CHAPTER 7 ftl-ch7-JUL2

Relations and Partial Orders

A relation is a mathematical tool for describing associations between elements of sets. Relations are widely used in computer science, especially in databases and scheduling applications. ~~Relations can~~ A relation can be defined ~~7.1 Binary Relations~~ across many ~~sets,~~ items in many sets, but in this text, we will focus ~~on the case with two items from one or two sets.~~ on binary relations which represent an association between two items in one or two sets.

7.1 Binary Relations

~~Definition F1:~~

~~Codomain~~

7.1.1 Definitions and Examples

Definition F1: Given sets A and B, a (binary) relation from A to B is a subset of $A \times B$. The ~~Rel~~ $R : A \to B$ ~~is a~~ ~~set A is called the domain of R and B is called the codomain~~ sets A and B are called the domain and codomain of R, respectively. We commonly use the notation aRb or $a \sim_R b$ to denote that $(a,b) \in R$. We also say that the relation is between A and B, or on A if $B = A$.

A relation is similar to a function. In fact every function $f: A \to B$ is a relation. In general, the ~~...~~ difference between ~~them~~ a function and a relation is that a relation ~~can~~ might associate ~~multiple~~ multiple elements of B with a single element of A, whereas a function can only associate at most one element of B ~~with~~ (namely $f(a)$) with each element $a \in A$.

We have already encountered examples of relations in earlier chapters. For example, in Section ~~5.3~~ 5.2, we talked about ~~two different~~ a relation between the set of men and the set of women, ~~In one case mRw if the man~~ where mRw if man m likes woman w.

In Section 5.3, we talked about a relation on the set of ~~course~~ MIT courses where $c_1 R c_2$ if the exams for c_1 and c_2 cannot be given at the same time. ~~We did not use the form~~ In Section ~~...~~ 6.3, we talked about a relation on the set of switches in a network where $s_1 R s_2$ if s_1 and s_2 are directly connected by a wire. We did not use the ~~relation~~ formal definition of a relation in any of these cases, but they are all examples of relations.

As another example, we can define ~~a relation~~

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~~It is common to use the infix notation~~

~~aRb to denote that (a,b) ∈ R~~

~~or $a \sim_R b$~~

an "in charge of" relation T for spring '10 from the set of MIT faculty F to the set of subject numbers in the 2010 ~~cat~~ catalogue. This relation contains

For example, "equality" and "less-than" are very familiar examples of binary relations on sets such as $\mathbb{N}$, $\mathbb{Q}$ or $\mathbb{R}$. As a less familiar example,

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a relation

~~B", or "from A to B." When the domain and codomain are the same set, A, we simply say the relation is~~ "a on A." ~~It's common to use infix notation "$a\ R\ b$" to mean that the pair (a, b) is in the graph of R.~~

— Insert N1 goes here —

~~For example, we can define an "in-charge of" relation, T, for MIT in Spring '10 to have domain equal to the set, F, of names of the faculty and codomain equal to all the set, N, of subject numbers in the current catalogue. The graph of T contains precisely the~~ pairs of the form

$$(\langle\text{instructor-name}\rangle, \langle\text{subject-num}\rangle)$$

where ~~such that~~ the faculty member named ⟨instructor-name⟩ is in charge of the subject with number ⟨subject-num⟩ in Spring '10. So ~~graph(T)~~ T contains pairs like those shown in Figure FA.

```
(A. R. Meyer,  6.042),
(A. R. Meyer,  18.062),
(A. R. Meyer,  6.844),
(T. Leighton,  6.042),
(T. Leighton,  18.062),
(G. Freeman,   6.011),
(G. Freeman,   6.881)
(G. Freeman,   6.882)
(G. Freeman,   6.UAT)
(T. Eng,       6.UAT)
(J. Guttag,    6.00)
      ⋮
```

Figure FA: some "in charge of" items in the relation T between faculty and subject numbers.

This is a surprisingly complicated relation: Meyer is in charge of subjects with three numbers. Leighton is also in charge of subjects with two of these three numbers —because the same subject, Mathematics for Computer Science, has two numbers: 6.042 and 18.062, and Meyer and Leighton are co-in-charge of the subject. Freeman is in-charge of even more subjects numbers (around 20), since as Department Education Officer, he is in charge of whole blocks of special subject numbers. Some subjects, like 6.844 and 6.00 have only one person in-charge. Some faculty, like Guttag, are in charge of only one subject number, and no one else is co-in-charge of his subject, 6.00.

Some subjects in the codomain, N, do not appear in the list —that is, they are not an element of any of the pairs in the graph of T; these are the Fall term only subjects. Similarly, there are faculty in the domain, F, who do not appear in the list because all their in-charge subjects are Fall term only.

—— INSERT FA goes here ——

7.1.3 ~~5.5.2~~ Relational Images

The idea of the image of a set under a function extends directly to relations.

INSERT FA

7.1.2 Representation as a Bipartite Graph

Every ~~binary~~ relation $R : A \to B$ can be easily represented as a bipartite graph $G = (V, E)$ by ~~setting $V = A \cup B$ and $E = R$. In the speci~~ creating a "left" node for each element of A ~~on the left~~ and a "right" node for each element of B. ~~on the~~ We then create an edge ~~for a left~~ between a left node u and a right node v ~~if~~ whenever $a R b$. Similarly, every bipartite graph (and every partition of the nodes into a "left" and "right" set for which no edge connects a pair of left or right nodes) determines a relation between the nodes on the left and the nodes on the right.

For example, we have shown part of the bipartite graph for the "in charge of" relation from Figure FA in Figure FB. In this case, there is an edge between <instructor-name> and <subject-number> if <instructor-name> is in charge of <subject-number> in Spring '10.

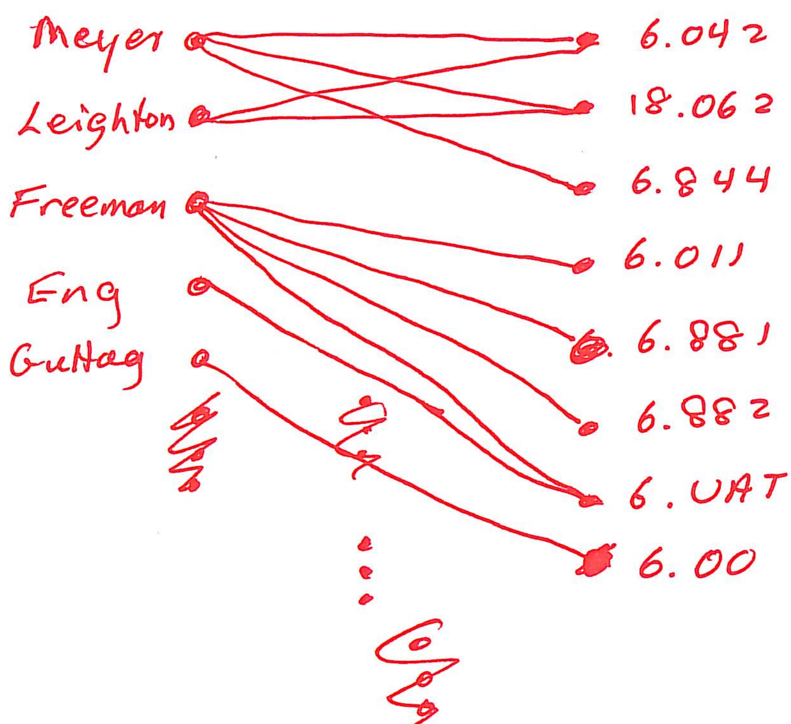

Figure FB : ~~The~~ Part of the bipartite graph for the "in-charge of" relation T from Figure FA.

A relation $R:A\to B$ can also be represented as ~~an~~ a ~~adjacency~~ 0,1 matrix $A = \{a_{i,j}\}$ where

$$a_{ij} = \begin{cases} 1 & \text{if the } i\text{th element of } A \text{ is related to the } j\text{th element of } B \\ 0 & \text{otherwise} \end{cases}$$

for $1 \le i \le |A|$ and $1 \le j \le |B|$. For example, the matrix for the ~~of the~~ relation in Figure FB ~~is~~ (but restricted to the~~se~~ ~~five~~ 5 faculty and 8 subject numbers shown in Figure FB, and ~~listing~~ ordering them ~~in the~~ as they appear ~~order~~ top-to-bottom ~~order they appear~~ in Figure FB) is shown in Figure FC.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

<u>Figure FC</u>: The matrix for the "in charge of" ~~rest~~ relation T restricted to the ~~ca~~ 5 faculty and 8 subject numbers shown in Figure FB. The (3,4) entry of this matrix is 1 since the 3rd professor (Freeman) is in charge of the 4th subject ~~course~~ ~~c~~ number (6.011).

~~In the case of a relation R: A → B~~

~~where B~~

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$R: A \to B$,

Definition 5.5.2. The *image* of a set, Y, under a relation, ~~R~~, written $R(Y)$, is the set of elements of ~~the codomain~~ codomain, B, ~~of R~~ that are related to some element in Y, namely,

$$R(Y) ::= \{b \in B \mid yRb \text{ for some } y \in Y\}.$$

The image of the domain, $R(A)$, is called the range of R.

For example, to find the subject numbers that Meyer is in charge of in Spring '10 ~~'09~~, we can look for all the pairs of the form

$$(\cancel{\text{A.}}\ \text{Meyer}, \langle\text{subject-number}\rangle)$$

in the graph of the teaching relation, T, and then just list the right hand sides of these pairs. These righthand sides are exactly the image T(~~A.~~ Meyer), which happens to be $\{6.042, 18.062, 6.844\}$. ~~Similarly, to find the subject numbers that either Freeman or Eng are in charge of, we can collect all the pairs in T of the form~~

~~(G. Freeman, ⟨subject-number⟩)~~

~~or~~

~~(T. Eng, ⟨subject-number⟩);~~

~~and list their right hand sides. These right hand sides are exactly the image T({G. Freeman, T. Eng}).~~

~~So the partial list of pairs in T given above implies that~~

$$\cancel{\{\texttt{6.011}, \texttt{6.881}, \texttt{6.882}, \texttt{6.UAT}\} \subseteq T(\{\text{G. Freeman}, \text{T. Eng}\})}.$$

Similarly, ~~Finally~~, since the domain, F, is the set of all in-charge faculty, the range of T, $T(F)$, is exactly the set of *all* Spring '~~09~~ '10 subjects being taught.

7.1.4 ~~5.5.3~~ Inverse Relations and Images

Definition 5.5.3. The *inverse*, R^{-1} of a relation $R : A \to B$ is the relation from B to A defined by the rule

$$bR^{-1}a \ \cancel{\text{iff}}\ \text{if and only if}\ a\ R\ \cancel{B}\,b.$$

The image of a set under the relation, R^{-1}, is called the *inverse image* of the set. That is, the inverse image of a set, X, under the relation, R, is $R^{-1}(X)$.

Continuing with the in-charge example above, we can find the faculty in charge of 6.UAT in Spring '10 ~~can be found~~ by taking the pairs of the form

$$(\langle\text{instructor-name}\rangle\,, 6.UAT)$$

~~in the graph of~~ for the teaching relation, T, and then just listing the left hand sides of

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these pairs; these turn out to be just Eng and Freeman. These left hand sides are exactly the inverse image of $\{6.\text{UAT}\}$ under T.

Now let D be the set of introductory ~~course 6~~ subject numbers. These are the subject numbers that ~~start with 6.0~~ have a 0 immediately following the decimal point. Now we can likewise find out all the instructors who were in-charge of introductory ~~course 6~~ subjects in Spring ~~'09~~ '10, by taking all the pairs of the form ($\langle$instructor-name$\rangle$, ~~6.0~~ $\{1.0\ldots, 2.0\ldots, 3.0\ldots, \ldots\}$ . . .) and list the left hand sides of these pairs. These left hand sides are exactly the inverse image ~~of~~ of D under T. From the part ~~of the graph~~ of T shown above, we can see that

$$\{\text{Meyer, Leighton, Freeman, Guttag}\} \subseteq T^{-1}(D).$$

That is, Meyer, Leighton, Freeman, and Guttag were among the instructors in charge of introductory subjects in Spring '10. Finally, the inverse image under T of the set, N, of all subject numbers is the set of all instructors who were in charge of a Spring ~~'09~~ '10 subject.

It gets interesting when we write composite expressions mixing images, inverse images and set operations. For example, $T(T^{-1}(D))$ is the set of Spring ~~'09~~ '10 subjects

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that have an instructor in charge who also is ~~in~~ charge of an introductory ~~course~~ subject.

So $T(T^{-1}(D)) - D$ are the advanced subjects with someone in-charge who is also in-charge of an introductory subject. Similarly, $T^{-1}(D) \cap T^{-1}(N - D)$ is the set of faculty in charge of both an introductory *and* an advanced subject in Spring '~~09~~ 10.

~~Insert N2 goes here~~

7.1.5 ~~5.5.4~~ Surjective and Injective Relations

There are ~~a few~~ some properties of relations that will be useful when we take up the topic of counting in Part III because they imply certain relations between the *sizes* of domains and codomains. ~~We say a~~ In particular, we say that a binary relation $R : A \to B$ is:

- *surjective* when every element of B is mapped to *at least once*; more concisely, R is surjective iff $R(A) = B$ (i.e., if the range of R is the codomain of R).

- *total* when every element of A is assigned to some element of B; more concisely, R is total iff $A = R^{-1}(B)$.

- *injective* if every element of B is mapped to *at most once*, and

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and a function[5].

- *bijective* if R is total, surjective, ~~and~~ injective ~~function.~~[5]

~~Note that this definition of R being total agrees with the definition in Section 5.4 when R is a function.~~

~~If R is a binary relation from A to B, we define $R(A)$ to to be the *range* of R. So~~ a ~~relation is surjective iff its range equals its codomain. Again,~~ Note: ~~In the case that R is a function, these definitions of "range" and "total" agree with the definitions in Section 5.4.~~

5.5.4

~~5.5.5 Relation Diagrams~~

illustrate the

We can ~~explain all~~ these properties of a relation $R : A \to B$ in terms of ~~a diagram~~

corresponding bipartite graph G for the relation,

~~where all the elements of the domain, A, appear in one column (a very long one if~~

where nodes on the left side of G correspond to

~~A is infinite) and all the elements of the codomain, B, appear in another column,~~

elements of A and nodes on the right side of G correspond to elements of B. For example:

~~and we draw an arrow from a point a in the first column to a point b in the sec-~~

[5]These words "surjective," "injective," and "bijective" are not very memorable. Some authors use the possibly more memorable phrases *onto* for surjective, *one-to-one* for injective, and *exact correspondence* for bijective.

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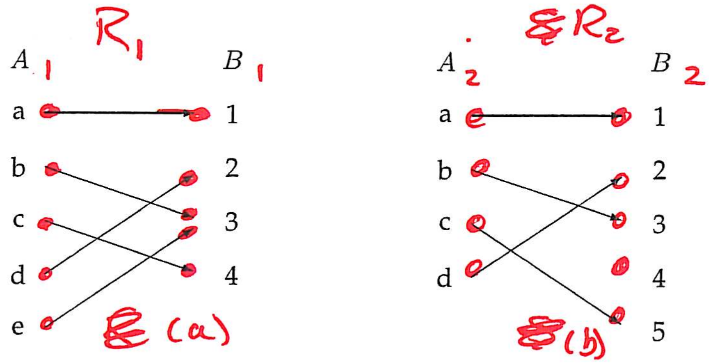
David: remove arrow heads & move fig one later to next page

~~ond column when a is related to b by R. For example, here are diagrams for two functions:~~

Figure FD: Relation $R_1: A_1 \to B_1$ is shown in (a) and relation $R_2: A_2 \to B_2$ is shown in (b).



~~OK~~ ~~was~~

~~Here is what the definitions say about such pictures:~~

- "R is a function" means that every ~~point in the domain column, A, has at most one arrow out of it.~~ node on the left has ~~at most~~ ~~precisely~~ ~~edge~~ is incident to at most one edge.

- "R is total" means that *every* ~~point in the A column has *at least one arrow out of it*.~~ on the left is incident to at least one edge. So if R is a function, being total really means ~~every point in the A column has *exactly one arrow out of it*.~~ that every node on the left is is incident to exactly one edge.

- "R is surjective" means that *every* ~~point in the codomain column, B, has *at least one arrow into it*.~~ node on the right is incident to at least one edge.

- "R is injective" means that every ~~point in the codomain column, B, has *at most one arrow into it*.~~ node on the right is incident to at most one edge.

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- "R is bijective" means that ~~*every* point in the A column has exactly one arrow out of it, and *every* point in the B column has exactly one arrow into it.~~ every node on both sides is incident to precisely one edge (i.e., that there is a perfect matching between A and B).

For example, consider the relations R_1 and R_2 shown in Figure FD. R_1 ~~So in the diagrams above, the relation on the left~~ is a total, surjective function (every ~~element~~ node in the ~~A~~ left column ~~has exactly one arrow out~~ is incident to exactly one edge, and every ~~element~~ node in the ~~B~~ right column ~~has at least one arrow in~~ is incident to at least one edge), but not injective (~~element~~ node 3 ~~has two arrows going into it~~ is incident to 2 edges). ~~The relation on the right~~ R_2 is a total, injective function (every ~~element~~ node in the ~~A~~ left column ~~has exactly one arrow out~~ is incident to exactly one edge, and every ~~element~~ node in the ~~B~~ right column ~~has at most one arrow in~~ is incident to at most one edge), but not surjective (~~element~~ node 4 ~~has no arrow going into it~~ is not incident to any edges).

~~Notice that the arrows in a diagram for R precisely correspond to the pairs in the graph of R. But graph(R) does not determine by itself whether R is total or surjective; we also need to~~ # Notice that we need to know what the domain is to determine if ~~R~~ a relation is total, and we need to know the codomain to tell if it's surjective. For example, the

~~*Example* 5.5.4. The~~ function defined by the formula $1/x^2$ is total if its domain is $\mathbb{R}^+$ but partial if its domain is some set of real numbers including 0. It is bijective if its domain and codomain are both $\mathbb{R}^+$, but neither injective nor surjective if its domain and co-domain are both R.

7.1.6 Combining Relations

There ~~a~~ ~~al~~ are at least two ~~a couple~~ ~~of~~ natural ways to combine ~~pair~~ relations to form new relations. For example, ~~we can~~ given relations $R : B \to C$ and $S : A \to B$, the <u>composition</u> of R with S is the relation $(R \circ S) : A \to C$ defined by the rule

$$a\,(R \circ S)\,c ::= \exists b \in B.\ (b\,R\,c) \text{ AND } (a\,S\,b)$$

where $a \in A$ and $c \in C$.

As a special case, ~~we de~~ ~~given two~~ ~~we define~~ the composition of two functions $f : B \to C$ and $g : A \to B$ ~~to be~~ is the function $(f \circ g) : A \to C$ defined by

$$(f \circ g)(a) = f(g(a))$$

for all $a \in A$. For example, if $A = B = C = \mathbb{R}$, $g(x) = x + 1$ and $f(x) = x^2$, then ~~$f \circ g : \mathbb{R} \to$~~

$$(f \circ g)(x) = (x+1)^2 = x^2 + 2x + 1.$$

One can also define the product of two relations $R_1 : A_1 \to B_1$ and $R_2 : A_2 \to B_2$ to be the relation $S = R_1 \times R_2$ where

$$S : A_1 \times A_2 \to B_1 \times B_2$$

and

$$(a_1, a_2)\, S\, (b_1, b_2) \text{ iff } a_1 R_1 b_1 \text{ and } a_2 R_2 b_2.$$

~~For example,~~

7.2 Relations on One Set

For the rest of this chapter, we are going to ~~for~~

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Relations

1 Relations

A "relation" is a mathematical tool used to describe relationships between set elements. Relations are widely used in computer science, especially in databases and and scheduling applications.

Formally, a ***relation*** from a set A to a set B is a subset $R \subseteq A \times B$. For example, suppose that A is a set of students, and B is a set of classes. Then we might consider the relation R consisting of all pairs (a, b) such that student a is taking class b:

$$R = \{(a, b) \mid \text{student } a \text{ is taking class } b\}$$

Thus, student a is taking class b if and only if $(a, b) \in R$. There are a couple common, alternative ways of writing $(a, b) \in R$ when we're working with relations: aRb and $a \sim_R b$. The motivation for these alternative notations will become clear shortly.

~~1.1 Relations on One Set~~

~~We're mainly going to~~ focus on relationships between elements of a single set; that is, relations from a set A to a set B where $A = B$. Thus, a ***relation*** on a set A is a subset $R \subseteq A \times A$. Here are some examples:

- Let A be a set of people and the relation R describe who likes whom; that is, $(x, y) \in R$ if and only if x likes y.
- Let A be a set of cities. Then we can define a relation R such that xRy if and only if there is a nonstop flight from city x to city y.
- Let $A = \mathbb{Z}$, and let xRy hold if and only if $x \equiv y \pmod 5$.
- Let $A = \mathbb{N}$, and let xRy hold if and only if $x \mid y$.
- Let $A = \mathbb{N}$, and let xRy hold if and only if $x \leq y$.

The last examples clarify the reason for using xRy or $x \sim_R y$ to indicate that the relation R holds between x and y: many common relations ($<$, $\leq$, $=$, $\mid$, $\equiv$) are expressed with the relational symbol in the middle.

7.2.1 Representation as a Digraph

Every ~~binary~~ relation on a single set A can be modeled as a directed graph (albeit one that ~~with the~~ may ~~the possibility~~ contain loops). For example, the graph in Figure FE describes the "likes" relation for a particular set of 3 people.

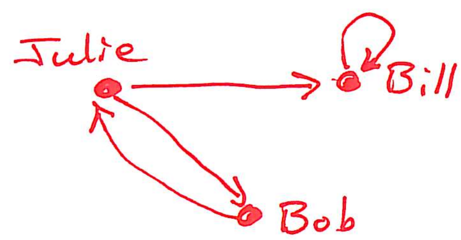

Figure FE: The directed graph for the "likes" relation on the set {Bill, Bob, Julie}.

In this case, we see that:

- Julie likes Bill and Bob, but not herself,
- Bill likes only himself,
- Bob likes Julie, but not Bill nor himself.

Everything about the relationship is conveyed by the graph and nothing more. This is no coincidence; a set A together with a relation R is precisely the same thing as a directed graph $G = (V, E)$ with vertex set $V = A$ and edge set $E = R$ (where E may have loops).

~~As another~~

As another example, we have illustrated the directed graph for the divisibility relationships on the set $\{1, 2, \ldots, 12\}$ in Figure FF. In this graph, every node has a loop (since every positive number divides itself) and the ~~primes are the number~~ ~~composit~~ composite ~~for~~ numbers are the nodes with indegree ~~g~~ more than 1 (not counting the loop).

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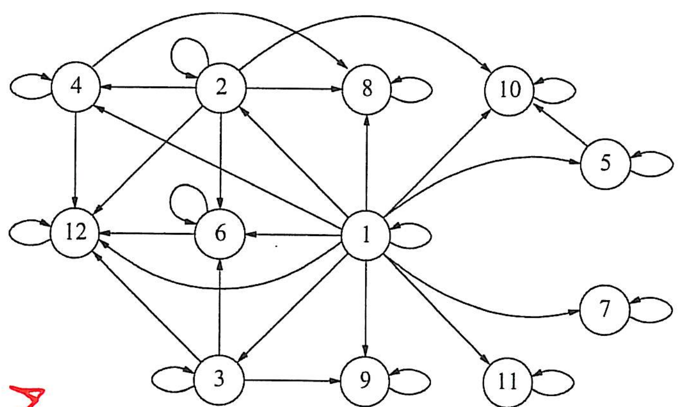
are *directed edges*. Writing $a \to b$ is a more suggestive alternative for the pair (a, b). Directed edges are also called *arrows*.

For example, the divisibility relation on $\{1, 2, \ldots, 12\}$ is could be pictured by the digraph:

Redraw in standard format where 2 looks like 2



FF

Figure 7.1: The Digraph for Divisibility on $\{1, 2, \ldots, 12\}$.

7.1.1 Paths in Digraphs

Picturing digraphs with points and arrows makes it natural to talk about following a *path* of successive edges through the graph. For example, in the digraph of Figure 7.1, a path might start at vertex 1, successively follow the edges from vertex 1 to vertex 2, from 2 to 4, from 4 to 12, and then from 12 to 12 twice (or as many times

Relations on a single set can also be represented as a 0,1 matrix. In this case, the matrix is identical to the adjacency matrix for the corresponding digraph. For example, the matrix for the relation shown in Figure FE is simply

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

~~where Julie is that~~

where v_1 = Julie, v_2 = Bill and v_3 = Bob. ~~Since~~

7.2.2 Symmetry, Transitivity, and other Special Properties

~~Relation~~

Many relations on a single set that arise in practice possess ~~some or certain noticeable~~ one or more noteworthy properties.

~~In particular, we say that~~

These properties are summarized in the box on the following page. In each case, we provide the formal definition of the property, explain what the property looks like in a digraph & for the relation, and give an example of what the property means for the "likes" relation.

~~David:~~ David: use box format

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Properties of Relations

Reflexivity: $R: A \to A$ is *reflexive* if

$\forall x \in A.\ xRx$

"Everyone likes themselves."

Every node in ~~the digraph~~ G has a loop.

Irreflexivity: $R: A \to A$ is *irreflexive* if

$\neg \exists x \in A.\ xRx$

"No one likes themselves."

There are no loops in ~~the digraph~~ G.

Symmetry: $R: A \to A$ is *symmetric* if

$\forall x, y \in A.\ xRy$ IMPLIES yRx

"If x likes y, then y likes x."

If there is an edge from x to y in G, then there is an edge from y to x in G as well.

Antisymmetry: $R: A \to A$ is *antisymmetric* if

$\forall x, y \in A.\ (xRy$ AND $yRx)$ IMPLIES $x = y$

"No pair of distinct people ~~both~~ like each other."

There is at most one directed edge between any pair of distinct nodes.

cont on next page

<u>Asymmetry</u>: $R: A \to A$ is <u>asymmetric</u> if

$\neg \exists x, y \in A.\ xRy$ AND yRx

"No one likes themselves and no pair of people like each other."

There are no loops and ~~at~~ there is at most one directed edge between any pair of nodes.

<u>Transitivity</u>: $R: A \to A$ is <u>transitive</u> if

$\forall x, y, z \in A.\ (xRy$ AND $yRz)$ IMPLIES xRz

"If x likes y and y likes z, then x likes z too."

For any walk $v_0, v_1, \ldots, v_k$ in G where $k \geq 2$, ~~there is a directed edge from~~ $v_0 \to v_k$ is in G (and, hence, $v_i \to v_j$ is also in G for all $i < j$).

For example, the congruence relation ~~on~~ modulo 5 on $\mathbb{Z}$ is reflexive, symmetric and transitive, but not irreflexive, antisymmetric or asymmetric. ~~The same is true of the relation~~ The same is true for the "connected" relation $R: V \to V$ on an undirected graph $G = (V, E)$: where uRv if u and v are in the same connected component of ~~an undirected~~ graph G ~~$= (V, E)$~~. In fact, relations that ~~so~~ have these three properties are so common that we give them a special name: <u>equivalence relations</u>. We will discuss them in greater detail in just a moment.

As another example, the "divides" relation on $\mathbb{Z}^+$ is reflexive, antisymmetric and transitive, but not irreflexive, ~~asy~~ symmetric or asymmetric. The same is true for the "$\leq$" relation on $\mathbb{R}$. Relations that have these three properties are also very common and they ~~also have a special name~~ fall into a special class of relations ~~known~~: called a <u>partial order</u>. We will ~~dis~~ ~~focus on them~~ discuss ~~them~~ partial orders at length in sections 7.4 – 7.?? (last section).

~~As The~~

~~Note that the "<" relation on $\mathbb{R}$ is different from the "$\leq$" relation since "<" is not reflexive. Hence, "<" on $\mathbb{R}$ is irreflexive, asymmetric and transitive (as well as irreflexive and anti-symmetric, of course).~~

As a final example, consider the ~~"the~~ ~~"adjency" relationship R on the simple 4-node cycle graph C_4. This rel relation is~~ "likes" relation on the set {Julie, Bill, Bob} ~~relation~~ illustrated in Figure FE. This relation has <u>none</u> of the six properties ~~that~~ described in the box.

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4. ***transitive*** if for all $x, y, z \in A$, xRy and yRz imply xRz.
   (If x likes y and y likes z, then x also likes z.)

Let's see which of these properties hold for some of the relations we've considered so far:

	reflexive?	symmetric?	antisymmetric?	transitive?
$x \equiv y \pmod 5$	yes	yes	no	yes
$x \mid y$	yes	no	yes	yes
$x \leq y$	yes	no	yes	yes

The two different yes/not patterns in this table are both extremely common. A relation with the first pattern of properties (like $\equiv$) is called an "equivalence relation", and a relation with the second pattern (like $\mid$ and $\leq$) is called a "partially-ordered set". The rest of this ~~lecture focuses on just these two types of relation.~~

7.3 ~~3~~ Equivalence Relations

not bold

A relation is an ***equivalence relation*** if it is reflexive, symmetric, and transitive. Congruence modulo n is a excellent example of an equivalence relation:

- It is reflexive because $x \equiv x \pmod n$.
- It is symmetric because $x \equiv y \pmod n$ implies $y \equiv x \pmod n$.
- It is transitive because $x \equiv y \pmod n$ and $y \equiv z \pmod n$ imply that $x \equiv z \pmod n$.

There is an even more well-known example of an equivalence relation: equality itself. Thus, an equivalence relation is a relation that shares some key properties with "=."

7.3.1 ~~3.1~~ Partitions

There is another way to think about equivalence relations, but we'll need a couple definitions to understand this alternative perspective.

Definition Given an equivalence relation $R : A \to A$, ~~Suppose that R is an equivalence relation on a set A. Then~~ the ***equivalence class*** of an element $x \in A$ is the set of all elements in A related to x by R. The equivalence class of x is denoted $[x]$. Thus, in symbols:

$$[x] = \{y \mid xRy\}.$$

For example, suppose that $A = \mathbb{Z}$ and xRy means that $x \equiv y \pmod 5$. Then:

$$[7] = \{\ldots, -3, 2, 7, 12, 17, 22, \ldots\}$$

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Notice that 7, 12, 17, etc. all have the same equivalence class; that is, $[7] = [12] = [17] = \ldots$.

Definition : A ***partition*** of a set A is a collection of disjoint, nonempty subsets $A_1, A_2, \ldots, A_n$ whose union is all of A. For example, one possible partition of $A = \{a, b, c, d, e\}$ is:

#

$$A_1 = \{a, c\} \qquad A_2 = \{b, e\} \qquad A_3 = \{d\}$$

These subsets are usually called the ***blocks*** of the partition.[1]

Here's the connection between all this stuff: there is an exact correspondence between *equivalence relations on A* and *partitions of A*. We can state this as a theorem:

Theorem ~~1~~ ??. *The equivalence classes of an equivalence relation on a set A form a partition of A.*

We won't prove this theorem (too dull even for us!), but let's look at an example. The congruent-mod-5 relation partitions the integers into five equivalance classes:

$$\begin{gathered}
\{\ldots, -5, 0, 5, 10, 15, 20, \ldots\} \\
\{\ldots, -4, 1, 6, 11, 16, 21, \ldots\} \\
\{\ldots, -3, 2, 7, 12, 17, 22, \ldots\} \\
\{\ldots, -2, 3, 8, 13, 18, 23, \ldots\} \\
\{\ldots, -1, 4, 9, 14, 19, 24, \ldots\}
\end{gathered}$$

In these terms, $x \equiv y \pmod 5$ is equivalent to the assertion that x and y are both in the same block of this partition. For example, $6 \equiv 16 \pmod 5$, because they're both in the second block, but $2 \not\equiv 9 \pmod 5$ because 2 is in the third block while 9 is in the last block.

In social terms, if "likes" were an equivalence relation, then everyone would be partitioned into cliques of friends who all like each other and no one else.

7.4 ~~4~~ Partial Orders

not bold

A relation is a ***partial order*** if it is reflexive, antisymmetric, and transitive. In terms of properties, the only difference between an equivalence relation and a partial order is that the former is symmetric and the latter is antisymmetric. But this small change makes a big difference; equivalence relations and partial orders are very different creatures.

As an example, the "divides" relation on the natural numbers is a partial order:

- It is reflexive because $x \mid x$.
- It is antisymmetric because $x \mid y$ and $y \mid x$ implies $x = y$.
- It is transitive because $x \mid y$ and $y \mid z$ implies $x \mid z$.

[1] ~~I~~ we think they should be called the ***parts*** of the partition. Don't you think that makes a lot more sense?

DAVID - keep the footnote

7.4 Partial Orders

7.4.1 ~~Weak and St~~ ~~Definitions and Examples~~ Strong and Weak Partial Orders

Definition ??: A relation ~~$R: A \to A$~~ R on a set A is a weak partial order if it is transitive, ~~reflexive~~ antisymmetric and reflexive. ~~antisymmetric.~~ The relation is said to be a strong partial order if it is transitive, ~~and asymmetric,~~ antisymmetric and irreflexive.[1]

Some authors define partial orders to be what we call weak partial orders, but we'll use the phrase partial order to mean either a weak or strong partial order. The difference between a weak partial order and a strong one has to do with the reflexivity property: in a weak partial order, every element is related to itself, but in a strong partial order, no element is related two itself. Otherwise, they are the same in that they are both transitive and antisymmetric.

~~IMPLIES there is no pair of nodes~~

1 Equivalently, the relation is a strong partial order if it is transitive and asymmetric, but stating it this way might have ~~obscured~~ obscured the irreflexivity property.

~~1 we as asymmetry~~
~~Antisymmetry and together with irreflexivity is the same as asymmetry, but defining it that way could~~

Examples of weak partial orders include "$\leq$" on $\mathbb{R}$, "$\subseteq$" on ~~the set of~~ the set of subsets of (say) $\mathbb{Z}$, and the "divides" relation on $\mathbb{Z}$. Examples of strict partial orders include "$<$" on $\mathbb{R}$ ~~and~~ and "$\subset$" on the set of subsets of $\mathbb{Z}$.[1] ~~and the "prerequisite" relation on the set of MIT subjects in the MIT Catalogue.[1,2] If you are not feeling comfortable~~

~~All of the preceding relations are transitive~~

~~As another example,~~

~~For weak partial orders, we often~~

We often denote a weak partial order with a symbol such as $\preceq$ or $\sqsubseteq$ instead of a letter such as R. This makes sense from one perspective since the symbols call to mind $\leq$ and $\subseteq$ which define ~~are associated with~~ common partial orders. On the other hand, a partial order is really a set of related ~~items~~ pairs of items, and

1 If you are not feeling comfortable with all the definitions that we've been throwing at you, it's probably a good idea to verify that each of these relations are indeed partial orders by checking that they have the transitivity and antisymmetry properties.

~~2 MIT's Committee on Curricula has the responsibility for making sure that no subject x is a prerequisite for itself or for another subject y that is a prerequisite for x. It's already hard enough to find your way to a degree without encountering such traps!~~

David: we'll use this footnote 2 on the next page so a letter like R would be more ~~important~~ normal.

Likewise, we will often ~~use~~ use a symbols like $\prec$ or $\sqsubset$ to denote a strong partial order.

——— INSERT FG goes here ———

~~7.4.2 Total Orders~~

~~7.4~~ 7.5 ~~Partially Ordered Sets~~ Posets and DAGs

7.5.1 Partially Ordered Sets

Definition ??: Given a partial order $\preceq$ on a set A, the pair $(A, \preceq)$ is called a partially ordered set or poset.

~~Since every~~ In terms of graph theory, ~~In other words~~ a poset is simply ~~the digraph corresponding to a relation. The nodes of the graph are the elements in the poset and~~ the directed graph $G=(A,\preceq)$ with vertex set A and edge set $\preceq$. For example, Figure FF shows the graph form of the poset for the "divides" relation on $\{1, 2, \ldots, 12\}$. ~~If you go back and look closely, you will see that other than the loops, this graph does not contain any cycles. In fact, this is true for any poset.~~

We have shown the ~~poset~~ graph form of the ~~FF~~ poset ~~poset on~~ for the "<" on $\{1, 2, 3, 4\}$ in Figure FH.

INSERT FG

7.4.2 Total Orders

A partial order is "partial" because there can be two elements with no relation between them. For example, in the "divides" partial order on $\{1, 2, \ldots, 12\}$, there is no relation between 3 and 5 (since neither divides the other). ~~In general~~

In general, we say that two elements a and b in a poset are *incomparable* if neither $a \preceq b$ nor $b \preceq a$. Otherwise, if $a \preceq b$ or $b \preceq a$, then we say that a and b are *comparable*.

Definition ??: A *total order* is a partial order ~~where~~ in which every pair of distinct elements is comparable.

For example, the "$\leq$" partial order on $\mathbb{R}$ is a total order ~~but~~ because ~~the natural numbers are totally ord~~ for any pair of ~~non~~ real numbers x and y, either $x \leq y$ or $y \leq x$. The "divides" partial order on $\{1, 2, \ldots, 12\}$ is not a total order because $3 \nmid 5$ and $5 \nmid 3$.

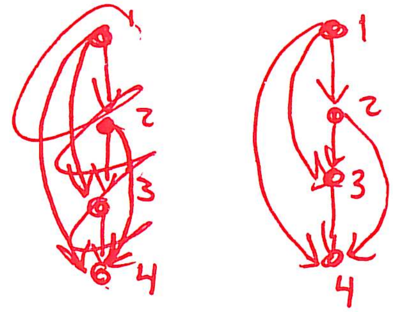


Figure FH: ~~The digraph for the~~ ~~form of the~~ Representing the poset ~~of the~~ for the "<" relation on $\{1,2,3,4\}$ as a digraph.

~~A quick inspection of Figure~~

7.5.2 ~~The Correspondence with DAGs~~ Posets are Acyclic

Do you notice anything that is common in Figures FF and FH? Of course, they both exhibit the transitivity and antisymmetry properties. And, except for the loops in Figure FF, they both do not contain any cycles. This is not a coincidence. In fact, the combination of the transitivity and asymetry properties imply the digraph ~~graph~~ for any poset is an ~~directed a~~ acyclic graph (i.e., a DAG), at least if you don't count loops as cycles. We prove this fact in the following theorem.

7 - 33

of elements $a_1, a_2, \ldots, a_n$ that are all related to one another by R: $a_1 R a_2, \ldots, a_{n-1} R a_n$. For example, the transitive closure of the relation $R = (1,3),(1,4),(2,1),(3,2)$ on the set $A = 1,2,3,4$ is $R' = (1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4)$.

4.1 Directed Acyclic Graphs

Notice that there are no ***directed cycles*** in the getting-dressed poset. In other words, there is no sequence of $n \geq 2$ distinct elements $a_1, a_2, \ldots, a_n$ such that:

$$a_1 \preceq a_2 \preceq a_3 \preceq \ldots \preceq a_{n-1} \preceq a_n \preceq a_1$$

This is a good thing; if there were such a cycle, you could never get dressed and would have to spend all day in bed reading books and eating fudgesicles. This lack of directed cycles is a property shared by all posets.

Theorem ~~2~~ 4. *A poset has no directed cycles other than self-loops.*

F6

Proof. We use proof by contradiction. Let $(A, \preceq)$ be a poset. Suppose that there exist $n \geq 2$ distinct elements $a_1, a_2, \ldots, a_n$ such that:

$$a_1 \preceq a_2 \preceq a_3 \preceq \ldots \preceq a_{n-1} \preceq a_n \preceq a_1$$

Since $a_1 \preceq a_2$ and $a_2 \preceq a_3$, transitivity implies $a_1 \preceq a_3$. Another application of transitivity shows that $a_1 \preceq a_4$ and a routine induction argument establishes that $a_1 \preceq a_n$. Since we also know that $a_n \preceq a_1$, antisymmetry implies $a_1 = a_n$ contradicting the supposition that $a_1, \ldots, a_n$ are distinct and $n \geq 2$. Thus, there is no such directed cycle. $\square$

Thus, deleting the self-loops from a poset leaves a directed graph without cycles, which makes it a ***directed acyclic graph*** or ***DAG***.

4.2 Partial Orders and Total Orders

A partially-ordered set is "partial" because there can be two elements with no relation between them. For example, in the getting-dressed poset, there is no relation between the left sock and the right sock; you could put them on in either order. In general, elements a and b of a poset are ***incomparable*** if neither $a \preceq b$ nor $b \preceq a$. Otherwise, if $a \preceq b$ or $b \preceq a$, then a and b are ***comparable***.

A ***total order*** is a partial order in which every pair of elements is comparable. For example, the natural numbers are totally ordered by the relation $\leq$; for every pair of natural numbers a and b, either $a \leq b$ or $b \leq a$. On the other hand, the natural numbers are *not* totally ordered by the "divides" relation. For example, 3 and 5 are incomparable under this relation; 3 does not divide 5 and 5 does not divide 3. The Hasse diagram of a total order is distinctive:

7.5.3 Transitive Closure

Theorem FG tells us that every poset corresponds to a DAG. Is the reverse true? I.e., does every DAG correspond to a poset? The answer is "Yes," but we may need to modify the ~~poset to~~ DAG to make sure that it satisfies the transitivity property. ~~We can do this by means of taking the transitive closure~~ For example, consider the DAG shown in Figure FJ. ~~This graph satisfies the antis.~~ As any DAG must, ~~it~~ this graph satisfies the antisymmetry property[1] but it does not satisfy the transitivity property because ~~1-2 are~~ $v_1 \to v_2$ and $v_2 \to v_3$ are in the graph but ~~and~~ $v_1 \to v_3$ is not in the graph.

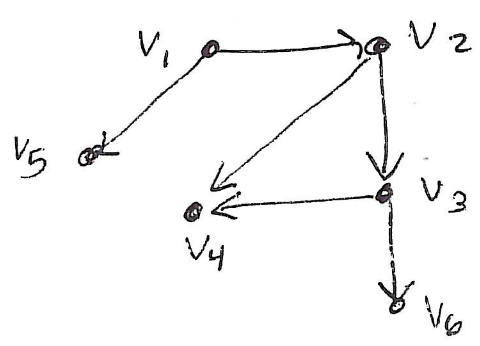


Figure FJ: A 6-node digraph that does not satisfy the transitivity property.

---

1 If $u \to v$ and $v \to u$ are in ~~the~~ a digraph G, ~~then the~~ then G would have a cycle of length 2 and it could not be a DAG.

<u>Definition ??</u>: Given a digraph $G = (V, E)$, the <u>transitive closure</u> of G is the ~~gra~~ digraph $G^+ = (V, E^+)$ where

$$E^+ = \{ u \to v \mid \text{there is a directed path of positive length from } u \text{ to } v \text{ in } G \}.$$

Similarly, if R is the relation corresponding to G, the <u>transitive closure</u> of R (denoted R^+) is the relation corresponding to G^+.

For example, the transitive closure for the graph in Figure FJ is shown in Figure FK.

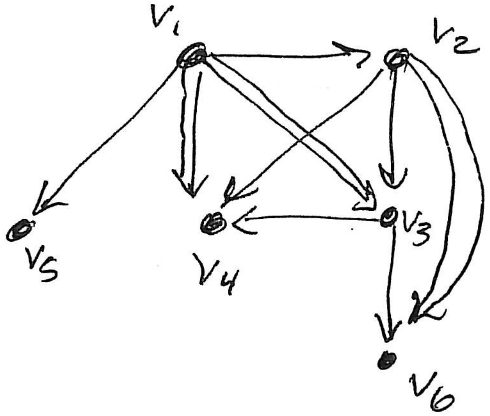


Figure FK: The transitive closure for the digraph in Figure FJ, ~~Added~~ the edges ~~are shown~~ that were ~~in bold.~~ added to form the transitive closure are shown in bold.

~~at In the problem section, Then~~
~~If G is a DAG, then cone~~

If G is a DAG, then the transitive closure of G ~~correspon~~ is a strong partial order. ~~That's because~~ the proof of this fact is left as an exercise in the problem section.

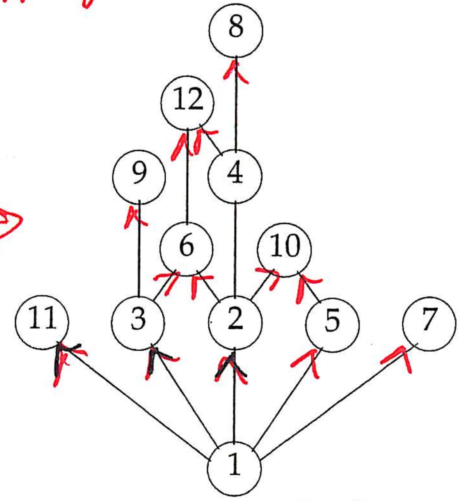
7.5.4 The Hasse Diagram

One problem with viewing a poset as a digraph is that there tend to be lots of edges due to the transitivity property. Fortunately, we do not necessarily have to draw all the edges if we know that the digraph corresponds to a poset. For example, we could choose not to draw any edge which would be implied by the transitivity property, knowing that ~~we could~~ it is really there by implication. In general, a Hasse diagram for a poset $(A, \preceq)$ is a digraph with vertex set A and edge set $\preceq$ ~~set~~ minus all self-loops and edges implied by transitivity. For example, the Hasse diagrams of the posets shown in Figures FF and FH are shown in Figure FM.

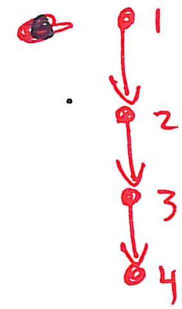
7-38

Also, flip it upside down so arrows point downward

Redraw in standard form (2) should be •2 & use directed edges



(a)



(b)

~~Figure 7.2: A DAG whose Path Relation is Divisibility on $\{1, 2, \ldots, 12\}$.~~

Figure FM: The Hasse diagrams for the posets in Figure FF and FH.

~~If we're using a DAG to represent a partial order —so all we care about is the the path relation of the DAG —we could replace the DAG with any other DAG with the same path relation. This raises the question of finding a DAG with the same path relation but the *smallest* number of edges. This DAG turns out to be unique and easy to find (see Problem ??).~~

7.6 Topological Sort

A total order that is consistent with a partial order is called a topological sort. More precisely, a topological sort of a poset $(A, \preceq)$ is a total order $(A, \preceq_T)$ such that

$$x \preceq y \text{ IMPLIES } x \preceq_T y.$$

For example, consider the poset that describes how a guy might get dressed for a formal occasion. The Hasse diagram for such a poset is shown in Figure FP. In this poset, the set is ~~the~~ all the garments and the partial order specifies which items must precede others when getting dressed.

7-40

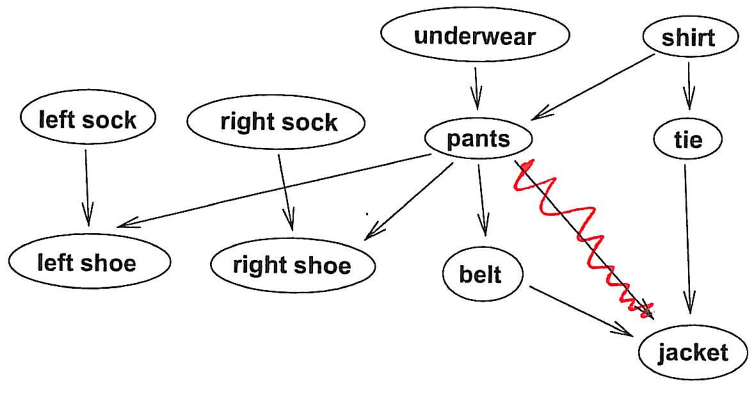
The $\leq$ relation on the natural numbers is also a partial order. However, the $<$ relation is not a partial order, because it is not reflexive; no number is less than itelf.[2]

Often a partial order relation is denoted with the symbol $\preceq$ instead of a letter, like R. This makes sense from one perspective since the symbol calls to mind $\leq$, which is one of the most common partial orders. On the other hand, this means that $\preceq$ actually denotes the *set* of all related pairs. And a set is usually named with a letter like R instead of a cryptic squiggly symbol. (So $\preceq$ is kind of like Prince.)

Anyway, if $\preceq$ is a partial order on the set A, then the pair $(A, \preceq)$ is called a ***partially-ordered set*** or ***poset***. Mathematically, a poset *is* a directed graph with vertex set A and edge set $\preceq$. So posets can be drawn just like directed graphs.

An an example, here is a poset that describes how a guy might get dressed for a formal occasion:

Figure FP: The Hasse diagram for a poset that describes which items must precede others when getting dressed.



In this poset, the *set* is all the garments and the *partial order* specifies which items must precede others when getting dressed. Not every edge appears in this diagram; for example, the shirt must be put on before the jacket, but there is no edge to indicate this. This edge would just clutter up the diagram without adding any new information; we already know that the shirt must precede the jacket, because the tie precedes the jacket and the shirt precedes the tie. We've also not not bothered to draw all the self-loops, even though $x \preceq x$ for all x by the definition of a partial order. Again, we know they're there, so the self-loops would just add clutter.

In general, a ***Hasse diagram*** for a poset $(A, \preceq)$ is a directed graph with vertex set A and edge set $\preceq$ minus all self-loops and edges implied by transitivity. The diagram above is *almost* a Hasse diagram, except we've left in one extra edge. Can you find it?

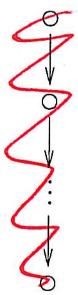
Note that the directed graph of a poset that includes all of the self-loops and all the edges implied by transitivity is the ***transitive closure*** of the graph. The transitive closure of a relation R on A is a new relation R' on A, where R' contains all pairs $(a_1, a_n), n \geq 1$ such that there is a "path" between a_1 and a_n in R. By "path" we mean that exists some sequence

texts

[2] Some sources omit the requirement that a partial order be reflexive and thus would say that $<$ *is* a partial order. The convention in this course, however, is that a relation *must* be reflexive to be a partial order.

There are several total orders that are consistent with the partial order shown in Figure FP. We have shown two of them in list form in Figure FQ. ~~For simplicity, we have listed the items in~~ Each such list is a topological sort for the partial order in Figure FP. In what follows, we will prove that every finite poset has a topological sort. You can think of this as a mathematical proof that you can get dressed in the morning (and then show up for ~~6.042~~ math lecture).

7-~~402~~ 41b



A total order defines a complete ranking of elements, unlike other posets. Still, for every poset there exists some ranking of the elements that is consistent with the partial order, though that ranking might not be unique. For example, you can put your clothes on in several different orders that are consistent with the getting-dressed poset. Here are a couple:

underwear	left sock
pants	shirt
belt	tie
shirt	underwear
tie	right sock
jacket	pants
left sock	right shoe
right sock	belt
left shoe	jacket
right shoe	left shoe
(a)	(b)

Figure FQ: Two possible topological sorts of the poset shown in Figure FP. In each case, the elements are listed ~~from~~ so that $x \preceq y$ iff x is above y in the list.

A total order consistent with a partial order is called a "topological sort". More precisely, a ***topological sort*** of a poset $(A, \preceq)$ is a total order $(A, \preceq_T)$ such that:

$$x \preceq y \quad \text{implies} \quad x \preceq_T y$$

So the two lists above are topological sorts of the getting-dressed poset. We're going to prove that *every* finite poset has a topological sort. You can think of this as a mathematical proof that you *can* get dressed in the morning (and then show up for 6.042 lecture).

Theorem 3. *Every finite poset has a topological sort.*

We'll prove the theorem constructively. The basic idea is to pull the "smallest" element a out of the poset, find a topological sort of the remainder recursively, and then add a back into the topological sort as an element smaller than all the others.

The first hurdle is that "smallest" is not such a simple concept in a set that is only partially ordered. In a poset $(A, \preceq)$, an element $x \in A$ is ***minimal*** if there is no other element $y \in A$ such that $y \preceq x$. For example, there are *four* minimal elements in the getting-dressed poset: left sock, right sock, underwear, and shirt. (It may seem odd that the minimal elements are at the top of the Hasse diagram rather than the bottom. Some people

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adopt the opposite convention. If you're not sure whether minimal elements are on the top or bottom in a particular context, ask.) Similarly, an element $x \in A$ is ***maximal*** if there is no other element $y \in A$ such that $x \preceq y$.

Proving that every poset *has* a minimal element is extremely difficult, because this is actually false. For example the poset $(\mathbb{Z}, \leq)$ has no minimal element. However, there is at least one minimal element in every *finite* poset.

Lemma 4. *Every finite poset has a minimal element.*

Proof. Let $(A, \preceq)$ be an arbitrary poset. Let $a_1, a_2, \ldots, a_n$ be a maximum-length sequence of distinct elements in A such that:

$$a_1 \preceq a_2 \preceq \ldots \preceq a_n$$

The existence of such a maximum-length sequence follows from the well-ordering principle and the fact that A is finite. Now $a_0 \preceq a_1$ can not hold for any element $a_0 \in A$ not in the chain, since the chain already has maximum length. And $a_i \preceq a_1$ can not hold for any $i \geq 2$, since that would imply a cycle

$$a_i \preceq a_1 \preceq a_2 \preceq \ldots \preceq a_i$$

and no cycles exist in a poset by Theorem 2. Therefore, a_1 is a minimal element. □

Now we're ready to prove Theorem 3, which says that every finite poset has a topological sort. The proof is rather intricate; understanding the argument requires a clear grasp of all the mathematical machinery related to posets and relations!

Proof (of Theorem 3). We use induction. Let $P(n)$ be the proposition that every n-element poset has a topological sort.

Base case. Every 1-element poset is already a total order and thus is its own topological sort. So $P(1)$ is true.

Inductive step. Now we assume $P(n)$ in order to prove $P(n+1)$ where $n \geq 1$. Let $(A, \preceq)$ be an $(n+1)$-element poset. By Lemma 4, there exists a minimal element $a \in A$. Remove a and all pairs in $\preceq$ involving a to obtain an n-element poset $(A', \preceq')$. This has a topological sort $(A', \preceq'_T)$ by the assumption $P(n)$. Now we construct a total order $(A, \preceq_T)$ by adding back a as an element smaller than all the others. Formally, let:

$$\preceq_T \quad = \quad \preceq'_T \cup \{(a, z) \mid z \in A\}$$

All that remains is the check that this total order is consistent with the original partial order $(A, \preceq)$; that is, we must show that:

$$x \preceq y \qquad \text{implies} \qquad x \preceq_T y$$

We assume that the left side is true and show that the right side follows. There are two cases:

Case 1: If $x = a$, then $a \preceq_T y$ holds, because $a \preceq_T z$ for all $z \in A$.

Case 2: If $x \neq a$, then y can not equal a either, since a is a minimal element in the partial order $\preceq$. Thus, both x and y are in A' and so $x \preceq' y$. This means $x \preceq'_T y$, since $\preceq'_T$ is a topological sort of the partial order $\preceq'$. And this implies $x \preceq_T y$, since $\preceq_T$ contains $\preceq'_T$.

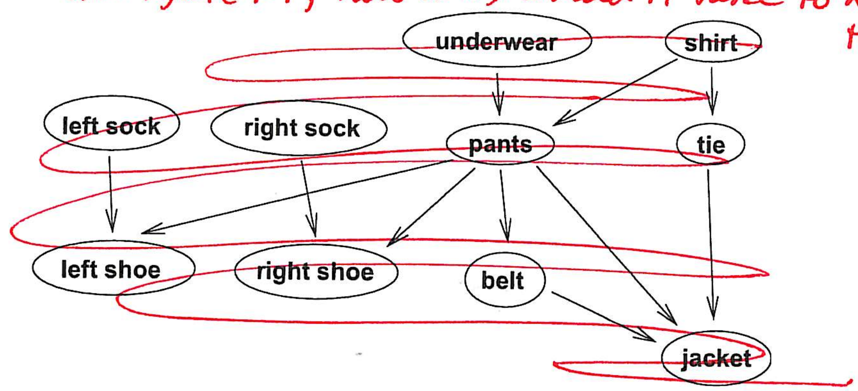
Thus, $(A, \preceq_T)$ is a topological sort of $(A, \preceq)$. This shows that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Therefore, $P(n)$ is true for all $n \geq 1$ by the principle of induction, which prove the theorem. $\square$

7.7 ~~5~~ Parallel Task Scheduling

When items of a poset are tasks that need to be done and the partial order is a precedence constraint, topological sorting provides us with a ~~legal~~ way to execute the tasks sequentially ~~i.e.,~~ without violating the precedence constraints.

But what if we have the ability to execute more than one task at the same time? For example, suppose that the ~~say~~ tasks are programs, the partial order indicates data dependence, and we have a parallel machine with lots of processors instead of a sequential machine with only one processor. How should we schedule the tasks~~?~~ so as to minimize the total time used? ~~For simplicity, we will assume that all tasks take the same amount of time and that all processors are identical.~~

For simplicity, assume ~~Assume~~ all tasks take 1 unit of time and we have an unlimited number of identical processors. ~~Given a poset of tasks, how long does it take to do all of them?~~ For example, in the clothes example ~~that we saw in lecture, how should we schedule the tasks?~~ in Figure FP, how long would it take to handle all the garments?



In the first unit of time we should do all minimal items, so we would put on our left sock, our right sock, our underwear, and our shirt. In the second unit of time, we should put on our pants and our tie. Note that we cannot put on our left or right shoe yet, since we have not yet put on our pants. In the third unit of time we should put on our left shoe, our right shoe, and our belt. Finally, in the last unit of time we can put on our jacket. ~~For~~ This schedule is illustrated in Figure FS.

INSERT FS goes here

The total time to do these tasks is 4 units. We cannot do any better than 4 units of time because there is a sequence of 4 tasks, each needing to be done before the next, of length

INSERT FS

~~Figure FS~~

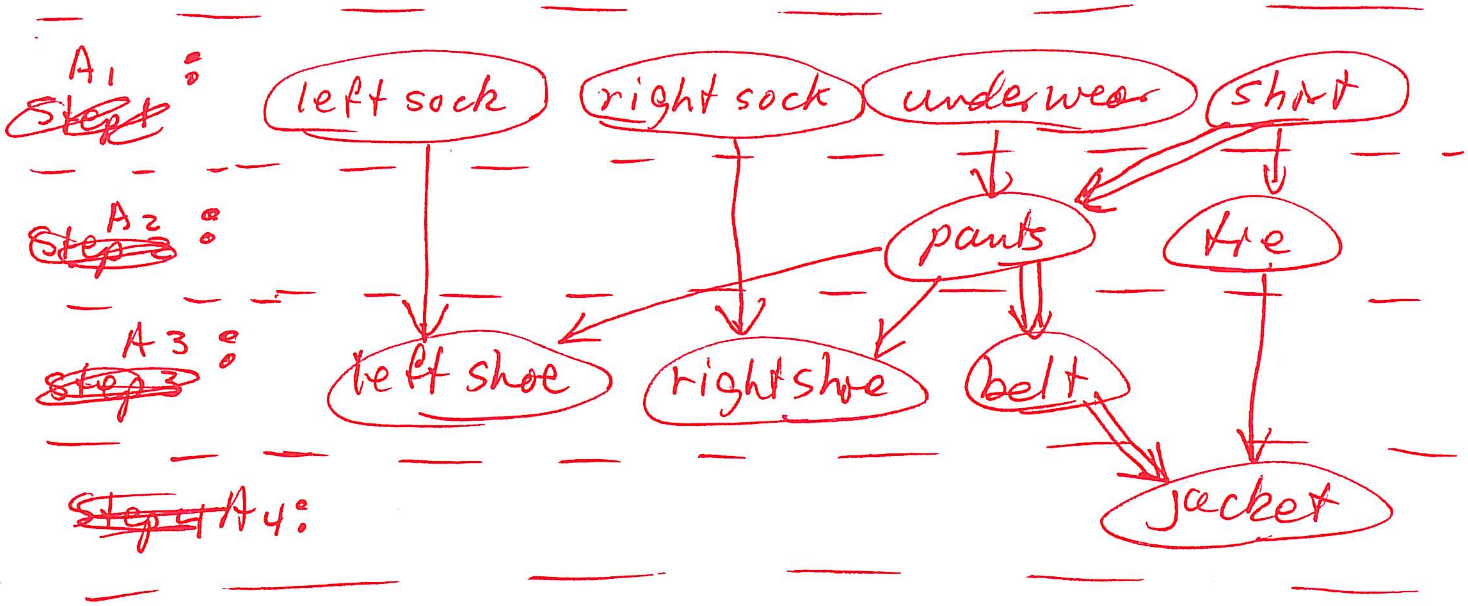


Figure FS: A parallel ~~schedu~~ schedule for the tasks in the getting dressed poset in Figure FP. The tasks in A_i can be performed in Step i for ~~(illegible)~~ $1 \le i \le 4$. ~~The cr A critical path co~~ A chain of length 4 (the critical path in this example) is shown with bold edges.

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4. Here, for example, we must put on our shirt before our pants, our pants before our belt, and our belt before our jacket. Such a sequence of items is known as a *chain*.

Definition 1. *A* chain *is a sequence $a_1 \preceq a_2 \preceq a_3 \cdots \preceq a_t$, where $a_i \neq a_j$ for all $i \neq j$, such that each item is comparable to the next in the chain, and it is smaller with respect to $\preceq$.* The length of the chain is the number of elements in the chain, t.

Thus, the time it takes to schedule tasks, even with ~~multiple~~ an unlimited number of processors, is at least the length of the longest chain. Indeed, if we used less time, then two items from the chain would have to be done at the same time, which contradicts the precedence constraints. For this reason, the longest chain is also known as a *critical path*. For example, ~~see Figure FS~~ Figure FS shows the critical path for the getting dressed poset.

~~Now~~ In this example, we were in fact able to schedule all the tasks in t steps, where t is the length of the longest chain. The really nice thing about posets is that this is always possible! In other words, for any poset, there is a legal parallel schedule that runs in t steps, where t is the length of the longest chain.

There are a lot of ways to prove this fact. Our proof will also give us the corresponding schedule in t time steps, and allow us to obtain some nice corollaries.

Theorem 5. *Given any finite poset $(A, \preceq)$ for which the longest chain has length t, it is possible to partition A into t subsets $A_1, A_2, \ldots, A_t$ such that for all $i \in \{1, 2, \ldots, t\}$ and for all $a \in A_i$, we have that all $b \preceq a$ appear in the set $A_1 \cup \cdots \cup A_{i-1}$.*

Before proving this theorem, first note that for each i, all items in A_i can be scheduled in time step i. This is because all preceding tasks are scheduled in preceding time steps, and thus are already completed. So the theorem implies that,

Corollary 6. *The total amount of parallel time needed to complete the tasks is the same as the length of the longest chain.*

For example, the A_i ~~we have done this~~ for the getting dressed poset are shown in Figure FS.

Proof. (*of Theorem* 5) For all $a \in A$, put a in A_i, where i is the length of the longest chain ending at a. In what follows, we ~~We~~ show that for all i, for all $a \in A_i$, and for all $b \preceq a$ with $b \neq a$, we have $b \in A_1 \cup A_2 \cup \cdots \cup A_{i-1}$.

We prove this by contradiction. Assume there is some i, $a \in A_i$, and $b \preceq a$ with $b \neq a$ and $b \notin A_1 \cup A_2 \cup \cdots \cup A_{i-1}$. By the way we defined A_i, this implies there is a chain of length at least i ending at b. Since $b \preceq a$ and $b \neq a$, we can extend this chain to a chain of length at least $i + 1$, ending at a. But then a could not be in A_i. This is a contradiction. □

If we have an unlimited number of processors, then the time to complete all tasks is equal to the length of the longest chain of dependent tasks. ~~In recitation you'll see what we can do if~~ The case when there are only a limited number of processors is covered in the problem section.

~~It turns out that the theorem we just proved can be used to do a lot more than schedule tasks for parallel machines. First, we need a definition.~~

~~**Definition 2.** *An antichain is a set of incomparable items, e.g., things that can be scheduled at the same time.*~~

for $\prec$ *consists of the sets* $A_0, A_1, \ldots,$ *where*

$$A_k ::= \{a \mid \text{depth}(a) = k\}.$$

We'll leave to Problem **??** the proof that the sets A_k are a parallel schedule according to Definition 6.5.6.

The minimum number of steps needed to schedule a partial order, $\prec$, is called the *parallel time* required by $\prec$, and a largest possible chain in $\prec$ is called a *critical path* for $\prec$. So we can summarize the story above by this way: with an unlimited number of processors, the minimum parallel time to complete all tasks is simply the size of a critical path:

Corollary 6.5.9. *Parallel time = length of critical path.*

7.8 ~~6.6~~ Dilworth's Lemma

Definition 6.6.1. An *antichain* in a partial order is a set of elements such that any two elements in the set are incomparable.

For example, ~~it's easy to verify that each set A_k is an antichain (see Problem ??)~~. each Ai in the proof of Theorem 5 and in Figure FS is an antichain since they ~~can be executed at the same time.~~ have no dependencies between them (which is why they could be executed at the same time).

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~~So our~~ Our conclusions about scheduling also tell us something about antichains.

Corollary 6.6.2. *If the largest chain in a partial order on a set, A, is of size t, then A can be partitioned into t antichains.*

Proof. Let the antichains be the sets $A_2, A_2, \ldots, A_t$. ■

Corollary 6.6.2 implies a famous result[4] about partially ordered sets:

Lemma 6.6.3 (Dilworth). *For all $t > 0$, every partially ordered set with n elements must have either a chain of size greater than t or an antichain of size at least n/t.*

By contradiction. Assume that the largest ~~an~~chain has length at most t and the largest antichain has size less than n/t.

Proof. Suppose the largest chain is of size $\leq t$. Then by Corollary 6.6.2, the n elements can be partitioned into at most t antichains. Hence, there are fewer than $t \cdot n/t$ ~~elements~~ $= n$ elements, ~~which~~ in the poset, which is a contradiction. Hence there must be a chain longer than t or an antichain with at least n/t elements. ~~Let ℓ be the size of the largest antichain. Since every element belongs to exactly one antichain, and there are at most t antichains, there can't be more than ℓt elements, namely, $\ell t \geq n$. So there is an antichain with at least $\ell \geq n/t$ elements.~~ ■

[4] Lemma 6.6.3 also follows from a more general result known as Dilworth's Theorem which we will not discuss.

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Corollary 6.6.4. *Every partially ordered set with n elements has a chain of size greater than $\sqrt{n}$ or an antichain of size at least $\sqrt{n}$.*

Proof. Set $t = \sqrt{n}$ in Lemma 6.6.3. ∎

Example 6.6.5. In the dressing partially ordered set, $n = 10$.

Try $t = 3$. There is a chain of size 4.

Try $t = 4$. There is no chain of size 5, but there is an antichain of size $4 \geq 10/4$.

Example 6.6.6. Suppose we have a class of 101 students. Then using the product partial order, Y, from Example 6.4.2, we can apply Dilworth's Lemma to conclude that there is a chain of 11 students who get taller as they get older, or an antichain of 11 students who get taller as they get younger, which makes for an amusing in-class demo.

As an application, ~~For example~~ consider a permutation of the numbers from 1 to n arranged as a sequence from left to right on a line. ~~For ex~~ Corollary 6.6.4 can be used to show that there must be a $\sqrt{n}$-length subsequence of these numbers that is completely increasing or completely decreasing as you move from left to right. ~~The~~ For example, the sequence 7, 8, 9, 4, 5, 6, 1, 2, 3, has an increasing subsequence of length 3 (e.g., 7, 8, 9) and a decreasing sequence of length 3 (e.g., 9, 6, 3). The proof of this result is left as an exercise ~~to help you~~ that will test your ability to ~~model the sequence~~ find the right partial order on the numbers in the sequence.

7.9 Problems