Black-Litterman Model

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The Black-Litterman model is a portfolio allocation framework in which asset returns are based on market equilibrium returns adjusted by subjective forecasts called *views*. The model was initially introduced in 1992 by Fischer Black and Robert Litterman in their paper *Global Portfolio Optimization*. The framework proposed in the paper is a natural combination of Modern Portfolio Theory(MPT) and Capital Asset Pricing Model(CAPM) with a Bayesian perspective.

Introduction

One of the fundamental strategies for risk management is diversification of a portfolio. This strategy was quantified initially by Harry Markowitz in Modern Portfolio Theory, which suggests that the risk associated with an asset class can be measured by the historical standard deviation of the returns. The model enables investors to construct a portfolio made up of multiple asset classes that has the highest expected return given a chosen level of risk. However, the model has it's shortcomings. Specifically, it assumes that the historical returns of asset classes are stationary. For example, if Turkish equities class had an annualized return of 10% in the historically available time period, we can expect that it will have a similar annualized return in the future. Research in the field shows that this is not a realistic assumption, mainly because the return characteristics of each asset vary significantly in different economic periods. The historical returns of asset classes are insufficient to predict the future distribution of asset returns.

Another portfolio allocation strategy proposed shortly after the MPT is the CAPM, which proposed that, rather than trying to predict the individual returns and risks of asset classes, investors can track the money invested in each asset class and calculate the allocation of the general population. This allocation is called the market portfolio and essentially CAPM proposes that the optimal portfolio is the market portfolio.

The Black-Litterman model combines these two strategies by proposing that the allocation generated by CAPM is a good starting point for a strategy and it can be improved by adding our views for the performance of asset classes in the future. This enables investors to have a much more flexible framework than both of the previous investment strategies. Instead of only relying on a single source of information about the performance of asset classes, investors can incorporate other sources of information into the system utilizing the investor's expectation of future returns.

We can summarize the contributions of the Black-Litterman model with answering two important questions for portfolio allocation:

- What is a good starting point for portfolio allocation?
- How can I update the starting point in terms of my personal views about the asset classes?

The Black-Litterman Framework

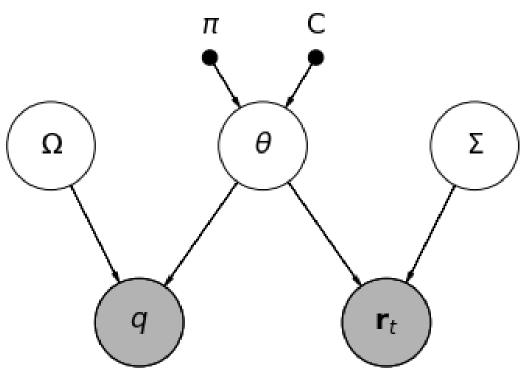


Figure 1: Black-Litterman Model as a Probabilistic Graphical Model

The Black-Litterman model can be considered as a Bayesian model since we use the asset returns implied by the CAPM as the priors to our expected future returns. If we have no additional view of how the markets will perform, the Black-Litterman model will yield the same portfolio as CAPM. On the other hand, if we have particular views about how an asset class will perform individually or relative to another asset class, we incorporate this information to our model as noisy observations of the future. The views are treated as noisy observations since there may be different amounts of uncertainty associated with any expectations of the future. These noisy observations correspond to the likelihood function in a Bayesian model.

We show the Black-Litterman model as a probabilistic graphical model in Figure 1. The main variable we are trying to understand is the distribution of returns of asset classes, shown as $\mathbf{r}_t$. We assume that the expected value of $\mathbf{r}_t$ is θ which is also treated as a variable, and the covariance of $\mathbf{r}_t$ is Σ. We assume that Σ can be reasonably estimated using the historical performance of asset classes. However, a better estimate of the Σ matrix can be easily implemented inside the

Black-Litterman framework. Notice that we have hyperparameters for the variable θ, which shows that θ has an expected value of π and covariance of C. These values are calculated using the asset performances implied by the CAPM. On the other hand, q is the noisy observations we add to the system. Notice that the expected value of both $\mathbf{r}_t$ and q are assumed to be the same. However, q has a different covariance matrix, Ω, since the error in measurement for q involves human judgment.

The model shown in the graph in Figure 1 implies that there is dependence between $\mathbf{r}_t$ and q. Assuming we observed some returns of asset classes in the form of q, these noisy observations give us more information about the distribution of $\mathbf{r}_t$. Thus, for a complete calculation of the model the distribution of $\mathbf{r}_t$ given q, $p(\mathbf{r}_t|q)$ is required. This corresponds to the following integral:

$$p(\mathbf{r}_t|q) = \int p(\mathbf{r}_t|\theta)p(\theta|q)d\theta \tag{1}$$

The likelihood of θ given q, $p(\theta|q)$, is not defined by our initial model and must be first calculated using the Bayes' theorem:

$$p(\theta|q) = \frac{p(q|\theta)p(\theta)}{\int p(q|\theta)p(\theta)d\theta} \tag{2}$$

In equation (2), the prior probability density of θ, namely $p(\theta)$, is defined by the mean π and covariance matrix C as mentioned in the above discussion. The relationship between θ and the views of the investor q is provided by the fact that the views are given in a way that describes the difference in expected asset returns. The mathematical formulas used for these densities for the original Black-Litterman model will be given in the following sections. However, equations (1) and (2) together with the probabilistic graphical model given in Figure 1 enables investors to arbitrarily define the relationships between the aforementioned variables and calculate the full distribution of the asset returns, $\mathbf{r}_t$ in a personalized manner.

After we obtain the estimated distribution of asset returns, $\mathbf{r}_t$, the asset allocation problem becomes an optimization problem. A portfolio is a linear combination of assets. We can calculate the return of a portfolio by simply taking a weighted sum of the individual asset classes where ω_i is the weight given to the i^{th} asset class. The simplest objective function for portfolio allocation is the expected returns of a portfolio defined as:

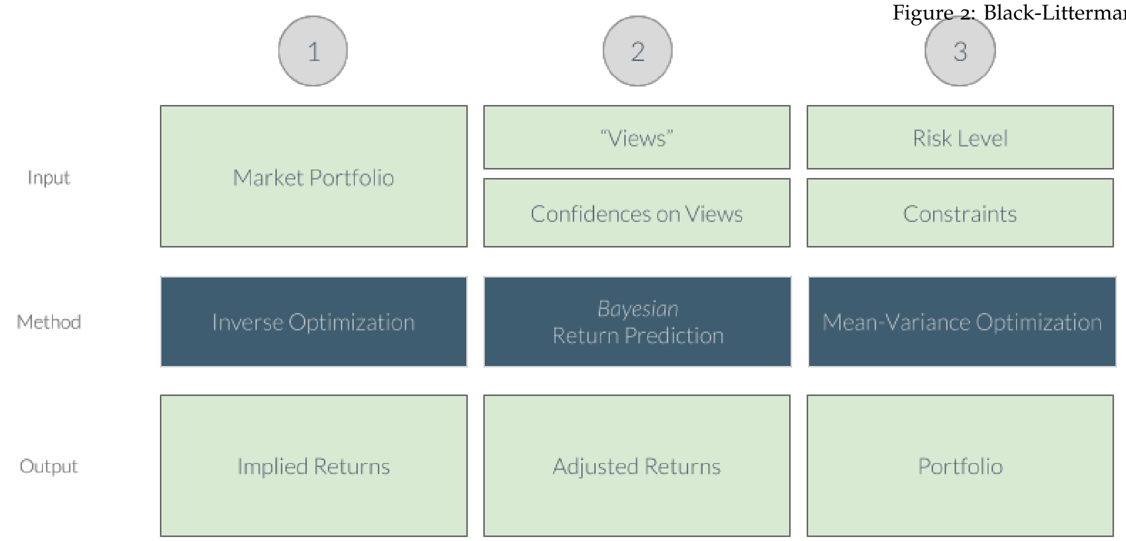
$$f(w) = E[\mathbf{r}_p] \tag{3}$$

where $\mathbf{r}_p$ is the return of our portfolio. We want to maximize this function with the constraint that the variance of the portfolio is lower than some constant L defined by the preferences of the investor. This optimization problem was initially introduced in MPT and is called the mean-variance optimization.

The Original Model

The original Black-Litterman model is a simplified version of the framework shown above. Since the expected value and covariance matrices are explicitly used to define the random variables q, θ and $\mathbf{r}_t$, using Gaussian distributions for every probability density is the simplest and most natural. The investor is not interested in any statistics of $\mathbf{r}_t$ other than it's expected value θ and the covariance matrix Σ. In such a model we only need to calculate $p(\theta|q)$, since the posterior of θ will be used in mean-variance optimization. The Black-Litterman algorithm can be summarized into three steps and are shown in Figure 2. These assumptions enable the investor to calculate a closed form solution for each of the steps shown in Figure 2.

Figure 2: Black-Litterman Model Steps



Step 1: Return Predictions of the Market

As a starting point we observe the market capitalization of each asset class and compute the market-equilibrium premiums implied by the market capitalization by solving an inverse portfolio optimization problem. The details of this process are shown in [1].

[1]

$$\pi = 2\delta\Sigma w \tag{4}$$

where π is the market-equilibrium premiums, δ is the risk-aversion rate, Σ is the covariance matrix and w is the weights in the market portfolio.

Step 2: Return Predictions by Blending Market and Views

The views given to the system can be summarized as:

$$\mathbf{P}\theta = q + \epsilon_i \tag{5}$$

where the ϵ_i term indicates the level of uncertainty about the view and it is assumed to be independent for every view. Consequently, we can interpret these views as noisy observations of the future. **P** is a matrix that is made up of linear equations relating the expected returns θ_i of an asset i and θ_j of an asset j. For example, we might expect that asset i will outperform asset j for about 500 basis points. This view can be shown as:

$$\theta_i - \theta_j = 0.05 \tag{6}$$

In terms of a statistical model, the likelihood function relating the expected return parameter θ and the investor's views q is assumed to be a Gaussian distribution:

$$f(q|\theta) \sim exp(-\frac{1}{2}(P\theta - q)^T \Omega^{-1}(P\theta - q)) \tag{7}$$

For a more detailed discussion of blending the market's expected return values and the investor's views, the reader is referred to [2]. From an operational point of view the expected value of the posterior distribution of θ given q, shown in equation (2), is calculated using the following formula:

[2]

$$E[\theta|q] = \left[(\tau\Sigma)^{-1} + P^T\Omega^{-1}P\right]^{-1} \left[(\tau\Sigma)^{-1}\pi + P^T\Omega^{-1}q\right] \tag{8}$$

where θ is the expected asset premium means, q is the estimated premiums from views, Ω is the (diagonal) view uncertainty matrix, P is the link matrix and τ is a scalar to adjust uncertainty matrix. There is an implicit assumption that the uncertainty associated with market capitalization based expected returns is a scaled version of the uncertainty we observe in historical returns.

Step 3: Portfolio Optimization

Once we obtain the adjusted expected returns of individual asset classes, θ is plugged in to the optimization framework introduced in

Modern Portfolio Theory. The optimal weights w are the weights that maximize the following optimization problem:

$$\begin{aligned}
&\max_{w} \theta^T w \\
&\text{s.t.} \\
&w^T \Sigma w = L^2 \\
&w^T e = 1 \\
&LB \leq w \leq UB
\end{aligned} \tag{9}$$

where e is a vector of ones, L is the target volatility and LB, UB are lower and upper bound vectors for asset weights.