David: I'll send the Intro to Part III later ~~with ch 10~~ after I have gone thru the other chapters.

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7 JUL 2010

~~11~~ 9½ Sums & Asymptotics

~~EDITING NOTE:~~

~~Closed Forms and Approximations~~

Sums and products arise regularly in the analysis of algorithms, financial applications, ~~and in other technical areas such as finance~~ physical problems, and probabilistic systems. For example, ~~we used Well Ordering to prove in Theorem 3.2.1 that~~ we have already encountered the sum $1+2+4+\cdots+N$ when counting the number of nodes in a complete binary tree with N inputs. Although

$$\cancel{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.} \qquad \cancel{(11.1)}$$

such a sum can be represented compactly using the sigma notation

$$\sum_{i=0}^{\log N} 2^i, \qquad (9.1)$$

it is alot easier ~~to keep~~ and more helpful to

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INSERT A

express the sum by its closed form ~~the~~ value

2N − 1.

~~Similarly, it is we~~

~~the~~ ~~# the sum~~

~~Proving that~~

~~That is because 2N-1 is still easier to compute~~

~~given N and you~~

By closed form, we mean an expression that does not make use of summation or product symbols or otherwise need those ~~han~~ handy (but sometimes troublesome) dots .... Expressions in closed form are usually easier to evaluate (it doesn't get much simpler than 2N-1, for example) and it is usually easier to get a feel for their magnitude than expressions ~~doing~~ involving large sums and products. ~~symbols~~

But how do you find a closed form for a sum or product? Well, it's ~~a bit of~~ part ~~so~~ math ~~an art of~~ and part art. And it is the subject of this chapter. # We will start the chapter with a motivating example involving annuities. Figuring out the value of the annuity will involve a large and nasty looking ~~the kind of sum that~~ sum. We will then describe several methods ~~a whole bunch of ways to~~ for finding ~~find~~ closed forms for all sorts ~~the~~ ~~annuity sum as well as~~ of sums, including the annuity sum. In some cases, a closed form for a sum may not exist and so we will provide a general method for finding good ~~closed f clos bou~~ upper and lower bounds on the sum (which are closed form, of course).

The methods we develop for sums will also work for products since you can convert any ~~sum~~ product into a sum by taking a ~~the~~ logarithm of the product. As an example, we will use this approach to find a good closed form approximation ~~the value~~ to ~~$n! = 1 \cdot 2 \cdot 3 \cdots n$~~ $n!$.

~~We car~~

$$n! ::= 1 \cdot 2 \cdot 3 \cdots n .$$

We conclude the chapter with a discussion of asymptotic notation. Asymptotic notation is often used to bound the error terms when there is no exact closed form expression known for a sum or product. It also provides a convenient way

A-B

to express the growth rate or order of magnitude of a sum or product.

The simple *closed form* expression $n(n+1)/2$ makes the sum a lot easier to understand and evaluate. But the proof of equation (11.1) by Well Ordering does not explain where it came from in the first place. In Section 11.3, we'll discuss ways to find such closed forms. Even when there are no closed forms exactly equal to a sum, we may still be able to find a closed form that *approximates* a sum with useful accuracy.

The product we focus on in this chapter is the familiar factorial:

$$n! ::= 1 \cdot 2 \cdots (n-1) \cdot n = \prod_{i=1}^{n} i.$$

We'll describe a closed form approximation for it called *Stirling's Formula*.

Finally, when there isn't a good closed form approximation for some expression, there may still be a closed form that characterizes its growth rate. We'll introduce *asymptotic notation*, such as "big Oh", to describe growth rates. ■

9.1

~~11.1~~ The Value of an Annuity

Would you prefer a million dollars today or $50,000 a year for the rest of your life? On the one hand, instant gratification is nice. On the other hand, the total dollars received at $50K per year is much larger if you live long enough.

Formally, this is a question about the value of an annuity. An *annuity* is a financial instrument that pays out a fixed amount of money at the beginning of every year for some specified number of years. In particular, an n-year, m-payment annuity pays m dollars at the start of each year for n years. In some cases, n is finite, but not always. Examples include lottery payouts, student loans, and home mortgages. There are even Wall Street people who specialize in trading annuities.

"What is the annuity worth?"

A key question is, ~~what an annuity is worth~~. For example, lotteries often pay out jackpots over many years. Intuitively, \$50,000 a year for 20 years ought to be worth less than a million dollars right now. If you had all the cash right away, you could invest it and begin collecting interest. But what if the choice were between \$50,000 a year for 20 years and a *half* million dollars today? Now it is not clear which option is better.

9.1.1 The Future Value of Money

In order to answer such questions, we need to know what a dollar paid out in the future is worth today. To model this, let's assume that money can be invested at a fixed annual interest rate p. We'll assume an 8% rate[1] for the rest of the discussion.

Here is why the interest rate p matters. Ten dollars invested today at interest rate p will become $(1 + p) \cdot 10 = 10.80$ dollars in a year, $(1 + p)^2 \cdot 10 \approx 11.66$ dollars in two

[1]U.S. interest rates have dropped steadily for several years, and ordinary bank deposits now earn around 1.5%. But just a few years ago the rate was 8%; this rate makes some of our examples a little more dramatic. The rate has been as high as 17% in the past thirty years.

In Japan, the standard interest rate is near zero%, and on a few occasions in the past few years has even been slightly negative. It's a mystery why the Japanese populace keeps any money in their banks.

years, and so forth. Looked at another way, ten dollars paid out a year from now ~~are~~ is only really worth $1/(1+p) \cdot 10 \approx 9.26$ dollars today. The reason is that if we had the \$9.26 today, we could invest it and would have \$10.00 in a year anyway. Therefore, p determines the value of money paid out in the future.

11.1.1 The Future Value of Money

So for an n-year, m-payment annuity, the first payment of m dollars is truly worth m dollars. But the second payment a year later is worth only $m/(1+p)$ dollars. Similarly, the third payment is worth $m/(1+p)^2$, and the n-th payment is worth only $m/(1+p)^{n-1}$.

The total value, V, of the annuity is equal to the sum of the payment values. This gives:

$$\begin{aligned}
V &= \sum_{i=1}^{n} \frac{m}{(1+p)^{i-1}} \\
&= m \cdot \sum_{j=0}^{n-1} \left(\frac{1}{1+p}\right)^{j} && (\text{substitute } j ::= i-1) \\
&= m \cdot \sum_{j=0}^{n-1} x^{j} && (\text{substitute } x = \frac{1}{1+p}). \qquad (11.2)
\end{aligned}$$

INSERT B goes here

~~The summation in (11.2) is a geometric sum that has a *closed form*, making the evaluation a lot easier, namely[2],~~

$$\sum_{i=0}^{n-1} x^{i} = \frac{1-x^{n}}{1-x}. \qquad (11.3)$$

~~(The phrase "closed form" refers to a mathematical expression without any summation or product notation.)~~

Equation (11.3) ~~was proved by induction in Problem ??~~ can be verified by induction, but, as is often the case, the proof by induction ~~gave~~ gives no hint about how the formula was found in the first place. So

[2]To make this equality hold for $x = 0$, we adopt the convention that $0^0 ::= 1$.

1 By "closed form", we mean an expression that does not have any summation, ~~or~~ product ~~notation nor any "..." note~~ or "..." notation. ~~The expression could~~

INSERT B

The goal of the preceding substitutions was to get the summation into a simple, special form so that we can solve it with a general formula. In particular, the terms of the sum

$$\sum_{j=0}^{n-1} x^j = 1 + x + x^2 + x^3 + \dots + x^{n-1}$$

form a geometric series, which means that ~~each term is a constant times the pre~~ ratio of consecutive terms is always the same. (In this case, the ratio is always x.) It turns out that there is a ~~simple formula for evalua~~ ~~clo~~ nice closed form[1] expression ~~for~~ any geometric series, namely[2] now you could figure it out for yourself called the Perturbation method.

~~a way that you could~~

~~INSERT C~~

INSERT C goes here

we'll take this opportunity to explain ~~where it comes from. The trick is to let S be the value of the sum and then observe what $-xS$ is:~~

$$\begin{array}{rcllllll} S & = & 1 & +x & +x^2 & +x^3 & + \cdots & +x^{n-1} \\ -xS & = & & -x & -x^2 & -x^3 & - \cdots & -x^{n-1} - x^n. \end{array}$$

The result of the subtraction is

~~Adding these two equations gives:~~

$$S - xS = 1 - x^n.$$

~~so~~ Solving for S gives the desired closed form expression in Equation 11.3:

$$S = \frac{1 - x^n}{1 - x}.$$

see more examples of this method

We'll ~~look further into this method of proof in a few weeks~~ when we introduce *generating functions* in Chapter ~~16~~. 12

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Etl

9.1.2 The Perturbation Method

Given a sum S that has a nice structure, ~~such as~~

$$\cancel{S = 1+x+x^2+\cdots+x^{n-1},}$$

~~then~~ it is often useful to "perturb" the sum ~~in a way that~~ so that we can somehow combine the sum with the perturbation to get something much simpler. For example, suppose

$$S = 1+x+x^2+\cdots+x^{n-1}.$$

An example of a perturbation would be ~~to multiply by~~

$$xS = x+x^2+\cdots+x^n.$$

The difference between S and xS is not ~~too large~~ so great, ~~and if we and if we~~ ~~and so if we were subtract xS from S, we get something much simpler.~~ ~~there is massive cancellation:~~ and so if we were to subtract xS from S, there would be massive cancellation:

$$\cancel{S - xS = 1 - x^n}.$$

~~Solving for S results in the closed form~~

9.1.3 A

~~11.1.2~~ A Closed Form for the Annuity Value

Using ~~Form~~ Equation 11.3, we can derive

~~So now we have~~ a simple formula for V, the value of an annuity that pays m dollars at the start of each year for n years.

$$V = m\frac{1-x^n}{1-x} \qquad \text{(by (11.2) and (11.3))} \qquad (11.4)$$

$$= m\frac{1+p-(1/(1+p))^{n-1}}{p} \qquad \text{(substituting } \cancel{\$}\, x = 1/(1+p)). \qquad (11.5)$$

The formula (11.5) is much easier to use than a summation with dozens of terms. For example, what is the real value of a winning lottery ticket that pays \$50,000 per year for 20 years? Plugging in $m = \$50,000$, $n = 20$, and $p = 0.08$ gives $V \approx \$530,180$. So because payments are deferred, the million dollar lottery is really only worth about a half million dollars! This is a good trick for the lottery advertisers!

9.1.4 ~~11.1.3~~ Infinite Geometric Series

The question we began with was whether you would prefer a million dollars today or $\$50,000$ a year for the rest of your life. Of course, this depends on how long you live, so optimistically assume that the second option is to receive $\$50,000$ a year *forever*. This sounds like infinite money! But we can compute the value of an annuity with an infinite number of payments by taking the limit of our geometric sum in (11.3) as n tends to infinity.

Theorem 11.1.1. *If* $|x| < 1$, *then*

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}.$$

Proof.

$$\begin{aligned}
\sum_{i=0}^{\infty} x^i &::= \lim_{n\to\infty} \sum_{i=0}^{n-1} x^i \\
&= \lim_{n\to\infty} \frac{1-x^n}{1-x} && \text{(by (11.3))} \\
&= \frac{1}{1-x}.
\end{aligned}$$

The final line follows from that fact that $\lim_{n\to\infty} x^n = 0$ when $|x| < 1$. ■

In our annuity problem, $x = 1/(1+p) < 1$, so Theorem 11.1.1 applies, and we get

$$\begin{aligned}
V &= m \cdot \sum_{j=0}^{\infty} x^j && \text{(by (11.2))} \\
&= m \cdot \frac{1}{1-x} && \text{(by Theorem 11.1.1)} \\
&= m \cdot \frac{1+p}{p} && (x = 1/(1+p)).
\end{aligned}$$

Plugging in $m = \$50,000$ and $p = 0.08$, the value, V, is only $\$675,000$. Amazingly, a million dollars today is worth much more than $\$50,000$ paid every year forever! Then again, if we had a million dollars today in the bank earning 8% interest, we could take out and spend \$80,000 a year forever. So on second thought, this answer really isn't so amazing.

~~EDITING NOTE:~~

9.1.5 Examples

~~Examples~~

~~We now have closed form formulas for geometric sums and series. Some examples are given below. In each case, the solution follows immediately from either equation (11.3)~~

Equation 11.3 and Theorem 11.1.1 are incredibly useful in computer science. In fact, we already used Equation 11.3 implicitly when we claimed in chapter 6 that an N-input complete binary tree has

$$1 + 2 + 4 + \cdots + N = 2N - 1$$

nodes. Here are some other common sums that can be ~~evaluated~~ put into closed form using Equation 11.3 and Theorem 11.1.1:

David: fix spacing around all the =

(for finite sums) or Theorem 11.1.1 (for infinite series).

$$1 + 1/2 + 1/4 + 1/8 + \cdots = \sum_{i=0}^{\infty} (1/2)^i = \frac{1}{1-(1/2)} = 2 \tag{11.6}$$

$$0.999999999\ldots = 0.9 \sum_{i=0}^{\infty} (1/10)^i = 0.9 \frac{1}{1-1/10} = 0.9 \frac{10}{9} = 1 \tag{11.7}$$

$$1 - 1/2 + 1/4 - 1/8 + \cdots = \sum_{i=0}^{\infty} (-1/2)^i = \frac{1}{1-(-1/2)} = 2/3 \tag{11.8}$$

$$1 + 2 + 4 + 8 + \cdots + 2^{n-1} = \sum_{i=0}^{n-1} 2^i = \frac{1-2^n}{1-2} = 2^n - 1 \tag{11.9}$$

$$1 + 3 + 9 + 27 + \cdots + 3^{n-1} = \sum_{i=0}^{n-1} 3^i = \frac{1-3^n}{1-3} = \frac{3^n - 1}{2} \tag{11.10}$$

If the terms in a geometric sum or series grow smaller, as in equation (11.6), then the sum is said to be *geometrically decreasing*. If the terms in a geometric sum grow progressively larger, as in (11.9) and (11.10), then the sum is said to be *geometrically increasing*. In either case, the sum is ~~always~~ usually

~~Here is a good rule of thumb: *a geometric sum or series is*~~ *approximately equal to the term* in the sum

take out italics

with the *greatest absolute value*. For example, in ~~In~~ equations (11.6) and (11.8), the largest term is equal to 1 and the sums are 2 and 2/3, both relatively close to 1. In equation (11.9), the sum is about twice the largest term. In the final equation (11.10), the largest term is 3^{n-1} and the sum is $(3^n - 1)/2$, which is only about a factor of 1.5 greater. You can see why this rule of thumb works by looking carefully at Equation 11.3 and Theorem 11.1.1.

~~Related Sums~~

9.1.6 Variants of Geometric Sums

We now know all about geometric sums — if you have one, life is easy. But in practice one often encounters sums that cannot be transformed by simple variable substitutions to the form $\sum x^i$.

A non-obvious, but useful way to obtain new summation formulas from old is by differentiating or integrating with respect to x. As an example, consider the following sum:

$$\sum_{i=1}^{n} ix^i = x + 2x^2 + 3x^3 + \cdots + nx^n$$

This is not a geometric sum, since the ratio between successive terms is not ~~constant~~ fixed, and so ~~Our~~ our formula for the sum of a geometric sum cannot be directly applied. But suppose that we differentiate that formula:

$$\begin{aligned}
\frac{d}{dx}\sum_{i=0}^{n} x^i &= \frac{d}{dx}\,\frac{1-x^{n+1}}{1-x} \\
\sum_{i=1}^{n} i x^{i-1} &= \frac{-(n+1)x^n(1-x)-(-1)(1-x^{n+1})}{(1-x)^2} \\
&= \frac{-(n+1)x^n+(n+1)x^{n+1}+1-x^{n+1}}{(1-x)^2} \\
&= \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}.
\end{aligned}$$

Often differentiating or integrating messes up the exponent of x in every term. In this case, we now have a formula for a sum of the form $\sum ix^{i-1}$, but we want a formula for the series $\sum ix^i$. The solution is simple: multiply by x. This gives:

$$\sum_{i=1}^{n} i x^i = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2} \qquad \text{(eqnG-1)}$$

~~and our problem~~ and we have ~~a clos~~ the desired closed form

expression for our sum. 1 It's a little complicated looking, but it's a lot easier to work with than the sum.

Since we could easily have made a mistake ~~it is~~ in the calculation, it is always a good idea to go back and validate a formula obtained this way with a proof by induction.

Notice that if $|x| < 1$, then this series converges to a finite value even if there are

1

infinitely many terms. Taking the limit of Equation 6.1 as n tends infinity gives the following theorem:

Theorem 11.1.2. *If* $|x| < 1$, *then*

$$\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}.$$

As a consequence, suppose that there is an annuity that pays im dollars at the *end* of each year i forever. For example, if $m = \$50,000$, then the payouts are $\$50,000$ and then $\$100,000$ and then $\$150,000$ and so on. It is hard to believe that the value of this annuity is finite! But we can use ~~the preceding~~ Theorem 11.1.2 to compute the value:

$$\begin{aligned}
V &= \sum_{i=1}^{\infty} \frac{im}{(1+p)^i} \\
&= m \cdot \frac{\frac{1}{1+p}}{(1-\frac{1}{1+p})^2} \\
&= m \cdot \frac{1+p}{p^2}.
\end{aligned}$$

The second line follows by an application of Theorem 11.1.2. The third line is obtained by multiplying the numerator and denominator by $(1+p)^2$.

For example, if $m = \$50,000$, and $p = 0.08$ as usual, then the value of the annuity is $V = \$8,437,500$. Even though payments increase every year, the increase is only additive with time; by contrast, dollars paid out in the future decrease in value exponentially with time. The geometric decrease swamps out the additive increase. Payments in the distant future are almost worthless, so the value of the annuity is finite.

The important thing to remember is the trick of taking the derivative (or integral) of a summation formula. Of course, this technique requires one to compute nasty derivatives correctly, but this is at least theoretically possible!

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9.2 Power Sums

In Chapter 3, we verified the formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \qquad \text{(Eqn 6.26)}$$

But the source of this formula is still a mystery. Sure, we can prove it is true using well ordering or induction, but where did the expression on the right come from in the first place? Even more inexplicable is the closed form expression for the sum of consecutive squares:

$$\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}. \qquad \text{(Eqn 6.27)}$$

~~There is a nice~~ It turns out that there is a way to ~~solve~~ derive these expressions but before we explain it, we thought it would be fun[1] to show you how Gauss proved Equation 6.26 when he was a young boy.[2]

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1 ~~Remem we are~~ Remember that we are mathematicians, so our definition of "fun" may be different than yours.

2 We suspect that Gauss was probably not an ordinary boy.

Gauss' idea ~~was~~ is ~~similar~~ related to the perturbation method we used in ~~the last~~ Subsection 9.1.2. ~~Suppose~~ Let

$$S = \sum_{i=1}^{n} i .$$

Then we can write the sum in two orders:

$$S = 1 + 2 + \cdots + n-1 + n, \cancel{\text{and}}$$

$$S = n + n-1 + \cdots + 2 + 1.$$

Adding these two equations gives

$$2S = (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

$$= n(n+1).$$

Hence,

$$S = \frac{n(n+1)}{2}.$$

Not bad for a young child. Looks like Gauss had some potential...

Unfortunately, the same trick does not work for ~~the~~ summing consecutive squares. However, we can observe that the result might be a 3rd degree polynomial in n, since ~~it is~~ the sum contains n terms that average out to a value that grows quadratically in n. So we might guess that

—— INSERT M goes here ——
(it is text on pp 720-722)

End of

9.~~2~~3 Approximating Sums

Unfortunately, it is not always possible to find a closed form expression for a sum. For example, consider the sum

$$S = \sum_{i=1}^{n} \sqrt{i} .$$

No closed form expression is known for S.

~~However~~

~~In such cases, we will~~

~~The good news~~

~~But it is possible to find bo good bounds on for S that~~

In such cases, we need to resort to approximations for S if we need to have a closed form. The good news is that there ~~is~~ is a general ~~for~~ method to find ~~good~~ ~~are good upper and lower~~ ~~closed form bound~~ close (closed form) upper ~~an~~

and lower bounds ~~for most any sum~~ that work for most any sum. ~~These The~~ Even better, the method is simple and easy to remember. It works by replacing the sum by an integral and then adding either the ~~first or the~~ first or last term in the sum.

~~Theorem 63: Let $S = \sum$ Given any sum in which the terms are steadily~~ Let f: ~~$R^+ \to R$ be a~~ nondecreasing ~~function and let let $S = \sum_{i=1}^{n} f(i)$ continuous function~~

~~f is an increasing function of for which the terms increase as i~~

~~for which f is a nondecreasing function of i~~

~~$S = \sum_{i=1}^{n} f(i)$~~

Theorem 63: Let $f: \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing[1] (footnote on next page) continuous function and let

$$S = \sum_{i=1}^{n} f(i).$$

~~Then~~ Let

$$I = \int_1^n f(x)\,dx.$$

Then

$$I + f(1) \le S \le I + f(n).$$

Similarly, if f is a nonincreasing ~~function~~ then

$$I + f(n) \le S \le I + f(1).$$

Proof: Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function. For example, $f(x) = \sqrt{x}$ is such a function. ~~The value of~~ [1]Consider the graph shown in Figure 64. The value of

$$S = \sum_{i=1}^{n} f(i)$$

is ~~the area~~ represented by the shaded area in this figure. This is because the ~~height of the~~ ith rectangle in the figure (counting from left to right) has width 1 and height $f(i)$.

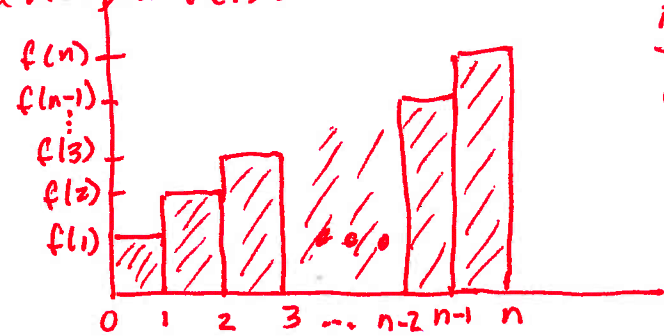

Figure 64: The area of the ith rectangle is $f(i)$. The shaded region has area $S = \sum_{i=1}^{n} f(i)$.

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1 A function f is nondecreasing if $f(x) \ge f(y)$ whenever $x \ge y$. It is nonincreasing if $f(x) \le f(y)$ whenever $x \ge y$.

The value of

$$\mathcal{I} = \int_1^n f(x)\,dx$$

is the shaded area under the curve of f(x) from 1 to n shown in Figure 6-5. Comparing the shaded regions in Figures 6-4 and 6-5, we see that S is at least $\mathcal{I}$ plus the area of the left most rectangle. Hence, $\cancel{\mathcal{I} + f(1) \le S}$ $\cancel{S \ge \mathcal{I} + f(1)}$.

$$S \ge \mathcal{I} + f(1). \qquad \text{(Eqn 6-7)}$$

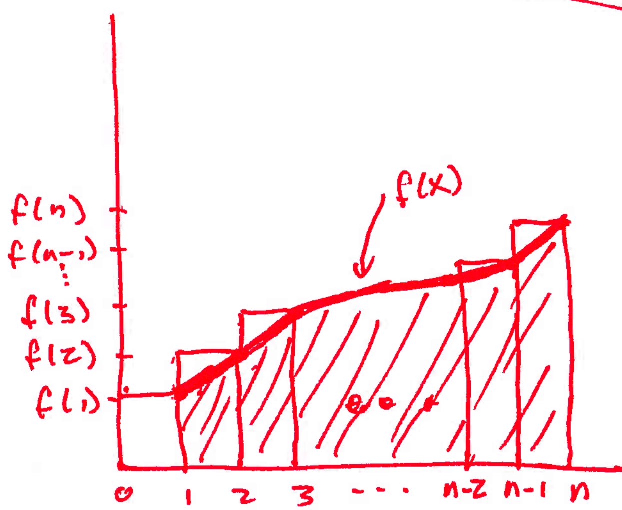


Figure 6-5: The shaded area under the curve of f(x) from 1 to n (shown in bold) is $\mathcal{I} = \int_1^n f(x)\,dx$.

~~We have~~ This is the lower bound for S. We next derive the upper bound.

Figure 6.6 shows the curve of $f(x)$ from 1 to n shifted left by 1. This is the same as the curve $f(x+1)$ from 0 to $n-1$ and it has the same area I.

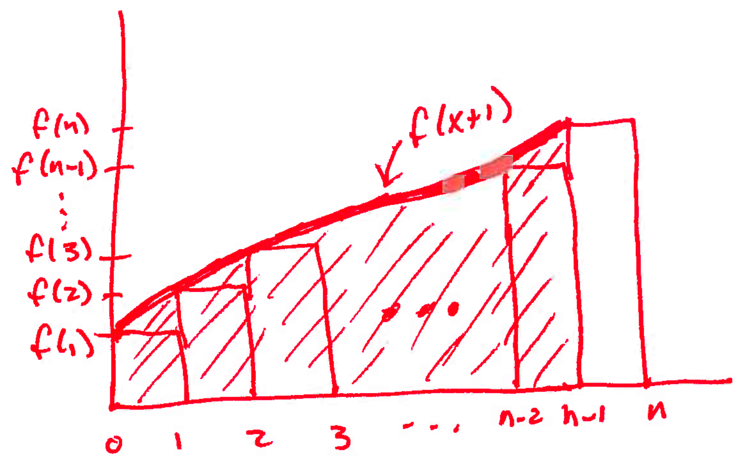


Figure 6.6: The shaded area under the curve of $f(x+1)$ from 0 to $n-1$ is I, the same as the area under the curve of $f(x)$ from 1 to n. This curve is the same as the curve in Figure 6.5 except that it has been shifted left by 1.

Comparing the shaded regions in Figures 6-4 and 6-6, we see that S is at most $\mathcal{I}$ plus the area of the rightmost rectangle. Hence,

$$S \le \mathcal{I} + f(n). \qquad \text{(Eqn 6-8)}$$

Combining Equations 6-7 and 6-8, we find that

$$\mathcal{I} + f(1) \le S \le \mathcal{I} + f(n),$$

~~as claimed~~
for any nondecreasing function f, as claimed.

The argument for the case when f is nonincreasing is very similar. The analogous graphs to those shown in Figures 6-4–6-6 are provided in Figure 6-9. As you can see by comparing the shaded regions in Figures 6-9(a) and 6-9(b),

$$S \le \mathcal{I} + f(1).$$

~~Comparing the~~

Similarly, comparing the shaded regions in Figures 6.9(a) and 6.9(b) reveals that

$$S \geq \mathcal{I} + f(n).$$

Hence, if f is nonincreasing,

$$\mathcal{I} + f(n) \leq S \leq \mathcal{I} + f(1),$$

as claimed. $\square$

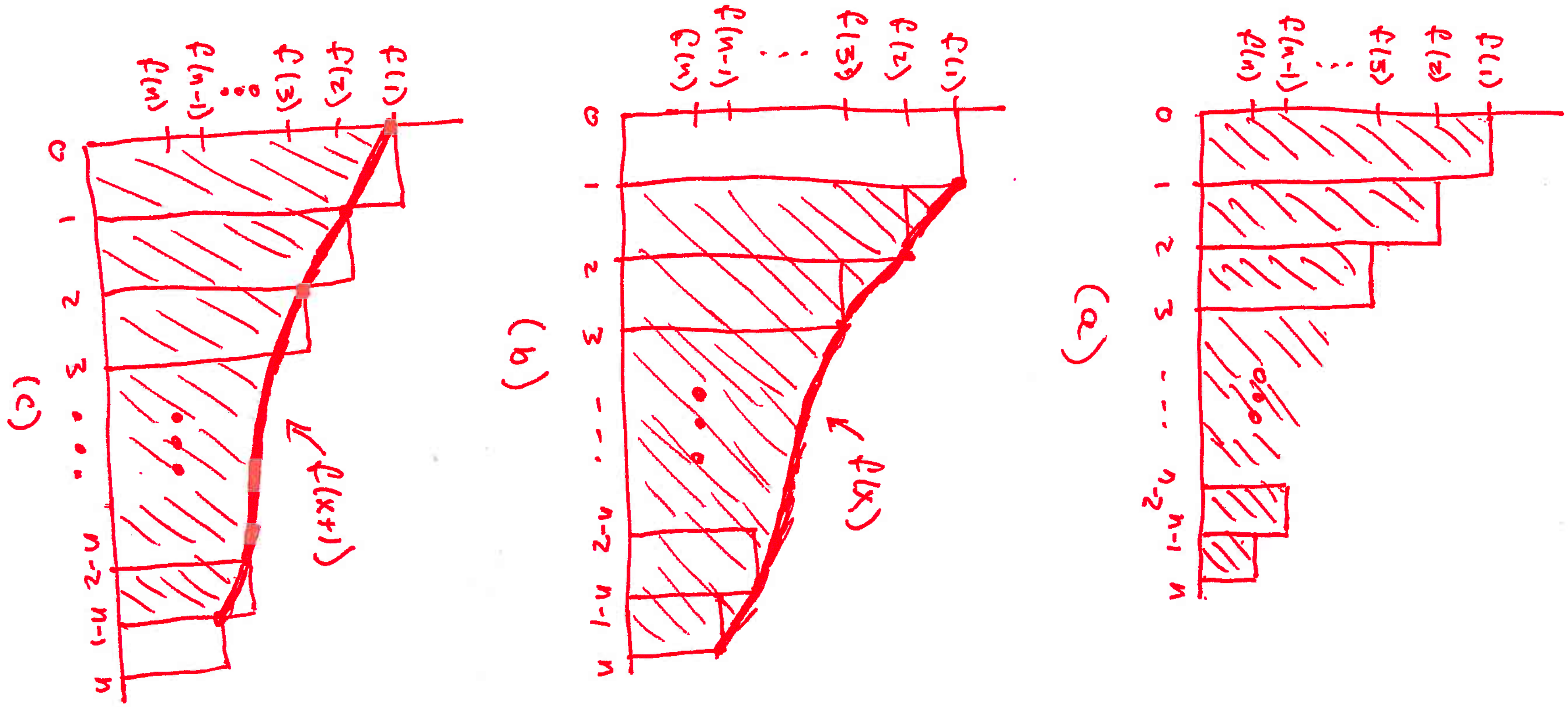


Figure 69: The area of the shaded region in (a) is $S = \sum_{i=1}^{n} f(i)$.
The area in the shaded region in (b) and (c) is $\mathcal{I} = \int_1^n f(x)\,dx$.

Theorem 6.3 provides good bounds for most sums. At worst, the bounds will be off by ~~at m~~ the largest term in the sum. For example, we can use them to bound the sum

$$S = \sum_{i=1}^{n} \sqrt{i}$$

as follows.

We begin by computing

$$\begin{aligned} I &= \int_1^n \sqrt{x}\,dx \\ &= \frac{x^{3/2}}{3/2} \Big|_1^n \\ &= \frac{2}{3}\left(n^{3/2} - 1\right). \end{aligned}$$

We then apply Theorem 6.3 to conclude that

$$\frac{2}{3}\left(n^{3/2}-1\right) + 1 \le S \le \frac{2}{3}\left(n^{3/2}-1\right) + \sqrt{n}$$

and thus that

$$\frac{2}{3}n^{3/2} + \frac{1}{3} \le S \le \frac{2}{3}n^{3/2} + \sqrt{n} - \frac{2}{3}.$$

In other words, the sum is very close to $\frac{2}{3} n^{3/2}$.

We'll be using ~~this~~ Theorem 6.3 extensively going forward. ~~We~~ At the end of this chapter, we will also introduce some notation that expresses phrases like "the sum is very close to" in a more precise mathematical manner.

But first, we'll see how Theorem 6.3 can be used to ~~solve~~ resolve a classic paradox in ~~physics~~ structural engineering.

1 (footnote) We will assume that the blocks are rectangular, uniformly weighted, ~~INSERT B~~ and of length 1. 6-11

9.4 ~~How Far~~ Hanging Out Over the Edge

Suppose that you have n identical ~~uniformly weighted~~ ~~(with unit of length)~~ blocks[1] and that you stack them one on top of the next ~~as~~ on a table as shown in Figure 6.11. Is there some value of n for which ~~Is it possible to~~ it is possible to arrange the stack so that one of the blocks ~~(say the top block)~~ hangs out completely over the edge of the table ~~(without holding onto the stack holding the stack~~ without having the stack fall over? (You are not allowed to use glue or to otherwise hold the stack in position.)

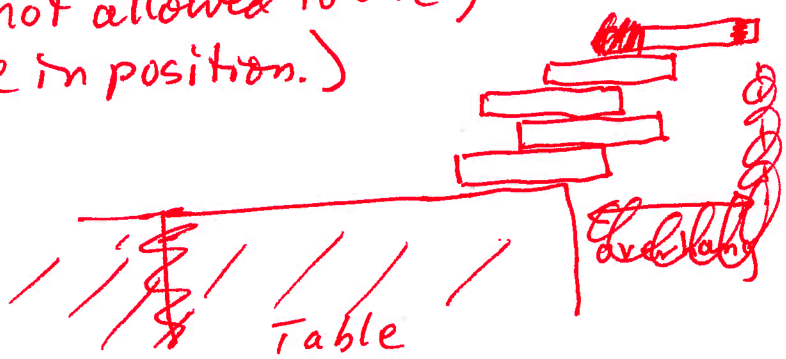


Figure 6.11: A stack of 5 identical blocks on a table. The top block is hanging out over the edge of the table, but if you try stacking the blocks this way, the stack would fall over.

11.3 How far ~~is too far?~~ can you go?

~~11.2 Book Stacking~~

collection of identical ~~better~~ blocks that ~~to~~ blocks that

Suppose you have ~~a pile of books~~ and you want to stack them on a table ~~in some off-center way~~ so the top book sticks out past books below it. How far past the edge of the table do you think you could get the top book to go without having the stack fall over? Could the top book stick out completely beyond the edge of table?

Most people's first response to this question—sometimes also their second and third responses—is "No, the top ~~book~~ block will never get completely past the edge of the table." But in fact, if n is large enough, you can get the top ~~book~~ block to stick out as far as you want: one ~~book~~ blocklength, two ~~book~~ blocklengths, any number of booklengths!

— INSERT H goes here —

9.~~3~~4.1 Stability

A stack of ~~books~~ blocks is said to be <u>stable</u> if it will not fall over of its own accord. For example, the stack illustrated in Figure 6.11 is not stable because the top block is sure to fall over. This is because the center of mass of the top ~~book~~ block is hanging out over air.

In general, a stack of n blocks will be stable if and only if the center of mass of the top i blocks sits over the $(i+1)$st block for $i = 1, 2, \ldots, n-1$, and over the table for $i = n$.

~~Let's~~ We define the <u>overhang</u> ~~of a~~ of a stable stack to be the distance between the edge of the table and the rightmost end of the rightmost block in the stack.[1] ~~For example, we have illustrated the overhang~~ Our goal is thus to maximize the overhang of a stable stack.

---

[1] For simplicity, we have assumed ~~every~~ that every block has unit length so that the overhang will be a real number of block lengths.

For example, the maximum possible overhang for a single block is $1/2$. That is because the center of mass of ~~the~~ a single block is in the middle of the block (which is distance $1/2$ from the right edge of the block). If we were to place the block so that it's right edge is more than $1/2$ from the edge of the table, the center of mass would be over air and the block would tip over. But we can place the block so the center of mass is at the edge of the table, thereby achieving overhang $1/2$. This position is illustrated in Figure 11.1

9.3.1

11.2.1 Formalizing the Problem

We'll approach this problem recursively. How far past the end of the table can we get one book to stick out? It won't tip as long as its center of mass is over the table, so we can get it to stick out half its length, as shown in Figure 11.1.

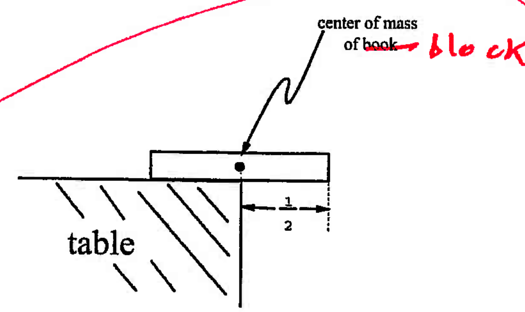


Figure 11.1: One ~~book~~ block can overhang half a ~~book~~ block length.

Now suppose that we have a stack of ~~books~~ blocks that will stick out past the table edge without tipping over—call that a *stable stack*. Let's define the ~~*overhang*~~ spread of a stable stack to be the largest horizontal distance from the center of mass of the stack to the furthest edge of

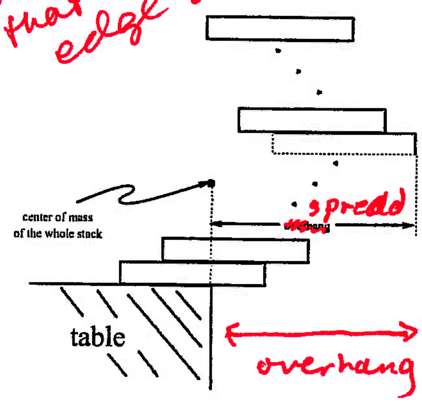
— INSERT I goes here —

INSERT I

~~In general~~

In general, the overhang of a stack of blocks is maximized ~~by~~ by sliding the entire stack rightward until its center of mass is at the edge of the table. The overhang will then be equal to the ~~amount by which~~ distance between the center of mass of the stack and the rightmost edge of the rightmost block. We call this distance the *spread* of the stack. Note that the spread does not depend on the location of the stack on the table — it is purely a property of the blocks in the stack. Of course, as we just observed, the maximum possible overhang is equal to the maximum possible spread. This relationship is illustrated in Figure 11.2

The overhang is maximized by maximizing the spread and then placing the stack so that the center of mass is at the edge of the table.



~~Example~~

Figure 11.2: ~~Overhanging the edge of the table.~~

block. note that the definition of spread does not depend in any way on the location of the stack with respect to the table. But

a ~~book~~ block ~~If we place the center of mass of the stable stack at the edge of the table as in~~

~~Figure 11.2, that's how far we can get a book in the stack to stick out past the edge.~~

~~So we want a formula for the maximum possible overhang, B_n, achievable with a stack of n books.~~

~~INSERT I goes here~~

~~We've already observed that the overhang of one book is 1/2 a book length. That is,~~

$$\cancel{B_1 = \frac{1}{2}.}$$

J

— INSERT J goes here —

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~~9.3.2 A Recursive Solution~~ 9.4.2 A Recursive Solution

Our goal is to find a formula for the maximum possible spread, S_n, that is achievable with a stable stack of n blocks. ~~For example:~~

We already know that $S_1 = 1/2$ since the right edge of a single block with length 1 is always 1/2 from its center of mass. Let's see if we can use a recursive approach to determine S_n for all n. This means that we need to find a formula for S_n in terms of S_i where $i < n$.

Suppose we have a stable stack of n blocks with maximum possible spread S_n. There

~~Then it~~ ~~Reasoning as before, it must be that the~~

~~Then the center of mass of the top n blocks must be at the rightmost edge of the $(n+1)$st block.~~

~~otherwise~~

are two cases to consider depending on where the rightmost block is in the stack.

Case 1: The rightmost block in S is the bottom block.

Since the center of mass of the top $n-1$ blocks must be over the bottom block for stability, the spread is maximized by having the center of mass of the top $n-1$ blocks be ~~over~~ directly over the left edge of the bottom block. ~~For example~~ In this case ~~see Fi~~ the center of mass of ~~the~~ S is[1]

$$\frac{(n-1)\cdot 1 + 1\cdot\frac{1}{2}}{n} = 1 - \frac{1}{2n}$$

to the left of the right edge of the bottom block and so the spread for S is

$$1 - \frac{1}{2n}. \qquad \text{(eqn 6.15)}$$

For example, see Figure 6.14.

---

1 The center of mass of ~~all~~ a stack of blocks is the average of the centers of mass of the individual blocks. ~~In this calculation, we have n~~

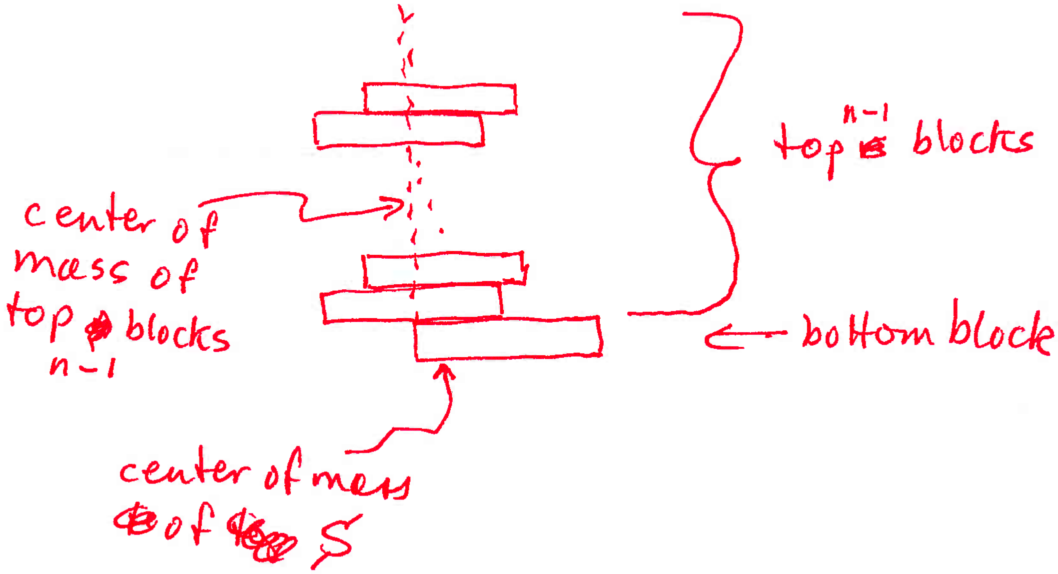


Figure 6-14: The scenario when the ~~If the right~~ bottom block is the rightmost block. In this case, the spread is maximized by having the center of mass of the top $n-1$ blocks be directly over the left edge of the bottom block.

In fact, ~~this~~ the scenario just described ~~is achievable~~ is easily achieved by arranging the blocks as shown in Figure 6.15, in which case we ~~achieve have the spreads shown in Figure 15b.~~ have the spreads given by Equation 6.15. For example the spread is 3/4 for 2 blocks, 5/6 for 3 blocks, 7/8 for 4 blocks, etc.

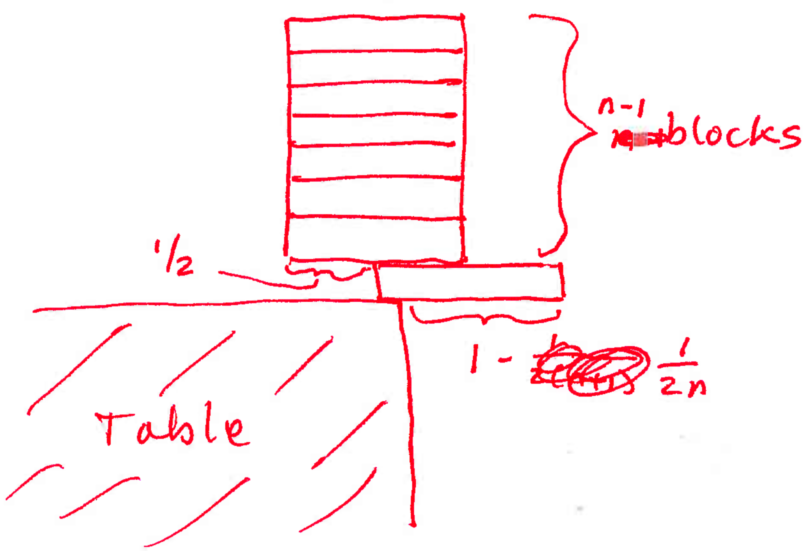


Figure 6.15 : A method for achieving spread (and hence overhang) $1-\frac{1}{2n}$ with n blocks, where the ~~right~~ bottom block is the rightmost block.

J-S

b) Can we do any better? The ~~sp~~ best ~~If not, then it will not be possible the get a full block~~ spread in case 1 is always less than 1, which means that we cannot get a block fully out over the table in this scenario. Maybe our intuition ~~that~~ was right that we can't do better. Before we jump to any false conclusions, however, lets see ~~Let's see,~~ what happens in the other case.

Case 2: The rightmost block in S is among the top $n-1$ blocks. In this case, the spread is maximized by placing the top $n-1$ blocks so that their center of mass is directly over the right end of the bottom block. ~~otherwise, moreover, moreover, the top n blocks should be arranged in a way that maximizes~~ ~~the top n blocks should be arranged in a way that maximizes their spread. Hence the center of mass for~~

This means that the center of mass for S is

$$\frac{(n-1)\cdot C + 1\cdot\left(C-\frac{1}{2}\right)}{n} = C-\frac{1}{2n}$$

~~David: Capital C here~~

where C is the center of mass of the top $n-1$ blocks. In other words, ~~the center of bottom block of S~~ the center of mass of S is $\frac{1}{2n}$ ~~shifts~~ to the left of the center of mass of the top $n-1$ blocks. The difference is due to the effect of the bottom block, whose center of mass is $1/2$ unit to the left of ~~the center~~ C, ~~by the position~~. This means that the spread of S is $\frac{1}{2n}$ greater than the spread of the top $n-1$ blocks (because we are in the case where the rightmost block is among the top $n-1$ blocks).

If the rightmost block is among the top $n-1$ blocks, ~~Since~~ since ~~In order to maximize~~ the spread for S, is maximized by maximizing ~~it must be that~~ the spread ~~is also maximized~~ among the top $n-1$ blocks. Hence the maximum spread for S in this case is

$$S_{n-1} + \frac{1}{2n} \qquad \text{(Eqn 6.16)}$$

where S_{n-1} is the maximum possible spread for $n-1$ blocks ~~Together~~ (using any strategy).

We are now almost done. There are only two cases to consider when designing a stack with maximum spread and we have analyzed both of them. This means that we can combine Equation 6.15 from Case 1 with Equation 6.16 from Case 2 to conclude that

$$S_n = \max\left\{1 - \frac{1}{2n},\ S_{n-1} + \frac{1}{2n}\right\} \quad \text{(Eqn 6.17)}$$

for any $n > 1$.

Uh-oh. This looks complicated. Maybe we are not almost done after all!

~~Combining~~ ~~There are only two cases to~~ ~~consider and by taking the one with~~ ~~@ Case~~ ~~the most spread for any n, we can see~~

~~We are now almost done. Combining Equation 617 from Case 1 with Equation 616 from Case 2, we find that for any $n \geq 1$,~~

$$\delta_{n+1} = \max\left\{1-\frac{1}{2(n+1)},\ \delta_n + \frac{1}{2(n+1)}\right\} \qquad \text{(Eqn 617)}$$

~~where~~ for any $n \geq 1$.

~~Whoa. This looks complicated. Maybe we are not almost done after all...~~

Equation 617 is an example of a <u>recurrence</u>. We will describe numerous techniques for solving recurrences in Chapter 10, but ~~for now~~, fortunately, Equation 617 is simple enough that we can solve it ~~with bra~~ without waiting for all the hardware in Chapter 10.

One of the first things to do when you have a recurrence is to get a feel for it by ~~looki~~ computing the first few terms. This often gives ~~a~~ clues about a way to solve the recurrence, as it will in this case.

We already know that $S_1 = 1/2$. What about S_2? From Equation 6.17, we find that

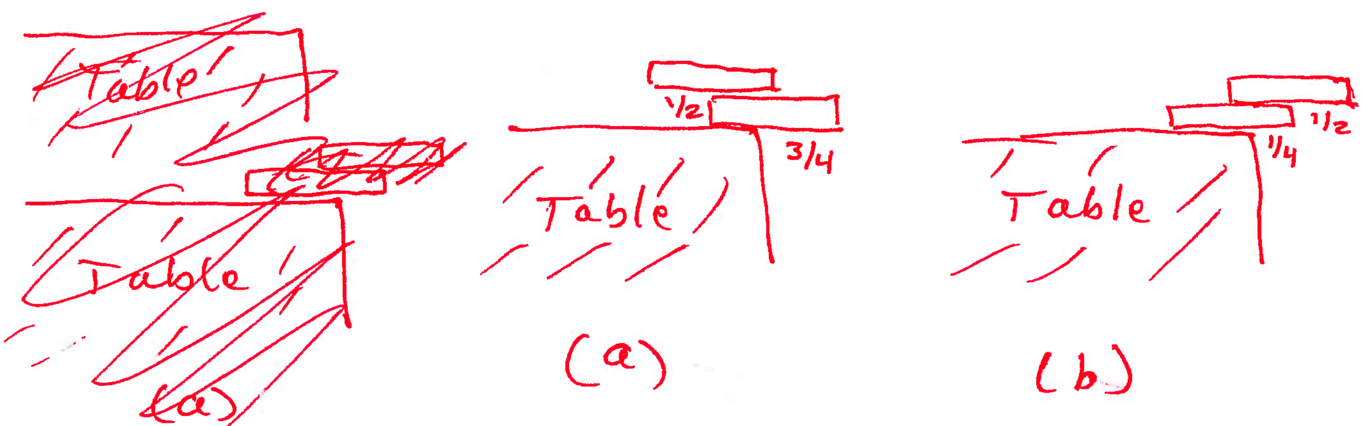
$$S_2 = \max\left\{1 - \frac{1}{4}, \frac{1}{2} + \frac{1}{4}\right\}$$

$$= 3/4.$$

Both cases give the same ~~result~~ spread, albeit by different ~~a~~ ~~That was easy enough. Now what about~~ ~~S_3?~~ approaches. For example, see Figure 6.20.

$$\cancel{S_3 = \max\left\{1 - \frac{1}{6}, \frac{3}{4} + \frac{1}{6}\right\}}$$

$$\cancel{= \frac{11}{12}}$$



~~approaches. For example se~~

Figure 6.20: Two ways to achieve spread (and ~~over~~ hence overhang) 3/4 with n=2 blocks. The first way (a) is from case 1 and the second (b) is from case 2.

\# That was easy enough. What ~~a~~ about S_3?

$$S_3 = \max\left\{1 - 1/6,\ \frac{3}{4} + \frac{1}{6}\right\}$$

$$= \max\left\{5/6,\ 11/12\right\}.$$

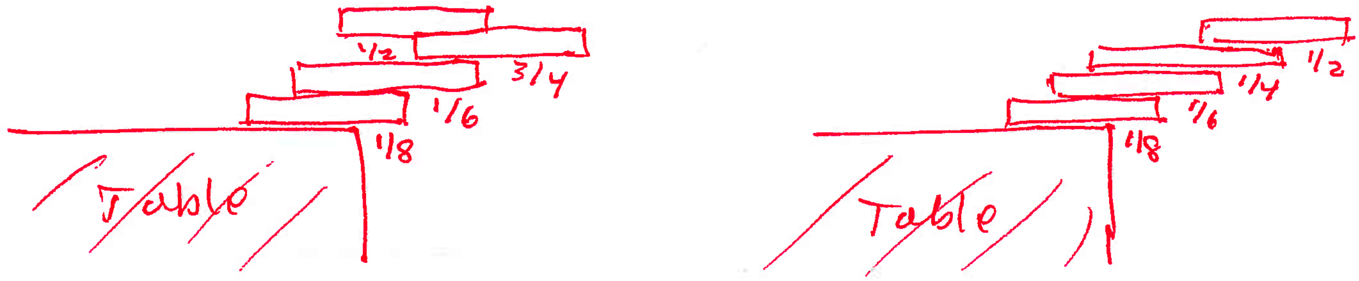
$$= 11/12$$

As we can see, ~~For n=3,~~ ~~In this case~~ the method provided by case 2 is the best. Let's check $n=4$.

$$S_4 = \max\left\{1 - 1/8,\ \frac{11}{12} + \frac{1}{8}\right\}$$

$$= \frac{25}{24} \text{ Q.E.} \qquad \text{(Eqn 6.21)}$$

Wow! This is a breakthrough — for two reasons. First, Equation 6.21 tells us that ~~we can get a book out~~ block ~~totally out~~ over ~~the edge of the table with only~~ 4 blocks, we can make a stack by using only 4 blocks, we can make a stack so that one of the blocks is hanging out completely over the edge of the table. The two ways to do this are ~~illustrat~~ shown in Figure 6.22.



(a) (b)

Figure 6-21: The two ways to achieve spread (and overhang) $\frac{25}{24}$. ~~Th~~ The method in (a) uses case 1 for the ~~block next to the~~ top 2 blocks and ~~both~~ case 2 for the others. The method in (b) uses case 2 for every block that is added to the stack.

The second reason that Equation 6.21 is important is that ~~once~~ we now ~~that we~~ know that $S_4 > 1$, ~~we no~~ which means that we no longer have to worry about case 1 for ~~larger values of n > 4~~ ~~In fact, since the spread using~~ $n > 4$ since case ~~one~~ 1 never ~~gets~~ achieves spread greater than 1. Moreover, ~~since the spread~~ even for $n \le 4$, we ~~obt ach~~ have now seen that the spread achieved by case 1 never exceeds the spread achieved by case 2. ~~(It can equal)~~ (And they can be equal only ~~they can be equal~~ for $n = 1$ and $n = 2$.)

This means that

$$S_n = S_{n-1} + \frac{1}{2n} \qquad \text{(Eqn 6.23)}$$

$$\cancel{S_{n+1} = S_n + \frac{1}{2(n+1)}}$$

for all $n > 1$, since ~~the~~ we have shown that the best spread can always be achieved using case 2.

The recurrence ~~is now ne~~ in Equation 6.23 is much easier to solve ~~is much easier to solve~~ ~~is easy to solve by just plugging in~~ than the one we started with in Equation 6.17. We can solve it ~~recursively~~ by expanding the equation as follows

$$\cancel{S_{n+1} = S_n + \frac{1}{2(n+1)}}$$

$$\cancel{= S_{n-1} + \frac{1}{2n} + \frac{1}{2(n+1)}}$$

$$\cancel{= S_{n-2} + \frac{1}{2(n-1)} + \frac{1}{2n} + \frac{1}{2(n+1)}}$$

$$S_n = S_{n-1} + \frac{1}{2n}$$

$$= S_{n-2} + \frac{1}{2(n-1)} + \frac{1}{2n}$$

$$= S_{n-3} + \frac{1}{2(n-2)} + \frac{1}{2(n-1)} + \frac{1}{2n}$$

and so on. This suggests that

$$S_n = \sum_{i=1}^{n} \frac{1}{2i} \qquad \text{(Eqn 6-24)}$$

~~In fact,~~ which is, indeed, the case. # Equation 6-24 can be verified by induction. The base case when $n=1$ is true since we know that $S_1 = \frac{1}{2}$. The inductive step follows from Equation 6-23.

~~So we're done~~

So we now know the maximum possible spread and hence the maximum possible overhang for any stable stack of books. ~~still~~ Are we done? Not ~~Quit~~ quite. ~~we still~~ Although we know that $S_4 > 1$, we still don't know how big the sum $\sum_{i=1}^{n} \frac{1}{2i}$ can get.

It turns out that S_n is very close to a famous sum known as the nth Harmonic number H_n.

9.4.3 Harmonic Numbers

~~9.3.2 A Recursive Solution~~

① ~~We will now use recursion to find a formula for B_n.~~

~~Now~~ S~~s~~uppose that we have a stable stack of $n + 1$ ~~books~~ blocks with maximum overhang. If the overhang of the n books on top of the bottom book was not maximum, we could get a book to stick out further by replacing the top stack with a stack of n books with larger overhang. So the maximum overhang, B_{n+1}, of a stack of $n + 1$ books is obtained by placing a maximum overhang stable stack of n books on top of the bottom book. And we get the biggest overhang for the stack of $n + 1$ books by placing the center of mass ~~of the n books right over the edge of the bottom book as in Figure 11.3.~~

So we know where to place the $n + 1$st book to get maximum overhang, and all we have to do is calculate what it is. The simplest way to do that is to let the center of mass of the top n books be the origin. That way the horizontal coordinate of the center of mass of the whole stack of $n + 1$ books will equal the increase in the overhang. But now the center of mass of the bottom book has horizontal coordinate $1/2$, so the horizontal

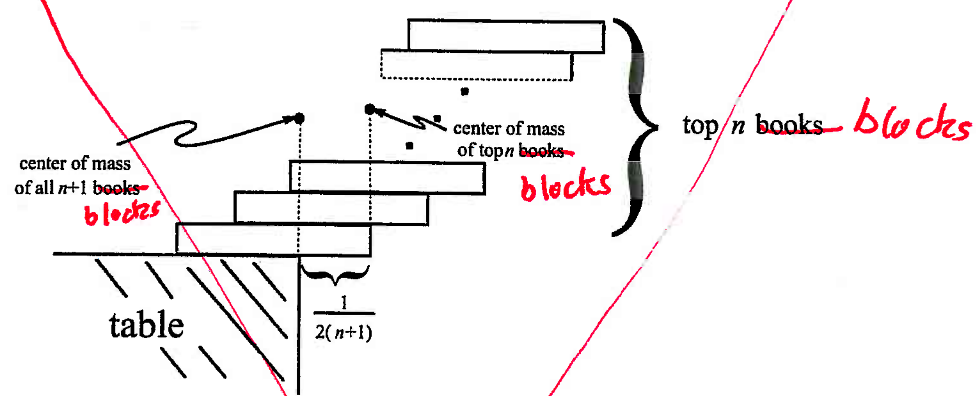


Figure 11.3: Additional ~~overhang with $n+1$ books~~. spread (and overhang) by adding an (n+1)st block.

coordinate of center of mass of the whole stack of $n+1$ books is

$$\frac{0 \cdot n + (1/2) \cdot 1}{n+1} = \frac{1}{2(n+1)}.$$

In other words,

$$B_{n+1} = B_n + \frac{1}{2(n+1)}, \tag{11.11}$$

as shown in Figure 11.3.

~~Expanding equation (11.11), we have~~

$$\begin{aligned}
B_{n+1} &= B_{n-1} + \frac{1}{2n} + \frac{1}{2(n+1)} \\
&= B_1 + \frac{1}{2\cdot 2} + \cdots + \frac{1}{2n} + \frac{1}{2(n+1)} \\
&= \frac{1}{2}\sum_{i=1}^{n+1}\frac{1}{i}. \qquad (11.12)
\end{aligned}$$

~~The nth *Harmonic number*, H_n, is defined to be~~

Definition 11.2.1. The nth Harmonic number is (Eqn 6.25)

$$H_n ::= \sum_{i=1}^{n}\frac{1}{i}.$$

So Equation 6.24 ~~(11.12)~~ means that

$$S_n \; \cancel{B_n} = \frac{H_n}{2}.$$

There is good news and bad news about Harmonic numbers. The bad news is that

The first few Harmonic numbers are easy to compute. For example, $H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$. ~~The fact that H_4 is greater than 2 has special significance; it implies that the total extension of a 4-book stack is greater than one full book! This is the situation~~

~~Unfortunately,~~ there is no closed form expression known for the Harmonic numbers.

The good news is that we can use Theorem 6.3 to get close upper and lower bounds on H_n. In particular, since

&

INSERT K goes here

$$\int_1^n \frac{1}{x}\,dx = \ln(x)\Big|_1^n$$

$$= \ln(n),$$

Theorem 6.3 means that

$$\ln(n) + 1 \le H_n \le \ln(n) + \frac{1}{n}. \qquad \text{(Eqn 6.30)}$$

In other words, the nth Harmonic number is very close to ln n. ~~This means that~~ ~~Because B~~

~~S_n is within~~

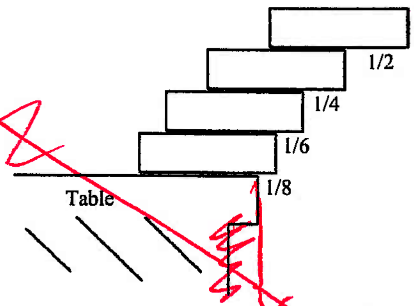
$$\cancel{\frac{\ln(n)}{2} + \frac{1}{2} \le S_n \le \frac{\ln(n)}{2} + \frac{1}{2n}}.$$

Because the Harmonic numbers frequently arise in practice, mathematicians have worked hard to get even better approximations for them. In fact, it is now known that

— INSERT L goes here —
(text on pp 716-717)

We are now finally done with our analysis of the block stacking problem. Plugging the value of H_n into Equation 6.25, we find that the maximum overhang for n blocks is ~~very close to $\frac{1}{2}\ln(n)$~~ ~~within~~ ~~$\frac{1}{2}$ of~~ ~~$S_n = \frac{1}{2}\ln(n)$~~ ~~$\frac{1}{2}$ of $\frac{1}{2}\ln(n)$~~ very close to $\frac{1}{2}\ln(n)$. ~~This mea~~ Since $\ln(n)$ grows to infinity as n increases, this means that ~~we~~, if we are given enough ~~books~~ blocks (in theory anyway), we can get a ~~bo~~ block to hang out arbitrarily far over the edge of the table. ~~For example, with~~ Of course, the number of blocks we need will grow as an exponentially function of the overhang, so it ~~c~~ ~~m~~ will probably take you a long time to achieve an overhang of 2 or 3, never mind an overhang of 100.

shown in Figure 11.4.



David: this is close to Fig 6.21 (b).

Figure 11.4: Stack of four books with maximum overhang.

11.2.2 Evaluating the Sum—The Integral Method

It would be nice to answer questions like, "How many books are needed to build a stack extending 100 book lengths beyond the table?" One approach to this question would be to keep computing Harmonic numbers until we found one exceeding 200. However, as we will see, this is not such a keen idea.

Such questions would be settled if we could express H_n in a closed form. Unfortunately, no closed form is known, and probably none exists. As a second best, however, we can find closed forms for very good approximations to H_n using the Integral Method. The idea of the Integral Method is to bound terms of the sum above and below by simple functions as suggested in Figure 11.5. The integrals of these functions then bound the value of the sum above and below.

The Integral Method gives the following upper and lower bounds on the harmonic number H_n:

$$H_n \leq 1 + \int_1^n \frac{1}{x}\,dx = 1 + \ln n \tag{11.13}$$

$$H_n \geq \int_0^n \frac{1}{x+1}\,dx = \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1). \tag{11.14}$$

These bounds imply that the harmonic number H_n is around $\ln n$.

But $\ln n$ grows —slowly —but without bound. That means we can get books to overhang *any distance* past the edge of the table by piling them high enough! For example,

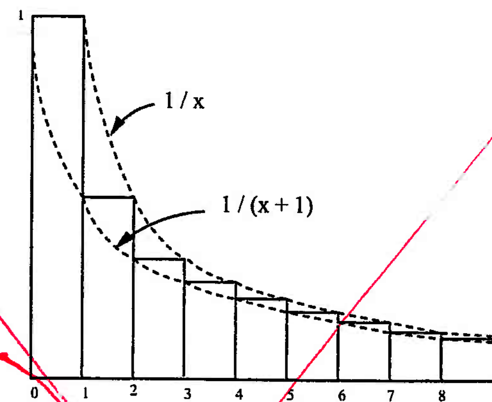


Figure 11.5: This figure illustrates the Integral Method for bounding a sum. The area under the "stairstep" curve over the interval $[0, n]$ is equal to $H_n = \sum_{i=1}^{n} 1/i$. The function $1/x$ is everywhere greater than or equal to the stairstep and so the integral of $1/x$ over this interval is an upper bound on the sum. Similarly, $1/(x+1)$ is everywhere less than or equal to the stairstep and so the integral of $1/(x+1)$ is a lower bound on the sum.

to build a stack extending three book lengths beyond the table, we need a number of books n so that $H_n \geq 6$. By inequality (11.14), this means we want

$$H_n \geq \ln(n+1) \geq 6,$$

so $n \geq e^6 - 1$ books will work, that is, 403 books will be enough to get a three book overhang. Actual calculation of H_6 shows that 227 books is the smallest number that will work.

11.2.3 More about Harmonic Numbers

In the preceding section, we showed that H_n is about $\ln n$. An even better approximation is known:

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& goes in INSERT K

$$H_n = \ln(n) + \gamma + \frac{1}{2n} + \frac{1}{12n^2} + \frac{\epsilon(n)}{120n^4} \quad \text{(eqn K2)}$$

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Here γ is a value $0.577215664\ldots$ called *Euler's constant*, and $\epsilon(n)$ is between 0 and 1 for all n. We will not prove this formula.

Asymptotic Equality

9.4.4 ← make a subsection

~~The shorthand~~ $H_n \sim \ln n$ ~~is used~~ to indicate that the leading term of H_n is $\ln n$. More precisely:

Definition 11.2.2. For functions $f, g : \mathbb{R} \to \mathbb{R}$, we say f is *asymptotically equal* to g, in symbols,

$$f(x) \sim g(x)$$

iff

$$\lim_{x \to \infty} f(x)/g(x) = 1.$$

Although it is

this is

~~It's~~ tempting to ~~might~~ write $H_n \sim \ln n + \gamma$ to indicate the two leading terms, ~~but it is~~

For cases like Equation K2 where we ~~have an approximate value for a value like Hn and where we~~ understand the growth of a function like Hn up to some (unimportant) error terms, we use a special notation, ~, to denote the leading term of the function. For example, we say that

Rest of Insert 2

not really right. According to Definition 11.2.2, $H_n \sim \ln n + c$ where c is *any constant*. The correct way to indicate that γ is the second-largest term is $H_n - \ln n \sim \gamma$.

The reason that the $\sim$ notation is useful is that often we do not care about lower order terms. For example, if $n = 100$, then we can compute $H(n)$ to great precision using only the two leading terms:

$$|H_n - \ln n - \gamma| \leq \left| \frac{1}{200} - \frac{1}{120000} + \frac{1}{120 \cdot 100^4} \right| < \frac{1}{200}.$$

~~11.3 Finding Summation Formulas~~

~~The Integral Method offers a way to derive formulas like~~

~~EDITING NOTE: equation (11.1)~~

~~for the sum of consecutive integers,~~

$$\cancel{\sum_{i=1}^{n} i = n(n+1)/2,}$$

we will spend a lot more time talking about asymptotic notation at the end of the chapter. But for now, lets get back to sums.

or for the sum of squares,

$$\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}$$
$$= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}. \tag{11.15}$$

These equations appeared in Chapter 3.1 as equations (3.1) and (??) where they were proved using the Well-ordering Principle. But those proofs did not explain how someone figured out in the first place that these were the formulas to prove.

Here's how the Integral Method leads to the sum-of-squares formula, for example. First, get a quick estimate of the sum:

$$\int_0^n x^2\,dx \le \sum_{i=1}^{n} i^2 \le \int_0^n (x+1)^2\,dx,$$

so

$$n^3/3 \le \sum_{i=1}^{n} i^2 \le (n+1)^3/3 - 1/3. \tag{11.16}$$

~~and the upper and lower bounds (11.16) imply that~~

$$\sum_{i=1}^{n} i^2 \sim n^3/3.$$

~~To get an exact formula, we then guess the general form of the solution. Where we are uncertain, we can add parameters $a, b, c, \ldots$. For example, we might make the guess:~~

This is INSERT M and goes in to INSERT X

$$\sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d.$$

If the guess is correct, then we can determine the parameters a, b, c, and d by plugging in a few values for n. Each such value gives a linear equation in a, b, c, and d. If we plug in enough values, we may get a linear system with a unique solution. Applying this is insert M

this method to our example gives:

$$n = 0 \quad \text{implies} \quad 0 = d$$

$$n = 1 \quad \text{implies} \quad 1 = a + b + c + d$$

$$n = 2 \quad \text{implies} \quad 5 = 8a + 4b + 2c + d$$

$$n = 3 \quad \text{implies} \quad 14 = 27a + 9b + 3c + d.$$

Solving this system gives the solution $a = 1/3$, $b = 1/2$, $c = 1/6$, $d = 0$. Therefore, *if* our initial guess at the form of the solution was correct, then the summation is equal to $n^3/3 + n^2/2 + n/6$, which matches Equation (~~11.15~~) 6.27.

The point is that if the desired formula turns out to be a polynomial, then once you get an estimate of the *degree* of the polynomial, ~~—by the Integral Method or any other way—~~, all the coefficients of the polynomial can be found automatically.

This is insert M

Be careful! This method let's you discover formulas, but it doesn't guarantee they are right! After obtaining a formula by this method, it's important to go back and *prove* it using induction or some other method, because if the initial guess at the solution was not of the right form, then the resulting formula will be completely wrong!

~~11.3.1 Double Sums~~ 9.5 Double ~~Sums~~ Trouble

Sometimes we have to evaluate sums of sums, otherwise known as *double summations*. ~~This looks hairy, and sometimes it is.~~

This sounds hairy, and sometimes it is. But usually, it is straightforward — you just

~~This can be easy:~~ evaluate the inner sum, replace it with a closed form, and then evaluate

1

the outer sum (which no longer has a summation inside it). For example,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(y^n \sum_{i=0}^{n} x^i\right) \\
&= \sum_{n=0}^{\infty}\left(y^n \frac{1-x^{n+1}}{1-x}\right) && \text{(geometric sum formula (11.3))} \\
&= \frac{\sum_{n=0}^{\infty} y^n}{1-x} - \frac{\sum_{n=0}^{\infty} y^n x^{n+1}}{1-x} \\
&= \frac{1}{(1-y)(1-x)} - \frac{x\sum_{n=0}^{\infty}(xy)^n}{1-x} && \text{(infinite geometric sum, Theorem 11.1.1)} \\
&= \frac{1}{(1-y)(1-x)} - \frac{x}{(1-xy)(1-x)} && \text{(infinite geometric sum, Theorem 11.1.1)} \\
&= \frac{(1-xy)-x(1-y)}{(1-xy)(1-y)(1-x)} \\
&= \frac{1-x}{(1-xy)(1-y)(1-x)} \\
&= \frac{1}{(1-xy)(1-y)}.
\end{aligned}$$

When there's no obvious closed form for the inner sum, a special trick that is often useful is to try *exchanging the order of summation*. For example, suppose we want to

---

1 ok, so maybe this one is a little hairy, but it is also fairly straightforward. Wait 'till you see the next one!

compute the sum of the ~~harmonic~~ first n Harmonic numbers

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n}\sum_{j=1}^{k} 1/j .$$

For intuition about this sum, we ~~can try the integral method~~ can apply Theorem 6.3 to Equation 6.30 to conclude that the sum is close to:

$$\cancel{\sum_{k=1}^{n} H_k \approx} \int_1^n \ln x \, dx \cancel{\approx n \ln n - n.} = x \ln(x) - x \Big|_1^n$$
$$= n \ln(n) - n + 1 .$$

Now let's look for an exact answer. If we think about the pairs (k, j) over which we are summing, they form a triangle:

		j 1	2	3	4	5	…	n
k	1	1						
	2	1	1/2					
	3	1	1/2	1/3				
	4	1	1/2	1/3	1/4			
		…						
	n	1	1/2		…			$1/n$

The summation above is summing each row and then adding the row sums. Instead, we can sum the columns and then add the column sums. Inspecting the table we see that this double sum can be written as

$$\begin{aligned}
\sum_{k=1}^{n} H_k &= \sum_{k=1}^{n} \sum_{j=1}^{k} 1/j \\
&= \sum_{j=1}^{n} \sum_{k=j}^{n} 1/j \\
&= \sum_{j=1}^{n} 1/j \sum_{k=j}^{n} 1 \\
&= \sum_{j=1}^{n} \frac{1}{j}(n-j+1) \\
&\cancel{= \sum_{j=1}^{n} \frac{n-j+1}{j}} \\
&= \sum_{j=1}^{n} \frac{n+1}{j} - \sum_{j=1}^{n} \frac{j}{j} \\
&= (n+1)\sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{n} 1 \\
&= (n+1)H_n - n. \qquad (11.17)
\end{aligned}$$

— INSERT P goes here —

9.6 Products

We've covered several techniques for finding closed forms for sums but ~~none~~ no methods for dealing with products. Fortunately, we do not need to develop ~~a~~ an entirely new set of tools when ~~we~~ we encounter a product such as

$$n! ::= \prod_{i=1}^{n} i. \qquad \text{(Eqn P1)}$$

That's because, we ~~can~~ can convert any product into a sum by taking a logarithm. ~~and then apply~~

For example, if

$$P = \prod_{i=1}^{n} f(i),$$

then

$$\ln(P) = \sum_{i=1}^{n} \ln(f(i)).$$

We can then apply our summing tools to ~~to~~ find a closed form (or approximate closed form) for $\ln(P)$ and then exponentiate at the end to undo the logarithm.

For example, lets see how this works ~~for~~ the factorial ~~produ~~ function $n!$. We start by taking the logarithm:

$\cancel{\ln(n!)}$

9.6 Approximating n!

~~11.4 Stirling's Approximation for n!~~

~~The familiar factorial notation, $n!$, is an abbreviation for the product~~

$$\prod_{i=1}^{n} i.$$

~~This is by far the most common product in discrete mathematics. In this section we describe a good closed-form estimate of $n!$ called *Stirling's Approximation*. Unfortunately, all we can do is estimate: there is no closed form for $n!$ —though proving so would take us beyond the scope of this text.~~

~~11.4.1 Products to Sums~~

~~A good way to handle a product is often to convert it into a sum by taking the logarithm.~~

~~In the case of factorial, this gives~~

$$\ln(n!) = \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n)$$

$$= \ln 1 + \ln 2 + \ln 3 + \cdots + \ln(n-1) + \ln n$$

$$= \sum_{i=1}^{n} \ln i.$$

Unfortunately, we've ~~We've~~ not seen a summation containing a logarithm before, Even worse, ~~Fortunately, one tool that we used in evaluating sums is still applicable: the Integral Method. We can bound the terms of this sum with $\ln x$ and $\ln(x+1)$ as shown in Figure 11.6. This gives bounds on~~

— INSERT a goes here —

INSERT Q

no closed form for this sum is known. However, we can apply Theorem 6.3 to find good closed-form bounds on the sum. To do this, we first compute

$$\int_1^n \ln(x)\,dx = \left(x\ln(x) - x\right)\Big|_1^n$$

$$= n\ln(n) - n + 1.$$

Plugging ~~Using~~ into Theorem 6.3, this means that

$$n\ln(n) - n + 1 \le \sum_{i=1}^{n} \ln(i) \le n\ln(n) - n + 1 + \ln(n).$$

Exponentiation then gives

$$\frac{n^n}{e^{n-1}} \le n! \le \frac{n^{n+1}}{e^{n-1}}. \qquad \text{(eqn Q1)}$$

This means that $n!$ is within a factor of n of n^n/e^{n-1}.

9.6.1 Stirling's Formula

~~The~~ $n!$ is probably the most commonly used product in discrete mathematics, and so mathematicians have put in the effort to find ~~very close~~ ~~much closer~~ ~~app~~ much better closed form bounds on its value. Perhaps the most useful bounds are ~~those known as Stirling's Approximation~~ given in Theorem Q2.

Theorem Q2. ~~(Stirling's Approximation)~~ For all $n \geq 1$,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/(12n+1)} \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}.$$

Theorem Q2 can be proved by induction on n but the details are ~~nasty~~ a bit painful (even for us) ~~so we won't do~~ and so we will not ~~do~~ go through them here.

One important thing to notice about Theorem Q2 is that both $e^{1/(12n+1)}$ and $e^{1/12n}$ tend to 1 as n ~~go~~ goes to infinity. This means that[1]

~~n! tends to~~

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \qquad \text{~~Equation~~ (eqn Q4)}$$

~~as n tends to infinity~~

1. The $\sim$ notation was defined in ~~sec~~ subsection 9.4.4.

Equation Q4 is known as <u>Stirling's Formula</u> and is widely used in discrete mathematics. It is a rather surprising result. After all, who would ~~who would have~~ expected that both π and e would show up ~~to~~ in ~~the~~ the closed form expression that is asymptotically equal to $n!$?

Stirling's Formula gives a very good approximation to $n!$ For example, if $n = 100$, we know from Theorem Q2 that

$\ln(n!)$ as follows:

$$
\begin{aligned}
\int_1^n \ln x \, dx \leq &\ \textstyle\sum_{i=1}^n \ln i &&\leq \int_0^n \ln(x+1)\, dx \\
n \ln(\frac{n}{e}) + 1 \leq &\ \textstyle\sum_{i=1}^n \ln i &&\leq (n+1)\ln\left(\frac{n+1}{e}\right) + 1 \\
\left(\frac{n}{e}\right)^n e \leq &\ n! &&\leq \left(\frac{n+1}{e}\right)^{n+1} e.
\end{aligned}
$$

The second line follows from the first by completing the integrations. The third line is obtained by exponentiating.

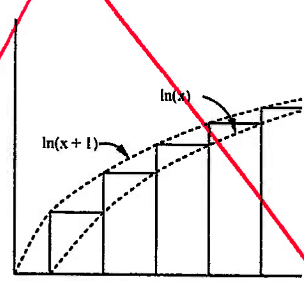

Figure 11.6: This figure illustrates the Integral Method for bounding the sum $\sum_{i=1}^n ln\ i$.

So $n!$ behaves something like the closed form formula $(n/e)^n$. A more careful analysis yields an unexpected closed form formula that is asymptotically exact:

Lemma (Stirling's Formula).

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}, \tag{11.18}$$

Stirling's Formula describes how $n!$ behaves in the limit, but to use it effectively, we need to know how close it is to the limit for different values of n. That information is given by the bounding formulas:

Fact (Stirling's Approximation).

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/(12n+1)} \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}.$$

Stirling's Approximation implies the asymptotic formula (11.18), since $e^{1/(12n+1)}$ and $e^{1/12n}$ both approach 1 as n grows large. These inequalities can be verified by induction, but the details are nasty.

~~The bounds in Stirling's formula are very tight. For example, if $n = 100$, then Stirling's bounds are:~~

$$\cancel{100! \geq} \sqrt{200\pi}\left(\frac{100}{e}\right)^{100} e^{1/1201} \leq$$

~~and~~

$$\cancel{100! \leq} \sqrt{200\pi}\left(\frac{100}{e}\right)^{100} e^{1/1200}.$$

The only difference between the upper bound and the lower bound is in the final term. In particular $e^{1/1201} \approx 1.00083299$ and $e^{1/1200} \approx 1.00083368$. As a result, the upper bound is no more than $1 + 10^{-6}$ times the lower bound. This is amazingly tight! Remember Stirling's formula; we will use it often.

~~EDITING NOTE:~~

Bounds by Double Summing

Another way to derive Stirling's approximation is to remember that $\ln n$ is roughly the same as H_n. This lets us use the result we derived before for $\sum H_k$ via double summation. Our approximation for H_k told us that $\ln(k+1) \leq H_k \leq 1 + \ln k$. Rewriting, we find that $H_k - 1 \leq \ln k \leq H_{k-1}$. It follows that (leaving out the $i = 1$ term in the sum, which contributes 0),

$$\begin{aligned}
\sum_{i=2}^{n} \ln i &\leq \sum_{i=2}^{n} H_{i-1} \\
&= \sum_{i=1}^{n-1} H_i \\
&= nH_{n-1} - (n-1) && \text{by (11.17)} \\
&\leq n(1 + \ln(n-1)) - (n-1) && \text{by (11.13)} \\
&= n\ln(n-1) + 1,
\end{aligned}$$

~~roughly the same bound as we proved before via the integral method. We can derive a similar lower bound.~~

■

9.7

~~11.5~~ Asymptotic Notation

Asymptotic notation is a shorthand used to give a quick measure of the behavior of a function $f(n)$ as n grows large.

9.7.1

~~11.5.1~~ Little Oh

For example, the ~~The~~ asymptotic notation, $\sim$, of Definition 11.2.2 is a binary relation indicating that two functions grow at the *same* rate. There is also a binary relation indicating that one function grows at a significantly *slower* rate than another. ~~Namely,~~

Definition 11.5.1. For functions $f, g : \mathbb{R} \to \mathbb{R}$, with g nonnegative, we say f is *asymptot-*

ically smaller than g, in symbols,

$$f(x) = o(g(x)),$$

iff

$$\lim_{x \to \infty} f(x)/g(x) = 0.$$

For example, $1000x^{1.9} = o(x^2)$, because $1000x^{1.9}/x^2 = 1000/x^{0.1}$ and since $x^{0.1}$ goes to infinity with x and 1000 is constant, we have $\lim_{x \to \infty} 1000x^{1.9}/x^2 = 0$. This argument generalizes directly to yield

~~Proposition~~

OK as was **~~Lemma~~ 11.5.2.** *$x^a = o(x^b)$ for all nonnegative constants $a < b$.*

Using the familiar fact that $\log x < x$ for all $x > 1$, we can prove

~~Proposition~~

OK as was **~~Lemma~~ 11.5.3.** $\log x = o(x^\epsilon)$ *for all* $\epsilon > 0$ ~~*and* $x > 1$~~.

Proof. Choose $\epsilon > \delta > 0$ and let $x = z^\delta$ in the inequality $\log x < x$. This implies

$$\log z < z^\delta/\delta = o(z^\epsilon) \qquad \text{by Lemma 11.5.2.} \tag{11.19}$$

■

Corollary 11.5.4. $x^b = o(a^x)$ *for any* $a, b \in \mathbb{R}$ *with* $a > 1$.

Proof. From (11.19),

$$\log z < z^\delta/\delta$$

for all $z > 1$, $\delta > 0$. Hence

$$\begin{aligned}
(e^b)^{\log z} &< (e^b)^{z^\delta/\delta} \\
z^b &< \left(e^{\log a (b/\log a)}\right)^{z^\delta/\delta} \\
&= a^{(b/\delta \log a) z^\delta} \\
&< a^z
\end{aligned}$$

for all z such that

$$(b/\delta \log a) z^\delta < z.$$

~~But choosing $\delta < 1$, we know $z^\delta = o(z)$, so this last inequality holds for all large enough z.~~ ■

Lemma 11.5.3 and Corollary 11.5.4 can also be proved ~~easily in several other ways, for example,~~ using L'Hopital's Rule or the McLaurin Series for $\log x$ and e^x. Proofs can be found in most calculus texts.

9.7.2

~~11.5.2~~ Big Oh

Big Oh is the most frequently used asymptotic notation. It is used to give an upper bound on the growth of a function, such as the running time of an algorithm.

Definition 11.5.5. Given nonnegative functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that

$$f = O(g)$$

iff

$$\limsup_{x \to \infty} f(x)/g(x) < \infty.$$

This definition[3] makes it clear that

~~Proposition~~

OK as was **~~Lemma~~ 11.5.6.** *If $f = o(g)$ or $f \sim g$, then $f = O(g)$.*

Proof. $\lim f/g = 0$ or $\lim f/g = 1$ implies $\lim f/g < \infty$. ∎

It is easy to see that the converse of Lemma 11.5.6 is not true. For example, $2x = O(x)$, but $2x \not\sim x$ and $2x \neq o(x)$.

[3]We can't simply use the limit as $x \to \infty$ in the definition of $O()$, because if $f(x)/g(x)$ oscillates between, say, 3 and 5 as x grows, then $f = O(g)$ because $f \leq 5g$, but $\lim_{x \to \infty} f(x)/g(x)$ does not exist. So instead of limit, we use the technical notion of lim sup. In this oscillating case, $\limsup_{x \to \infty} f(x)/g(x) = 5$.

The precise definition of lim sup is

$$\limsup_{x \to \infty} h(x) ::= \lim_{x \to \infty} \mathrm{lub}_{y \geq x} h(y),$$

where "lub" abbreviates "least upper bound."

The usual formulation of Big Oh spells out the definition of lim sup without mentioning it. Namely, here is an equivalent definition:

Definition 11.5.7. Given functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that

$$f = O(g)$$

iff there exists a constant $c \geq 0$ and an x_0 such that for all $x \geq x_0$, $|f(x)| \leq cg(x)$.

This definition is rather complicated, but the idea is simple: $f(x) = O(g(x))$ means $f(x)$ is less than or equal to $g(x)$, except that we're willing to ignore a constant factor, namely, c, and to allow exceptions for small x, namely, $x < x_0$.

We observe,

~~Proposition~~

OK as was **~~Lemma~~ 11.5.8.** *If $f = o(g)$, then it is* not *true that $g = O(f)$.*

Proof.

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \frac{1}{\lim_{x\to\infty} f(x)/g(x)} = \frac{1}{0} = \infty,$$

so $g \neq O(f)$.

■

Proposition 11.5.9. $100x^2 = O(x^2)$.

Proof. Choose $c = 100$ and $x_0 = 1$. Then the proposition holds, since for all $x \geq 1$, $\left|100x^2\right| \leq 100x^2$. ■

Proposition 11.5.10. $x^2 + 100x + 10 = O(x^2)$.

Proof. $(x^2 + 100x + 10)/x^2 = 1 + 100/x + 10/x^2$ and so its limit as x approaches infinity is $1 + 0 + 0 = 1$. So in fact, $x^2 + 100x + 10 \sim x^2$, and therefore $x^2 + 100x + 10 = O(x^2)$. Indeed, it's conversely true that $x^2 = O(x^2 + 100x + 10)$. ■

Proposition 11.5.10 generalizes to an arbitrary polynomial:

Proposition 11.5.11. *For* $\cancel{a_k \neq 0}$, $a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 = O(x^k)$.

We'll omit the routine proof.

Big Oh notation is especially useful when describing the running time of an algorithm. For example, the usual algorithm for multiplying $n \times n$ matrices requires proportional to n^3 operations in the worst case. This fact can be expressed concisely by saying that the running time is $O(n^3)$. So this asymptotic notation allows the speed of the algorithm to be discussed without reference to constant factors or lower-order terms that might be machine specific. It turns out that ~~In this case~~ there is another, ingenious matrix multiplication procedure that requires $O(n^{2.55})$ operations. This procedure will therefore be much more efficient on large enough matrices. Unfortunately, the $O(n^{2.55})$-operation multiplication procedure is almost never used in practice because it happens to be less efficient than the usual $O(n^3)$ procedure on matrices of practical size.[1]

---

1 ~~EDITING NOTE:~~ It is even conceivable that there is an $O(n^2)$ matrix multiplication

procedure, but none is known. 4.4

— INSERT S goes here —

~~9.7.4~~

~~11.5.3 Theta~~

Definition 11.5.12.

~~$f = \Omega(g)$~~. ← OK as was

$$f = \Theta(g) \quad \text{iff} \quad f = O(g) \text{ and } \cancel{g = O(f)}.$$

The statement $f = \Theta(g)$ can be paraphrased intuitively as "f and g are equal to within a constant factor." ~~Equivalently, we could have defined~~

¶ The Theta notation allows us to ~~The value of these~~ notation~~s is that they~~ highlight growth rates and allow suppression of distracting factors and low-order terms. For example, if the running time of an algorithm is

$$T(n) = 10n^3 - 20n^2 + 1,$$

Indeed, ~~And~~ by Theorem 52, we know that ~~$f = O(g)$~~ ~~if and only~~ iff

~~$f = O(g)$ and $f = \Omega(g)$~~

$f = \Theta(g)$ iff $f = O(g)$ and ~~$g = O(f)$~~ $f = \Omega(g)$.

9.7.3 Omega

Suppose you want to say a statement of the form "the running time of the algorithm is at least ...". Can you say it is "at least $\Theta(n^2)$"? No! This statement is meaningless since big-Oh can only be used for upper bounds. For lower bounds, we use a different symbol called "big-Omega."

<u>Definition S1</u>: Given functions $f, g : \mathbb{R} \to \mathbb{R}$, we ~~say f = Ω(g~~ say that

$$f = \Omega(g)$$

iff there exists a constant $c > 0$ and an x_0 such that for all $x \geq x_0$, we have $f(x) \geq c|g(x)|$.

In other words, $f(x) = \Omega(g(x))$ means that $f(x)$ is greater than or equal to $g(x)$, except that we are willing to ignore a constant factor and to allow for exceptions for small x. ~~In fact, Big-Omega is precisely the opposite of big-Oh.~~

If all this sounds a lot like big-Oh, only in reverse, that is because big-Omega is ~~precisely~~ the opposite of big-Oh. More precisely,

Theorem S2: $f(x) = O(g(x))$ if and only if $g(x) = \Omega(f(x))$.

Proof: ~~By Defini~~ ~~$f(x) = O(g(x))$ iff~~ ~~$\exists c > 0, \exists x_0$ such~~

$f(x) = O(g(x))$ iff $\exists\, c>0, x_0.\ \forall x \geq x_0.$ ~~$f(x) \leq g$~~ $|f(x)| \leq c\, g(x)$ (Definition 11.9.7)

iff $\exists\, c>0, x_0.\ \forall x \geq x_0:\ g(x) \geq \frac{1}{c}|f(x)|$

iff $\exists\, c'>0, x_0.\ \forall x \geq x_0.\ g(x) \geq c' |f(x)|$ (set $c' = \frac{1}{c}$)

iff $g(x) = \Omega(f(x))$ (Definition S1) $\square$

For example, $x^2 = \Omega(x)$, $2^x = \Omega(x^2)$, and $\frac{x}{100} = \Omega(100x + \sqrt{x})$. ~~So~~

So if the running time of your algorithm on inputs of size n is $T(n)$, and you want to say it is at least quadratic, say

$$T(n) = \Omega(n^2).$$

Little Omega ⟸ subsubsection

There is also an analogous symbol to little-oh[#] ~~but it is not widely used~~ to denote that one function grows strictly faster than another function called little-omega.

Definition #53: ~~$f(x) = \omega(g(x))$ if and only if $g(x) = o(f(x))$~~ For functions $f, g : \mathbb{R} \to \mathbb{R}$ with f nonnegative, we say that

$$f(x) = \omega(g(x))$$

iff ~~lim x→∞~~

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0 .$$

In other words, ~~the~~

$$f(x) = \omega(g(x))$$

iff

$$g(x) = o(f(x)) .$$

For example, $x^{1.5} = \omega(x)$ and $\sqrt{x} = \omega(\ln^2(x))$.

~~the~~ # The little-omega symbol is not as widely used ~~and so on~~ as the other asymptotic symbols we have been discussing.

9.7.4 Theta

Sometimes, we want to specify ~~a running~~ that a ~~time~~ running time T(n) is precisely quadratic up to constant factors (both upper bound <u>and</u> lower bound).

~~For this~~ we could do this by saying that $T(n) = O(n)$ and $T(n) = \Omega(n)$, but rather than have to say both, mathematicians have devised yet another symbol, ~~theta big-that~~ Theta, to do the job. ~~more~~

then we can more simply write ~~say that~~

$$T(n) = \Theta(n^3).$$

In this case, we would say that T *is of order* n^3 or that $T(n)$ *grows cubically*, which is probably what we really want to know.

Another such example is

$$\pi^2 3^{x-7} + \frac{(2.7x^{113} + x^9 - 86)^4}{\sqrt{x}} - 1.08^{3x} = \Theta(3^x).$$

Just knowing that the running time of an algorithm is $\Theta(n^3)$, for example, is useful, because if n doubles we can predict that the running time will *by and large*[4] increase by a factor of at most 8 for large n. In this way, Theta notation preserves information about the scalability of an algorithm or system. Scalability is, of course, a big issue in the design of algorithms and systems.

[4]Since $\Theta(n^3)$ only implies that the running time, $T(n)$, is between cn^3 and dn^3 for constants $0 < c < d$, the time $T(2n)$ could regularly exceed $T(n)$ by a factor as large as $8d/c$. The factor is sure to be close to 8 for all large n only if $T(n) \sim n^3$.

~~EDITING NOTE:~~

~~Figure 11.7 illustrates the relationships among the asymptotic growth notations we have considered.~~

9.7.5

11.5.4 Pitfalls with ~~Big Oh~~ Asymptotic Notation

There is a long list of ways to make mistakes with ~~Big Oh~~ asymptotic notation. This section presents some of the ways that Big Oh notation can lead to ruin and despair. with minimal effort, you ~~You~~ can cause just as much chaos with the other symbols. ~~if you try~~

The Exponential Fiasco

Sometimes relationships involving Big Oh are not so obvious. For example, one might guess that $4^x = O(2^x)$ since 4 is only a constant factor larger than 2. This reasoning is incorrect, however; actually 4^x grows much faster than 2^x.

Proposition 11.5.13. $4^x \neq O(2^x)$

Proof. $2^x/4^x = 2^x/(2^x 2^x) = 1/2^x$. Hence, $\lim_{x\to\infty} 2^x/4^x = 0$, so in fact $2^x = o(4^x)$. We observed earlier that this implies that $4^x \neq O(2^x)$. ■

Constant Confusion

Every constant is $O(1)$. For example, $17 = O(1)$. This is true because if we let $f(x) = 17$ and $g(x) = 1$, then there exists a $c > 0$ and an x_0 such that $|f(x)| \leq cg(x)$. In particular, we could choose $c = 17$ and $x_0 = 1$, since $|17| \leq 17 \cdot 1$ for all $x \geq 1$. We can construct a false theorem that exploits this fact.

False Theorem 11.5.14.

$$\sum_{i=1}^{n} i = O(n)$$

False proof. Define $f(n) = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$. Since we have shown that every constant i is $O(1)$, $f(n) = O(1) + O(1) + \cdots + O(1) = O(n)$. ■

Of course in reality $\sum_{i=1}^{n} i = n(n+1)/2 \neq O(n)$.

The error stems from confusion over what is meant in the statement $i = O(1)$. For any *constant* $i \in \mathbb{N}$ it is true that $i = O(1)$. More precisely, if f is any constant function, then $f = O(1)$. But in this False Theorem, i is not constant but ranges over a set of values $0, 1, \ldots, n$ that depends on n.

And anyway, we should not be adding $O(1)$'s as though they were numbers. We never even defined what $O(g)$ means by itself; it should only be used in the context "$f = O(g)$" to describe a relation between functions f and g.

Lower Bound Blunder

Sometimes people incorrectly use Big Oh in the context of a lower bound. For example, they might say, "The running time, $T(n)$, is at least $O(n^2)$," when they probably mean

"$T(n) = \Omega(n^2)$".

~~something like "$O(T(n)) = n^2$," or more properly, "$n^2 = O(T(n))$."~~

Equality Blunder

The notation $f = O(g)$ is too firmly entrenched to avoid, but the use of "=" is really regrettable. For example, if $f = O(g)$, it seems quite reasonable to write $O(g) = f$. But doing so might tempt us to the following blunder: because $2n = O(n)$, we can say $O(n) = 2n$. But $n = O(n)$, so we conclude that $n = O(n) = 2n$, and therefore $n = 2n$. To avoid such nonsense, we will never write "$O(f) = g$."

Similarly, you will often see statements like

$$H_n = \ln(n) + \gamma + O\left(\frac{1}{n}\right)$$

or

$$n! = (1 + o(1))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

In such cases, the true meaning is

$$H_n = \ln(n) + \gamma + f(n)$$

~~and~~ $\cancel{n! = (1 + f(n))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$

— INSERT T goes here —

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for some $f(n)$ where $f(n) = O(1/n)$, and

$$n! = (1+g(n))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

where $g(n) = o(1)$. These transgressions are OK as long as you (and your reader) know what you mean.

9.8 Problems

where ~~$f(n) = o(1)$~~

~~where $f(n) = O(1/n)$~~

~~and~~ and

$n!$

Class Problems

Homework Problems

Class Problems

Homework Problems

Practice Problems

Homework Problems

Class Problems

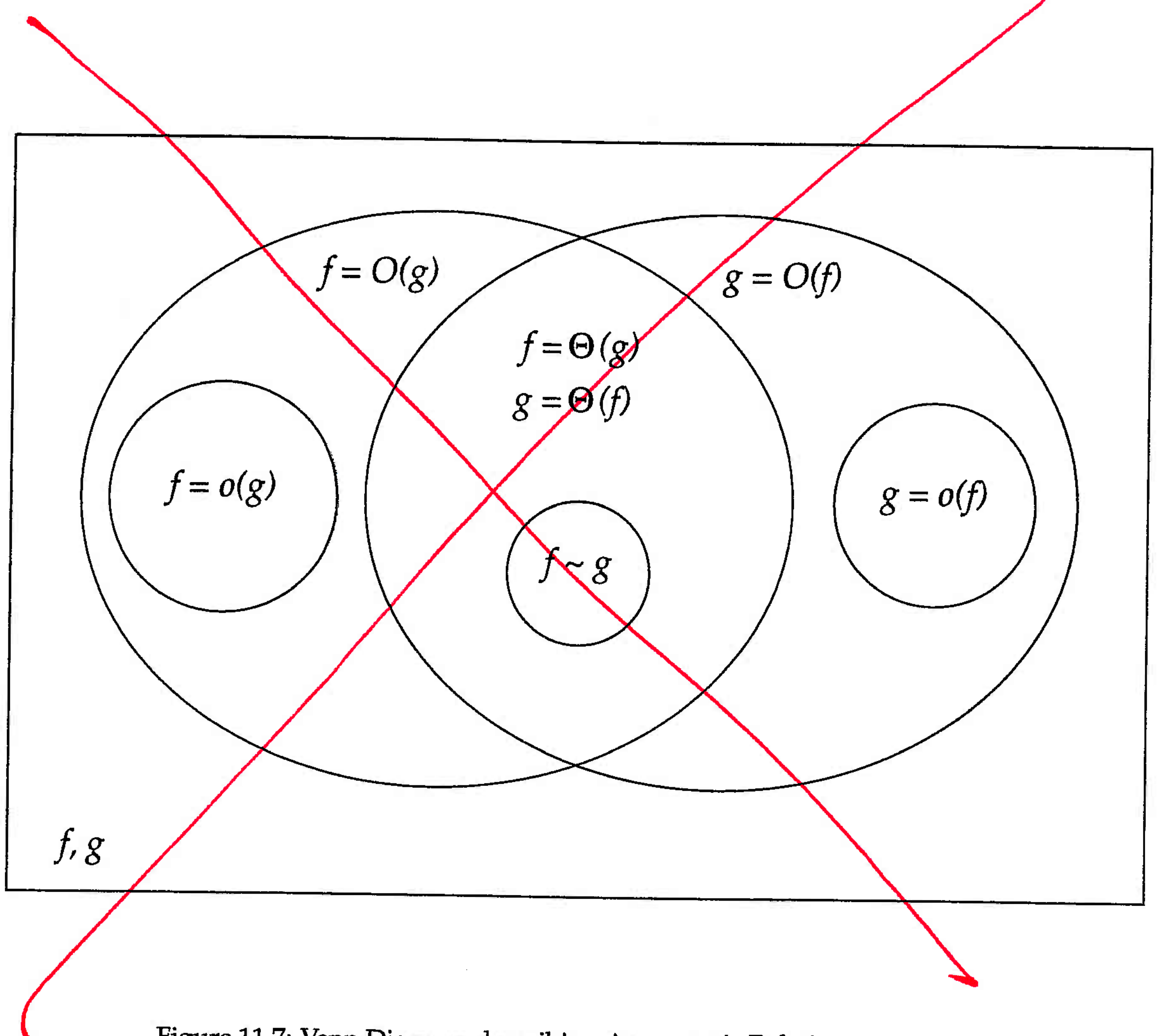

Figure 11.7: Venn Diagram describing Asymptotic Relations