Matrix Course Cheat Sheet

Given Formulas

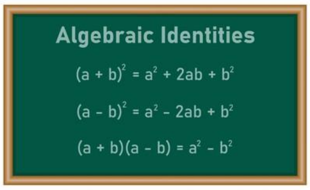
$\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$ $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ $\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$ $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$ $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta.$ $\operatorname{rank}(A) + \operatorname{nullity}(A) = \#\text{ columns of } A$

$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$ $\det(AB) = (\det A)(\det B)$ $\det A = \det A^T$ $\det(A - \lambda I) = 0$ $D = P^{-1}AP$

	Normal Form	General Form	Vector Form	Parametric Form
Lines	$\begin{cases}\mathbf{n}_1\cdot\mathbf{x} = \mathbf{n}_1\cdot\mathbf{p}_1 \\ \mathbf{n}_2\cdot\mathbf{x} = \mathbf{n}_2\cdot\mathbf{p}_2\end{cases}$	$\begin{cases}a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2\end{cases}$	$\mathbf{x} = \mathbf{p} + t\mathbf{d}$	$\begin{cases}x = p_1 + td_1 \\ y = p_2 + td_2 \\ z = p_3 + td_3\end{cases}$
Planes	$\mathbf{n}\cdot\mathbf{x} = \mathbf{n}\cdot\mathbf{p}$	$ax + by + cz = d$	$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$	$\begin{cases}x = p_1 + su_1 + tv_1 \\ y = p_2 + su_2 + tv_2 \\ z = p_3 + su_3 + tv_3\end{cases}$

Useful Matlab formulas and functions



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dot_product = dot(u, v);

cross(u,v); # cross product

norm(u); # Length of vector u

proj_u_v = (dot(u, v) / norm(u)^2) * u; # Projection of vector v onto u:

A_transpose = A';

Element_wise_mul → C = A .* B;

determinant_A = det(A);

A_inversion = inv(A);

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

x = A \ b; # Solving Linear Systems: Instead of inverting matrices, solve systems of equations:

[V, D] = eig(A); # V contains the eigenvectors. D contains the eigenvalues along the diagonal.

N = null(A); # null space

RowSpace = null(A')';

ColSpace = orth(A);

r = rank(A);

[R, pivot_cols] = rref(A); #R gives the reduced row echelon form, and pivot_cols shows the indices of pivot columns.

Plotting a vector

x = 1:10;
y = [1,2,3,4,5,6,7,8,9,10];
plot(x, y);

Week1 - Vector Operations

- Displacement: Describes the change in position of a point, represented as vectors.
- Linear Combination: Represents one vector as a combination of other vectors.
  v = a * u + b * w (where a, b are scalars)
- Dot product: Scalar product of two vectors, **yielding a real number**.
  $$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$
  $$it\ is\ commutitative\ and\ distributive \rightarrow u.v = v.u \,\&\, u.(w+v) = u.w + u.v$$
- Length: magnitude of a vector, found using the dot product. ǁuǁ = sqrt(dot(u, u))
- Normalization: Converts a vector into a unit vector (length = 1). u_normalized = u / norm(u)
- Cauchy-Schwarz Inequality: Relationship between the dot product and vector lengths.
- Triangle Inequality: ǁu + vǁ ≤ ǁuǁ + ǁvǁ
- Projection: proj_u(v) = dot(u, v) * u (if u is a unit vector)
- Orthogonality: Two vectors are orthogonal if their dot product is zero.
- Pythagoras Theorem: If two vectors are orthogonal, their magnitudes follow Pythagoras' theorem. **ǁu + vǁ = ǁuǁ + ǁvǁ if u ⊥ v**
- Equation of a Line**: X = P + td** where P is a point on the line, and d is the direction vector.

  The basic algebraic properties of vectors in $\mathbb{R}^n$

  Theorem 1.1

  a. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutativity
  b. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ Associativity
  c. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ additive identity
  d. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ additive inverse
  e. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ Distributivity
  f. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ Distributivity
  g. $c(d\mathbf{u}) = (cd)\mathbf{u}$
  h. $1\mathbf{u} = \mathbf{u}$

Week 2: Linear Equations

- General equation of a plane: **ax + by + cz = d** where a, b, and c are coefficients representing the normal vector of the plane.
- Intersection of planes: a line (if the system is consistent) or no solution (if inconsistent).
- Solving Linear Equations: A * x = b In Matlab: x = A\b
- Gaussian Elimination: A systematic method for solving linear equations by transforming the matrix to row-echelon form.
- Inverse of a Matrix: A matrix is invertible if it can be transformed into the identity matrix by row operations.
- Elementary row operations: row swaps, multiplying a row by a scalar, and adding/subtracting multiples of rows.
- Matrices are row equivalent if and only if they have the same RREF.
- The span of a set of vectors is the set of all linear combinations of the vectors.
  Span(d) = {***t***d: t of R}
- To show S is a spanning set, should be solvable, rank(S) less than the dimension of space.

Week 3 – Matrix operations

- Matrix Operations: addition, multiplication, and scalar multiplication.
- Transpose: flipping it over its diagonal (rows become columns). In matlab: A'
- Transpose Properties- Matrices

  $$(A^T)^T = A$$
  $$(A+B)^T = A^T + B^T$$
  $$(k * A)^T = k * A^T$$
  $$(A^k)^T = (A^T)^k$$

- **Properties of matrix multiplication**

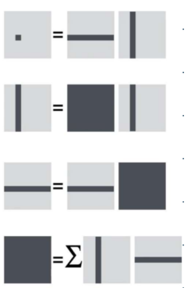
  In this table, A, B, and C are $n \times n$ matrices, I is the $n \times n$ identity matrix, and O is the $n \times n$ zero matrix

Property	Example
The commutative property of multiplication does not hold!	$AB \neq BA$
Associative property of multiplication	$(AB)C = A(BC)$
Distributive properties	$A(B+C) = AB + AC$
	$(B+C)A = BA + CA$
Multiplicative identity property	$IA = A$ and $AI = A$
Multiplicative property of zero	$OA = O$ and $AO = O$
Dimension property	The product of an $m \times n$ matrix and an $n \times k$ matrix is an $m \times k$ matrix.

  - Matrix multiplication is not commutative → AB <> BA

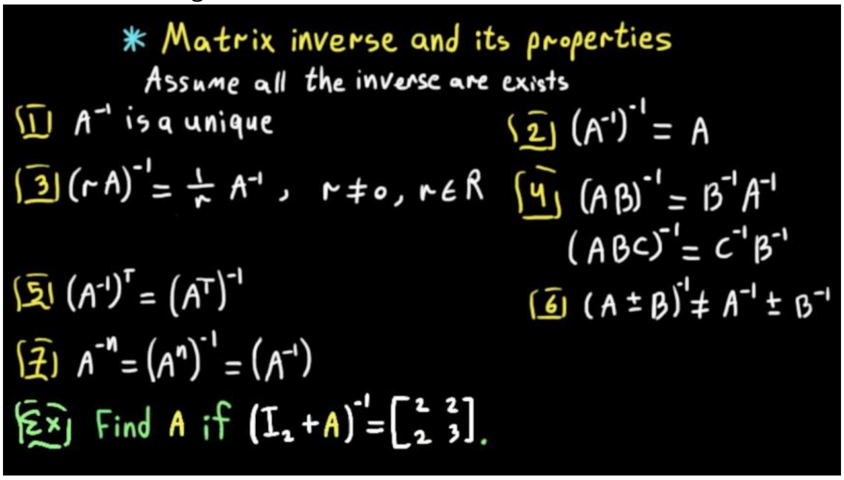
- Diagonal Matrix: Multiplication of any matrix by a diagonal matrix scales the rows or columns accordingly.
- Inverse: A matrix A is invertible if there exists a matrix B such that A * B = I.
- Matrix Determinant: Is a scalar value, if non-zero then the matrix is invertible.
- Matrix Rank: The number of linearly independent rows or columns.
- Null Space: The null space (kernel) of a matrix A consists of all vectors x such that A * x = 0.
- **We have four representations of the product $C = AB$**

row-column dot product expansion	$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$	
matrix-column	$\mathbf{c}_j = A\mathbf{b}_j$	
row-matrix	$\mathbf{C}_i = \mathbf{A}_i B$	
column-row	$\mathbf{C} = \sum_k \mathbf{a}_k \mathbf{B}_k$	



- $A^r A^s = A^{r+s}$
  $(A^r)^s = A^{rs}$
- If AA' and A'A are symmetric, (AB)' = B'A'

Week 4 – Inverse and rank

- Inverse: A transformation that undoes transformation A and is undone by A is an inverse of A X(A**z**) = **z** and A(X**z**)=**z**
- Inverse is both right and left → AX = I also XA = I if X is the inverse of A.
- 

- Conditions for Invertibility: matrix is square, and determinant is non-zero.
- Rank-Nullity Theorem: rank(A) + nullity(A) = number of columns.
- Singular and Non-Singular Matrices: A matrix is singular if it is not invertible (i.e., det(A) = 0), and non-singular if it is invertible (det(A) ≠ 0).

  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad - cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
-
- A is invertible if → Ax = b has a unique solution for any b; Ax = 0 has a nontrivial solution; rref(A)=I; A is the product of elementary matrices.
- To calculate inv, augment with I. $\left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{1\ 0\ 0}}^{I}} & \multicolumn{3}{c}{\overbrace{\hphantom{9\ -\tfrac{3}{2}\ -5}}^{A^{-1}}} \\ 1 & 0 & 0 & 9 & -\frac{3}{2} & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right]$
- Basis for null is the solution of Ax = 0.

Week 5 – Determinants

- The determinant is a scalar value that can be computed from a square matrix. It helps determine whether a matrix is invertible and has important applications in geometry and linear transformations. Determinant is important because of eigenvalues.

- Properties of determinant

[redacted]

Rules of Determinants

1) c = constant, A is nxn matrix $|cA| = c^n |A|$
2) nxn determinant $|-A| = (-1)^n |A|$
3) distributive property $|AB| = |A|\,|B|$
4) identity matrix $|\mathbf{I}| = |AA^{-1}| = |A|\,|A^{-1}| = 1$
5) $|A| = \frac{1}{|A^{-1}|}$
6) $|BAB^{-1}| = |B|\,|A|\,|B^{-1}| = |B|\,|A|\,\frac{1}{|B|} = \frac{|B|}{|B|}|A| = |A|$
7) $|A| = |A^{-1}|$
8) $|\overline{A}| = \overline{|A|}$
9) if 2 rows are identical $|A| = 0$
10) if A ha a row of zeros $|A| = 0$

⋮

[redacted]

$$\det(A^{-1}) = \frac{1}{\det A}$$

$$\det(AB) = (\det A)(\det B)$$

Theorem 4.10

$$\boxed{\det(A) = \det(A^T)}$$

det(A^T) = det(A)
det(I) = 1
If A is 2x2 → det(A)=ad−bc
Det(triangular matrix) = product of the diagonal elements حاصلضرب عناصر روی قطر

- Eigenvalues and Eigenvectors: Eigenvalues are scalars **λ** such that for a matrix A and a non-zero **vector v**, we have: $\boldsymbol{A*v = \lambda*v}$. Eigenvectors are the corresponding vectors. Eigenvalues give important information about the transformation properties of a matrix.

- Finding Eigenvalues: roots of the characteristic equation: $\boldsymbol{det(A-\lambda I)=0}$.

- Geometric Interpretation of Eigenvectors: Eigenvectors represent directions along which the linear transformation associated with matrix A stretches or compresses the space. Eigenvalues determine the factor by which the stretching or compressing happens.

- Diagonalization: A matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that: $\boldsymbol{A=P*D*P^{-1}}$ where the diagonal elements of D are the eigenvalues of A, and the columns of P are the corresponding eigenvectors.

- **The Fundamental Theorem of Invertible Matrices**

  a. A is invertible.
  b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
  c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  d. The reduced row echelon form of A is I_n.
  e. A is a product of elementary matrices.
  f. $\operatorname{rank}(A) = n$
  g. $\operatorname{nullity}(A) = 0$
  h. The column vectors of A are linearly independent.
  i. The column vectors of A span $\mathbb{R}^n$.
  j. The column vectors of A form a basis for $\mathbb{R}^n$.
  k. The row vectors of A are linearly independent.
  l. The row vectors of A span $\mathbb{R}^n$.
  m. The row vectors of A form a basis for $\mathbb{R}^n$.

- **The determinant of a 3×3 matrix can be computed using determinants of 2×2 submatrices**

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = +a_{11}\det A_{11} \quad - a_{12}\det A_{12} \quad + a_{13}\det A_{13}$$

Week 6 – Eigenvalue and eigenvectors

- Finding Eigenvectors: After calculating eigenvalues, substitute each eigenvalue λ into the equation $(A-\lambda I)v=0$ to find the corresponding eigenvectors. Matlab: [V, D] = eig(A)
- Diagonalization Criteria: A matrix A can be diagonalized if it has n linearly independent eigenvectors, where n is the dimension of the matrix.
- Geometric Multiplicity: The geometric multiplicity of an eigenvalue λ is the number of linearly independent eigenvectors corresponding to λ. For a matrix to be diagonalizable, the geometric multiplicity of each eigenvalue must equal its algebraic multiplicity (the number of times it appears as a root of the characteristic equation).
- Defective Matrix: A matrix is called defective if it cannot be diagonalized, meaning it does not have enough linearly independent eigenvectors. This happens when the geometric multiplicity of some eigenvalues is less than their algebraic multiplicity.
- Symmetric Matrices: Symmetric matrices are always diagonalizable, and their eigenvalues are always real.
- To find eigenvectrs, should replace $(A-\lambda I)x=0$
- $\boxed{\text{If } A = PDP^{-1} \text{ then } A^n = PD^nP^{-1}}$
- **A is <u>similar</u> to B if $P^{-1}AP = B$ for some P**
- Similarity is
  reflexive: $A \sim A$
  symmetric: $A \sim B \Rightarrow B \sim A$
  transitive: $A \sim B \;\&\; B \sim C \Rightarrow A \sim C$
- **A square matrix is invertible if and only if it doesn't have a zero eigenvalue**