standard dynamic programming shows that the value of (12) is $V_{N+1}(x_0, 0)$, where V_k is defined by (8)-(9). This proves that (1) has a finite value if and only if the sequence $\{V_k(x, 0)\}_{k=0}^{\infty}$ is upper bounded. The limit $V_*(x_0, 0)$ is equal to the value of (1). The relationship $V_k(x, Z) \leq V_k(x, 0)$ for $Z \succeq 0$ shows that the limit then exists for all $Z \succeq 0$.

If (7) is finite, then V_k is bounded above by (7), so also $V_* := \lim_{k \to \infty} V_k$ is finite. Conversely, if $V_0 \leq \bar{V} \leq \infty$ and $\mathcal{F}_{\bar{\eta}} \bar{V} \leq \bar{V}$, we may define the sequence $W_0, W_1, W_2, \ldots$ recursively by $W_0 := V_0$ and

$$W_{k+1}(x, Z) := \mathcal{F}_{\bar{\eta}(x,Z)} W_k(x, Z).$$

By dynamic programming,

$$W_N(x, 0) = \max_{v} \left[\sum_{t=0}^{N} \left(|x_t|_Q^2 + |u_t|_R^2 \right) - \gamma^2 \min_{A,B} \| \begin{bmatrix} I & A & B \end{bmatrix}^\top \|_{Z_{N+1}}^2 \right]$$

where x, Z, u are generated by (5) and (6). Hence (7) is bounded above by $\lim_{k \to \infty} W_k(x_0, 0)$. The definitions of V_k and W_k give by induction $\bar{V} \geq W_k \geq V_k$ for all k, so $V_* \leq \lim_{k \to \infty} W_k \leq \bar{V}$. This proves that $V_*(x_0, 0) \leq J_{\bar{\mu}}(x_0) \leq \bar{V}(x_0, 0)$ when the control law $\bar{\mu}$ is defined by $u_t = \bar{\eta}(x_t, Z_t)$. In particular, the control law is optimal if $\bar{V} = V_*$. $\qquad \square$

We conclude the section by pointing out that even though all our results assume full state measurements, they are immediately applicable also to input-output models. For example, the input-output model

$$y_t = -a_1 y_{t-1} - \cdots - a_n y_{t-n} + b_1 u_{t-1} + \cdots + b_n u_{t-n} \tag{13}$$

has the (non-minimal) state realization $x_{t+1} = Ax_t + Bu_t$ where

$$x_t = \begin{bmatrix} y_{t-1} \\ \vdots \\ y_{t-n} \\ u_{t-1} \\ \vdots \\ u_{t-n} \end{bmatrix} \qquad A = \begin{bmatrix} -a_1 & \cdots & -a_n & b_1 & \cdots & b_n \\ 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{14}$$

where all states are known from the inputs and outputs. This also shows that it is not really restrictive to consider B as known. Example 1 will use this state realization for a double integrator where only the output of the second integrator is available for measurement.

5

5. Systems with unknown input direction

The following result gives an explicit expression for an adaptive controller satisfying a pre-specified bound on the L_2-gain for any stabilizable model set of the form

$$\mathcal{M} := \{(A, B), (A, -B))\} \subset \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}.$$

Theorem 2 *Consider $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and positive definite $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$. Suppose there exists $K \in \mathbb{R}^{m \times n}$ and symmetric $P, T \in \mathbb{R}^{n \times n}$ satisfying $0 \prec P \prec T \prec \gamma^2 I$ and*

$$P \succeq Q + K^\top RK + (A - BK)^\top (P^{-1} - \gamma^{-2}I)^{-1}(A - BK) \tag{15}$$

$$T \succeq Q + K^\top RK + (A + BK)^\top (P^{-1} - \gamma^{-2}I)^{-1}(A + BK) \tag{16}$$

$$T \succeq Q + K^\top (R - \gamma^2 B^\top B)K + A^\top (T^{-1} - \gamma^{-2}I)^{-1}A. \tag{17}$$

Then the Bellman inequality $\bar{V} \leq \mathcal{F}\bar{V}$ is satisfied by

$$\bar{V}(x, Z) := \max \left\{ |x|_P^2 - \gamma^2 \left\| \begin{bmatrix} I & A & \pm B \end{bmatrix}^\top \right\|_Z^2, |x|_T^2 - \gamma^2 \operatorname{trace}\left(\begin{bmatrix} [I \ A]^\top [I \ A] & 0 \\ 0 & B^\top B \end{bmatrix} Z \right) \right\} \tag{18}$$

and the bound $J_\mu(x_0) \leq |x_0|_T^2$ is valid for the control law (3) defined by

$$u_t = \begin{cases} -Kx & \text{if } \sum_{\tau=0}^{t-1}(x_{\tau+1} - Ax_\tau)^\top Bu_\tau \geq 0 \\ Kx & \text{otherwise.} \end{cases} \tag{19}$$

Proof. Theorem 2 is a special case of Theorem 3, to be proved later. Defining

$$(A_1, B_1) = (A, B) \qquad\qquad (A_2, B_2) = (A, -B)$$
$$(P_{11}, K_1) = (P, K) \qquad\qquad (P_{22}, K_2) = (P, -K)$$
$$P_{12} = T \qquad\qquad\qquad P_{21} = T$$

turns (21)-(23) into (15)-(19). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 1 *See Figure 2 for an illustration. Following (13)-(14), we model a double integrator with unknown sign of the input gain as*

$$x_{t+1} = \underbrace{\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A} x_t \pm \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u_t + w_t$$

where the state is $x_t = \begin{bmatrix} y_t & y_{t-1} & u_{t-1} \end{bmatrix}^\top$. Theorem 3 can be applied with $\mathcal{M} = \{(A, \pm B)\}$. By first solving the Riccati equation for P and K, then solving the matrix inequalities for T by convex optimization, we get

$$P = \begin{bmatrix} 20.61 & -11.09 & 11.09 \\ -11.09 & 7.83 & -6.83 \\ 11.09 & -6.83 & 7.83 \end{bmatrix} \qquad\quad T = \begin{bmatrix} 155.0 & -84.4 & 84.4 \\ -84.4 & 89.0 & -87.5 \\ 84.4 & -87.5 & 89.0 \end{bmatrix}$$

$$\gamma = 19 \qquad\qquad\qquad\qquad\qquad\qquad K = \begin{bmatrix} 1.786 & -1.288 & 1.288 \end{bmatrix}$$
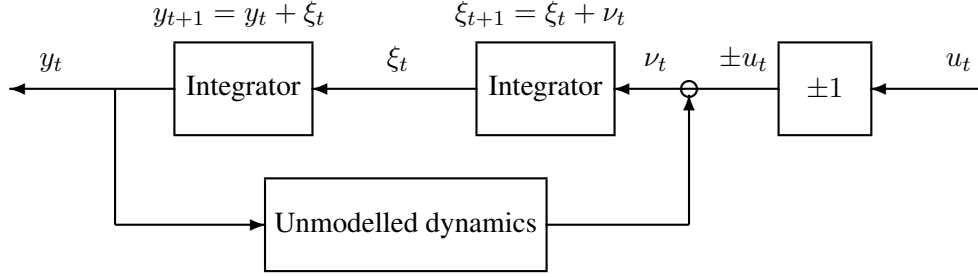
Figure 2: No linear feedback can stabilize a double integrator with unknown input sign. However, Example 1 illustrates how a minimax adaptive controller can do this and even get robustness to a certain amount of unmodelled dynamics. Only the output of the second integrator is measured, but using (13)-(14) it is still possible to write a double integrator on state space form with a fully accessible state.

Simulations with the control law (19) are shown in Figure 3, first for $w = 0$, then for w being a white noise sequence. The case with worst case disturbance w is not plotted, since it actually makes the state grow exponentially according to the formula $x_{t+1} = (I - \gamma^{-2}T)^{-1}Ax_t$. This does not mean that the system is unstable. Instead, what happens is that the worst disturbance w grows exponentially, to keep the current state large compared to past data and prevent the adaptive controller from becoming confident about the input matrix sign. The construction can be used to verify that the the l_2-gain from w to $(Q^{1/2}x, R^{1/2}u)$ when using the adaptive control law (19) is somewhere between $\sqrt{\|T\|} = 16.8$ and $\gamma = 19$.

6. Explicit solution of the Bellman inequality in the general case

The following result gives an explicit expression for an adaptive controller satisfying a pre-specified bound on the L_2-gain for model sets of the form

$$\mathcal{M} := \{(A_1, B_1), \ldots, (A_N, B_N))\} \subset \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}. \tag{20}$$

Theorem 3 *Given $A_1, \ldots, A_N \in \mathbb{R}^{n \times n}$, $B_1, \ldots, B_N \in \mathbb{R}^{n \times m}$ and positive definite $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$, suppose there exist $K_1, \ldots, K_N \in \mathbb{R}^{m \times n}$ and $P_{ij} \in \mathbb{R}^{n \times n}$ with $0 \prec P_{ij} \prec \gamma^2 I$ and*

$$|x|_{P_{ik}}^2 \geq |x|_Q^2 + |K_k x|_R^2 + \left| (A_i - B_i K_k + A_j - B_j K_k)x/2 \right|_{(P_{ij}^{-1} - \gamma^2 I)^{-1}}^2$$

$$- \gamma^2 |(A_i - B_i K_k - A_j + B_j K_k)x/2|^2 \tag{21}$$

for $x \in \mathbb{R}^n$ and $i, j, k \in \{1, \ldots, N\}$. Then the Bellman inequality $\bar{V} \leq \mathcal{F}\bar{V}$ is satisfied by

$$\bar{V}(x, Z) := \max_{i,j} \left\{ |x|_{P_{ij}}^2 - \frac{\gamma^2}{2} \left\| \begin{bmatrix} I & A_i & B_i \end{bmatrix}^\top \right\|_Z^2 - \frac{\gamma^2}{2} \left\| \begin{bmatrix} I & A_j & B_j \end{bmatrix}^\top \right\|_Z^2 \right\} \tag{22}$$
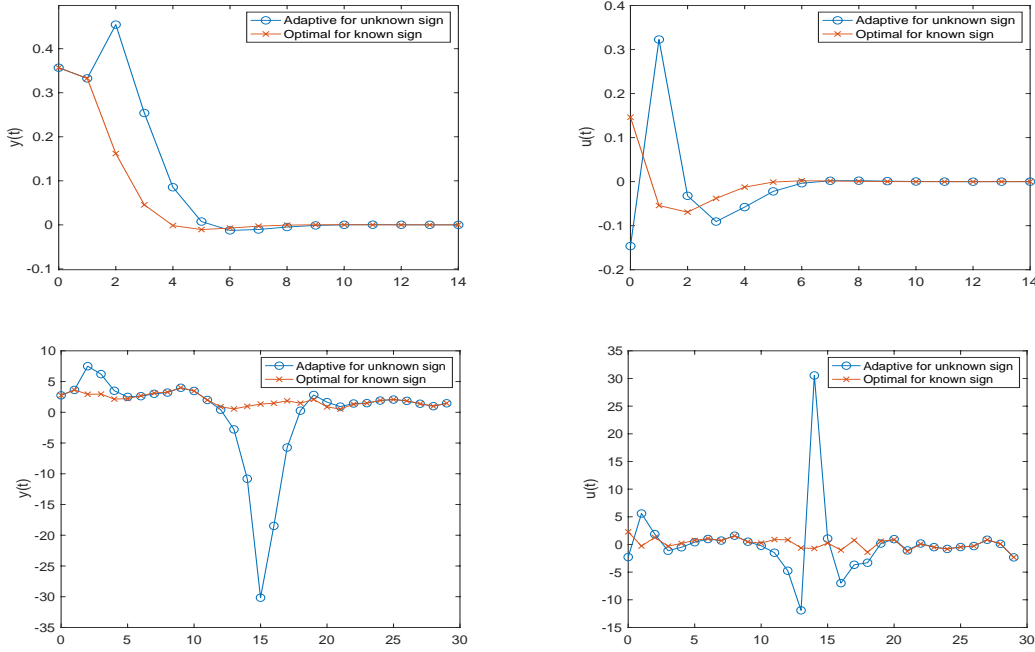
7

Figure 3: Output (left) and input (right) are plotted for the double integrator with input matrix sign uncertainty, controlled using a minimax adaptive controller (blue plots with circles). For comparison, trajectories of the optimal controller with known input matrix sign are also given (red plots). The top plots have been generated with $w = 0$. The minimax controller automatically increases control activity for the purpose of exploration in the beginning. In the lower plots, w is a white noise sequence and the sign of the input matrix is changed at $t = 10$. The adaptation takes longer the second time, since data collected before $t = 10$ makes the adaptive controller reluctant to change its internal model.

and the bound $J_\mu(x_0) \leq \max_{i,j} |x_0|^2_{P_{ij}}$ is valid for the control law μ defined by

$$u_t = -K_{k_t} x_t, \text{ where } k_t = \arg\min_i \sum_{\tau=0}^{t-1} |A_i x_\tau + B_i u_\tau - x_{\tau+1}|^2. \tag{23}$$

Remark 1. One way to approach the solution of (21) is to first solve the Riccati equations

$$|x|^2_{P_{ii}} = \min_u \max_w \left[|x|^2_Q + |u|^2_R - \gamma^2 |w|^2 + |A_i x + B_i u + w|^2_{P_{ii}} \right]$$

for P_{ii} and the minimizing control laws $u = -K_i x$, then use semi-definite programming to determine P_{ij} for $i \neq j$. If no solution exists, then increase γ. However, it should be noted that this approach is not guaranteed to find a solution with the smallest possible γ.

8

Remark 2. The inequalities (21) have interesting interpretations. First, for $i = j = k$ it is the standard H_∞ Riccati inequality, which specifies that $|x|^2_{P_{ii}}$ is an upper bound on the minimax cost function for the system (A_i, B_i) when the model is known. Second, for $i = j \neq k$ it shows that the difference in step cost due to use of a controller that does not match the model is upper bounded by $|x|^2_{P_{ik}} - |x|^2_{P_{ii}}$. Third, (21) also shows that if improved model knowledge is taken into account, then cost-to-go is decreasing even when the "wrong" controller is used. This is due to the last term of the inequality, which represents improved knowledge about the system due to learning.

Proof. Define for $i, j \in \{1, \ldots, N\}$

$$V^{ij}(x, Z) := |x|^2_{P_{ij}} - (z_i + z_j)/2 \qquad\qquad z_i := \gamma^2 \left\| \begin{bmatrix} I & A_i & B_i \end{bmatrix}^\top \right\|^2_Z$$

Then $\bar{V} = \max_{i,j} V^{ij}$. Define $k := \arg\min_i z_i$. Lemma 5 in the Appendix together with (21) give

$$\mathcal{F}_{-K_k x} V^{ij}(x, Z) = \max_v \left\{ |x|^2_Q + |K_k x|^2_R + V^{ij}\left(v, Z + \begin{bmatrix} -v \\ x \\ -K_k x \end{bmatrix} \begin{bmatrix} -v \\ x \\ -K_k x \end{bmatrix}^\top \right) \right\}$$

$$= \max_v \left\{ |x|^2_Q + |K_k x|^2_R + |v|^2_{P_{ij}} - \frac{1}{2} \sum_{l \in \{i,j\}} \left(\gamma^2 |A_l x + B_l u - v|^2 + z_l \right) \right\}$$

$$= |x|^2_Q + |K_k x|^2_R + \left| (A_i - B_i K_k + A_j - B_j K_k) x/2 \right|^2_{(P_{ij}^{-1} - \gamma^{-2} I)^{-1}}$$

$$\quad - \gamma^2 |(A_i - B_i K_k - A_j + B_j K_k) x/2|^2 - (z_i + z_j)/2$$

$$\leq |x|^2_{P_{ik}} - (z_i + z_j)/2$$

$$= V^{ik}(x, Z) + (z_k - z_j)/2$$

$$\leq V^{ik}(x, Z).$$

Hence

$$\mathcal{F}\bar{V}(x, Z) \leq \mathcal{F}_{-K_k x} \bar{V}(x, Z) = \max_{i,j} \mathcal{F}_{-K_k x} V^{ij}(x, Z) \leq \max_i V^{ik}(x, Z) \leq \bar{V}(x, Z).$$

so the Bellman inequality $\bar{V} \leq \mathcal{F}\bar{V}$ is proved. Moreover, the control law $u = -K_k x$ can equivalently be written as (23), since (4) gives

$$\left\| \begin{bmatrix} I & A_i & B_i \end{bmatrix}^\top \right\|^2_{Z_t} = \sum_{\tau=0}^{t-1} |A_i x_\tau + B_i u_\tau - x_{\tau+1}|^2.$$

Finally, the inequality $J_\mu(x_0) \leq \max_{i,j} |x_0|^2_{P_{ij}}$ follows by application of Theorem 1. $\qquad \square$

7. Conclusions

Two main contributions have been given in the is paper. First, Theorem 1 shows that the minimax adaptive control problem can be stated as a standard zero sum dynamic game with a finite-dimensional state space. Second, Theorem 3 gives and explicit upper bound for the value of the

game and a corresponding adaptive control scheme achieving the upper bound. We believe that the approach has tremendous potential for applications and generalizations in various directions. However, we also believe that upper bound of Theorem 3 may not be the last word, but rather a first step towards an exact formula for the minimal achievable gain and a seed for a rich new theory of the interplay between learning and control.

Acknowledgments

8. Appendix: Quadratic optimization.

Lemma 4 *For $P \in \mathbb{R}^{n \times n}$, $y, v \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, it holds that*

$$\max_v \left\{ |v|_P^2 - \gamma^2 |y - v|^2 \right\} = |y|_{(P^{-1} - \gamma^{-2} I)^{-1}}^2.$$

Proof.

$$
\begin{aligned}
\max_v \left\{ |v|_P^2 - \gamma^2 |y - v|^2 \right\} &= \max_v \left\{ |v|_{(P - \gamma^2 I)}^2 - \gamma^2 |y|^2 + 2\gamma^2 y^\top v \right\} \\
&= -\gamma^2 |y|^2 - \gamma^4 y^\top (P - \gamma^2 I)^{-1} y \\
&= \gamma^2 y^\top (P - \gamma^2 I)^{-1} \left[\gamma^2 I - P - \gamma^2 I \right] y \\
&= y^\top (P^{-1} - \gamma^{-2} I)^{-1} y
\end{aligned}
$$

$\square$

Lemma 5 *For $T \in \mathbb{R}^{n \times n}$, $y_1, y_2, v \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, it holds that*

$$
\max_v \left\{ |v|_T^2 - \gamma^2 |y_1 - v|^2 / 2 - \gamma^2 |y_2 - v|^2 / 2 \right\}
$$
$$
= |(y_1 + y_2)/2|_{(T^{-1} - \gamma^{-2} I)^{-1}}^2 - \gamma^2 |(y_1 - y_2)/2|^2
$$

Proof. Application of Lemma 4 gives

$$
\max_v \left\{ |v|_T^2 - \gamma^2 |y_1 - v|^2 / 2 - \gamma^2 |y_2 - v|^2 / 2 \right\}
$$
$$
= \max_v \left\{ |v|_T^2 - \gamma^2 |(y_1 + y_2)/2 - v|^2 + \gamma^2 |(y_1 + y_2)/2|^2 - \gamma^2 (|y_1|^2 + |y_2|^2)/2 \right\}
$$
$$
= |(y_1 + y_2)/2|_{(T^{-1} - \gamma^{-2} I)^{-1}}^2 + \gamma^2 |(y_1 + y_2)/2|^2 - \gamma^2 (|y_1|^2 + |y_2|^2)/2
$$
$$
= |(y_1 + y_2)/2|_{(T^{-1} - \gamma^{-2} I)^{-1}}^2 - \gamma^2 |(y_1 - y_2)/2|^2.
$$

$\square$

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