

Review PASSIVE rotations and Euler rates.

In this Tutorial we'll review the concepts of PASSIVE rotation matrices. We'll then look at an application of these rotation matrices which involves relating Euler angle rates to the angular velocity of a rigid body. In preparation for the "Wheel on a Pole" example, we'll look at the specific case where:

- a BODY frame undergoes 3 successive rotations
- a BODY frame undergoes 2 successive rotations

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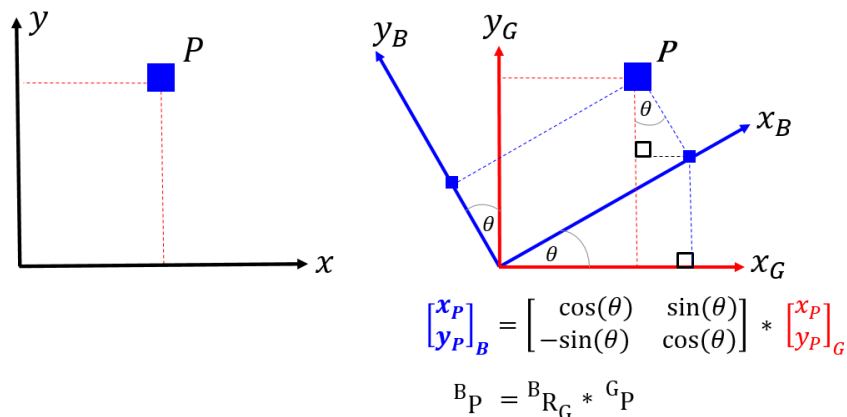
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Review the concept of PASSIVE rotation matrices :

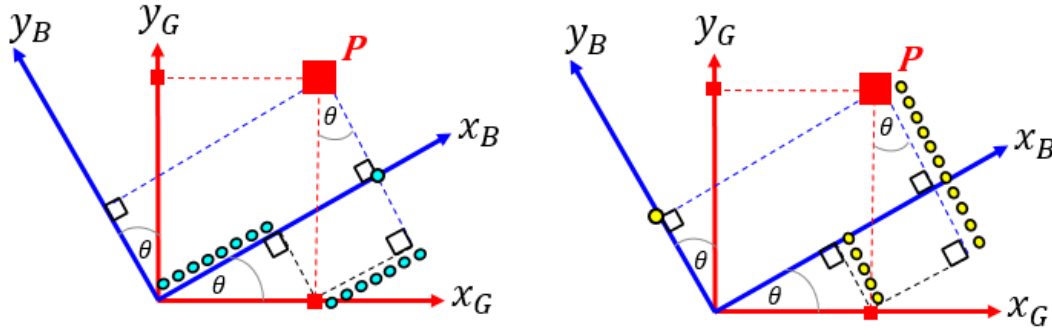
Consider the following scenario:

- We have a data point P .
- We have a fixed frame called the **G-frame**.
- We know the (x,y) co-ordinates of the point P in this **G-frame** and refer to this as ${}^G P$.
- We then rotate the **B-frame** by θ relative to the fixed **G-frame**.

We now want to know what the co-ordinate of the point P is relative to this new **B-frame**, ie: what is ${}^B P$? This scenario is shown in the figure below:



A **PASSIVE** rotation matrix, converts the co-ordinates of a point expressed in a fixed **G-frame**, into the co-ordinates of the same point expressed in the new **B-frame**.



$$\begin{bmatrix} x_P \\ y_P \end{bmatrix}_B = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} * \begin{bmatrix} x_P \\ y_P \end{bmatrix}_G$$

An example of 3 successive PASSIVE rotations

Say we have a fixed G-frame. We start by having our B-frame co-incident with G, and then we start to rotate the B-frame. Specifically, we're going to apply 3 LOCAL axes rotations which will result in a newly orientated B-frame. Assume that we apply these 3 successive rotations in the following order:

1. R1Z occurs 1st about the LOCAL **Z** body axis (ϕ), aka **YAW**
2. R2Y occurs 2nd about the LOCAL **Y** body axis (θ), aka **PITCH**
3. R3X occurs 3rd about the LOCAL **X** body axis (ψ), aka **ROLL**

We can express a vector defined in the G axis to it's corresponding description in the B axis, using a sequence of **PASSIVE** rotation matrices, ie:

$${}^B_V = R3X(\psi_x) \times R2Y(\theta_y) \times R1Z(\phi_z) \times {}^G_V$$

OR, in a more compact form as:

$${}^B_V = {}^B R_G \times {}^G_V$$

So let's get ready to explore:

```
clear phi      theta      psi
clear phi_dot  theta_dot  psi_dot
clear phi_ddot theta_ddot psi_ddot

syms phi      theta      psi
syms phi_dot  theta_dot  psi_dot
syms phi_ddot theta_ddot psi_ddot
```

Create a passive rotation object

```
OBJ_P = bh_rot_passive_G2B_CLS({'D1Z', 'D2Y', 'D3X'}, [phi, theta, psi], 'SYM')
```

```
OBJ_P =
  bh_rot_passive_G2B_CLS with properties:
```

```
    ang_units: SYM
  num_rotations: 3
    dir_1st: D1Z
    dir_2nd: D2Y
    dir_3rd: D3X
    ang_1st: [1x1 sym]
    ang_2nd: [1x1 sym]
    ang_3rd: [1x1 sym]
```

Here are the **PASSIVE** rotation matrices:

Here's **R1Z**(ϕ_z):

```
R1Z = OBJ_P.get_R1
```

R1Z =

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here's **R2Y**(θ_y):

```
R2Y = OBJ_P.get_R2
```

R2Y =

$$\begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

Here's **R3X**(ψ_x):

```
R3X = OBJ_P.get_R3
```

R3X =

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & \sin(\psi) \\ 0 & -\sin(\psi) & \cos(\psi) \end{pmatrix}$$

Calculate the Direction Cosine Matrix ${}^B R_G$

Recall we earlier said: ${}^B \mathbf{v} = {}^B R_G * {}^G \mathbf{v}$

```
bRg = R3X * R2Y * R1Z
```

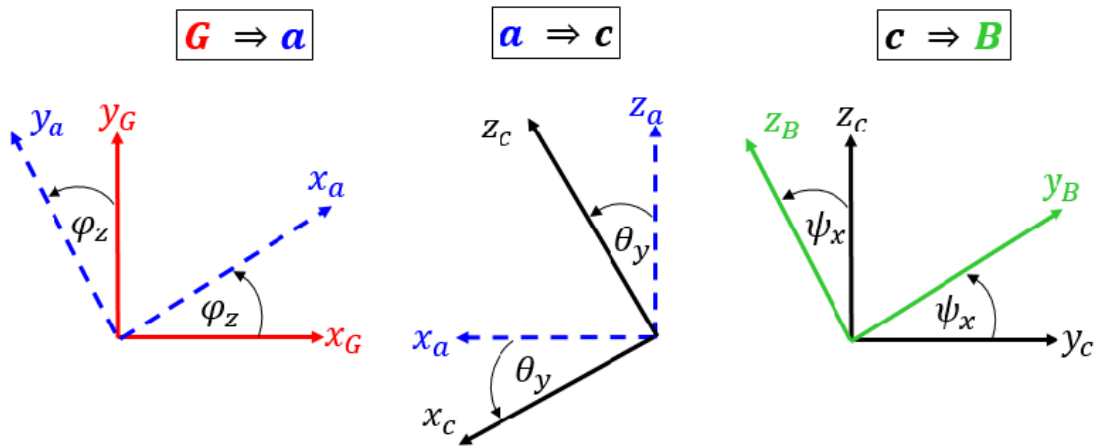
bRg =

$$\begin{pmatrix} \cos(\phi) \cos(\theta) & \cos(\theta) \sin(\phi) & -\sin(\theta) \\ \cos(\phi) \sin(\psi) \sin(\theta) - \cos(\psi) \sin(\phi) & \cos(\phi) \cos(\psi) + \sin(\phi) \sin(\psi) \sin(\theta) & \cos(\theta) \sin(\psi) \\ \sin(\phi) \sin(\psi) + \cos(\phi) \cos(\psi) \sin(\theta) & \cos(\psi) \sin(\phi) \sin(\theta) - \cos(\phi) \sin(\psi) & \cos(\psi) \cos(\theta) \end{pmatrix}$$

Explore the relationship between BODY rates and EULER rates

As we apply these local frame rotations of (ϕ, θ, ψ) , we can represent the **angular rates** $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ of the rotating frames into the LOCAL frame co-ordinates. These local frame co-ordinates can then be converted into co-ordinates expressed in the final B frame. For example, during each of the local axes rotations we can think of there being a START frame and an END frame:

Rotation	START frame	End frame	Angular rate vector Associated with Rotation
$R1Z(\phi)$	G-frame	a-frame	$\begin{pmatrix} 0 \\ 0 \\ \phi_{DOT} \end{pmatrix}$ in G
$R2Y(\theta)$	a-frame	c-frame	$\begin{pmatrix} 0 \\ \theta_{DOT} \\ 0 \end{pmatrix}$ in a
$R3X(\psi)$	c-frame	B-frame (the body frame)	$\begin{pmatrix} \psi_{DOT} \\ 0 \\ 0 \end{pmatrix}$ in c



We can express each of the local frame angular velocities into their corresponding components in the final B frame - and we'll use PASSIVE rotation matrices to do this:

```
aRg = R1Z;
cRa = R2Y;
bRc = R3X;
```

Here's the portion of the BODY frame rate associated with our $\dot{\phi}$ rate:

```
wb_part_1 = bRc * cRa * aRg * [0;0;phi_dot]    % convert local G into B
wb_part_1 =
```

$$\begin{pmatrix} -\dot{\phi} \sin(\theta) \\ \dot{\phi} \cos(\theta) \sin(\psi) \\ \dot{\phi} \cos(\psi) \cos(\theta) \end{pmatrix}$$

Here's the portion of the BODY frame rate associated with our $\dot{\theta}$ rate:

```
wb_part_2 = bRc * cRa * [0;theta_dot;0] % convert local a into B
```

$$\text{wb_part_2} = \begin{pmatrix} 0 \\ \dot{\theta} \cos(\psi) \\ -\dot{\theta} \sin(\psi) \end{pmatrix}$$

Here's the portion of the BODY frame rate associated with our $\dot{\psi}$ rate:

```
wb_part_3 = bRc * [psi_dot;0;0] % convert local c into B
```

$$\text{wb_part_3} = \begin{pmatrix} \dot{\psi} \\ 0 \\ 0 \end{pmatrix}$$

The total angular velocity expressed in the BODY B frame is therefore

We can now construct the total angular velocity vector expressed in components of the final B frame.

- ${}^B_G\omega_b \equiv \omega_b = f(\dot{\phi}, \dot{\theta}, \dot{\psi}, \phi, \theta, \psi)$

```
wb = wb_part_1 + wb_part_2 + wb_part_3
```

$$\text{wb} = \begin{pmatrix} \dot{\psi} - \dot{\phi} \sin(\theta) \\ \dot{\theta} \cos(\psi) + \dot{\phi} \cos(\theta) \sin(\psi) \\ \dot{\phi} \cos(\psi) \cos(\theta) - \dot{\theta} \sin(\psi) \end{pmatrix}$$

We can write the angular velocity vector ω_b as a MATRIX equation

Let's say that: $\omega_b = [\omega_x \ \omega_y \ \omega_z]^T = [p \ q \ r]^T$, we can write a matrix equation of the form **A.x = b** that describes the relationship between the body rates ω_b and the Euler rates:

- $A \times \begin{pmatrix} \phi_{dot} \\ \theta_{dot} \\ \psi_{dot} \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

```
syms p_b q_b r_b
```

```
x = [phi_dot, theta_dot, psi_dot].';

[A,b] = equationsToMatrix( wb(1)==p_b, ...
                           wb(2)==q_b, ...
                           wb(3)==r_b, ...
                           x);
```

Just to clarify this, let me put each of the 3 elements side by side as $[A, x, b]$, where:

$$\begin{array}{ccc} A & \times & x & = & b \\ \downarrow & & \downarrow & & \downarrow \\ A & \times & \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} & = & \begin{pmatrix} {}^B\omega_x \\ {}^B\omega_y \\ {}^B\omega_z \end{pmatrix} \end{array}$$

```
[A, x, b]
```

```
ans =
```

$$\begin{pmatrix} -\sin(\theta) & 0 & 1 & \dot{\phi} & p_b \\ \cos(\theta) \sin(\psi) & \cos(\psi) & 0 & \dot{\theta} & q_b \\ \cos(\psi) \cos(\theta) & -\sin(\psi) & 0 & \dot{\psi} & r_b \end{pmatrix}$$

So our Euler rates can be written as:

```
the_euler_rates = A\b;
the_euler_rates = simplify(the_euler_rates)
```

```
the_euler_rates =
```

$$\begin{pmatrix} \frac{r_b \cos(\psi) + q_b \sin(\psi)}{\cos(\theta)} \\ q_b \cos(\psi) - r_b \sin(\psi) \\ \frac{p_b \cos(\theta) + r_b \cos(\psi) \sin(\theta) + q_b \sin(\psi) \sin(\theta)}{\cos(\theta)} \end{pmatrix}$$

An example of 2 successive PASSIVE rotations

In the Modified Euler analysis of the "Wheel on the pole", we'll allow the body frame to yaw (ϕ) and pitch (θ) only - the spinning of the wheel will be handled separately from the angular motion of the "semi" attached body frame. So our body frame is described by the 2 rotations:

1. R1Z occurs 1st about the LOCAL **Z** body axis (ϕ), aka **YAW**
2. R2Y occurs 2nd about the LOCAL **Y** body axis (θ), aka **PITCH**

So what is the angular velocity of the body frame

```
aRg = R1Z;
```

```

cRa      = R2Y;

wb_part_1 = R2Y * R1Z * [0; 0; phi_dot]; % convert local G into B
wb_part_2 = R2Y *      [0; theta_dot; 0]; % convert local a into B

```

So the body frame angular velocity is:

- ω_b

```
wb_for_only_2_rots = wb_part_1 + wb_part_2
```

```
wb_for_only_2_rots =
```

$$\begin{pmatrix} -\dot{\phi} \sin(\theta) \\ \dot{\theta} \\ \dot{\phi} \cos(\theta) \end{pmatrix}$$

And the Direction Cosine matrix is:

- ${}^B_V = {}^B R_G * {}^G_V$

```
bRg_for_only_2_rots = R2Y * R1Z
```

```
bRg_for_only_2_rots =
```

$$\begin{pmatrix} \cos(\phi) \cos(\theta) & \cos(\theta) \sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \\ \cos(\phi) \sin(\theta) & \sin(\phi) \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Finally let's explore the ω_b :

```
syms t my_phi(t) my_theta(t)
```

```

my_phi_dot(t)      = diff(my_phi(t) , t);
my_theta_dot(t)     = diff(my_theta(t) , t);
my_phi_dot_dot(t)   = diff(my_phi(t) , t, 2);
my_theta_dot_dot(t) = diff(my_theta(t) , t, 2);

```

```

plain_symbols = [ phi, theta, phi_dot, theta_dot, phi_ddot, theta_ddot,
t_symbols      = [ my_phi(t) my_theta(t), my_phi_dot(t), my_theta_dot(t), my_phi_dot_dot(t), my_theta_dot_dot(t)

```

```
my_wb = subs(wb_for_only_2_rots, plain_symbols, t_symbols );
```

```
my_wb_dot = diff(my_wb, t)
```

```
my_wb_dot =
```

$$\begin{pmatrix} -\sin(\text{my}_\theta(t)) \frac{\partial^2}{\partial t^2} \text{my}_\phi(t) - \cos(\text{my}_\theta(t)) \frac{\partial}{\partial t} \text{my}_\theta(t) \frac{\partial}{\partial t} \text{my}_\phi(t) \\ \frac{\partial^2}{\partial t^2} \text{my}_\theta(t) \\ \cos(\text{my}_\theta(t)) \frac{\partial^2}{\partial t^2} \text{my}_\phi(t) - \sin(\text{my}_\theta(t)) \frac{\partial}{\partial t} \text{my}_\theta(t) \frac{\partial}{\partial t} \text{my}_\phi(t) \end{pmatrix}$$

Substitute in the original symbols for a nicer display of the results:

- $\dot{\omega}_b$

```
my_wb_dot = subs(my_wb_dot,t_symbols , plain_symbols )
```

```
my_wb_dot =
```

$$\begin{pmatrix} -\ddot{\phi} \sin(\theta) - \dot{\phi} \dot{\theta} \cos(\theta) \\ \ddot{\theta} \\ \ddot{\phi} \cos(\theta) - \dot{\phi} \dot{\theta} \sin(\theta) \end{pmatrix}$$