Linear Algebra

A Gentle Introduction

Screenshots from 3blue1brown’s animations:
https://www.youtube.com/playlist?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab

Algebra

« Algebra is the intellectual instrument which has been created for rendering clear the quantitative aspect of the world » — Alfred North Whitehead

Math tends to be characterized by an extensive use of symbolism :

Generally, the use of these symbols mark the transition between arithmetic and algebra.

$\Rightarrow$ Linear algebra can be understood as a language applied to *lines* and objects which go beyond a single dimension (numbers).

Vectors

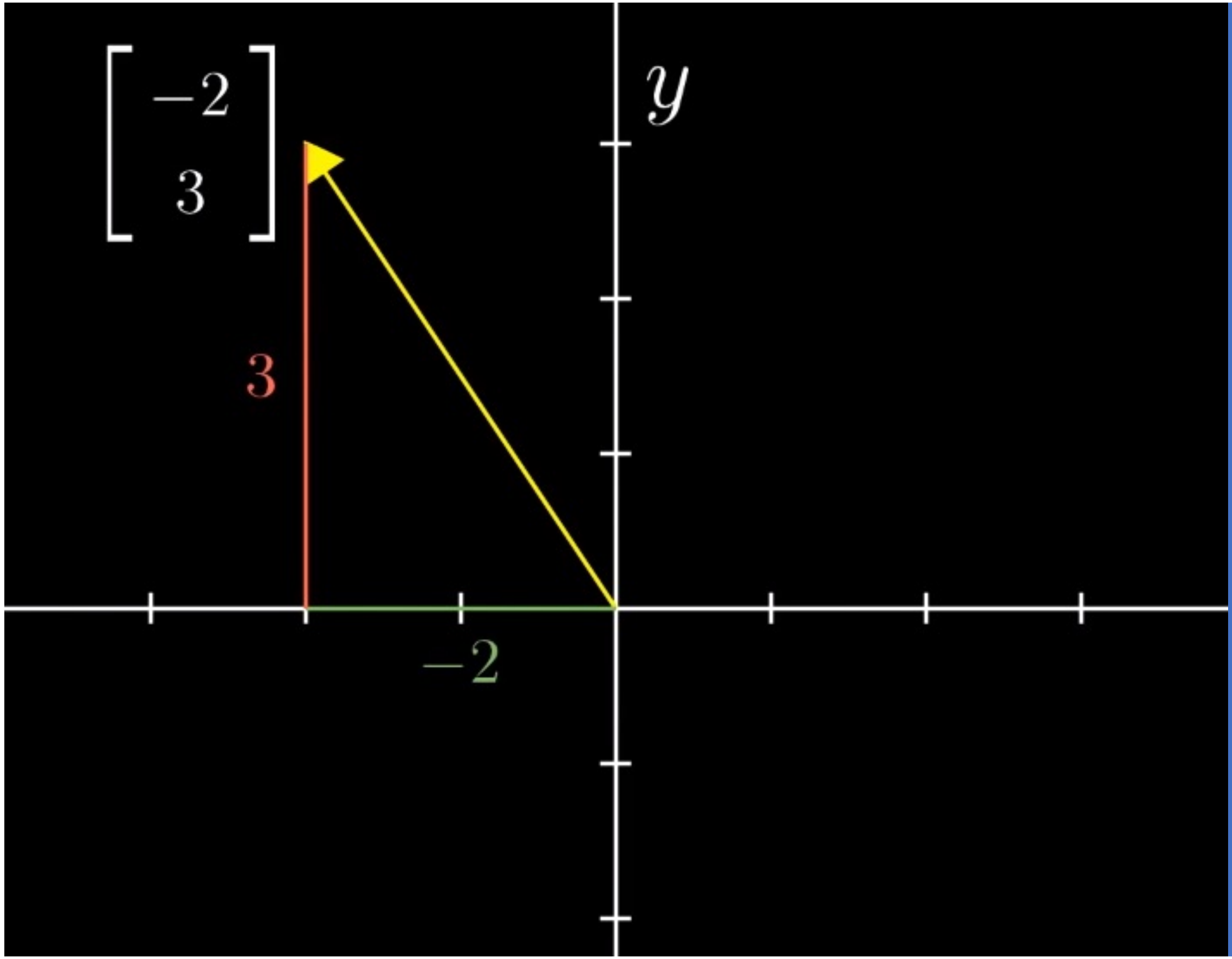
- Fundamental building block for linear algebra
- 3 perspectives :
  - Physics student
    - Vectors are arrows that we can move around
  - Computer Science student
    - Vectors are ordered lists of numbers => think of feature vectors
    - [200 sqm, 1.5M] vs. [500K, 150 sqm] => order matters
  - Math student
    - Generalizes both of these views

Vectors (2D for now)

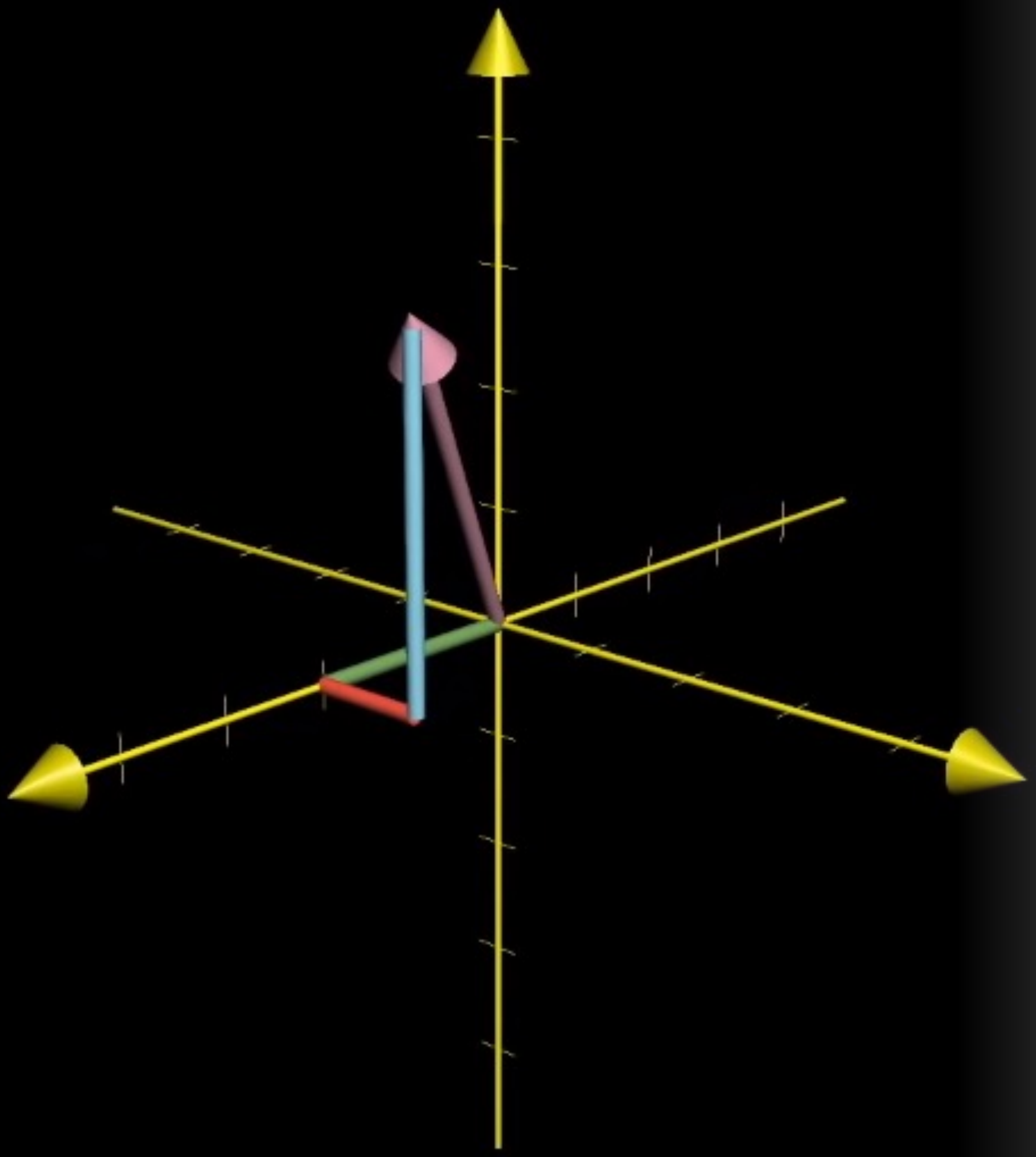
- What do you think about when someone talks about vectors:
  - An arrow ?
  - A set of coordinates/list of numbers ?

- An arrow that sits inside a coordinate system, with its tail on the origin => where is the origin ?

- What are coordinates ?
  - Can be seen as a set of **instructions** to go from the base to the tip of the vector
  - Go along the x axis a certain amount and then along the y axis a certain amount



- A point's coordinates are traditionally written horizontally(-2, 3)
- Unlike vector coordinates which are vertical with square brackets $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

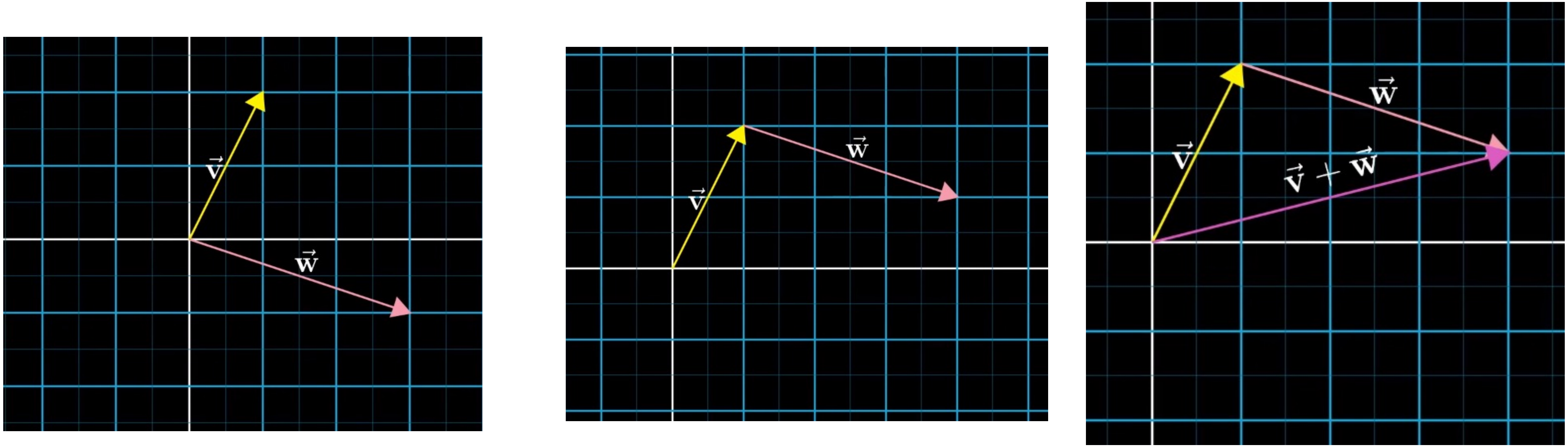


3rd Dimension ?

- Ordered Triplet of numbers
- Same « instruction » principle
  - move along the x axis, then parallel to the y axis then parallel to the z axis
- Gives you one unique vector in space.

2 Central Operations

- Addition
- Multiplication
- You know how to apply those to numbers
- What happens when we enter 2D space ?

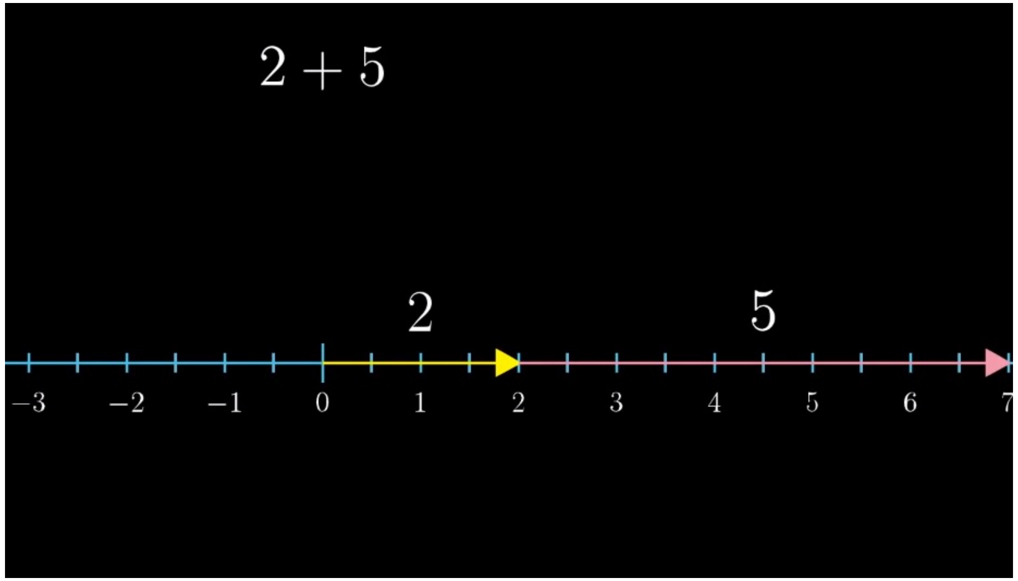


Vector Addition

$$\boldsymbol{v} + \boldsymbol{w}$$

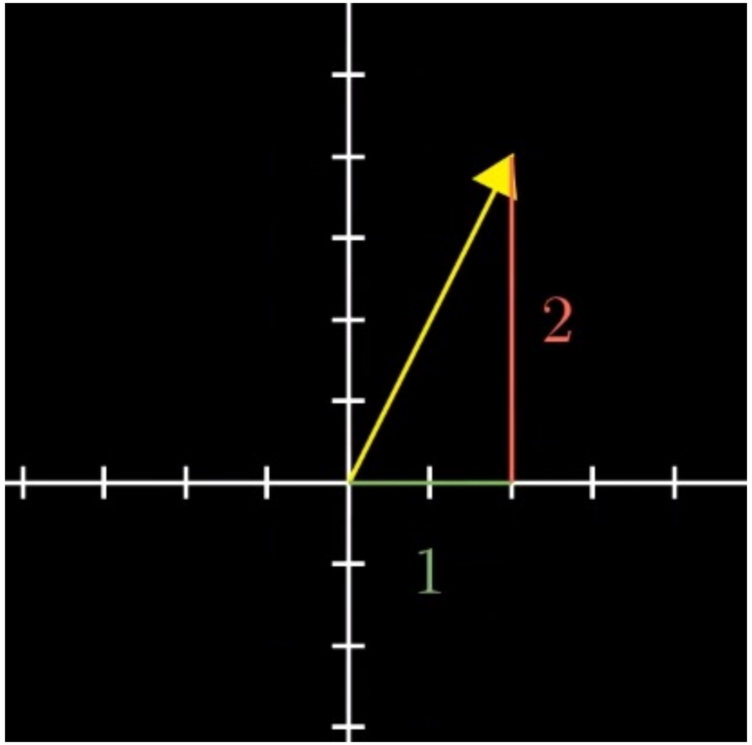
Why this definition of addition ?

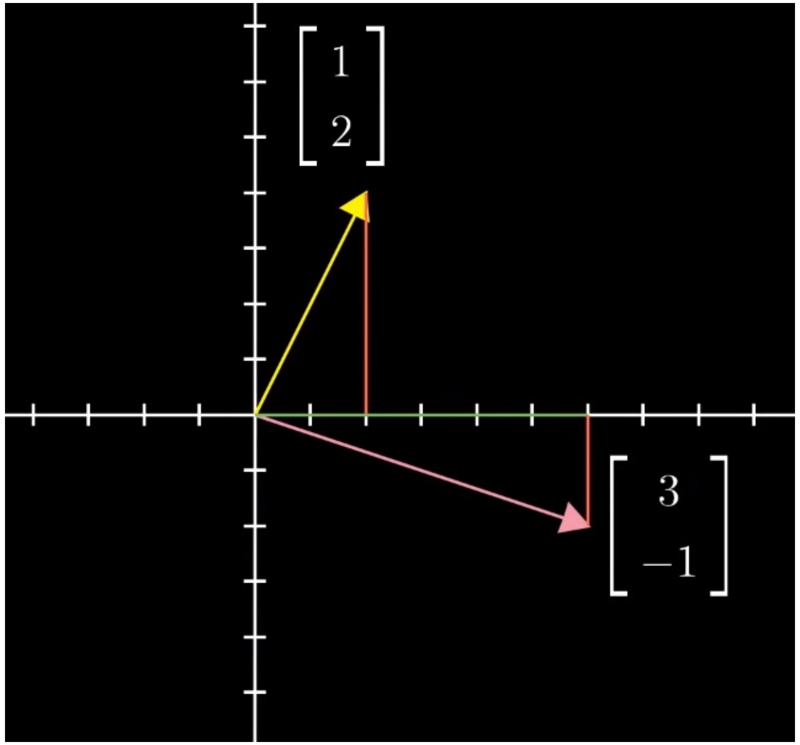
- Each vector represents a certain movement, a step in a direction
- So going along the sum of vectors which starts from the origin takes you exactly to the place where you end up by going along one vector and then the other
- Same thing in 1-dimension when adding 2 + 5 for example.

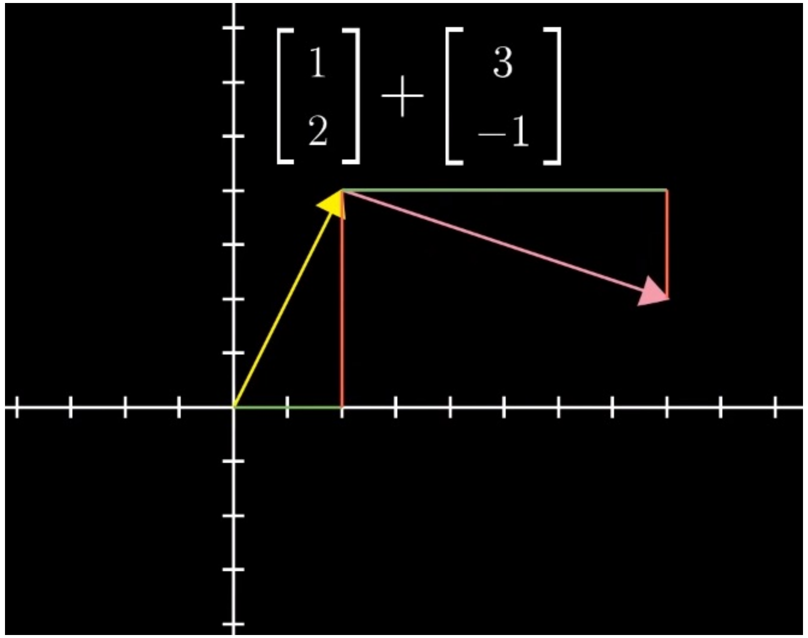


Example

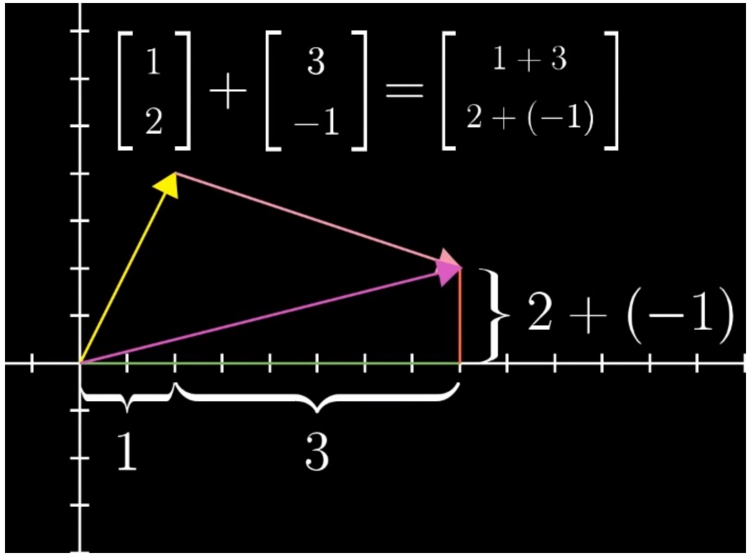
Can think of vector addition as a *4 step path* from the origin to the tip of the second vector.







Example and General Formula



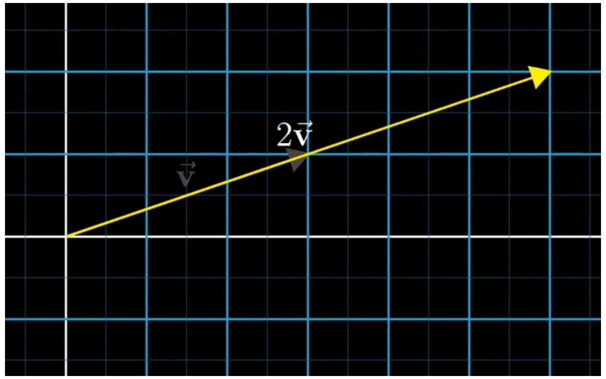
$$\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}=\begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix}$$

- This is the same as
  - moving along the x axis a certain number of steps
  - and then moving parallel to the y axis for a certain number of steps.

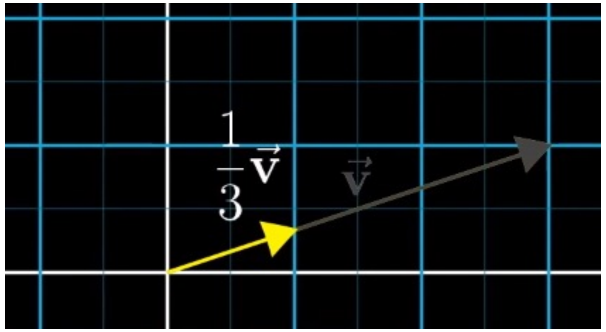
- What about subtracting one vector from another ?

Vector Multiplication

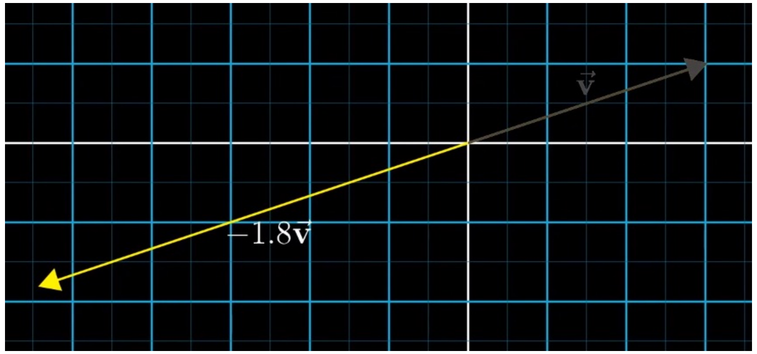
- A few examples :
  - $2\boldsymbol{v}$ => stretches the vector $\boldsymbol{v}$



  - $\frac{1}{3}\boldsymbol{v}$ => squishes vector $\boldsymbol{v}$

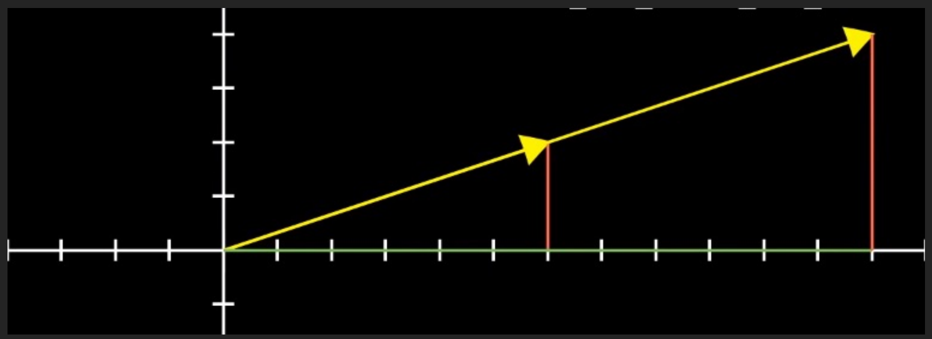


  - -1.8 $\boldsymbol{v}$ flips it around and stretches it



- This is called « scaling » a vector
- 2, -1.8 etc… are also known as « scalars »
- General Idea:

$$2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$



Exercises

$$\boldsymbol{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \ and \ \boldsymbol{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

Compute $\boldsymbol{u} + 2\boldsymbol{v}$ and $\boldsymbol{u} - \boldsymbol{v}$

Exercises

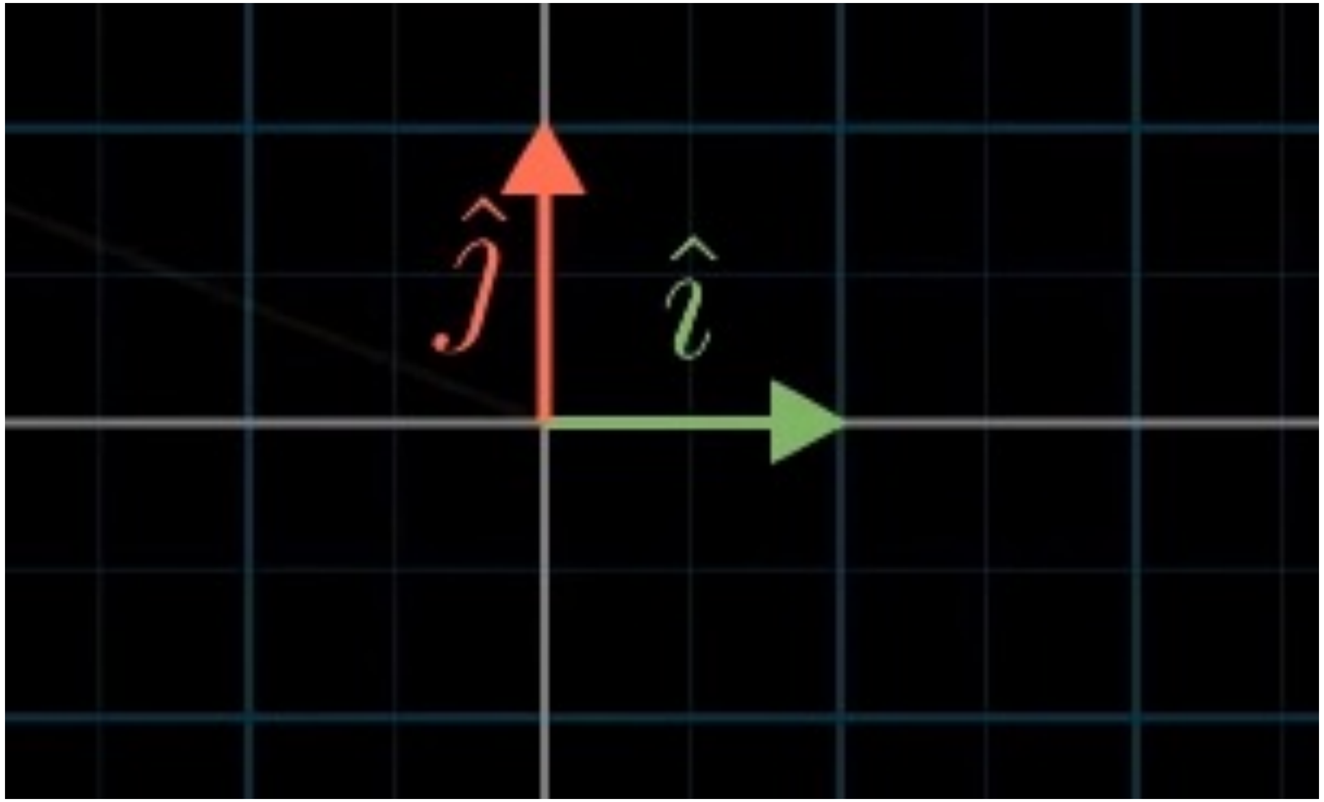
- Display the following vectors using arrows on an xy-graph:
- u,
- v,
- $-v$,
- $2v$,
- $u + 2v$
- $u - v$

How to reconcile the different views mentioned ?

- Linear Algebra is in fact about the ability to go back and forth between the geometric / spatial view and the ordered list of numbers view
- CS student can now visualize his many lists of numbers which can be very helpful to find patterns in data, see how it is distributed
- Physics students and computer graphics programmers can now describe space mathematically and manipulate it.

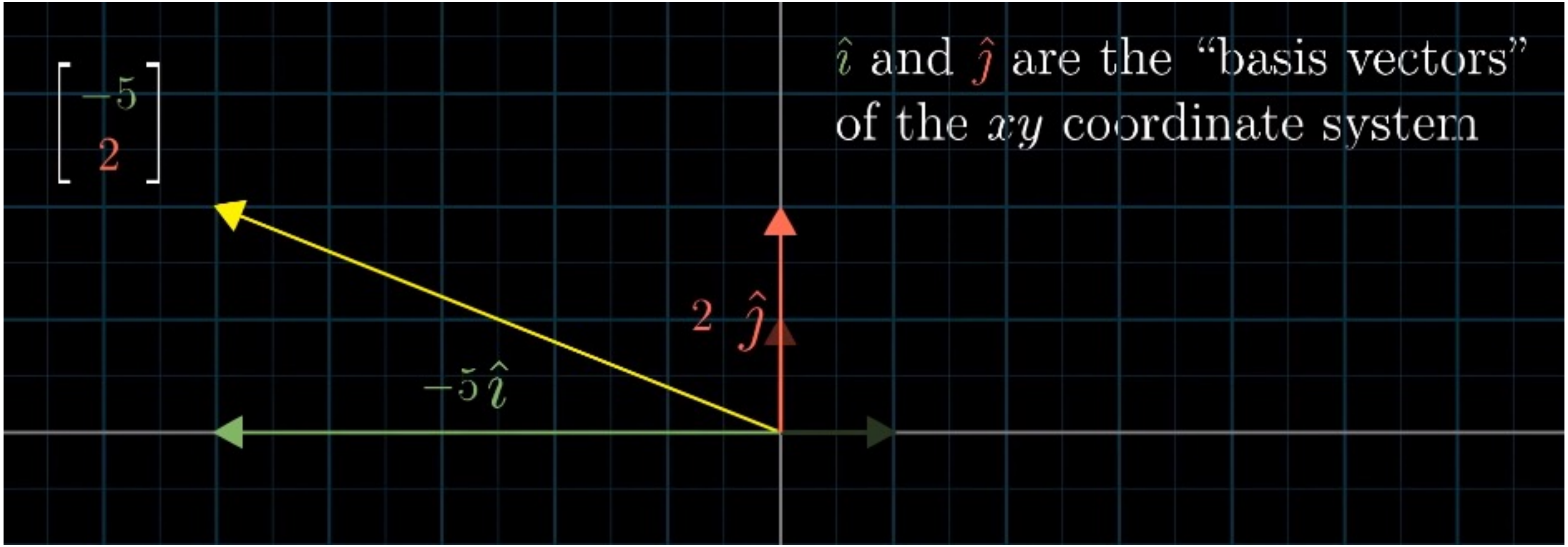
Basis Vectors

- Basis vectors : $i\ hat$ and $j\ hat$
- Vectors which determine the basic unit for the space
- $i_{hat} = \begin{bmatrix}1\\0\end{bmatrix}$
- $j_{hat} = \begin{bmatrix}0\\1\end{bmatrix}$



Linear Combinations

- Any vector in 2D space can be expressed using these 2 vectors :



Linear Combinations

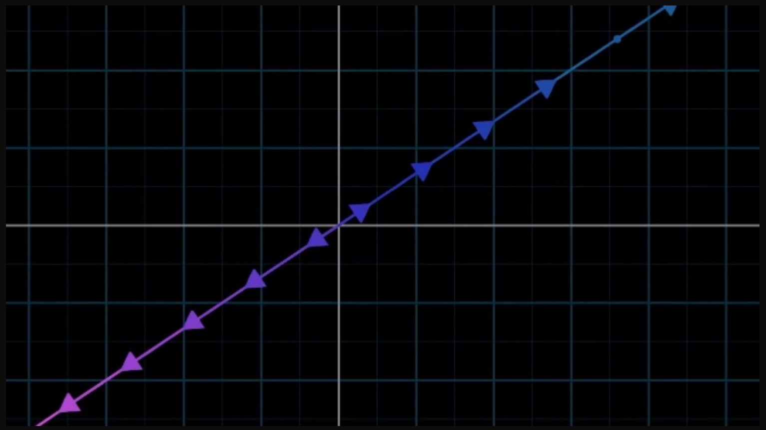
- So any time we describe vectors, it depends on an implicit choice of basis vectors.
- We can actually choose many other vectors as « basis » vectors, we are not limited to $i_{hat}\ and\ j_{hat}$

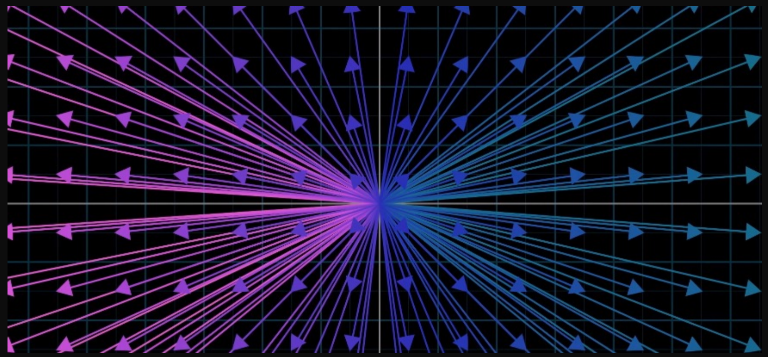
$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} = 5i_{hat} + (-2)j_{hat}$$

- Any time we scale and add vectors => we call this a **linear combination**

Span

- Can reach any point in space by using linear combinations (barring certain specific cases where vectors are colinear or one of the vectors is the 0 vector)
- This is called the *span* of 2 vectors, the amount of space they can cover basically.
- In 2D : this can be
    - a line if the vectors are colinear
    - or a plane, ie all of the 2D space





Mini exercise

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$$

Is the vector $\vec{b}$ a linear combination of the above vectors ?

$$\vec{b} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Solution

$$\vec{b} = 3\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3$$

Quick Recap

- Vector addition
- Vector multiplication
- Basis vectors can be any vectors really, not just $\hat{\imath}$ and $\hat{\jmath}$
- We can describe any vector in a particular space as a linear combination of basis vectors.

Linear Transformations and Matrices

More intuitive to figure out what they do *visually*, so let’s try and do this

> Unfortunately, no one can be told what the Matrix is. You have to see it for yourself.
>
> -Morpheus

- Get an intuition for
    - matrix / vector multiplication
    - and later on vector / vector multiplication or dot product
    - (so far we’ve only seen what happens when we multiply vectors by scalars)

- Goal is to try not to rely too much on memorization for these operations which neurals nets rely on and which are generally used in machine learning.

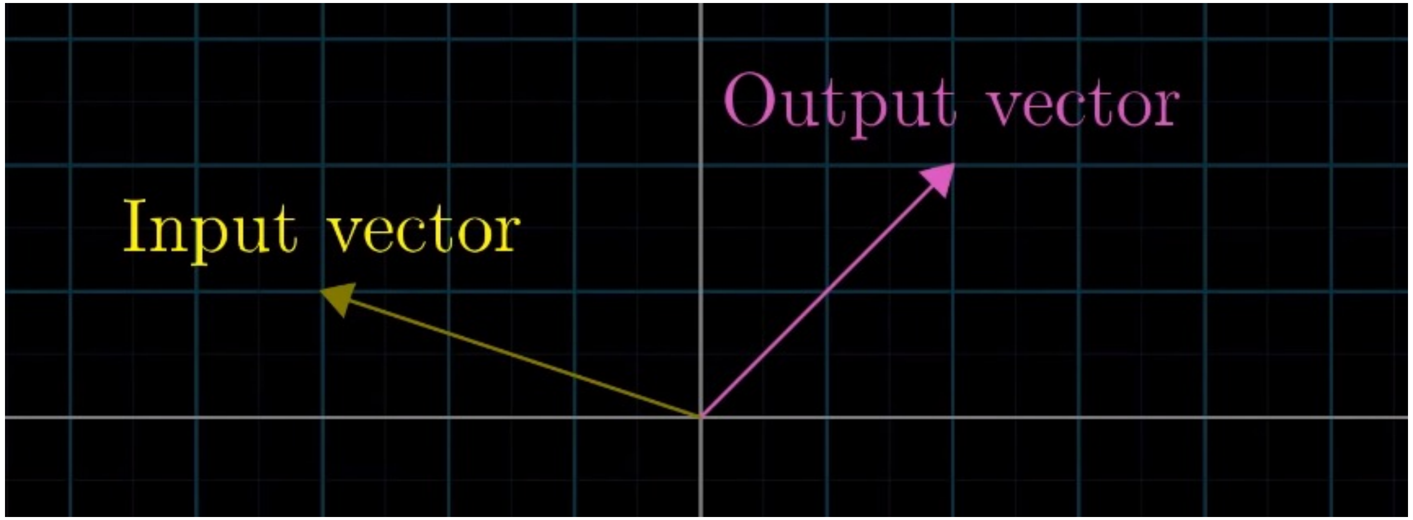
Transformation

- Fancy word for function:
  - Takes an input and spits out an output
  - In Linear Algebra, this means 1 input vector is mapped onto 1 output vector

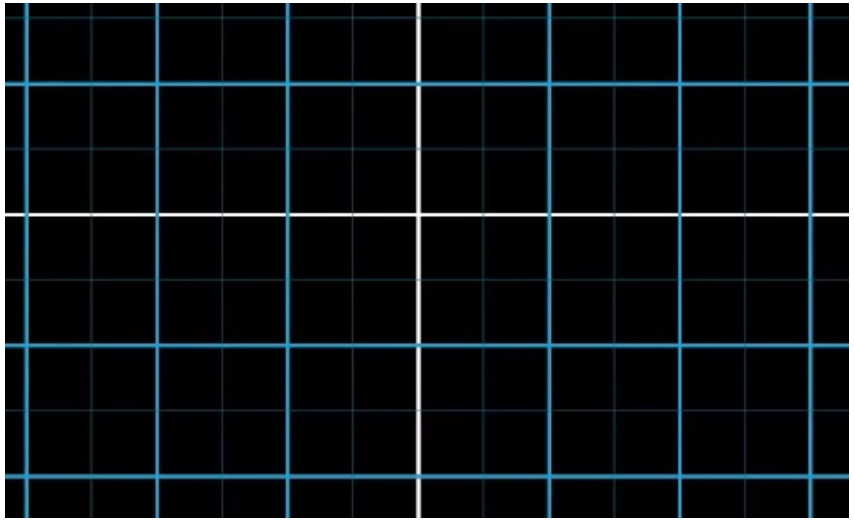
  - Transformation has maybe more of a visual connotation than function however…
  - If a transformation maps an input vector to an output vector, we can imagine that input vector *moving over* to the output vector.

Illustration

- For a single vector :



- For every vector in the space (represented by an infinite grid):



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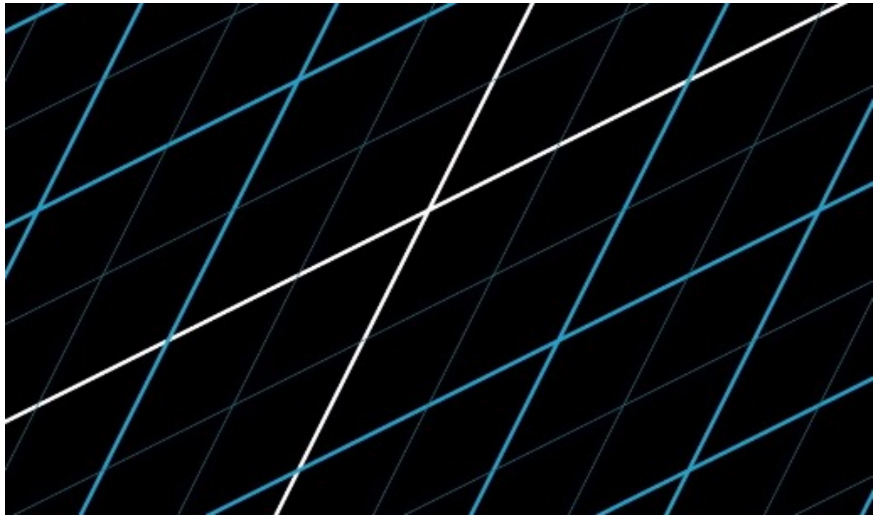


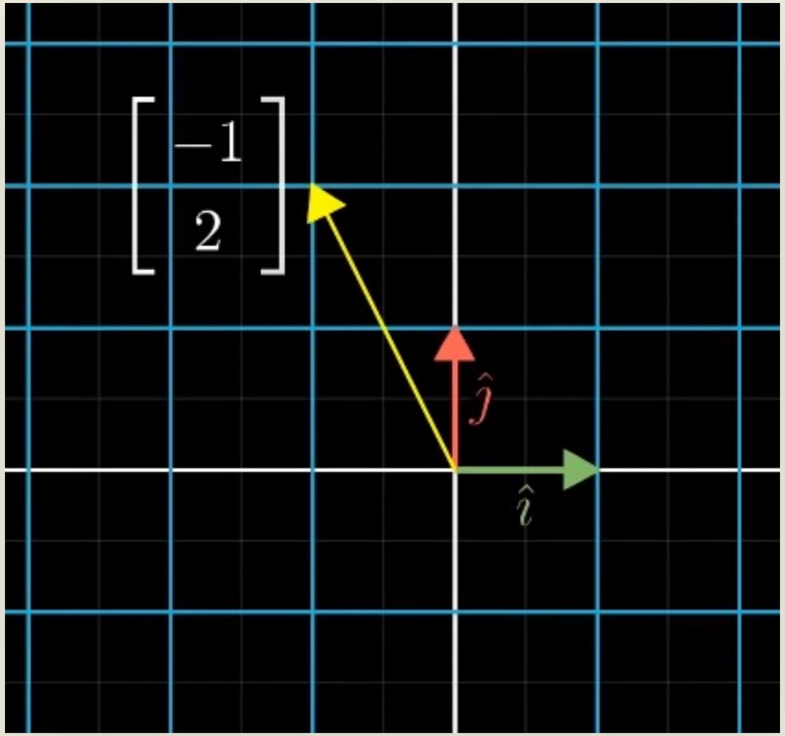
Linear (Transformation)

- All lines must remain lines without getting curved
- The origin must remain fixed in place

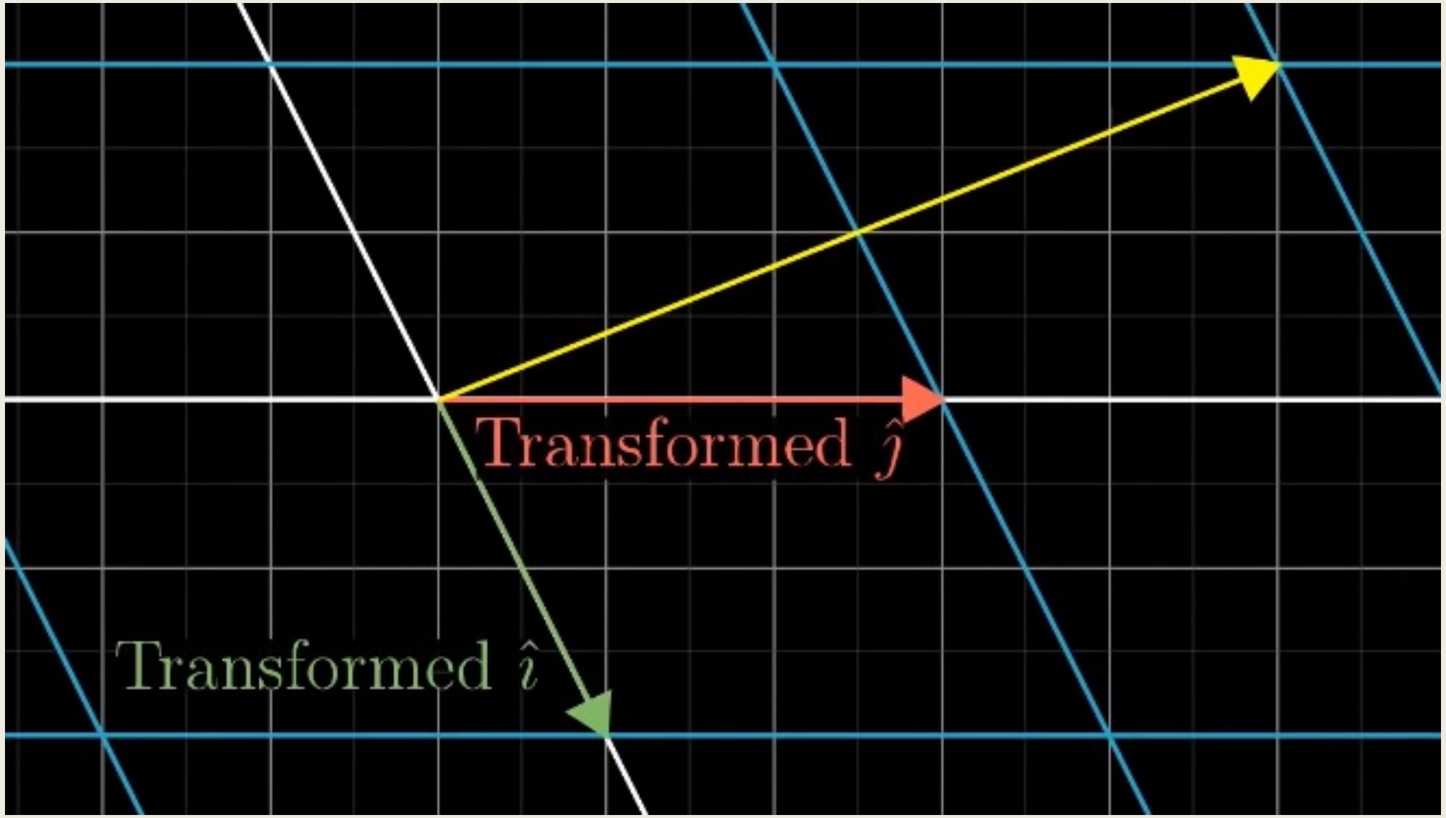
- Everything remains parallel and evenly spaced, which has some nice properties.

- To know where a vector is going to land after going through a linear function/linear transfromation, all we need to do is look at where $\hat{\imath}$ and $\hat{\jmath}$ land.
- Remember :
  - if a vector in the input space is a linear combination $\hat{\imath}$ and $\hat{\jmath}$
  - Then it will be equal to a similar linear combination in the transformed space, but with $\hat{\imath}$ and $\hat{\jmath}$ transformed (ie. where these basis vectors landed).

Example



$$\vec{\mathbf{v}} = -1\hat{\imath} + 2\hat{\jmath}$$



$$\text{Transformed } \vec{\mathbf{v}} = -1(\text{Transformed } \hat{\imath}) + 2(\text{Transformed } \hat{\jmath})$$

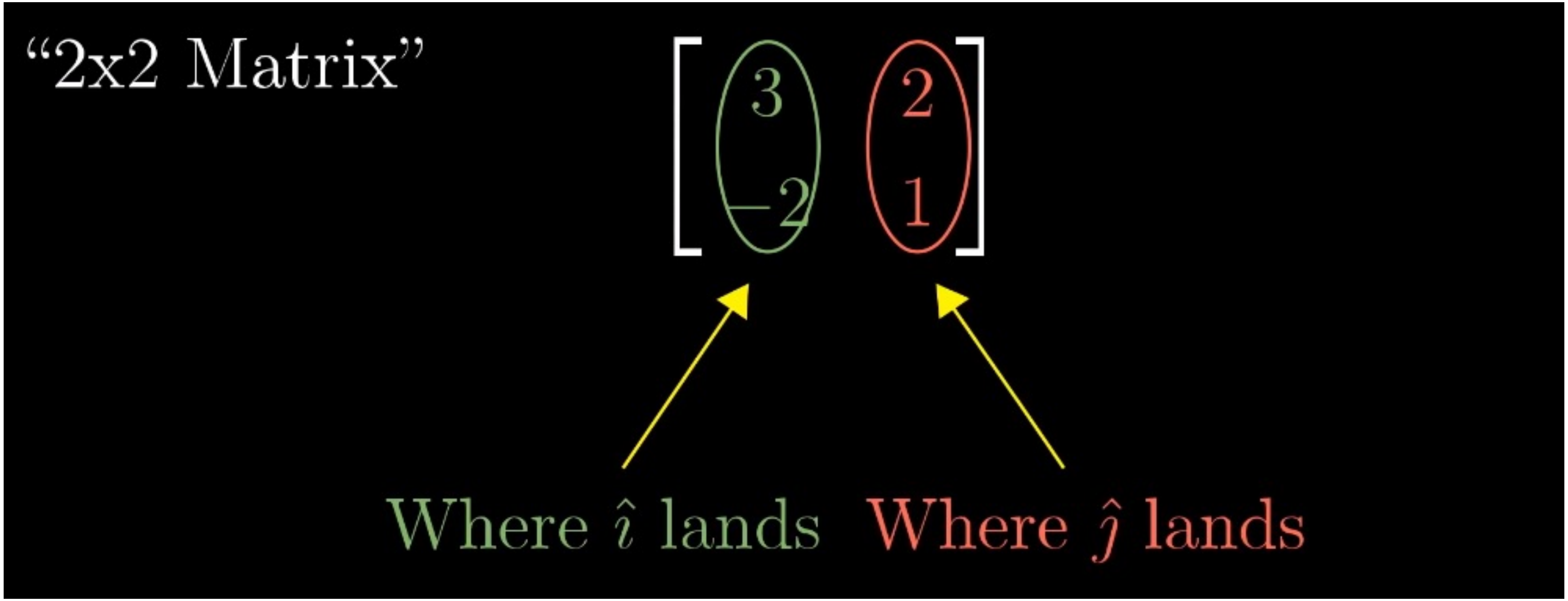
- In the example, we see where the vector landed…
- But by **only looking** at where the **basis vectors** land, we can deduce any transfromed vector's new position.
- So we basically only need these 2 new « basis » vectors :

$$\hat{\imath} \rightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \hat{\jmath} \rightarrow \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$$

Matrices

- We commonly package these coordinates into a 2x2 grid of numbers
- => a 2x2 Matrix



Generalized Formula of a Matrix / Vector Product

- A matrix / vector product is like applying a function to a vector where:
  - The columns of the matrix are the transformed basis vectors
  - The result of the operation is a linear combination of those column vectors.
- When learning about matrix vector products, seeing it as a linear combination is usually not made explicit… and you're left with having to remember by heart ax + by etc…

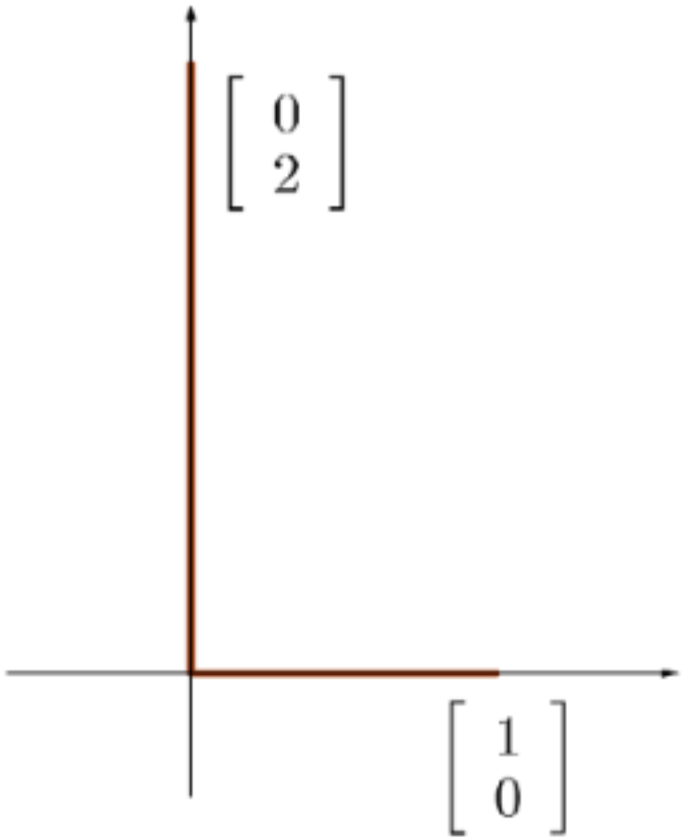
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}}_{\text{Where all the intuition is}} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

Transformation examples

- Rotation
  - 90° for example
- Shear (lateral shift)
- => so everytime you see a matrix, you can interpret it as a certain transformation of space (where everything stays linear, no curves!)

Exercise

- Show the effect of the linear transformation T on a letter L made up of the vectors



$$T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

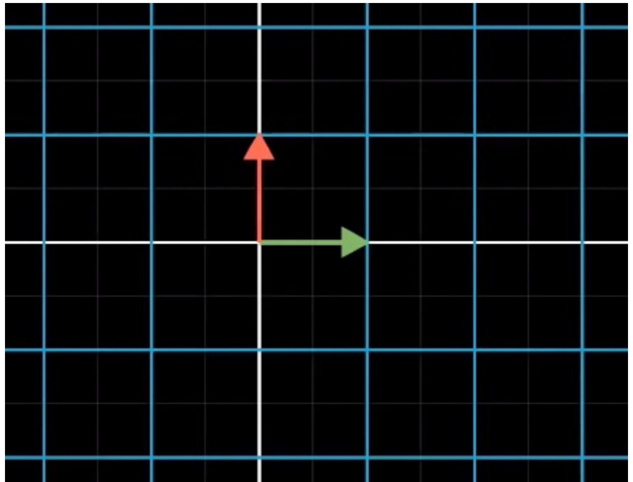
Matrix Multiplication

- What happens when we multiply 1 matrix by another ?
- What does it mean intuitively, what happens ?
- Always a good idea to think of the geometric meaning and not get lost in the numbers at first

Matrix Multiplication

- Apply one transformation to another: move space around twice !

- The combined transformation is called a « composition »
- Rotation then shear for example : new basis vectors are the column of the composition matrix



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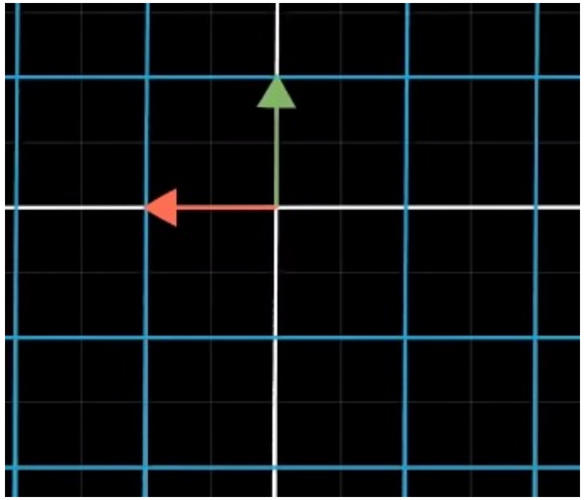

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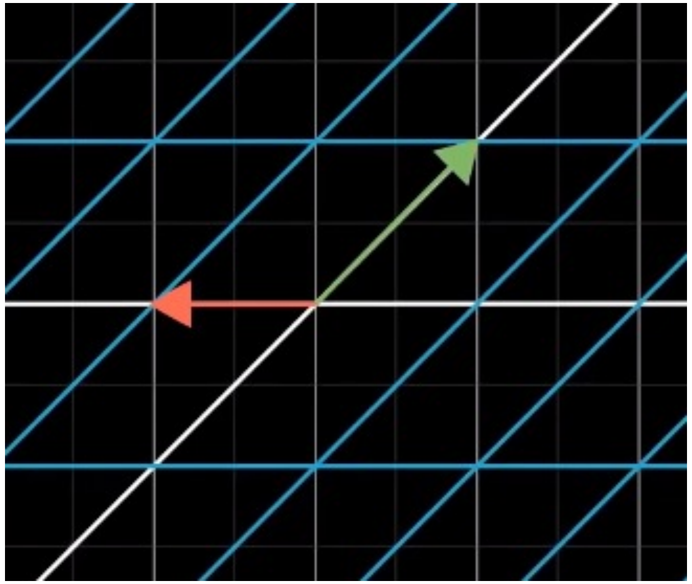


$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{Shear}} \left(\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}} \begin{bmatrix} x \\ y \end{bmatrix} \right) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Composition}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix Multiplication

- The composition matrix can be understood as the result of a product of matrices.
- That product should be read right to left, as with functions:

$$f(g(x))$$

- example :

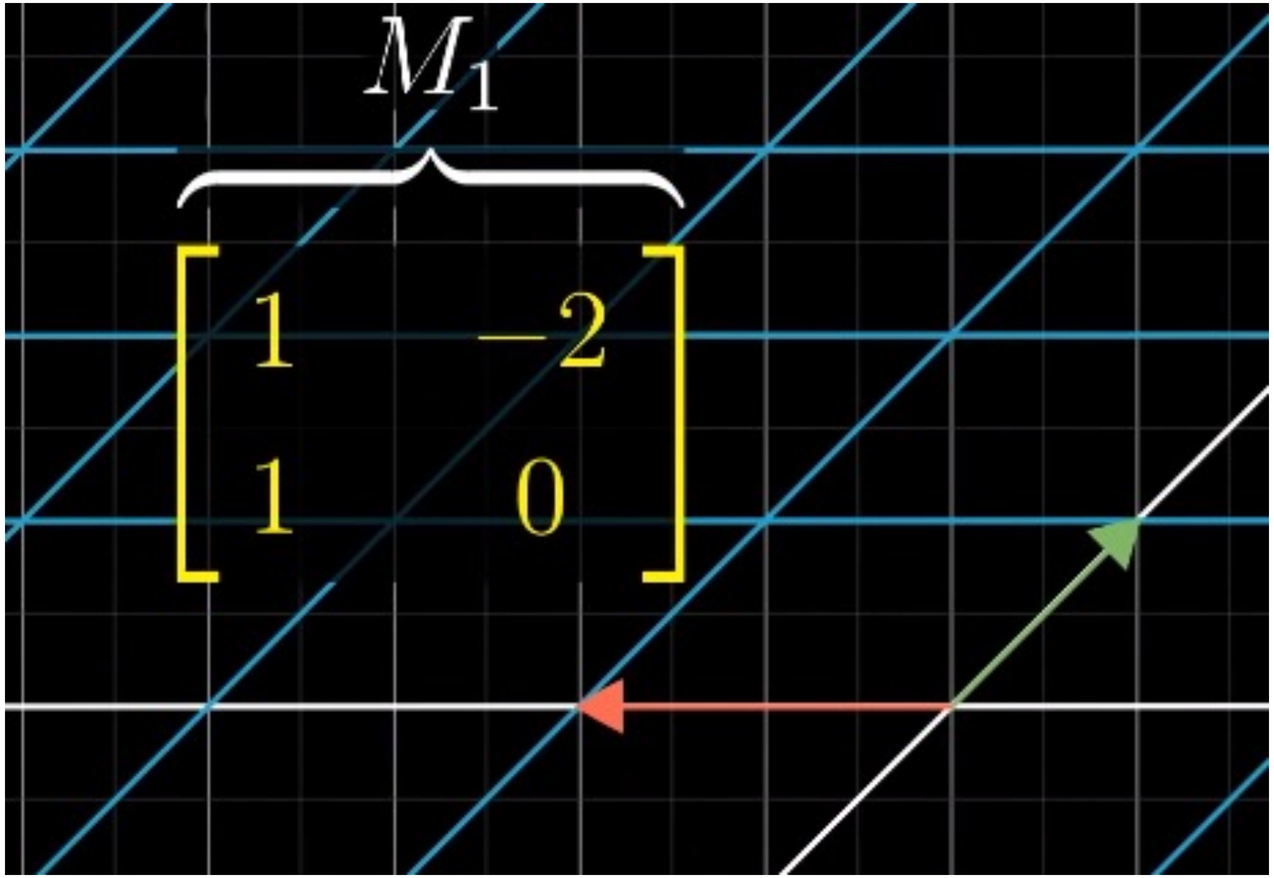
$$g(x) = 3x$$
$$f(x) = 2x$$
$$f\big(g(x)\big) = ?$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{Shear}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Composition}}$$

Example

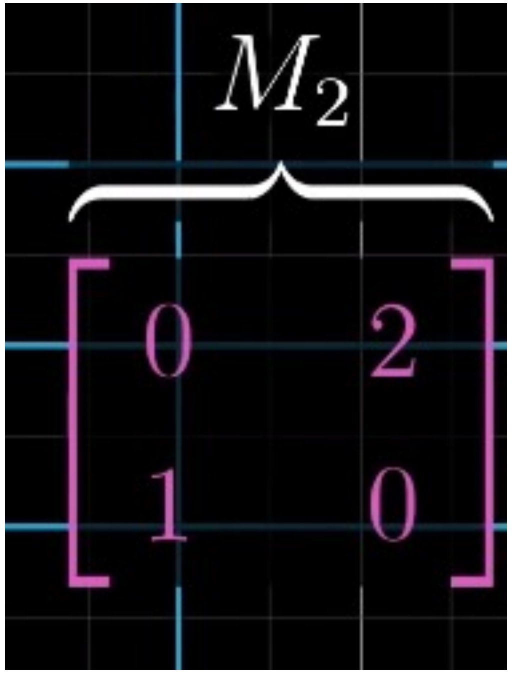
- We are going to apply 2 consecutive transformations, and follow the basis vectors:

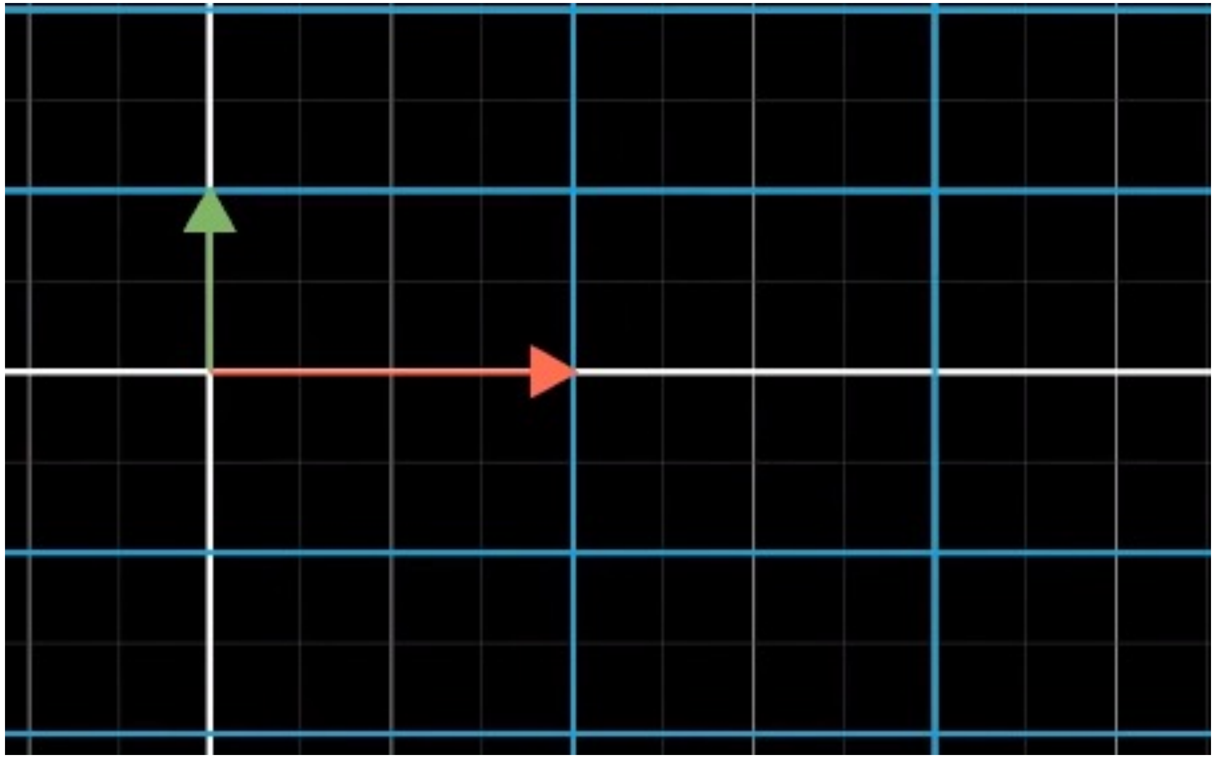
- Matrix M_1 and its transformation:



Example

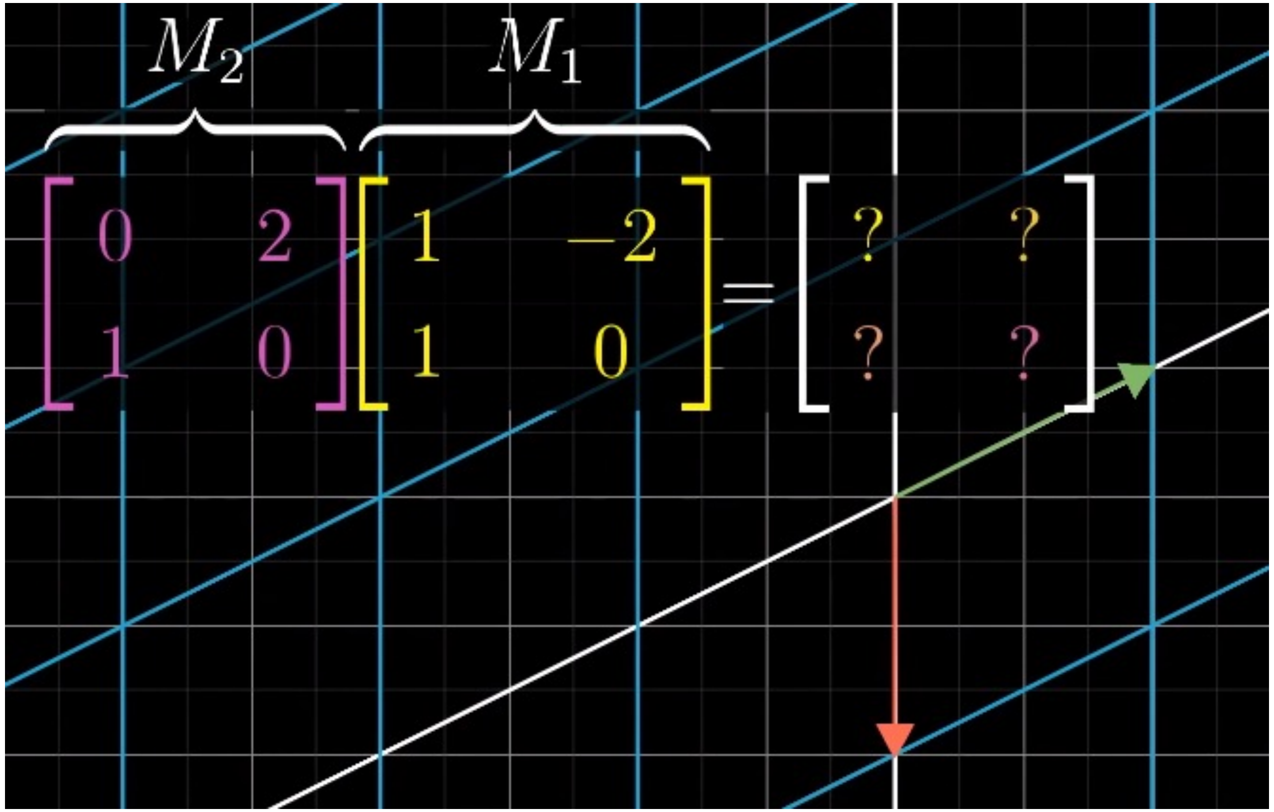
- Matrix M_2 and its transformation :





Example

- Total effect gives us a new transformation !
- Although we could figure this out geometrically/visually, this isn't necessarily immediately intuitive like in the previous example…
- So let's compute this composition numerically without looking at the vectors in space.



Example

- First column of M_1 tells us where $\hat{\imath}$ lands. Let's follow this vector to begin with.

$$\overbrace{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}^{M_2} \overbrace{\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}}^{M_1} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

Example

- To figure out what happens to this vector when we then apply the transformation defined by M_2,
- we can use a simple matrix vector product, just like in the previous slides !

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example

- And if you remember the intuition behind it,
- This is nothing but a linear combination :

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Mini Exercise

- Now try and figure out the values in the second column of the matrix!

$$\overbrace{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}^{M_2}\overbrace{\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}}^{M_1} = \begin{bmatrix} 2 & ? \\ 1 & ? \end{bmatrix}$$

Example

- Same thing as for the first column :

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

General formula

- As previously, let's first take care of $\hat{\imath}$ and its resulting transformation

$$\overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{M_2}\overbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}^{M_1} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e \\ g \end{bmatrix} = e\begin{bmatrix} a \\ c \end{bmatrix} + g\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ae + bg \\ ce + dg \end{bmatrix}$$

General Formula

- And then follow the same process for $\hat{\jmath}$!

$$\overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{M_2}\overbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}^{M_1} = \begin{bmatrix} ae + bg & ? \\ ce + dg & ? \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} f \\ h \end{bmatrix} = f\begin{bmatrix} a \\ c \end{bmatrix} + h\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} af + bh \\ cf + dh \end{bmatrix}$$

General Formula

- The final result :

$$\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

- It is **very** common to be taught this formula as something to memorize as a process…
- Without anyone really ever going through where it comes from and what it represents.
- Can be very useful to understand that we are actually applying 2 consecutive linear transformations.

Practice

- $E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
- $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

- Let $H = \mathrm{EA}$.
- Find H.
- Now compute AE

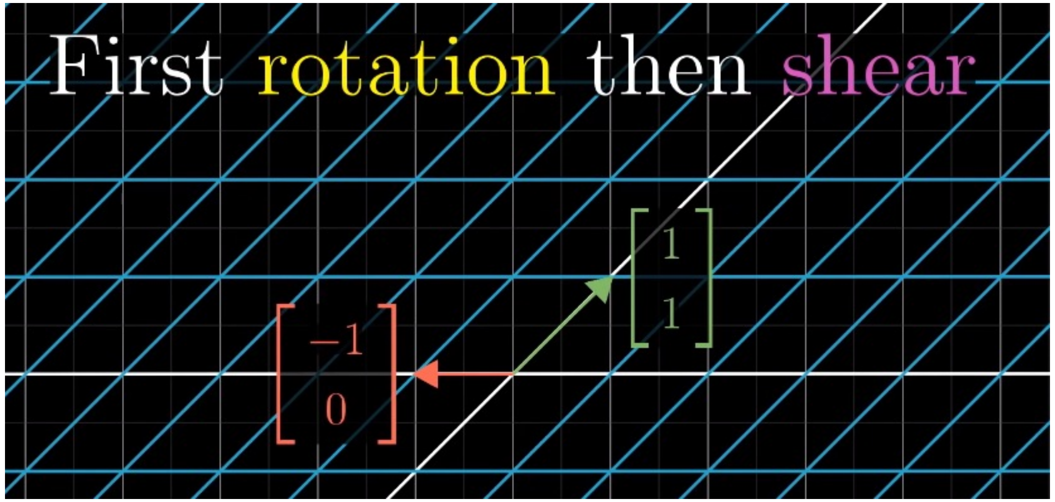
Matrix Multiplication is <u>not</u> commutative

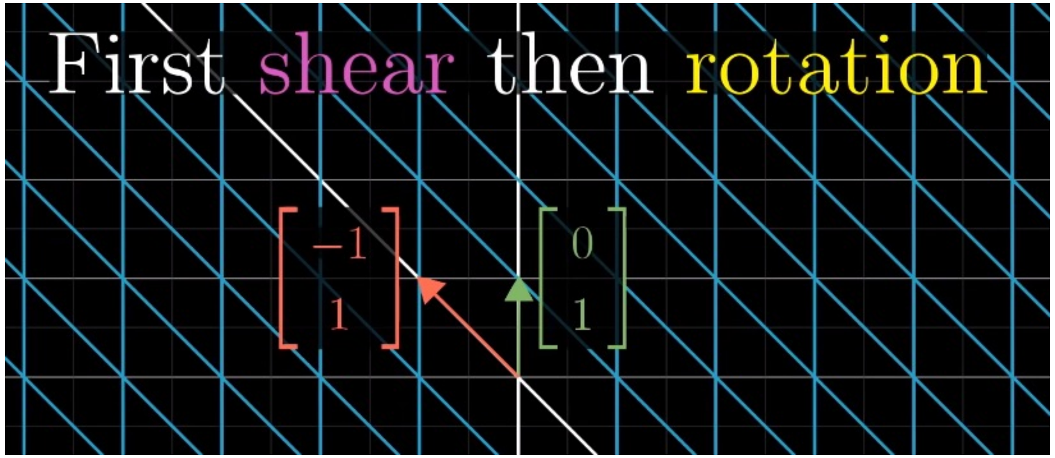
- Now you know what this operation represents, here’s a question :
- Does order matter ?

$$M_1 M_2 \overset{???}{=} M_2 M_1$$

Extra example

- Because you now understand what these matrices represent, it seems intuitive that those 2 transformations are not equal if you go through them in your head.
- No need to go do calculations an compare the results !
- $M_1 M_2 \neq M_2 M_1$





Non-square transformations

- Can go from 2d to 3d with a linear transformation, but this is slightly less intuitive as the spaces are unconnected compared to 2d -> 2

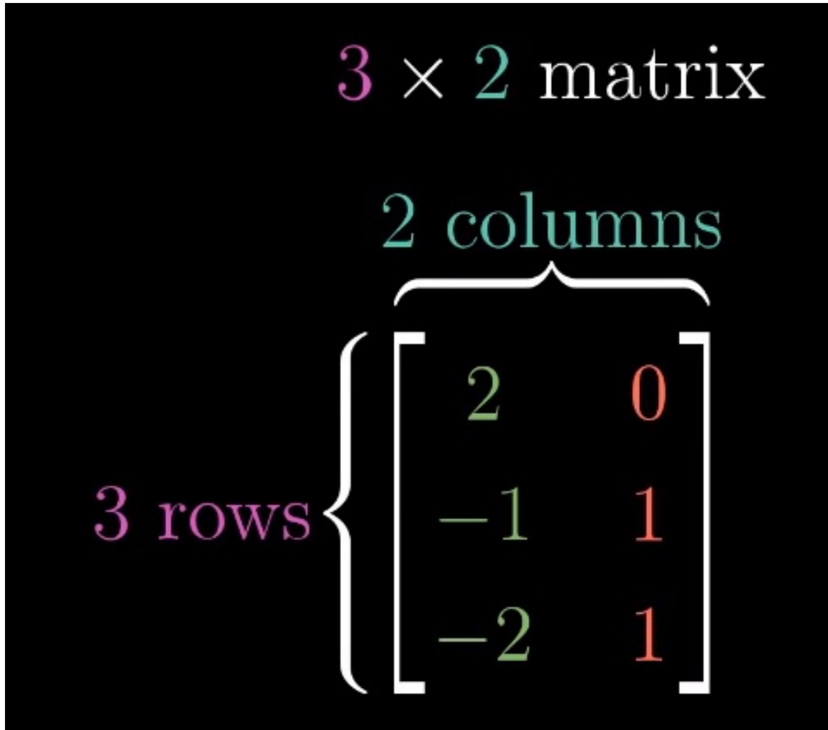
- # Columns : # of basis vectors in the input space
- # Rows = landing spot for the basis vectors has

3 separate coordinates

- These matrices map from (#columns) dimensions

to (#rows) dimensions

$\mathbb{R}^{3\times 2}$



Practice

- $\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ => ?

- $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ => ?

Recap

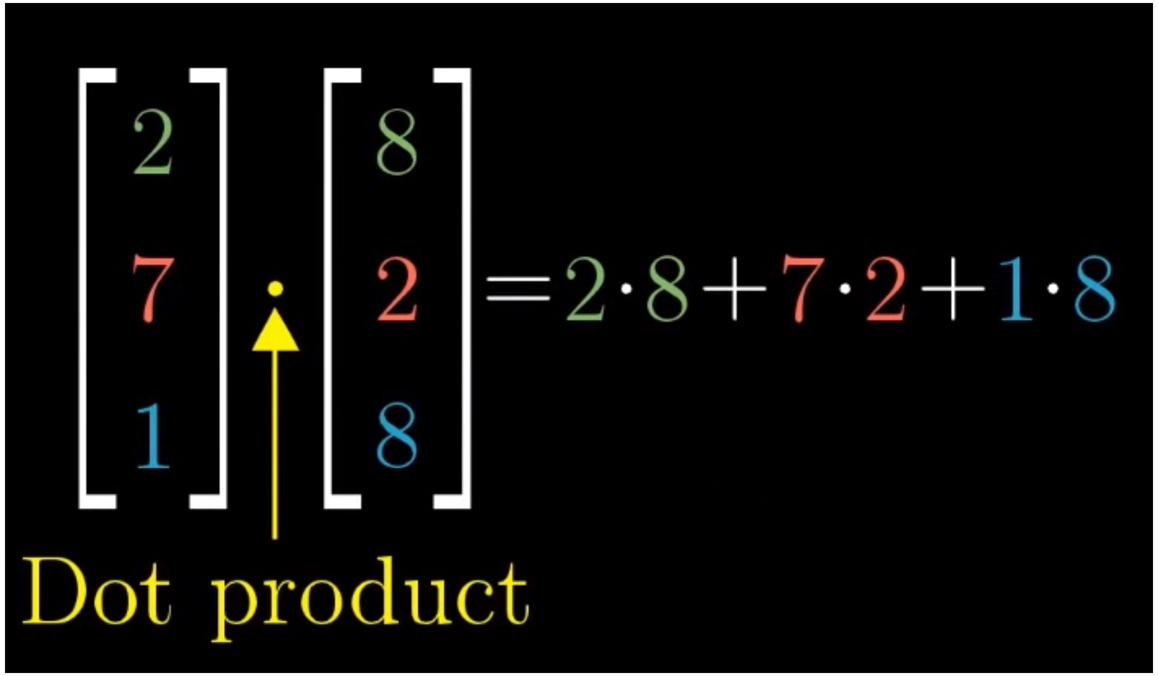
- Matrix matrix multiplication :
  - Consists in applying 1 transformation after another
  - The « product » of 2 matrices is called a « composition » and describes the combined, overall transformation.
  - Therefore the « product » of 2 matrices yields a new matrix.
  - Lastly, matrices can be « rectangular » and map 1 set of dimensions to another

Dot Product

- Now let’s look at the dot product, ie. « Vector / vector multiplication »
- Keep linear transformations in mind !

Standard view first

- If you have 2 vectors with the same dimensions, then the dot product is equal to :
- Summing the products of all the paired up coordinates.

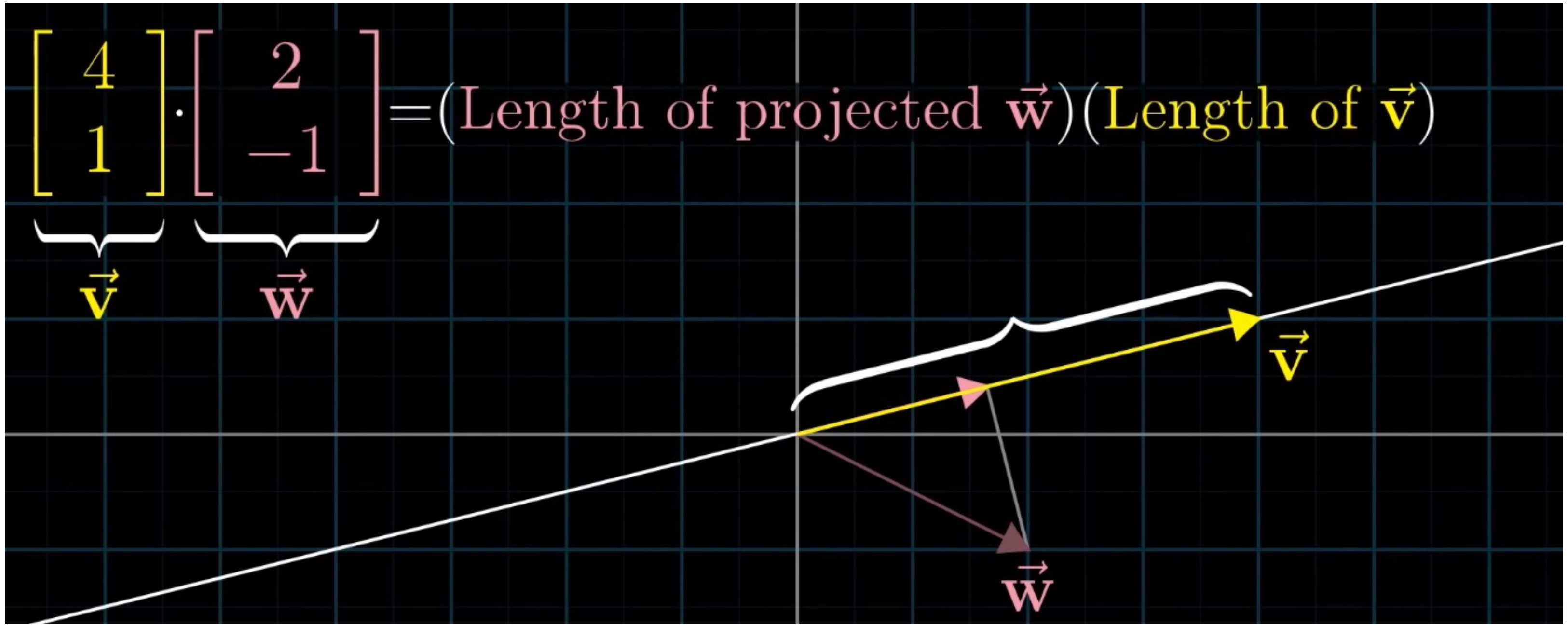


Okay so what about…

- $\begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}3\\4\end{bmatrix}$…?
- $\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$…?
- What do you notice about the result ?

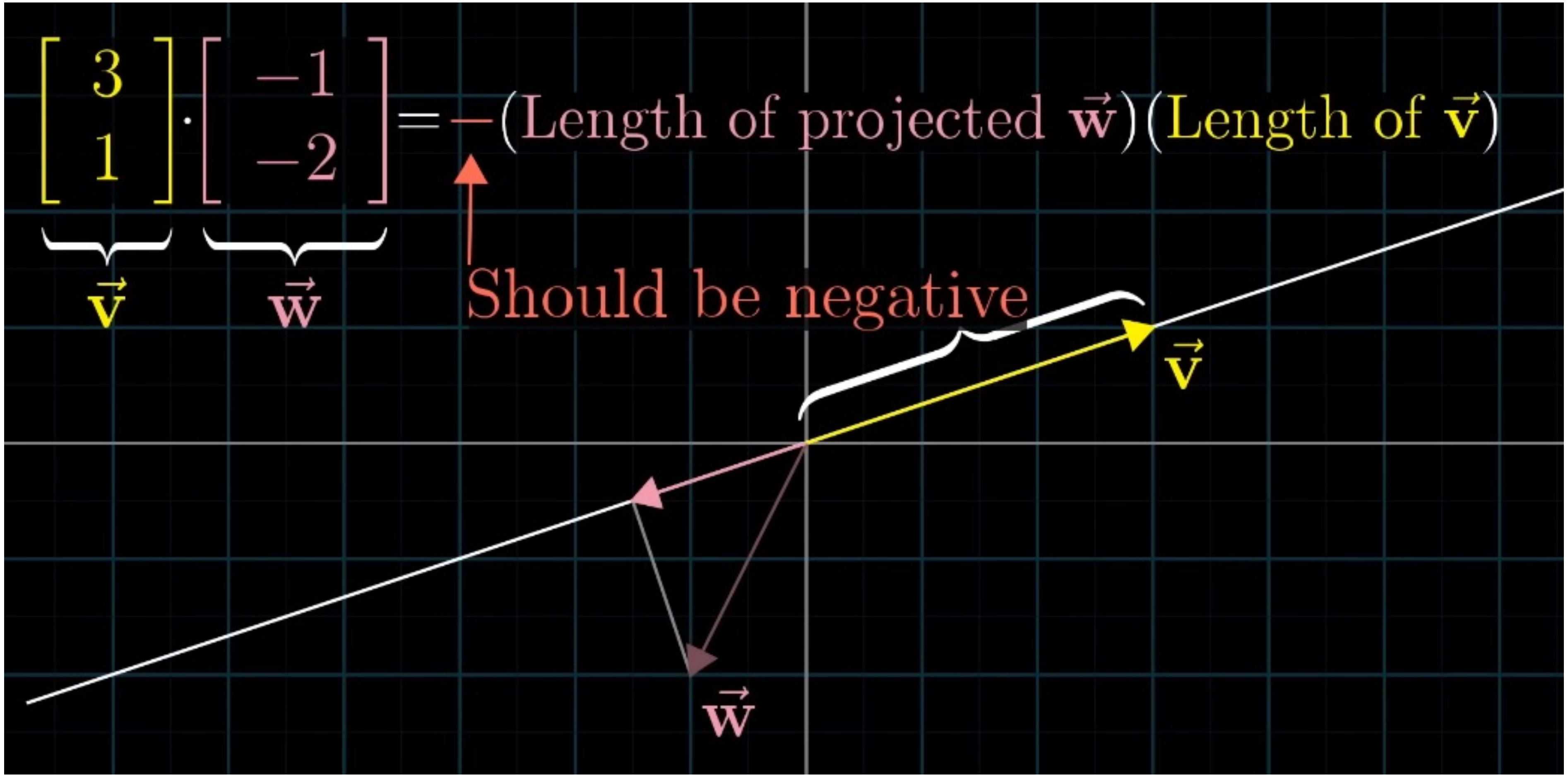
Geometric Interpretation

- This often accompanies the dot product's introduction.
- Projecting $\boldsymbol{w}$ onto the line which passes through the origin and $\boldsymbol{v}$'s tip, (imagine shining a torch, where the projection is $\boldsymbol{w}'$s shadow)
- the dot product is equal to the length of that projection multiplied by the length of $\boldsymbol{v}$.



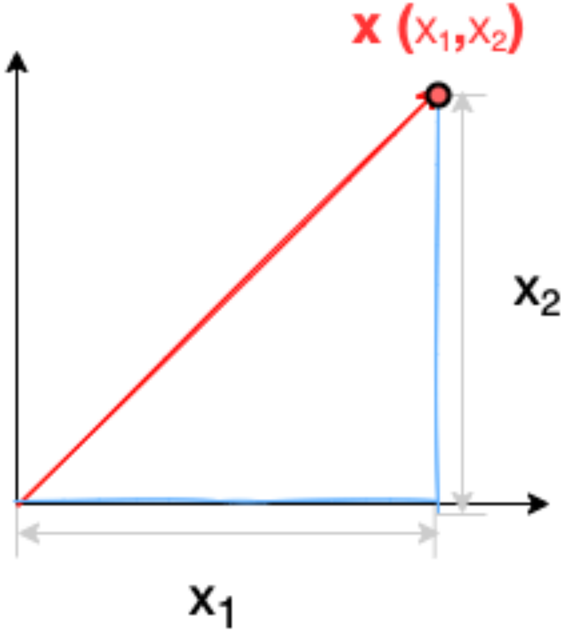
Negative dot product

When $\boldsymbol{w}$ points in the opposite direction, the dot product will actually be negative however.



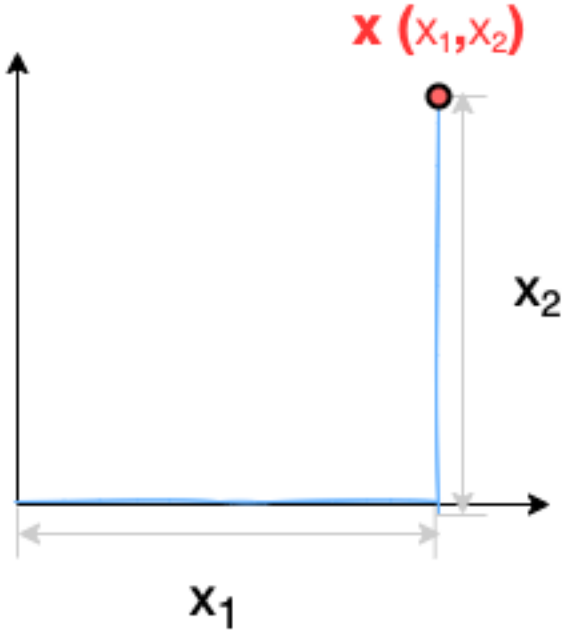
Vector Norm

- How can we calculate the length of a vector ?
- $\| \boldsymbol{a} \| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$
- Euclidian norm or L^2 norm
- Measures the shortest distance rom the origin



Vector Norm

- $\| \boldsymbol{a} \| = |a_1| + |a_2| + \cdots + |a_n|$

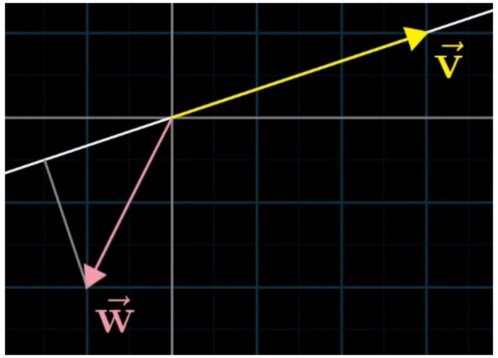


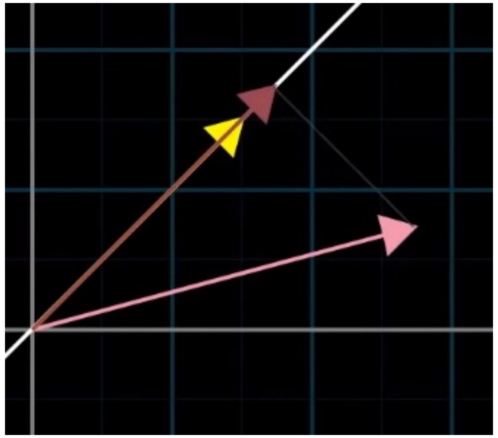
- L_1 norm or Manhattan norm

- Sum of the absolute values of the components of the vector

A couple important properties

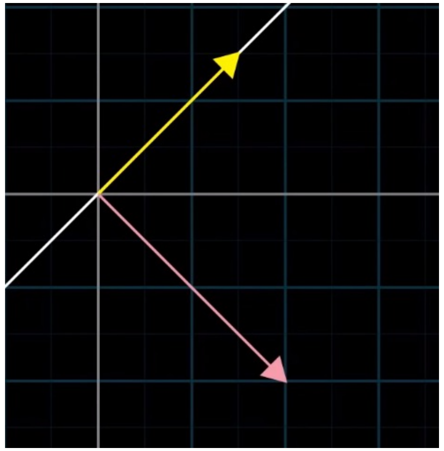
- The dot product between 2 vectors is :
- < 0, when the projection faces the opposite direction





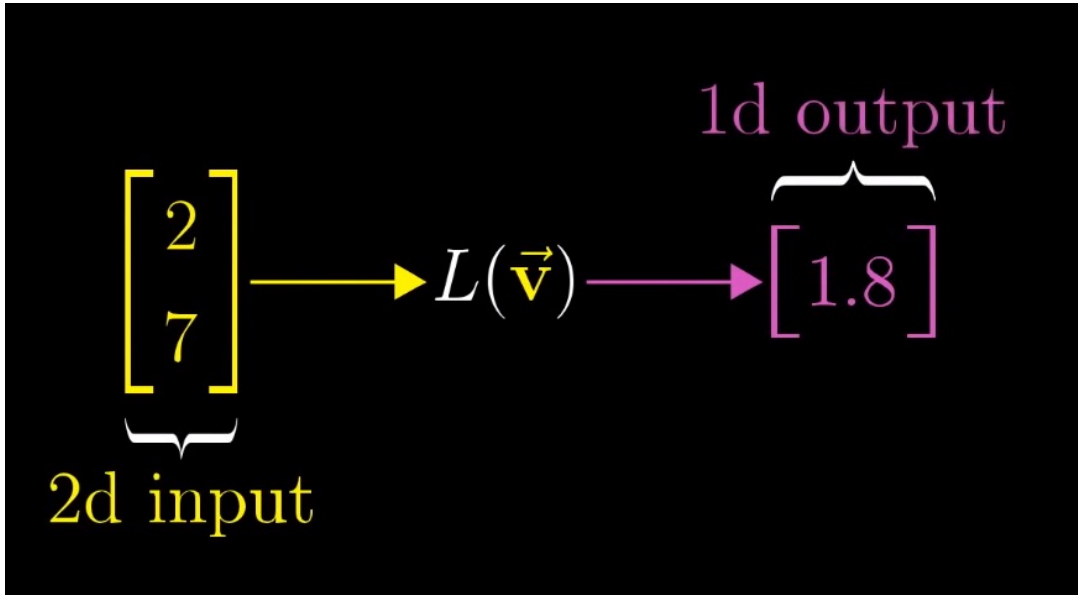
- > 0, when vectors point in the same general direction

- = 0 when perpendicular



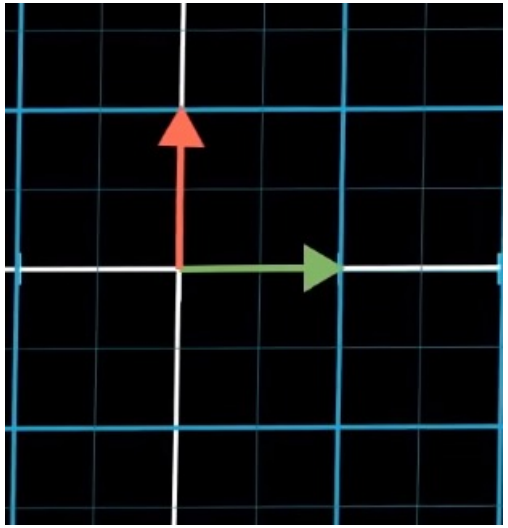
Back to linear transformations : Multi-D to 1D

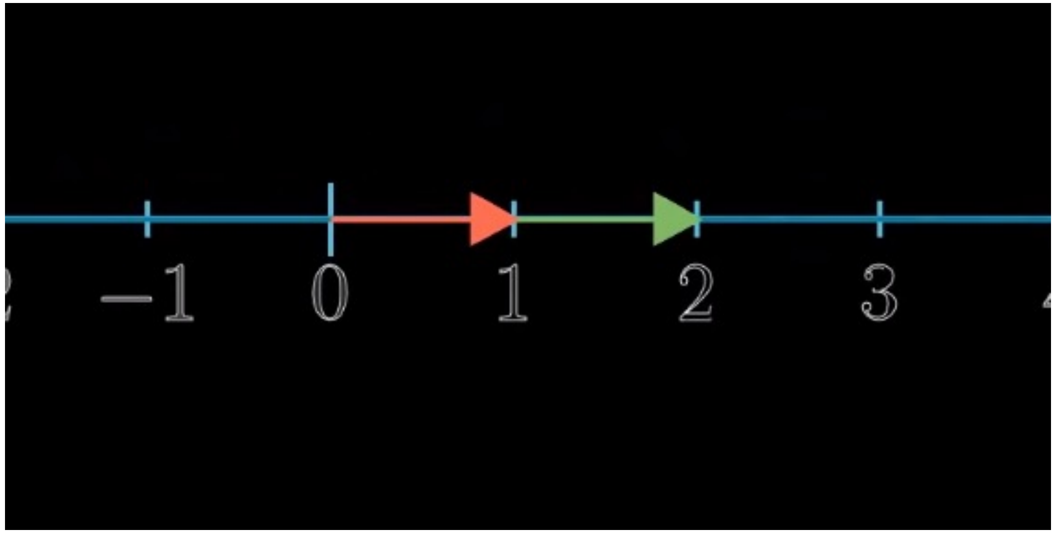
- This type of transformation is a function which takes in a **vector** and spits out a **number**



2D to 1D transformation

- Remember a transformation can be determined by looking at where $\hat{\imath}$ and $\hat{\jmath}$ land…
- In this case, each basis vector just lands on a number vs. A point in 2D space.

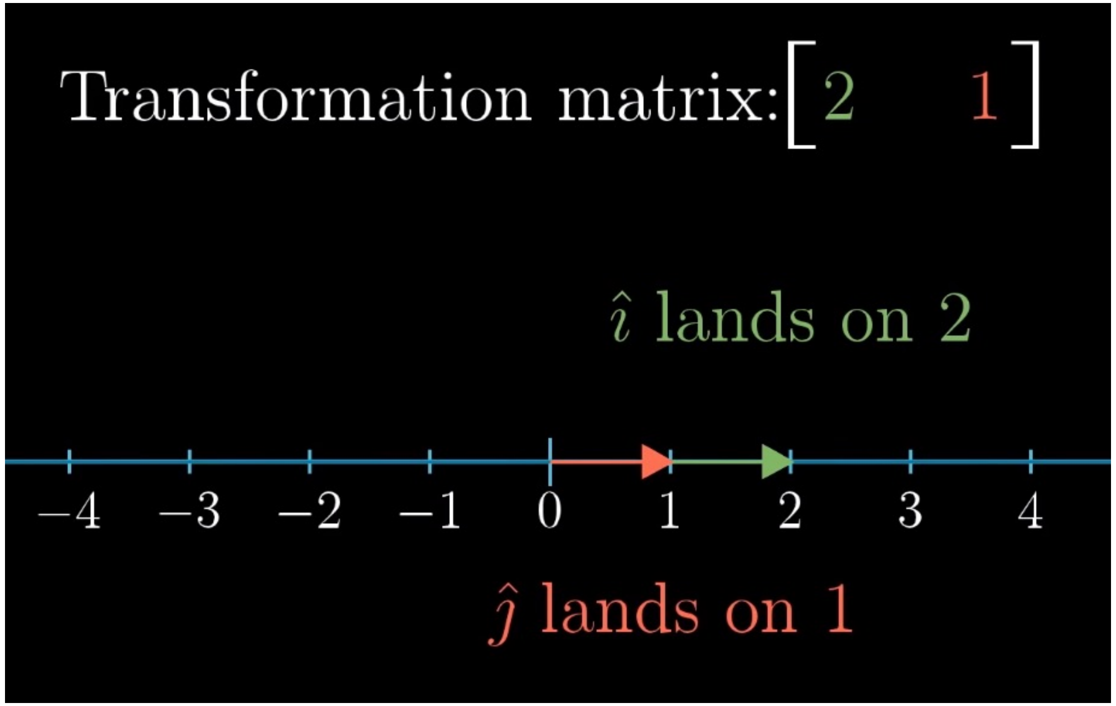




Matrix

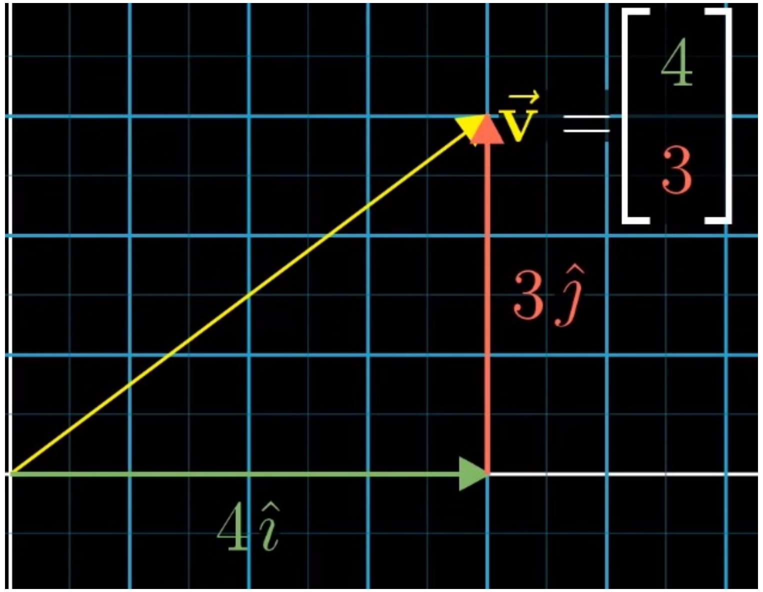
- So when we record where these vectors land as the columns of the matrix
- Each column just has a single number !

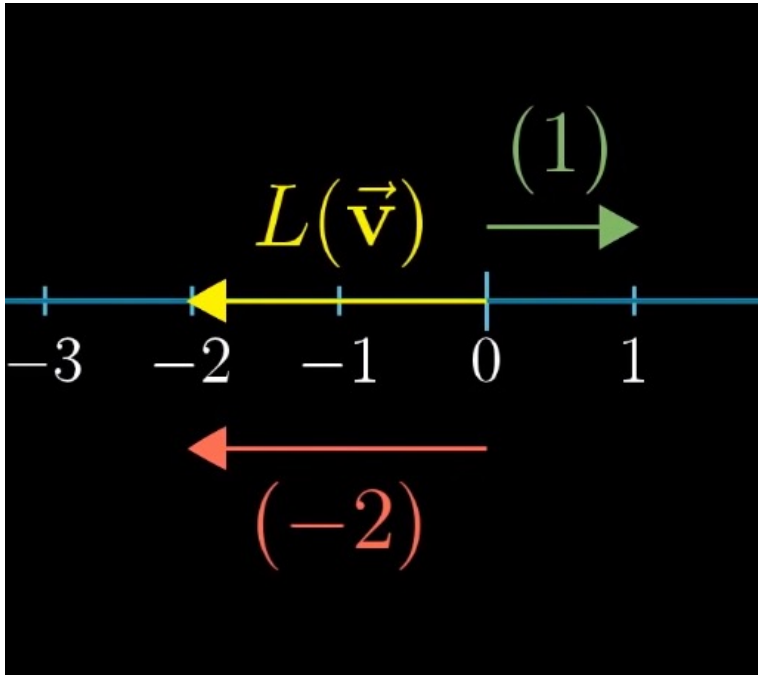
- This is a 1 x 2 matrix ($\mathbb{R}^{1\times 2}$)

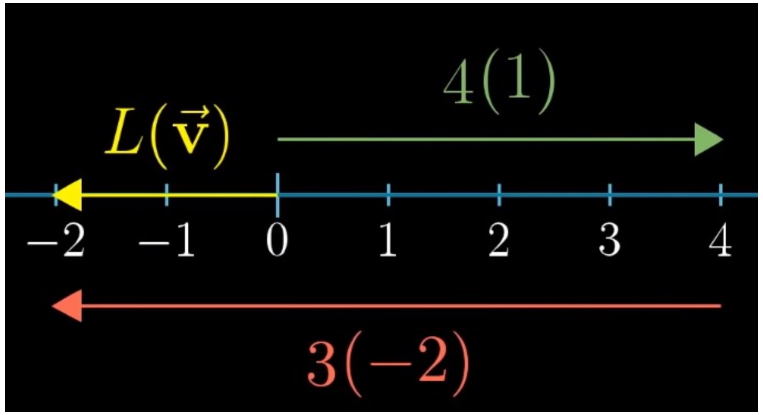


Applying the transformation to a vector $\boldsymbol{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

- The vector is defined as a linear combination of the basis vectors,
- and a consequence of linearity is that
- The same linear combination applies
- Using the « landing spots » of $\hat{\imath}$ and $\hat{\jmath}$.

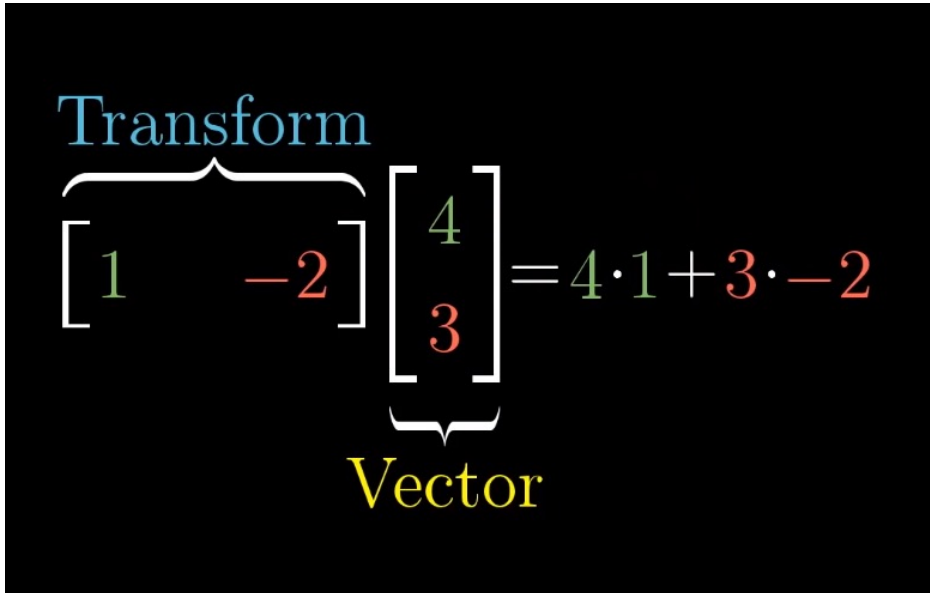


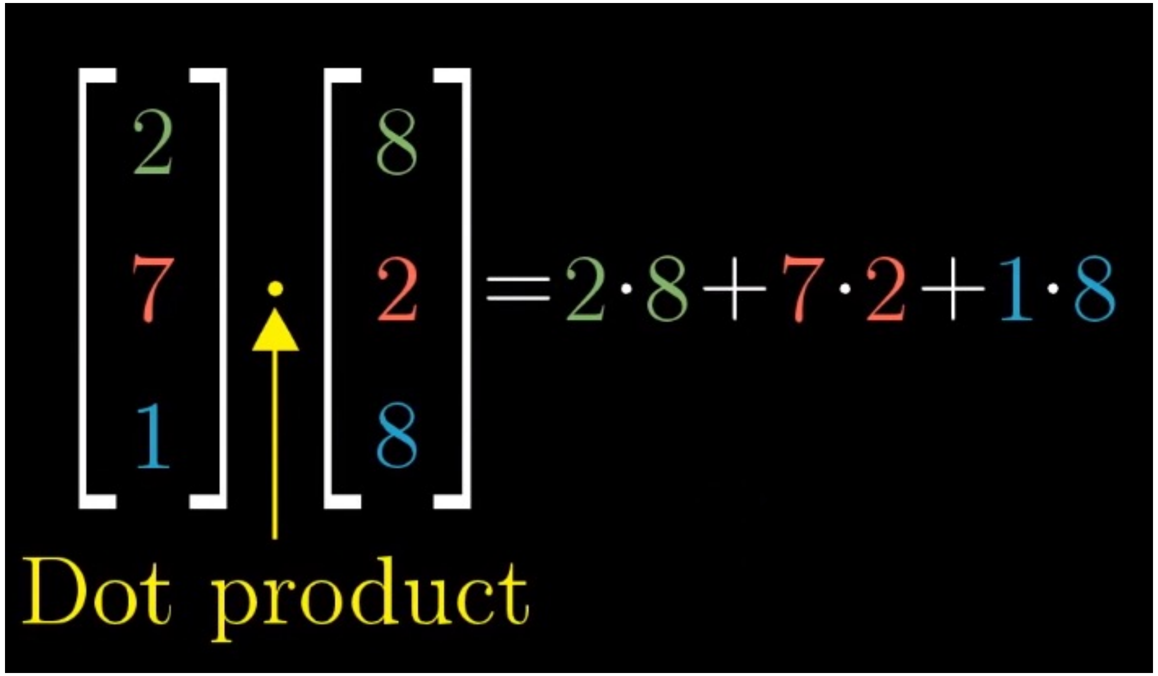




Matrix / vector operation vs. Dot Product

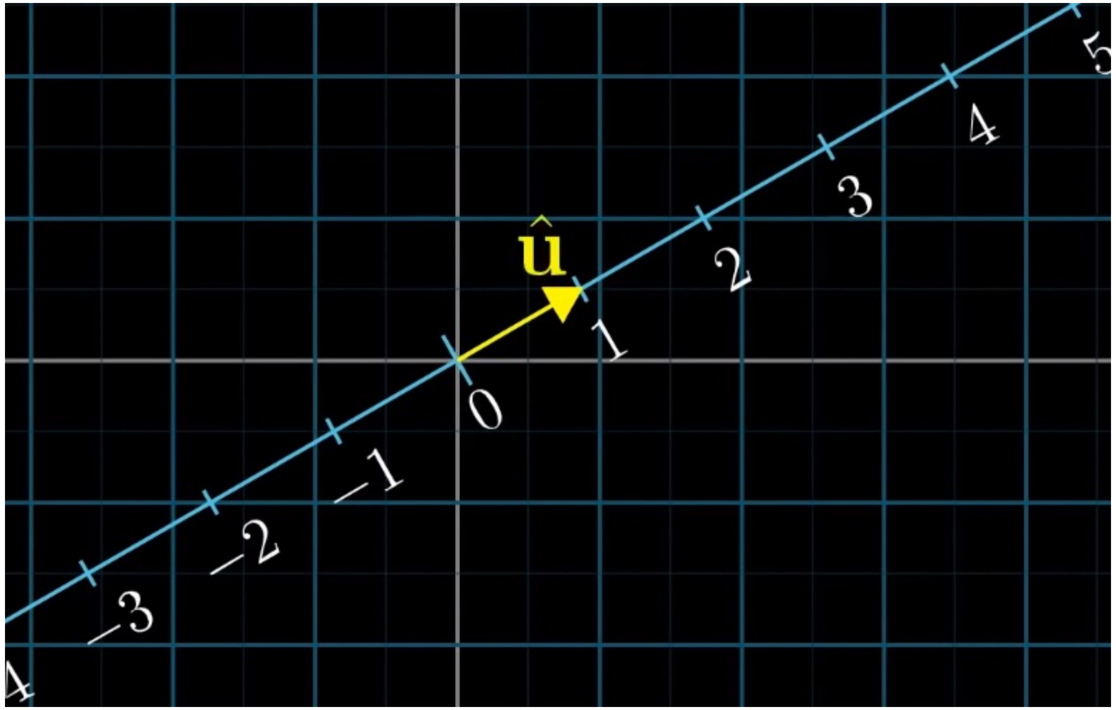
- Looking at things numerically, the matrix vector product in this case feels just like taking the dot product between 2 vectors !
- A 1 x 2 matrix just looks like a vector, but tipped on its side

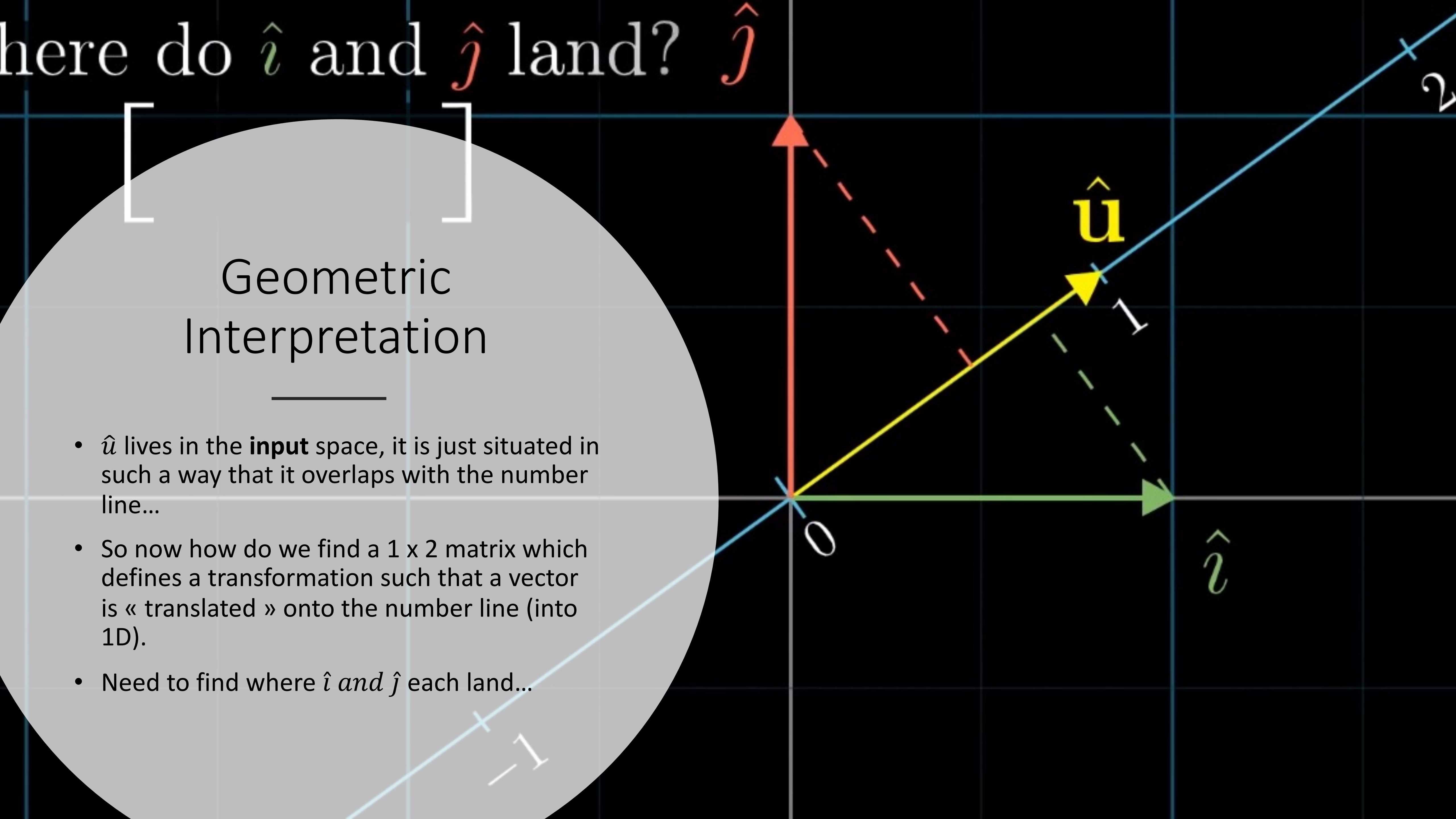




What about the projection of one vector onto another ? (optional)

- One clever way to determine where $\hat{\imath}$ and $\hat{\jmath}$ land is to take a copy of the number line and embed it diagonally in 2D space.
- Then add a 2D unit (length = 1) vector $\hat{u}$ whose tip sits on the number 1 on the number line.





Geometric Interpretation

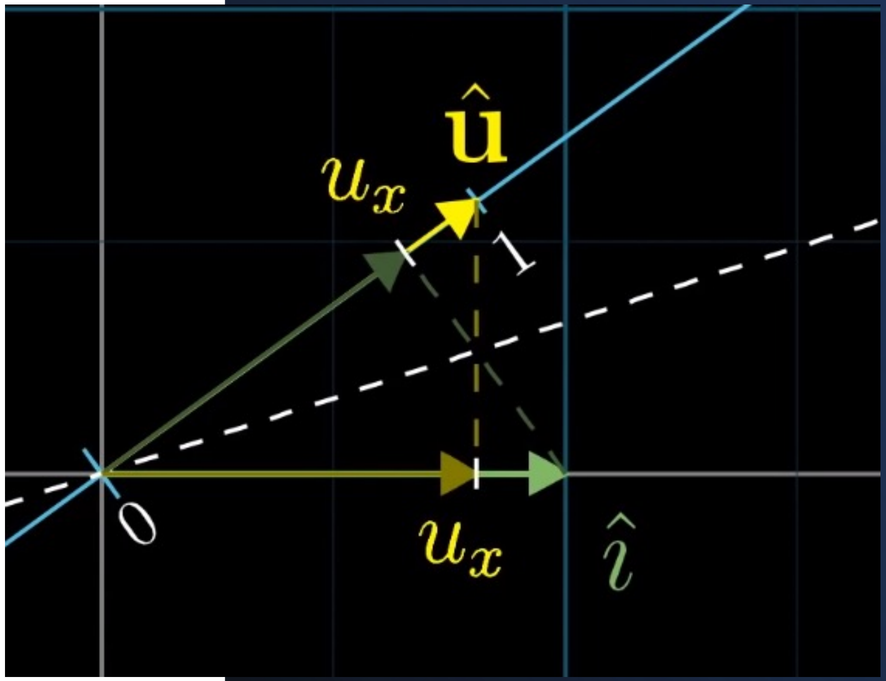
- $\hat{u}$ lives in the **input** space, it is just situated in such a way that it overlaps with the number line…
- So now how do we find a 1 x 2 matrix which defines a transformation such that a vector is « translated » onto the number line (into 1D).
- Need to find where $\hat{\imath}$ and $\hat{\jmath}$ each land…

Geometric Interpretation

- This is where projections come in.
- Can reason by symmetry because $\hat{\imath}$ and $\hat{u}$ are unit vectors (both have length 1):
    - Projecting $\hat{\imath}$ onto the line of numbers looks completely symmetric to projecting $\hat{u}$ onto the x axis…
    - So where does $\hat{\imath}$ land ? => Same place as $\hat{u}$ on the x axis !

$$T(\hat{\imath}) = u_x$$

- The same can be done for $\hat{\jmath}$



Geometric Interpretation

- So the entries of the 1x2 matrix describing the transformation are going to be the coordinates of $\hat{u}$…

- Therefore, projecting any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ onto the unit vector $\hat{u}$ is just the sum of the projections of $\hat{\imath}$ and $\hat{\jmath}$ scaled by x and y:

Input space : $x\hat{\imath} + y\hat{\jmath}$

Output space (number line) : $x\,T(\hat{\imath}) + y\,T(\hat{\jmath})$

Geometric Interpretaion

- What about when we project onto a non-unit vector ?
- If the vector we project onto has a magnitude $\neq 1$, then we just scale up that projection. This brings us exactly to our original geometric definition !

- This shows the close relationship between the dot product, projections and linear transformations.