

7 JUL 2010

7

Relations and Partial Orders

A relation is a mathematical tool for describing associations between elements of sets. Relations are widely used in computer science, especially in databases and scheduling applications. A relation can be defined across many items in many sets, but in this text, we will focus on *binary* relations, which represent an association between two items in one or two sets.

7.1 Binary Relations

7.1.1 Definitions and Examples

Definition 7.1.1. Given sets A and B , a *binary relation* $R : A \rightarrow B$ from¹ A

to B is a subset of $A \times B$. The sets A and B are called the *domain* and *codomain*

of R , respectively. We commonly use the notation aRb or $a \sim_R b$ to denote that

$$(a, b) \in R.$$

A relation is similar to a function. In fact, every function $f : A \rightarrow B$ is a rela-

tion. In general, the difference between a function and a relation is that a relation

might associate multiple elements of B with a single element of A , whereas a func-

tion can only associate at most one element of B (namely, $f(a)$) with each element

$$a \in A.$$

¹We also say that the relationship is *between* A and B , or *on* A if $B = A$.

We have already encountered examples of relations in earlier chapters. For example, in Section 5.2, we talked about a relation between the set of men and the set of women where mRw if man m likes woman w . In Section 5.3, we talked about a relation on the set of MIT courses where c_1Rc_2 if the exams for c_1 and c_2 cannot be given at the same time. In Section 6.3, we talked about a relation on the set of switches in a network where s_1Rs_2 if s_1 and s_2 are directly connected by a wire. *That can send a packet from s_1 to s_2 .*

We did not use the formal definition of a relation in any of these cases, but they are all examples of relations.

As another example, we can define an “in-charge-of” relation T from the set of MIT faculty F to the set of subjects in the 2010 MIT course catalog. This relation contains pairs of the form

$$(\langle \text{instructor-name} \rangle, \langle \text{subject-num} \rangle)$$

where the faculty member named $\langle \text{instructor-name} \rangle$ is in charge of the subject with

[A lot of the MIT-specific number (subject-num). So T contains pairs like those shown in Figure 7.1.

stuff should be dropped This is a surprisingly complicated relation: Meyer is in charge of subjects with

unless there's a clear three numbers. Leighton is also in charge of subjects with two of these three

pedagogic purpose. For numbers—because the same subject, Mathematics for Computer Science, has two

example, do we depend numbers (6.042 and 18.062) and Meyer and Leighton are jointly in-charge-of the

on the fact that 6.042 = subject. Freeman is in-charge-of even more subjects numbers (around 20), since

18.062? Do we care that as Department Education Officer, he is in charge of whole blocks of special sub-

not all MIT faculty are ject numbers. Some subjects, like 6.844 and 6.00 have only one person in-charge.

mentioned because some Some faculty, like Guttag, are in-charge-of only one subject number, and no one

are only in charge of Fall else is jointly in-charge-of his subject, 6.00.

courses? Do we care who Some subjects in the codomain, N , do not appear in the list—that is, they are

the Department Education not an element of any of the pairs in the graph of T ; these are the Fall term only

Officer is?—dmj] subjects. Similarly, there are faculty in the domain, F , who do not appear in the

(Meyer, 6.042),

(Meyer, 18.062),

(Meyer, 6.844),

(Leighton, 6.042),

(Leighton, 18.062),

(Freeman, 6.011),

(Freeman, 6.881)

(Freeman, 6.882)

(Freeman, 6.UAT)

(Eng, 6.UAT)

(Guttag, 6.00)

Figure 7.1: Some items in the “in-charge-of” relation T between faculty and subject numbers.

list because all their in-charge-of subjects are Fall term only.

7.1.2 Representation as a Bipartite Graph

Every relation $R : A \rightarrow B$ can be easily represented as a bipartite graph $G =$

(V, E) by creating a “left” node for each element of A and a “right” node for each

element of B . We then create an edge between a left node u and a right node v

whenever aRb . Similarly, every bipartite graph (and every partition of the nodes

into a “left” and “right” set for which no edge connects a pair of left \cancel{or} right nodes)

nodes or a pair of

determines a relation between the nodes on the left and the nodes on the right.

For example, we have shown part of the bipartite graph for the in-charge-of

relation from Figure 7.1 in Figure 7.2. In this case, there is an edge between

[Do you need to be

<instructor-name> and <subject-number> if <instructor-name> is in charge of <subject-number>.

explicit about assuming A

A relation $R : A \rightarrow B$ can also be represented as a matrix $A = \{a_{ij}\}$ where

and B are finite(?) and

$$a_{ij} = \begin{cases} 1 & \text{if the } i\text{th element of } A \text{ is related to the } j\text{th element of } B \\ 0 & \text{otherwise} \end{cases}$$

that you're imposing an

arbitrary order on them?

Especially since you use a

Figure FB, p. 7-7, fit-ch7-jul2.

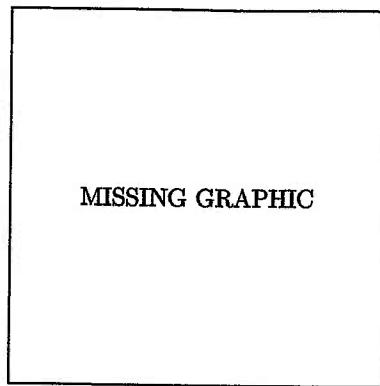


Figure 7.2: Part of the bipartite graph for the “in charge of” relation T from Figure 7.1.

for $1 \leq i \leq |A|$ and $1 \leq j \leq |B|$. For example, the matrix for the relation in

Figure 7.2 (but restricted to the five faculty and eight subject numbers shown in

Figure 7.2, ordering them as they appear top-to-bottom in Figure 7.2) is shown in

Figure 7.3.

7.1.3 Relational Images

The idea of the image of a set under a function extends directly to relations.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 7.3: The matrix for the “in charge of” relation T restricted to the five faculty

and eight subject numbers shown in Figure 7.2. The $(3, 4)$ entry of this matrix is 1 since the third professor (Freeman) is in charge of the fourth subject number (6.011).

Definition 7.1.2. The *image* of a set Y under a relation $R : A \rightarrow B$, written $R(Y)$,

is the set of elements that are related to some element in Y , namely,

$$R(Y) ::= \{ b \in B \mid yRb \text{ for some } y \in Y \}.$$

The image of the domain, $R(A)$, is called the *range* of R .

For example, to find the subject numbers that Meyer is in charge of, we can look

for all the pairs of the form

(Meyer, {subject-number})

in the graph of the teaching relation T , and then just list the right-hand sides of these

pairs. These right-hand sides are exactly the image $T(\text{Meyer})$, which happens to

be $\{6.042, 18.062, 6.844\}$. Similarly, since the domain F is the set of all in-charge

faculty, $T(F)$, the range of T , is exactly the set of *all* subjects being taught.

7.1.4 Inverse Relations and Images

Definition 7.1.3. The *inverse* R^{-1} of a relation $R : A \rightarrow B$ is the relation from B to A defined by the rule

$$bR^{-1}a \text{ if and only if } aRb.$$

The image of a set under the relation R^{-1} is called the *inverse image* of the set.

That is, the inverse image of a set X under the relation R is $R^{-1}(X)$.

Continuing with the in-charge-of example above, we can find the faculty in charge of 6.UAT by taking the pairs of the form

$$(\langle\text{instructor-name}\rangle, 6.\text{UAT})$$

for the teaching relation T , and then just listing the left-hand sides of these pairs;

these turn out to be just Eng and Freeman. These left-hand sides are exactly the

inverse image of {6.UAT} under T .

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this

Now let D be the set of introductory subjects. These are the subjects that have

an 0 immediately following the decimal point. Now we can likewise find all the instructors who were in-charge-of introductory subjects by taking all pairs of the form $(\langle \text{instructor-name} \rangle, \{1.0 \dots, 2.0 \dots, 3.0 \dots, \dots\})$ and list the left-hand sides of these pairs. These left-hand sides are exactly the inverse image of D under T .

From the part of T show above, we can see that

$$\{\text{Meyer, Leighton, Freeman, Guttag}\} \subseteq T^{-1}(D).$$

That is, Meyer, Leighton, Freeman, and Guttag are among the instructors in charge of introductory subjects. Finally, the inverse image under T of the set N of all subject numbers is the set of all instructors who are in charge of a subject.

It gets interesting when we write composite expressions mixing images, inverse images, and set operations. For example, $T(T^{-1}(D))$ is the set of subjects that have an instructor in charge who is also in charge of an introductory subject. So $T(T^{-1}(D)) - D$ is the set of advanced subjects with someone in-charge who is

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also in-charge-of an introductory subject. Similarly, $T^{-1}(D) \cap T^{-1}(N - D)$ is the set of faculty in charge of both an introductory *and* an advanced subject.

7.2.1 7.1.5 Surjective and Injective Relations

This is INSERT
goes to INSERT
(new 7.2)

There are some properties of relations that will be useful when we take up the topic

of counting in Part III because they imply certain relations between the *sizes* of domains and codomains. In particular, we say that a binary relation $R : A \rightarrow B$ is

- *surjective* if every element of B is mapped *at least once*,¹ more concisely, R

is surjective iff $R(A) = B$ (*that is*, if the range of R is the codomain of R),

- *total* when every element of A is assigned to some element of B ,² more concisely, R is total iff $A = R^{-1}(B)$,

- *injective* if every element of B is mapped *at most once*, and

- *bijective* if R is total, surjective, injective, and a function².

²These words *surjective*, *injective*, and *bijective* are not very memorable. Some authors use the

We can illustrate these properties of a relation $R : A \rightarrow B$ in terms of the corresponding bipartite graph G for the relation, where nodes on the left side of G correspond to elements of A and nodes on the right side of G correspond to elements of B . For example:

- " R is a function" means that every node on the left is incident to *at most* one edge.
- " R is total" means that *every* node on the left is incident to *at least* one edge.

So if R is a function, being total means that every node on the left is incident to exactly one edge.

- " R is surjective" means that *every* node on the right is incident to *at least* one edge.

possibly more memorable phrases *onto* for surjective, *one-to-one* for injective, and *exact correspondence* for bijective.

- " R is injective" means that *every* node on the right is incident to *at most* one edge.
- " R is bijective" means that every node on both sides is incident to *precisely* one edge (~~that is, that~~ ^{HA} there is a perfect matching between A and B). X

For example, consider the relations R_1 and R_2 shown in Figure 7.4. R_1 is a total surjective function (every node in the left column is incident to exactly one edge, C ^{in the}

and every node in the right column is incident to at least one edge), but not injective

(node 3 is incident to 2 edges). R_2 is a total injective function (every node ~~in~~ ^{one in}

~~column~~

left is incident to exactly one edge, and every node in the right column is incident ~~to~~ ¹

to at most one edge), but not surjective (node 4 is not incident to any edges).

Notice that we need to know what the domain is to determine whether a relation is total, and we need to know the codomain to determine whether it's surjective.

For example, the function defined by the formula $1/x^2$ is total if its domain is \mathbb{R}^+

Figure FD, page 7-14,

ftl-ch7-jul2.

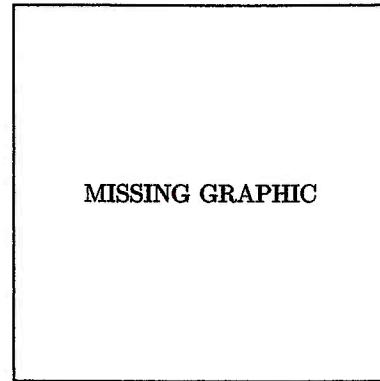


Figure 7.4: Relation $R_1 : A_1 \rightarrow B_1$ is shown in (a) and relation $R_2 : A_2 \rightarrow B_2$ is shown in (b).

but partial if its domain is some set of real numbers that includes 0. It is bijective

if its domain and codomain are both \mathbb{R}^+ , but neither injective nor surjective it is

domain and codomain are both \mathbb{R} .

INSERT B ends here

~~7.1.5~~ **7.1.6 Combining Relations**

There are at least two natural ways to combine relations to form new relations. For

example, given relations $R : B \rightarrow C$ and $S : A \rightarrow B$, the *composition* of R with S

[Should ::= be

is the relation $(R \circ S) : A \rightarrow C$ defined by the rule

"iff"—dmj]

$$a(R \circ S)c ::= \exists b \in B. (bRc) \text{ AND } (aSb)$$

where $a \in A$ and $c \in C$.

As a special case, the composition of two functions $f : B \rightarrow C$ and $g : A \rightarrow B$

is the function $f \circ g : A \rightarrow C$ defined by

$$(f \circ g)(a) = f(g(a))$$

for all $a \in A$. For example, if $A = B = C = \mathbb{R}$, $g(x) = x + 1$ and $f(x) = x^2$,

then

$$(f \circ g)(x) = (x + 1)^2$$

$$= x^2 + 2x + 1.$$

One can also define the *product* of two relations $R_1 : A_1 \rightarrow B_1$ and $R_2 : A_2 \rightarrow$

B_2 to be the relation $S = R_1 \times R_2$ where

$$S : A_1 \times A_2 \rightarrow B_1 \times B_2$$

and

$$(a_1, a_2)S(b_1, b_2) \text{ iff } a_1 R_1 b_1 \text{ and } a_2 R_2 b_2.$$

INSERT C goes here

~~7.3~~ 7.2 Relations on One Set

For the rest of this chapter, we are going to focus on relationships between elements

of a single set; that is, relations from a set A to a set B where $A = B$. Thus, a

relation on a set A is a subset $R \subseteq A \times A$. Here are some examples:

- Let A be a set of people and the relation R describe who likes whom: that is,

$(x, y) \in R$ if and only if x likes y .

- Let A be a set of cities. Then we can define a relation R such that xRy if and

only if there is a nonstop flight from city x to city y .

- Let $A = \mathbb{Z}$ and let xRy hold if and only if $x \equiv y \pmod{5}$.

INSERT C7.2 Relations and Cardinality

— INSERT B goes here —
 (text on pp 287 - 290)

7.2.2 ~~Properties~~ Cardinality

— INSERT D goes here —
 (this is ~~a~~ text from old version
 of ch 5 & ~~is a copy is on~~
 next few pages)

We will see many examples where Theorem PA
 is used to determine the cardinality of a ^{finite} set.

~~That's later~~
 Later, in Chapter 13, we will consider the case
~~when A and B~~ when the sets are infinite and we'll ~~use~~
~~that some~~ ~~surjective and injective relations to~~
~~prove that some infinite sets are "bigger"~~
~~than other infinite sets.~~

domain and codomain are both \mathbb{R} .

7.1.6

5.6 Cardinality

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this

5.6.1 Mappings and Cardinality

Subsection 7.2.1

The relational properties in Section 5.5 are useful in figuring out the relative sizes

of domains and codomains.

If A is a finite set, we let $|A|$ be the number of elements in A . A finite set may

this is also called
the cardinality of A ,
in general, a

have no elements (the empty set), or one element, or two elements, ... or any non-

negative integer number of elements, so def.: for any finite set $\rightarrow |A| \in \mathbb{N}$.

Now suppose $R : A \rightarrow B$ is a function. Then every arrow in the diagram for

edge

bipartite graph G

is incident to

R comes from exactly one element of A , so the number of arrows is at most the

edges

number of elements in A . That is, if R is a function, then

$$|A| \geq \# \text{arrows} = |E(G)|$$

edge incident to

Similarly, if R is surjective, then every element of B has an arrow into it, so there

is incident to an edge,

edges graph

must be at least as many arrows in the diagram as the size of B . That is,

$$|E(B)| \geq |B|.$$

$$R : A \rightarrow B$$

Combining these inequalities implies that if R is a surjective function, then $|A| \geq$

~~This fact and hence other three facts~~

~~|B|. In short, if we write $A \text{ surj } B$ to mean that there is a surjective function from~~

~~A to B, then we've just proved a lemma: if $A \text{ surj } B$, then $|A| \geq |B|$. The following~~

~~includes~~

~~two~~

~~definition and lemma lists this statement and three similar rules relating domain~~

~~and codomain size to relational properties, is captured in the following lemma.~~

Definition 5.6.1. Let A, B be (not necessarily finite) sets. Then

1. $A \text{ surj } B$ iff there is a surjective *function* from A to B .
2. $A \text{ inj } B$ iff there is a total injective *relation* from A to B .
3. $A \text{ bij } B$ iff there is a bijection from A to B .
4. $A \text{ strict } B$ iff $A \text{ surj } B$, but not $B \text{ surj } A$.

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is referenced
in CH11.

Theorem PA

Lemma 5.6.2. [Mapping Rules] Let A and B be finite sets.

If there is a surjection from A to B ,

1. If $A \text{ surj } B$, then $|A| \geq |B|$.

(Staff note:
shouldn't
we call
these
relational
rules?)

5.6. CARDINALITY

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If there is an injection from A to B,

2. ~~If $A \in B$, then $|A| \leq |B|$.~~

If there is a bijection between A and B,

3. ~~If R bij B , then $|A| = |B|$.~~

~~If there is a surjection from A to B but no~~

4. ~~If R strict B , then $|A| > |B|$.~~

~~surjection from B to A,~~

Mapping rule 2. can be explained by the same kind of *arrow reasoning* we

is an

the

used for rule 1. Rules 3. and 4. are immediate consequences of *the first two*

mapping rules.

5.6.2 The sizes of infinite sets

Mapping Rule 1 has a converse: if the size of a finite set, A , is greater than or equal

to the size of another finite set, B , then it's always possible to define a surjective

function from A to B . In fact, the surjection can be a total function. To see how this

works, suppose for example that

$$A = \{a_0, a_1, a_2, a_3, a_4, a_5\}$$

$$B = \{b_0, b_1, b_2, b_3\}.$$

- Let $A = \mathbb{N}$ and let xRy if and only if $x | y$.

- Let $A = \mathbb{N}$ and let xRy if and only if $x \leq y$.

The last examples clarify the reason for using xRy or $x \sim_R y$ to indicate that the relation R holds between x and y : many common relations ($<$, \leq , $=$, $|$, \equiv) are expressed with the relational symbol in the middle.

7.2.1 Representation as a Digraph

Every relation on a single set A can be modeled as a directed graph (albeit one that may contain loops). For example, the graph in Figure 7.5 describes the “likes” relation for a particular set of 3 people.

In this case, we see that:

- Julie likes Bill and Bob, but not herself.
- Bill likes only himself.

Figure FE, page 7-19,

ftl-ch7-jul2

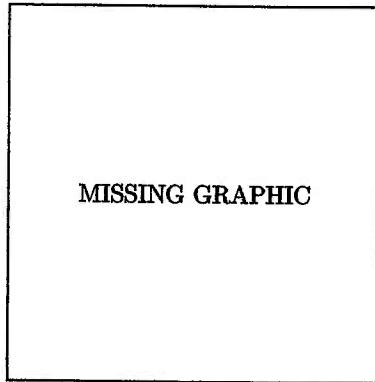


Figure 7.5: The directed graph for the “likes” relation on the set {Bill, Bob, Julie}.

- Bob likes Julie, but not Bill nor himself.

Everything about the relationship is conveyed by the directed graph and nothing

more. This is no coincidence; a set A together with a relation R is precisely the

same thing as directed graph $G = (V, E)$ with vertex set $V = A$ and edge set

$E = R$ (where E may have loops).

As another example, we have illustrated the directed graph for the divisibility

relationship on the set $\{1, 2, \dots, 12\}$ in Figure 7.6. In this graph, every node has

a loop (since every positive number divides itself) and the composite numbers are

Figure FF, p. 7-21, ftl-ch7-jul2.

Redraw in standard format with

solid nodes and labels next to

the nodes.

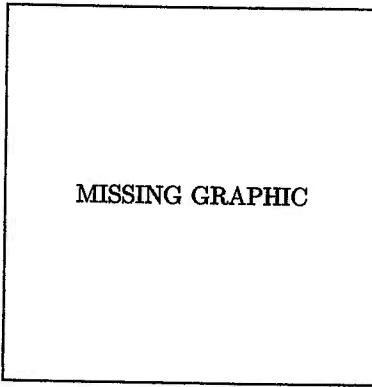


Figure 7.6: The digraph for divisibility on $\{1, 2, \dots, 12\}$.

[Easier to just say

the nodes with indegree more than 1 (not counting the loop).

↑
"greater than 2"?—dmj
No edges
don't count
in degree

Relations on a single set can also be represented as a 0, 1-matrix. In this case,

the matrix is identical to the adjacency matrix for the corresponding digraph. For

example, the matrix for the relation shown in Figure 7.5 is simply

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where $v_1 = \text{Julie}$, $v_2 = \text{Bill}$, and $v_3 = \text{Bob}$.

7.2.2 Symmetry, Transitivity, and Other Special Properties

Many relations on a single set that arise in practice possess one or more noteworthy properties. These properties are summarized in the box on the following page. In each case, we provide the formal of the definition of the property, explain what the property looks like in a digraph G for the relation, and give an example of what the property means for the "likes" relation.

For example, the congruence relation modulo 5 on \mathbb{Z} is reflexive symmetric, and transitive, but not irreflexive, antisymmetric, or asymmetric. The same is true for the "connected" relation $R : V \rightarrow V$ on an undirected graph $G = (V, E)$ where uRv if u and v are in the same connected component of graph G . In fact, relations that have these three properties are so common that we give them a special name: *equivalence relations*. We will discuss them in greater detail in just a moment.

As another example, the "divides" relation on \mathbb{Z}^+ is reflexive, antisymmetric,

Properties of a Relation $R : A \rightarrow A$

Reflexivity R is *reflexive* if $\forall x \in A. xRx$.

"Everyone likes themselves."

Every node in G has a loop.

Irreflexivity R is *irreflexive* if $\neg \exists x \in A. xRx$.

"No one likes themselves."

There are no loops in G .

Symmetry R is *symmetric* if $\forall x, y \in A. xRy \text{ IMPLIES } yRx$.

"If x likes y , then y likes x ."

If there is an edge from x to y in G , then there is an edge from y to x in G as well.

Antisymmetry R is *antisymmetric* if $\forall x, y \in A (xRy \text{ AND } yRx) \text{ IMPLIES } x = y$.

"No pair of distinct people like each other."

and transitive, but not irreflexive, symmetric, or asymmetric. The same is true for the " \leq " relation on \mathbb{R} . Relations that have these three properties are also very common and they fall into a special case of relations called a *partial order*. We will discuss partial orders at length in Sections 7.4–7.8.

As a final example, consider the "likes" relation on the set {Julie, Bill, Bob} illustrated in Figure 7.5. This relation has *none* of the six properties described in the box.

7.3 Equivalence Relations

A relation is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Congruence modulo n is an excellent example of an equivalence relation:

- It is reflexive because $x \equiv x \pmod{n}$.

- It is symmetric because $x \equiv y \pmod{n}$ implies $y \equiv x \pmod{n}$.
- It is transitive because $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n}$ imply that $x \equiv z \pmod{n}$.

There is an even more well-known example of an equivalence relation: equality itself. Thus, an equivalence relation is a relation that shares some key properties with “=”.

7.3.1 Partitions

There is another way to think about equivalence relations, but we’ll need a couple of definitions to understand this alternative perspective.

Definition 7.3.1. Given an equivalence relation $R : A \rightarrow A$, the *equivalence class* of an element $x \in A$ is the set of all elements of A related to x by R . The equivalence class of x is denoted $[x]$. Thus, in symbols:

$$[x] = \{ y \mid xRy \}.$$

For example, suppose that $A = \mathbb{Z}$ and xRy means that $x \equiv y \pmod{5}$. Then

$$[7] = \{\dots, -3, 2, 7, 12, 22, \dots\}.$$

Notice that 7, 12, 17, etc., all have the same equivalence class; that is, $[7] = [12] =$

$$[17] = \dots$$

Definition 7.3.2. A *partition* of a set A is a collection of disjoint, nonempty subsets

[" A_n " implies the A_1, A_2, \dots, A_n whose union is all of A . The subsets are usually called the *blocks*

partition is finite?—dmj] of the partition.³ For example, one possible partition of $A = \{a, b, c, d, e\}$ is

$$A_1 = \{a, c\} \quad A_2 = \{b, e\} \quad A_3 = \{d\}.$$

Here's the connection between all this stuff: there is an exact correspondence

between *equivalence relations on A* and *partitions of A*. We can state this as a

theorem:

³We think they should be called the *parts* of the partition. Don't you think that makes a lot more sense?

Theorem 7.3.3. *The equivalence classes of an equivalence relation on a set A form a partition of A.*

We won't prove this theorem (too dull even for us!), but let's look at an example.

The congruent-mod-5 relation partitions the integers into five equivalence classes:

$$\{\dots, -5, 0, 5, 10, 15, 20, \dots\}$$

$$\{\dots, -4, 1, 6, 11, 16, 21, \dots\}$$

$$\{\dots, -3, 2, 7, 12, 17, 22, \dots\}$$

$$\{\dots, -2, 3, 8, 13, 18, 23, \dots\}$$

$$\{\dots, -1, 4, 9, 14, 19, 24, \dots\}$$

In these terms, $x \equiv y \pmod{5}$ is equivalent to the assertion that x and y are both in the same block of this partition. For example, $6 \equiv 16 \pmod{5}$, because they're both in the second block, but $2 \not\equiv 9 \pmod{5}$ because 2 is in the third block while

9 is in the last block.

In social terms, if "likes" were an equivalence relation, then everyone would be partitioned into cliques of friends who all like each other and no one else.

7.4 Partial Orders

7.4.1 Strong and Weak Partial Orders

Definition 7.4.1. A relation R on a set A is a *weak partial order* if it is transitive, antisymmetric, and reflexive. The relation is said to be a *strong partial order* if it is transitive, antisymmetric, and irreflexive.⁴

Some authors defined partial orders to be what we call weak partial orders, but we'll use the phrase *partial order* to mean either a weak or a strong partial order.

⁴Equivalently, the relation is transitive and asymmetric, but stating it this way might have obscured the irreflexivity property.

The difference between a weak partial order and a strong one has to do with the reflexivity property: in a weak partial order, *every* element is related to itself, but in a strong partial order, *no* element is related to itself. Otherwise, they are the same in that they are both transitive and antisymmetric.

Examples of weak partial orders include " \leq " on \mathbb{R} , " \subseteq " on the set of subsets

*OKes
wes*

of (say) \mathbb{Z} , and the "divides" relation on \mathbb{N}^+ . Examples of strict partial orders in-

clude " $<$ " on \mathbb{R} , and " \subset " on the set of subsets of \mathbb{Z} .⁵

We often denote a weak partial order with a symbol such as \preceq or \sqsubseteq instead of a

letter such as R . This makes sense from one perspective since the symbols call to

OKeswes

mind \leq and \subseteq , which define common partial orders. On the other hand, a partial

order is really a set of related pairs of items, and so a letter like R would be more

⁵If you are not feeling comfortable with all the definitions that we've been throwing at you, it's

probably a good idea to verify that each of these relations are indeed partial orders by checking that

they have the transitivity and antisymmetry properties.

normal.

Likewise, we will often use a symbol like \prec or \sqsubset to denote a strong partial order.

7.4.2 Total Orders

A partial order is "partial" because there can be two elements with no relation

between them. For example, in the "divides" partial order on $\{1, 2, \dots, 12\}$, there

is no relation between 3 and 5 (since neither divides the other).

[You don't define "poset"
until the next
section.—dmj]

In general, we say that two elements a and b in a poset are *incomparable* if neither $a \leq b$ nor $b \leq a$. Otherwise, if $a \leq b$ or $b \leq a$, then we say that a and b are *comparable*.

Definition 7.4.2. A *total order* is a partial order in which every pair of distinct elements is comparable.

For example, the " \leq " partial order on \mathbb{R} is a total order because for any pair of real numbers x and y , either $x \leq y$ or $y \leq x$. The "divides" partial order on

$\{1, 2, \dots, 12\}$ is not a total order because $3 \nmid 5$ and $5 \nmid 3$.

7.5 Posets and DAGs

7.5.1 Partially Ordered Sets

Definition 7.5.1. Given a partial order \leq on a set A , the pair (A, \leq) is called a *partially ordered set* or *poset*.

In terms of graph theory, a poset is simply the directed graph $G = (A, \leq)$ with vertex set A and edge set \leq . For example, Figure 7.6 shows the graph form of the poset for the “divides” relation on $\{1, 2, \dots, 12\}$. We have shown the graph form of the poset for the “ $<$ ”-relation on $\{1, 2, 3, 4\}$ in Figure 7.7.

Figure FH, p. 7-32, ftl-ch7-jul2.

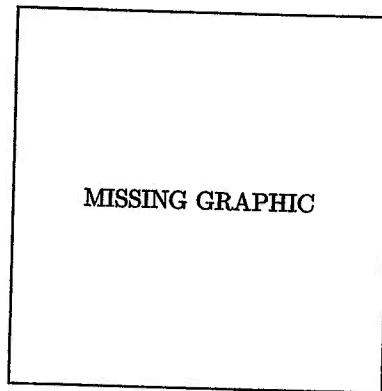


Figure 7.7: Representing the poset for the “ $<$ ”-relation on $\{1, 2, 3, 4\}$ as a digraph.

7.5.2 Posets Are Acyclic

Did you notice anything that is common to Figures 7.6 and 7.7? Of course, they both exhibit the transitivity and antisymmetry properties. And, except for the loops in Figure 7.6, they both do not contain any cycles. This is not a coincidence. In fact, the combination of the transitivity and asymmetry properties imply that the digraph for any poset is an acyclic graph (that is, a DAG), at least if you don't count loops as cycles. We prove this fact in the following theorem.

Theorem 7.5.2. *A poset has no directed cycles other than self-loops.*

Proof. We use proof by contradiction. Let (A, \preceq) be a poset. Suppose that there exist $n \geq 2$ distinct elements a_1, a_2, \dots, a_n such that

$$a_1 \preceq a_2 \preceq a_3 \preceq \cdots \preceq a_{n-1} \preceq a_n \preceq a_1.$$

Since $a_1 \preceq a_2$ and $a_2 \preceq a_3$, transitivity implies $a_1 \preceq a_3$. Another application of transitivity shows that $a_1 \preceq a_4$ and a routine induction argument establishes that $a_1 \preceq a_n$. Since we know that $a_n \preceq a_1$, antisymmetry implies $a_1 = a_n$, contradicting the supposition that a_1, \dots, a_n are distinct and $n \geq 2$. Thus, there is no such directed cycle. ■

Thus, deleting the self-loops from a poset leaves a directed graph without cycles, which makes it a *directed acyclic graph* or *DAG*.

7.5.3 Transitive Closure

Theorem 7.5.2 tells us that every poset corresponds to a DAG. Is the reverse true?

That is, does every DAG correspond to a poset? The answer is “Yes,” but we need to modify the DAG to make sure that it satisfies the transitivity property. For example, consider the DAG shown in Figure 7.8. As any DAG must, this graph satisfies the antisymmetry property⁶ but it does not satisfy the transitivity property because $v_1 \rightarrow v_2$ and $v_2 \rightarrow v_3$ are in the graph but $v_1 \rightarrow v_3$ is not in the graph.

Definition 7.5.3. Given a digraph $G = (V, E)$, the *transitive closure* of G is the digraph $G^+ = (V, E^+)$ where

$$E^+ = \{ u \rightarrow v \mid \text{there is a directed path of positive length from } u \text{ to } v \text{ in } G \}.$$

Similarly, if R is the relation corresponding to G , the *transitive closure* of R (denoted R^+) is the relation corresponding to G^+ .

⁶If $u \rightarrow v$ and $v \rightarrow u$ are in a digraph G , then G would have a cycle of length 2 and it could not be a DAG.

Figure FJ, p. 7-34, ftl-ch7-jul7.

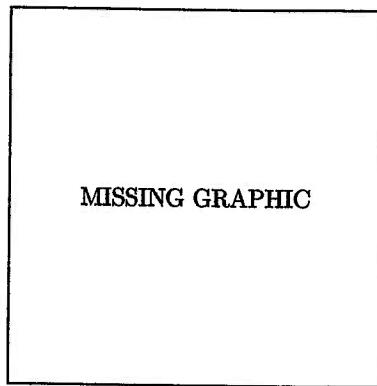


Figure 7.8: A 6-node digraph that does not satisfy the transitivity property.

For example, the transitive closure for the graph in Figure 7.8 is shown in Figure 7.9.

If G is a DAG, then the transitive closure of G is a strong partial order. The proof

of this fact is left as an exercise in the problem section.

7.5.4 The Hasse Diagram

One

On problem with viewing a poset as a digraph is that there tend to be lots of edges

due to the transitivity property. Fortunately, we do not necessarily have to draw

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Figure FK, p. 7-36, ftl-ch7-jul2.

Doubled lines should be

rendered as heavy lines.

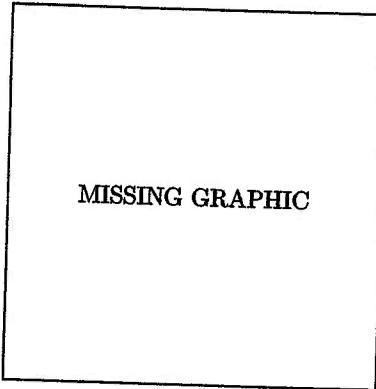


Figure 7.9: The transitivity closure for the digraph in Figure 7.8. The edges that

were added to form the transitive closure are shown in bold.

all the edges if we know that the digraph corresponds to a poset. For example,

we could choose not to draw any edge which would be implied by the transitivity

property, knowing that it is really there by implication. In general, a *Hasse diagram*

for a poset (A, \leq) is a digraph with vertex set A and edge set \leq minus all self-loops

and edges implied by transitivity. For example, the Hasse diagrams of the posets

shown in Figures 7.6 and 7.7 are shown in Figure 7.10.

Figure FM, p 7-38,

ftl-chap7-jul2. Render as two

figures.

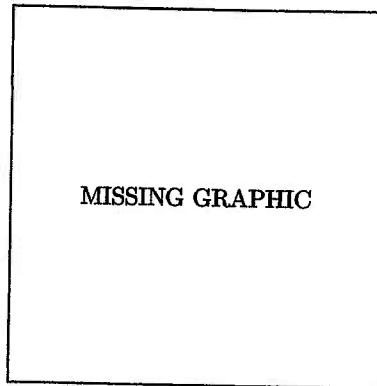


Figure 7.10: The Hasse diagrams for the posets in Figure 7.6 and 7.7.

7.6 Topological Sort

A total order that is consistent with a partial order is called a topological sort More

precisely, a *topological sort* of a poset (A, \leq) is a total order (A, \leq_T) such that

Definition :

$x \leq y$ IMPLIES $x \leq_T y$.

For example, consider the poset that describes how a guy might get dressed for

a formal occasion. The Hasse diagram for such a poset is shown in Figure 7.11.

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Figure FP, p. 7-40, ftl-ch7-jul2.

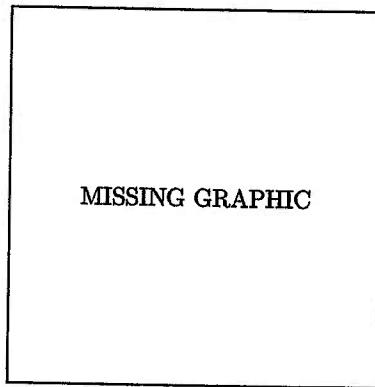


Figure 7.11: The Hasse diagram for a poset that describes which items much precede others when getting dressed.

In this poset, the *set* is all the garments and the *partial order* specifies which items much precede others when getting dressed.

There are several total orders that are consistent with the partial order shown in

Figure 7.11. We have shown ~~16~~ ^{two} of them in list form in Figure 7.12. Each such

list is a topological sort for the partial order in Figure 7.11. In what follows, we will prove that every *finite* poset has a topological sort. You can think of this as a mathematical proof that you *can* get dressed in the morning (and then show up for

math lecture).

Theorem 7.6.1. *Every finite poset has a topological sort.*

We'll prove the theorem constructively. The basic idea is to pull the “smallest” element a out of the poset, find a topological sort of the remainder recursively, and then add a back into the topological sort as an element smaller than all the others.

The first hurdle is that “smallest” is not such a simple concept in a set that is only partially ordered. In a poset (A, \leq) , an element $x \in A$ is *minimal* if there is no other element $y \in A$ such that $y \leq x$. For example, there are *four* minimal elements in the getting-dressed poset: left sock, right sock, underwear, and shirt. (It may seem odd that the minimal elements are at the top of the Hasse diagram rather than the bottom. Some people adopt the opposite convention. If you're not sure whether minimal elements are on the top or bottom in a particular context, ask.) Similarly, an element $x \in A$ is *maximal* if there is no other element $y \in A$ such that $x \leq y$.

underwear left sock

 pants shirt

 belt tie

 shirt underwear

 tie right sock

 jacket pants

 left sock right shoe

 right sock belt

 left shoe jacket

 right shoe left shoe

(a)

(b)

Figure 7.12: Two possible topological sorts of the poset shown in Figure 7.11. In

each case, the elements are listed so that $x \preceq y$ iff x is above y in the list.

X

Proving that every poset *has* a minimal element is extremely difficult, because
~~it is not true.~~
~~this is actually false.~~ For example, the poset (\mathbb{Z}, \leq) has no minimal element. However, there is at least one minimal element in every *finite* poset.

Lemma 7.6.2. *Every finite poset has a minimal element.*

Proof. Let (A, \leq) be an arbitrary poset. Let a_1, a_2, \dots, a_n be a maximum-length sequence of distinct elements in A such that

$$a_1 \leq a_2 \leq \dots \leq a_n.$$

The existence of such a maximum-length sequence follows from the Well Ordering Principle and the fact that A is finite. Now $a_0 \leq a_1$ cannot hold for any element $a_0 \in A$ not in the chain, since the chain already has maximum length. And $a_i \leq a_1$ cannot hold for any $i \geq 2$, since that would imply a cycle

$$a_i \leq a_1 \leq a_2 \leq \dots \leq a_i$$

and no cycles exist in a poset by Theorem 7.5.2. Therefore a_1 is a minimal element.

Now we're ready to prove Theorem 7.6.1, which says that every finite poset has a topological sort. The proof is rather intricate; understanding the argument requires a clear grasp of all the mathematical machinery related to posets and relations!

Proof of Theorem 7.6.1. We use induction. Let $P(n)$ be the proposition that every n -element poset has a topological sort.

Base case: Every 1-element poset is already a total order and thus is its own topological sort. So $P(1)$ is true.

Inductive step: Now we assume $P(n)$ in order to prove $P(n + 1)$ where $n \geq 1$.

Let (A, \leq) be an $(n + 1)$ -element poset. By Lemma 7.6.2, there exists a minimal element in $a \in A$. Remove a and all pairs in \leq involving a to obtain an n -element poset (A', \leq') . This has a topological sort (A', \leq'_T) by the assumption $P(n)$. Now we construct a total order (A, \leq_T) by adding a back as an element smaller than all

the others. Formally, let

$$\preceq_T = \preceq'_T \cup \{ (a, z) \mid z \in A \}.$$

All that remains is to check that this total order is consistent with the original partial order (A, \preceq) ; that is, we must show that

$$x \preceq y \text{ IMPLIES } x \preceq_T y.$$

We assume that the left side is true and show that the right side follows. There are two cases.

Case 1 If $x = a$, then $a \preceq_T y$ holds, because $a \preceq_T z$ for all $z \in A$.

Case 2 if $x \neq a$, then y can not equal a either, since a is a minimal element in the partial order \preceq . Thus, both x and y are in A' and so $x \preceq' y$. This means $x \preceq'_T y$, since \preceq'_T is a topological sort of the partial order \preceq' . And this implies $x \preceq_T y$ since \preceq_T contains \preceq'_T .

Thus, (A, \preceq_T) is a topological sort of (A, \preceq) . This shows that $P(n)$ implies $P(n + 1)$ for all $n \geq 1$. Therefore $P(n)$ is true for all $n \geq 1$ by the principle of induction, which proves the theorem. ■

7.7 Parallel Task Scheduling

When items of a poset are tasks that need to be done and the partial order is a precedence constraint, topological sorting provides us with a way to execute the tasks sequentially without violating the precedence constraints.

But what if we have the ability to execute more than one task at the same time? For example, suppose that the tasks are programs, the partial order indicates data dependence, and we have a parallel machine with lots of processors instead of a sequential machine with only one processor. How should we schedule the tasks so

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as to minimize the total time used?

For simplicity, assume all tasks take 1 unit of time and we have an unlimited

number of identical processors. For example, in the clothes example in Figure 7.11,

how long would it take to handle all the garments? *etc*

In the first unit of time, we should do all minimal items, so we would put on our

1
left sock, our right sock, our underwear, and our shirt. In the second unit of time,

we should put on our pants and our tie. Note that we cannot put on our left or right

shoe yet, since we have not yet put on our pants. In the third unit of time, we should

put on our left shoe, our right shoe, and our belt. Finally, in the last unit of time,

we can put on our jacket. This schedule is illustrated in Figure 7.13.

The total time to do these tasks is 4 units. We cannot do better than 4 units of

time because there is a sequence of 4 tasks, each needing to be done before the

next, of length 4. *With For*
why

1 Yes, we know that you can't actually put on ~~several~~
~~clothes at once, but for now~~
~~both socks at once, but imagine you are being~~
~~dressed by a bunch of robot processors and~~
~~you are in a big hurry. Still not working for you?~~
 OK, ~~assume these are programs for~~
~~forget about the clothes and imagine they are~~
~~programs with precedence constraints & shown in Figure 7.11.~~

Figure FS, p. 7-44, ftl-ch7-jul2.

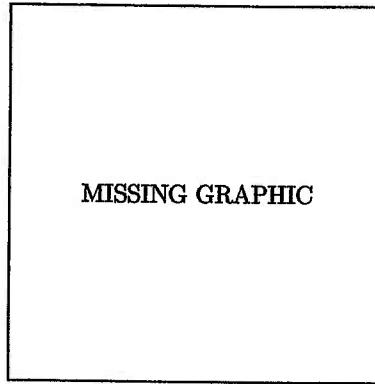


Figure 7.13: A parallel schedule for the tasks-in-getting-dressed poset in Fig-

ure 7.11. The tasks in A_i can be performed in step i for $1 \leq i \leq 4$. A chain

of length 4 (the critical path in this example) is shown with bold edges.

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pants before our belt, and our belt before our jacket. Such a sequence of items is known as a *chain*

Definition 7.7.1. A *chain* is a sequence $a_1 \preceq a_2 \preceq \dots \preceq a_t$, where $a_i \neq a_j$ for all $i \neq j$, such that each item is comparable to the next in the chain, and it is smaller with respect to \preceq . The *length* of the chain is t , the number of elements in the chain.

Thus, the time it takes to schedule tasks, even with an unlimited number of predecessors, is at least the length of the longest chain. Indeed, if we used less time,

[*a longest chain?*—dmj]

then two items from *a* chain would have to be done at the same time, which contradicts the precedence constraints. For this reason, *a* longest chain is also

known as a *critical path*. For example, Figure 7.13 shows the critical path for the getting-dressed poset.

In this example, we were in fact able to schedule all the tasks in t steps, where t is the length of the longest chain. The really nice thing about posets is that this

is always possible! In other words, for any poset, there is a legal parallel schedule that runs in t steps, where t is the length of the longest chain.

There are lots of ways to prove this fact. Our proof will also give us the corresponding schedule in t time steps, and allow us to obtain some nice corollaries.

Theorem 7.7.2. *Given any finite poset (A, \leq) for which the longest chain has length t , it is possible to partition A into t subsets A_1, A_2, \dots, A_t such that for all $i \in \{1, 2, \dots, t\}$ and for all $a \in A_i$, we have that all $b \leq a$ appear in the set*

$$A_1 \cup \dots \cup A_{i-1}.$$

Before proving this theorem, first note that for each i , all items in A_i can be scheduled in time step i . This is because all preceding tasks are scheduled in preceding time steps, and thus are already completed. So the theorem implies that

Corollary 7.7.3. *The total amount of parallel time needed to complete the tasks is the same as the length of the longest chain.*

X

A, put $a \in A_i$

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Proof of Theorem 7.7.2. For all $a \in A_i$, put $a \in A_i$, where i is the length of the longest chain ending at a . For example, the A_i for the getting-dressed poset are shown in Figure 7.13. In what follows, we show that for all i , for all $a \in A_i$ and for all $b \preceq a$ with $b \neq a$, we have $b \in A_1 \cup A_2 \cup \dots \cup A_{i-1}$.

We prove this by contradiction. Assume there is some i , $a \in A_i$, and $b \preceq a$ with $b \neq a$ and $b \notin A_1 \cup A_2 \cup \dots \cup A_{i-1}$. By the way we defined A_i , this implies there is a chain of length at least i ending at b . Since $b \preceq a$ and $b \neq a$, we can extend this chain to a chain of length at least $i + 1$, ending at a . But then a could not be in A_i . This is a contradiction. ■

If we have an unlimited number of processors, then the time to complete all tasks is equal to the length of the longest chain of dependent tasks. The case where there are only a limited number of processors is covered in the Problems section.

*very useful in practice
and it is*

7.8 Dilworth's Lemma

poset

Definition 7.8.1. An *antichain* in a partial order is a set of elements such that any two elements in the set are incomparable.

For example, each A_i in the proof of Theorem 7.7.2 and in Figure 7.13 is an

its elements have
~~they have~~ antichain since they have no dependencies between them (which is why they could

be executed at the same time).

Our conclusions about scheduling also tell us something about antichains.

Corollary 7.8.2. If the largest chain in a partial order on a set A is of size t , then

A can be partitioned into t antichains.

Let

Proof. Let the antichains be the sets A_1, A_2, \dots, A_t . ~~defined in~~ \rightarrow
Theorem 7.7.2.

Corollary 7.8.2 implies a famous result⁷ about partially ordered sets:

⁷Lemma 7.8.3 also follows from a more general result known as Dilworth's Theorem that we will

X

(Dilworth)

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Lemma 7.8.3. *For all $t > 0$, every partially ordered set with n elements must have*

either a chain of size greater than t or an antichain of size at least n/t .

Proof. By contradiction. Assume that the longest chain has length at most t and

the longest antichain has size less than n/t . Suppose that the largest chain is of

size $\leq t$. Then Corollary 7.8.2, the n elements can be partitioned into at most t antichains. Hence, there are fewer than $t \cdot n/t = n$ elements in the poset, which is

a contradiction. Hence there must be a chain longer than t or an antichain with at

least n/t elements. ■

Corollary 7.8.4. *Every partially ordered set with n elements has a chain of size*

greater than \sqrt{n} or an antichain of size at least \sqrt{n} .

Proof. Set $t = \sqrt{n}$ in Lemma 7.8.3. ■

As an application, consider a permutation of the numbers from 1 to n arranged
not discuss.

as a sequence from left to right on a line. Corollary 7.8.4 can be used to show

that there must be a \sqrt{n} -length subsequence of these numbers that is completely

increasing or completely decreasing as you move from left to right. For example,

the sequence

7, 8, 9, 4, 5, 6, 1, 2, 3

has an increasing subsequence of length 3 (for example, 7, 8, 9) and a decreasing

subsequence of length 3 (for example, 9, 6, 3). The proof of this result is left as an

exercise that will test your ability to find the right partial order on the numbers in

the sequence.

7.9 Problems