Lecture 17: More Counting Methods

Reading: 11.5 – 11.9

Goal: Practice mapping sets to sequences and back again

Introductory example:

n=100 do k=3

$A = \{a_1, a_2, \ldots a_n\}$. How many sets $S \subseteq A$ with $|S| = k$ are there?

$$\frac{n!}{(n-k)!\,k!} := \binom{n}{k} \text{, but why?}$$

The number of ordered sequences of k elements is

$$\overset{100}{n}\,\overset{99}{(n-1)}\,\overset{98}{(n-2)} \ldots (n-k+1)$$

by Generalized Product Rule. Then if we forget order, we get a set and have a $k!$-to-1 map

6-to-1

This by Division Rule we get

$$\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!}$$

This is the basis for many counting problems

Let's study poker hands

Q1 How many 5-card hands are there? $\binom{52}{5} = 2598960$

Q2 How many hands with four of a kind are there?

$(5\diamondsuit, 5\clubsuit, 5\heartsuit, 5\spadesuit, A\heartsuit)$

$\downarrow$

1. rank of four of a kind
2. remaining card

sequence of steps for how to generate such a hand

This is a bijection, thus the number is 13×48

why?

Q3 How many hands with every suit are there? $\frac{13^4 \times 48}{2}$

$(6\diamondsuit, Q\heartsuit, A\spadesuit, 2\clubsuit, J\diamondsuit)$

2 to 1 map

$\downarrow$

1. rank of $\diamondsuit, \heartsuit, \spadesuit, \clubsuit$ — 13^4
2. remaining card — 48

But $13^4 \times 48$ is not the right answer, why?

Every hand gives rise to two sequences, $6\diamondsuit$ first or $J\diamondsuit$ first

Another Key Example: How many ways are there of ~~arranging~~ partitioning $A = \{a_1, a_2 \ldots a_n\}$ into sets of size $k_1, k_2, \ldots k_m$?

When $k_1 = k$ and $k_2 = n-k$ we get

$$\frac{n!}{k!(n-k)!}$$

More generally (same proof) we get

$$\frac{n!}{k_1!\,k_2!\cdots k_m!} := \binom{n}{k_1, k_2 \ldots k_m}$$

Application: How many ways to arrange letters in BOOKKEEPER?

totals: 1 B, 2 O, 2 K, 3 E, 1 P, 1 R

positions	1	2	3	4	5	6	7	8	9	10
letters	B	E	E	P	O	K	E	R	O	K

arrangement as partition

$B = \{1\}$ $O = \{5, 9\}$ $K = \{6, 10\}$

$E = \{2, 3, 7\}$ $P = \{4\}$ $R = \{8\}$

Arrangement $\longleftrightarrow$ Partition of locations into sets of size 1, 2, 2, 3, 1, 1

$$\binom{10}{1,2,2,3,1,1} = \frac{10!}{2!\,2!\,3!}$$

Now let's see some algebraic applications:

Bookkeeper Rule: The number of distinct sequences of k_1 letters l_1, k_2 letters l_2, ... k_m letters l_m is $(k_1+k_2+\ldots+k_m)!\,/\,k_1!\,k_2!\ldots k_m!$

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$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

We can find the coefficients by counting:

$(x+1)(x+1)(x+1)(x+1) =$

$\ldots + C x^2 + \ldots$

$\uparrow$

$\#$ ways of choosing 2 x's, 2 1's $= \binom{4}{2}$
(what locations are x's $\longleftrightarrow$ 2 element subset of $\{1,2,3,4\}$)

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Let's get some more practice:

$(x+y+z)^7 = \ldots + C x^3 y^2 z^2 + \ldots$

What is C? $\binom{7}{3,2,2}$, because need to partition $\{1,2 \ldots 7\}$ into where we choose x, y and z

Special case of multinomial theorem (see book)

Now that we've had practice with binomial coeffs

~~Now~~ let's go back to combinatorial proofs

Let's prove $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

by showing l.h.s and r.h.s. are different ways to count the same thing

Proof:

Let B be all k element subsets of $A = \{a_1, a_2 \ldots a_n\}$

Then $|B| = \binom{n}{k}$ (l.h.s.)

Alternatively let B_1 be all k element subsets that contain a_1

let B_2 " " do not contain a_1

$$|B_1| = \binom{n-1}{k-1}, \quad |B_2| = \binom{n-1}{k}$$

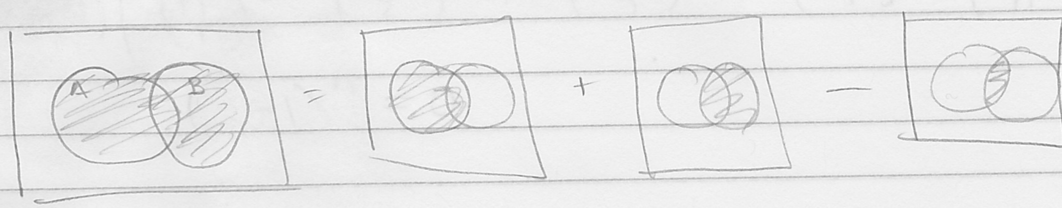
But B_1 and B_2 are disjoint, and $B = B_1 \cup B_2$

Thus $\binom{n}{k} = |B| = |B_1| + |B_2| = \binom{n-1}{k-1} + \binom{n-1}{k}$ ☑

More practice on pset.

Last major counting rule: Even when A and B intersect

$$|A \cup B| = |A| + |B| - |A \cap B|$$



example first

More generally:

I-E: $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k|$

$\ldots \; (-1)^{n-1} |A_1 \cap A_2 \ldots \cap A_n|$

Application: Let $n = p \cdot q$ where $p \neq q$, both prime

How many #'s in $\{1, 2 \ldots n\}$ are relatively prime to n?

$$\underset{\text{Euler's function}}{\phi(n)} = \left(pq - q - p + 1 = (p-1)(q-1)\right)$$

Let $A_p \subseteq \{1, 2 \ldots n\}$ divisible by p
A_q " by q

Then $|A_p| = \frac{n}{p}$, $|A_q| = \frac{n}{q}$

$$|A_p \cup A_q| = |A_p| + |A_q| - |A_p \cap A_q|$$

$$= \underbrace{\frac{n}{p}}_{q} + \underbrace{\frac{n}{q}}_{p} - 1$$

Why does $j \in A_1 \cap A_2 \ldots \cap A_n$ contribute only once?

$$\binom{n}{0} + (-1)\left(\binom{n}{1} - \binom{n}{2} + \binom{n}{3} \ldots (-1)^{n-1}\binom{n}{n}\right)$$

$$= (1-1)^n = 0$$