Lecture 18: Generating Functions

Reading: 12.1 – 12.6

Theme: Counting and solving recurrences by manipulating functions

infinite sequence $\qquad$ power series

$$(a_0, a_1, a_2, a_3, \ldots) \longleftrightarrow A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Examples:

$$(3, 2, 1, 0, 0 \ldots) \longleftrightarrow A(x) = 3 + 2x + x^2$$

$$(1, 1, 1, 1, 1 \ldots) \longleftrightarrow A(x) = 1 + x + x^2 + x^3 + \ldots$$

$$= \frac{1}{1-x} \quad (\text{for } |x| < 1)$$

$$(1, a, a^2, a^3, a^4, \ldots) \longleftrightarrow A(x) = \frac{1}{1-ax}$$

Let's build up a dictionary of operations

$$(a_0, a_1, a_2, \ldots) \longleftrightarrow a_0 + a_1 x + a_2 x^2 + \ldots = A(x)$$

$$(b_0, b_1, b_2 \ldots) \longleftrightarrow b_0 + b_1 x + b_2 x^2 + \ldots = B(x)$$

scaling:

$$(ca_0, ca_1, ca_2, \ldots) \longleftrightarrow ca_0 + ca_1 x + ca_2 x^2 \ldots = cA(x)$$

addition:

$$(a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots) \longleftrightarrow (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots = A(x) + B(x)$$

right shift:

$$(\underbrace{0, 0, \ldots 0}_{k}, a_0, a_1, a_2, \ldots) \longleftrightarrow a_0 x^k + a_1 x^{k+1} \ldots$$

$$= x^k A(x)$$

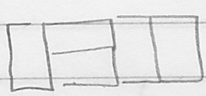
differentiation:

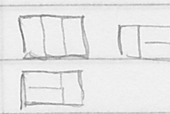
$$(a_1, 2a_2, 3a_3, \ldots) \longleftrightarrow \frac{d}{dx} A(x) = \sum_{i=1}^{\infty} (a_i) i x^{i-1}$$

$$= \sum_{i=0}^{\infty} (a_{i+1})(i+1) x^i$$

Now that we know how to operate on a function, as a proxy for manipulating a series, let's use it:

Example: Tile a $2 \times n$ board with dominos

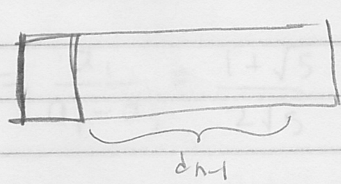
e.g.  or 

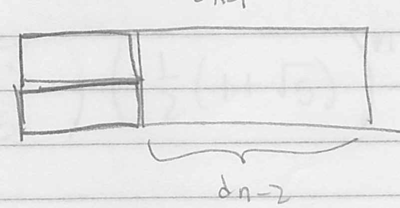
How many total ways to tile? Try out $n=3$ 

step#1 Find the recurrence

Let d_n = # ways for a $2 \times n$ board

Then two cases:

(1) 

(2) 

$f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0$, $f_1 = 1$

Hence $d_n = d_{n-1} + d_{n-2}$ with $d_0 = 1$, $d_1 = 1$

step #2 Find the generating function

Now let $D(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \dots$

$-1\left(x D(x) = d_0 x + d_1 x^2 + d_2 x^3 + \dots \right)$

$-1\left(x^2 D(x) = d_0 x^2 + d_1 x^3 + \dots \right)$

$D(x) - x D(x) - x^2 D(x) = d_0 + (d_1 - d_0) x = 1$

$$D(x) = \frac{1}{1-x-x^2}$$

step #3: Partial fractions

$$D(x) \stackrel{?}{=} \frac{c_1}{1-a_1 x} + \frac{c_2}{1-a_2 x} = \sum_{i=0}^{\infty} \left(c_1 (a_1)^i + c_2 (a_2)^i \right) x^i$$

$1-x-x^2 = (1-a_1 x)(1-a_2 x)$ where $a_1 = \frac{1}{2}(1+\sqrt{5})$

$a_2 = \frac{1}{2}(1-\sqrt{5})$

Finally solve $c_1(1-a_2 x) + c_2(1-a_1 x) = 1$

$c_1 + c_2 = 1 \qquad c_1 a_2 + c_2 a_1 = 0$

$$c_1 = \frac{a_1}{a_1 - a_2} = \frac{1+\sqrt{5}}{2\sqrt{5}} \qquad c_2 = \frac{-a_2}{a_1 - a_2} = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$\Rightarrow d_n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1}{2}(1+\sqrt{5})\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1}{2}(1-\sqrt{5})\right)^n$$

It's not obvious this is even an integer!

Let's try out $n=6$

$n=5 \quad d_n = 8$

$n=6 \quad d_n = 13$

Other ways to extract the coefficients

$$D(x) = d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \dots$$

$$D(0) = d_0$$

$$\left.\frac{d\,D(x)}{dx}\right|_{x=0} = d_1 \qquad \left.\frac{d^2\,D(x)}{dx^2}\right|_{x=0} = 2 d_2$$

More generally

$$\left.\frac{d^n\,D(x)}{dx^n}\right|_{x=0} = n!\, d_n$$

Thus another way to extract coefficients is the following:

<u>Lemma</u>: If $D(x)$ is the generating function for the sequence $(d_0, d_1, d_2, \ldots)$ then

$$\left(\frac{1}{n!}\right)\frac{d^n}{dx^n} D(x)\Big|_{x=0} = d_n$$

This rule is sometimes difficult to use, e.g. $D(x) = \frac{1}{1-x-x^2}$

<u>Example</u>: $D(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$

Then $\frac{d}{dx} D(x) = 2(1-x)^{-3}$

$$\frac{d^2}{dx^2} D(x) = 2\cdot 3(1-x)^{-4}$$

$$\frac{d^n}{dx^n} D(x) = (2\cdot 3\cdot 4 \cdots (n+1))(1-x)^{-(n+2)}$$

Thus $d_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{dx^n} D(x)\Big|_{x=0}\right) = n+1$

What about multiplying generating functions?

Again, let $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \ldots$$

then let $C(x) = A(x)B(x)$

$$= c_0 + c_1 x + c_2 x^2 + \ldots$$

what is c_n?

	b_0	$b_1 x$	$b_2 x^2$	$b_3 x^3$
a_0	$a_0 b_0$	$a_0 b_1 x$	$a_0 b_2 x^2$	
$a_1 x$	$a_1 b_0 x$	$a_1 b_1 x^2$		
$a_2 x^2$	$a_2 b_0 x^2$			
$a_3 x^3$				
$\vdots$				

terms with same power of x appear on same diagonal

$c_0 = a_0 b_0$, $c_1 = a_1 b_0 + a_0 b_1$,

$c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$, etc

Convolution: $A(x) B(x) = C(x)$

where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$

Interpretation:

- Let a_i = # ways of selecting i items from A
- Let b_j = " j " B

Then c_n = # ways of selecting n items from $A \cup B$

Example: Fruit salad

- # apples must be even
- # bananas must be multiple of 5

- # oranges at most 4
- # pears at most 1

How many ways are there with 6 total fruits?

A	6	4	4	2	2	0	0
B	0	0	0	0	0	5	5
O	0	2	1	4	3	1	0
P	0	0	1	0	1	0	1

Let's build a generating function for each fruit:

$$A(x) = 1 + x^2 + x^4 + x^6 \ldots = \frac{1}{1-x^2}$$

$$B(x) = 1 + x^5 + x^{10} + x^{15} \ldots = \frac{1}{1-x^5}$$

$$O(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$$

$$P(x) = 1 + x = 1 + x$$

Then $A(x)B(x)O(x)P(x) = \dfrac{1}{(1-x)^2} = 1 + 2x + 3x^3 + \ldots$

(from earlier example)