

# Coarse-graining for the Analysis of Soft Matter Scattering

submitted by

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for the degree of Doctor of Philosophy

of the

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Department of Chemistry

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*"Atticus told me to delete the adjectives and I'd have the facts."*

Scout Finch – To Kill a Mockingbird



UNIVERSITY OF BATH

*Abstract*

Department of Chemistry

Doctor of Philosophy

**Coarse-graining for the Analysis of Soft Matter Scattering**

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## *Acknowledgements*



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# List of Abbreviations

<b>BM</b>	bending magnet
<b>DLS</b>	Diamond Light Source
<b>ESRF</b>	European Synchrotron Radiation Facility
<b>ESS</b>	European Spallation Source
<b>GiSAS</b>	grazing incidence small angle scattering
<b>ID</b>	insertion device
<b>ILL</b>	Institut Laue-Langevin
<b>SAS</b>	small angle scattering
<b>ToF</b>	time-of-flight



# Physical Constants

$$\begin{array}{ll} \pi = 3.1415 \dots & \\ \text{Planck constant } h = 6.626 \dots \times 10^{-34} \text{ J s} & \end{array}$$





# List of Symbols

$a_0$	optimum head-group area	$\text{m}^2$
$b$	scattering length	$\text{m}$
$m$	mass	$\text{kg}$
$n_i$	refractive index	
$q$	scattering vector magnitude	$\text{m}^{-1}$
$r_{n,n+1}$	Fresnel equation coefficient	
$res(q)$	resolution function	
$t_F$	time of flight	$\text{s}$
$v$	velocity	$\text{m s}^{-1}$
$B$	resultant matrix	
$E_k$	kinetic energy	$\text{J}$
$I$	intensity	
$L_F$	distance of flight	$\text{m}$
$M$	layer matrix	
$N_P$	number of magnets	
$R$	reflected intensity	
$S$	nuclear spin	
$V$	volume	$\text{m}^3$
$\mathbf{k}_i$	incident wavevector	$\text{m}^{-1}$
$\mathbf{k}_f$	final wavevector	$\text{m}^{-1}$
$\mathbf{q}$	scattering vector	$\text{m}^{-1}$
$\beta_c$	fraction of the speed of light	
$\beta_n$	phase factor	
$\theta$	scattering angle	$\text{rad}$
$\theta_c$	critical angle	$\text{rad}$
$\theta_e$	angle between electron and photon	$\text{rad}$
$\lambda$	wavelength	$\text{m}$
$\rho$	scattering length density	$\text{m}^{-2}$
$\sigma$	interfacial roughness	$\text{m}$
$\sigma_{\text{coh}}$	coherent cross-section	$\text{m}^2$
$\sigma_{\text{incoh}}$	incoherent cross-section	$\text{m}^2$
$\phi$	scattering angle	$\text{rad}$
$\omega$	frequency	$\text{s}^{-1}$
$\omega_i$	initial frequency	$\text{s}^{-1}$
$\omega_f$	final frequency	$\text{s}^{-1}$
$\Lambda$	temperature factor	
$d\sigma(q)/d\Omega$	differential cross-section	



# 1 Theory

## 1.1 Scattering

The use of scattering techniques to probe soft condensed matter systems is commonplace. In this work, we have focussed on the use of small angle scattering (SAS), reflectometry, and grazing incidence small angle scattering (GiSAS) techniques. These are particularly appropriate for application to soft condensed matter systems due to the length scales capable of being probed being similar to the persistence length of the soft condensed matter systems. The length scales covered for such techniques is from around 1 nm to 100 nm, as is shown in Figure 1.1. Since it is the equilibrium structure(s) under study, there is no interest in the system dynamics. Therefore, the system can be studied using exclusively elastic scattering techniques, where there is no energy transfer between the probing radiation and the system. This is in contrast to inelastic scattering where energy transfer occurs; facilitating the measurement of system dynamics, such as the dynamical modes of polymers or lipid bilayers [1, 2]. The techniques mentioned above all involve the use of elastic scattering and therefore probe the system equilibrium structure.

Both X-ray and neutron scattering techniques are discussed and used in this work. From an experimental viewpoint, there are significant differences between an X-ray scattering and a neutron scattering experiment. However, there is little variation in terms of the data analysis, where the differences are limited to; the nature of the scattering lengths, and the higher background that is present in the neutron scattering experiments.

### 1.1.1 The scattering vector

The scattering of some probing radiation, by some sample can be represented as shown in Figure 1.2. Since only elastic scattering is being considered, there will be no change in the frequency of the radiation,  $\omega_i = \omega_f$ . This means that only the wavevector,  $\mathbf{k}$ , can change,  $\mathbf{k}_i \neq \mathbf{k}_f$ . The difference between the incident and final wavevectors is the scattering vector,  $\mathbf{q}$ , where,

$$\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f. \quad (1.1)$$

The scattering vector strictly has units of  $\text{m}^{-1}$ , however it is often more practical to use  $\text{nm}^{-1}$  or  $\text{\AA}^{-1}$ . Throughout this work, units of reciprocal Angstrom will be wherever possible. Since the frequency of the probing radiation does not change during an elastic scattering event, the wavelength,  $\lambda$ , will also not change, meaning that the moduli of the incident and final wavevectors are,

$$|\mathbf{k}_i| = |\mathbf{k}_f| = \frac{2\pi}{\lambda}. \quad (1.2)$$

This means that only the angle will change during the elastic scattering event. The vector diagram in Figure 1.3 can be used to describe the geometry of an elastic scattering event. From this, and Eqn 1.2, the value of  $q$ , where  $q = |\mathbf{q}|$  can be shown as,

$$q = \frac{4\pi \sin \theta}{\lambda}. \quad (1.3)$$

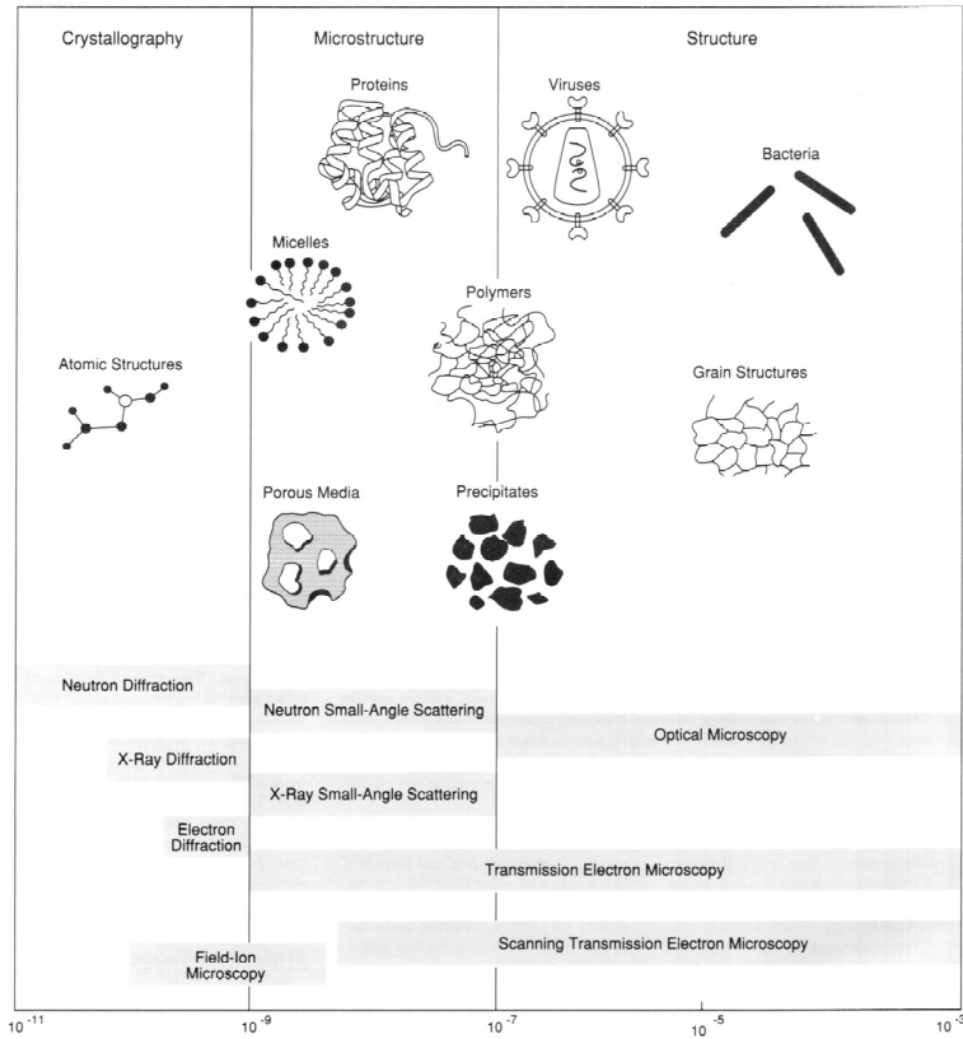


FIGURE 1.1: A representation of how different techniques can be used to probe various length scales, from Ref [3].

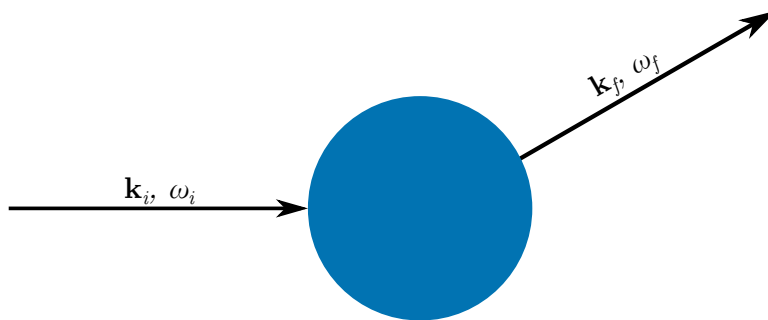


FIGURE 1.2: A schematic of the scattering of some probing radiation by a sample (blue circle), adapted from Ref [3].

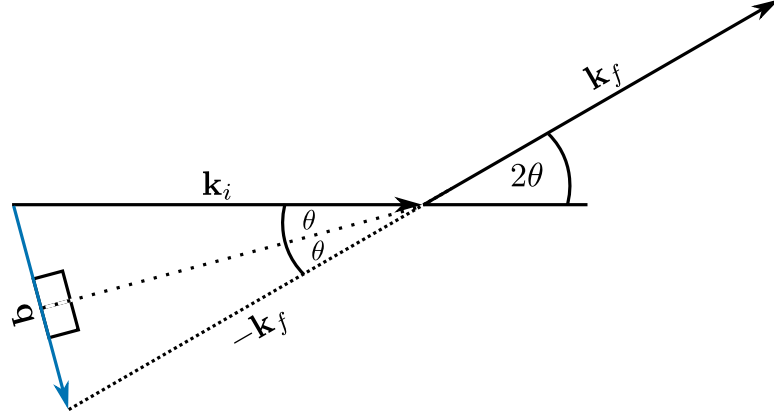


FIGURE 1.3: A vector diagram describing an elastic scattering event, adapted from Ref [3].

However, this fails to fully capture the three dimensional nature of the scattering event. Hence, it is necessary to describe the scattering with spherical coordinates,  $2\theta$ , and  $\phi$ , such that the incoming and outgoing radiation can be described as,

$$\begin{aligned} \mathbf{k}_i &= \left(0, 0, \frac{2\pi}{\lambda}\right), \\ \mathbf{k}_f &= \frac{2\pi}{\lambda}(\sin 2\theta \cos \phi, \sin 2\theta \sin \phi, \cos 2\theta), \end{aligned} \quad (1.4)$$

where,  $|\mathbf{k}_f| = 2\pi/\lambda$ . This allows the scattering vector to be written,

$$\mathbf{q} = \frac{4\pi \sin \theta}{\lambda}(-\cos \theta \cos \phi, -\cos \theta \sin \phi, \sin \theta). \quad (1.5)$$

For an isotropic scattering pattern, it is the magnitude of the scattering vector,  $q$ , that is measured. In practical terms, the scattering vector allows for easy comparison of measurements made at different radiation wavelengths.

The basic quantity measured in a scattering experiment is the differential cross section,  $d\sigma(q)/d\Omega$ . This is the fraction of particles of probing radiation that are scattered with a particular set of polar coordinates,  $2\theta$  and  $\phi$ ,

$$\frac{d\sigma(q)}{d\Omega} = \frac{R(2\theta, \phi)}{NV\Phi\Delta\Omega}, \quad (1.6)$$

where,  $R(2\theta, \phi)$  is the rate of arrival of the scattered particles at the position  $2\theta, \phi$ ,  $V$  is the volume of the sample,  $\Phi$  is incident flux,  $\Delta\Omega$  is some small solid angle, and  $N$  is the number of scattering particles of interest, in the case of elastically scattering particles,  $N = N\%_{\text{el}}$ , where  $\%_{\text{el}}$  is the fraction of elastically scattered particles.

### 1.1.2 Scattering from a single fixed particle

It is possible to describe a steady stream of particles of wavelength,  $\lambda$ , travelling through space as follows,

$$\psi_i = \psi_o \exp(\mathbf{i}kz), \quad (1.7)$$

where,  $z$  is the direction of travel, and the incident flux is the magnitude of the wave squared,  $\Phi = |\psi_o|^2$ . This wave then interacts with a single fixed particle elastically, propagating the wave radially outwards (Figure 1.4). This propagation is centred on

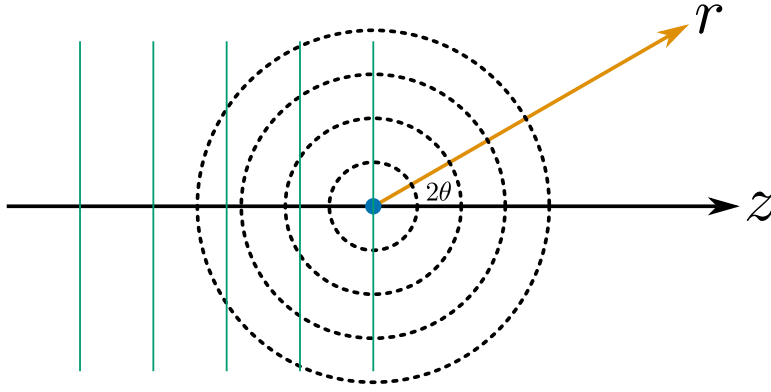


FIGURE 1.4: A schematic of the interaction between a single particle and a wave of probing radiation (green lines), adapted from Ref [3].

the atom, therefore the wavevector,  $\mathbf{k}_f$  is parallel to the displacement vector,  $\mathbf{r}$ , and the following holds,

$$\exp(\mathbf{i}\mathbf{k}_f \cdot \mathbf{r}) = \exp(\mathbf{i}kr). \quad (1.8)$$

This final wave is no longer collimated and therefore diminishes with distance,  $r$ . Hence the final scattered wave has the form,

$$\psi_f = \psi_o b \frac{\exp(\mathbf{i}kr)}{r}, \quad (1.9)$$

where,  $b$  is the scattering length discussed in Section 1.2.2.

### 1.1.3 Scattering from multiple particles

It is important to consider how the probing radiation would interact with a real system, consisting of many particles. If the incident beam has the form of Eqn. 1.7, with the wavevector  $\mathbf{k}_i = (0, 0, k)$ , each particle,  $j$ , will contribute the following to the total scattered wave,  $\psi_f$ , made up of the scattering from all,  $N$ , atoms,

$$[\delta\psi_f]_j = \psi_o \exp(\mathbf{i}\mathbf{k}_i \cdot \mathbf{R}_j) b_j \frac{\exp\{\mathbf{i}\mathbf{k}_f \cdot (\mathbf{r} - \mathbf{R}_j)\}}{|\mathbf{r} - \mathbf{R}_j|}, \quad (1.10)$$

where,  $\mathbf{R}_j$  is the position of particle  $j$ ,  $\mathbf{r}$  is some arbitrary position, and  $\mathbf{k}_f$  is the wavevector of the scattered wave (Figure 1.5). This allows the total scattered wave to be defined as a summation of the contributions from the individual waves,

$$\psi_f = \psi_o \exp(\mathbf{i}\mathbf{k}_f \cdot \mathbf{r}) \sum_{j=1}^N \left\{ b_j \frac{\exp(\mathbf{i}\mathbf{q} \cdot \mathbf{R}_j)}{|\mathbf{r} - \mathbf{R}_j|} \right\}. \quad (1.11)$$

Eqn. 1.11 holds true, within the Born approximation, where the scattered wave has no impact on the incident wave and each wave is scattered only once.

The sample-detector distance is usually much larger than the typical particle size, allowing for the following approximation,

$$|\mathbf{r} - \mathbf{R}_j| = |\mathbf{r}| = r. \quad (1.12)$$

This is termed the Fraunhofer, or far-field limit, and allows Eqn. 1.11 to be simplified,

$$|\psi_f|^2 = \frac{\Phi}{r^2} \left| \sum_{j=1}^N b_j \exp(\mathbf{i}\mathbf{q} \cdot \mathbf{R}_j) \right|^2. \quad (1.13)$$

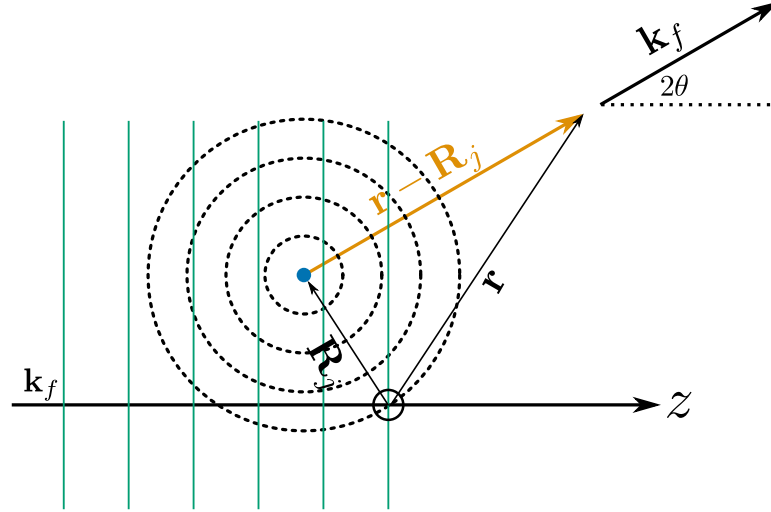


FIGURE 1.5: A schematic of the interaction between a particle,  $j$ , at position  $\mathbf{R}_j$  and a wave of probing radiation (green lines), adapted from Ref [3].

In the scattering experiment, particles are deflected elastically into a detector with a small area,  $\delta A$ , with the polar coordinates,  $2\theta$  and  $\phi$ , at a rate of  $R_{\text{el}}$ ,

$$R_{\text{el}}(2\theta, \phi) = |\psi_f|^2 \delta A = \Phi \delta \Omega \left| \sum_{j=1}^N b_j \exp(\mathbf{i}\mathbf{q} \cdot \mathbf{R}_j) \right|^2, \quad (1.14)$$

where,  $\delta \Omega = \delta A / r^2$ . Therefore, the differential cross section, defined in Eqn. 1.6 can be related to the scattering from the sample as,

$$\left( \frac{d\sigma(q)}{d\Omega} \right)_{\text{el}} = \frac{1}{V} \left| \sum_{j=1}^N b_j \exp(\mathbf{i}\mathbf{q} \cdot \mathbf{R}_j) \right|^2. \quad (1.15)$$

#### 1.1.4 Scattering length density

While it may be helpful to consider the scattering of multiple particles individually, where each particle has a scattering length,  $b$ . In practice, it is more common to consider the scattering length density,  $\rho$ , of the system,

$$\rho = \frac{1}{V} \sum_{i=0}^N b_i, \quad (1.16)$$

where  $N$  is the total number of particles in the volume  $V$ . A result of this equation is the ability to rewrite 1.15 as,

$$\left( \frac{d\sigma(q)}{d\Omega} \right)_{\text{el}} = \frac{1}{V} \left| \iiint_V \rho \exp(\mathbf{i}\mathbf{q} \cdot \mathbf{R}) d^3\mathbf{R} \right|^2. \quad (1.17)$$

This above equation shows that the scattering differential cross-section from some object is related to the scattering length density profile of that object by a Fourier transform.

### 1.1.5 Model-dependent analysis

All types of scattering patterns can be analysed by one of two methods; model independent and model-dependent. The nature of this work means that it will focus on model-dependent analysis methods, often where the model is derived from some atomistic, or coarse-grained simulation. Model-dependent analysis, has significant benefits over model-independent methods, such as improved resolution and more detailed information of the structure. However, the necessity of the inclusion of *a priori* information within model-dependent analysis may act to bias the result, while this is undesirable, these assumptions can, and should, be educated based on the chemical information present – such as the propensity for twin-tailed lipid molecules to form monolayers at an air-water interface [4].

The scattering from the model system is determined, using technique specific methods that are discussed in detail in later sections. This is then compared with the experimental data using some goodness-of-fit metric, as this the model is varied the quality of the fit, and therefore the value of the metric, will improve. In order to accurately reproduce the experimental measurement, it is necessary to include some instrumental resolution function,  $res(q)$ , in the modelling procedure. This is instrument specific, although it may be approximated by convolving the experimental dataset with some Gaussian smearing function, the modelled intensity can then be determined from,

$$I(q) = res(q) * \frac{d\sigma(q)}{d\Omega}, \quad (1.18)$$

where,  $d\sigma(q)/d\Omega$  is the differential cross-section, a measure of the number of scattering particles hitting a given solid angle of the detector.

The aim of model-dependent analysis is to obtain a model for the system which agrees with the experimentally measured scattering data, while producing something that is chemically, and physically relevant. This means that optimisation algorithms are often applied to these analysis problems, which are discussed, in detail, in Section 1.4.

### 1.1.6 Reflectometry

Reflectometry involves the interaction of the probing radiation with some interface, from which the radiation is reflected. The geometry of a reflectometry experiment is shown in Figure 1.6, where the reflectometry instrument is in the horizontal configuration, ideal for the study of liquid interfaces. Reflectometry measurements give information about the structure perpendicular to the interface, the  $z$ -dimension in Figure 1.6, and therefore the analysis of reflectometry data is founded on the assumption that the layers will be completely homogeneous in the plane of the interface, the  $xy$ -plane in Figure 1.6. In reality, since the layers are usually not completely homogeneous, an average is obtained for the area in the radiation beam. A reflectometry instrument operates by measuring the intensity of specular radiation at a series of different angles,  $\theta$ , or wavelengths,  $\lambda$ . The reflected intensity is defined in terms of  $q$  (by Eqn 1.1.7), and is defined as follows,

$$R(q) = \frac{\text{rate of specular reflective scattering}}{\text{rate of incidence}}. \quad (1.19)$$

It is clear from Eqn 1.19 that the value of the measured reflectometry cannot be greater than one, as this would mean that more particles of probing radiation were being reflected than were incident.



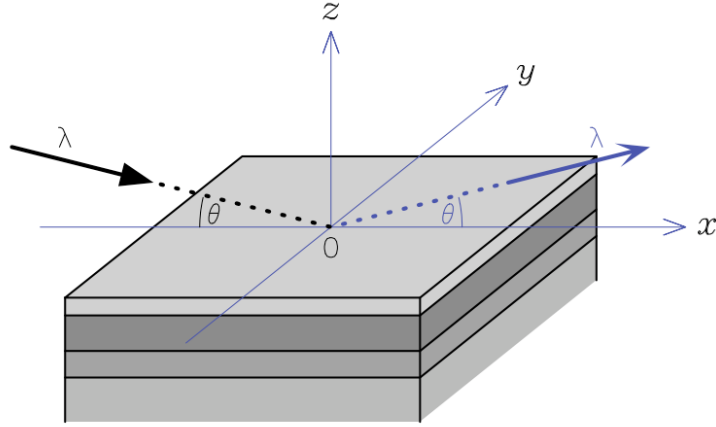


FIGURE 1.6: A schematic of specular reflectometry from a layered sample, from Ref [3].

### Analysis

There are two model-dependent analysis techniques that can be applied to the rationalisation of a reflectometry dataset. The first is the kinematic approach, which can be obtained from Eqn. 1.15, from the assumption that  $q_x = 0$  and  $q_y = 0$ , as we are only measuring the specular scattering. This approach models the reflectometry as a function of the scattering length density profile in the  $z$ -dimension,  $\rho(z)$ ,

$$R(q) \approx \frac{16\pi^2}{q^4} \left| \int_{-\infty}^{+\infty} \frac{d\rho(z)}{dz} \exp(-izq_z) dz \right|^2, \quad (1.20)$$

where,  $d\rho(z)/dz$  is the first derivative of the scattering length density profile. However, this method has a significant problem, which can be described by applying Eqn 1.20 to the scattering length density profile of a bare silicon substrate, which can be modelled as a Heaviside function (Figure 1.7a),

$$\rho(z) = \begin{cases} 0, & \text{if } z < 0 \\ \rho_{\text{Si}}, & \text{otherwise} \end{cases} \quad (1.21)$$

where,  $\rho_{\text{Si}}$  is the scattering length density of pure silicon ( $2.1 \times 10^{-6} \text{ \AA}^{-2}$  for neutrons). The derivative of a stepwise Heaviside function is a scaled  $\delta$ -function (Figure 1.7b),

$$\rho'(z) = \rho_{\text{Si}} \delta(z). \quad (1.22)$$

Then, as in Eqn 1.20, the Fourier transform of this  $\delta$ -function is taken,

$$\rho_{\text{Si}} \int_{-\infty}^{+\infty} \delta(z) \exp(-izq_z) dz = \rho_{\text{Si}} \exp(0) = \rho_{\text{Si}}. \quad (1.23)$$

This means that the reflectometry profile could be calculated from the following relationship,

$$R(q) \approx \frac{16\pi^2 \rho_{\text{Si}}^2}{q_z^4}. \quad (1.24)$$

The curve from this relationship is shown in Figure 1.7, where it is clear that the agreement with an experimental profile would be poor as  $q \rightarrow 0$ . It can be seen that for low

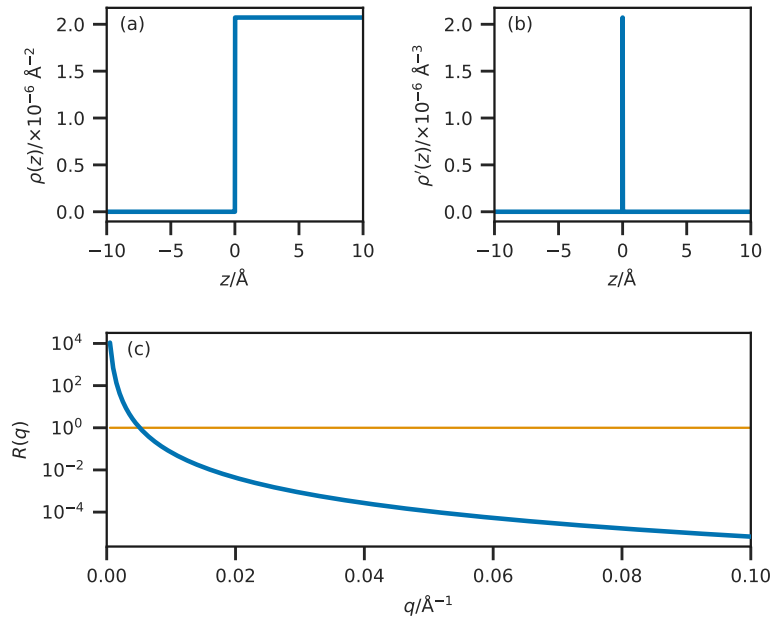


FIGURE 1.7: A graphical representation of the kinematic approach; (a) the Heaviside function describing the scattering length density profile of a bare silicon substrate, (b) the  $\delta$ -function arising from the first derivative of the function in (a), and (c) the reflectometry profile resulting from Eqn 1.20, where the orange line at  $R = 1$  identifies the break down between experimental and theory in the kinematic approach, adapted from Ref [3].

values of  $q$  the calculated reflectometry is greater than 1, which violates the physical constraint imposed with Eqn 1.19. This break down of the kinematic approach is due to the assumption present in this approach that the Born approximation (mentioned in Section 1.1.3) will hold. However, in the reflectometry scattering geometry, this is no longer true rendering the kinematic approach invalid.

This breakdown of the kinematic approach has lead to the application of the Abeles, or Parratt, model for the reflection of light at a given number of stratified interfaces (also known as dynamical theory) [5, 6]. This methods involves considering the system as a layered structure at the interfaces of which the probing radiation can either be reflected or refracted, by some refractive index,  $n_i$ . Figure 1.8 shows this process for a system of two layers, where the layer 0 is the air or vacuum above the sample, it is clear to see how the two waves labelled  $r$  could interfere constructively or destructively depending on the thickness of layer 1,  $d$ . This means that for a single interface, such as that between layers 0 and 1 in Figure 1.8, the reflectometry can be described by the Fresnel equation,

$$R(q) = \left| \frac{n_0 \sin \theta_0 - n_1 \sin \theta_1}{n_0 \sin \theta_0 + n_1 \sin \theta_1} \right|^2. \quad (1.25)$$

Additionally at the point of total reflection, where  $\theta_0 = \theta_c$ , the critical angle, there will be no transmitted wave so,

$$n_1 \sin \theta_1 = 0, \quad (1.26)$$

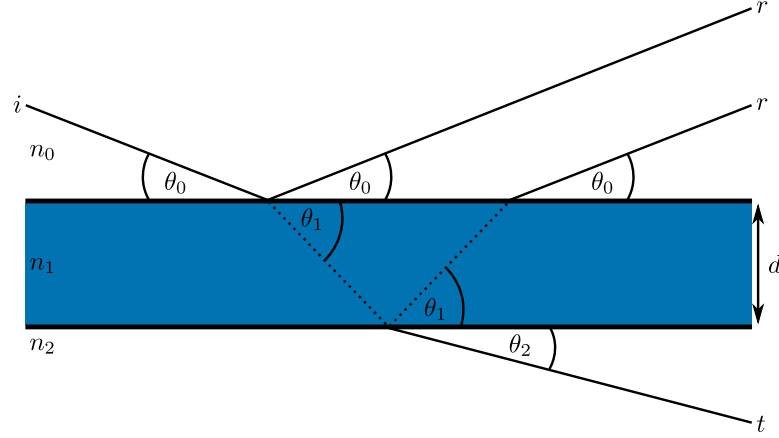


FIGURE 1.8: A schematic diagram showing the reflected ( $r$ ) and transmitted ( $t$ ) waves when an incident ( $i$ ) wave enters an interface of thickness  $d$ , where the refractive indices of each layer are  $n_0$ ,  $n_1$ , and  $n_2$ , adapted from Ref. [7].

and therefore the reflected radiation will never be greater than 1, while the critical angle can be defined as,

$$\cos^2 \theta_c = \frac{n_1^2}{n_0^2}. \quad (1.27)$$

This is the angle below which a reflectometry profile will be measured.

The above method can then be generalised to a structure of an arbitrary number of layers, as shown in Code Block 1.1. For each value of  $q$  for which the reflectometry is to be calculated, the system is considered in terms of  $n_{\max}$  layers. The incident radiation beam will be refracted by each of the layers, giving wavevectors values for each layer,  $k_n$ ,

$$k_n = \sqrt{k_0^2 + 4\pi(\rho_n - \rho_0)}, \quad (1.28)$$

where,  $k_0 = q/2$ . The Fresnel equation coefficient between layers  $n$  and  $n+1$ ,  $r_{n,n+1}$  can then be found along with the phase factor,  $\beta_n$ , which is dependent on the thickness of the layer,  $d_n$ ,

$$r_{n,n+1} = \frac{k_n - k_{n+1}}{k_n + k_{n+1}}, \quad (1.29)$$

$$\beta_n = k_n d_n. \quad (1.30)$$

This means that a matrix can be evaluated for each layer,  $M_n$ ,

$$M_n = \begin{bmatrix} \exp \beta_n & r_{n,n+1} \exp -\beta_n \\ r_{n,n+1} \exp \beta_n & \exp -\beta_n \end{bmatrix} \quad (1.31)$$

The resultant matrix,  $B_n$ , is then found as a product of the matrix from each layer,

$$B = \prod_{n=0}^{n_{\max}} M_n, \quad (1.32)$$

and from this the reflected intensity at the given value of  $q$  can be found,

$$R(q_z) = \frac{B_{1,2}}{B_{1,1}}. \quad (1.33)$$

This algorithm models the layers as perfectly flat layers, which will not be strictly true in the case of soft matter systems such as a bilayer. This resulted in the correction term

CODE BLOCK 1.1: An example code block for the Abelès method for the calculation of reflectometry, adapted from Ref. [8].

```
import numpy as np

def abeles(q_values, sld, d):
    R = np.zeros_like(q_values)
    kn = np.sqrt(q_values[:, np.newaxis] ** 2. / 4. - 4. * np.pi * sld)
    B = np.zeros((2, 2, q_values.size))
    B[0, 0, :] = 1
    B[1, 1, :] = 1
    k = kn[:, 0]
    nmax = sld.size
    for n in range(1, nmax):
        kn1 = kn[:, n]
        r = (k - kn1) / (k + kn1)
        betan = k * d[n]
        if n > 0:
            Mn = np.array([[np.exp(betan * 1j), r * np.exp(betan * 1j)],
                           [r * np.exp(-betan * 1j), np.exp(-betan * 1j)]])
        else:
            Mn = np.array([[1, r],
                           [r, 1]])
        p0 = B[0, 0, :] * Mn[0, 0, :] + B[1, 0, :] * Mn[0, 1, :]
        p1 = B[0, 0, :] * Mn[1, 0, :] + B[1, 0, :] * Mn[1, 1, :]
        B[0, 0, :] = p0
        B[1, 0, :] = p1
        p0 = B[0, 1, :] * Mn[0, 0, :] + B[1, 1, :] * Mn[0, 1, :]
        p1 = B[0, 1, :] * Mn[1, 0, :] + B[1, 1, :] * Mn[1, 1, :]
        B[0, 1, :] = p0
        B[1, 1, :] = p1
        k = kn1
    R = (B[0, 1, :] * np.conj(B[0, 1, :])) / (B[0, 0, :] * np.conj(B[0, 0, :]))
    R[np.where(np.isnan(R))] = 1.
    return np.real(R)
```

being added to Eqn 1.29 to account for the roughness of the layers. This adapts Eqn 1.29 to the form,

$$r_{n,n+1} = \frac{k_n - k_{n+1}}{k_n + k_{n+1}} \exp(-2k_n k_{n+1} \sigma_{n,n+1}^2), \quad (1.34)$$

where,  $\sigma_{n,n+1}$  is the interfacial roughness between layers  $n$  and  $n + 1$ . This has the effect of Gaussian broadening the layers into each other, as a result. Code Block 1.1 is currently implemented in a variety of reflectometry modelling software packages, such as refnx, MOTOFIT, RasCAL, and Aurore [8–12]. Applying this method to the scattering length density profile shown in Figure 1.7 gives the reflectometry profile shown with the dashed green line in Figure 1.9.

### 1.1.7 Small angle scattering

Eqn. 1.17 identified that the scattering differential cross-section for some object was related to the scattering length density by a Fourier transform, which is shown graphically in Figure 1.10. This figure shows that there is a reciprocal relationship between the size of the object and the scattered intensity, decaying significantly up to values of  $2\pi/d_x$ , where  $d_x$  is the size of the object. This means that in order to probe the large-scale structural features that are of interest in the study of soft materials, it is necessary to consider small values of  $q$ . When considering the nature of  $q$  in Eqn. , it is clear that such experiments would benefit from small values of  $\theta$  and large values of  $\lambda$ . Hence, the application of small angle scattering (SAS).

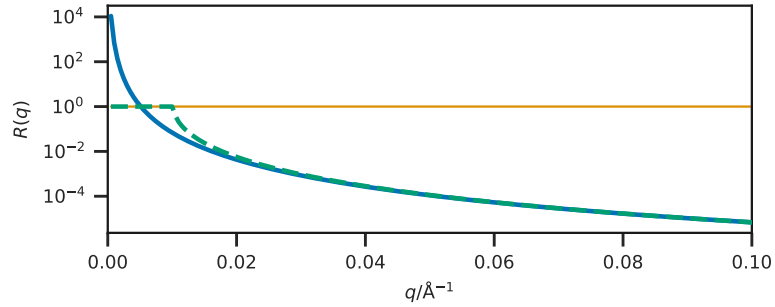


FIGURE 1.9: A comparison of the kinematic approach (blue solid line), and the dynamical approach (green dashed line), to determine the reflected intensity from the material with the scattering length density profile given in Figure 1.7(a). It is clear that at low  $q$ , there is a noticeable deviation between the two.

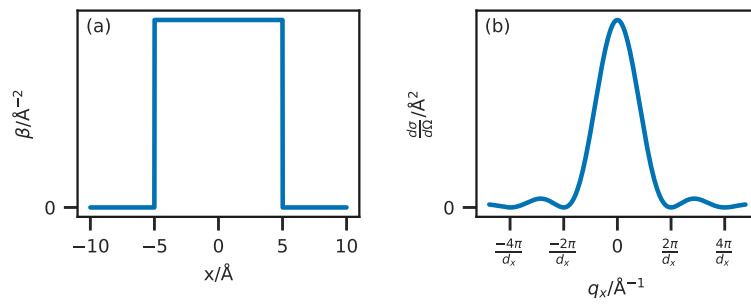


FIGURE 1.10: The effect of a Fourier transform (a) the scattering length density profile for some object with a width of  $10 \text{ \AA}$ , (b) the Fourier transform of this object showing the minima in the differential cross section at values of  $2n\pi/10$ , where  $n$  is some integer.

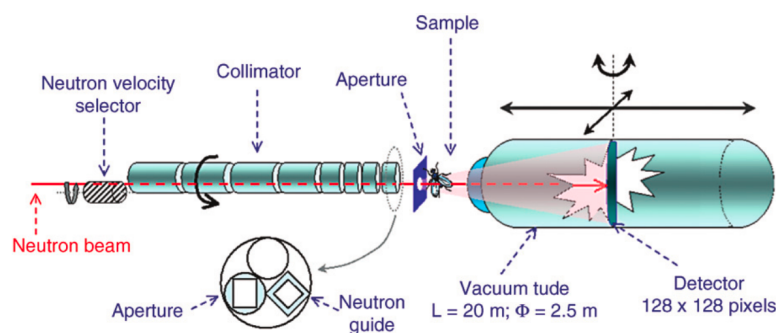


FIGURE 1.11: A schematic of the D22 instrument of the ILL, from Ref. [15].

A SAS experiment generally involves some sample being placed in the path of the probing radiation; and the scattering pattern that results from this transmission is measured at some distance, as is shown in Figure 1.11 for the D22 SANS instrument of the ILL. SAS instruments are usually very large, due to the large post sample flight path that is necessary to reach the small angles being measured; a longer flight path allows more space for angular divergence. Transmission SAS can provide information about the size, shape and orientation of the sample's components [13]. The range in  $q$  that is typically covered by a SAS instrument is usually around  $5 \times 10^{-3} \text{ \AA}^{-1}$  to  $0.5 \text{ \AA}^{-1}$ , which corresponds to  $10 \text{ \AA}$  to  $1000 \text{ \AA}$  in real-space. The neutron or X-ray detector of a SAS instrument is often two-dimensional, meaning that for an isotropic scattering profile, the detector image is radially averaged to give an  $I(q)$  scattering profile. It is possible to increase the  $q$ -range of a SAS instrument through the introduction of wide- $q$  detector banks close to the sample or small- $q$  detector banks further away. This allows the SANS2D instrument, at the ISIS Neutron and Muon Source, to have a total range from  $2 \times 10^{-3} \text{ \AA}$  to  $2 \text{ \AA}$  [14].

## Analysis

### 1.1.8 Grazing incidence small angle scattering

## 1.2 Probing radiation

This work is focussed on the use of X-ray and neutron scattering, therefore it is pertinent to discuss how each of these probing radiation is produced and detail the advantages of each with respect to the other.

### 1.2.1 The generation of X-ray and neutrons

#### X-rays

X-rays are a form of electromagnetic radiation similar to visible light, albeit with a much shorter wavelength – from  $0.01 \text{ nm}$  to  $10 \text{ nm}$ . There are three common ways to produce X-rays; two are available within the laboratory, while the other is exclusive to large scale facilities.

The two laboratory source X-ray generation techniques are the X-ray tube and the rotating anode. An X-ray tube consists of a filament and an anode within a vacuum chamber, by passing a high voltage electrical current across the filament electrons are emitted which accelerate towards the anode. On collision with the anode, the rapid

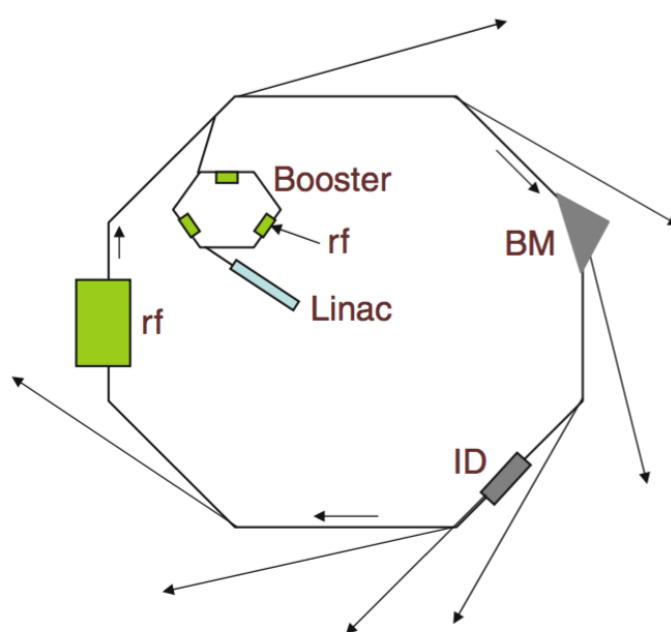


FIGURE 1.12: A schematic representation of a synchrotron radiation source, identifying the Linac, the booster ring, the radio-frequency cavities (rf), the bending magnet (BM) and the insertion device (ID), from Ref [17].

deceleration results in the emission of X-rays of a characteristic wavelength based on the anode material.[16] The most common material for an X-ray tube anode is copper which gives off radiation of about 8 keV.

The other common laboratory method for the generation of X-rays is the rotating anode, which is an improvement on the X-ray tube. In the X-ray tube, each time that an electron contacts the anode there is some energy transfer, this means that over many millions of collisions, the temperature of the anode can raise significantly – leading to a temperature limitation on the X-ray flux available. This led to the development of the rotating anode, this is simply where the anode is made from a rotating wheel, so that the bombardment is spread across the whole wheel reducing the energy localisation. This allows an increase in the photon flux by about an order of magnitude.[16]

The third method of X-ray generation is at a synchrotron facility, this method has the drawback that it requires access to a national or international facility; such as Diamond Light Source (DLS) or the European Synchrotron Radiation Facility (ESRF). The way in which X-rays are generated at the synchrotron involves the acceleration of an electron, rather than the deceleration as with the laboratory sources. This is achieved by having relativistic electrons travel in around a curve, from Newtonian mechanics it is known that travelling on a curve at constant speed is equivalent to acceleration. This is achieved by firstly accelerating the electrons, produced in an linear accelerator (Linac), to near the speed of light in a booster synchrotron before injecting them into the storage ring. In the storage ring, the electrons are kept at relativistic speeds with bending magnets (BM) and straight sections making up a ring (Figure 1.12). The circularity of the ring is dependent on the number of bending magnets that make up the ring; for example, DLS has 48 bending magnets with 48 straight sections.

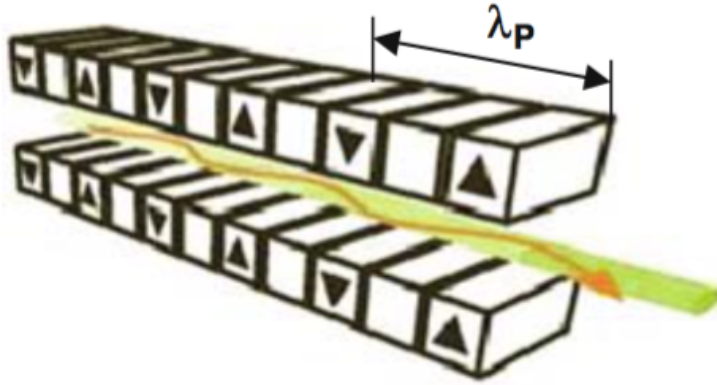


FIGURE 1.13: A diagram of an undulator insertion device such as that on I07 or I22 where  $\lambda_P$  is the period length between opposing magnets, from Ref [17].

TABLE 1.1: A comparison of the photon brilliance from different light sources, adapted from Ref [3].

Light source	Approximate brilliance/ photons $s^{-1} mrad^{-2} 0.1\% \text{ bandwidth}^{-1}$
Candle	$10^5$
X-ray tube	$10^8$
Sun	$10^{10}$
Bending magnet	$10^{15}$
Undulator	$10^{20}$

When an electron accelerates (or travels on a curve), Cherenkov radiation is emitted in accordance with the Cherenkov relation,

$$n_i \beta_c \cos \theta_e = 1, \quad (1.35)$$

where,  $n_i$  is the refractive index for the dielectric medium,  $\beta_c$  is the fraction of the speed of light at which that electron is travelling, and  $\theta_e$  is the angle between the electron trajectory and the trajectory of the resulting photon.[17] The curve is the result of a bending magnet, meaning that at each bending magnet there can be a beamline which gives out synchrotron light. The light this is given off from a bending magnet is continuous and broad, covering a wide range of the electromagnetic spectrum. The alternative to a bending magnet beamline is a beamline which is served by an insertion device (ID). An insertion device is able to offer more specific radiation characteristics (photon energy, narrower band) than a bending magnet, and are placed on the magnet-free straight sections of the synchrotron. Common insertion devices include wavelength shifters, wigglers, and undulators.

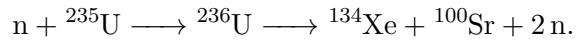
The type of insertion device that is present at both I07 and I22 at DLS is an undulator. An undulator consists of a series of magnets of opposing polarity which causes the electrons to ‘wiggle’ back and forth (Figure 1.13). This results in a superposition of radiation from  $N_P$  sources, where  $N_P$  is the number of magnets, yielding quasi-monochromatic radiation. The brilliance of different X-ray sources are compared in Table 1.1, this shows the significant benefit that an undulator offers in terms of photon brilliance.



### Neutrons

Neutrons hold an advantage over X-rays, particularly for application to the study of soft matter, in the ability to utilise contrast variation to increase the quantity of information from the sample, this is discussed in detail in Section 1.2.2. However, neutrons cannot be produced safely on a laboratory scale, therefore it is always necessary to visit large scale facilities to harness neutrons for scattering experiments. These facilities come in two flavours; the reactor source and the spallation source, each offering unique benefits.

Neutron reactor sources, such as the Institut Laue-Langevin (ILL) in Grenoble, France, as currently the most common format of neutron source and are capable of producing the highest average neutron flux, the number of neutrons per second per unit area, for example the High-Flux Reactor at the ILL is capable of producing a neutron flux of  $1.5 \times 10^{15}$  neutrons  $\text{s}^{-1}\text{cm}^{-2}$ . [18] A reactor source operates on the principle of nuclear fission, where an atomic nucleus is capable of breaking down into smaller nuclei, overcoming the strong nuclear force. This often involves using uranium enriched with its fissile isotope,  $^{235}\text{U}$ , which after the initial absorption of a stray neutron, from a cosmic ray, or spontaneous fission, will undergo fission to release, on average, 2.5 daughter neutrons, an example of a possible uranium fission mechanism is:



This type of mechanism is the basis for research, and nuclear power, reactors. [3] One of the major drawbacks for reactor neutron sources is the perceived public opinion towards such facilities. Major safety concerns, such as “nuclear meltdown” and the resulting nuclear waste, mean that reactor sources are often unpopular and therefore struggle to obtain funding required for operation.

The other form of neutron source is a spallation source, this is much less controversial as it does not require fissile materials and hence there is no risk of a nuclear disaster. The ISIS neutron and muon source (Oxfordshire, UK) is an example of a spallation source, where high energy protons, 800 MeV, [19] are accelerated towards a tungsten target. When the protons strike the target, they can cause the release of a series of neutrons, the first batch of neutrons are given off with too high an energy to be useful, however, less excited neutrons are given off by secondary emissions. In addition to the public perception benefit, spallation sources also have a technological advantage in the time-of-flight technique. The time-of-flight (ToF) technique is based on the fact that at a spallation source, it is possible to know the time at which the neutron was ejected by the target to a high level of precision and accuracy, and therefore it is possible to measure the time taken for the neutron to reach the instrument. Since the neutron is a particle of a finite mass,  $m$ , it is possible to correlate the velocity,  $v$ , of the particle with the kinetic energy,  $E_k$ ,

$$E_k = \frac{mv^2}{2}, \quad (1.36)$$

and with knowledge of the energy of the particle, its wavelength  $\lambda$ , can be determined by the de Broglie relation,

$$E = h\omega = \frac{hv}{\lambda}, \quad (1.37)$$

where,  $h$  is Planck’s constant and  $\omega$  is the neutron frequency. Therefore, the wavelength of the neutron is proportional to the inverse of the particle’s velocity, and hence the time-of-flight,  $t_F$ ,

$$\lambda = \frac{h}{mv} = \frac{ht_F}{mL_F}, \quad (1.38)$$

where,  $L_F$  is the distance between the target and the instrument. The fact that the neutrons can spread out in the flight from the target means that wavelength-dispersive techniques, where the neutron wavelength is measured rather than the scattering angle, are possible at spallation sources which cannot be carried out at reactor sources. The negative side-effect of current spallation sources is that they have a lower average flux than reactor sources, however the building of the European Spallation Source (ESS) will change this as it offers an average flux similar to that of a reactor source, but with the benefits of the spallation technique.

A problem that is inherent for both reactor and spallation sources is that the energy of the neutrons given off is usually too high to be used to study condensed materials, such as soft matter. This means that moderation must be used to reduce the energy of the neutrons passing through the sample. The neutrons which are considered to be optimal for the study of condensed materials are thermal neutrons, named because their energy is approximately that of ambient temperature. Thermal neutrons are achieved by allowing the neutrons to pass through a large volume of moderator material, usually graphite or heavy water ( $D_2O$ ), stored at 300 K before they reach the instrument.[3]

### 1.2.2 Contrast variation

The scattering profile generated by the interaction of some system with radiation depends on three factors:

- the spatial arrangement of the atoms in the system,
- the instrument being used to measure the pattern – instrumental resolution function, and
- the interaction between the radiation and the matter under investigation.

This final factor is perhaps better known as the ‘scattering contrast’, this is an extremely important factor in the study of soft matter, particularly when the probing radiation is the neutron. The scattering contrast makes it possible to select individual components of the system and investigate their structural properties.[20] The differential cross-section,  $d\sigma/d\Omega$  of a point scatterer, as shown in Eqn. 1.15, varies only with respect to the scattering length of the species,  $b$ ,

$$\frac{d\sigma}{d\Omega} \propto b^2. \quad (1.39)$$

However, as discussed in Section 1.1.4, it is often easier to use the scattering length density,  $\rho$ .

When an X-ray interacts with an atom, it is scattered by the interaction with the electrons, this is due to the X-ray being a form of electromagnetic radiation. Furthermore, it means that the scattering length of an atom by an X-ray is directly proportional to the number of electrons in the atom, so it is therefore difficult to discern between the scattering from a carbon atom (6 electrons) and a nitrogen atom (7 electrons), furthermore the scattering from hydrogen atoms is practically non-existent.

The scattering length a neutron by an atom varies unsystematically with respect to the atomic number of a species, this is shown in Figure 1.14. Furthermore to the apparently random variation with changes in atomic number, there is also significant variation with mass number, e.g. between isotopes of the same atom. This is also dependance due to the magnetic state of the atom, however this is normally unimportant for soft matter. The scattering lengths differ with the nuclear spin energy level, this

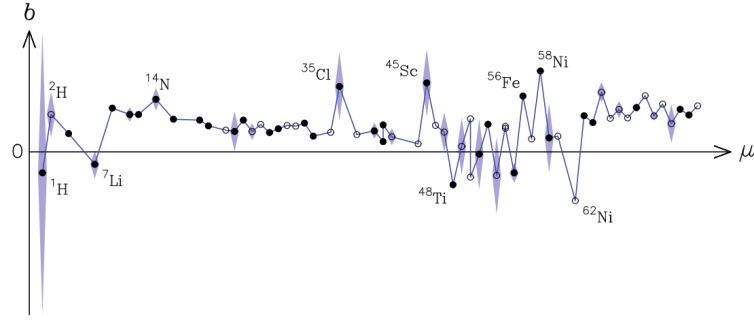


FIGURE 1.14: The variation of the average neutron scattering length,  $\langle b \rangle$  (circles), with atomic mass,  $\mu$ . The standard deviation,  $\Delta b$ , is indicated with the shaded regions, from Ref [3].

TABLE 1.2: Examples of coherent and incoherent scattering cross-sections, from Ref [20].

Isotope	$S$	$\sigma_{\text{coh}}/10^{-28} \text{ m}$	$\sigma_{\text{incoh}}/10^{-28} \text{ m}$
$^1\text{H}$	$1/2$	1.8	79.7
$^2\text{H}$	1	5.6	2.0
$^{12}\text{C}$	0	5.6	—
$^{14}\text{N}$	1	11.6	0.3
$^{16}\text{O}$	0	4.2	—

leads to an average scattering length,  $\langle b \rangle$ , for isotopes where the nuclear spin is non-zero ( $S \neq 0$ ). There are two forms of scattering, coherent and incoherent, for which the scattering cross-sections,  $\sigma$ , are determined by,

$$\begin{aligned}\sigma_{\text{coh}} &= 4\pi \langle b \rangle^2 \\ \sigma_{\text{incoh}} &= 4\pi (\langle b^2 \rangle - \langle b \rangle^2)\end{aligned}\tag{1.40}$$

The coherent scattering is the scattering from nuclei that all have the same value of  $\langle b \rangle$ , and leads to the important scattering pattern. Whereas, the incoherent scattering is caused by the ‘disorder’ between the isotopes, and is the cause of the background present in the measurement. Examples of these scattering cross-sections for nuclei relevant to soft matter are shown in Table 1.2. It can be seen that the incoherent scattering from the  $^1\text{H}$  nuclei is more than forty times the coherent scattering. This leads to a large, intrusive background present in the scattering pattern of hydrogenous samples.

The difference between the scattering of  $^1\text{H}$  and  $^2\text{H}$ , evident in Table 1.2, can lead to a very useful technique if soft matter scattering, known as contrast variation. The idea of contrast variation is based on the substitution of one isotope of an atom for another, while not introducing significant change to the properties of the material. Traditionally the benefit of this came in terms of contrast matching out a part of the system to reduce the dimensionality of the problem for analysis. For example, by matching the solvent scattering length density to that of the tails of the surfactants at the centre of a micelle there would only be scattering from the heads, and conversely there would only be scattering from the tails if the solvent had the same scattering length density as the head groups. This means that the problem becomes more straightforward as there are fewer variable parameters when fitting the data. This idea is represented graphically in Figure 1.15. The technique of contrast variation may also be used in terms of

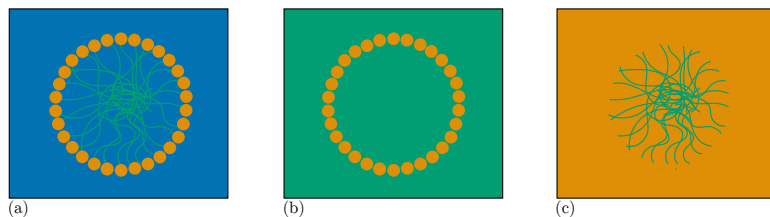


FIGURE 1.15: The effect of varying the scattering length density of the solvent in a micelle system, (a) the system in pure solvent, (b) the solvent is contrast matched to the surfactant tails, and (c) the solvent is contrast matched to the surfactant heads.

data analysis. By increasing the number of data sets corresponding to a single model – at different contrasts, the solution for the true structure of the model from the scattering data becomes more robust. This is due to the fact that each different contrast can be considered as an independent measurement of the same system, and hence each set of scattering data can be used within the data analysis procedure to obtain the best global agreement to the experiment. This co-refinement of multiple experiments can, under the right conditions, be used to simultaneously consider both neutron and X-ray datasets.[9]

There is also the possibility of using contrast variation when the probing radiation is the X-ray, through the use of anomalous scattering. This is where different wavelengths of radiation give different scattering, when the wavelengths are on opposite sides of an X-ray absorption edge. This is not frequently used for soft matter species, as the X-ray absorption edges for elements common in soft matter (H, C, N, O, etc.) are at very low X-ray energies so generally outside of the accessible range. [20]

### 1.3 Classical simulation

### 1.4 Optimisation & Sampling

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