Function: A function f is a rule that associates a unique output with each input. If the input is denoted by x, then the output is denoted by $f(x)$.

One-one function: Let $f: X \to Y$ be a function. Then f is one-to-one (injective) if f maps every element of X to a unique element in Y.. In other words no element of Y are mapped to by two or more elements of X. Symbolically, $\forall\, x_1, x_2 \in X$ *if* $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Onto function: Let $f: X \to Y$ be a function. Then f is onto (surjective) if every element of Y is mapped to by some element of A. In other words, nothing is left out. Symbolically, $\forall\, y \in Y$ there exist $x \in X$ such that $y = f(x)$.

Question:

Let $f: R - \left\{\frac{5}{4}\right\} \to R - \left\{\frac{1}{2}\right\}$ be a function defined by $f(x) = \dfrac{2x+3}{4x-5}$. Find the inverse of $f(x)$.

Let $f: R - \left\{\frac{-d}{c}\right\} \to R - \left\{\frac{a}{c}\right\}$ be a function defined by $f(x) = \dfrac{ax+b}{cx+d}$. Find the inverse of $f(x)$.

Solution:

Let $y = f(x) = \dfrac{2x+3}{4x-5}$

Then we get $y = \dfrac{2x+3}{4x-5}$

$$\Rightarrow 4xy - 5y = 2x + 3$$
$$\Rightarrow 4xy - 2x = 5y + 3$$
$$\Rightarrow x(4y-2) = 5y + 3$$
$$\Rightarrow x = \frac{5y+3}{4y-2}$$
$$\Rightarrow f^{-1}(y) = \frac{5y+3}{4y-2}$$
$$\therefore f^{-1}(x) = \frac{5x+3}{4x-2}$$

Let $y = f(x) = \dfrac{ax+b}{cx+d}$

Then we get $y = \dfrac{ax+b}{cx+d}$

$$\Rightarrow cxy + dy = ax + b$$
$$\Rightarrow cxy - ax = -dy + b$$
$$\Rightarrow x(cy-a) = -dy + b$$
$$\Rightarrow x = \frac{-dy+b}{cy-a}$$
$$\Rightarrow f^{-1}(y) = \frac{-dy+b}{cy-a}$$
$$\therefore f^{-1}(x) = \frac{-dx+b}{cx-a}$$

Even and odd functions: A function f is said to be an ***even function*** if $f(-x) = f(x) \cdots (1)$ and is said to be an ***odd function*** if $f(-x) = f(x) \quad \cdots (2)$. Geometrically, the graphs of even functions are symmetric about the y-axis because replacing x by $-x$ in the equation $y = f(x)$ yields $y = f(-x)$, which is equivalent to the original equation $y = f(x)$ by (1). Similarly, it follows from (2) that graphs of odd functions are symmetric about the origin. Some examples of even functions are x^2, x^4, x^6 and $\cos x$; and some examples of odd functions are x^3, x^5, x^7 and $\sin x$.

Limits: If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write $\lim_{x\to a} f(x) = L$ ......(3) which is read "the limit of $f(x)$ as x approaches a is L".

Evaluate the following limits

a) $\lim_{x\to 0}\frac{\tan x - \sin x}{x^3}$ b) $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{\cos 3x}{x^2}\right)$

Solution:

$$\lim_{x\to 0}\frac{\frac{\sin x}{\cos x} - \sin x}{x^3}$$

$$= \lim_{x\to 0}\frac{\frac{\sin x - \sin x\cos x}{\cos x}}{x^3}$$

$$= \lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}$$

$$= \lim_{x\to 0}\frac{\sin x(1-\cos x)(1+\cos x)}{x^3\cos x(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{\sin x(1-\cos^2 x)}{x^3\cos x(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{\sin^3 x}{x^3\cos x(1+\cos x)}$$

$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3 \lim_{x\to 0}\frac{1}{\cos x(1+\cos x)}$$

$$= 1^3 \times \frac{1}{1\times(1+1)} = \frac{1}{2}$$

$$\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{\cos 3x}{x^2}\right)$$

$$= \lim_{x\to 0}\frac{1-\cos 3x}{x^2}$$

$$= \lim_{x\to 0}\frac{3\sin 3x}{2x} \quad [\textit{differentiating with respect to } x]$$

$$= \lim_{x\to 0}\frac{9\cos 3x}{2} \quad [\textit{differentiating with respect to } x]$$

$$= \frac{9\times\cos 0}{2}$$

$$= \frac{9\times 1}{2}$$

$$= \frac{9}{2}$$

Exercise:

(i) $\lim_{x\to 0}\frac{1-2\cos x+\cos 2x}{x^2}$ (ii) $\lim_{x\to 0}\frac{\sin 7x - \sin x}{\sin 6x}$ (iii) $\lim_{x\to 0}\frac{\cos 7x - \cos 9x}{\cos 3x - \cos 5x}$ (iv) $\lim_{x\to 0}\frac{1-\cos 7x}{3x^2}$

(v) $\lim_{x\to 0}\frac{\cos 3x - \cos 5x}{x^2}$ (vi) $\lim_{x\to \frac{\pi}{2}}\frac{1-\sin x}{\cos x}$ (vii) $\lim_{x\to 0}\frac{\sec^3 x - \tan^3 x}{\tan x}$. (vii) $\lim_{x\to 0}\frac{\tan x - \sin x}{\sin^3 x}$

One-sided limits: If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write $\lim_{x\to a^+} f(x) = L$ ......(4) and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write $\lim_{x\to a^-} f(x) = L$ ......(5). Expression (4) is read "the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right." Similarly, expression (5) is read "the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches from a the left."

Continuity: A function f is said to be ***continuous at*** $x = a$ provided the following conditions are satisfied:

1. $f(a)$ is defined.
2. $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists
3. $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$

Question: Check the continuity of the following function at point $x = 0,1,2$

$$f(x) = \begin{cases} -x^2 & \text{if} \quad x \le 0 \\ 5x-4 & \text{if} \quad 0 < x \le 1 \\ 4x^2-3x & \text{if} \quad 1 < x < 2 \\ 3x+4 & \text{if} \quad x \ge 2 \end{cases}$$

Solution: Continuity at point $x = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(5x-4) = 5\times 0 - 4 = -4$ and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x^2) = -0^2 = 0$

Since $\lim_{x\to 0^+} f(x) \ne \lim_{x\to 0^-} f(x)$, so the function $f(x)$ is not continuous at $x = 0$

Continuity at point $x = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4x^2-3x) = 4\times 1^2 - 3\times 1 = 4-3 = 1$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(5x-4) = 5\times 1 - 4 = 1$

Since $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$, so the function $f(x)$ is continuous at $x = 1$

Continuity at point $x = 2$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3x+4) = 3\times 2 + 4 = 10$ & $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(4x^2-3x) = 4\times 2^2 - 3\times 2 = 16-6 = 10$

Since $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$, so the function $f(x)$ is continuous at $x = 2$

Question: Check the continuity of the following function at point $x=0,2$

$$f(x)=\begin{cases}3+2x & \text{if} \quad -2<x\le 0\\ 3-2x & \text{if} \quad 0<x<2\\ -3-2x & \text{if} \quad x\ge 2\end{cases}$$

Solution: **Continuity at point** $x=0$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}(3-2x)=3-2\times 0=3$ and $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}(3+2x)=3+2\times 0=3$

Since $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^-} f(x)$, so the function $f(x)$ is continuous at $x=0$

Continuity at point $x=2$

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}(-3-2x)=-3-2\times 2=-7$ and $\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}(3-2x)=3-2\times 2=-1$

Since $\lim\limits_{x\to 2^+} f(x)\ne\lim\limits_{x\to 2^-} f(x)$, so the function $f(x)$ is not continuous at $x=2$

Exercise:

1. Check the continuity of the following function at point $x=1,2$

$$f(x)=\begin{cases}2x^3+x-7 & \text{if} \quad x\le 1\\ 5x+9 & \text{if} \quad 1<x<2\\ x^2-4x+3 & \text{if} \quad x\ge 2\end{cases}$$

2. Check the continuity of the following function at point $x=0,1$

$$f(x)=\begin{cases}x^2+1 & \text{if} \quad x<0\\ x & \text{if} \quad 0\le x\le 1\\ 1/x & \text{if} \quad x>1\end{cases}$$

Maxima and Minima: V.V.I

Find the maximum and minimum value of the following functions

$(i)\quad f(x)=2x^3-6x^2-18x+7$

$(ii)\quad f(x)=x^4-4x^3+4x^2$

$(iii)\quad f(x)=3x^4-20x^3-6x^2+60x+15$

$(iv)\quad f(x)=x^4+2x^3-3x^2-4x+4$

(i) Given $f(x)=2x^3-6x^2-18x+7$

Therefore $f'(x)=6x^2-12x-18$ and $f''(x)=12x-12$

For maximum or minimum value $f'(x)=0$

$\Rightarrow 6x^2-12x-18=0$

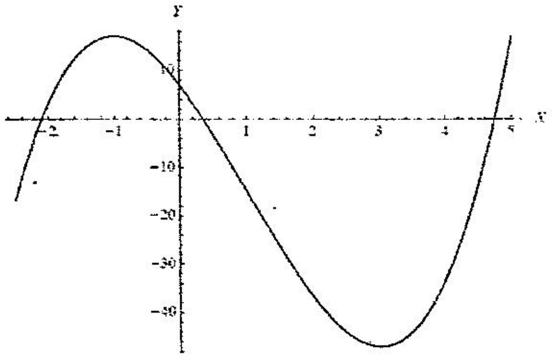
$\Rightarrow x^2-2x-3=0$

$\Rightarrow x^2-3x+x-3=0$

$\Rightarrow x(x-3)+1(x-3)=0$

$\Rightarrow (x-3)(x+1)=0$

$\therefore\quad x=-1,3$



At point $x=-1$, $f''(-1)=-12-12=-24<0$. So the function $f(x)$ is maximum at point $x=-1$

Maximum value at $x=-1$ is $f(-1)=2(-1)^3-6(-1)^2-18(-1)+7=-2-6+18+7=17$

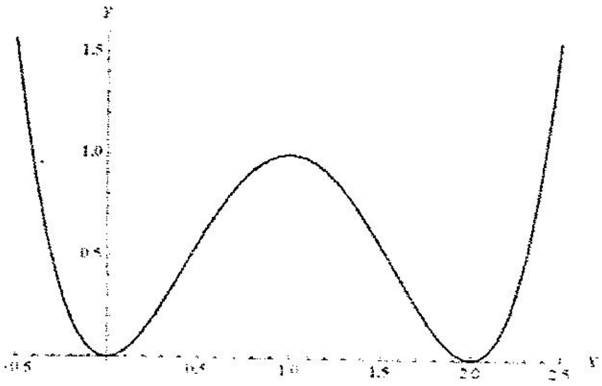
At poin $x=3$, $f''(3)=12\times3-12=24>0$. So the function $f(x)$ is minimum at point $x=3$.

Minimum value at $x=3$ is $f(3)=2(3)^3-6(3)^2-18(3)+7=54-54-54+7=-47$

(ii) Given $f(x)=x^4-4x^3+4x^2$

Therefore $f'(x)=4x^3-12x^2+8x$ and $f''(x)=12x^2-24x+8$

For maximum or minimum value $f'(x)=0$



$\Rightarrow 4x^3-12x^2+8x=0$

$\Rightarrow x^3-3x^2+2x=0$

$\Rightarrow x(x^2-3x+2)=0$

$\Rightarrow x(x^2-2x-x+2)=0$

$\Rightarrow x\{x(x-2)-1(x-2)\}=0$

$\Rightarrow x(x-2)(x-1)=0$

$\therefore\ x=0,1,2$

At point $x=0$, $f''(0)=12\times0-24\times0+8=8>0$. So the function $f(x)$ is minimum at point $x=0$

Minimum value at $x=0$ is $f(0)=0^4-4\times0^3+4\times0^2=0$

At point $x=1$, $f''(1)=12\times1-24\times1+8=-4<0$. So the function $f(x)$ is maximum at point $x=1$. Maximum value at $x=1$ is $f(1)=1^4-4\times1^3+4\times1^2=1-4+4=1$

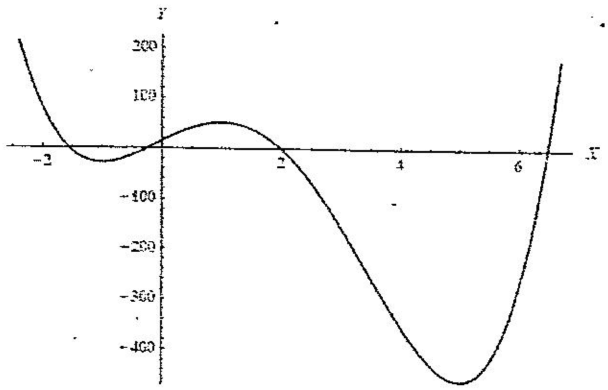
At poin $x=2$, $f''(2)=12\times2^2-24\times2+8=48-48+8=8>0$. So the function $f(x)$ is minimum at point $x=2$.

Minimum value at $x=2$ is s $f(2)=2^4-4\times2^3+4\times2^2=16-32+16=0$

(iii) Given $f(x)=3x^4-20x^3-6x^2+60x+15$

Therefore $f'(x)=12x^3-60x^2-12x+60$ and $f''(x)=36x^2-120x-12$

For maximum or minimum value $f'(x)=0$



$\Rightarrow 12x^3-60x^2-12x+60=0$

$\Rightarrow x^3-5x^2-x+5=0$

$\Rightarrow x^2(x-5)-1(x-5)=0$

$\Rightarrow (x^2-1)(x-5)=0$

$\Rightarrow (x+1)(x-1)(x-5)=0$

$\therefore\ x=-1,1,5$

At point $x=-1$, $f''(-1)=36\times(-1)^2-120\times(-1)-12=36+120-12=144>0$.

So the function $f(x)$ is minimum at point $x=-1$. Minimum value at $x=-1$ is

$f(-1)=3(-1)^4-20\times(-1)^3-6\times(-1)^2+60(-1)+15=3+20-6-60+15=-28$

At poin $x=1$, $f''(1)=36\times1^2-120\times1-12=-96<0$. So the function $f(x)$ is maximum at point $x=1$. Maximum value at $x=1$ is
$f(1)=3\times1^4-20\times1^3-6\times1^2+60\times1+15=3-20-6+60+15=52$

At poin $x=5$, $f''(5)=36\times5^2-120\times5-12=288>0$. So the function $f(x)$ is minimum at point $x=5$. Minimum value at $x=5$ is
$f(5)=3\times5^4-20\times5^3-6\times5^2+60\times5+15=1875-2500-150+300+15=-460$

(iv) Given $f(x)=x^4+2x^3-3x^2-4x+4$

Therefore $f'(x)=4x^3+6x^2-6x-4$ and $f''(x)=12x^2+12x-6$

For maximum or minimum value $f'(x)=0$

$\Rightarrow 4x^3+6x^2-6x-4=0$

$\Rightarrow 2x^3+3x^2-3x-2=0$

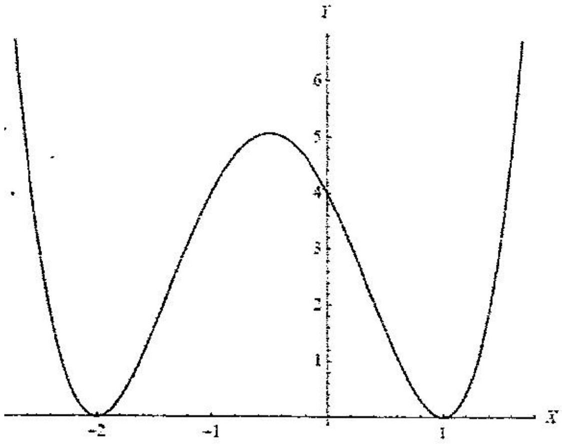
$\Rightarrow 2x^3-2x^2+5x^2-5x+2x-2=0$

$\Rightarrow 2x^2(x-1)+5x(x-1)+2(x-1)=0$

$\Rightarrow (x-1)(2x^2+5x+2)=0$

$\Rightarrow (x-1)(2x+1)(x+2)=0$

$\therefore \quad x=-2,-\frac{1}{2},1$



At point $x=-2$, $f''(-2)=12(-2)^2+12(-2)-6=48-24-6=18>0$.

So the function $f(x)$ is minimum at point $x=-2$. Minimum value at $x=-2$ is
$f(-2)=(-2)^4+2\times(-2)^3-3\times(-2)^2-4(-2)+4=16-16-12+8+4=0$

At poin $x=-\frac{1}{2}$, $f''\left(-\frac{1}{2}\right)=12\left(-\frac{1}{2}\right)^2+12\left(-\frac{1}{2}\right)-6=3-6-6=-9<0$.

So the function $f(x)$ is maximum at point $x=-\frac{1}{2}$. Maximum value at $x=-\frac{1}{2}$ is

$f\left(-\frac{1}{2}\right)=\left(-\frac{1}{2}\right)^4+2\times\left(-\frac{1}{2}\right)^3-3\times\left(-\frac{1}{2}\right)^2-4\left(-\frac{1}{2}\right)+4=\frac{1}{16}-\frac{1}{4}-\frac{3}{4}+2+4=\frac{81}{16}$

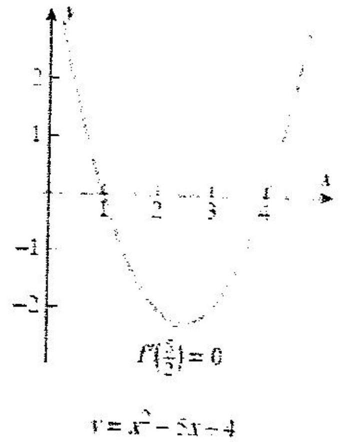
At point $x=1$, $f''(1)=12\times1^2+12\times1-6=18>0$.

So the function $f(x)$ is minimum at point $x=1$.

Minimum value at $x=1$ is $f(1)=(1)^4+2\times(1)^3-3\times(1)^2-4(1)+4=1+2-3-4+4=0$

Rolle's Theorem: Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a)=0$ and $f(b)=0$ then there is at least one point c in the interval (a, b) such that $f'(c)=0$

Example-1: Verify that the function $f(x)=x^2-5x+4$ satisfied the hypotheses of Rolle's Theorem are on the interval $[1,4]$, and find all values of c in that interval that satisfy the conclusion of the theorem.



Solution: The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $[1,4]$ and differentiable on $(1,4)$.

Also $f(1)=1^2-5\times1+4=0$ and $f(4)=4^2-5\times4+4=0$

So the hypotheses of Rolle's Theorem are satisfied on the interval [1, 4].

Thus, we are guaranteed the existence of at least one point c in the interval $(1,4)$ such that $f'(c)=0$. Differentiating $f(x)$ we get $f'(x)=2x-5$. Therefore $f'(c)=2c-5$

Now $f'(c)=0 \Rightarrow 2c-5=0 \quad \therefore c=\dfrac{5}{2}$

So $c=\dfrac{5}{2}$ is a point in the interval $(1,4)$ at which $f'(c)=0$

Example-2: Verify that the function $f(x)=2x^3+x^2-4x-2$ satisfied the hypotheses of Rolle's Theorem are on the interval $\left[-\sqrt{2},\sqrt{2}\right]$, and find all values of c in that interval that satisfy the conclusion of the theorem.

Solution: The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $\left[-\sqrt{2},\sqrt{2}\right]$ and differentiable on $\left(-\sqrt{2},\sqrt{2}\right)$.

Also $f(a)=f\left(-\sqrt{2}\right)=\left(-\sqrt{2}\right)^3+\left(-\sqrt{2}\right)^2-4\left(-\sqrt{2}\right)-2=-4\sqrt{2}+2+4\sqrt{2}-2=0$

and $f(b)=f\left(\sqrt{2}\right)=\left(\sqrt{2}\right)^3+\left(\sqrt{2}\right)^2-4\sqrt{2}-2=4\sqrt{2}+2-4\sqrt{2}-2=0$

So the hypotheses of Rolle's Theorem are satisfied on the interval $\left[-\sqrt{2},\sqrt{2}\right]$.

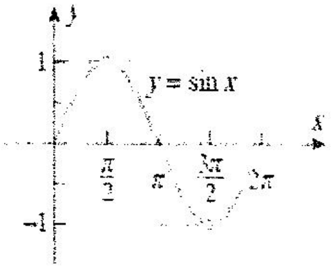
Here $f'(x)=6x^2+2x-4$. Therefore $f'(c)=6c^2+2c-4$

From Rolle's Theorem, we have $f'(c)=0$

$$\Rightarrow 6c^2+2c-4=0$$
$$\Rightarrow 3c^2+c-2=0$$
$$\Rightarrow 3c^2+3c-2c-2=0$$
$$\Rightarrow 3c(c+1)-2(c+1)=0$$
$$\Rightarrow (c+1)(3c-2)=0$$
$$\therefore c=-1,\frac{2}{3}$$

Thus $c=-1,\frac{2}{3}$ are lying in the interval $\left(-\sqrt{2},\sqrt{2}\right)$ at which $f'(c)=0$

Example-3: The function $f(x)=\sin x$ is continuous and differentiable everywhere, so the hypotheses of Rolle's Theorem are satisfied on the interval $[0,\ 2\pi]$ whose endpoints are roots of f. As indicated in Figure, there are two points in the interval $[0,\ 2\pi]$ at which the graph of f has a horizontal tangent, $c_1=\frac{\pi}{2}$ and $c_2=\frac{3\pi}{2}$.



Exercise: Verify that the hypotheses of Rolle's Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

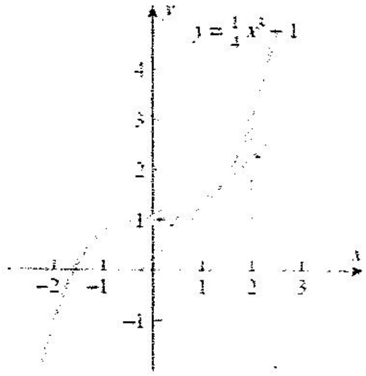
(i) $f(x)=x^3+2x^2-5x-10;$ $\left[-\sqrt{5},\sqrt{5}\right]$

(iii) $f(x)=\cos x;$ $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$

Mean-Value Theorem: Let f be continuous on the closed interval $[a,\ b]$ and differentiable on the open interval $(a,\ b)$. Then there is at least one point c in $(a,\ b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Example-1: Show that the function $f(x)=\frac{1}{4}x^3+1$ satisfies the hypotheses of the Mean-Value Theorem over the interval $[0,2]$, and find all values of c in the interval $(0,2)$ at which the tangent line to the graph of f is parallel to the secant line joining the points $(0,f(0))$ and $(2,f(2))$.

Solution: The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $[0,2]$ and differentiable on $(0,2)$, so the hypotheses of the Mean-Value Theorem are satisfied with $a = 0$ and $b = 2$.



But $f(a) = f(0) = 1$ and $f(b) = f(2) = 3$, $f'(x) = \frac{3}{4}x^2$, $f'(c) = \frac{3}{4}c^2$ so in this case Equation (1) becomes $\frac{3c^2}{4} = \frac{3-1}{2-0}$ *or,* $3c^2 = 4$ which has the two solutions $c = \pm 2/\sqrt{3} \approx \pm 1.15$. However, only the positive solution lies in the interval $(0,2)$; this value of c is consistent with Figure.

Example-2: Verify that the function $f(x) = x^3 - 6x^2 + 11x - 6$ hypotheses of the Mean-Value Theorem are satisfied on the interval $[0, 4]$ and find all values of c in that interval that satisfy the conclusion of the theorem.

Solution: The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $[0,4]$ and differentiable on $(0, 4)$. So the hypotheses of the Mean-Value Theorem are satisfied with $a = 0$ and $b = 4$.

Now $f(a) = f(0) = 0^3 - 6\times 0^2 + 11\times 0 - 6 = -6$ and $f(b) = f(4) = 4^3 - 6\times 4^2 + 11\times 4 - 6 = 6$

Here $f'(x) = 3x^2 - 12x + 11$. Therefore $f'(c) = 3c^2 - 12c + 11$

From Mean Value Theorem, we have $f'(c) = \frac{f(b) - f(a)}{b - a}$

$$\Rightarrow 3c^2 - 12c + 11 = \frac{6+6}{4-0} = \frac{12}{4} = 3$$

$$\Rightarrow 3c^2 - 12c + 8 = 0$$

$$\Rightarrow c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4\times 3\times 8}}{2\times 3}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$\therefore c = 0.845, 3.155$$

Thus $C = 0.845,\ 3.155$ are lying in the interval $(0, 4)$.

Exercise: Verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

$(i)\ \ f(x)=x^3-4x;\ \ [-2,1]$

$(ii)\ \ f(x)=x^3+x-2;\ \ [-1,2]$

$(iii)\ \ f(x)=x^3+3x^2-5x;\ \ [1,2]$

LEIBNITZ'S THEOREM:

If u and v are any two functions of x such that all their desired differential coefficients exist, then the n-th differential coefficient of their product is given by

$$\frac{d^n}{dx^n}(uv)=\frac{d^nu}{dx^n}v+{}^nc_1\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx}+{}^nc_2\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2}+\cdots\cdots+{}^nc_r\frac{d^{n-r}u}{dx^{n-r}}\frac{d^rv}{dx^r}+\cdots\cdots+u\frac{d^nv}{dx^n}\ \text{ or,}$$

$$\frac{d^n}{dx^n}(uv)=\frac{d^nu}{dx^n}v+n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx}+\frac{n(n-1)}{2!}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2}+\cdots\cdots+\frac{n!}{r!(n-r)!}\frac{d^{n-r}u}{dx^{n-r}}\frac{d^rv}{dx^r}+\cdots\cdots+u\frac{d^nv}{dx^n}$$

Example 1: If $y=a\cos(\ln x)+b\sin(\ln x)$, where a and b are constant, then show that

$x^2y_2+xy_1+y=0$ and $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$

Solution: Given $y=a\cos(\ln x)+b\sin(\ln x)$

Differentiating both sides with respect to x, we get

$$y_1=-a\sin(\ln x)\frac{1}{x}+b\cos(\ln x)\frac{1}{x}\ \text{ or, } xy_1=-a\sin(\ln x)+b\cos(\ln x)$$

Now again differentiating both sides, we get

$$xy_2+y_1=-a\cos(\ln x)\frac{1}{x}-b\sin(\ln x)\frac{1}{x}$$

or, $x^2y_2+xy_1=-a\cos(\ln x)-b\sin(\ln x)=-\{a\cos(\ln x)+b\sin(\ln x)\}=-y$

$\therefore\ \ x^2y_2+xy_1+y=0$

Again differentiating both sides in times by Leibnitz's theorem,

$$\frac{d^n}{dx^n}(x^2y_2)+\frac{d^n}{dx^n}(xy_1)+\frac{d^n}{dx^n}(y)=0$$

$$\Rightarrow x^2\frac{d^n}{dx^n}(y_2)+n\frac{d}{dx}(x^2)\frac{d^{n-1}}{dx^{n-1}}(y_2)+\frac{n(n-1)}{2}\frac{d^2}{dx^2}(x^2)\frac{d^{n-2}}{dx^{n-2}}(y_2)+x\frac{d^n}{dx^n}(y_1)+n\frac{d^{n-1}}{dx^{n-1}}(y_1)+\frac{d^n}{dx^n}(y)=0$$

$$\Rightarrow x^2y_{n+2}+n2xy_{n+1}+\frac{n(n-1)}{2}2y_n+xy_{n+1}+ny_n+y_n=0$$

$\therefore\ \ x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$

Example 2: If $y=\sin\left(m\sin^{-1}x\right)$, then show that $\left(1-x^2\right)y_2-xy_1+m^2y=0$ and deduced that

$\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-\left(n^2-m^2\right)y_n=0$

Solution: Given $y=\sin\left(m\sin^{-1}x\right)$

Differentiating both sides with respect to x, we get

$y_1=\cos\left(m\sin^{-1}x\right)\frac{m}{\sqrt{1-x^2}}$ or, $\left(1-x^2\right)y_1^2=m^2\cos^2\left(m\sin^{-1}x\right)$ [squaring both sides]

Or, $\left(1-x^2\right)y_1^2=m^2-m^2\sin^2\left(m\sin^{-1}x\right)=m^2-m^2y^2$

Or, $\left(1-x^2\right)y_1^2+m^2y^2=m^2$

Now again differentiating both sides, we get

$\left(1-x^2\right)2y_1y_2+(-2x)y_1^2+2m^2yy_1=0$

$\therefore\ \left(1-x^2\right)y_2-xy_1+m^2y=0$

Again differentiating both sides in times by Leibnitz's theorem,

$$\frac{d^n}{dx^n}\left\{\left(1-x^2\right)y_2\right\}-\frac{d^n}{dx^n}(xy_1)+m^2\frac{d^n}{dx^n}(y)=0$$

$$\Rightarrow\left(1-x^2\right)\frac{d^n}{dx^n}(y_2)+n(-2x)\frac{d^{n-1}}{dx^{n-1}}(y_2)+\frac{n(n-1)}{2}(-2)\frac{d^{n-2}}{dx^{n-2}}(y_2)-x\frac{d^n}{dx^n}(y_1)-n\frac{d^{n-1}}{dx^{n-1}}(y_1)+m^2y_n=0$$

$$\Rightarrow\left(1-x^2\right)y_{n+2}-2nxy_{n+1}-\frac{n(n-1)}{2}2y_n-xy_{n+1}-ny_n+m^2y_n=0$$

$$\therefore\ \left(1-x^2\right)y_{n+2}-(2n-1)xy_{n+1}-\left(n^2-m^2\right)y_n=0$$

Exercise:

1. If $y=e^{a\sin^{-1}x}$, then show that $\left(1-x^2\right)y_2-xy_1-a^2y=0$ and deduced that

   $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-\left(n^2+a^2\right)y_n=0$

2. If $y=e^{b\tan^{-1}x}$, then show that $\left(1+x^2\right)y_2+(2x-b)y_1=0$ and deduced that

   $\left(1+x^2\right)y_{n+2}+(2nx+2x-b)y_{n+1}+n(n+1)y_n=0$

Example-1: A store selling 200 Lenovo laptop a week at $\$350$ each. A market survey indicates that for each $\$10$ rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the profit function. How large a rebate should the store offer to maximize its profit?

Solution: If x is the number of Lenovo laptop sold per week, then the weekly increase in sales is $(x-200)$. For each increase of 20 units sold, the price is decrease by $\$10$. So for each additional unit sold, the decrease in price will be $\frac{10}{20}$ and the demand function is

$$d(x)=350-\frac{10}{20}(x-200)=450-\frac{1}{2}x$$

The profit function is $p(x)=x\,d(x)=450\,x-\frac{1}{2}x^2$.

Since $p'(x)=450-x$, we see that $p'(x)=0$ when $x=450$. This value of x gives an absolute maximum because $p''(x)=-1$ is always negative.

The corresponding price is $d(450)=450-\frac{1}{2}450=225$ and the rebate is $350-225=125$

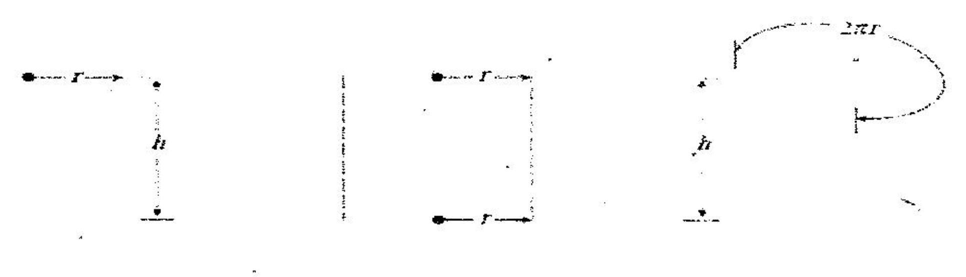
Therefore, to maximize profit the store should offer a rebate of $\$125$.

Example 3: A closed cylindrical can need to be made up with fixed volume. How should we choose the height and radius to minimize the amount of material needed to manufacture the can?

Solution: Let h be the height, r be the radius, V be the volume and S be the total surface area of the can. The total surface area is $S=2\pi r^2+2\pi rh \quad \cdots\cdots \quad (1)$

Volume ot the cylinder is $\pi r^2 h=V$ (constant). Therefore $h=\frac{V}{\pi r^2} \quad \cdots\cdots \quad (2)$

Substituting (2) in (1) yields $S=2\pi r^2+\frac{2V}{r} \quad \cdots\cdots \quad (3)$



Area $2\pi r^2$ Area $2\pi rh$

Differentiating (3) with respect to r, we get $\frac{dS}{dr}=4\pi r-\frac{2V}{r^2}$ and $\frac{d^2S}{dr^2}=4\pi+\frac{4V}{r^3}$

For maximum or minimum $\frac{dS}{dr}=0 \Rightarrow 4\pi r-\frac{2V}{r^2}=0 \Rightarrow 2\pi r=\frac{V}{r^2} \Rightarrow r^3=\frac{V}{2\pi} \quad \therefore r=\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$

For $r=\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$, $\frac{d^2S}{dr^2}=4\pi+\frac{4V\times 2\pi}{V}=12\pi>0$

Therefore the total surface area of the cylinder will minimum for $r=\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$

Now , $\pi r^2 h=V=2\pi r^3 \quad \therefore h=2r$

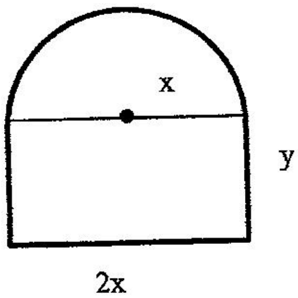
Thus the total surface area will minimum if height is equal to the diameter of the cylinder.

Question: The upper part of the frame of windows is like a half circle. Show that maximum light will enter in room if the height of the rectangular is equal to the radius of the half circle.

Solution: Let y be the height of rectangular part, x be the radius of the half circular part and S be the perimeter of the windows. Then $2x$ is its width. Now $S=2x+y+\pi x+y=2y+(2+\pi)x$

$$2y=S-(2+\pi)x \quad \cdots\cdots\cdots \quad (1)$$

If A is the area of the windows, then $A=2xy+\frac{\pi x^2}{2}=xS-2x^2-\frac{\pi x^2}{2}$



Differentiating A with respect to x we get, $\frac{dA}{dx}=S-4x-\pi x$ and $\frac{d^2A}{dx^2}=-4-\pi$

For maximum area of the windows $\frac{dA}{dx}=0 \Rightarrow S-4x-\pi x=0 \quad \therefore x=\frac{S}{4+\pi}$

Since $\frac{d^2A}{dx^2}<0$ for all x. Therefore the area of the windows will maximum for $x=\frac{S}{4+\pi}$.

That is, maximum light enter in room. Substituting the value of x in (1) we get $y=\frac{S}{4+\pi}$

Thus the radius of the half circle is equal to the height of the rectangle.