Matrix Course Detailed Cheat Sheet

Day 1: Core and Intermediate Concepts

Vector Operations

- **Vector Addition**:
  Formula:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

  - **Commutative**:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

  - **Associative**:

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

- **Scalar Multiplication**:
  Formula:

$$c\mathbf{u} = (cu_1, cu_2, \dots, cu_n)$$

  - **Distributive**:

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$$

  - **Associative with Scalars**:

$$c(d\mathbf{u}) = (cd)\mathbf{u}$$

- **Dot Product**:
  Formula:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- **Commutative**:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

- **Distributive**:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

Length and Angle of Vectors

- **Length (Magnitude)**:
  Formula:

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

- **Cosine of the Angle Between Vectors**:
  Formula:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

- **Cauchy-Schwartz Inequality**:
  Formula:

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$$

- **Triangle Inequality**:
  Formula:

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$

Linear Combinations and Span

- **Linear Combination**:
  A vector ( ) can be written as a linear combination of other vectors.
  Formula:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

- **Span**:
  The span of vectors (_1, _2, , _n) is the set of all possible linear combinations of those vectors.

Projections and Orthogonality

- **Projection of ( ) onto ( )**:
  Formula:

$$\text{Proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

- **Orthogonality**:
  Two vectors ( ) and ( ) are orthogonal if their dot product is zero:

$$\mathbf{u} \cdot \mathbf{v} = 0$$

- **Pythagorean Theorem (in (^n))**:
  If ( ), then:

$$|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$$

Gaussian Elimination

1. **Forming the Augmented Matrix**:
   Write the system of linear equations in matrix form.

2. **Row Operations**:

   - **Swap rows**.
   - **Multiply a row by a non-zero scalar**.
   - **Add/subtract multiples of one row to/from another**.

3. **Back Substitution**:

   - Solve the system by simplifying to row echelon form (upper triangular matrix) and using back substitution.

Matrix Operations

- **Matrix Addition**:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

- **Matrix Multiplication**:
  Formula for the element in the (i)-th row and (j)-th column of the product matrix:

$$(AB)_{ij} = \sum_k a_{ik} b_{kj}$$

- **Transpose of a Matrix**:
  Formula:

$$A_{ij}^T = A_{ji}$$

Key Properties

- **Distributive**:

$$A(B + C) = AB + AC$$

- **Associative** (Matrix Multiplication):

$$A(BC) = (AB)C$$

- **Non-Commutative** (Matrix Multiplication):

$$AB \neq BA$$

  in general.

Day 2: Advanced Concepts

Matrix Inverses

- **Matrix Inverse** (for square matrix (A)):

$$A^{-1}A = I = AA^{-1}$$

  - A matrix (A) is invertible if and only if ((A)).

Determinants

- **Determinant of a 2x2 Matrix**:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- **Properties of Determinants**:
  - ((AB) = (A)(B))
  - ((A^T) = (A))
  - If ((A) = 0), then (A) is not invertible.

Eigenvalues and Eigenvectors

- **Eigenvalue Equation**:

$$A\mathbf{v} = \lambda\mathbf{v}$$

  - Where ( ) is the eigenvalue, and ( ) is the corresponding eigenvector.
- **Finding Eigenvalues**:
  Solve ((A - I) = 0) for ( ).
- **Finding Eigenvectors**:
  Solve ((A - I) = 0) for ( ).

Diagonalization

- A matrix (A) is diagonalizable if there exists an invertible matrix (P) such that:

$$A = PDP^{-1}$$

  - Where (D) is a diagonal matrix whose diagonal entries are the eigenvalues of (A), and the columns of (P) are the corresponding eigenvectors of (A).

Practice Problems

Problem 1: Vector Orthogonality

- Let ( ) and ( ) be n-dimensional vectors. Define the vectors:

$$p = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}, \ h = \mathbf{v} - p$$

- Prove that (h) is orthogonal to ( ).

- **Solution**: Compute (h) and show that it equals zero.

Problem 2: Gaussian Elimination

Solve the system of equations using Gaussian Elimination:

$$\begin{aligned} 2x + 3y + z &= 1 \\ 4x + y - z &= 2 \\ -2x + 2y + 3z &= 4 \end{aligned}$$

Problem 3: Determinants and Eigenvalues

Find the determinant and eigenvalues of the matrix:

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

This cheat sheet now has a **more expanded version**, following the requested format with LaTeX-like syntax that can be used in platforms that