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Riemann Sums for Generalized Integrals

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Any student has once in his life encountered this kind of problem: find the limit, when $n \to \infty$ of the sums $\sum_{k=1}^{n} \frac{n}{k^2+n^2}$. The well-known answer consists in writing this sum as $\frac{1}{n}\sum_{k=1}^{n} \frac{n^2}{n^2+k^2} = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+(\frac{k}{n})^2}$, which is a Riemann sum for the function $f(x) = \frac{1}{1+x^2}$ over the interval [0, 1]. According to Riemann's definition of the integral (cf. [3, p. 125]), this sum approaches $\int_0^1 \frac{dt}{1+t^2} = \arctan(1) - \arctan(0) = \frac{\pi}{4}$ as $n \to \infty$.

In this note, we would like to give an answer to the following questions, which frequently arise on mathematical forums: from where come these particular sums in the theory of integration? Is this technique valid not only for definite integrals but also for generalized integrals?

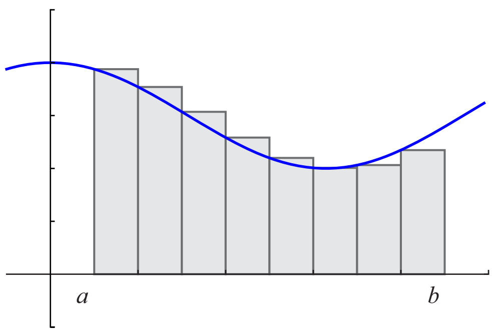

Figure 1. Geometric interpretation of a Riemann sum for a positive function.

A brief history of the Riemann sums

The so-called Riemann sums (see Figure 1) have their origin in the efforts of Greek mathematicians to find the center of gravity or the volume of a solid body. These researches led to the method of exhaustion, discovered by Archimedes and described using modern ideas by MacLaurin in his *Treatise of Fluxions* in 1742. At this time the sums were only a practical method for computing an area under a curve, and the

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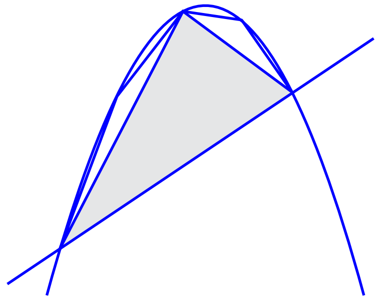

Figure 2. The method of exhaustion.

existence of this area was considered geometrically obvious. The method of exhaustion consists in almost covering the space enclosed by the curve with n geometric objects with well-known areas such as rectangles or triangles, and finding the limit (though this topic was very blurry at these early times) when n increases. One of its most remarkable application is squaring the area $\mathcal{A}$ enclosed by a parabola and a line (see Figure 2).

Archimedes wrote about this subject: "it is possible to inscribe in this segment (of parabola) a polygon such that the remaining segments are less than any given areas." This method was often used by the ancients to compare two areas. For instance Archimedes (cf. [1]) proved that the area $\mathcal{A}$ was $\frac{4}{3}S$, where S denotes the area of the inscribed triangle of maximal height constructed over the line (Figure 2). For this purpose, he constructed a sequence of inscribed triangles, each one built on the side of the preceding triangle.

A new way appears with the *method of indivisibles* proposed by Cavalieri. In 1635 Cavalieri conjectured the formula for the area under the curve $y = x^k$ for $a \leq x \leq b$ to be $(b-a)^{k+1}/(k+1)$. Instead of using two-dimensional domains, Cavalieri filled the space with lines. Blaise Pascal was a strong defender of this new method, contested by many geometers (cf. [6]).

These two methods lead to computing sums, which could be considered today as the ancestors of Riemann sums.

With the increasing progress of calculus due to Newton and Leibniz, Riemann sums were largely abandoned for computing integrals in favor of antiderivatives. They did not disappear completely; Euler in 1768 (*Institutiones calculi integralis*) wrote two chapters on them, but he seems to have considered these sums essentially as a tool for numerical computation of integrals. Later, in the eighteenth century, Legendre and Poisson studied the error committed by replacing an integral by a Riemann sum. Riemann sums bursted into the theory of integration in 1823, when Cauchy wrote his *Résumé des leçons données à l'École Royale Polytechnique sur le calcul infinitésimal* ([2]): "I consider any integral as the sum of infinitely small quantities of the differential expression placed under the integral sign, corresponding to the different values of the variable between the limits considered." In the twenty-first lesson [2, p. 81] Cauchy detailed precisely how to construct such a sum between two finite limits x_0 and X and wrote the sum as:

$$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \ldots + (X - x_{n-1})f(x_{n-1}).$$

He gave a partial proof of the convergence of the Riemann sums for a continuous function but omitted to mention the essential uniform continuity of the function on the segment $[a, b]$. Riemann himself gave the final touch in his memoir for his habilitation

(1854), *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe,* published in 1867 after his death. In seven pages, he established the theory of definite integrals based on the convergence of Riemann sums when the mesh size of the partition tends to zero.

Nowadays, Riemann sums remain a useful tool to study some sequences involving sums.

In this note, we intend to investigate, for a given function f on $(0, 1]$, whether the particular Riemann sums with equidistant points, $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$, converge toward $\int_0^1 f(t)dt$. Here, the integral is a convergent Riemann generalized integral, defined as $\lim_{x\to 0}\int_x^1 f(t)dt$. Then we shall apply this result to the study of some particular series.

The simplest case

Our teaching experience has shown that it easier for students to begin with this very simple case, which contains all the main ideas of the general case, but does not present too many technical details in calculations.

In this section, p denotes a real number such that $0 < p < 1$, so that the generalized integral $I_p = \int_0^1 \frac{dx}{x^p}dx$ is convergent. Let f be the function $f\ :\ x \mapsto \frac{1}{x^p}$. This function is continuous over $(0, 1]$, but not uniformly continuous, since if x and y are two real numbers such that $0 < x < y < \delta < 1$, the ratio $\frac{f(x)-f(y)}{x-y}$ tends to infinity as $\delta \to 0$. So, the usual proof of the convergence of the Riemann sums is no longer valid. In order to prove the convergence of the regular Riemann sums defined for $n \in \mathbb{N}^*$ by $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$, we shall use the fact that this function is decreasing on $(0, 1]$. It follows that

$$\forall k \in [[1, n-1]]\ :\ \frac{1}{n}f\left(\frac{k+1}{n}\right) \le \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt \le \frac{1}{n}f\left(\frac{k}{n}\right),$$

where the double-bracket notation means that we consider an interval of integers. Adding these inequalities and using the additivity of the integral, we get:

$$\frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k+1}{n}\right) \le \int_{\frac{1}{n}}^{1} f(t)dt \le \frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right).$$

Notice that we have used the additivity property (sometimes called Chasles relation) for a proper integral, namely $\int_{\frac{1}{n}}^{1} f(t)dt$, which is legitimate (see [5, p. 131], or [7, p. 100]).

We note that $I_{n,p} = \int_{\frac{1}{n}}^{1} f(t)dt$, and rewrite the previous estimate as:

$$\sigma_n - \frac{1}{n}f\left(\frac{1}{n}\right) \le I_{n,p} \le \sigma_n - \frac{f(1)}{n}. \tag{1}$$

Now, we observe that

$$\frac{1}{n}f\left(\frac{1}{n}\right) = \frac{1}{n^{p+1}},$$

and since $p + 1 > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} f\left(\frac{1}{n}\right) = 0. \tag{2}$$

According to (1), we obtain

$$I_{n,p} + \frac{f(1)}{n} \leq \sigma_n \leq I_{n,p} + \frac{1}{n} f\left(\frac{1}{n}\right). \tag{3}$$

When $n \to \infty$, the convergence of the integral I_p leads to $\lim_{n\to\infty} I_{n,p} = I_p$. Then, it follows from (2) and (3) that: $\lim_{n\to\infty} \sigma_n = I_p$. □

This result can be stated as follows:

Theorem 1. *For any $p < 1$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{-p} = \lim_{n\to\infty} \frac{1}{n^{1-p}} \sum_{k=1}^{n} \frac{1}{k^p} = \frac{1}{1-p}.$$

(We have proved the result for $0 < p < 1$, and for $p \leq 0$ the integral is definite, so the Riemann sums are clearly convergent.)

An application to series

Theorem 1 can be very convenient to solve special problems involving series and sums. We have tested the following one in a "Mathématiques Spéciales" (CPGE) classroom. This exercise was proposed in a "khôlle" to one of my students. In the CPGE classes, students spend several hours each week completing exams and colles (very often written "khôlles" to look like a Greek word). The so-called "colles" are unique to French academic education in scientific CPGEs. They consist of oral examinations twice a week in maths, physics, chemistry. After her examination, the student came to see me after my course, and asked for additional explanations. This led us to use Riemann sums, as follows.

Theorem 2. *Let $(p, q) \in \mathbb{R}^2$. The series $\sum_n n^p \sum_{k=1}^n k^q$ has the following behaviour:*

- *If $q > -1$, it is convergent if $p + q < -2$ and divergent if $p + q \geq -2$.*
- *If $q \leq -1$, it is convergent if $p < -1$ and divergent if $p \geq -1$.*

Proof. We shall consider three different cases :

(a) First case: Assuming $q > -1$, we rewrite the general term of the series as:

$$u_n = n^{p+q+1}\left(\frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^q\right). \tag{4}$$

From Theorem 1, we see that the positive term u_n satisfies $u_n \sim \frac{n^{p+q+1}}{q+1}$ as $n \to \infty$. Therefore the series $\sum u_n$ has the same nature as the series $\sum \frac{1}{n^{-(p+q+1)}}$ which converges if and only if $-(p + q + 1) > 1$, which is the same as : $p + q < -2$.

(b) Second case: Assuming $q < -1$, we notice that the series $\sum k^q$ is convergent. Let $C = \sum_{k=0}^{+\infty} k^q$. This time: $u_n \sim Cn^p$ and the series is convergent if and only if $p < -1$.
(c) Third case: Let $q = -1$, then the general term is : $u_n = n^p \sum_{k=1}^{n} \frac{1}{k}$ and it is well known that $\sum_{k=1}^{n} \frac{1}{k}$ is asymptotically equivalent to $\ln n$, so $u_n \sim n^p \ln n = \frac{\ln n}{n^{-p}}$, which is the general term of a Bertrand's series (cf. [7, p. 177]). It is convergent if $-p > 1$, which is the same as $p < -1$ and divergent if $p \geq -1$. ■

Generalization

Using the same method as in the proof of Theorem 1, we may establish the following result:

Theorem 3. *Let f be a decreasing function on $(0, 1]$ that is Riemann integrable on every segment $[a, 1]$ for $0 < a < 1$. Assume that the integral $\int_0^1 f(t)dt$ converges. Then*

$$\lim_{n\to\infty} \sigma_n = \int_0^1 f(t)dt,$$

where

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

Proof. Invoking the decreasing behaviour of f, we obtain the same inequalities as in (3):

$$\sigma_n - \frac{1}{n} f\left(\frac{1}{n}\right) \leq \int_{\frac{1}{n}}^{1} f(t)dt \leq \sigma_n - \frac{f(1)}{n}. \tag{5}$$

We have now to prove that the quantity $\frac{1}{n} f\left(\frac{1}{n}\right)$ tends to zero as $n \to \infty$. Notice that $\forall t \in (0, \frac{1}{n}]$: $f(t) \geq f(\frac{1}{n})$. Integrating on $(0, \frac{1}{n}]$, we obtain

$$R\left(\frac{1}{n}\right) = \int_0^{\frac{1}{n}} f(t)dt \geq \frac{1}{n} f\left(\frac{1}{n}\right) \geq 0,$$

where R is the remainder of the convergent integral $\int_0^1 f(t)dt$, defined by $R(x) = \int_0^x f(t)dt$. Recalling that $\lim_{x\to 0} R(x) = 0$, we have $\lim_{n\to\infty} R\left(\frac{1}{n}\right) = 0$. It ensues that $\lim_{n\to\infty} \frac{1}{n} f\left(\frac{1}{n}\right) = 0$, which proves the theorem. ■

Example. For instance, let f be the function $t \mapsto -\ln t$. As f is decreasing on $(0, 1]$ and $\int_0^1 \ln t dt = -1$, we may write that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n}\right) = -1$. Thus, when $n \to \infty$, we have

$$\sum_{k=1}^{n} \ln k \sim n \ln n. \tag{6}$$

Remark. Of course the theorem is still valid for the generalized integral $\int_a^b f(t)dt$, with f decreasing on $(a, b]$ and $\sigma_n = \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right)$.

First counterexample

Let g denote the continuous piecewise linear function on $[0, \infty)$, such that $\forall n \in \mathbb{N}^*$,

$$g(x) = \begin{cases} 2n^2(x - n + \frac{1}{2n^2}) & \text{if } n - \frac{1}{2n^2} \le x \le n \\ -2n^2(x - n - \frac{1}{2n^2}) & \text{if } n \le x \le n + \frac{1}{2n^2} \end{cases}$$

and $g(x) = 0$ elsewhere. Figure 3 shows the graph of this function for $0 \le x \le 5.5$.

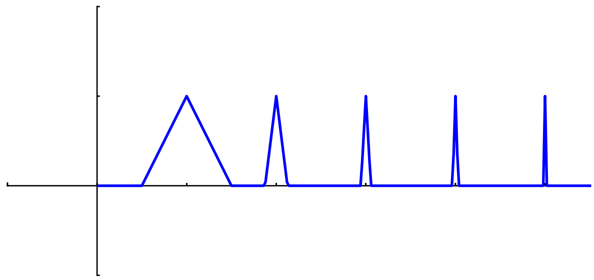

Figure 3. The function g.

As the area under the curve consists of a sequence of triangles, each one of area $\frac{1}{2n^2}$, we conclude that g is Riemann integrable on $[0, \infty)$, and the value of its integral is $\int_0^{+\infty} g(t)dt = \int_{1/2}^{+\infty} g(t)dt = \sum_1^{+\infty}\frac{1}{2n^2} = \frac{\pi^2}{12}$. The change of variable $u = \frac{1}{2t}$ transforms this integral in another generalized (at $u = 0$) integral of the first kind: $\int_0^1 g\left(\frac{1}{2u}\right)\frac{du}{2u^2}$. Setting $f(u) = g\left(\frac{1}{2u}\right)\frac{1}{2u^2}$, we define a positive continuous function f, integrable on $(0, 1]$.

Let σ_n be the Riemann sum for f on $(0, 1]$ defined by

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} g\left(\frac{n}{2k}\right)\frac{n^2}{2k^2}.$$

Simplifying this expression leads to

$$\sigma_n = \frac{n}{2}\sum_{k=1}^{n} g\left(\frac{n}{2k}\right)\frac{1}{k^2}.$$

Since g is positive, we may retain the first term ($k = 1$) to obtain a lower bound: $\sigma_n \ge \frac{n}{2} g\left(\frac{n}{2}\right)$. For n even ($n = 2p$), since $g(p) = 1$, it follows that

$$\sigma_{2p} \ge p. \tag{7}$$

It follows that $\lim_{n\to\infty}\sigma_{2p} = +\infty$, and the Riemann sums for f on $(0, 1]$ do not converge towards $\int_0^1 f(t)dt$. The shape of the representative curve of f appears on Figure 4. Notice that, for reasons of computability, the curve is sketched for $0.02436884686616627 \leq x \leq 0.96153846153846145$ only. The Python code is available on demand.

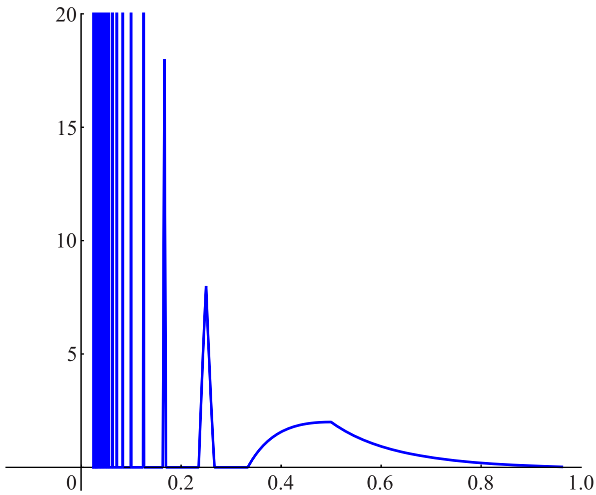

Figure 4. The function f.

Second counterexample (Hardy's integral)

In this section, following a private communication of Raymond Cordier, we consider the generalized integrals:

$$H_0 = \int_0^{+\infty} \frac{x^\alpha dx}{1 + x^\beta \sin^2 x}, \qquad H_1 = \int_\pi^{+\infty} \frac{x^\alpha dx}{1 + x^\beta \sin^2 x}, \tag{8}$$

where $\alpha > 0$, $\beta > 0$. The integral H_0 was proposed as an exercise by G.H. Hardy, who showed a great interest in Integral Calculus (cf. [4]). For $\alpha > 0$ and $\beta > 0$, the integral H_0 is convergent if and only if the real numbers α and β satisfy $\beta > 2(\alpha + 1)$. Of course, H_1 is defined if these conditions are satisfied.

Though the detailed study of Hardy's integral is not our topic in this note, we shall give some hints about its nature. Since there is no problem at the zero lower bound, this integral has the same nature as the series $\sum u_n$, where

$$u_n = \int_{n\pi}^{(n+1)\pi} \frac{x^\alpha dx}{1 + x^\beta \sin^2 x},$$

and the change of variable $t = x - n\pi$ leads to

$$u_n = \int_0^\pi \frac{(t + n\pi)^\alpha dt}{1 + (t + n\pi)^\beta \sin^2 t}.$$

Since $\int_0^\pi \frac{dt}{1+x \sin^2 t} = \frac{\pi}{\sqrt{1+x}}$, it is possible to bound u_n between two other series and to show that $u_n \sim \pi (n\pi)^{\alpha - \frac{\beta}{2}}$. The result follows.

The change of variables $t = \frac{\pi}{x}$ transforms H_1 into

$$H_1 = \pi^{\alpha+1} \int_0^1 f(t)dt, \tag{9}$$

where f is defined on (0, 1] by

$$f(t) = \frac{t^{\beta-\alpha-2}}{t^\beta + \pi^\beta \sin^2(\frac{\pi}{t})}. \tag{10}$$

Let us now consider the Riemann sum σ_n. As the function is positive, we have

$$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \geq \frac{1}{n} f\left(\frac{1}{n}\right).$$

This first term is easy to compute as the sine term vanishes for $k = 1$:

$$\frac{1}{n} f\left(\frac{1}{n}\right) = \frac{n^\beta}{n^{\beta-\alpha-1}} = n^{\alpha+1}.$$

It follows that $\lim_{n\to\infty} \sigma_n = +\infty$. Thus, the Riemann sums do not converge to the generalized integral. Of course, the reader will observe that the function f is not decreasing on (0, 1].

Riemann sums today

Fast Fourier transform. Thousands of engineers and scientists make computations every day, using DFT (discrete Fourier transform) and FFT (fast Fourier transform), without thinking about the mathematical engine of these powerful tools. Looking under the hood, they may notice that Riemann sums play a central role! Let f be a signal assumed to vanish for $t < 0$. Furthermore, suppose that we only know some values of f, regularly spaced. We may suppose (eventually with a change of unit) that these values are the $y_k = f(k)$, $0 \leq k \leq N$, for some integer N large enough. Instead of computing the complete Fourier transform $\hat{f}(x) = \int_{-\infty}^{+\infty} f(t)e^{-2i\pi xt}dt = \int_0^{+\infty} f(t)e^{-2i\pi xt}dt$, we may think of computing the approached value $\hat{f}(x) \simeq \int_0^N f(t)e^{-2i\pi xt}dt$. Since we don't know many values of the signal, it will be convenient to replace this integral by a Riemann sum s_N (for the same N) on $[0, N]$, using the integers as subdivision points:

$$s_N(x) = \sum_{k=0}^{N-1} y_k e^{-2i\pi kx}.$$

Since $x \mapsto s_N(x)$ is a periodic function of period $T = 1$, it is sufficient to know it on [0, 1], and as we practice discrete computations, it will be enough to obtain a sample of values, namely the $s_N(x_p)$, for $x_p = \frac{p}{N}$, $p = 0...N-1$. Thus, the Riemann sums have led us to the DFT formula:

$$\hat{y}_p = \sum_{k=0}^{N-1} y_k \omega_N^{pk}, \tag{11}$$

where $\omega = e^{-\frac{2i\pi}{N}}$. For more details, the reader may refer to [7].

Computational complexity. Estimating the magnitude of some finite sums like (6) may be useful in computer science for evaluating the computational complexity of an algorithm. For instance, with $\alpha \geq 0,\ \beta > 0$, the following estimates hold:

1. $\sum_{k=1}^{n} k^\alpha = \Theta(n^{\alpha+1})$.
2. $\sum_{k=1}^{n} k^\alpha (\ln k)^\beta = \Theta(n^{\alpha+1}(\ln n)^\beta)$.

(We recall that $u_n = \Theta(v_n)$ means $u_n = O(v_n)$ and $v_n = O(u_n)$, with Landau notation).

For the first estimate for instance, we write

$$S_n = \sum_{k=1}^{n} k^\alpha = n^{\alpha+1}\Big[\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{k}{n}\Big)^\alpha\Big].$$

The term between the brackets is a Riemann sum converging towards $\int_0^1 t^\alpha dt$ as its limit. We have $S_n \sim \frac{n^{\alpha+1}}{\alpha+1}$, which is stronger than $\Theta(n^{\alpha+1})$.

Suppose, to simplify, that $\beta = 1$. The second estimate becomes

$$\sum_{k=0}^{n} k^\alpha \ln k = \Theta(n^{\alpha+1}\ln n).$$

To prove this, we may show that the difference between this sum and $n^{\alpha+1}\ln n$ is $o\Big(n^{\alpha+1}\ln n\Big)$. For this purpose, we shall make use of a Riemann sum for the integral $\int_0^1 t^\alpha \ln t\ dt$. Let $\alpha_n = \sum_{k=1}^{n} k^\alpha \ln k$ and $\beta_n = \sum_{k=1}^{n} k^\alpha \ln n$. Then

$$\alpha_n - \beta_n = n^{\alpha+1}\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{k}{n}\Big)^\alpha \ln\Big(\frac{k}{n}\Big).$$

The result follows, since the Riemann sum converges toward $\int_0^1 t^\alpha \ln t\ dt$, the integral of a continuous function on [0, 1], because $\alpha > 0$.

For instance, the complexity of an algorithm constructed with a loop "for k from 1 to n" and computing at each step the value of a polynomial function $P_k(x)$ of degree k with a naive method (complexity $\Theta(k^2)$) would be $\Theta(n^3)$. Using Horner's method would lead to $\Theta(n^2)$.

The complexity of an algorithm constructed with a loop "for k from 1 to n" and searching at each step an element in a list of length k, using dichotomy (complexity $\Theta(\ln k)$) would be $\Theta(n \ln n)$.

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Summary. It is well known that Riemann sums converge toward the integral of any Riemann-integrable function f on the segment $[a, b]$. In some cases this property is still valid for generalized integrals and can be useful to solve problems involving series and sums. In this note we prove (Theorem 3) that the result still holds for a generalized integral under the assumption that the function is decreasing on the interval. We give two counterexamples: the first one with a linear piecewise function, and a second one based on an integral suggested by Hardy. The last section shows recent applications of the Riemann sums in computer science and applied mathematics.

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