ECE368: Probabilistic Reasoning
Lab 3: Hidden Markov Model

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You should hand in: 1) A scanned .pdf version of this sheet with your answers (file size should be under 2 MB); 2) one Python file inference.py that contains your code. The files should be uploaded to Quercus.

1. (a) Write down the formulas of the forward-backward algorithm to compute the marginal distribution $p(\mathbf{z}_i|(\hat{x}_0,\hat{y}_0),\ldots,(\hat{x}_{N-1},\hat{y}_{N-1}))$ for $i=0,1,\ldots,N-1$. Your answer should contain the initializations of the forward and backward messages, the recursion relations of the messages, and the computation of the marginal distribution based on the messages. (1.5 **pt**)

$p(z_i \mid (\hat{x_0}, \hat{y_0}), \ldots (\hat{x_{N-1}}, \hat{y_{N-1}})) = \gamma(z_i) = \frac{\alpha(z_i)\beta(z_i)}{\sum_{n=0} \alpha(z_n)\beta(z_n)}$ (normalized)

Where the initialization of the forward and backward messages is:

$\alpha(z_0) = p(z_0)\, p((\hat{x_0}, \hat{y_0}) \mid z_0)$ and $\beta(z_{N-1}) = 1$

And the recursion relations are:

$\alpha(z_i) = p((\hat{x_i}, \hat{y_i}) \mid z_i) \cdot \sum_{n=0}^{i-1} \alpha(z_n)\, p(z_i \mid z_n)$

$\beta(z_i) = \sum_{n=1}^{i+1} \beta(z_n)\, p((\hat{x_n}, \hat{y_n}) \mid z_n)\, p(z_n \mid z_i)$

(b) After you run the forward-backward algorithm on the data in test.txt, write down the obtained marginal distribution of the state at $i=99$ (the last time step), i.e., $p(\mathbf{z}_{99}|(\hat{x}_0,\hat{y}_0),\ldots,(\hat{x}_{99},\hat{y}_{99}))$. Only include states with non-zero probability in your answer. (2.5 **pt**)

$p(z_{99} \mid (\hat{x_0}, \hat{y_0}), \ldots (\hat{x_{99}}, \hat{y_{99}})) = \begin{cases} 0.8103 & (11, 0, \text{'stay'}) \\ 0.1796 & (11, 0, \text{'right'}) \\ 0.0101 & (10, 1, \text{'down'}) \end{cases}$

2. Modify your forward-backward algorithm so that it can handle missing observations. After you run the modified forward-backward algorithm on the data in test_missing.txt, write down the obtained marginal distribution of the state at $i=30$, i.e., $p(\mathbf{z}_{30}|(\hat{x}_0,\hat{y}_0),\ldots,(\hat{x}_{99},\hat{y}_{99}))$. Only include states with non-zero probability in your answer. (2 **pt**)

$p(z_{30} \mid (\hat{x_0}, \hat{y_0}), \ldots (\hat{x_{99}}, \hat{y_{99}})) = \begin{cases} 0.9130 & (6, 7, \text{'right'}) \\ 0.0435 & (5, 7, \text{'right'}) \\ 0.0435 & (5, 7, \text{'stay'}) \end{cases}$

3. (a) Write down the formulas of the Viterbi algorithm using $\mathbf{z}_i$ and $(\hat{x}_i, \hat{y}_i), i = 0, 1, \ldots, N-1$. Your answer should contain the initialization of the messages and the recursion of the messages in the Viterbi algorithm. (2 **pt**)

$w_0(z_0) = \ln p(z_0) + \ln p((\hat{x}_0, \hat{y}_0) \mid z_0)$

$\forall i \in [1, N-1]: \quad w_i(z_i) = \ln p((\hat{x}_i, \hat{y}_i \mid z_i)) + \max_{z_{i-1}} [\ln p(z_i \mid z_{i-1}) + w_{i-1}(z_{i-1})]$

Using backtracking, need to find:

$z_{i-1}^* = \phi_i(z_i) = \arg\max_{z_{i-1}} (w_i(z_i^*))$, solved recursively

(b) After you run the Viterbi algorithm on the data in test_missing.txt, write down the last 10 hidden states of the most likely sequence (i.e., $i = 90, 91, 92, \ldots, 99$) based on the MAP estimate. (3 **pt**)

$z_{90} = (11, 5,$ 'down'$)$
$z_{91} = (11, 6,$ 'down'$)$
$z_{92} = (11, 7,$ 'down'$)$
$z_{93} = (11, 7,$ 'stay'$)$
$z_{94} = (11, 7,$ 'stay'$)$
$z_{95} = (10, 7,$ 'left'$)$
$z_{96} = (9, 7,$ 'left'$)$
$z_{97} = (8, 7,$ 'left'$)$
$z_{98} = (7, 7,$ 'left'$)$
$z_{99} = (6, 7,$ 'left'$)$

4. Compute and compare the error probabilities of $\{\tilde{\mathbf{z}}_i\}$ and $\{\check{\mathbf{z}}_i\}$ using the data in test_missing.txt. The error probability of $\{\tilde{\mathbf{z}}_i\}$ is $\boxed{0.03}$. The error probability of $\{\check{\mathbf{z}}_i\}$ is $\boxed{0.02}$. (1 **pt**)

5. Is sequence $\{\check{\mathbf{z}}_i\}$ a valid sequence? If not, please find a small segment $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$ that violates the transition model for some time step i. You answer should specify the value of i as well as the corresponding states $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$. (1 **pt**)

$i = 64$: $z_{64} = (3, 7,$ 'stay'$)$
$z_{65} = (2, 7,$ 'stay'$)$

Since the action from z_{64} to z_{65} is 'stay', the robot's position should be the same, however, it is apparent that the robot has moved left