6.046	Lecture 8	Mar. 5, 2013

TODAY: Skip Lists
- one list, two lists, ...
- insert algorithm
- with-high-probability analysis

Skip lists [Pugh 1989]
- a simple, efficient dynamic search structure you'll never forget

Experiment in 2005: implement skip lists
≈ 10 minutes for linked list
≈ 30 minutes for skip list
[≈60 minutes debugging]
} not bad

- randomized: $O(\lg n)$ time/op. w.h.p.

With (polynomially) high probability: $\geq 1 - O(\frac{1}{n^\alpha})$ for desired constant $\alpha > 0$
⇒ if we do polynomially many operations, (↳ say n^k)
w.h.p. they all take $O(\lg n)$:
- $\Pr\{\text{any fail}\} \leq \sum \Pr\{\text{fail}\}$ (Union Bound)
$\leq n^k \cdot O(1/n^\alpha)$
$\leq O(n^{\alpha'})$ (set $\alpha = \alpha' + k$)

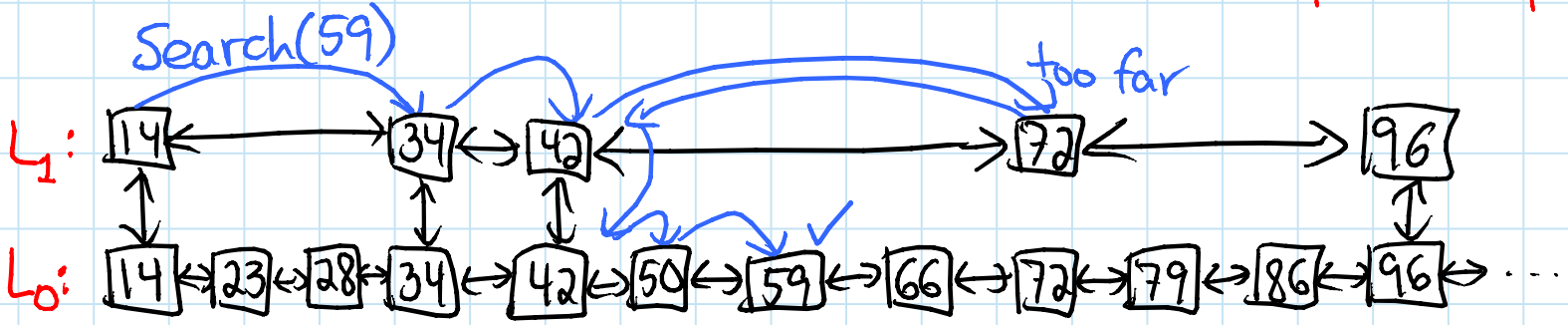
One sorted linked list: (starting from scratch)

- worst-case search: $\Theta(n)$
- how to improve?

Example:

1 14, 23, 28, 34, 42, 50, 59, 66, 72, 79, 86, 96, 103, 110, ...

2&3 express stops



(New York City 7th Ave. subway line)

Two sorted lists:

- L_0 stores all elements (as before)
- L_1 stores some elements, including first
- link between copies of same element

Search(x):

- express: go right in L_1 until going right would go too far
- transfer: go down to L_0
- local: go right in L_0 until find x or successor

Which elements should go in L_1?

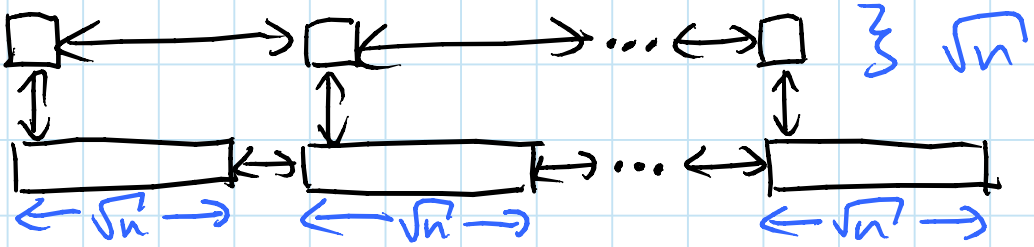
- in subway: "popular stations"
- here want to minimize worst-case performance
- best to evenly space nodes in L_1
- but how many should be in L_1?

Analysis: search cost $\approx \underbrace{|L_1|}_{\text{express cost}} + \underbrace{|L_0|/|L_1|}_{\text{local cost}}$

- minimized (up to constant factors) when $|L_1| = |L_0|/|L_1|$
  - i.e. $|L_1|^2 = |L_0| = n$ (everyone is in L_2)
  - i.e. $|L_1| = \sqrt{n}$

$\Rightarrow$ search cost $\approx \sqrt{n} + n/\sqrt{n} = 2\sqrt{n}$

Structure:

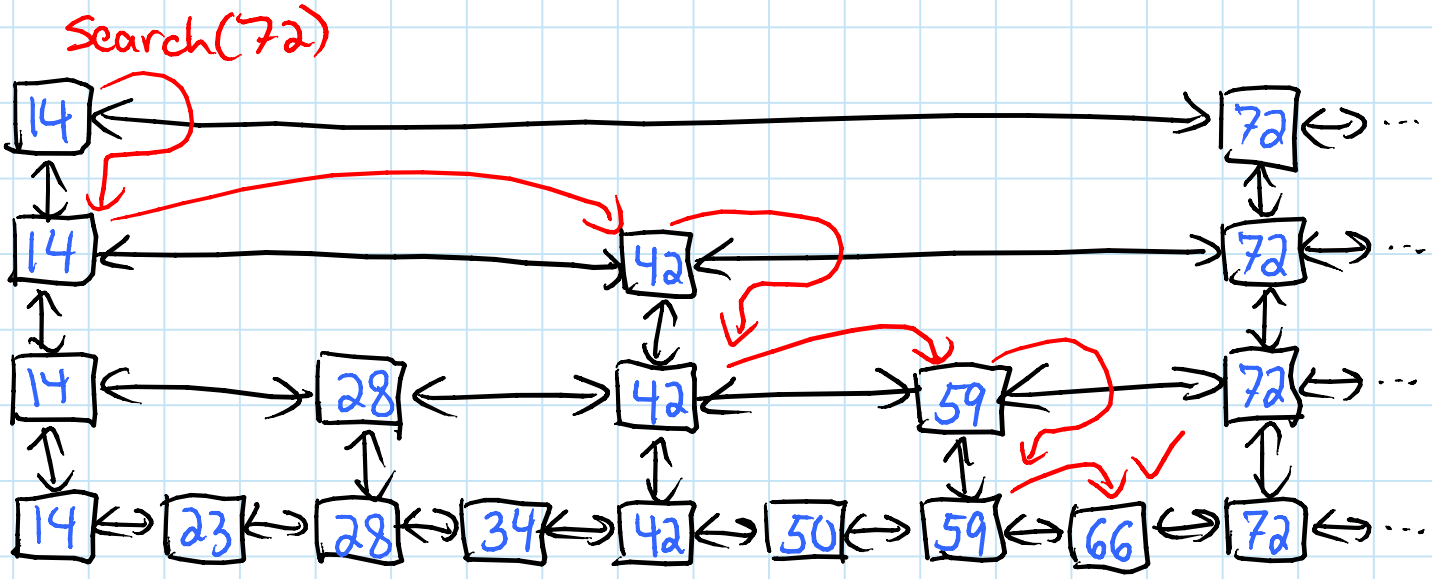


More lists:

- 2 sorted lists $\Rightarrow 2\sqrt{n}$
- 3 sorted lists $\Rightarrow 3 \cdot \sqrt[3]{n}$
- k sorted lists $\Rightarrow k \cdot \sqrt[k]{n}$
- $\lg n$ sorted lists $\Rightarrow \lg n \cdot \sqrt[\lg n]{n} = \lg n \cdot \underbrace{2^{\frac{\lg n}{\lg n}}}_{=2}$
  $= 2 \lg n$

lg n lists: IDEAL SKIP LIST

- like a binary tree where key(x) = min(subtree)
  (actually a level-linked B^+-tree) (data in leaves)
- bottom list L_0 contains all elements
- each list L_i contains subset of list L_{i-1} below
- first element in all lists
- vertical linked list of all copies of an element



Search(x):

- start at top left (first node of top list)
- until we fall below bottom list:
  if going right would go too far: go down
  else: go right

Skip list maintains roughly this idealized structure using randomization during inserts

- idea: Insert(x) always adds x to bottom list L_0, then promotes x up some i levels
- when should $i=0$ / $i \geq 1$? $1/2$
- when should $i=1$ / $i \geq 2$? $1/4$
- when should $i=2$ / $i \geq 3$? $1/8$

etc.

Insert(x):

- Search(x) to find where x fits in L_0
- insert x in L_0
- flip fair coin
- if heads: promote x to next level up (L_{i+1}) (possibly newly created)
  flip again

Detail: add special $-\infty$ value to every list
⇒ first element is in every list (needed for Search)

EXERCISE: build a skip list with a real coin
- write H/T coin flip at each node

Delete(x): just remove x from all lists

Why are skip lists good?

Warmup Lemma: # levels in n-element skip list is $O(\lg n)$ w.h.p.
($c \lg n$; ↖ prob. $1 - 1/n^\alpha$)

Proof: failure probability (not $\le c \lg n$ levels)
$= \Pr\{ > c \lg n \text{ levels}\}$
$= \Pr\{\text{some element got promoted } > c \lg n \text{ times}\}$
$\le n \cdot \Pr\{\text{element } x \text{ got promoted } > c \lg n \text{ times}\}$
by Union Bound
$= n \cdot (1/2)^{c \lg n}$
$= n / n^c$
$= 1 / n^{c-1}$
$= 1 / n^\alpha$ for $\alpha = c - 1$ i.e. $c = \alpha + 1$. □

Theorem: any search in an n-element skip list costs $O(\lg n)$ w.h.p.

Cool idea: analyze search backwards (up)
- search starts [ends] at node in bottom list
- at each node visited:
  - if node wasn't promoted higher (tails here) then we go [came from] left
  - if node was promoted higher (heads here) then we go [came from] up
- stop [start] when we reach top level or $-\infty$

Proof of theorem:

- Search makes "up" & "left" moves, each with probability $1/2$
- number of "up" moves $<$ # levels $\leq c \lg n$ w.h.p. (by Warmup Lemma)

$\Rightarrow$ w.h.p., number of moves $\leq$ number of coin flips before we get $\geq c \lg n$ tails

Claim: $\geq c \lg n$ tails in $O(\lg n)$ coin flips, w.h.p.
↳ constant d

Proof: Chernoff bound

- $E[\#\text{ tails in } d \lg n \text{ coin flips}] = \frac{1}{2} d \lg n$
- $\Pr\{< c \lg n \text{ tails in } cd \lg n \text{ coin flips}\}$
- $X_i = \begin{cases} 1 & \text{if } i\text{th coin flip is heads} \\ 0 & \text{else (tails)} \end{cases}$
- $E[\sum_i X_i] = \frac{1}{2} d \lg n$
- $\Pr\{\sum_i X_i \geq (d-c) \lg n\}$ (failure: too many heads / too few tails)

$= \Pr\{\sum_i X_i \geq (1+b) \cdot \frac{1}{2} d \lg n\}$

where $d - c = (1+b)\frac{1}{2} d$ i.e. $b = 1 - 2\frac{c}{d}$

$\leq e^{-b^2 (1/2 \cdot d \lg n)/3}$

$< e^{-b^2 d/6}$ ($2 < e$)

$< n^{-\alpha}$

$= n^{-\alpha}$ for $\alpha = (d - 4c + 4c^2/d)/6$

e.g. $d = 6(\alpha + 4c + 4c^2)$.

□

Space clearly $O(n \lg n)$ by Warmup Lemma...
In fact:

Theorem: space used by n-element skip list is $O(n)$ with exponentially high prob. (& in expectation)
↳ $1 - O(1/2^{\alpha n})$
α = specified constant

Proof: can think of skip-list construction as a big sequence of coin flips
- distance between consecutive heads = # times to promote next element
- space = # nodes = total # coin flips
- finished after n heads

$\Rightarrow$ space = # coin flips until $\geq n$ heads
= $O(n)$ w. exp. h.p. (by same claim)

Also: $E[\text{space}] = E[\sum_x \#\text{ lists containing } x]$

$= \sum_x E[\#\text{ lists containing } x]$
(linearity of expectation)

$= \sum_x (1 + 1/2 + 1/4 + \cdots)$

$= \sum_x 2$

$= 2n$ □