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Date 23/08/2021

1. Introduction to Linear Algebra

1. Basics of Matrices
2. Rank of a Matrix
3. System of Linear equations $\rightarrow Ax = B$, $\rightarrow Ax = 0$
4. Eigen Values & Eigen Vectors
5. Special Matrices
6. LU Decomposition
7. Minimal Polynomial

1.1 Basics of Matrix

→ **Matrix multiplication**

let ~~Amn~~ $A_{m\times n}$, $B_{p\times q}$

- $(AB)_{m\times q}$ exists if $n = p$
- $(BA)_{p\times n}$ exists if $q = m$

- $AB \neq BA$ (Not Commutative)
- $(AB)C = A(BC)$ (it is Associative)
- $AB = 0$ need not imply $A = 0$ or $B = 0$

Eg: $A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ $AB = 0$ // $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- Let $A_{m\times n}$ and $B_{n\times p}$

The product $(AB)_{m\times p}$ involves

- $mnp \rightarrow$ ~~basic~~ no. of multiplications
- $m(n-1)p \rightarrow$ ~~basic~~ no. of additions.

Eg:- $A_{3\times 4}$, $B_{4\times 5}$

$AB_{3\times 5}$ contains $3\times 4\times 5 = 60$ multiplications

$3\times 3\times 5 = 45$ additions //

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Find min add's & mult's involved.

Eg:- $A_{2\times3}\, B_{3\times4}\, C_{4\times3}$

$ABC \to A(BC)$: $(BC)_{3\times3} = 3\times4\times3 = 36$ mult, $3\times3\times3 = 27$ add.

$A(BC)_{2\times3} = 2\times3\times3 = 18$ mult, $2\times2\times3 = 12$ add.

$ABC \to (AB)C$: $(AB)_{2\times4} = 2\times3\times4 = 24$ mult, $2\times2\times4 = 16$ add.

$(AB)C_{2\times3} = 2\times4\times3 = 24$ mult, $= 2\times3\times3 = 18$ add.

minimum

$\therefore$ min no. of multiplications involved $= 48$//

min no. of add's involved $= 34$//

• Trace of a Matrix

Let A be $n\times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

principal diagonal.

Trace (A) = Sum of principal diagonal elements.

$= a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$//

Eg:- $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \\ 3 & 8 & 9 \end{bmatrix}_{3\times3}$, Trace (A) $= 1+5+9 = \underline{\underline{15}}$

Properties: $Tr(A \pm B) = Tr(A) \pm Tr(B)$

$Tr(A) = Tr(A^T)$

$Tr(AB) = Tr(BA)$

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• Diagonal Matrix

Let A be Matrix of order $n\times n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

A is said to be a diagonal Matrix, if $a_{ij} = 0$ for $i \neq j$

Eg:- $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times3}$ (or) $D = diag(2, -3, 4)$//

Usually diagonal Matrices are denoted by D, and

$D = diag(d_1, d_2 \cdots d_n)$

Note: $D^k = diag(d_1^k, d_2^k, \cdots, d_n^k)$

Eg:- $D = diag(2, -3, 4)$

$D^2 = diag(4, 9, 16) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}_{3\times3}$//

$D^3 = diag(8, -27, 64)$

• Upper Triangular Matrix:

Let $A_{n\times n}$

$$\begin{bmatrix} a_{11} & \cdots & \cdots \\ \text{Zeros} & \ddots & \vdots \\ & & a_{nn} \end{bmatrix}_{n\times n}$$

Eg:- $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 7 & 8 \\ 0 & 0 & 6 \end{bmatrix}_{3\times3}$

$a_{ij} = 0$ for $i > j$

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- Lower Triangular Matrix

$a_{ij} = 0$ for $i < j$

Eg:- $A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 5 \end{bmatrix}_{3\times3}$

- Inverse of a Matrix

1. $|A| = 0$ then A is called Singular Matrix
   $|A| \neq 0$ then A is called non-Singular Matrix

2. A^{-1} exists $\iff$ $|A| \neq 0$ (Only for non-singular matrix)

3. if $\boxed{AB = BA = I}$ then
   B is called ~~Transpose~~ Inverse of A and denoted by
   $\boxed{B = A^{-1}}$
   i.e. $\boxed{AA^{-1} = A^{-1}A = I}$

4. $A^{-1} = \dfrac{adj(A)}{|A|}$ $\qquad$ $adj(A) = (\text{Cofactor Matrix})^T$

Eg:- $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}$

$|A| = 1\times(3+2) \quad -3(6-2) \quad +4(-2-1)$
$= 5 - 12 - 12 = -19$

~~adj(A)~~ Cofactor Matrix $= \begin{bmatrix} +(3+2) & -(6-2) & +(-2-1) \\ -(9+4) & +(3-4) & -(-1-3) \\ +(6-4) & -(2-8) & +(1-6) \end{bmatrix} = \begin{bmatrix} 5 & -4 & -3 \\ -13 & -1 & 4 \\ 2 & 6 & -5 \end{bmatrix}$

$adj(A) = (\text{Cofactor Matrix})^T$

$= \begin{bmatrix} 5 & -13 & 2 \\ -4 & -1 & 6 \\ -3 & 4 & -5 \end{bmatrix}$

$\dfrac{adj(A)}{|A|} = \dfrac{-1}{19}\begin{bmatrix} 5 & -13 & 2 \\ -4 & -1 & 6 \\ -3 & 4 & -5 \end{bmatrix}$ //

* Shortcut to Find the determinant & Inverse of 3x3 Matrix

$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}$

det(A) =

$\begin{matrix} 1 & 3 & 4 & 1 & 3 \\ 2 & 1 & 2 & 2 & 1 \\ 1 & -1 & 3 & 1 & -1 \end{matrix}$

$20 = (4 + -2 + 18)$ $(3 + 6 + -8)$

$1 - 20 = -19$ → Det(A)

Inverse(A)

$A^{-1} = \dfrac{-1}{19}\begin{bmatrix} d & c \\ b & a \end{bmatrix} = \dfrac{-1}{19}\begin{bmatrix} 1 & 2 & 2 & 1 \\ -1 & 3 & 1 & -1 \\ 3 & 4 & 1 & 3 \\ 1 & 2 & 2 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}$ $\qquad = \dfrac{-1}{19}\begin{bmatrix} 5 & -13 & 2 \\ -4 & -1 & 6 \\ -3 & 4 & 5 \end{bmatrix}$

Properties

1. $(AB)^T = B^T A^T$ 2. $(A^T)^T = A$

3. $(A^{-1})^T = (A^T)^{-1}$ 4. $|A^{-1}| = \frac{1}{|A|}$

5. $(AB)^{-1} = B^{-1}A^{-1}$ 6. $(A^{-1})^{-1} = A$

7. $|adj(A)| = |A|^{n-1}$, $|adj(adj(A))| = |A|^{(n-1)^2}$

$|\underbrace{adj(\cdots adj(A))}_{k\,times}| = |A|^{(n-1)^k}$

29/07/22

* Determinant of a Matrix.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 9 \\ 1 & -1 & 4 \end{bmatrix}, \quad |A| = 1(-1)^{1+1}\begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 4(-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(7) - 2(5) + 4(-3) = -15$$

$|A| = det(A) =$ Sum of products of any row (column) element and corresponding cofactors.

Properties

1. $det(A) = det(A^T)$, $|A| = |A^T|$

2. If a Matrix has zero row (column) then $|A| = 0$

Eg:- $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \underline{|A| = 0}$

zero row

3. If two rows/columns of a Matrix are equal or proportional, then $|A| = 0$

Eg:- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$; $\underline{\underline{|A| = 0}}$

4. If two Rows/Columns of the Matrix are interchanged, then det is to be multiplied with -1

Eg:- $A = \begin{bmatrix} 1 & 4 \\ 3 & 15 \end{bmatrix}$, $|A| = 3$

$B = \begin{bmatrix} 3 & 15 \\ 1 & 4 \end{bmatrix}$, $|B| = -3$

5. The determinant of diagonal / lower triangular / upper triangular matrix is just product of diagonal elements only.

6. If all elements of a Row/Column of Matrix are Scalar multiple of k, then $|A| = k|A|$

Eg:- $\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = \underline{\underline{-2}}$

7. If all elements of the Matrix are multiplied by a scalar k then $|kA| = k^n|A|$

Eg:- $A = \begin{bmatrix} 2 & 4 \\ 6 & 10 \end{bmatrix}$, $|A| = \begin{vmatrix} 2 & 4 \\ 6 & 10 \end{vmatrix} = 2^2\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -4$

8. If every element of a Row/Column is multiplied by a scalar and added to another row/column, then the determinant remains the same.

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$R_j \to R_j + \alpha R_i \;/\; C_j \to C_j + \alpha C_i$

Eg:- $A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}$, $|A| = 2$

$B = R_2 \to R_2 + 3R_1$

$B = \begin{bmatrix} 1 & 3 \\ 5 & 17 \end{bmatrix}$, $|B| = 2$

Eg:- $C = C_2 \to C_2 + 2C_1$

$C = \begin{bmatrix} 1 & 5 \\ 2 & 12 \end{bmatrix}$; $|C| = 2$ $\therefore |A| = |B| = |C| = 2$

9. $|AB| = |A||B|$

10. $|A^k| = |A|^k$

* Problems

Q1 If $P_{10\times5}$, $Q_{5\times20}$, $R_{20\times10}$ are multiplied, then the min. no. of multiplications involved in computing the product PQR is ____

Ans $PQR \to (PQ)R \to \underset{PQ}{10\times5\times20} + \underset{(PQ)R}{10\times20\times10} = 1000 + 2000 = \underline{3000}$

$P(QR) \to QR = 5\times20\times10 + P(QR) = 10\times5\times10$

$= 1000 + 500$

$= \underline{1500}$

$\underline{1500}$

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Q2 If $\theta \in R$ then the maximum value of

$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ is ____.

Ans $[(1+\sin\theta)(1+\cos\theta) - 1] - [(1+\cos\theta) - 1] + [1 - 1 - \sin\theta]$

$\cancel{1} + \cancel{\cos\theta} + \cancel{\sin\theta} + \sin\theta\cos\theta - \cancel{1} - \cancel{\cos\theta} - \cancel{\sin\theta}$

$= \sin\theta\cos\theta$

$\frac{d}{d\theta}\sin\theta\cos\theta = \frac{1}{2} \times 2\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

max of $\Delta = \frac{1}{2} \times 1 = \frac{1}{2}$

$2\sin\theta\cos\theta = \sin 2\theta$

(or)

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$ $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \cos\theta \end{bmatrix}$ $\det(A) = 1 \times \sin\theta \cos\theta$

Q3 If $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & x \end{vmatrix}$ then $\lim_{x\to0} \frac{f(x)}{x^2} =$ ____

Ans $\lim_{x\to0} \frac{\sin x}{x} = 1$ $\quad \lim_{x\to0} \frac{\tan x}{x} = 1$

$\frac{f(x)}{x^2} = \frac{1}{x^2} \begin{vmatrix} \cos x & x & 1 \\ \frac{2\sin x \cdot x}{x} & x^2 & 2x \\ \frac{\tan x \cdot x}{x} & x & x \end{vmatrix} = \frac{x^2}{x^2} \begin{vmatrix} \cos x & x & 1 \\ \frac{2\sin x}{x} & x & 2 \\ \frac{\tan x}{x} & 1 & 1 \end{vmatrix}$

$$\lim_{x\to 0} \frac{f(x)}{x^2} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \cancel{-2+1=-1} \; 0$$

By $i=6$ property 3

(Q4) if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ then $|adj(A^{-1})| =$ ______.

Ans $\frac{adj(A)}{|A|} = A^{-1}$

$\frac{adj(A^{-1})}{|A^{-1}|} = A$

$adj(A^{-1}) = A|A^{-1}|$ $\left(|A^{-1}| = \frac{1}{|A|}\right)$

$= \frac{A}{|A|}$

$|A| = (1-4) - 2(2-4) + 2(4-2)$

$= -3+4+4 = 5$

$adj(A) = \frac{A}{5} = \begin{bmatrix} 1/5 & 2/5 & 2/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix}$

But Q is for $|adj(A^{-1})|$

$|adj(A)| = |A|^{n-1}$

$|adj(A^{-1})| = |A^{-1}|^{n-1} = \frac{1}{|A|^{n-1}} = \frac{1}{5^{3-1}} = \frac{1}{5^2} = \frac{1}{25}$

Ans

(Q5) If the Matrix M_r is given by $M_r = \begin{bmatrix} r & r-1 \\ r-1 & r \end{bmatrix}$

$r = 1, 2, 3, \ldots$ Then

$\text{Det}M_1 + \text{Det}M_2 + \cdots + \text{Det}M_{18} =$ ______.

Ans $r^2 - (r-1)^2$

$= r^2 - (r^2 + 1 - 2r)$

$= 2r - 1$

Ans $= 2(1+2+\cdots+18) - 18$

$= 2\left(\frac{18\times 19}{2}\right) - 18$

$= 18(19-1) = 18^2 = 324$

$\begin{array}{r} 18 \\ 18 \\ \hline 144 \\ 180 \\ \hline 324 \end{array}$

(or)

$2r-1$

$1 + 3 + 5 + \cdots 35$ (Arithmetic Progression)

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$\frac{18}{2}\left[2\times(1) + (18-1)2\right] = 324$

(Q6) If $A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then the determinant of $adj(adj A)$ is ______.

A $|adj(adj A)| = |A|^{(n-1)^2} = |A|^4$

$|A| = 3 - 2\times 4 = -5$

$(-5)^4 = 625$

Gate 1999

(Q7) The no. of terms in the expression of general determinant of order n is ____.

(a) n^2 (b) $n!$ (c) n (d) $(n+1)^2$.

Ans let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = ad - bc = 2$

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A| = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{2\text{ terms}} - a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{2\text{ terms}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{2\text{ terms}}$ = 6 terms

$= 6 = 3!$

$\therefore$ (b) $n!$

(Q8) Consider the Matrix $J_6 = \begin{bmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \end{bmatrix}$ which is obtained by reversing the order of the columns of the identity Matrix I_6.

Let $P = I_6 + \alpha J_6$, where α is non-negative Real No., the value of α for which $\det(P) = 0$ is ____.

Ans $P = \begin{bmatrix} 1&0&0&0&0&\alpha \\ 0&1&0&0&\alpha&0 \\ 0&0&1&\alpha&0&0 \\ 0&0&\alpha&1&0&0 \\ 0&\alpha&0&0&1&0 \\ \alpha&0&0&0&0&1 \end{bmatrix}$

$\alpha = 1$ (property 3 R no. 4)

Gate 2014

(Q9) Given that the determinant of Matrix $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ is -12

then the determinant of the Matrix $\begin{bmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{bmatrix}$ is ____ (a) -96 (b) -24 (c) 24 (d) 96

Ans $2^3 \times -12 = 8 \times -12 = \underline{\underline{-96}}$ (a) (Page 1, Property 6 & 3)

(Q10) Perform the following operations on the matrix $\begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix}$

1. Add the third row to the second row.
2. Subtract the third column from the first column.

The determinant of the Resultant Matrix is ____.

Ans (Page no. 7, Property 8 & 3) det = 15 $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\frac{45}{135}{60}$$

$= \begin{vmatrix} 3 & 4 & 45 \\ -2 & -3 & -30 \\ 4 & -10 & 60 \end{vmatrix}$

$3(-180 + 300) - 4(\cancel{60} - 120 + 120) + 45(20 + 12)$

$\cancel{-1440}$ $\cancel{2913}$ $+ 45 \times 32$

$-14\cancel{0}0 + 1440 = \cancel{0}$

$$\begin{array}{r} 1440 \\ 360 \\ \hline 1800 \end{array} \qquad \begin{array}{r} 45 \\ \times 32 \\ \hline 90 \\ 1350 \\ \hline 1440 \end{array}$$

4×360, 1440

$3 \times 15 = 45$

$7 \times 15 = 105$

$13 \times 15 = 195$

15 $\frac{15}{7}{105}$ 13

$\therefore$ det = 0

(Q11) If $A_n = \begin{vmatrix} 1 & k & k \\ 2n & k^2+k+1 & k^2+k \\ 2n-1 & k^2 & k^2+k+1 \end{vmatrix}$ and $\sum_{n=1}^{k} A_n = 72$.

Then $k =$ ______.

A) $A_1 + A_2 + \cdots A_k = 72$

$C_3 \to C_3 - C_2$

$A_n = \begin{vmatrix} 1 & k & 0 \\ 2n & k^2+k+1 & 1 \\ 2n-1 & k^2 & k+1 \end{vmatrix} =$

$= \left[(k^2+k+1)(k+1) - k^2 \right] - k\left[2n(k+1) - \cancel{2n(\ldots)} (2n-1) \right]$

$= k^3 + k^2 + k + k^2 + k + 1 - \cancel{k^2} - k\left[2nk + 2n - 2n + 1 \right]$

$= k^3 + k^2 + 2k + 1 - 2nk^2 + k$

$= k^3 + 3k + k^2 - 2nk^2 + 1$

$= 1$

$R_2 = R_2 - R_3$

$\begin{vmatrix} 1 & k & 0 \\ -1 & k+1 & -k \\ 2n-1 & k^2 & k+1 \end{vmatrix} = \left[(k+1)^2 + k^3 \right] - k\left[-(k+1) + k(2n-1) \right]$

$R_1 \to R_1 + R_2$

$= (k+1)^2 + k^3 - k\left[-k+1 \right] + -k^2(2n-1)$

$= (k+1)^2 + k^3 + \cancel{k} - \cancel{k} - 2nk^2 + k^2$

$0 \longrightarrow 2k+1$

$A_n = \cancel{k^3 + (k+1)^2 + k^2 - 2nk^2}$

$A_n = k^2(k$

$\sum_{n=1}^{k} A_n = \begin{vmatrix} \sum_{n=1}^{k} 1 & k & k \\ \sum_{n=1}^{k} 2n & k^2+k+1 & k^2+k \\ \sum_{n=1}^{k} (2n-1) & k^2 & k^2+k+1 \end{vmatrix}$

$= \begin{vmatrix} k & k & k \\ \frac{2k(k+1)}{2} & k^2+k+1 & k^2+k \\ \frac{2k(k+1)}{2} - k & k^2 & k^2+k+1 \end{vmatrix}$

$= \begin{vmatrix} k & k & k \\ k^2+k & k^2+k+1 & k^2+k \\ k^2 & k^2 & k^2+k+1 \end{vmatrix}$

12×6

$4 \times 3 \times 6 \, 3 \times 2$

$\begin{vmatrix} 0 & 0 & k \\ 0 & 1 & k^2+k \\ -k-1 & -k-1 & k^2+k+1 \end{vmatrix} = \cancel{k(-k-1)}$

$k(0 - (-k-1))$

$= k(k+1) = 72$

$\underline{8 \times 9} = 72$

$k = 8$

(Q.2) If each element of a 2×2 determinant is either 0 or 1, then the probability that the determinant is positive is ______

Ans

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} \Big\} \text{ 3 possibilities for positive.}$$

$$3/2^4 = \frac{3}{16}$$

24/02/2021 **1.2 Rank of a Matrix.**

Let A, be any non-zero Matrix of order $m \times n$

$r = \rho(A)$ = The order of largest non-zero minor.
minor = det of SubMatrix (Square)
Submatrix = Matrix obtained by deleting Rows & Columns

Eg:- $A = \begin{bmatrix}1 & 2 & 3 & 4\\2 & 4 & 6 & 8\\1 & 2 & 9 & 4\end{bmatrix}_{3\times4}$

3×3 Minors

$$\underbrace{\begin{vmatrix}1 & 2 & 3\\2 & 4 & 6\\1 & 2 & 9\end{vmatrix} = 0,\ \begin{vmatrix}2 & 3 & 4\\4 & 6 & 8\\2 & 9 & 4\end{vmatrix} = 0,\ \begin{vmatrix}1 & 3 & 4\\2 & 6 & 8\\1 & 9 & 4\end{vmatrix} = 0}_{\text{proportional columns}} \text{ - 4}$$

2×2 Minors

$$\begin{vmatrix}1 & 2\\2 & 4\end{vmatrix}, \begin{vmatrix}2 & 3\\4 & 6\end{vmatrix}, \begin{vmatrix}6 & 8\\9 & 4\end{vmatrix} \ldots$$

∴ atleast one 2×2 non-zero minor exists. ∴ $\rho(A) = 2$

(Rank never becomes zero, the least rank possible - 1) for 1×1 minor.
Only for null matrix all become zero.

Eg: $A = \begin{bmatrix}0 & 0 & -3\\9 & 3 & 5\\3 & 1 & 1\end{bmatrix}_{3\times3}$

3×3 minors
$|A| = 0$ (proportional column)
∴ $\rho(A) \neq 3$

2×2 minors

$\begin{vmatrix}9 & 3\\3 & 1\end{vmatrix} = 0$, $\begin{vmatrix}0 & -3\\3 & 5\end{vmatrix}$ [illegible]
atleast

$\begin{vmatrix}3 & 5\\1 & 1\end{vmatrix} = 3-5 = -2$ atleast one non-zero minor exists ∴ $\rho(A) = 2$

Properties.

1. $\rho(A_{m\times n}) \le \min(m, n)$
2. $\rho(A_{n\times n}) = n$ if $(|A| \neq 0)$
   $< n$ if $(|A| = 0)$
3. $\rho(A) = \rho(A^T)$
4. $\rho(A+B) \le \rho(A) + \rho(B)$
5. $\rho(AB) \le \min(\rho(A), \rho(B))$
6. $\rho(A) \neq 0$
   $= 0$ only for null matrix
7. if $\rho(A) = n \Rightarrow \rho(\operatorname{adj} A) = n$
   $= n-1 \Rightarrow \rho(\operatorname{adj} A) = 1$
   $\le n-2 \Rightarrow \rho(\operatorname{adj} A) = 0$
   adj A will be a null matrix.

(Note: The Rank of Matrix is not affected by elementary Row operations or Column operations)

* Row Echelon form (alternate approach to find the rank of a Matrix)

Let A be $m \times n$ Matrix.
A is said to be in Row Echelon form if
1. Zero rows (if any) should be below the non-zero rows.
2. Zeros before first non-zero number in a row should be less than zeros before first non-zero number in next row.

Then $\boxed{r = \rho(A) = \text{no. of non-zero rows in Echelon form}}$

General form of Row Echelon form

$$\begin{bmatrix} \text{nonzero} & \cdots & \cdots & \cdots \\ 0 & \text{nonzero} & \cdots & \cdots \\ 0 & 0 & \text{nonzero} & \cdots \\ 0 & 0 & 0 & \text{nonzero} \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ $\underline{\rho(A) = 2}$

$B = \begin{bmatrix} 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 5 & 3 & 1 \\ 0 & 0 & 0 & 7 & 8 \end{bmatrix}_{3\times5}$, $\rho(B) = 3$

$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}_{4\times3} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$, $\underline{\underline{\rho(C) = 3}}$

Eg: $A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}_{3\times4} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & -3 & 3 & 9 \end{bmatrix} \xrightarrow{R_3 \to 2R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$ //

Problems

Gate
(Q1) A 5×7 Matrix has all its entries equal to 1. Then the Rank of Matrix is ______.
ⓐ 7 ⓑ 5 ⓒ 1 ⓓ zero

Ans $\underline{\underline{1}}$ ⓒ

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & & \\ \vdots & & & \text{zeros} & & & \\ 0 & & & & & & \end{bmatrix}_{5\times7}$$ ⓒ

Gate
(Q2) The Rank of the following Matrix
$$\begin{bmatrix} 1 & a & a^2 & a^3 & \cdots & a^n \\ 1 & a & a^2 & a^3 & \cdots & a^n \\ \vdots & & & & & \\ 1 & a & a^2 & a^3 & \cdots & a^n \end{bmatrix}_{(n+1)(n+1)}$$ here a is a real no.
is ______
ⓐ 1 ⓑ 2 ⓒ n ⓓ depends on the value of a.

Ans ⓐ 1 (Similar to previous Q logic) $\rightarrow \begin{bmatrix} 1 & a & a^2 & a^3 & \cdots & a^n \\ 0 & 0 & 0 & 0 & \cdots & - \\ \vdots & & & \text{zeros} & & \\ 0 & & & & & \end{bmatrix}$

$\underline{\underline{\rho(A) = 1}}$

Gate 2014
(Q3) $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ $B = \begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix}$. If the Rank of Matrix A is N, then the rank of B is ______.
ⓐ $\frac{N}{2}$ ⓑ $N-1$ ⓒ N ⓓ $2N$

Ans $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\rho(A) = 2$ — $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$ $\rho(B) = 2$

$\underline{\underline{ⓒ}}$

(Q4) The rank of the Matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is ______.

Ans

$\boxed{e(A) = e(A^T)} \rightarrow$ $\begin{bmatrix} 6 & -2 & 14 \\ 0 & 14 & -14 \\ 4 & 8 & 0 \\ 4 & 18 & -10 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 6 & -2 & 14 \\ 0 & 14 & -14 \\ 0 & -10 & 0 \\ 4 & 18 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & -2 & 14 \\ 4 & 18 & -10 \\ 0 & -10 & 0 \\ 0 & 14 & -14 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 6 & -2 & 14 \\ 4 & 18 & -10 \\ 0 & 14 & -14 \\ 0 & -10 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 12 & 14 \\ 4 & 8 & -10 \\ 0 & 0 & -14 \\ 0 & -10 & 0 \end{bmatrix}$

$R_2 \rightarrow 3R_2 + R_1$
$R_3 \rightarrow 3R_3 - 7R_1 \rightarrow \begin{bmatrix} 6 & 0 & 4 & 4 \\ 0 & 42 & 28 & 58 \\ 0 & -42 & -28 & -58 \end{bmatrix}$

$\begin{bmatrix} 6 & 0 & 4 & 4 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow e(A) = 2$

Gate 2015

(Q5) Let $A = (a_{ij})$, $1 \le i, j \le n$, with $n \ge 3$ and $a_{ij} = i \cdot j$. Then Rank of A is ______.
(a) 0 (b) 1 (c) $n-1$ (d) n

Ans $i = 3, j = 3$

$A = \begin{bmatrix} 1\cdot1 & 1\cdot2 & 1\cdot3 \\ 2\cdot1 & 2\cdot2 & 2\cdot3 \\ 3\cdot1 & 3\cdot2 & 3\cdot3 \end{bmatrix}_{3\times3} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}_{3\times3}$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$e(A) = 1$ (b)

Gate 2017

(Q6) Let $P = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ $Q = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ be two Matrices.

Then the Rank of $(P+Q)$ is ______.

A $e(P+Q) \le e(P) + e(Q)$ (property 2, gate 2017)

$P+Q = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 & -2 \\ 0 & 1 & 2 \\ 8 & 8 & 8 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 9 & 10 \end{bmatrix}$ $\det(A) = 0$, $\therefore e(A) < 3$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 8 & 8 & 8 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 8 & 8 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $e(A) = 2$

$\begin{pmatrix} 8 & 9 & 10 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ $e(A) = 2$

(Q8) The Rank of the Matrix $A = \begin{bmatrix} 1 & -1 & 0 & -2 \\ 2 & 0 & 2 & -2 \\ 4 & 1 & 3 & -8 \end{bmatrix}$ is ______.

Sol $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$= \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 5 & 3 & 0 \end{bmatrix}_{3\times 4}$

$\rho(A) \le 3$

$R_3 \to 2R_3 - 5R_2$

$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -4 & -10 \end{bmatrix}$ Row Echelon form

$\cancel{\rho(A) \le 3}$

$\rho(A) = 3$

(Q9) The Rank of the Matrix $\begin{bmatrix} -1 & 2 & 5 \\ 2 & -4 & a-4 \\ 1 & -2 & a+1 \end{bmatrix}$ is

(I) 1 if $a = -6$ (II) is 2 if $a \ne -6$

Which of these statements are correct.

Ans $R_3 \to R_3 + R_1$
$R_2 \to R_2 + 2R_1$

$\begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & (a-4)+10 \\ 0 & 0 & (a+1)+5 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} -1 & 2 & 5 \\ 0 & 0 & a+6 \\ 0 & 0 & a+6 \end{bmatrix}$ // (I) & (II) are correct

(Q10) The Rank of Matrix $\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{5\times 5}$ is ______.

Ans $R_2 \leftrightarrow R_4$

$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1}$

$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

$R_3 \to R_3 + R_2$

$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_4 \to R_4 + R_3} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

$R_5 \to R_5 - R_4$

$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 4$

1.5 System of Linear Equations

- $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$
  $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$
  $\vdots$
  $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$ — m = no. of equations, n = no. of unknowns

Matrix form :

$$\underbrace{\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{B}$$

$AX = B$

$A \to m \times n$
$X \to n \times 1$
$B \to m \times 1$

- Solution : The value of vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$ satisfying $AX = B$
  Consistent : The system has atleast one solution
  Inconsistent : The system has no solutions
  Homogeneous : $AX = 0$
  Non-homogeneous : $AX = B$

- System $a_{11}x_1 + a_{12}x_2 = b_1$
  $a_{21}x_1 + a_{22}x_2 = b_2$, $m = 2$, $n = 2$

There are 2 line segments. This can be : intersecting exactly at one point, unique solution

- (parallel lines, no solution)

- (infinitely many solutions, coinciding lines)

∴ The possibilities are (i) Unique solution
(ii) No solution
(iii) Infinitely many solutions

- 3×3 System

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

To Solve this $A \quad X \quad = \quad B$

1) Matrix Inverse Method : $X = A^{-1}B$
2) By Cramer's Rule Method :

$$x_1 = \frac{\Delta_1}{\Delta} \quad x_2 = \frac{\Delta_2}{\Delta} \quad x_3 = \frac{\Delta_3}{\Delta} \quad , \Delta = |A|$$

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

(i) Unique Solution : if $\Delta \neq 0$, unique solution exists
$x_1 = \frac{\Delta_1}{\Delta}$, $x_2 = \frac{\Delta_2}{\Delta}$, $x_3 = \frac{\Delta_3}{\Delta}$

(ii) Infinitely many Solutions : if $\Delta = 0$ and all Δ_i are also zero

(iii) No Solutions : if $\Delta = 0$ and atleast one of Δ_i is non-zero.

→ Disadvantages of these Methods

(1) Only applicable when $m = n$ (no. of unknowns = no. of equations)
(2) In Inverse Method, $|A|$ should not be equal to 0
(3) In Cramer's Rule, we need to calculate $(n+1)$ determinants of order n, [illegible]

Eg: $x + 2y + z = 4$
$2x + y + 2z = 5$
$x - y + z = 1$ has

(a) Unique Solution $(x=1, y=1, z=1)$
(b) Only two solutions $(x=1, y=1, z=1)$, $(x=2, y=1, z=0)$
(c) Infinite no. of solutions
(d) No Solution

Ans $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$

$A \; X = B$

$|A| = 3 + -3 = 0$ //

$\Delta_1 = \begin{vmatrix} 4 & 2 & 1 \\ 5 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 4\times3 - 2\times3 + 1\times -6 = 12 - 6 - 6 = 6 - 18 = -12$

$\Delta_1 \neq 0$

$\therefore$ (d) No Solution $\longrightarrow \begin{vmatrix} 4 & 2 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$ //

$\because \Delta_2 = \Delta_3 = 0$

i.e. (c)

→ Due to the before mentioned disadvantages, we look into a generalized method to find the solution of a system of linear equations.

→ Gaussian Elimination Method

* Gaussian Elimination Method

Let $AX = B$, be the given system of linear equations.

- Augmented Matrix: $[A/B]$ = A and B together.

Eg:- $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} \longrightarrow [A/B] = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 2 & 5 \\ 1 & -1 & 1 & 1 \end{bmatrix}$

$A \; X = B$

- Methodology

(n = no. of unknowns
m = no. of equations)

$\boxed{AX = B}$

$\boxed{e(A/B) = e(A)}$ (Consistent)

$\boxed{e(A/B) \neq e(A)}$ No Solution (Inconsistent)

$\boxed{e(A/B) = e(A) = r = n}$ Unique Solution

$\boxed{e(A/B) = e(A) = r < n}$ Infinitely many solutions exists

Eg: $-x + 5y = -1$
$x - y = 2$
$x + 3y = 3$ has

(a) infinitely many solⁿ (b) Two distinct solⁿ (c) Unique Solⁿ (d) None

Ans $\begin{bmatrix} -1 & 5 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}_{3\times2} \begin{bmatrix} x \\ y \end{bmatrix}_{2\times1} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$ $m = 3$, $n = 2$ //

$[A/B] = \begin{bmatrix} -1 & 5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \therefore$ (a)

$\rho(A|B) = 2$

$$\begin{bmatrix} -1 & 5 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}_{3\times 2} \qquad \rho(A) \le 2$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} -1 & 5 \\ 0 & 4 \\ 0 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 5 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \quad \rho(A) = 2.$$

$\rho(A|B) = \rho(A) = n = 2$ (no: of unknowns)

$\therefore$ Unique Solution (a)

(a)

$$\begin{bmatrix} -1 & 5 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$-x + 5y = -1$

$4y = 1$

$y = 1/4$

$x = 1 + \frac{5}{4} = 9/4$

- **Homogeneous Equations**

General form of linear Homogeneous Equations $\Rightarrow AX = 0$

eg: $x + 2y = 0$, $2x + 4y = 0$ — infinite sol^{n}s: $(0,0)$, $(2,-1)$, $(4,-2)$, $(-2,1)$ ... — Non Trivial Solutions

$x + y = 0$, $3x + 2y = 0$ $\rightarrow$ Only one solution $(0,0)$

$\therefore$ By default every homogeneous system has one solution, $(0,0)$ — Trivial Solution

- There are 2 types of Solutions for the homogeneous system.
  1. Trivial Solution / Zero Solution $(X = 0)$
  2. Non-Trivial Solutions (Non-Zero Solutions)

Note: Every homogeneous system is always consistent. Non-trivial solutions may/may not exist. If exists infinitely many solutions exist.

- Methodology

$\boxed{AX=0}$ (m = no: of eq. n = no: of unknowns)

$\boxed{m = n}$

1. $\rho(A) = r = n$ (or) $|A| \neq 0$ then only Trivial Sol^{n}
2. $\rho(A) = r < n$ (or) $|A| = 0$ then infinitely many non-trivial sol^{n} exist.

$\boxed{m \neq n}$

1. $m < n$ always posess infinitely many non-trivial sol^{n}
2. $m > n$
   $\rho(A) = r = n$ only trivial sol^{n}
   $\rho(A) = r < n$ Infinitely many non-trivial sol^{n}.

- **Null Space:** Set of all Solutions of $AX = O$
- **Nullity:** Dimension of Null Space
  = No. of linearly independent solutions
  = $n - r$, n = no. of unknowns
  r = Rank of coefficient Matrix A

$$\boxed{\text{Rank} + \text{Nullity} = n}$$

Q) $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has

(a) No Solution (b) Only One Solution (c) Non-zero unique Solution (d) Multiple Soln

Ans (a) & (c) can be instantly eliminated, because for homogeneous eqs. both the ops. are impossible.

$n = 2, m = 2$ (or) $|A| = 0$
$\rho(A) = 1 = r$
$n > r$ then infinite (d)

No. of linearly independent Solutions = nullity = $n - r$
$= 2 - 1 = 1$

$\therefore$ One independent soln exists
$2x - 2y = 0$
$x - y = 0$
$\therefore x = y$
$(x, y) = k(1, 1) \longrightarrow (1,1), (2,2), (3,3) —$
linearly independent solution

* **Linearly Dependent Vectors:**
Two vectors $X = (\ )$
$Y = (\ )$
are said to Linearly Dependent (LD) if ~~X=2Y~~
$X = \alpha Y$
where α is a scalar.

* **Linearly independent Vectors:** (LI)
Otherwise if $X \neq \alpha Y$
Then linearly independent.

Eg: $X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $Y = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $\Rightarrow X = 2Y$
$\therefore$ LD

$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $Y = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$ $\longrightarrow$ X & Y are LI

① $AX = O$, $n = 3$, $r = \rho(A) = 2$
$n > r$ $\therefore$ infinitely many solns exist.
$r + \text{Nullity} = n$
$2 + N = n$
$N = n - 2$
$= 3 - 2 = 1$ $\therefore$ 1 independent soln exists.

② $AX = O$, $n = 4$, $r = \rho(A) = 2$
$n > r$ $\therefore$ infinite soln
$n - r = 4 - 2 = 2$ independent soln exist.

③ $AX = O$, $n = 3$, $r = \rho(A) = 3$
$n = r$ (only trivial solns exist)
$n - r = 0$ independent soln.

30/08/2021

Problems

(Q1) For what values of a the following system of equations have a solution?

$$2x+3y=4$$
$$x+y+z=4$$
$$x+2y-z=a$$

(a) any real number (b) 0 (c) 1 (d) There is no such value.

Ans:
$$\begin{bmatrix}2&3&0\\1&1&1\\1&2&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\4\\a\end{bmatrix}$$

$\cancel{\Delta = 2(-1-2)-3(-1-1)}$
$\cancel{= 2\times 3 - 3\times(-2)}$
$\cancel{= -6+6=0}$

$\cancel{\Delta_1 = \begin{vmatrix}2&3\\1&1\end{vmatrix} = 2-3=-1}$

$\cancel{\Delta_1 \neq 0}$

~~No solution~~

Find the value of 'a' in the Q.

A = $\begin{bmatrix}2&3&0\\1&1&1\\1&2&-1\end{bmatrix} \xrightarrow[R_3\to 2R_3-R_1]{R_2\to 2R_2-R_1} \begin{bmatrix}2&3&0\\0&-1&2\\0&1&-2\end{bmatrix} \rightarrow \begin{bmatrix}2&3&0\\0&-1&2\\0&0&0\end{bmatrix}$

$\rho(A)=2.$

$(A|B) = \begin{bmatrix}2&3&0&4\\1&1&1&4\\1&2&-1&a\end{bmatrix}$

$R_2\to 2R_2-R_1$
$R_3\to 2R_3-R_1$
$R_3\to R_3+R_2$

$\begin{bmatrix}2&3&0&4\\0&-1&2&4\\0&0&0&2a\end{bmatrix}$ for $\overbrace{\rho(A|B)=\rho(A)=2}^{\text{constant}}$.

remain $2a=0$

$\Rightarrow a=0$ (b)

Gate 2014

(Q2) The system of equations

$$x+2y+4z=2$$
$$4x+3y+z=5$$
$$3x+2y+3z=1$$

has (a) unique solution (b) Two solutions (c) No solution (d) More than Two.

Ans
$$\begin{bmatrix}1&2&4\\4&3&1\\3&2&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\5\\1\end{bmatrix}$$
$$A \quad X = B$$

(d instantly eliminated)

$R_2\to R_2-4R_1$
$R_3\to R_3-3R_1$

$\begin{bmatrix}1&2&4\\0&-5&-15\\0&-4&-9\end{bmatrix} \xrightarrow{R_3\to 5R_3-4R_2} \begin{bmatrix}1&2&4\\0&-5&-15\\0&0&15\end{bmatrix}$

$-45-(-60)$
$60-45$

$-25-(4\times\ldots)$
$-25+12$

$\rho(A)=3$

$(A|B) = \begin{bmatrix}1&2&4&2\\4&3&1&5\\3&2&3&1\end{bmatrix} \xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-4R_1} \begin{bmatrix}1&2&4&2\\0&-5&-15&-3\\0&-4&-9&-5\end{bmatrix}$

$R_3\to 5R_3-4R_2$

$\rho(A|B)=3 \quad \begin{bmatrix}1&2&4&2\\0&-5&-15&-3\\0&0&15&-13\end{bmatrix}$

$\rho(A)=\rho(A|B)=r=3=n$ ∴ (a)

Gate 2014

(Q3) Consider the system of equations

$$3x + 2y = 1$$
$$4x + 7z = 1$$
$$x + y + z = 3$$
$$x - 2y + 7z = 0$$

The no. of solutions for this system is ______.

Ans $(A|B)$. $\cancel{3 \quad 2 \quad 1}$ $\cancel{4 \quad 7 \quad 1}$

$$\begin{bmatrix} 3 & 2 & 0 & 1 \\ 4 & 0 & 7 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & -2 & 7 & 0 \end{bmatrix}$$

$n = 3$, $m = 4$

$R_2 \to 3R_2 - 4R_1$
$R_3 \to 3R_3 - R_1$
$R_4 \to 3R_4 - R_1$

$$\begin{bmatrix} 3 & 2 & 0 & 1 \\ 0 & -8 & 21 & -1 \\ 0 & 1 & 3 & 8 \\ 0 & -8 & 21 & -1 \end{bmatrix}$$

$\cancel{3 \quad 2 \quad 0 \quad 1}$ $\cancel{0 \quad -8 \quad 21 \quad -1}$

$$\downarrow$$

$$\begin{bmatrix} 3 & 2 & 0 & 1 \\ 0 & 1 & 3 & 8 \\ 0 & -8 & 21 & -1 \\ 0 & -8 & 21 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 2 & 0 & 1 \\ 0 & 1 & 3 & 8 \\ 0 & -8 & 21 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Big\downarrow R_3 \to R_3 + 8R_2$$

$$\begin{bmatrix} 3 & 2 & 0 & 1 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 45 & 63 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A|B) = \rho(A)$

$r = 3$

$r = 3 = n$

$\therefore$ Unique soln (1)

(Q4) The system of linear equations

$$t - x + 2y - z = 1$$
$$t + y + z = 0$$
$$-x + y - 2z = -1$$

has

(a) unique solution (b) No solution (c) Infinitely many solutions
(d) exactly two distinct solutions.

Ans $n = 4$, $m = 3$ (d instantly eliminated)

$$(A|B) = \begin{bmatrix} 1 & -1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -2 & -1 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -1 & 1 & -2 & -1 \end{bmatrix}$$

$$\Big\downarrow R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \to \rho(A|B) = 3$$

$\rho(A) = 2$

$\rho(A) \neq \rho(A|B)$

No solution (b)

(Q5) The system

$$kx + y + z = 1$$
$$x + ky + z = 1$$
$$x + y + kz = 1$$

has no solution when $k =$ ______.

Ans PTO

$$(A|B) = \begin{bmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{l} R_2 \to kR_2 - R_1 \\ R_3 \to kR_3 - R_1 \end{array}$$

$$\begin{bmatrix} k & 1 & 1 & 1 \\ 0 & k^2-1 & k-1 & k-1 \\ 0 & k-1 & k^2-1 & k-1 \end{bmatrix} \xrightarrow{R_3 \to (k+1)R_3 - R_2}$$

$$\cancel{\begin{bmatrix} k & 1 & 1 & 1 \\ 0 & k^2-1 & k+1 & k+1 \\ 0 & 0 & (k^2-1)(k+1) & k^2-1 \end{bmatrix}}$$

$\cancel{e(A|B) \neq e(A)}$

$\cancel{\text{for that } k = -1}$

$$\cancel{(k+1)^2 \begin{bmatrix} k & 1 & 1 & 1 \\ 0 & k+1 & 1 & 1 \\ 0 & 0 & (k+1)^2 & (k+1) \end{bmatrix}}$$

$$\cancel{\begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & \end{bmatrix}}$$

~~for no solution $e(A|B) \neq e(A)$~~

$$\begin{bmatrix} k & 1 & 1 & 1 \\ 0 & k^2-1 & k-1 & k-1 \\ 0 & 0 & (k+1)(k^2-1) - (k-1) & (k^2-1) - (k-1) \end{bmatrix}$$

$(k+1)^2(k-1) - (k-1)$

$(k-1)((k+1)^2 - 1)$

$(k-1)(k^2 + 2k)$

$k^2 - 1 + k + 1$

$$= \begin{bmatrix} k & 1 & 1 & 1 \\ 0 & k^2-1 & k-1 & k-1 \\ 0 & 0 & (k-1)(k^2+2k) & k^2-k \end{bmatrix}$$

$(k-1)$ cant be zero

Then R_3 become 0, $\therefore k^2 + 2k = 0$

$k + 2 = 0$, $k = -2$

$$k = -2 \Rightarrow \begin{bmatrix} -2 & 1 & 1 & 1 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$e(A) = 2 \neq e(A|B) = 3$

$\therefore k = -2$

(Q6) Consider $x + 2y - 3z = a$

$2x + 3y + 3z = b$

$5x + 9y - 6z = c$.

The system of equations is inconsistent if

(a) $(3a + b - c) = 0$ (b) $(3a + b - c) \neq 0$

(c) $(7a - b - c) = 0$ (d) $(7a - b - c) \neq 0$.

Ans

$$(A|B) = \begin{bmatrix} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{bmatrix}$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & -1 & 9 & c-5a \end{bmatrix}$$

$-6 - (5 \times -3)$

$$\Big\downarrow R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & 0 & 0 & c-5a-b+2a \end{bmatrix}$$

$c - 5a - b + 2a \neq 0$

$c - 3a - b \neq 0$

$3a + b - c \neq 0$

(b)

(P1) For the system $x+y+z=6$

$x+2y+3z=10$

$x+2y+\lambda z=\mu$

Which of the following statement is correct?

(a) The system has unique solution if $\lambda=3$ and $\mu=10$

(b) " " $\lambda=3$ & $\mu\neq10$

(c) " " $\lambda\neq3$

(d) " " $\lambda=3$

Sol: $(A|B)=\begin{bmatrix}1&1&1&6\\1&2&3&10\\1&2&\lambda&\mu\end{bmatrix} \Rightarrow \begin{bmatrix}1&1&1&6\\0&1&2&4\\0&1&\lambda-1&\mu-6\end{bmatrix}$

$\downarrow$

$\begin{bmatrix}1&1&1&6\\0&1&2&4\\0&0&\lambda-3&\mu-10\end{bmatrix}$

$\lambda\neq3$ & μ any no:

$\Rightarrow \rho(A)=\rho(A|B)=r=n=3$

(c)

* 1.4 Eigen Values & Eigen Vector

- **Characteristic Equation**: Let A be $n\times n$ matrix. Then $\boxed{|A-\lambda I|=0}$ is called characteristic equation of A.

Eg:- $A=\begin{bmatrix}1&-2\\1&4\end{bmatrix}$

$$\left|\begin{pmatrix}1&-2\\1&4\end{pmatrix}-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=0$$

$$\left|\begin{pmatrix}1-\lambda&-2\\1&4-\lambda\end{pmatrix}\right|=0 \Rightarrow 4-\lambda-4\lambda+\lambda^2+2=0$$

$$6-5\lambda+\lambda^2=0$$

$\boxed{\lambda^2-5\lambda+6=0}$ → characteristic eq: for the given matrix.

The degree of the characteristic eq. of a given matrix $A_{n\times n}$ will be equal to $\boxed{n}$ (here 2)

$\boxed{\text{Roots of this eq. are called Eigen Values}}$

here 2, 3 are the eigen values.

$\lambda^2-5\lambda+6=(\lambda-2)(\lambda-3)$

$\lambda=2$ or 3

- In general for $n\times n$ matrix 'n' eigen values exist.

- **Cayley-Hamilton Theorem**: Every square matrix of order 'n' satisfies its own characteristic eq. $(n>1)$

Eg:- for the earlier eg: $A=\begin{pmatrix}1&-2\\1&4\end{pmatrix} \Rightarrow \lambda^2-5\lambda+6=0$

Then $A^2-5A+6I=O$

- **Advantage of Cayle Hamilton Theorem**

1. We can find A^{-1} easily :

Eg:- from the earlier eq: we got

$A^2 - 5A + 6I = 0$

$(A^2 - 5A + 6I)A^{-1} = A - 5I + 6A^{-1} = 0$

$\cancel{A^{-1} \Rightarrow 5A}$ $\quad A^{-1} = \dfrac{5I - A}{6}$ //

2. We can express A^k in terms of A and I :

$A^2 = 5A - 6I$

$A^3 = 5A^2 - 6A$
$= 5(5A - 6I) - 6A$
$= 25A - 6A - 30I$
$= 19A - 30I$

Thus if $A_{2\times2}$, then we can express A^k in terms of A and I.
if $A_{3\times3}$, then we can express A^k in terms of A^2, A and I
if $A_{4\times4}$, " " A^k " " A^3, A^2, A and I
in general if $A_{n\times n}$, " " A^k " " $A^{n-1}, A^{n-2}, \ldots$ And I.

- **Shortcut method to find the characteristic eq. of 2x2 & 3x3 Matrix**

For 2x2 Matrix : Characteristic Eq $=$ $\lambda^2 - S_1\lambda + S_2 = 0$
S_1 = Trace (sum of diagonals)
S_2 = determinant (Δ)

Eg:- $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$

$S_1 = 1 + 4 = 5$
$S_2 = 4 - (-2) = 6$ $\quad \therefore \lambda^2 - 5\lambda + 6 = 0$ //

For 3x3 Matrix

Characteristic Eq $=$ $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$
S_1 = Trace of A
S_2 = Sum of principal minors
S_3 = det A

Eg:

$A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$

$S_1 = 2 + -1 + 0 = 1$
$S_3 = 2(0 - 12) + 2(0 - 6) + 3(-4 + 1)$
$= -24 - 12 - 9$
$= -45$

$S_2 = \begin{vmatrix} -1 & 6 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ -2 & -1 \end{vmatrix}$

delete 1st row 1st column, delete 2nd row 2nd column, delete 3rd row 3rd column

$= -12 + -3 + -6$
$= -21$

$\therefore \lambda^3 - \lambda^2 - 21\lambda + 45 = 0$ //

Eigen Values

$\cancel{1 \quad -1 \quad -21 \quad -45}$

	1	-1	-21	+45
3	+0	3x1	3x2	-45
	1	2	-15	0

$\therefore$ 3 is a factor

$(\lambda - 3)(\lambda^2 + 2\lambda - 15) = 0$ $\quad = (\lambda - 3)(\lambda - 3)(\lambda + 5)$
$\Rightarrow \lambda = 3, 3, -5$ // (eigenvalues)

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- **Algebraic multiplicity of eigen value**

$\boxed{\text{AM of } \lambda = \text{no. of times eigen value occurred.}}$

- for the above example we got $\lambda = 3, 3, -5$.

$\therefore$ AM of $3 = 2$

AM of $-5 = 1$

* **Eigen Vectors :** A non-zero Vector X is said to be eigen vector corresponding to eigen value λ of a matrix A if $\boxed{(A-\lambda I)X = 0}$ → homogeneous System,

So always consistent,

$\therefore$ Eigen vector always exists

Note : ① Corresponding to one eigenvalue infinitely many eigen vectors exists.

② Geometric multiplicity of λ.

$\boxed{\text{GM of } \lambda = \text{No. of Linearly Independent eigenvectors} = n - r}$

$\boxed{\text{here } n = \text{order of } A,\ r = \text{Rank of } (A-\lambda I)}$

③ $\boxed{\text{GM} \le \text{AM for any eigenvalue } \lambda}$

- **Properties of Eigen Values & Eigen Vectors** (imp)

1. Sum of the eigenvalues = Trace of A

2. Product of eigenvalues = det of A

3. If Zero is one of the eigen values of A, then $\det A = 0$

then $\rho(A) = r < n$

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4. The eigenvalues of A and A^T are same.

5. The eigen values of upper / lower triangular and diagonal matrix are just the diagonal elements only.

6. If $a + ib$ is an eigenvalue of real Matrix A, then $a - ib$ is also eigen value

(May not be true for Complex matrices, only for real Matrices)

7. If $a + \sqrt{b}$ is an eigenvalue of Real matrix A, then $a - \sqrt{b}$ is also an eigen value.

Eg:- $A_{3\times3} \rightarrow 1+i, 2, 1-i$

$A_{3\times3} \rightarrow 2+\sqrt{3}, -1, 2-\sqrt{3}$ } eigen values

8. Eigen vectors corresponding to distinct eigen values are Linearly Independent

9. Eigen values are $\lambda_1, \lambda_2, \dots \lambda_n$

Eigen vectors are $X_1, X_2, \dots X_n$ } of A.

$A^2 \longrightarrow$ Eigen Values are $\lambda_1^2, \lambda_2^2, \dots \lambda_n^2$

$A^m \longrightarrow$ Eigen Values are $\lambda_1^m, \lambda_2^m, \dots \lambda_n^m$

$f(A) = a_0 A^n + a_1 A^{n-1} + \dots + a_n I$

then eigenvalues of $f(A) \Rightarrow f(\lambda_1), f(\lambda_2), \dots f(\lambda_n)$

$A^{-1} \longrightarrow$ Eigen Values are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots \frac{1}{\lambda_n}$

if A is a (non-singular Matrix)

Non Singular Matrix means det of the Matrix $\neq 0$

$\Longrightarrow$ No eigen values are zero.

For all these, eigen vectors remain the same, $X_1, X_2, \dots X_n$

Eg: $A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$ → Upper triangular Matrix
∴ Eigen Values = $\underline{1, 4}$

Eg: $A = \begin{pmatrix} -1 & 0 \\ 2 & 5 \end{pmatrix}$ → lower Δ Matrix
∴ Eigen Values = $\underline{-1, 5}$

Eg: $A = \begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}$ → diagonal Matrix
∴ Eigen Values $\underline{3, -7}$

10. If λ is an eigenvalue of non-singular Matrix.
(a) Then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}

(b) $\frac{|A|}{\lambda}$ is an eigenvalue of adj(A)

Eg: $A = \begin{pmatrix} 1 & -2 \\ & 4 \end{pmatrix}$, 2, 3 are eigenvalues
$|A| \neq 0 = 6$
∴ $\frac{1}{2}, \frac{1}{3}$ are eigenvalues of A^{-1}

$6/2$ and $6/3$ are eigenvalues of adj(A).

Eg: $A_{3\times3}$ has eigenvalues → 2, 8, 4
∴ Non-singular matrix.
A^{-1} → eigen values = $\frac{1}{2}, \frac{1}{8}, \frac{1}{4}$.

$|A| = 2\times8\times4 = \underline{64}$
adj(A) eigen values → $\frac{64}{2}$, $\frac{64}{8}$ and $\frac{64}{4}$

11. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigenvalues of A, then
(a) $\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$ " " " A^2
(b) $\lambda_1^{-1}, \lambda_2^{-1}, \ldots \lambda_n^{-1}$ " " " A^{-1}
(c) $\lambda_1 \pm k, \lambda_2 \pm k, \ldots \lambda_n \pm k$ " " " $A \pm kI$
(d) $(\lambda_1 \pm k)^{-1}, (\lambda_2 \pm k)^{-1}, \ldots (\lambda_n \pm k)^{-1}$ " " $(A \pm kI)^{-1}$
(e) If $f(A) = a_0A^k + a_1A^{k-1} \ldots + a_nI$ then $f(\lambda)$ is eigenvalue of $\underline{f(A)}$.

Eg: $A_{3\times3}$ with eigenvalues → 2, 3, -2
∴

Then → Matrix	Eigenvalues
A^2	4, 9, 4
A^3	8, 27, -8
$A + 2I$	4, 5, 0
$f(A) = A^2 + 3A + I$	(f(2), f(3), f(-2) ; 6, 14, -6) //
$(A+I)^2$	$(2+1)^2, (3+1)^2, (-2+1)^2$; 9, 16, 1

(5A) • Types of Problems in Gate from this Topic (Most Standard Models)

Model 1: For a given Matrix find eigenvalues
Model 2: For a given Matrix A & eigenvalue λ / eigenvector x finding eigenvector / eigenvalue.

Model 3: If eigenvalues and eigen vectors are given find Matrix A
Model 4: Problems related to Cayley-Hamilton Theorem
Model 5: Problems related to properties (Miscellaneous)

• **Model 1**

Gate 2012 (Q) The eigenvalues of Matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$ are.

(a) 0, -1, -3 (b) 0, -2, -3
(c) 0, 2, 3 (d) 0, 1, 3

A: $|A-\lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -3 & -4-\lambda \end{vmatrix} = 0$$

$$-\lambda\left(\lambda(4+\lambda)+3\right) - 1\left(0\right) + 0 = 0$$

$$\lambda^2(4+\lambda) + 3\lambda = 0$$
$$\lambda(\lambda(4+\lambda)+3) = 0$$
$$\lambda(4\lambda + \lambda^2 + 3) = 0$$

(a) $\underline{\underline{-1, -3, 0}}$

(or)

Shortcut
Trace(A) = 0 + 0 + -4 = -4
det(A) = Since the Matrix has one column full '0' the determinant will be $\underline{\underline{0}}$.

now look at the opts: Sum of eigenvalues = Trace(A) = -4,
product of eigenvalues = det(A) = 0

$\therefore$ (a)

• **Model 2**

Gate 2005 (Q2) For Matrix $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ one of the eigenvalue is -2 which of the following is an eigenvector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ (d) $\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$

A: $(A-\lambda I)X = 0 \; = (A+2I)X$

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} X = 0$$

$5x - 2y + 2z = 0$ $\qquad$ $5x = 2y$
$z = 0$ $\qquad$ $z = 0$
$3z = 0$ $\qquad$ $\therefore$ (d)

(or)

$(A-\lambda I)X = 0 \Rightarrow AX - \lambda X = 0$
$\boxed{AX = \lambda X}$

$$\begin{bmatrix} 3 & -2 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

substitute options

(d)

Qb 2010

(3) If $(1, 0, -1)^T$ is an eigenvector of the Matrix $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ corresponding eigenvalue is ______

(a) 1 (b) 2 (c) 3 (d) 5.

Sol: $(A - \lambda I)(X) = 0$

$$\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$$

$(1-\lambda) = 0$

$\underline{\underline{\lambda = 1}}$ (a)

(or)

$\boxed{AX = \lambda X}$ check that

• Model 3

Qb 2006

(4) For 2x2 Matrix

eigen Values	eigenVectors
$\lambda_1 = 8$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\lambda_2 = 4$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Then Matrix is ______

(a) $\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$

Ans: $A = 8 \times 4 = \underline{32}$

$\therefore$ (a) $6\times6 - 2\times2 = 36-4 = \underline{\underline{32}}$

(or)

Sum $= 12 = $ Trace (A)

$\underline{\underline{(a)}}$

* Diagonalization Method

Given Matrix A is said to be diagonalizable if there exist a non-singular Matrix P such that.

$AP = PD \Longrightarrow \boxed{A = PDP^{-1}}$

here P = Matrix with columns which are eigenvectors of A

D = diagonal Matrix, eigenvalues of A are diagonal elements

1. $A^k = PD^kP^{-1}$

Proof $A^2 = (PDP^{-1})(PDP^{-1})$

$= PD\cancel{P^{-1}P}DP^{-1}$

$= \underline{\underline{PD^2P^{-1}}}$

2. A is diagonalizable iff. $|P| \neq 0$

i.e. A has 'n' linearly independent eigenvectors

Ans for (Q4) Solving by this method.

$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $|P| = -1-1 = \underline{-2}$

$P^{-1} = \frac{-1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

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$$D = \begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix}$$

$A = PDP^{-1}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 8 & 4 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 12 & 4 \\ 4 & 12 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} \text{ (a)}$$

- Model 4

(P5) If $A =$ $\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 2i & i \\ 0 & 0 & 0 & -2i \end{pmatrix}$ then $A^4 =$ ____

(a) I (b) $4I$ (c) $16I$ (d) $64I$

Ans $A \to$ Upper Δ Matrix $\leftarrow A - \lambda I$

$|A| \to$ product of diagonals $\leftarrow |A - \lambda I|$

$$|A - \lambda I| = (2-\lambda)(-2-\lambda)(2i-\lambda)(-2i-\lambda) = 0$$

$$-(2^2 - \lambda^2)((2i)^2 - \lambda^2) = 0$$

$$= (4-\lambda^2)(-4-\lambda^2)$$

$$= -(4-\lambda^2)(4+\lambda^2)$$

$$= -(16-\lambda^4)$$

$$= \lambda^4 - 16 = 0$$

$$\Rightarrow A^4 - 16I = 0$$

$$A^4 = 16I \text{ (c)}$$

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* Problems

(Q1) The value of x for which all the eigenvalues of the matrix given below are real is ____.

$$\begin{pmatrix} 10 & 5+i & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{pmatrix}$$ (a) $5+i$ (b) $5-i$ (c) $1-5i$ (d) $1+5i$

Ans ~~$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$~~

Eigenvalues of Hermitian Matrix are real.

$$\left[\begin{matrix} A^{\theta} = A \\ (\bar{A})^T = A \end{matrix}\right]$$

$$\begin{pmatrix} 10 & 5-i & 4 \\ \bar{x} & 20 & 2 \\ 4 & 2 & -10 \end{pmatrix}^T$$

$$\Downarrow$$

Clearly $x = \overline{5+i} = 5-i$ (b)

$$A = \begin{pmatrix} 10 & \bar{x} & 4 \\ 5-i & 20 & 2 \\ 4 & 2 & -10 \end{pmatrix} = (\bar{A})^T$$

(Q2) Let the eigenvalues of 2x2 matrix A be 1, -2, with eigenvectors x_1, x_2 respectively. Then the eigenvalues and eigenvectors of $A^2 - 3A + 4I$ would respectively be.

(a) $2, 14, x_1, x_2$ (b) $2, 14, x_1 + x_2, x_1 - x_2$
(c) $2, 0, x_1, x_2$ (d) $2, 0, x_1 + x_2, x_1 - x_2$

Ans $S_1 = 1 + -2 = -1$ $\quad \lambda^2 - S_1\lambda + S_2 = 0$

$S_2 = 1 \times -2 = -2$ $\quad \lambda^2 + \lambda - 2 = 0$

By Cayley-Hamilton theorem. $A^2 + A - 2I = 0$

$$A^2 - 3A + 4I - 0 = A^2 - 3A + 4I$$

$$= A^2 - 3A + 4I - A^2 - A + 2I$$

$$= 6I - 4A = A^2 - 3A + 4I$$

pg. 43, property 9
$f(A) = a_0A^n + a_1A^{n-1} + \dots a_nI$
$f(A) = A^2 - 3A + 4I$
λ of $f(A) = f(\lambda)$
$= f(1) = 1-3+4 = 2$
$f(-2) = 4+6+4 = \underline{14}$
and eigenvectors remain the same, X_1, X_2
$\therefore$ (b) $2, 14, X_1, X_2$

(Q3) Two eigen values of 3×3 matrix are $2+\sqrt{-1}$ and 3, the determinant of the Matrix is ______.

Ans pg 43, Property 3.
$\therefore (2+\sqrt{-1}) \times (2-\sqrt{-1}) \times 3$
$= (2^2 - \sqrt{-1}^2)3$
$= (4+1)3 = 15$

(Q4) Consider $A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$ whose eigen vectors corresponding to eigen values λ_1 and λ_2 are $X_1 = \begin{pmatrix} 70 \\ \lambda_1 - 50 \end{pmatrix}$ and $X_2 = \begin{pmatrix} \lambda_2 - 80 \\ 70 \end{pmatrix}$ respectively, the value of $X_1^T X_2$ is ______.

Ans $\lambda_1 + \lambda_2 = 50+80 = 130$
$\lambda_1\lambda_2 = 50 \times 80 - 70 \times 70$
$= 4000 - 4900$
$= -900$

$\begin{pmatrix} 70 \\ \lambda_1 - 50 \end{pmatrix}^T \begin{pmatrix} \lambda_2 - 80 \\ 70 \end{pmatrix} = (70 \quad \lambda_1 - 50)\begin{pmatrix} \lambda_2 - 80 \\ 70 \end{pmatrix}$

$= \cancel{70(\lambda_2 - 80)\ 4900}$ $\quad 70(\lambda_2 - 80) + (\lambda_1 - 50)70$
$= 70(\lambda_2 - 80 + \lambda_1 - 50)$
$= 70(\lambda_1 + \lambda_2 - 130)$
$= 70(130 - 130) = \underline{0}$

(Q5) Consider $\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$ which of the following statements is true?

(a) Eigenvalue of 3 has multiplicity of 2, and only one independent eigenvector exists.
(b) " 3 " 2 " two "
(c) " 3 " 2 " no "
(d) " 3, -3 " and has 2 independent eigenvectors exist

Ans (c) can be eliminated directly, atleast one independent eigenvector exists.
$\Delta = 5+4 = 9$, 2×2 Matrix, $\therefore$ 2 eigenvalues
$\lambda_1 = 3$
so $\lambda_2 = 3$
3, multiplicity = 2, only one independent.
(a)

$A - \lambda I$
$\cancel{X(A-3I)=0} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$A - 3I = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \xrightarrow{R_2 - 2R_1} = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \rightarrow$ Rank of $A-3I = 1$

Pg 42 and 30
GM of ($\lambda = 3$)
$n - r = 2 - 1 = 1$ // Only one Independent Solution.

(56) Consider a non-singular Matrix of order 2×2. If Trace(A) = 4 and Trace(A^2) = 5 then the determinant of A is ____

Ans: $A \to \lambda_1, \lambda_2$ $\quad \lambda_1+\lambda_2 = 4$ $\quad |A| = \lambda_1\lambda_2$

$A^2 \to \lambda_1^2, \lambda_2^2$ $\quad \lambda_1^2+\lambda_2^2 = 5$

$16 = (\lambda_1+\lambda_2)^2 = \lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2$

$16-5 = 2\lambda_1\lambda_2$

$\frac{11}{2} = \lambda_1\lambda_2 = |A|$

$= \underline{\underline{5.5}}$

(57) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ and $B = A^3 - A^2 - 4A + 5I$ then the determinant of B is ____

Ans:

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 1+2-2 = 1$

$S_2 = -4 + -2 + 2 = -4$

$S_3 = -4-(0) = -4$ $\quad (|A|)$

By Cayley-Hamilton: $A^3 - A^2 - 4A + 4I = 0$

$B = A^3 - A^2 - 4A + 5I = I$

$|B| = |I| = \underline{\underline{1}}$

(58) The eigenvalues of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ are

(a) 0,0,0 (b) 0,0,1 (c) 0,0,3 (d) 1,1,1

Ans: Trace(A) = 1+1+1 = 3 ∴ (c) 0+0+3 = 3

(59) If the eigenvalues and eigenvectors of 2×2 matrix are

eigenvalues	eigenvectors
-2	$\begin{pmatrix} -4 \\ 3 \end{pmatrix}$
5	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The Matrix A is ____

(a) $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & 6 \\ 1 & 4 \end{pmatrix}$

Ans: $(A - \lambda I)X = 0$

$\boxed{AX = \lambda X}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -4 \\ 3 \end{pmatrix} = -2\begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$

$\left.\begin{array}{l} -4a + 3b = 8 \\ -4c + 3d = -6 \end{array}\right\} \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ satisfy the given

(c)

(or)

$P = \begin{pmatrix} -4 & 1 \\ 3 & 1 \end{pmatrix}$ $\quad P^{-1} = \frac{1}{-7}\begin{pmatrix} 1 & -1 \\ -3 & -4 \end{pmatrix}$

$D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$

$PDP^{-1} = \begin{pmatrix} -4 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}\frac{1}{-7}\begin{pmatrix} 1 & -1 \\ -3 & -4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ (c) Ans

Q10 The eigen values of the matrix

$$\begin{bmatrix} 1 & 2 & 34 & 49 \\ 0 & 2 & 43 & 94 \\ 0 & 0 & -2 & 104 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$ are.

(a) 1, 2, -2, -1 , (b) -1, -2, -1, -2 (c) 1, 2, 2, 1 (d) None.

Ans (a) 1, 2, -2, -1 (upper A matrix, ∴ diagonal elements)

Q11 Let A be 3x3 matrix with Rank 2, then $AX = 0$ has

(a) only Trivial soln $X = 0$ (c) Two independent solⁿ

(b) One independent solⁿ (d) 3 independent solⁿ.

Ans $\rho(A) = 2 = r$ $(m = n = 3)$

$n = 3$

$n \neq r$

$r < n$ $n - r = 3 - 2 = \underline{\underline{1}}$ independent solⁿ,

(b)

Q12 If $A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}$ then A^9 equals to ______.

(a) $511A + 510I$ (b) $309A + 104I$ (c) $154A + 155I$ (d) e^{9A}.

A. $|A - \lambda I| = 0$

$\begin{vmatrix} -3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = -(-3-\lambda)\lambda + 2 = 0$

$= \cancel{-3\lambda}\ +3\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 3\lambda + 2 = 0$

Cayley Hamilton Theorem $A^2 + 3A + 2I = 0$

$A^2 = \cancel{3A + 2I}\ -3A - 2I$

$A^3 = 3A^2 + 2A$

$= 3(3A + 2I) + 2A$

$= 9A + 2A + 6I = 11A + 6I$

$A^4 = A(11A + 6I)$

$= 11A^2 + 6A$

$= 11(3A + 2I) + 6A$

$= 39A + 22I$

~~$A^3 \times A^3 = A^6 = (11A + 6I)(11A + 6I)$~~

~~$= 121A^2 + 66A + 66A + 36I^2$~~

$A^3 = -3A^2 - 2A$

$A^3 = -3(-3A - 2I) - 2A$

$= 9A - 2A + 6I$

$= 7A + 6I$

⋮

Similarly calculate upto A^9

(·/·) Shortcut

$A \rightarrow \lambda_1, \lambda_2$

$\lambda^2 + 3\lambda + 2 = 0$

$\lambda^2 + 2\lambda + \lambda + 2 = 0$

$\lambda(\cancel{\lambda+2}) + 1 (\lambda + 1)(\lambda + 2) = 0.$

$\underline{\underline{\lambda_1 = -1}}, \underline{\underline{\lambda_2 = -2}}$

$A^9 \rightarrow \lambda_1^9, \lambda_2^9$

$= -1, -512$

$S_1 = -1 + -512 = -513$

$S_2 = -512 \times -1 = 512$

$\lambda^2 - S_1\lambda + S_2 = 0$

$\lambda^2 + 513\lambda + 512 = 0$

$(A^9)^2 + 513A^9 + 512 = 0$

$A^9 = x$ $x^2 + 512x + x + 512 = 0$

$x(x + 512) + (x + 512)$

$(x + 1)(x + 512) = 0.$

$A^3 \rightarrow \lambda_1^3, \lambda_2^3$

$-1, -8$

$S_1 = -9$

$S_2 = 8$

$(A^3)^2 - S_1A^3 + S_2 = 0$

$A^9 + 9A^3 + 8 = 0$

$A^9 = -8 - 9A^3$

$= -8 - 9(7A + 6I)$

$A^9 = -8I - 63A - 54I$

Req. form: $A^k = aA + bI$.
to find a & b, solve:

$(\lambda_1)^k = a\lambda_1 + b$

$(\lambda_2)^k = a\lambda_2 + b$

$(-1)^9 = a \times -1 + b \quad = -1 = -a + b$

$(-2)^9 = a \times -2 + b \quad = -512 = -2a + b$

$a = \cancel{513}\ 511$

$b = \cancel{512}\ 510$

$\therefore$ (a)

Q13) How many of the following matrices have an eigen value of 1?

$\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}1&-1\\1&1\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix}$

(a) 1 (b) 2 (c) 3 (d) none

Ans (b) 2 $\begin{pmatrix}1&0\\0&0\end{pmatrix} \longrightarrow 1, 0$ (diagonal)

$\begin{pmatrix}0&1\\0&0\end{pmatrix} \longrightarrow 0, 0$ upper Δ

$\begin{pmatrix}-1&0\\0&-1\end{pmatrix} \longrightarrow -1, -1$ (diagonal)

$\begin{pmatrix}1&-1\\1&1\end{pmatrix} \longrightarrow Tr(A) = 2 = \lambda_1 + \lambda_2$

$\Delta(A) = 2 \quad \lambda_1\lambda_2$

$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\lambda_1 + \dfrac{2}{\lambda_1} = 2$

$\lambda_1^2 + 2 = 2\lambda_1$

$\lambda_1 = \dfrac{2 \pm \sqrt{4-8}}{2} \longleftarrow \lambda_1^2 - 2\lambda_1 + 2 = 0$

not 1

$= \cancel{1 \pm 2i}\ 1 \pm i$

$\therefore$ (a) 1

Q14) Consider the following system of equations

$2x_1 + x_2 + x_3 = 0$

$x_2 - x_3 = 0$

$x_1 + x_2 = 0$ then the system has

(a) unique solution (b) no solution (c) infinite no. of solutions
(d) two solutions

Ans $\begin{bmatrix}2&1&1\\0&1&-1\\1&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$AX = 0$

$n = m$

$\xrightarrow{R_3 \to 2R_3 - R_1} \begin{bmatrix}2&1&1\\0&1&-1\\0&1&-1\end{bmatrix} \longrightarrow \begin{bmatrix}2&1&1\\0&1&-1\\0&0&0\end{bmatrix}$

$\rho(A) = 2 = r$

$r \neq n$

$r < n$ (c)

Q15) For the matrix A satisfying the equation given below

$[A]\begin{pmatrix}1&2&3\\7&8&9\\4&5&6\end{pmatrix} = \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$ The eigenvalues of A are:

(a) $1, -i, i$ (b) $1, 1, 0$
(c) $1, 1, -1$ (d) $1, 0, 0$

Ans $AB = C$

$|AB| = |A||B| = |C|$

$|A| 0 = 0 \quad A = CB^{-1}$

$|B| = (48-45) - 2(42-36) + 3(35-32)$

$= 3 - 2\times6 + 3\times3$

$= 3 - 12 + 9$

$= 0$

$AB = C$

C can be ~~obtained~~ obtained by interchanging R_2 and R_3

$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} B = C$

A

$(1-\lambda)(\lambda^2-1) = 0$ $\quad \det(A) = -1$, Trace $= 1$

$\lambda^2 - 1 - \lambda^3 + \lambda = 0$

$-\lambda^3 + \lambda^2 + \lambda - 1 = 0$ $\quad$ (c)

Q16) For the following system of equations

$ax + by + cz = 0$

$bx + cy + az = 0$

$cx + ay + bz = 0$ non-trivial solⁿ existing

(a) $(a-b+c) = 0$ or $a = b = c$

(b) $(a+b-c) = 0$ or $a = -b = c$

(c) $(a+b+c) = 0$ or $a = b = c$

(d) $(a-b+c) = 0$ or $a = -b = c$.

Ans $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$AX = 0$

~~$m = n$~~

~~$n = 3$~~

~~$r = 2$ or 1~~ $\quad$ if $a = b = c$, $r = 1$

$b - a = 0$

$c - a = 0$

$a - (c-b) = 0$

$a - c + b = 0$

$= a + b - c = 0$

~~$a - b = c - a$~~

$b - a = c - a = 0$

$c - b = a - b = 0$ and $b - a = 0$

$a - c = b - c$

$\underline{\underline{a = b = c}}$

$\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \xrightarrow{C_1 \to C_1 + C_2 + C_3} \begin{pmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{pmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$\begin{pmatrix} a+b+c & b & c \\ 0 & b-c & a-c \\ 0 & a-c & b-a \end{pmatrix}$

~~now $b - c = a - c \Rightarrow a = b$~~

~~$a - c = b - a \Rightarrow a - c = 0$~~

~~$a = c \Rightarrow a = b = c$ for $\rho(A) = 2$~~

For non-trivial solⁿ $|A| = 0$

$a(cb - a^2) - b(b^2 - ac) + c(ab - c^2)$

$= abc - a^3 - b^3 + abc + abc - c^3$

$= -a^3 - b^3 - c^3 + 3abc = 0$

$a^3 + b^3 + c^3 = 3abc$

$|A| = (a+b+c)\left((b-c)(b-a) - (a-c)^2\right)$

$= (a+b+c)(a^2 - b^2 - c^2 + bc + ac + ab) = 0$.

so $a + b + c = 0$

or $a = b = c = 0$ // (c) //

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(Q17) Let $A = \begin{pmatrix} 1 & 3 & 1 & -2 \\ 1 & 4 & 3 & -1 \\ 3 & 3 & -4 & -7 \\ 3 & 8 & 1 & -7 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ then

The system $AX=0$ has
(a) unique soln (b) only one linearly independent soln
(c) 2 " "
(d) 3 " "

Ans $m=4$
$n=4$ $(m=n)$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 3R_1$

$\begin{pmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & -6 & -7 & -1 \\ 0 & -1 & -2 & -1 \end{pmatrix} \xrightarrow[R_4 \to R_4 + R_2]{R_3 \to R_3 + 6R_2} \begin{pmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\rho(A) = 3$

$4 - 3 = \underline{1}$

(b)

(Q18) Find the independent eigen vectors corresponding to a real eigen value of the matrix
$\begin{pmatrix} 5 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix}$ = ____

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Ans $\begin{vmatrix} 5-\lambda & 0 & 0 & 1 \\ 0 & 5-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix} = 0$

$= (5-\lambda)\left[(5-\lambda)(2-3) - 0 + 0\right] - 0 + 0 - 1\left[0 - (5-\lambda)(0) + 0\right]$

$= (5-\lambda)(5-\lambda) \times -1$

$= (5-\lambda)^2 \times -1$

$= -(25 + \lambda^2 - 10\lambda)$

$= -\lambda^2 - 25 + 10\lambda = 0$

$= \cancel{-\lambda^2 + 5\lambda + 5\lambda - 25}$

$= \cancel{\lambda(\lambda+5)}$

$\lambda^2 + 25 - 10\lambda = 0$

$\lambda^2 - 10\lambda + 25 = 0$

$\lambda = \dfrac{10 \pm \sqrt{10^2 - 4 \times 1 \times 25}}{2} = \dfrac{10 \pm \sqrt{100 - 100}}{2}$

$= \underline{5}$

$\lambda^2 - 5\lambda - 5\lambda + 25 = 0$

$\lambda(\lambda - 5) - 5(\lambda - 5)$

$\lambda = \underline{5, 5}$

$\rho(A) = 4 = r$

$\underline{n = 0}$

$(5-\lambda)\left[(5-\lambda)\left[(2-\lambda)(1-\lambda) - 3\right]\right]$

$(5-\lambda)^2 (2 - 3\lambda + \lambda^2 - 3) = (5-\lambda)^2 (\lambda^2 - 3\lambda - 1)$

eigen values $(5-\lambda)^2$ $\lambda = 5, 5$, $\dfrac{3 \pm \sqrt{9 + 4}}{2}$

$\lambda = 5, 5, \dfrac{3 \pm \sqrt{13}}{2}$

No. of independent eigen vectors corresponding to $\lambda=5$
$= n-r$

Here $n=4$
$r =$ Rank $(A-\lambda I)$. Rank $(A-5I)$

$$(A-5I) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{pmatrix}$$

$$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \rightarrow \begin{pmatrix} 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{Interch.}$$

$$\begin{pmatrix} 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_3 \rightarrow R_3 + \frac{R_2}{3}$$

$$\begin{pmatrix} 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rho(A-5I) = r = \underline{2}$$

$r < n$
$2 < 4$

Geometric Multiplicity (GM) $= n-r = 4-2 = \underline{2}$
$\therefore$ 2 linearly independent eigen vectors

AM $= 2$ (for $\lambda=5$)

$\boxed{\text{always GM} \le \text{AM}}$

Q19 $A_{4\times4}$ Matrix is such that $A^4 = A$ then the eigen values of A are.
(a) $1, 1, i, -i$ (b) $1, -1, i, -i$
(c) $1, 1, -1, -1$ (d) $0, 1, -0.5 \pm (0.866)i$

Ans $A \longrightarrow \lambda_1, \lambda_2, \lambda_3, \lambda_4$
$A^4 \longrightarrow \lambda_1^4, \lambda_2^4, \lambda_3^4, \lambda_4^4 = \lambda_1, \lambda_2, \lambda_3, \lambda_4$
because $\underline{A = A^4}$

Now check options $\rightarrow$ only (d) is possible

(or)

$A^4 - A = 0$
$f(A) = A^4 - A$
Property: if λ is eigen value of A,
f(A) is polynomial in A, then $f(\lambda)$ is eigen value of f(A)

$\therefore \lambda^4 - \lambda = 0$
$= \lambda(\lambda^3 - 1) = 0$
$= \lambda(\lambda-1)(\lambda^2+\lambda+1) = 0.$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$\lambda = 0, \lambda = 1, \lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -0.5 \pm i\sqrt{3}$

(d)

866
866
5196
51960
692800
749956

Q20 The eigen vectors of the matrix $\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ are $\begin{pmatrix} a \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ b \end{pmatrix}$ then $a+b =$ ____.

Ans $-(3-\lambda)(3+\lambda) - 16 = 0$
$(3^2 - \lambda^2) + 16 = 0$
$9 - \lambda^2 + 16 = 0$
$\lambda^2 - \cancel{7} = 0 \quad = \cancel{(\lambda - \sqrt{7})(\lambda + \sqrt{7})}$
$\cancel{\lambda = -\sqrt{7}, +\sqrt{7}}$ $\quad \lambda = \underline{-5, +5}$

$$\begin{bmatrix} 3-\sqrt{7} & 4 \\ 4 & -3-\sqrt{7} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$(3-\sqrt{7})a + 4b = 0$

$4a - b(3+\sqrt{7}) = 0$

$(3-\sqrt{7})a = -4b$

$4a = b(3+\sqrt{7})$

$(A-\lambda I)X = 0$

$AX - \lambda X = 0$

$\boxed{AX = \lambda X}$

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} a \\ 1 \end{bmatrix} = 5\begin{bmatrix} a \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} 3a+4 \\ 4a-3 \end{pmatrix} = \begin{pmatrix} 5a \\ 5 \end{pmatrix}$$

$4(3a+4 = 5a)$, $12a + 16 = 4\cdot 5a$

$-\ 3(4a-3 = 5)$, $12a - 9 = 3\cdot 5$

$25 = 4\sqrt{7}a - 3\sqrt{7}$

$\frac{25}{\sqrt{7}} = 4a-3$

$\frac{1}{4}\left(\frac{25}{\sqrt{7}}+3\right) = a.$

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ b \end{pmatrix} = -5\begin{pmatrix} 1 \\ b \end{pmatrix}$$

$$\begin{pmatrix} 3+4b \\ 4-3b \end{pmatrix} = \begin{pmatrix} -5 \\ -5b \end{pmatrix}$$

$3(3+4b = -5)$

$4(4-3b) = -5b$

$~~9 + 12b = -3\sqrt{7}~~$

$~~16 - 12b~~$

$4a - 3 = 5$

$3 + 4b = -5$

$4a + 4b = 0$

$\underline{a+b=0}$ //

(or)

Correct method

$~~AX = 0~~$

$\underline{\underline{\lambda = 5}}$ $(A-\lambda I)X = 0$

$(A-5I)X = 0$

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 + 2R_1$ $\begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2x + 4y = 0$

$x - 2y = 0.$

$\frac{x}{2} = y.$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} k \\ k/2 \end{bmatrix} = k\begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$ $\begin{pmatrix} 1 \\ b \end{pmatrix}$

$~~k~~\ k$

$\therefore$ $\underline{\underline{\lambda = -5}}$ $(A+5I)X = 0$

we get $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -k/2 \\ k \end{bmatrix} = k\cdot\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$ $\begin{pmatrix} a \\ 1 \end{pmatrix}$

$\underline{a+b=0}$

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Q21) The eigen vectors of pair of the Matrix $\begin{pmatrix}1&2\\0&2\end{pmatrix}$ is ____

(a) $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}$ (b) $\begin{pmatrix}2\\0\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}$ (c) $\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}2\\81\end{pmatrix}$ (d) $\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}$

Sol: $\lambda_1 = 1$
$\lambda_2 = 2$

AX

$\lambda = 1$ $\begin{bmatrix}0&2\\0&1\end{bmatrix}X = \lambda X$

$\begin{pmatrix}0&2\\0&1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

$\begin{pmatrix}2\\1\end{pmatrix}$

$\lambda = 2$ $\begin{bmatrix}-1&2\\0&0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}2a\\2b\end{bmatrix}$

$\begin{pmatrix}-a+2b\\0\end{pmatrix} = \begin{bmatrix}2a\\2b\end{bmatrix}$

$b=0$,
$AX = \lambda X$.

$\lambda = 1$ $\begin{pmatrix}1&2\\0&2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

$\begin{pmatrix}a+2b\\2b\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

$b = 0$
$a = a$

$\begin{pmatrix}a\\0\end{pmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

(a)

$\lambda = 2$
$\begin{pmatrix}1&2\\0&2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}2a\\2b\end{pmatrix}$

$a+2b = 2a$
$2b = 2b$

$a = 2b$

$\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}2b\\b\end{pmatrix} = b\begin{pmatrix}2\\1\end{pmatrix}$

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Q22) Let $A = \begin{pmatrix}3&5&2\\5&12&7\\2&7&5\end{pmatrix}$. If A is singular and the eigenvalues are non-negative real nos. Then the eigenvector corresponding to the minimum eigenvalue is ____

(a) $\begin{pmatrix}1\\1\\1\end{pmatrix}$ (b) $\begin{pmatrix}1\\1\\0\end{pmatrix}$ (c) $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ (d) $\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

Sol: A is singular, $|A| = 0$
$\therefore$ $\lambda_1 = 0$
Since eigen values are given as non-negative.
min eigen value $= \lambda_1 = 0$

$\boxed{AX = \lambda X}$

$\begin{bmatrix}3&5&2\\5&12&7\\2&7&5\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = 0X = 0$

Substitute values (c)

$3a + 5b + 2c = 0$
$5a + 12b + 7c = 0$
$2a + 7b + 5c = 0$

$R_2 \to 3R_2 - 5R_1$
$R_3 \to 3R_3 - 2R_1$

36 − 25
21 − 10
21 − 10
15 − 4

$\begin{bmatrix}3&5&2\\0&11&11\\0&11&4\end{bmatrix}$

$\begin{bmatrix}3&5&2\\0&11&11\\0&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = 0$

$3a + 5b + 2c = 0$
~~$3a + 7b = 0$~~

$11b + 11c = 0$
$b = -c$

~~$3a + 5b + 2c = 0$~~
$-7b + 3a = 0$

$a = \frac{-7b}{3}$

$-7b/3$
b
b

$-7/3$

$-3a/7$

$3a + 5b + 2c$
$= 3a + 3b = 0$ $a = -b$

$-b\begin{pmatrix}1\\-1\\1\end{pmatrix}$

(c)

$\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-b\\b\\-b\end{pmatrix}$

The product of non-zero eigen values of

[illegible]

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(Q24) The value of k such that the vector $\begin{pmatrix}1\\2\\k\end{pmatrix}$ is eigen vector of the Matrix $\begin{pmatrix}4 & 1 & 2\\17 & 2 & 1\\14 & -4 & 10\end{pmatrix}$ is ______

Ans $AX = \lambda X$

$$\begin{bmatrix}4 & 1 & 2\\17 & 2 & 1\\14 & -4 & 10\end{bmatrix}\begin{bmatrix}1\\2\\k\end{bmatrix} = \lambda\begin{bmatrix}1\\2\\k\end{bmatrix}$$

$$\begin{bmatrix}4+2+2k\\17+4+k\\14-8+10k\end{bmatrix} = \lambda\begin{bmatrix}1\\2\\k\end{bmatrix}$$

$$\begin{bmatrix}6+2k\\21+k\\6+10k\end{bmatrix} = \lambda\begin{bmatrix}1\\2\\k\end{bmatrix}$$

λ^3 $S_1 = Tr(A) = 16$

$S_3 = \Delta(A) = 4(24) - (170-14) + 2(-34-28)$

$= 96 - 156 + -124$

$= -156 - 28$

$= -184$

$S_2 = 24 + 12 + -9 = 27$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = \lambda^3 - 16\lambda^2 + 27\lambda + 184 = 0$

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$6 + 2k = \lambda$ $12 + 4k = 21 + k$
$21 + k = 2\lambda$ $9 = 3k$
$6 + 10k = \lambda k$ $k = 3$

$12 = \lambda$

Ans 3

(Q25) For matrix $A = \begin{bmatrix}a & 0 & 3 & 7\\2 & 5 & 1 & 3\\0 & 0 & 2 & 4\\0 & 0 & 0 & b\end{bmatrix}$ if sum of the eigen values is 14 and product of eigen values is 100. Then $|a-b| =$ ______

Ans $7 + a + b = 14$
$a + b = 7$

$$b\begin{vmatrix}a & 0 & 3\\2 & 5 & 1\\0 & 0 & 2\end{vmatrix} = 100$$

$b\left[2 \times 5a\right] = 10ab = 100$
$ab = 10$

$5, 2$ $5 - 2 = 3$

1.5 Special Matrices

Generalized Terminology

- Symmetric Matrix $\rightarrow$ $\boxed{A^T = A}$
- Skew Symmetric Matrix $\rightarrow$ $\boxed{A^T = -A}$ {The diagonal elements of skew symmetric matrices are 0}
- Orthogonal Matrix $\rightarrow$ $\boxed{AA^T = A^TA = I}$

$\rightarrow$ Let A be ~~an~~ n×n Matrix. A can be expressed as sum of Symmetric & Skew Symmetric Matrices.

$$A = \underbrace{\left[\frac{A+A^T}{2}\right]}_{P = \text{Symmetric}} + \underbrace{\left[\frac{A-A^T}{2}\right]}_{Q = \text{Skew Symmetric}}$$

$$P = \frac{A+A^T}{2} \Rightarrow P^T = \left(\frac{A+A^T}{2}\right)^T$$

$$= \frac{A^T + (A^T)^T}{2} = \frac{A+A^T}{2} = P$$

$$Q = \frac{A-A^T}{2} \Rightarrow Q^T = \left(\frac{A-A^T}{2}\right)^T = \frac{A^T - (A^T)^T}{2}$$

$$= -\left(\frac{A-A^T}{2}\right) = -Q$$

$\rightarrow$ If A is an orthogonal Matrix then $\boxed{|A| = \pm 1}$

$$AA^T = A^TA = 1$$
$$|AA^T| = |I| = 1$$
$$|A||A^T| = 1 \qquad (|A| = |A^T|)$$
$$|A|^2 = 1$$
$$\Rightarrow |A| = \pm 1$$

$\rightarrow$ If A is an orthogonal matrix, then $\boxed{A^{-1} = A^T}$

- Hermitian Matrix $\rightarrow$ $A^\theta = A$
- Skew Hermitian Matrix $\rightarrow$ $A^\theta = -A$
- Unitary Matrix $\rightarrow$ $AA^\theta = A^\theta A = I$

Conjugate of a Complex no.

$z = x + iy$

$\bar{z} = x - iy$

Reflection of z on x-axis

- Conjugate Matrix

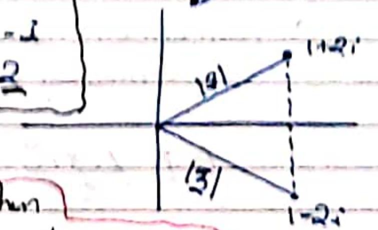
Let A be the given Matrix.

$\bar{A}$ = Replace elements of A by corresponding conjugates

Eg

$z = i, \quad \bar{z} = -i$

$z = 2, \quad \bar{z} = 2$



i) $\bar{z} = -z$, then z is purely imaginary

ii) $\bar{z} = z$, then, z is Real No.

Eg: $A = \begin{pmatrix} 1 & i \\ 2i & -3 \end{pmatrix}$

$$\Downarrow$$

$$\bar{A} = \begin{pmatrix} 1 & -i \\ -2i & -3 \end{pmatrix}$$

- Conjugate Transpose of a Matrix (A^θ)

$$\boxed{A^\theta = (\bar{A})^T \text{ or } \overline{(A^T)}}$$

$$A = \begin{pmatrix} -2 & 3i \\ 4+i & 5 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} -2 & -3i \\ 4-i & 5 \end{pmatrix}$$

$$A^\theta = (\bar{A})^T = \overline{(A^T)} = \begin{pmatrix} -2 & 4-i \\ -3i & 5 \end{pmatrix}$$

Properties

1. $\overline{(\bar{A})} = A$
2. $\overline{A+B} = \bar{A} + \bar{B}$
3. $(\overline{kA}) = \bar{k}\bar{A}$
4. $\overline{AB} = \bar{A}\,\bar{B}$
5. $(A^\theta)^\theta = A$
6. $(A+B)^\theta = A^\theta + B^\theta$
7. $(AB)^\theta = B^\theta A^\theta$

→ A matrix A can be expressed as sum of Hermitian & Skew Hermitian Matrices

$\cancel{A \neq \frac{A+A^\theta}{2}}$

$$A = \underbrace{\left(\frac{A+A^\theta}{2}\right)}_{\text{Hermitian Matrix}} + \underbrace{\left(\frac{A-A^\theta}{2}\right)}_{\text{Skew Hermitian Matrix}}$$

• Unitary Matrix $(AA^\theta = A^\theta A = I)$

Eg:- $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ $\quad A^\theta = (\bar{A})^T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

$$AA^\theta = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Diagonal elements of
→ Hermitian Matrix $(A^\theta = A)$ are Real.
The diagonal elements of
→ Skew-Hermitian Matrix $(A^\theta = -A)$ are purely imaginary or zero

• Positive integral power of $A_{n \times n}$

• $A^k = A \cdot A \cdot A \cdots A$ (k times).

• Idempotent Matrix → $\boxed{A^2 = A}$

Eg:- $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A$

• Nill Potent Matrix : If there exists a positive integer m, such that $\boxed{A^m = O}$, The least positive integer value of m is called index of nill potent Matrix A.

Eg: $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ $\quad A^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

∴ m=2 index = 2

Eg:- $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

m=2
index=2

• Involutory Matrix : Let A be a square matrix of order n. A is said to be involutory matrix if $\boxed{A^2 = I}$

Eg $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

• Periodic Matrix : if $\boxed{A^{k+1} = A}$ here k is positive integer

The least positive integer value of k is called periodicity of A

→ if $k=1$, $A^2 = A$ → A is idempotent Matrix.

31/08/2021

- Eigen Values of Special Matrices

Matrix	Eigen Values
1. Hermitian / Symmetric	Always Real.
2. Skew-Hermitian / Skew Symmetric	either zero / purely imaginary
3. Unitary Matrix / Orthogonal Matrix	Are of absolute value 1, ie $\lvert\lambda\rvert = 1$ (or) real or complex conjugate pairs. Eg:- $\left\lvert \frac{1}{2} + i\frac{\sqrt{3}}{2} \right\rvert = 1$, $\lvert i \rvert = 1$, $\lvert 1 \rvert = 1$, $\lvert -i \rvert = 1$ ---
4. Idempotent Matrix ($A^2 = A$)	if λ is eigenvalue of A, $f(\lambda)$ is polynomial function in A, $f(\lambda)$ is eigenvalue of $f(A)$. $A^2 - A = 0$; $\lambda^2 - \lambda = 0$, $\lambda(\lambda - 1) = 0$; $\boxed{\lambda = 0, \text{ or } 1}$
5. Nill Potent Matrix ($A^m = 0$)	Same explanation. 0 is the eigen value.
6. Involuntary Matrix ($A^2 = I$)	Same explanation. $A^2 - I = 0$; $\lambda^2 - 1 = 0$; $\boxed{\lambda = \pm 1}$

→ Let A be an orthogonal matrix
Let λ be an eigen value of A.
$\Rightarrow \frac{1}{\lambda}$ is also an eigen value of A.

$AA^T = I = A^TA$
By no: 4 → $\lvert A \rvert = \pm 1$
$\lvert A \rvert$ = product of eigen values.
$\therefore \lambda \times \frac{1}{\lambda} = 1$

* Orthogonal Vectors

Two vectors $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

are said to be orthogonal if $X^TY = 0$ (or) $Y^TX = 0$

$X^TY = (x_1\ x_2\ x_3)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$= \underline{x_1y_1 + x_2y_2 + x_3y_3} = 0$

inner product or dot product of the vectors

Eg1: $x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad X^TY = (1\ 0\ {-1})\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$

$\therefore$ X & Y are orthogonal vectors

Eg2: $X = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ $\quad X^TY = (1\ 2\ 1)\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = 2 + 3 = 5 \neq 0$.

$\therefore$ X & Y are not Orthogonal vectors

- Set of orthogonal Vectors.

$S = \{X_1, X_2, \text{---} X_n\}$ such that $X_iX_j^T = X_i^TX_j = 0$.

i.e, every pair from is orthogonal.

- Norm of Vector : $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, then Norm is $\boxed{\lVert X \rVert = \sqrt{x_1^2 + x_2^2 + x_3^2}}$

- Normalized Vector is $\boxed{\frac{X}{\lVert X \rVert}}$

Eg:- $X = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad \lVert x \rVert = \sqrt{1 + 4 + 9} = \sqrt{14}$ (norm of vector)

Normalized Vector = $\begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix}$

- Set of orthonormal vectors

Let $S = \{x_1, x_2, \dots, x_n\}$ be the set of vectors.

Then if →

1. $x_i x_j^T = x_i^T x_j = 0$
2. $\|x_i\| = 1$

or $x_i x_j^T = 0$ if $i \neq j$, $= 1$ if $i = j$

Then it is called orthonormal vectors.

→ If A is an orthogonal matrix, then its rows/columns are orthonormal.

Eg: $A = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{pmatrix}$ is an orthogonal matrix.

$X_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$ $X_2 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$ $X_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ -1/3 \end{pmatrix}$

1. $X_1^T X_2 = \left(\frac{1}{3} \quad \frac{2}{3} \quad \frac{-2}{3}\right) \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} = \left(\frac{2}{9} + \frac{2}{9} - \frac{4}{9}\right) = 0$ //

∴ X_1 & X_2 are orthogonal.

Similarly, (X_1, X_3), (X_2, X_3) are also orthogonal.

2. $\|X_1\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$ // $\|X_2\| = 1$, $\|X_3\| = 1$

∴ The columns of given orthogonal Matrix A are orthonormal. //

Problems

Gate 2017 (Q1) $P = \begin{pmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix}$ which of the following is not correct?

(a) Determinant of P is 1
(b) P is orthogonal
(c) Inverse of P is equal to P^T
(d) All of eigenvalues of P are real

Ans $|P| = 1$, a correct., all columns are orthonormal ∴ P is orthogonal, so b & c correct

Ans : (d) wrong

Gate 2017 (Q2) $A = \begin{pmatrix} 3/2 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 3/2 \end{pmatrix}$ has three distinct eigenvalues and one of its eigen ~~values~~ vectors is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Which one of the following can be another eigen vector of A.

(a) $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Ans $AX = \lambda X$

$\begin{pmatrix} 3/2 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 3/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ \lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \Rightarrow \lambda = 2$ //

$\left(\frac{3}{2} - \lambda\right)(-1-\lambda)\left(\frac{3}{2} - \lambda\right) + \frac{1}{2}\left(0 - (-1-\lambda)\frac{1}{2}\right) = 0$

$-\left(\frac{3}{2} - \lambda\right)^2 (1+\lambda) + \frac{1}{4}(1+\lambda) = 0$

$(\lambda+1)\left(\frac{1}{4} - \left(\frac{3}{2} - \lambda\right)^2\right) = 0$

$\lambda = -1, 1, 2$

$\lambda = -1, 1, 2$

$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\frac{1}{4} = \left(\frac{3}{2} - \lambda\right)\left(\frac{3}{2} - \lambda\right)$

$\lambda^2 + 2 - 3\lambda = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda^2 - 2\lambda - \lambda + 2 = 0$

$\lambda^2 2 - 3\lambda . 0$

1, $\frac{1}{2}$, 0, $-\frac{1}{2}$

$$\begin{pmatrix} 3/2 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 3/2 \end{pmatrix} X = X \quad \text{or} \quad -X$$

(a) $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ ✗ (c) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ ✓

(b) $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ ✗ (c)

(or)

$A^T = A$ (Symmetric Matrix)

A has 3 distinct eigen values (given) $\Rightarrow$ eigen vectors are pairwise ~~orthogonal~~ orthogonal.

$(1\ 0\ 1)\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -1 \neq 0$ a ✗

$(1\ 0\ 1)\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -1+0+0 = -1 \neq 0$ b ✗

$(1\ 0\ 1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1+0-1 = 0$ (c) ✓

$\therefore$ (c) is correct.

(1.6) LU Decomposition

- Objective: 1. We solve a given system of linear equations $AX = B$
  2. We can find A^{-1}
  3. We express the Matrix A as product of lower triangular Matrix and Upper Triangular Matrix.
  i.e., $\boxed{A = LU}$

- General Form.
  Let A be a $n \times n$ Matrix, then the LU Decomposition is given as follows :-

$$A = \begin{bmatrix} l_{11} & 0 & 0 & \cdots \\ l_{21} & l_{22} & 0 & \cdots \\ \vdots & & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ 0 & 0 & u_{33} \cdots & u_{3n} \\ \vdots & & & \\ 0 & 0 & 0 \cdots & u_{nn} \end{bmatrix}$$

$$A_{n\times n} = L_{n\times n} \qquad U_{n\times n}$$

→ Each Matrix contains $\frac{n(n+1)}{2}$ unknowns

→ Total no. of unknowns $= n(n+1)$

1. $A = LU$
By comparing the elements we get n^2 equations.

Eg $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{23} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Total no. of unknowns $= n(n+1) = 3\times 4 = 12$

We get $n^2 = 3^2 = 9$ equations to be solved

Eg. $l_{11}u_{11} = a_{11}$, $l_{11}u_{12} = a_{12}$, $l_{11}u_{13} = a_{13}$
$l_{21}u_{11} = a_{21}$, $l_{21}u_{12} + l_{22}u_{22} = a_{22}$ ...

→ If A is a non-singular Matrix then the $A = LU$ can be obtained by using Row addition operations.

Here,

L = lower Δ matrix with diagonal elements as 1

U = Upper Δ matrix

Eg. Find LU decomposition of

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 8 & 13 \\ 6 & 27 & 31 \end{bmatrix}$$

$A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \longrightarrow \text{Solve}$$

Shortcut Method

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 8 & 13 \\ 6 & 27 & 31 \end{bmatrix} \xrightarrow[\text{Convert to upper } \Delta \text{ matrix}]{R_2 \to R_2 - (2)R_1,\ R_3 \to R_3 - (3)R_1} \begin{bmatrix} 2 & 1 & 5 \\ 0 & 6 & 3 \\ 0 & 24 & 16 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - (4)R_2} \begin{bmatrix} 2 & 1 & 5 \\ 0 & 6 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 0 & 6 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$A = L \quad U$

$\therefore R_i \to R_i - (c)R_j$

Eg. $R_2 \to R_2 - (3)R_1$ (−3 [illegible])

* Minimal Polynomial

1. If $f(A) = 0$ then we say that the polynomial $f(x)$ annihilates the Matrix A.

2. Monic Polynomial : The coefficient of highest power of x is unity.

3. Minimal Polynomial : 1. lowest degree monic polynomial
2. Annihilates Matrix A.

Eg : $A_{3\times3} \longrightarrow$ eigenvalues 1, 1, 3.

char. then char. eq. ← 1. $(x-1)^2(x-3) \to f(A) = 0 = (A-I)^2(A-3I)$

2. $(x-1)(x-3) \to f(A) = 0 = (A-I)(A-3I)$

↓

(lowest) Minimum degree polynomial and annihilates Matrix A.

→ Minimal polynomial of A

$A_{4\times4} \to 1, 2, 2, 3.$

char. then char. eq. ← 1. $(x-1)(x-2)^2(x-3)$

2. $(x-1)(x-2)(x-3)$ } $f(A) = 0$

↓

Minimal Polynomial of A

$A_{5\times5} \longrightarrow$ eigenvalues $-1, -1, 3, 3, 4.$

possibility of Minimal polynomial are:

char. then char. eq. ← 1. $(x+1)^2(x-3)^2(x-4)$

2. $(x+1)^2(x-3)(x-4)$

3. $(x+1)(x-3)^2(x-4)$

4. $(x+1)(x-3)(x-4)$

} $f(A) = 0$

→ Minimal polynomial → if 4 for here $f(A) \neq 0$ then: 3. [illegible] or 4 will be the minimal polynomial

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1. Minimal polynomial always divides the characteristic equation
2. The characteristic equation and minimal polynomial are having same irreducible factors.
3. If λ is eigenvalue of A $\iff$ λ is root of minimal polynomial.

Problems

(Q1) Consider the system of linear equations

$4x_1 + 5x_2 = 7$

$12x_1 + 14x_2 = 18$ If we apply LU decomposition for solving the system where diagonal elements of L are unity. Then U = ____.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 5 \\ 0 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 5 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & -5 \\ 0 & 1 \end{pmatrix}$

Ans $\begin{pmatrix} 4 & 5 \\ 12 & 14 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 18 \end{pmatrix}$

A X = B

A = LU

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{pmatrix} 4 & 5 \\ 12 & 14 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix}\begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix}$$

$4 = u_1$ $\quad u_2 = 5$

$u_1 l = 12$ $\quad u_2 l + u_3 = 14$

$l = 3$ $\quad u_3 = -1$

$U = \begin{pmatrix} 4 & 5 \\ 0 & -1 \end{pmatrix}$ (b)

(or)

Shortcut

$A = \begin{pmatrix} 4 & 5 \\ 12 & 14 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{pmatrix} 4 & 5 \\ 0 & -1 \end{pmatrix}$ (b)

$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

Note : Don't apply interchange operations, while applying the short cut method.

(Q2) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{bmatrix}$ if $A = LU$ where L is Lower Δ matrix with diagonal elements as 1. and U is upper Δ matrix, Then U = ____.

(a) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$.

Ans $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -4 & 5 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - (4)R_2}$

(b) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix}$