Universal Turing Machines and Undecidability

Nabil Mustafa

Computational Complexity

Turing machine recap

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- 1 input tape, 1 output tape and 1 work tape

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No!

If function f is computable by a **TM** using alphabet Γ, then it is computable in time $4 \log |\Gamma| \cdot T(n)$ by a **TM** using the alphabet $\{0, 1, \square, \triangleright\}$.

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Church-Turing Hypothesis

Every physically realizable computation device can be simulated by a **TM** .

Alphabet size doesn't matter

Claim

*If function f is computable by a **TM** M using alphabet Γ, then it is computable in time $4 \log |\Gamma| \cdot T(n)$ by a **TM** M' using the alphabet $\{0, 1, \Box, \rhd\}$.*

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Have to decide:

- How to represent each symbol of Γ using $\{0, 1, \square, \triangleright\}$.

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- How to simulate each step of M.
  - Read the encoded symbol via $\log |\Gamma|$ bits. Problem: remember the bits read.
  - Modify the transition table appropriately
  - Lookup the (remembered) read bits in the (modified) table

Remembering bits read

Problem: How to remember the bits read?

$\mathbf{Q}$	Γ	$\mathbf{Q}$	Γ	{L,R,S}
q	i	q'	j	R

n	x	y	i	b	j	3

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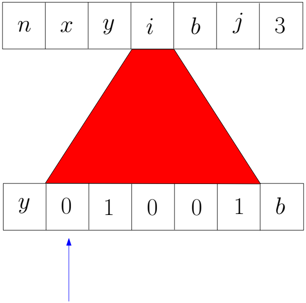

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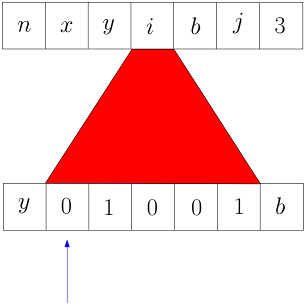
q	01001	q'	01010	R



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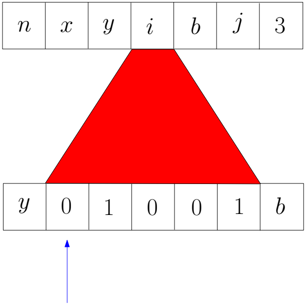

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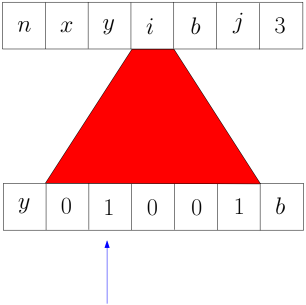

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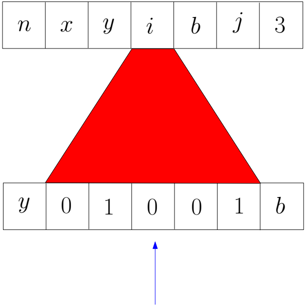

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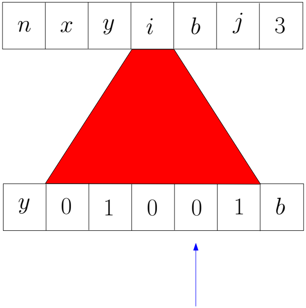

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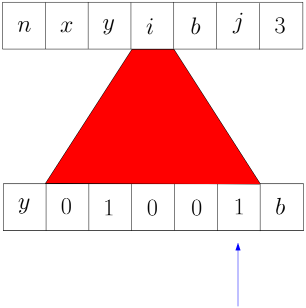

Lets say that $i \rightarrow 01001$ and $j \rightarrow 01010$. Then,

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q_{010}	0	q_{0100}	0	R
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q_{010}	0	q_{0100}	0	R
q_{0100}	1	$qb_{1.01001}$	1	L
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$qb_{4.01001}$	$-$	$qw_{1.01001}$	$-$	S
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$qw_{4.01001}$	$-$	$qw_{5.01001}$	1	R
$qw_{5.01001}$	$-$	q'	0	R

Note: need to move back $2 \log |\Gamma|$ steps as well!

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- Each step of M requires $5 \log |\Gamma|$ steps of M'.
- Total time taken: $5 \log |\Gamma| T(n)$ steps.

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Proof

- Lets consider that a **TM** M computes the function f and has k tapes (plus additional input and output tapes)
- Next we consider a single work tape Turing machine $\hat{M}$
- $\hat{M}$ encodes the k tapes of M on a single tape by using locations $1, k+1, 2k+1, \ldots$ to encode the first tape, locations $2, k+2, 2k+2, \ldots$ to encode the second tape etc.
- For every symbol a in M's alphabet, $\hat{M}$ will contain both the symbol a and the symbol $\hat{a}$. In the encoding of each tape, exactly one symbol will be of the '^ type', indicating that the corresponding head of M is positioned in that location.

More work tapes do not make much difference

Proof Cont'd

- To simulate one step of M, the machine makes two passes of its work tape: first it traverses the tape in the left-to-right direction and records (via additional states) the k symbols of the form $\hat{a}$
- Then $\hat{M}$ uses M's transition function to determine the new state, symbols, and head movements and sweeps the tape back in the right-to-left direction to update the encoding accordingly.
- $\hat{M}$ will never reach more than location $kT(n)$ of its work tape, meaning that for each of the at most $T(n)$ steps of M, $\hat{M}$ performs at most $5kT(n)$ work (sweeping back and forth requires about $2T(n)$ steps, and some additional steps needed for updating head movement and book keeping).

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- So UTM takes α and x, and simulates M_α on x.

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- So we just have to design a **TM** that can run the above algorithm.

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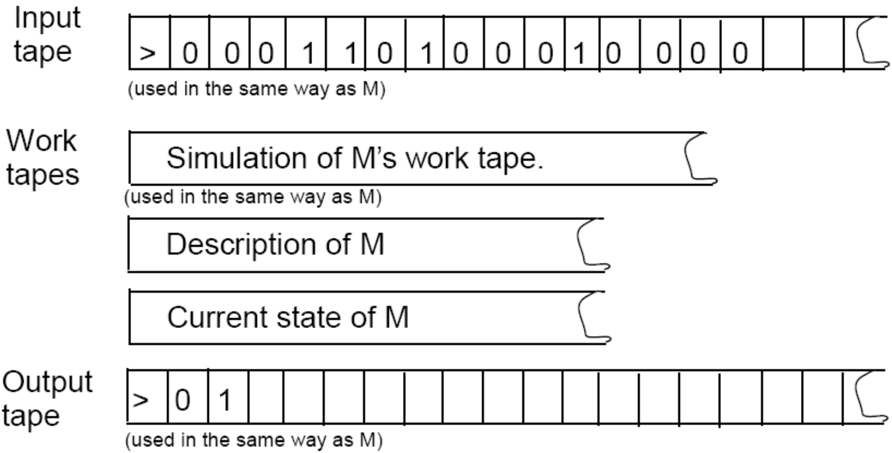
- Copy new symbol to T_4 and move heads of T_1 and T_3.

Simulating M_α



A Quote

"If we were so clever that we could understand our brain, our brain would be so complex that we couldn't understand it."

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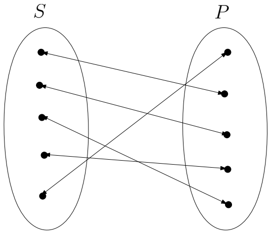

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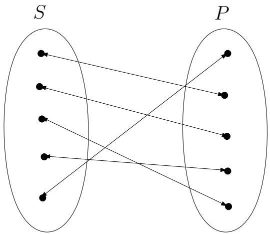

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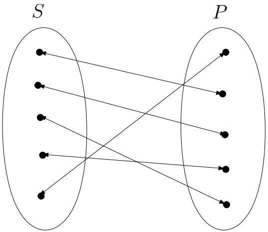

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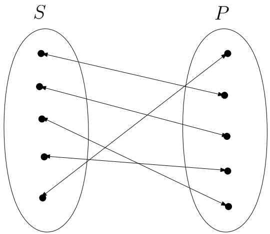

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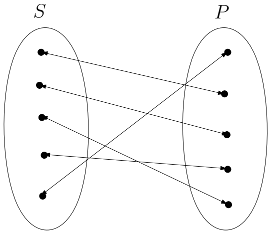

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Question: What about an infinite set, and its power set?

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$$A' = \{i \in S : i \notin f(i)\}$$

- A': All i which are not present in the set $f(i)$

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Important Corollary

Given *any* function $f : S \longrightarrow P$, there always exists a $p_j \in P$ such that **no** $s_i \in S$ maps to p_j.

A Quote

"It seemed unworthy of a grown man to spend his time on such trivialities, but what was I to do?" –Bertrand Russell.

Uncountable Sets and Uncomputable functions

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- By previous corollary, $\exists\ p_k \in P$ s.t. no s_i maps to p_k
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  - When M_H would halt with a **'No'**, D also halts.

□

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- This type of argument is called a *diagonalization* argument.

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