Regularization

In the beginning, we have motivated the use of a loss function with a Bayesian formulation combining the probability of the data given the model and the probability of the model

$$\log \mu_W(w|\mathcal{D} = \mathbf{d}) = \log \mu_{\mathcal{D}}(\mathbf{d}|W = w) + \log \mu_W(w) - \log Z$$

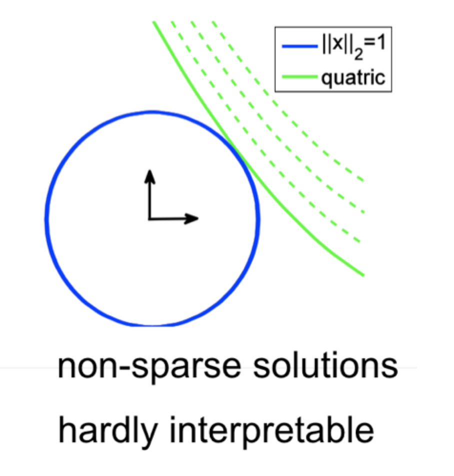
If μ_W is a Gaussian density with an isotropic covariance matrix $Cov = \lambda\mathbf{1}$ the log-prior $\log \mu_W(w)$ results in a quadratic penalty

$$\lambda\|w\|^2$$

Since this is convex, its sum with a convex functional is convex

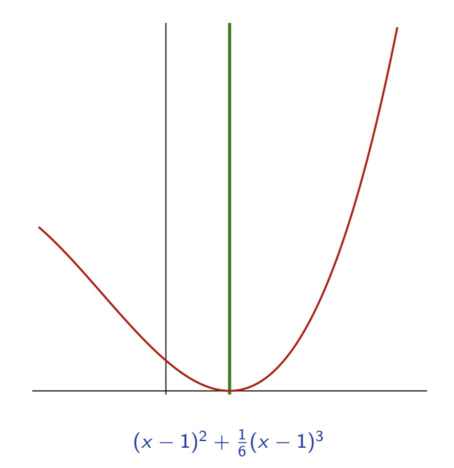
This is called the $\boldsymbol{\ell_2}$-**regularization**, or 'weight decay'

ℓ_2-norm visualization:



Increasing λ moves the optimum closer to 0 and away from the optimum for the loss alone

Since the derivative of $\|x\|^2$ is zero at zero, the optimum will never move there if it was not already there



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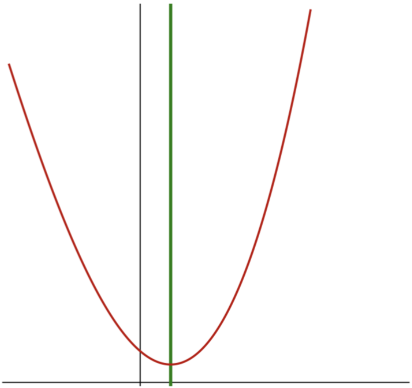

$(x-1)^2 + \frac{1}{6}(x-1)^3 + x^2$

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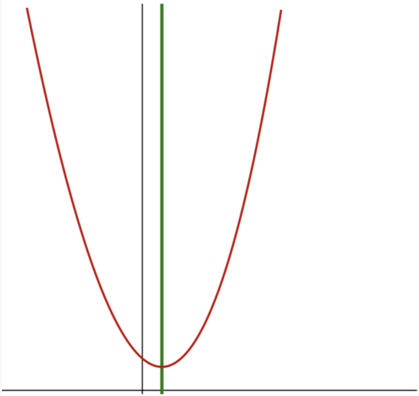

$(x-1)^2 + \frac{1}{6}(x-1)^3 + 2x^2$

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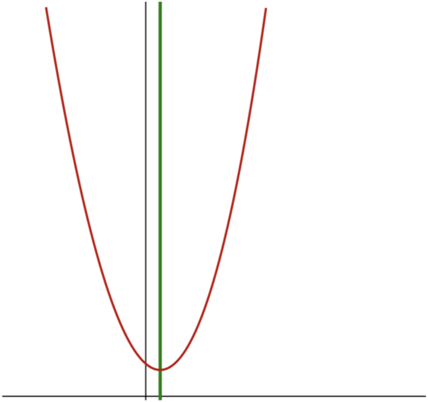

$(x-1)^2 + \frac{1}{6}(x-1)^3 + 3x^2$

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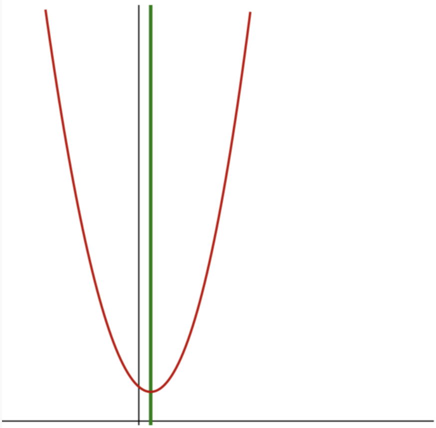


$(x-1)^2 + \frac{1}{6}(x-1)^3 + 4x^2$

We can apply the exact same scheme with a Laplace prior

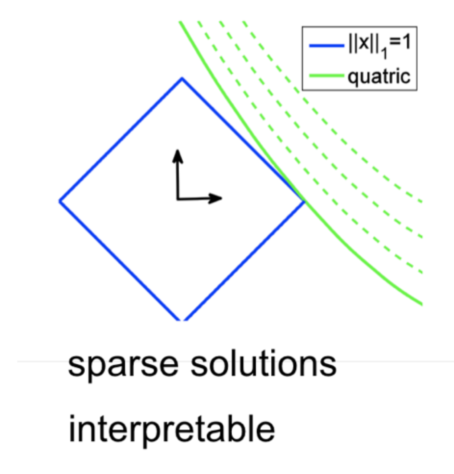
$$\begin{aligned}\mu(w) &= \frac{1}{(2b)^D} \exp\left(-\frac{\|w\|_1}{b}\right) \\ &= \frac{1}{(2b)^D} \exp\left(-\frac{1}{b}\sum_{d=1}^{D} |w|\right)\end{aligned}$$

which results in a penalty of the form

$$\lambda \|w\|_1$$

and is known as the **ℓ_1-regularization**. Similar to its ℓ_2 counterpart, the penalty is convex and its sum with a convex function also convex

ℓ_1-norm visualization:



An important property of the ℓ_1 regularization is that if E is convex and

$$w^* = \operatorname*{argmin}_w E(w) + \lambda \|w\|_1$$

then

$$\forall d, \left| \frac{\partial E}{\partial w_d} \right| < \lambda \Rightarrow w_d^* = 0$$

In practice it means that this penalty pushes some of the variables to zero, but in contrast to the ℓ_2 penalty, they remain there. ℓ_1 regularization requires the optimal solution to lie on a simplex

The λ parameter controls the sparsity of the solution.

With the ℓ_1 regularization, the update rule becomes

$$w_{t+1} = w_t - \eta g_t - \lambda \operatorname{sign}(w_t)$$

where sign is applied per component. This is almost identical to

$$\begin{aligned} w'_t &= w_t - \eta g_t \\ w_{t+1} &= w'_t - \lambda \operatorname{sign}(w'_t) \end{aligned}$$

This update may overshoot and result in a component of w'_t strictly on one side of 0 while the same component in w_{t+1} is strictly on the other

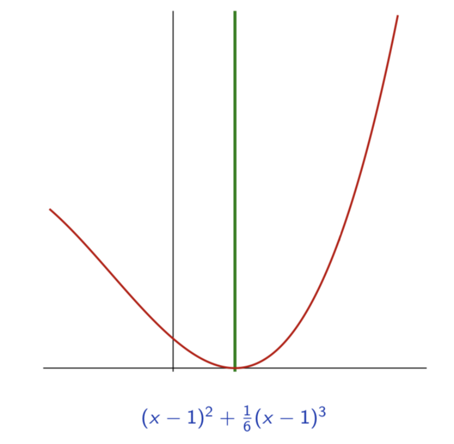
While this is not a problem in principle, since w_t will fluctuate around zero, it can be an issue if the zeroed weights are handled in a specific manner (e.g. sparse coding to reduce memory footprint or computation).

The **proximal operator** takes care of preventing parameters from 'crossing zero', by adapting λ when it is too large

$$\begin{aligned} w'_t &= w_t - \eta g_t \\ w_{t+1} &= w'_t - \min(\lambda, |w'_t|) \odot \operatorname{sign}(w'_t) \end{aligned}$$

where min is component-wise and $\odot$ is the Hadamard component-wise product

Increasing λ moves the optimum closer to 0, and away from the optimum of the loss without penalty



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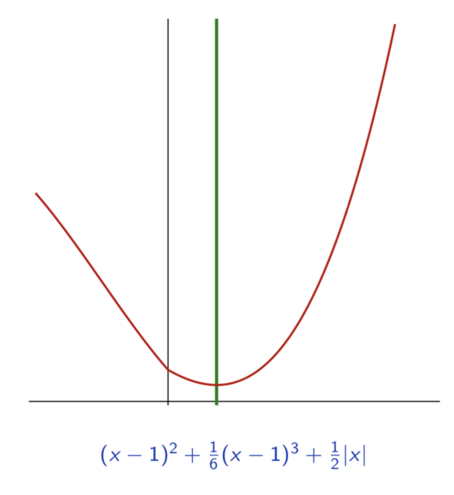

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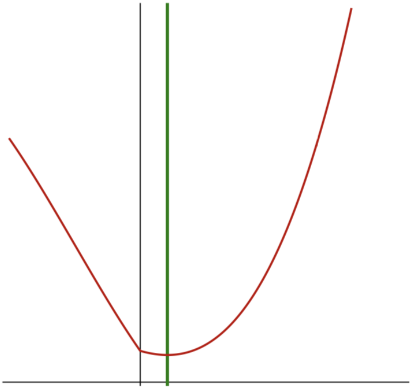

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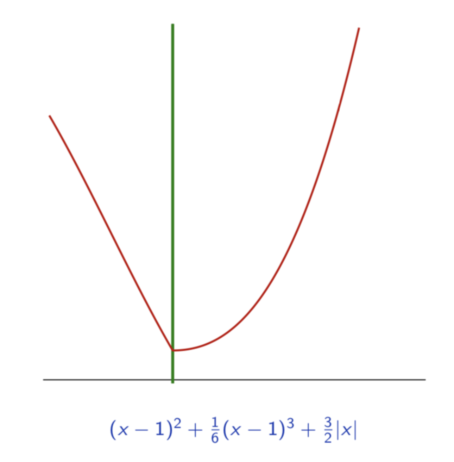

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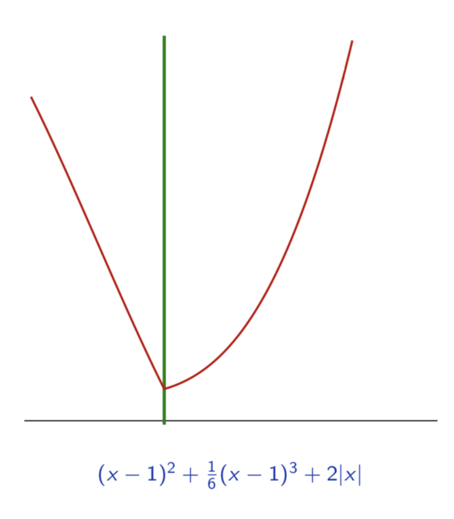

In general, regularization is often useful when dealing with complex models and data at small scales. While they have a limited impact for large-scale deep learning, they may still provide the little push needed to beat baselines