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Classwork

Question 1

Let the original instance space be $\mathbb{R}$ and consider the mapping ψ where for each nonnegative integer $n \geq 0$ there exists an element $\psi(x)_n$ that equals $\frac{1}{\sqrt{n!}} e^{-\frac{x^2}{2}} x^n$. Then find out the kernel function corresponding to this mapping function.

Solution

$$\begin{aligned}
\langle \psi(x), \psi(x') \rangle &= \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{n!}} e^{-\frac{x^2}{2}} x^n \right) \left(\frac{1}{\sqrt{n!}} e^{-\frac{(x')^2}{2}} (x')^n \right) \\
&= e^{-\frac{x^2+(x')^2}{2}} \sum_{n=0}^{\infty} \frac{(xx')^n}{n!} \\
&= e^{-\frac{\|x-x'\|^2}{2}}.
\end{aligned}$$

Question 2

Given a set of data points $x_1, \ldots, x_N$, we define the convex hull to be the set of all points $\mathbf{x}$ given by $\mathbf{x} = \sum_n \alpha_n x_n$ where $\alpha_n \geq 0$ and $\sum_n \alpha_n = 1$ (Intuitively, the convex hull of a set of points is the solid region that they enclose). Consider a second set of points $y_1, \ldots, y_M$ together with their corresponding convex hull. Show that the set of $\mathbf{x}$ and the set of $\mathbf{y}$ are linearly separable if and only if the convex hulls do not intersect.

Solution

Let C_X and C_Y be the convex hulls corresponding to points $X = \{x_i\}_{i=1}^N$ and $Y = \{y_i\}_{i=1}^N$ respectively.

$$C_X := \left\{ \mathbf{x} = \sum_{n=1}^{N} \alpha_n x_n : \alpha_i \geq 0, \sum_i \alpha_i = 1 \right\}$$

$$C_Y := \left\{ \mathbf{y} = \sum_{n=1}^{N} \beta_n y_n : \beta_i \geq 0, \sum_i \beta_i = 1 \right\}$$

Definition 1. The sets of points X and Y are said to be linearly separable if $\exists$ a line (hyperplane in high dimensional space) $L := \{x \in \mathbb{R}^D : m^T x + b = 0\}$ such that $m^T x_i + b \geq 0$, and $m^T y_i + b < 0 \ \forall i = 1, 2, \ldots, N$.

We have to show that X and Y are linearly separable iff $C_X \cap C_Y = \emptyset$.

($\Rightarrow$) Assume that X and Y are linearly separable. Suppose for contradiction that $C_X \cap C_Y \neq \emptyset$. This implies that $\exists \mathbf{z} \in C_X \cap C_Y$. Thus, by definition,

$$\mathbf{z} = \sum_{n=1}^{N} \alpha_n x_n = \sum_{n=1}^{N} \beta_n y_n$$

where $\alpha_i \geq 0, \sum_i \alpha_i = 1, \beta_i \geq 0, \sum_i \beta_i = 1$. Since, X and Y are linearly separable, there exists a line L satisfying $m^T x_i + b \geq 0$, and $m^T y_i + b < 0 \ \forall i = 1, 2, \ldots, N$.

Now, since $\alpha_i \geq 0, \ \forall i$,

$$\alpha_i m^T x_i + \alpha_i b \geq 0, \ \forall i = 1, 2, \ldots, N$$

$$\Rightarrow \sum_{n=1}^{N} \alpha_n m^T x_n + \sum_{n=1}^{N} \alpha_n b = \sum_{n=1}^{N} \alpha_n m^T x_n + b \sum_{n=1}^{N} \alpha_n = m^T \mathbf{z} + b \geq 0 \tag{1}$$

Similarly, since $\beta_i m^T y_i + \beta_i b < 0 \ \forall i = 1, 2, \ldots, N$,

$$\sum_{n=1}^{N} \beta_n m^T y_n + \sum_{n=1}^{N} \beta_n b = \sum_{n=1}^{N} \beta_n m^T y_n + b \sum_{n=1}^{N} \beta_n = m^T \mathbf{z} + b < 0 \tag{2}$$

The equations (1) and (2) present a contradiction. Thus, we have $C_X \cap C_Y = \emptyset$.

The second part of the proof is given as homework in Exercise 3.