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Double Integrals Expressed as Single Integrals or Interpolatory Functionals*

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1. INTRODUCTION

A phenomenon that pervades the whole of mathematics is that one and the same function or functional can masquerade in several different guises. As examples that are relevant to interpolatory function theory one can cite the following: Over $\mathscr{P}_3$, the set of polynomials of degree $\leqslant 3$, the functional $L_1(f) = (1/6)(f(0) + 4f(1/2) + f(0))$ is identical to the functional $L_2(f) = \int_0^1 f(x)\,dx$. Over the space of single-valued analytic functions $f(z)$ in $|z| < 1$, the functional $L_3(f) = \int_c f(z)\,dz$, where $c : |z| = \frac{1}{2}$, is identical to the functional $L_4(f) \equiv 0$.

Behind the identity of ostensibly different functionals, there lurks the notion of an appropriate space of functions for which the identity is valid. Thus, while $L_1 = L_2$ for $f \in \mathscr{P}_3$, the identity may fail for $f \in \mathscr{P}_4$. L_3 fails to equal L_4 for all continuous complex-valued functions in $|z| < 1$.

Given two functionals L_1 and L_2, one can consider a subspace M of all functions on which L_1 and L_2 are identical. That is, $L_1(f) = L_2(f), f \in M$ or $L_1 - L_2 \perp M$. But very often what one may really want to know is whether or not M contains some *particular* space of importance to analysis.

For reasons outlined in [2] and [8] having to do with the construction of "complete quadrature rules," we have been concerned with the question of when and how a double integral of an analytic function over a region B can be expressed, either as an integral over an open arc contained in B or as a differential operator acting at certain points of B, or a combination of these.

This paper extends results previously obtained in this direction. It is felt

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that the results are interesting in their own right, and applications will not be indicated in any detail.

2. Some Useful Formulas

In this section we collect some useful formulas. In the complex plane, write

$$\begin{aligned} z &= x + iy, & \bar{z} &= x - iy, \\ x &= (1/2)(z + \bar{z}), & y &= (1/2i)(z - \bar{z}). \end{aligned} \tag{2.1}$$

Introduce the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{2.2}$$

$$dz = dx + i\,dy, \qquad \overline{dz} = dx - i\,dy = d\bar{z}. \tag{2.2.1}$$

If $f(z) = u(x, y) + iv(x, y)$, then

$$\begin{aligned} (\partial f/\partial z) &= (1/2)(u_x + iv_x - iu_y + v_y), \\ (\partial f/\partial \bar{z}) &= (1/2)(u_x + iv_x + iu_y - v_y). \end{aligned} \tag{2.3}$$

If f is analytic then $u_x = v_y$, $u_y = -v_x$ and

$$\begin{aligned} (\partial f/\partial z) &= u_x + iv_x = f'(z), \\ (\partial f/\partial \bar{z}) &= 0. \end{aligned} \tag{2.4}$$

Writing $\overline{f(z)}$ for the antianalytic function $u - iv$, we obtain, similarly,

$$\frac{\partial \overline{f(z)}}{\partial z} = 0, \qquad \frac{\partial \overline{f(z)}}{\partial \bar{z}} = \overline{f'(z)}. \tag{2.5}$$

If B is a point set in the complex plane, the point set $\bar{B}$ (the reflection of B) is given by $\bar{B} = \{\bar{z} : z \in B\}$.

If $f(z)$ is analytic in the region B, the "reflected" function $\bar{f}(z)$ is that analytic function defined in $\bar{B}$ and given by

$$\bar{f}(z) = \overline{f(\bar{z})}. \tag{2.6}$$

Note that we have

$$\overline{f(z)} = \bar{f}(\bar{z}). \tag{2.7}$$

Green's theorem, written in complex form and valid for analytic functions f and g, is given by the alternate forms

$$\int_B\int \overline{f'(z)}\, g(z)\, dx\, dy = \frac{1}{2i}\int_{\partial B} \overline{f(z)}\, g(z)\, dz. \tag{2.8}$$

Here ∂B designates the boundary of B traversed in the positive sense.

$$\int_B\int \overline{f(z)}\, g'(z)\, dx\, dy = -\frac{1}{2i}\int_{\partial B} \overline{f(z)}\, g(z)\, \overline{dz} \tag{2.9}$$

or

$$\int_B\int \overline{f'(z)}\, g'(z)\, dx\, dy = \frac{1}{2i}\int_{\partial B} \overline{f(z)}\, g'(z)\, dz = -\frac{1}{2i}\int_{\partial B} \overline{f'(z)}\, g(z)\, \overline{dz}. \tag{2.10}$$

In particular, with $f(z) \equiv z$,

$$\int_B\int g(z)\, dx\, dy = \frac{1}{2i}\int_{\partial B} \bar{z} g(z)\, dz. \tag{2.11}$$

Sufficient conditions for the validity of these formulas are that f and g be analytic in B and continuously differentiable on ∂B.

The Schwarz Function of a Plane Curve

Suppose that a curve C in the complex plane is given by an equation in rectangular coordinates

$$\phi(x, y) = 0. \tag{2.12}$$

Introducing z and $\bar{z}$ we obtain as an equation

$$\phi\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = 0. \tag{2.13}$$

The coordinates z and $\bar{z}$ are sometimes referred to as *minimal* or *conjugate* coordinates. Assume now that we can solve (2.13) for $\bar{z}$ in terms of z:

$$\bar{z} = S(z). \tag{2.14}$$

If ϕ is analytic, then except in the vicinity of certain singular points, we obtain an analytic function as a solution of (2.13). We shall call $S(z)$ the *Schwarz function* of the curve C. See [6, 15]. No allusion to the Schwarz functions that arise in the theory of automorphic functions is intended here. If ϕ is an algebraic function of x, y then $S(z)$ is an algebraic function: it is an analytic multivalued function of z. If C is a closed analytic curve then $S(z)$ has a single-valued analytic branch in an annulus-like region that contains C in its interior.

The following is a list of elementary properties of the Schwarz function. All equations are to be understood as *holding along* C.

$$\bar{z} = S(z). \tag{2.15}$$

$$z = re^{i\theta}, \qquad r^2 = z\bar{z} = |S(z)|^2, \qquad \theta = \frac{i}{2}\log\left(\frac{\bar{z}}{z}\right) = \frac{i}{2}\log\left(\frac{S(z)}{z}\right). \tag{2.16}$$

$$S'(z) = \frac{d\bar{z}}{dz} = \frac{dx - i\,dy}{dx + i\,dy} = \frac{1 - iy'}{1 + iy'}. \tag{2.17}$$

$$y' = -i\frac{1 - S'(z)}{1 + S'(z)}. \tag{2.18}$$

$$|S'(z)| = 1, \tag{2.19}$$

If a curve with Schwarz function $S(z)$ passes through z_1, the equation of the tangent line at z_1 is

$$\bar{z} = S'(z_1)(z - z_1) + \bar{z}_1. \tag{2.20}$$

The angle of intersection θ between two curves with Schwarz functions S and T is given by

$$\tan\theta = i\left(\frac{S' - T'}{S' + T'}\right). \tag{2.21}$$

In particular, if at a common point, $S' = T'$, the two curves are tangent, while if $S' = -T'$, the curves are orthogonal.

$$\frac{d^2y}{dx^2} = \frac{4iS''(z)}{(1 + S'(z))^3}, \tag{2.22}$$

$$dz\,d\bar{z} = (dx + i\,dy)(dx - i\,dy) = dx^2 + dy^2 = ds^2 = S'(z)\,dz\,dz, \tag{2.23}$$

hence

$$ds = \sqrt{S'(z)}\,dz, \tag{2.24}$$

$$k = \text{curvature of } C = iS''/2(S')^{3/2}, \tag{2.25}$$

$$|k| = \tfrac{1}{2}|S''|. \tag{2.26}$$

One may therefore speak of $S''(z)$ as the "complex curvature" of C.

The Schwarz function for a closed analytic curve C can be expressed in terms of the mapping function of C. Let $w = M(z)$ be any analytic function which performs a 1–1 conformal map of C and its interior onto the closed unit disc. Let $z = m(w)$ designate its inverse. Then

$$S(z) = \bar{m}(1/M(z)). \tag{2.27}$$

The point $z_R = \overline{S(z)}$ is the Schwarzian reflection of the point z in the analytic curve C. In virtue of this fact, $S(z)$ satisfies the functional equation

$$\bar{S}(S(z)) \equiv z. \tag{2.28}$$

(See Davis and Pollak [6, p. 7, 23].)

3. Some Special Curves and Regions

There are a number of ways in which special curves are commonly and conveniently represented. These include equations in rectangular and polar coordinates, level lines of analytic functions, mapping functions from a straight line or a circle onto the curve, etc. In this section, we shall exhibit a number of specific curves where the Schwarz function can be obtained as an explicit elementary algebraic function. The examples are intended to be representative and not exhaustive.

3.1. *Rectangular Coordinates*

(a) *Circle*: $(x - x_0)^2 + (y - y_0)^2 = (z - z_0)(\bar{z} - \bar{z}_0) = r^2$.

$$\bar{z} = S(z) = \bar{z}_0 + \frac{r^2}{z - z_0}. \tag{3.1.1}$$

Thus, $S(z)$ has a simple pole at z_0. It can be shown (see Davis and Pollak [6]) that the circle is the only curve whose Schwarz function is a rational function of z.

(b) *Ellipse*: $(x^2/a^2) + (y^2/b^2) = 1$. Assume $a > b$. Setting

$$x = (1/2)(z + \bar{z}), \qquad y = (1/2i)(z - \bar{z}),$$

and solving for $\bar{z}$ we obtain

$$\bar{z} = S(z) = \frac{a^2 + b^2}{a^2 - b^2} z + \frac{2ab}{b^2 - a^2} \sqrt{z^2 + b^2 - a^2}. \tag{3.1.2}$$

Thus, $S(z)$ has bracnh points at $z = \pm\sqrt{a^2 - b^2}$, the foci of the ellipse. In the z plane cut from $-\sqrt{a^2 - b^2}$ to $\sqrt{a^2 - b^2}$, $S(z)$ can be defined as a single-valued analytic function.

(c) *The L^4 gauge curve*: $x^4 + y^4 = 1$. Using (2.1) and solving for $\bar{z}$, we obtain

$$\bar{z} = S(z) = (-3z^2 + 2^{3/2}(z^4 + 1)^{1/2})^{1/2}. \tag{3.1.3}$$

This curve bounds a convex region and inside the curve $S(z)$ has singularities only at the points where $z^4 + 1 = 0$, i.e., at $z = \pm(1/2)(\sqrt{2} \pm \sqrt{2}\, i)$.

3.2. *Polar Coordinates*

In preparation for the next group of curves, observe the identity

$$\cos n\theta = \frac{1}{2}(e^{in\theta} + e^{-in\theta}) = \frac{1}{2r^n}[(re^{i\theta})^n + (re^{-i\theta})^n]$$
$$= \frac{z^n + \bar{z}^n}{2r^n}.$$

Hence, if n is even,

$$\cos n\theta = \frac{z^n + \bar{z}^n}{2(z\bar{z})^{n/2}}. \tag{3.2.1}$$

(a) *The rose R_{2m}*.

$$r^{2m} = a + b\cos 2m\theta, \qquad 0 < |b| < a, \qquad m = 1, 2, \ldots. \tag{3.2.2}$$

Using (3.2.1) we obtain

$$z^m\bar{z}^m = a + b\left(\frac{z^{2m} + \bar{z}^{2m}}{2z^m\bar{z}^m}\right).$$

Hence,

$$\bar{z} = S(z) = z\left[\frac{a \pm\sqrt{a^2 - b^2 + 2bz^{2m}}}{2z^{2m} - b}\right]^{1/m}. \tag{3.2.3}$$

As special instances of this curve, we cite

The Bicircular Quartic Q,

($m = 1$) which we write in the form

$$r^2 = a^2 + 4\epsilon^2\cos^2\theta \equiv (a^2 + 2\epsilon^2) + 2\epsilon^2\cos 2\theta, \tag{3.2.4}$$

$$\bar{z} = S(z) = \frac{z(a^2 + 2\epsilon^2) + z\sqrt{a^2 + 4a^2\epsilon^2 + 4\epsilon^2z^2}}{2(z^2 - \epsilon^2)}. \tag{3.2.5}$$

The quantiy under the radical vanishes when $z = \pm i\sqrt{a^2 + (a^4/4\epsilon^2)}$. Since

$a^4/4\epsilon^2 > 0$ and $r = a$ when $\theta = \pi/2, 3\pi/2$, these points lie outside the curve. Hence, inside the curve, $S(z)$ can be defined as a single-valued analytic function with simple poles at $z = \pm\epsilon$.

(b) *The rose* R_4.

$$r^4 = a^4 + 2b^4 \cos 4\theta, \qquad a^4 > 2b^4. \tag{3.2.6}$$

This leads to

$$\bar{z}^2 = S^2(z) = \frac{a^4z^2 + z^2\sqrt{4b^4z^4 + a^8 - 4b^8}}{2(z^4 - b^4)}. \tag{3.2.7}$$

The quantity under the radical vanishes at the points

$z = \sqrt[4]{-1}\ \sqrt[4]{[(a^8 - 4b^8)/4b^4]}$. Since $[(a^8 - 4b^8)/4b^4] > a^4 - 2b^4$, it follows

that these points lie exterior to the curve. Hence, there is a single-valued branch of $S^2(z)$ inside the curve with simple poles at $z = \pm b, \pm ib$.

3.3. *Level Lines*

Let $\phi(z)$ be a function of a complex variable. A *level line* of ϕ is the locus of z's such that

$$|\phi(z)| = c > 0. \tag{3.3.1}$$

(a) A common family of level lines are the generalized lemniscates $\mathscr{L}_r$ given by

$$|p_n(z)| = r^n \qquad \text{or} \qquad |(z - z_1) \cdots (z - z_n)| = r^n, \tag{3.3.2}$$

where the polynomial $p_n(z)$ has been given in its factored form on the right. On the geometry of these curves, see Walsh [19, p. 55]. We can write this as

$$|p_n(z)|^2 = p_n(z)\,\overline{p_n(z)} = p_n(z)\,\overline{p_n}(\bar{z}) = r^{2n}. \tag{3.3.3}$$

If $p_n(z) = \sum_{k=0}^{n} a_k z^k$, we write $\overline{p_n}(z) = \sum_{k=0}^{n} \overline{a_k} z^k$.

Hence, on $\mathscr{L}_r$, we have

$$\overline{p_n}(S(z)) = r^{2n}/p_n(z). \tag{3.3.4}$$

The function $\overline{p_n}(S(z))$ is the rational function $r^{2n}/p_n(z)$ with poles at $z_1, \ldots, z_n$ while the Schwarz function $S(z)$ for $\mathscr{L}_r$ is the algebraic function of z obtained by solving (3.3.4) for S.

As a particular instance, if $p_n(z) \equiv \overline{p_n}(z) \equiv z^n - 1$, we obtain

$$\bar{z} = S(z) = \sqrt[n]{\frac{z^n + r^{2n} - 1}{z^n - 1}}. \tag{3.3.5}$$

For $n = 2$, we obtain the ovals of Cassini.

4. Conversion of Double Integrals to Other Functionals

Using the complex form of Green's theorem and an explicit representation of the Schwarz function, a double integral of an analytic function may often be given another representation. The exact nature of the representation depends crucially on the singularity structure of $S(z)$ inside its curve. From (2.2.1) we obtain for analytic $f(z)$,

$$\iint_B f(z)\,dx\,dy = \frac{1}{2i}\int_{\partial B} \bar{z}f(z)\,dz = \frac{1}{2i}\int_{\partial B} S(z)f(z)\,dz. \tag{4.1}$$

We have more generally from (2.8), for integer $n \geqslant 0$,

$$\begin{aligned}\iint_B \bar{z}^n f(z)\,dx\,dz &= \frac{1}{2i(n+1)}\int_{\partial B} \bar{z}^{n+1}f(z)\,dz \\ &= \frac{1}{2i(n+1)}\int_{\partial B} (S(z))^{n+1}f(z)\,dz.\end{aligned} \tag{4.2}$$

We are now at liberty to change the path of integration by Cauchy's theorem or to apply residue calculus.

We shall work out some examples.

(a) *Circle C*: $|z| \leqslant r$, $S(z) = r^2/z$.

$$\begin{aligned}\iint_C \bar{z}^n f(z)\,dx\,dy &= \frac{\pi r^{2n+2}n!}{2\pi i(n+1)!}\int_{\partial C} \frac{f(z)}{z^{n+1}}\,dz \\ &= \frac{\pi r^{2n+2}}{(n+1)!}f^{(n)}(0) \qquad (n = 0, 1, \ldots).\end{aligned} \tag{4.3}$$

The case $n = 0$ yields the mean-value theorem for analytic functions.

(b) *The half circle* HC: $x^2 + y^2 \leqslant r^2$, $y \geqslant 0$.

$$\iint_{\partial \mathrm{HC}} f(z)\,dx\,dy = \frac{1}{2i}\int_{\mathrm{HC}} \bar{z}f(z)\,dz = \frac{1}{2i}\int_{-r}^{r} xf(x)\,dx + \frac{1}{2i}\int_C \bar{z}f(z)\,dz.$$

Here C designates the circular arc traversed positively. Now

$$\frac{1}{2i}\int_C \bar{z}f(z)\,dz = \frac{r^2}{2i}\int_C \frac{f(z)}{z}\,dz$$

and we replace C by a deleted x axis augmented by a half circle C' of radius ϵ traversed positively.

$$\frac{r^2}{2i}\int_C \frac{f(z)}{z}\,dz = \frac{r^2}{2i}\int_r^{\epsilon} \frac{f(x)}{x}\,dx + \frac{r^2}{2i}\int_{-\epsilon}^{-r} \frac{f(x)}{x}\,dx + \frac{r^2}{2i}\int_{C'} \frac{f(z)}{z}\,dz.$$

Now,

$$\frac{r^2}{2i}\int_{C'} \frac{f(z)}{z}\,dz = \frac{r^2}{2}\int_0^{\pi} f(\epsilon e^{i\theta})\,d\theta.$$

Hence,

$$\lim_{\epsilon\to 0}\frac{r^2}{2i}\int_{C'} \frac{f(z)}{z}\,dz = \frac{\pi r^2}{2} f(0).$$

Thus, finally,

$$\iint_{\mathrm{HC}} f(z)\,dx\,dy = \frac{1}{2i}\int_{-r}^{r} xf(x)\,dx + \frac{r^2 i}{2}\int_{-r}^{r} \frac{f(x)}{x}\,dx + \frac{\pi r^2}{2} f(0). \tag{4.4}$$

In the second integral on the right, the Cauchy principal value is meant.

(b) *Ellipse* $\mathscr{E}$: $(x^2/a^2) + (y^2/b^2) \leqslant 1 \qquad a > b.$

$$S(z) = \frac{a^2+b^2}{a^2-b^2}\,z + \frac{2ab}{b^2-a^2}\sqrt{z^2+b^2-a^2}.$$

The first term of S is a regular function in $\mathscr{E}$. Hence

$$\iint_B f(z)\,dx\,dy = \frac{ab}{i(b^2-a^2)}\int_{\partial\mathscr{E}} \sqrt{z^2+b^2-a^2}\,f(z)\,dz$$
$$= \frac{ab}{b^2-a^2}\int_{\partial\mathscr{E}} \sqrt{a^2-b^2-z^2}\,f(z)\,dz.$$

The function $\sqrt{a^2-b^2-z^2}$ is single valued in the plane slit along $-\sqrt{a^2-b^2} \leqslant x \leqslant \sqrt{a^2-b^2}$. Hence, we may shrink the curve $\partial\mathscr{E}$ until it coincides with the slit traversed twice; the lower edge from $-\sqrt{a^2-b^2}$ to $\sqrt{a^2-b^2}$ and the upper edge back. On the first traversing, $dy = dx$ and the radical is $-\sqrt{a^2-b^2-x^2}$; on the return, $dz = -dx$ and the radical is $+\sqrt{a^2-b^2-x^2}$.

Hence,

$$\iint_{\mathscr{E}} f(z)\,dx\,dy = \frac{2ab}{a^2-b^2}\int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}} \sqrt{a^2-b^2-x^2}\,f(x)\,dx. \tag{4.5}$$

It is often convenient to place the foci of $\mathscr{E}$ at ± 1. Write $a = \frac{1}{2}(\rho + \rho^{-1})$, $b = \frac{1}{2}(\rho - \rho^{-1})$ and designate by $\mathscr{E}_\rho$ the ellipse with foci at ± 1 and semiaxis sum $a + b = \rho$. Then, $2ab = \frac{1}{2}(\rho^2 - \rho^{-2})$ and

$$\iint_{\mathscr{E}_\rho} f(z)\,dx\,dy = \frac{1}{2}(\rho^2 - \rho^{-2})\int_{-1}^{+1} \sqrt{1-x^2}\,f(x)\,dx. \tag{4.5'}$$

This formula was first obtained in Davis [2] through the double orthogonality of the Tschebycheff polynomials of the 2nd kind.

(c) *Bicircular quartic* Q: $r^2 \leqslant a^2 + 4\epsilon^2\cos^2\theta$.

$$S(z) = \frac{z(a^2+2\epsilon^2) + z\sqrt{a^2 + 4a^2\epsilon^2 + 4\epsilon^2 z^2}}{2(z^2-\epsilon^2)},$$

$$\iint_Q f(z)\,dx\,dy = \frac{\pi}{2\pi i}\int_{\partial Q} S(z)f(z)\,dz.$$

Since the only singularities of $S(z)$ are simple poles at $z = \pm\epsilon$, we need only evaluate the residues at $\pm\epsilon$. For $z = \epsilon$ we have

$$\begin{aligned}(z-\epsilon)\,S(z) &= \frac{\epsilon(a^2+2\epsilon^2) + \epsilon\sqrt{a^4 + 4a^2\epsilon^2 + 4\epsilon^4}}{2(2\epsilon)} \\ &= \frac{\epsilon a^2 + 2\epsilon^3 + \epsilon a^2 + 2\epsilon^3}{4\epsilon} = \frac{a^2}{2} + \epsilon^2.\end{aligned}$$

A similar result holds for $z = -\epsilon$. Hence,

$$\iint_Q f(z)\,dx\,dy = \pi\left(\frac{a^2}{2} + \epsilon^2\right)(f(\epsilon) + f(-\epsilon)). \tag{4.6}$$

The selection $f(z) \equiv z^{2m}$, and evaluation of the left-hand integral in polar coordinates yields the integral identity

$$\int_0^{2\pi} e^{i2m\theta}(a^2 + 4\epsilon^2\cos^2\theta)^{m+1}\,d\theta = (2m+2)\,\pi(a^2+2\epsilon^2)\,\epsilon^{2m}. \tag{4.7}$$

For $m = 0$ we obtain

$$\iint_Q dx\,dy = \text{area}(Q) = \pi(a^2 + 2\epsilon^2). \tag{4.8}$$

More generally, we have for $n \geqslant 0$ by (4.2),

$$\iint_Q \bar{z}^n f(z)\,dx\,dy = \frac{1}{2i(n+1)} \int_{\partial Q} S^{n+1}(z) f(z)\,dz.$$

For simplicity, write

$$N(z) = \frac{z}{2}\left((a^2 + 2\epsilon^2) + \sqrt{a^4 + 4a^2\epsilon^2 + 4\epsilon^2 z^2}\right) \tag{4.9}$$

so that $S^{n+1}(z) = [N^{n+1}(z)/(z-\epsilon)^{n+1}(z+\epsilon)^{n+1}]$ and this function is regular inside ∂Q except at $z = \pm\epsilon$ where it has poles of order $n+1$. Hence,

$$\begin{aligned}
\iint_Q \bar{z}^n f(z)\,dx\,dy &= \frac{\pi}{(n+1)!}\,\frac{n!}{2\pi i} \int_{\partial Q} \frac{N^{n+1}(z)}{(z-\epsilon)^{n+1}(z+\epsilon)^{n+1}} f(z)\,dz \\
&= \frac{\pi}{(n+1)!} \left\{ \frac{d^n}{dz^n}\left(\frac{N^{n+1}(z) f(z)}{(z+\epsilon)^{n+1}}\right)\Big|_{z=\epsilon} \right. \\
&\qquad \left. + \frac{d^n}{dz^n}\left(\frac{N^{n+1}(z) f(z)}{(z-\epsilon)^{n+1}}\right)\Big|_{z=-\epsilon} \right\} \qquad (4.10) \\
&= \frac{\pi}{(n+1)!} [a_{n0} f(\epsilon) + a_{n1} f'(\epsilon) + \cdots + a_{nn} f^{(n)}(\epsilon) \\
&\qquad + b_{n0} f(-\epsilon) + b_{n1} f'(-\epsilon) + \cdots + b_{nn} f^{(n)}(-\epsilon)],
\end{aligned}$$

where the constants a_{nk} and b_{nk} are independent of f and may be obtained explicitly by expanding the above bracket.

The formulas just developed have a harmonic counterpart. Let $u(x, y)$ be harmonic in Q. Then, as is well known, if we construct

$$v(x, y) = \int_{(x_0, y_0)}^{(x, y)} -u_y\,dx + u_x\,dy,$$

for an arbitrary (x_0, y_0) in Q, the function $f(z) = u(x, y) + iv(x, y)$ is single valued and analytic in Q and has u as its real part. Now by (4.6),

$$\iint_Q (u + iv)\,dx\,dy = \pi\left(\frac{a^2}{2} + \epsilon^2\right) [u(\epsilon, 0) + iv(\epsilon, 0) + u(-\epsilon, 0) + iv(-\epsilon, 0)].$$

Taking real parts we find

$$\iint_Q u(x, y)\,dx\,dy = \pi\left(\frac{a^2}{2} + \epsilon^2\right)(u(\epsilon, 0) + u(-\epsilon, 0)). \tag{4.11}$$

This is an extension of the mean-value theorem for harmonic functions. This identity suggests several questions. It is known that a necessary and sufficient condition for u to be harmonic in a region G is that

$$u(z_0) = \frac{1}{\pi r^2} \iint_{|z-z_0| \leqslant r} u(x, y)\, dx\, dy$$

for all $z_0 \in G$ and all sufficiently small r. Can a similar result be obtained for (4.11)? What about a theory of subharmonic functions based on (4.11)?

Inequalities for Harmonic Functions

Let C_r designate the circle $|z| \leqslant r$. In view of the obvious inequalities $a^2 + 4\epsilon^2 \geqslant a^2 + 4\epsilon^2 \cos^2\theta \geqslant a^2$, it follows that if we set $b = \sqrt{a^2 + 4\epsilon^2}$, we have $C_a \subseteq Q \subseteq C_b$. If now u is harmonic and nonnegative in C_b,

$$\iint_{C_b} u\, dx\, dy \geqslant \iint_{Q} u\, dx\, dy \geqslant \iint_{C_a} u\, dx\, dy.$$

Hence

$$\pi(a^2 + 4\epsilon^2)\, u(0, 0) + \pi\left(\frac{a^2}{2} + \epsilon^2\right)(u(\epsilon, 0) + u(-\epsilon, 0)) \geqslant \pi a^2 u(0, 0).$$

Therefore

$$\frac{a^2 + 4\epsilon^2}{a^2 + 2\epsilon^2}\, u(0, 0) \geqslant \frac{1}{2}(u(\epsilon, 0) + u(-\epsilon, 0)) \geqslant \frac{a^2}{a^2 + 2\epsilon^2}\, u(0, 0). \tag{4.12}$$

(d) *Rose* R_4: $r^4 \leqslant a^4 + 2b^4 \cos 4\theta,\ a^4 > 2b^4$.

$$\bar{z}^2 = S^2(z) = \frac{a^4 z^2 + z^2 \sqrt{4b^4 z^4 + a^8 - 4b^8}}{2(z^4 - b^4)}.$$

Inside R_4, $S^2(z)$ has only simple poles at $z = \pm b, \pm bi$.

$$\iint_{R_4} \bar{z} f(z)\, dx\, dy = \frac{1}{4i} \int_{\partial R_4} \bar{z}^2 f(z)\, dz = \frac{\pi}{2}\, \frac{1}{2\pi i} \int_{\partial R_4} S^2(z) f(z)\, dz.$$

The residue at $z = b$ is

$$\frac{a^4 z^2 + z^2 \sqrt{4b^4 z^4 + a^8 - 4b^8}}{2(z + b)(z^2 + b^2)} f(z)\,\Big|_{z=b}$$

and similarly for the other points. Hence,

$$\iint_{R_4} \bar{z} f(z)\, dx\, dy = \frac{\pi a^4}{8b^2}\,[f(b) - if(ib) - f(-b) + if(-ib)]. \tag{4.13}$$

In view of the singularity structure of $S(z)$, $\iint_{R_4} f(z)\,dx\,dy$ may be reduced to two line integrals extended from b to bi and from $-bi$ to $-b$. Thus, for even values of n, $\iint_{R_4} \bar{z}^n f(z)\,dx\,dy$ will be a functional of that type while for odd n, it can be reduced to an interpolation functional.

(e) *Lemniscates.* We begin with the lemniscate (3.3.5) with $n = 2$ and $r > 1$. Call it OC (oval of Cassini). We have

$$\bar{z} = S(z) = \sqrt{\frac{z^2 + r^4 - 1}{z^2 - 1}}$$

as the Schwarz function for OC. The zeros of the numerator are at $\pm i\sqrt{r^4 - 1}$ and these points are exterior to OC. Hence $S(z)$ is single valued in the interior minus the cut $-1 \leqslant x \leqslant 1$. We have

$$\iint_{\mathrm{OC}} f(z)\,dx\,dy = \frac{1}{2i}\int_{\partial\mathrm{OC}} \sqrt{\frac{z^2 + r^4 - 1}{z^2 - 1}}\,dz,$$

and shrinking ∂OC to the cut, there is obtained

$$\iint_{\mathrm{OC}} f(z)\,dx\,dy = \int_{-1}^{+1} \sqrt{\frac{x^2 + r^4 - 1}{1 - x^2}}\,f(x)\,dx. \tag{4.14}$$

If $r < 1$, the Cassinian oval consists of two lobes. Looking at the right lobe, $S(z)$ will have branch points at $z = 1$ and at $z = \sqrt{1 - r^4}$. An analogous formula to (4.14) extends the real integral over $\sqrt{1 - r^4} \leqslant x \leqslant 1$.

(f) *Lemniscates, continued.* Let $z_1, \ldots, z_n$ be n points in the complex plane, not necessarily distinct. Let $p(z) \equiv (z - z_1) \cdots (z - z_n)$, and let r be selected so large that the locus $|p(z)| = r^n$ consists of one closed curve containing $z_1, \ldots, z_n$ in its interior. Designate the set $|p(z)| \leqslant r^n$ by $\mathscr{L}_r$. On $\partial\mathscr{L}_r$ we have $\overline{p(z)} = r^{2n}/p(z)$. From (2.8) we have

$$\iint_{\mathscr{L}_r} \overline{p'(z)}\,f(z)\,dx\,dy = \frac{1}{2i}\int_{\partial\mathscr{L}_r} \overline{p(z)}\,f(z)\,dz = \frac{\pi r^{2n}}{2\pi i}\int_{\partial\mathscr{L}_r} \frac{f(z)}{p(z)}\,dz. \tag{4.15}$$

Now, if z_i are distinct,

$$\frac{1}{2\pi i}\int_{\partial\mathscr{L}_r} \frac{f(z)}{p(z)}\,dz = \frac{f(z_1)}{p'(z_1)} + \cdots + \frac{f(z_n)}{p'(z_n)} = [f(z_1), f(z_2), \ldots, f(z_n)]$$

$=$ the n-th divided difference of $f(z)$ with respect to the points $z_1, \ldots, z_n$.

In the case of multiple points, the contour integral equals the generalized divided difference. (See Davis [3, Chapter 3].)

Hence,

$$\iint_{\mathscr{L}_r} \overline{p'(z)} f(z)\, dx\, dy = \pi r^{2n}[f(z_1),\dots, f(z_n)]. \tag{4.16}$$

If the points z_i are spaced uniformly, then the divided difference becomes the ordinary n-th-order difference. Let $p(z) = z(z-1)\cdots(z-n)$ and $\mathscr{L}_r : |p(z)| \leqslant r^{n+1}$. Then,

$$\begin{aligned}\iint_{\mathscr{L}_r} \overline{p'(z)} f(z)\, dx\, dy &= \pi r^{2n+2}[f(0), f(1),\dots, f(n)] \\ &= \pi r^{2n+2} \Delta^n f(0)/n!. \end{aligned}\tag{4.17}$$

Another special case arises if we select $f(z) = p'(z)$ in (4.15). From the argument principle we have

$$\frac{1}{2\pi i}\int_{\partial\mathscr{L}_r} \frac{p'(z)}{p(z)}\, dz = n.$$

Hence

$$\iint_{\mathscr{L}_r} |p'(z)|^2\, dx\, dy = n\pi r^{2n}. \tag{4.18}$$

This equality may be phrased in an alternate way. For fixed $z_1, z_2,\dots, z_n$, let $D(f)$ designate the divided difference of f at $z_1,\dots, z_n$. Then, from (4.16),

$$D(f) = \frac{1}{\pi r^{2n}} \iint_{\mathscr{L}_r} \overline{p'(z)} f(z)\, dx\, dy.$$

Thus, over the Hilbert space $L^2(\mathscr{L}_r)$ (see, e.g., Davis [3, p, 207] for this space), the function $p'(z)/\pi r^{2n}$ is the representer of the functional D, and the norm of D is given by

$$\| D \|^2_{\mathscr{L}_r} = \iint_{\mathscr{L}_r} \left|\frac{p'(z)}{\pi r^{2n}}\right|^2 dx\, dy = \frac{1}{\pi^2 r^{4n}} \iint_{\mathscr{L}_r} |p'(z)|^2\, dx\, dy = \frac{n}{\pi r^{2n}}.$$

Hence,

$$\| D \|_{\mathscr{L}_r} = r^{-n} \sqrt{(n/\pi)}. \tag{4.19}$$

Suppose that $p(z)$ has zeros at $z_1,\dots, z_n$ of multiplicity $\alpha_1,\dots, \alpha_n$. Let $N = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $p(z) = (z-z_1)^{\alpha_1}\cdots(z-z_n)^{\alpha_n}$, $\mathscr{L}_r : |p(z)| \leqslant r^N$. For analytic f, we have from the residue theorem,

$$\frac{1}{2\pi i}\int_{\partial\mathscr{L}_r} \frac{p'(z)}{p(z)} f(z)\, dz = \sum_{k=1}^{n} \alpha_k f(z_k).$$

Hence,

$$\iint_{\mathscr{L}_r} |p'(z)|^2 f(z)\, dx\, dy = \pi r^{2N} \sum_{k=1}^{n} \alpha_k f(z_k). \tag{4.20}$$

Divided differences are closely related to remainder formulas for polynomial interpolation. Let $z_0, z_1, \ldots, z_n$ be points of the complex plane (not necessarily distinct) and let $p_n(f; z)$ designate the unique polynomial of degree $\leqslant n$ which interpolates to an analytic function f at these points. (If there are multiple points present then the interpolation is understood in the generalized sense.) Let

$$R_n(f; z) = f(z) - p_n(f; z). \tag{4.21}$$

Then (see, e.g., Davis [3, p. 67])

$$R_n(f; z) = (z - z_0)(z - z_1) \cdots (z - z_n)[f(z), f(z_0), \ldots, f(z_n)]. \tag{4.22}$$

This leads to the following formula which can be regarded as an analogue of the well-known Hermite formula for the remainder in polynomial interpolation.

Let z be regarded as a fixed point and set

$$P(t) = (t - z)(t - z_0) \cdots (t - z_n), \qquad t = u + iv. \tag{4.23}$$

$P(t)$ is a polynomial of degree $n + 2$. Let r be selected so large that the set

$$\mathscr{L}_r : |P(t)| \leqslant r^{n+2} \tag{4.24}$$

contains $z, z_0, \ldots, z_n$ in its interior.

Hence from (4.16) we obtain

$$R_n(f; z) = \frac{(z - z_0)(z - z_1) \cdots (z - z_n)}{\pi r^{2n+4}} \iint_{\mathscr{L}_r} \overline{P'(t)} f(t)\, du\, dv. \tag{4.25}$$

Thus, over the space $L^2(\mathscr{L}_r)$, the function

$$h(t) = \frac{\overline{(z - z_0)(z - z_1) \cdots (z - z_n)}}{\pi r^{2n+4}} P'(t)$$

is the representer of the remainder functional $R_n(f)$.

We have

$$\begin{aligned}
\| R_n \|^2_{\mathscr{L}_r} &= \iint_{\mathscr{L}_r} | h(t)|^2 \, du \, dv \\
&= \frac{|(z - z_0) \cdots (z - z_n)|^2}{\pi^2 r^{4n+8}} \iint_{\mathscr{L}_r} | P'(t)|^2 \, du \, dv \\
&= \frac{|(z - z_0) \cdots (z - z_n)|^2}{\pi^2 r^{4n+8}} \cdot (n + 2) \, \pi r^{2n+4}
\end{aligned}$$

by (4.18). Hence

$$\| R_n \|_{\mathscr{L}_r} = \sqrt{\frac{n+2}{\pi}} \frac{| z - z_0 | \cdots | z - z_n |}{r^{n+2}}. \tag{4.26}$$

(g) *Maps.* Let B be a simply connected region of the z plane containing $z = 0$ and suppose that

$$z = m(w), \qquad m(0) = 0$$

maps the unit circle $| w | \leqslant 1$ one-to-one conformally onto B. We have

$$\begin{aligned}
\iint_B \bar{z}^p f(z) \, dx \, dy &= \frac{1}{2i(p+1)} \int_{\partial B} \bar{z}^{p+1} f(z) \, dz \\
&= \frac{1}{2i(p+1)} \int_{|w|=1} \overline{(m(w))}^{p+1} f(m(w)) \, m'(w) \, dw \\
&= \frac{\pi}{p+1} \frac{1}{2\pi i} \int_{|w|=1} \bar{m}^{p+1} \left(\frac{1}{w}\right) f(m(w)) m'(w) \, dw.
\end{aligned} \tag{4.27}$$

We now make the special hypothesis that the mapping function $m(w)$ is a polynomial of degree $q \geqslant 1$,

$$m(w) = a_1 w + a_2 w^2 + \cdots + a_q w_q \qquad a_1 \neq 0, \quad a_q \neq 0. \tag{4.28}$$

Note conversely, that if we start with a polynomial of the form (4.28) and if $a_2, a_3, \ldots, a_q$ are all sufficiently small with respect to a_1, $m(w)$ will be univalent in the unit circle and hence will map it onto a simply connected schlicht region. Now $m^{p+1}(w)$ is a polynomial of degree $s = q(p+1)$ which can be written in the form

$$m^{p+1}(w) = b_1 p! \, w^{p+1} + b_2 (p+1)! \, w^{p+2} + \cdots + b_{s-p} (s-1)! \, w^s,$$

where the b's are determined from the a's. Hence, by the residue theorem,

$$\begin{aligned}
&\iint_B \bar{z}^p f(z)\,dx\,dy \\
&= \frac{\pi}{p+1}\frac{1}{2\pi i}\int_{|w|=1}\left(\bar{b}_1\frac{p!}{w^{p+1}}+\cdots+\frac{\bar{b}_{s-p}(s-1)!}{w^s}\right)f(m(w))\,m'(w)\,dw \\
&= \frac{\pi}{p+1}\left[\bar{b}_1(f(m(w))\,m'(w))^{(p)}\,|_{w=0}+\cdots+\bar{b}_{s-p}(f(m(w))\,m'(w))^{(s-1)}\,|_{w=0}\right] \\
&\equiv Q(D)f(0), \qquad (4.29)
\end{aligned}$$

where $Q(D)$ is a linear differential operator of order $s-1$ whose coefficients depend upon $m(w)$ but are independent of $f(z)$.

EXAMPLE.

$$m(w) = w + aw^2, \quad m'(w) = 1 + 2aw, \quad m''(w) = 2a, \quad \bar{m}(w) = w + \bar{a}w^2.$$

Now for a sufficiently small ($|a| < \frac{1}{2}$), $m(w)$ is univalent in the unit circle and hence maps onto a region B. Now

$$\begin{aligned}
\iint_B f(z)\,dx\,dy &= \pi\frac{1}{2\pi i}\int_{|w|=1} f(m(w))\,m'(w)\left(\frac{1}{w}+\frac{\bar{a}}{w^2}\right)dw \\
&= \pi[(1+2\,|\,a\,|^2)f(0)+\bar{a}f'(0)]. \qquad (4.30)
\end{aligned}$$

We next investigate the case where the mapping function is a general rational function. Let

$$P(w) = (w-\alpha_1)(w-\alpha_2)\cdots(w-\alpha_n), \qquad (4.31)$$

where $0 < |\,\alpha_i\,| < 1$, and let

$$R(w) = (w-\beta_1)(w-\beta_2)\cdots(w-\beta_p), \qquad (4.32)$$

where the β_i are distinct from $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let

$$\begin{aligned}
Q(w) &= w^n P(1/w) = (1-\alpha_1 w)(1-\alpha_2 w)\cdots(1-\alpha_n w), \\
S(w) &= w^p R(1/w) = (1-\beta_1 w)(1-\beta_2 w)\cdots(1-\beta_p w),
\end{aligned} \qquad (4.33)$$

and consider for $a \neq 0$,

$$\begin{aligned}
m(w) &= \frac{awS(w)}{Q(w)} = \frac{aw(1-\beta_1 w)\cdots(1-\beta_p w)}{(1-\alpha_1 w)\cdots(1-\alpha_n w)} \\
&= aw + \cdots. \qquad (4.34)
\end{aligned}$$

For values of α_i and β_i sufficiently small, it is clear that $m(w)$ is univalent in $|w| \leqslant 1$ and hence $z = m(w)$ maps the circle 1–1 conformally onto a region B in the z plane.

The function $m'(w) = [awS(w)/Q(w)]'$ is regular in $|w| \leqslant 1$. Now

$$\bar{m}\left(\frac{1}{w}\right) = \frac{\bar{a}(1/w)(1 - \bar{\beta}_1/w) \cdots (1 - \bar{\beta}_p/w)}{(1 - \bar{\alpha}_1/w) \cdots (1 - \bar{\alpha}_n/w)}$$
$$= \frac{\bar{a}w^{n-p-1}(w - \bar{\beta}_1) \cdots (w - \bar{\beta}_p)}{(w - \bar{\alpha}_1) \cdots (w - \bar{\alpha}_n)} = \frac{\bar{a}w^{n-p-1}\bar{R}(w)}{\bar{P}(w)}.$$

Now if $n \geqslant p + 1$, the function $\bar{m}(1/w)$ has poles at $\bar{\alpha}_1, \dots, \bar{\alpha}_n$ and no other singularities. If $n < p + 1$, then $\bar{m}(1/w)$ also has a pole of order $p + 1 - n$ at $w = 0$.

We have

$$\iint_B f(z)\,dx\,dy$$
$$= \pi \frac{1}{2\pi i} \int_{|w|=1} f(m(w))\, m'(w)\bar{m}\left(\frac{1}{w}\right) dw$$
$$= \pi \frac{1}{2\pi i} \int_{|w|=1} f(m(w))\, m'(w) \frac{\bar{a}w^{n-p-1}(w - \bar{\beta}_1) \cdots (w - \bar{\beta}_p)}{(w - \bar{\alpha}_1) \cdots (w - \bar{\alpha}_n)}\, dw. \quad (4.35)$$

If we now assume that $n \geqslant p + 1$ and that the points α_i are distinct, then we have by the residue theorem

$$\iint_B f(z)\,dx\,dy = \pi\bar{a} \sum_{k=1}^{m} f(m(\bar{\alpha}_k))\, m'(\bar{\alpha}_k) \frac{\bar{\alpha}_k^{n-p-1}(\bar{\alpha}_k - \bar{\beta}_1) \cdots (\bar{\alpha}_k - \bar{\beta}_p)}{\bar{P}'(\alpha_k)}, \quad (4.36)$$

Thus, we have an identity of the form

$$\iint_B f(z)\,dx\,dy = \sum_{k=1}^{n} c_k f(z_k), \quad (4.37)$$

where the coefficients c_k and the abscissas $z_k = m(\bar{\alpha}_k)$, are independent of f.

If the α_i are not distinct, then each point of higher multiplicity τ_k contributes a differential operator of order τ_{k-1} evaluated at $\bar{\alpha}_k$.

If $n < p + 1$ then the point $\alpha = 0$ is a pole, and hence $f(0)$ $(m(0) = 0)$ is present by itself if $n - p - 1 = -1$ or with its higher derivatives if $n - p - 1 < -1$.

If we require the higher moments $\iint_B \bar{z}^r f(z)\,dx\,dy$, we have from (4.27)

$$\iint_B \bar{z}^r f(z)\,dx\,dy = \frac{\pi}{r+1} \frac{1}{2\pi i} \int_{|w|=1} f(m(w))\, m'(w)(\bar{a})^{r+1}\, w^{(n-p-1)(r+1)}$$
$$\times \frac{[(w - \bar{\beta}_1) \cdots (w - \bar{\beta}_p)]^{r+1}}{[(w - \bar{\alpha}_1) \cdots (w - \bar{\alpha}_n)]^{r+1}}\, dw.$$

We shall not write out an explicit formula for this, but merely observe that at each point $\bar{\alpha}_i$ and possibly at $\alpha = 0$, we obtain a differential operator of order r. Thus, we can write

$$\iint_B \bar{z}^r f(z)\,dx\,dy = \sum_{k=1}^{n} \sum_{i=0}^{r} c_{ki} f^{(i)}(m(\bar{\alpha}_n)) + \sum_{i=0}^{r} d_i f^{(i)}(0), \tag{4.38}$$

for constants c_{ki}, d_i independent of f. The d_i's will all vanish if $n \geqslant p + 1$.

EXAMPLE.

$$\begin{aligned} P(w) &= w - \alpha \qquad Q(w) = 1 - \alpha w, \qquad |\alpha| < 1, \\ R(w) &\equiv 1, \\ m(w) &= w/(1 - \alpha w) \qquad m'(w) = 1/(1 - \alpha w)^2, \\ \bar{m}(1/w) &= 1/(w - \bar{\alpha}), \end{aligned}$$

$$\begin{aligned} \iint_B f(z)\,dx\,dy &= \frac{\pi}{2\pi i} \int_{|w|=1} f\left(\frac{w}{1 - \alpha w}\right) \frac{1}{(1 - \alpha w)^2} \frac{1}{w - \bar{\alpha}}\,dw \\ &= \frac{\pi}{(1 - |\alpha|^2)^2} f\left(\frac{\bar{\alpha}}{1 - |\alpha|^2}\right) \end{aligned}$$

and we recover (4.3) with $n = 0$ in another form inasmuch as the image of $|w| = 1$ under m is a circle with center at $\bar{\alpha}/(1 - |\alpha|^2)$ and radius $1/(1 - |\alpha|^2)$.

EXAMPLE. Let $m(w) = w(1 - \beta^2 w^2)/(1 - \alpha^2 w^2)$, $0 < |\alpha| < 1$, $\beta \neq \pm\alpha$, $n = p = 2$, $n - p - 1 = -1$. Take α and β sufficiently close to 0 so that $m(w)$ is univalent in the circle $|w| \leqslant 1$ and maps it onto a B.

$$m'(w) = \frac{1 + \alpha^2 w^2 - 3\beta^2 w^2 + \alpha^2 \beta^2 w^4}{(1 - \alpha^2 w^2)^2},$$

$$\bar{m}\left(\frac{1}{w}\right) = \frac{1}{w} \cdot \frac{(w^2 - \bar{\beta}^2)}{(w^2 - \bar{\alpha}^2)}.$$

$$\iint_B f(z)\,dx\,dy = Af(z^*) + Bf(0) + Af(-z^*), \tag{4.39}$$

where

$$\begin{aligned} z^* &= \frac{\bar{\alpha}(1 - \beta^2 \bar{\alpha}^2)}{1 - |\alpha|^4}, \\ A &= \frac{1 + |a|^4 - 3\beta^2 \bar{\alpha}^2 + \beta^2 \bar{\alpha}^2 |\alpha|^4}{1 - |\alpha|^4}, \\ B &= (\bar{\beta}/\bar{\alpha})^2. \end{aligned}$$

5. Application of Kernel Functions

We now turn our attention to a general problem. We are given a linear functional L applicable to a class R of analytic functions and whose further requirements will be specified shortly. (Think of L as an integrodifferential functional.) We wish to find (if possible) a region B in the z plane with the property that L has a representation as a double integral over B. That is,

$$\iint_B f(z)\,dx\,dy = L(f), \qquad f \in K. \tag{5.1}$$

It simplifies our thinking if we work with $K =$ a Hilbert Space of analytic functions, for example $L^2(B)$. Let $K(z, \bar{w})$ be the Bergman kernel function of $L^2(B)$. If L is a bounded linear functional over $L^2(B)$, then its representer is given by

$$r(z) = \overline{L_w K(\bar{z}, w)}. \tag{5.2}$$

See Davis [3, p. 318]. The w in the subscript in L_w means that the operation is to be performed on the w variable.

In other words,

$$L(f) = \iint_B \overline{r(z)} f(z)\,dx\,dy = (r, f), \qquad f \in L^2(B). \tag{5.3}$$

If $r(z) \equiv 1$ as in (5.1), then we have

$$\overline{L_w K(\bar{z}, w)} \equiv 1. \tag{5.4}$$

This is a necessary and sufficient condition for (5.1), and our problem, therefore, is: can a region B be found for which (5.4) holds? This criterion can be recast in several different forms. Let $\{\zeta_n(z)\}$ be a complete orthonormal system for $L^2(B)$, then

$$K(\bar{z}, w) = \sum_{n=0}^{\infty} \overline{\zeta_n(z)}\, \zeta_n(w). \tag{5.5}$$

In view of the boundedness of L, L is applicable term by term to the right hand of (5.5) and hence we need

$$\sum_{n=0}^{\infty} \zeta_n(z)\, \overline{L(\zeta_n(w))} \equiv 1. \tag{5.6}$$

If, as frequently happens, ζ_0 can be taken as a constant; $c = 1/\sqrt{\text{area (B)}}$, then (5.6) is equivalent to

$$c\,\overline{L(c)} = 1, \qquad L(\zeta_1) = L(\zeta_2) = \cdots = 0.$$

Thus,

$$|c|^2\,\overline{L(1)} = 1, \qquad L(1) = \iint dx\,dy = \text{area }(B).$$

This fixes the area (B), and the higher moments.

Progress can be made in the following way. Assume B is a simply connected region. We have

$$K(\bar{z}, w) = (1/\pi)[\overline{m'(z)}\,m'(w)/(1 - \overline{m(z)}\,m(w))^2], \tag{5.7}$$

where $m(z)$ performs a 1–1 conformal map of B onto the unit circle. Thus,

$$1 = \overline{\frac{1}{\pi} L_w\left(\frac{\overline{m'(z)}\,m'(w)}{(1 - \overline{m(z)}\,m(w))^2}\right)}. \tag{5.8}$$

This is a functional-differential equation, to be solved for a mapping function $m(z)$. Assuming we have solved it, the inverse map will give us the region B. However, trivial examples show that this problem may not have a solution. For instance if $L(f) = f'(0) = \iint_B f\,dx\,dy$ we obtain a contradiction by setting $f \equiv 1$. Thus, an interesting and significant open problem is: how can you characterize those functionals L for which there is a solution to (5.1) or (5.4)? If there is a solution, is the solution unique in some sense? Is it unique if the region B is restricted to be simply connected?

Further reduction of (5.8) is useful. We have

$$1 \equiv \frac{1}{\pi}\sum_{n=0}^{\infty} (n+1)(m(z))^n\,m'(z)\,\overline{L[(m(w))^n\,m'(w)]}, \tag{5.9}$$

the convergence being absolute and uniform for $|z|, |w| \leqslant 1 - \delta$. Thus,

$$1 = \frac{1}{\pi}\frac{d}{dz}\sum_{n=0}^{\infty} (m(z))^{n+1}\,\overline{L[(m(w))^n\,m'(w)]}. \tag{5.10}$$

Integrating from $z = 0$ to $z = z$ in $|z| < 1$,

$$\pi z = m(z)\sum_{n=0}^{\infty} (m(z))^n\,\overline{L[(m(w))^n\,m'(w)]}, \tag{5.11}$$

$$\pi\bar{z} = \overline{m(z)}\,L_w\sum_{n=0}^{\infty} \overline{[m(z)]^n}\,(m(w))^n\,m'(w)$$

$$= \overline{m(z)}\,L_w\left(\frac{m'(w)}{1 - m(w)\,\overline{m(z)}}\right); \tag{5.12}$$

or,

$$\pi\bar{z} = -\overline{m(z)}\ L_w \frac{d}{dw} \frac{1}{\overline{m(z)}} \log(1 - \overline{m(z)}\, m(w))$$
$$-\pi\bar{z} = L_w \frac{d}{dw} \log(1 - \overline{m(z)}\, m(w)). \tag{5.13}$$

EXAMPLE (a). $L(f) = \pi R^2 f(0)$. With this functional, (5.11) becomes

$$\pi z = m(z)\, \pi R^2 \overline{m'(0)}.$$

Differentiating,

$$\pi = \pi R^2 \mid m'(0)|^2$$

so that

$$m'(0) = (1/R)\, e^{i\psi}.$$

Hence,

$$z = m(z)(R^2/R)\, e^{-i\psi}$$

and

$$m(z) = e^{i\psi}/R)z.$$

Thus, B is the circle $| z | \leqslant R$, and we recover (4.3) with $n = 0$.

EXAMPLE (b). Our second selection leads to a new result. Let

$$L(f) = \int_{-1}^{+1} f(x)\, dx.$$

With this functional, (5.13) becomes

$$-\pi\bar{z} = \int_{-1}^{+1} \frac{d}{dw} \log(1 - \overline{m(z)}\, m(w))\, dw$$
$$= \log(1 - \overline{m(z)}\, m(1)) - \log(1 - \overline{m(z)}\, m(-1)) \tag{5.14}$$
$$= \log\left(\frac{1 - \overline{m(z)}\, m(1)}{1 - \overline{m(z)}\, m(-1)}\right).$$

To obtain symmetry we shall make the specialization

$$m(0) = 0, \qquad m(-1) = -m(1) = \alpha. \tag{5.15}$$

Inserting successively $z = 1$, $z = -1$ in (5.14) yields

$$e^{-\pi} = (1 - |\alpha|^2)/(1 + |\alpha|^2); \qquad e^{\pi} = (1 + |\alpha|^2)/(1 - |\alpha|^2) \tag{5.16}$$

and these two are identical. Thus

$$|\alpha|^2 = (e^{\pi} - 1)/(e^{\pi} + 1) = (1 - e^{-\pi})/(1 + e^{-\pi}), \tag{5.17}$$

or

$$\alpha = e^{i\theta} \sqrt{(1 - e^{-\pi})/(1 + e^{-\pi})}, \qquad 0 \leqslant \theta \leqslant 2\pi. \tag{5.18}$$

We select $\theta = 0$, and then

$$\alpha = \sqrt{(1 - e^{-\pi})/(1 + e^{-\pi})} \approx 0.958 < 1. \tag{5.19}$$

Thus, from (5.14)–(5.16) and (5.19), we have as our mapping function

$$z = \frac{1}{\pi} \log \left(\frac{1 - \alpha w}{1 + \alpha w} \right), \tag{5.20}$$

or

$$\begin{aligned} w = m(z) &= \frac{1}{\alpha} \left(\frac{1 - e^{\pi z}}{1 + e^{\pi z}} \right) = -\frac{1}{\alpha} \tanh \frac{\pi}{2} z; \\ z &= -\frac{2}{\pi} \operatorname{arctanh} \alpha w. \end{aligned} \tag{5.21}$$

Observe that with $0 < \alpha \approx 0.958 < 1$, as w traces the unit circle, $\zeta = (1 - \alpha w/1 + \alpha w)$ traces a circle that lies in $\operatorname{Re} \zeta > 0$. Its center is at $(1 + \alpha^2/1 - \alpha^2, 0)$ and has radius $(2\alpha/1 - \alpha^2)$. The image in the z plane of $|w| = 1$ under (5.20) is therefore schlicht. Since $|m(\pm 1)| = \alpha < 1$ and $m(z)$ is real for z real, the segment $[-1, 1]$ is interior to this image. The map (5.21) therefore defines a simply connected region B which solves the problem.

The region B is an ellipse-like figure having a semimajor axis

$$a = 1/\pi \log(1 + \alpha/1 - \alpha) \approx 1.22$$

and a semiminor axis $b = 1/\pi \arctan(2\alpha/1 - \alpha^2) \approx 0.486$.

We have used the theory of the kernel function to obtain conveniently a solution to the problem of finding a B for which

$$\iint_B f(z)\, dx\, dy = \int_{-1}^{+1} f(x)\, dx\, dy.$$

But now that we have an answer, we can verify it directly and also obtain second representations for the higher moments. We shall work with the Schwarz function. We have

$$m(w) = \frac{1}{\pi} \log \left(\frac{1 - \alpha w}{1 + \alpha w}\right), \tag{5.22}$$

$$M(z) = \frac{1}{\alpha} \left(\frac{1 - e^{\pi z}}{1 + e^{\pi z}}\right). \tag{5.23}$$

Now from (2.27),

$$S(z) = \bar{m}\left(\frac{1}{M(z)}\right) = \frac{1}{\pi} \log \left(\frac{1 - e^{\pi} e^{\pi z}}{e^{\pi} - e^{\pi z}}\right). \tag{5.24}$$

Note that $S(z)$ has logarithmic singularities at $z = \pm 1 \pm 2ki$, $k = 0, 1, 2, \ldots$ and at no other place. The only singularities of $S(z)$ within B are therefore at $z = \pm 1$. By making a cut along the real axis from $z = 1$ to $z = -1$, we can define a single-valued branch of $S(z)$ inside B thus cut.

For $-1 \leqslant x \leqslant 1$ along the upper edge of the cut, we take

$$S_u(z) = S_{\text{upper}} = \frac{1}{\pi} \log \left(\frac{e^{\pi x} e^{\pi} - 1}{e^{\pi} - e^{\pi x}}\right) - i. \tag{5.25}$$

Along the lower edge of the cut we take

$$S_l(z) = S_{\text{lower}} = \frac{1}{\pi} \log \left(\frac{e^{\pi x} e^{\pi} - 1}{e^{\pi} - e^{\pi x}}\right) + i. \tag{5.26}$$

Now,

$$\iint_B f(z)\, dx\, dy = \frac{1}{2i} \int_{\partial B} S(z) f(z)\, dz.$$

We now replace ∂B by a circuit consisting of $-(1 - \epsilon) \leqslant x \leqslant 1 - \epsilon$ augmented by two circles of radius ϵ at $x = 1$ and $x = -1$. We obtain

$$\iint_B f(z)\, dx\, dy = \frac{1}{2i} \int_{-1}^{+1} (S_l(x) f(x) - S_u(x) f(x))\, dx. \tag{5.27}$$

The limiting process is valid since $\lim_{\epsilon \to 0} \epsilon \log \epsilon = 0$. Hence, from (5.25) and (5.26)

$$\iint_B f(z)\, dx\, dy = \int_{-1}^{+1} f(x)\, dx.$$

The higher moments now follows,

$$\iint_B \bar{z}^p f(z)\,dx\,dy = \frac{1}{2i(p+1)} \int_{\partial B} S^{p+1}(z) f(z)\,dz$$
$$= \frac{1}{2i(p+1)} \int_{-1}^{+1} (S_l^{p+1}(x) - S_u^{p+1}(x)) f(x)\,dx$$
$$= \frac{1}{2i(p+1)} \int_{-1}^{+1} \zeta_p(x) f(x)\,dx, \tag{5.28}$$

where

$$\zeta_p(x) = \left(\frac{1}{\pi}\log\left(\frac{e^{\pi x}e^{\pi}-1}{e^{\pi}-e^{\pi x}}\right)+i\right)^{p+1} - \left(\frac{1}{\pi}\log\left(\frac{e^{\pi x}e^{\pi}-1}{e^{\pi}-e^{\pi x}}\right)-i\right)^{p+1}$$
$$= 2i\,\mathrm{Im}\left(\frac{1}{\pi}\log\left(\frac{e^{\pi x}e^{\pi}-1}{e^{\pi}-e^{\pi x}}\right)+i\right)^{p+1}. \tag{5.29}$$

As particular examples, for $p = 1$,

$$\iint_B \bar{z} f(z)\,dx\,dy = \frac{1}{\pi}\int_{-1}^{+1} \log\left(\frac{e^{\pi x}e^{\pi}-1}{e^{\pi}-e^{\pi x}}\right) f(x)\,dx. \tag{5.30}$$

For $p = 2$,

$$\iint_B \bar{z}^2 f(z)\,dx\,dy = \frac{1}{\pi^2}\int_{-1}^{+1} \log^2\left(\frac{e^{\pi x}e^{\pi}-1}{e^{\pi}-e^{\pi x}}\right) f(x)\,dx - \frac{1}{3}\int_{-1}^{+1} f(x)\,dx. \tag{5.31}$$

6. Construction of a Schwarz Function Regular in a Slit Region

In the examples of (4.5) and (5.24) we have seen how, if the Schwarz function for B is regular in a slit B, then the integral $\iint_B f\,dx\,dy$ can be reduced to one of the form $\int_{-1}^{+1} \mu(x) f(x)\,dx$. We wish next to construct general regions B with this property. This can be done by the method of "opening up the slit."

Fix an α with $0 < \alpha < 1$. Let w_s be the w plane with two slits $[(1/\alpha), +\infty]$, $[-(1/\alpha), -\infty]$ along the real axis. The plane w_s is mapped 1–1 conformally onto the circle $|u| \leqslant 1$ in the u plane by means of the map

$$\begin{aligned} u &= \Gamma(w) = \frac{1-\sqrt{1-\alpha^2 w^2}}{\alpha w}, \\ w &= \gamma(u) = \frac{2u}{\alpha(u^2+1)}. \end{aligned} \tag{6.1}$$

For $u = e^{i\theta}$,

$$w = \frac{2}{\alpha(u + u^{-1})} = \frac{1}{\alpha \cos \theta}. \tag{6.2}$$

Hence for $-(\pi/2) \leqslant \theta \leqslant (\pi/2)$, w goes from $+\infty$ to $1/\alpha$ and back to $+\infty$ along the right slit. For $(\pi/2) \leqslant \theta \leqslant (3/2)\pi$, w goes from $-\infty$ to $-(1/\alpha)$ back to $-\infty$ along the left slit.

Let the image of $| w | = | e^{i\zeta} | = 1$ in the w plane be called C. C is of course contained in $| u | \leqslant 1$ and its equation is

$$C: \quad u = \frac{1 - \sqrt{1 - \alpha^2 e^{2i\zeta}}}{\alpha e^{i\zeta}}. \tag{6.3}$$

The image in the u plane of $w = \pm\alpha$ in the w plane is $\pm\beta$ where $\beta = [1 - \sqrt{(1 - \alpha^4)}]/\alpha^2$. We now introduce an arbitrary analytic function $H(u)$ subject to the following restrictions. (Some of the restrictions are unnecessary and are to simplify the situation.) The function H will control the shape of B.

(1) The function $z = H(u) = a_1 u + a_2 u^2 + \cdots$ is regular in $| u | < 1$ and has real coefficients.

(2) $H(u)$ is continuous in $| u | \leqslant 1$ or mildly singular on $| u | = 1$.

(3) $H(u)$ is univalent in and on C. Let $H(-\beta) = a$, $H(\beta) = b$ with $a < b$.

The image in the z plane of C under H will be called B. The inverse function of H will be called h. This inverse $u = h(z)$ performs a 1–1 conformal map of B onto C. B contains the segment $[a, b]$ of the real axis.

Consider now the function

$$z = m(w) \equiv H(\Gamma(w)). \tag{6.4}$$

This function is regular in w_s. It is univalent in $| w | \leqslant 1$ (because the image of $| w | = 1$ under Γ is C). The function $z = m(w)$ therefore maps $| w | \leqslant 1$ one-to-one conformally onto B and ∂B, the boundary of B, is an analytic curve.

Designate by B_s the region B with a slit $[a, b]$ along the real axis. We wish to show that $S(z)$, the Schwarz function for ∂B, is regular in B_s. The inverse of $z = m(w)$ is $w = M(z) = \gamma(h(z))$. Therefore

$$S(z) = H(\Gamma(1/\gamma(h(z)))) \tag{6.5}$$

(since $\bar{m} = m$).

Now $h(z)$ is regular in B_s and takes values in C_s (C with the slit $[-\beta, \beta]$ removed). $\gamma(h(z))$ is therefore regular in B_s and takes values in $|w| \leqslant 1$, slit; that is, $|w| \leqslant 1$ with the interval $[-\alpha, \alpha]$ along the real axis deleted. Therefore $1/\gamma(h(z))$ is regular in B_s and takes values in $w_s - (|w| \leqslant 1)$. $\Gamma(1/\gamma(h(z)))$ is therefore regular in B_s and takes values in $(|u| \leqslant 1) - C$. Thus, finally, $S(z) = H(\Gamma(1/\gamma(h(z)))$ is regular in B_s, as we were required to show.

In view of the continuity (or mild singularity) of $H(u)$ on $|u| = 1$, it follows that $S(z)$ will be continuous from the interior of B_s on the upper and lower edges of the slit.

We wish next to determine those upper and lower functions. Call them $S_l(x)$ and $S_u(x)$.

We have $z = x + iy$. As x goes from a to b back to a, $h(x)$ goes from $-\beta$ to β to $-\beta$. $\gamma(h(x))$ goes from $-\alpha$ to α to $-\alpha$. With an obvious notation, $1/\gamma(h(x))$ goes from $-(1/\alpha)$ to $-\infty$, $+\infty$ to $(1/\alpha)$, $(1/\alpha)$ to $+\infty$, $-\infty$ to $-(1/\alpha)$. For $u = e^{i\theta}$, $w = 1/\alpha \cos \theta$; hence $\Gamma(1/\gamma(h(x))) = e^{i\theta}$ with θ running from $-\pi$ to 0 as x goes from a to b and from 0 to π as x goes from b to a. Thus we can write

$$\begin{aligned} S_l(x) &= H(e^{i\theta}) \qquad -\pi \leqslant \theta \leqslant 0, \\ S_u(x) &= H(e^{i\theta}) \qquad 0 \leqslant \theta \leqslant \pi. \end{aligned} \tag{6.6}$$

Finally,

$$\iint_B f(z)\, dx\, dy = \frac{1}{2i} \int_{\partial B} S(z) f(z)\, dz,$$

and since $S(z)$ is regular in B and continuous or mildly singular on $[a, b]$ we may collapse the contour ∂B to $[a, b]$ and we have

$$\frac{1}{2i} \int_{\partial B} S(z) f(z)\, dx = \frac{1}{2i} \int_a^b (S_l(x) - S_u(x)) f(x)\, dx. \tag{6.7}$$

If, now, $H(u)$ is an odd function then

$$\begin{aligned} S_l(x) &= H(e^{i\theta}), \\ S_u(x) &= H(e^{-i\theta}), \end{aligned} \tag{6.8}$$

and therefore $S_l(x) - S_u(x) = H(e^{i\theta}) - H(e^{-i\theta}) = 2i \operatorname{Im} H(e^{i\theta})$. Thus,

$$\iint_B f(z)\, dx\, dy = \int_{-b}^b \operatorname{Im} H(e^{i\theta(x)}) f(x)\, dx. \tag{6.9}$$

Here θ is determined from

$$\cos\theta = \alpha^2(h^2(x)+1)/2h(x), \tag{6.10}$$

We have further, in the case of H odd and real on the real axis,

$$\iint_B \bar{z}^p f(x)\,dx\,dy = \frac{1}{2i(p+1)}\int_{-b}^{b} [H^{p+1}(e^{i\theta}) - H^{p+1}(e^{-i\theta})]f(x)\,dx. \tag{6.11}$$

Since $H^{p+1}(e^{-i\theta}) = \overline{H^{p+1}(e^{i\theta})}$,

$$\iint_B \bar{z}^p f(x)\,dx\,dy = \frac{1}{p+1}\int_{-b}^{b} \operatorname{Im} H^{p+1}(e^{i\theta})f(x)\,dx. \tag{6.12}$$

EXAMPLE. Select $z = H(u) \equiv u$; $u = h(z) \equiv z$. This function obviously has the required properties. Moreover it is odd. We have

$$\beta = b = \frac{1-\sqrt{1-a^4}}{a^2}.$$

The region B is identical to C. Now $[1/\gamma(h(x))] = [\alpha(x^2+1)/2x]$. Hence $\cos\theta = [\alpha^2(x^2+1)/2x]$ and

$$\sin\theta = \frac{\sqrt{(\alpha^2(x^2+1)+2x)(\alpha^2(x^2+1)-2x)}}{\alpha^2(x^2+1)} = \zeta(x). \tag{6.13}$$

Further, $\operatorname{Im} H(e^{i\theta}) = \sin\theta$ so that

$$\iint_B f(z)\,dx\,dy = \int_{-\beta}^{\beta} \zeta(x)f(x)\,dx. \tag{6.14}$$

7. Orthonormal Expansions

Suppose that B is a bounded, simply connected region of the complex plane, whose boundary ∂B is an analytic curve. There exists a complete orthonormal set of polynomials $\{p_n^*(z)\}$ (the degree of p_n^* is n) for the Hilbert space $L^2(B)$, and for any $f(z) \in L^2(B)$, we have the expansion

$$f(z) = \sum_{n=0}^{\infty} (f, p_n^*)\,p_n^*(z) = \sum_{n=0}^{\infty}\Big(\iint_B f(z)\overline{p_n^*(z)}\,dx\,dy\Big)p_n^*(z). \tag{7.1}$$

This Fourier series converges uniformly and absolutely in compact subregions of B. Since

$$p_n^*(z) = \alpha_{n0} + \alpha_{n1}z + \cdots + \alpha_{nn}z^n, \qquad \alpha_{nn} \neq 0, \tag{7.2}$$

it should be clear from the preceding work that for certain special regions B, the Fourier coefficients $\iint_B f\overline{p_n^*}\, dx\, dy$ have alternate expressions as linear differential operators or real integrals operating on f. For example, if we take $B =$ the bicircular quartic Q, then from (4.10) we have

$$\iint_Q \bar{z}^n f(z)\, dx\, dy = R_n(D) f(\epsilon) + S_n(D) f(-\epsilon), \tag{7.3}$$

where $R_n(D)$ and $S_n(D)$ are certain n-th-order differential operators. They are independent of f. In view of (7.2), we have

$$(f, p^*) = \iint_Q f(z) \overline{p_n^*(z)}\, dx\, dy = R_n^*(D)\, f(\epsilon) + S_n^*(D) f(-\epsilon), \tag{7.4}$$

where R_n^* and S_n^* are the "orthonormalized" operators obtained from (7.3) by substituting (7.2) in (7.4). The Fourier expansion (7.1) may now be written in the form

$$f(z) = \sum_{n=0}^{\infty} (R_n^*(D) f(\epsilon) + S_n^*(D) f(-\epsilon))\, p_n^*(z). \tag{7.5}$$

The orthonormality of the functions $\{p_n^*(z)\}$ can be expressed as the biorthonormality

$$R_m^*(D)\, p_n^*(\epsilon) + S_m^*(D)\, p_n^*(-\epsilon) = \delta_{mn}\,. \tag{7.6}$$

If $f(z) \in L^2(Q)$, then (7.5) coincides with (7.1). The series (7.1) is therefore simultaneously a complex orthogonal Fourier expansion as well as an interpolation series. That is to say, the Fourier segments

$$f_N(z) = \sum_{n=0}^{N} (f, p_n^*)\, p_n^*(z) \tag{7.7}$$

interpolate to f in the sense that

$$T_k f_N(z) = T_k f(z), \qquad k = 0, 1, ..., N, \tag{7.8}$$

where we have written

$$T_k g(z) = R_k^*(D)\, g(\epsilon) + S_k^*(D)\, g(-\epsilon). \tag{7.9}$$

The norm of $f \in L^2(Q)$ is given by

$$\|f\|^2 = \sum_{n=0}^{\infty} |\, T_k(f)|^2. \tag{7.10}$$

Furthermore, the characteristic minimum problem of finding

$$I = \min_{p \in \mathscr{P}_N} \iint_Q |f(z) - p(z)|^2 \, dx \, dy \tag{7.11}$$

($\mathscr{P}_N$ = class of all polynomials of degree $\leqslant N$), can be solved by interpolation. The minimizing polynomial p is given by

$$p(z) = \sum_{n=0}^{N} (f, p_n^*) \, p_n^*(z) = \sum_{n=0}^{N} T_n(f) \, p_n^*(z) \tag{7.12}$$

and is that polynomial $p(z)$ of $\mathscr{P}_N$ for which

$$R_n(D) \, p(\epsilon) + S_n(D) \, p(-\epsilon) = R_n(D) f(\epsilon) + S_n(D) f(-\epsilon) \qquad n = 0, 1, \ldots, N.$$

The minimum value itself is given by

$$I = \iint_Q |f|^2 \, dx \, dy - \sum_{n=0}^{N} | T_n(f)|^2 = \sum_{N+1}^{\infty} | T_n(f)|^2. \tag{7.13}$$

However, a formal series of type (7.5) can be defined for much wider classes of functions than $L^2(Q)$; it suffices for f to have all derivatives at $z = \pm\epsilon$. The convergence, summability, etc., of such generalized Fourier expansions is then open to discussion. Similar remarks apply to other special regions.

Let the function $z = \phi(w) = w + c_0 + c_1 w^{-1} + c_2 w^{-2} + \cdots$ map the region $| w | > r_0$ conformally onto the complement of B. If ∂B is an analytic Jordan curve, then $\phi(w)$ may be continued in an analytic and schlicht manner across the circle $| w | = r_0$ to certain values $r < r_0$. Let B_r designate the complement of the image of $| w | \geqslant r$ under ϕ.

It is known (see for example Smirnov and Lebedev [16], particularly Section 3.3), that if $f(z)$ is regular in merely B_r, then it may be expanded uniquely into a series of orthonormal polynomials for B: $f(z) = \sum_{n=0}^{\infty} c_n p_n^*(z)$. This expansion possesses the usual "Maclaurin" properties of power series.

The present paper serves to identify the coefficients c_n in terms of interpolatory or other familiar integral operators. A comprehensive theory would be useful here. The work of J. L. Walsh and P. J. Davis [7] and Walsh [18] is also to be consulted in this connection.

Finally, it should be pointed out that one and the same special region may have associated with it several different sets of orthonormal polynomials [not both of the form (7.2)] and the related inner products both admitting interpolatory- or moment-type interpretations.

Consider, for example, the oval of Cassini, OC,

$$| z^2 - 1 | = r, \qquad r < 1 \quad \text{(right-hand lobe)}. \tag{7.14}$$

If $p_n^*(z)$ are the orthogonal polynomials of type (7.2), it is clear from the work around (4.14) that the inner products (f, p_n^*) in $L^2(\mathrm{OC})$ may be expressed as a linear combination of differential operators at $z = 1$ and of integrals extended over the segment $\sqrt{1 - r^4} \leqslant x \leqslant 1$.

On the other hand, the function

$$w = (1/r)(z^2 - 1) \tag{7.15}$$

maps OC 1–1 conformally onto the circle $|w| \leqslant 1$. Hence, (see, e.g., Davis [3, p. 325]) the polynomials of degree $2n + 1$,

$$q_n(z) = \frac{2\sqrt{n+1}}{\sqrt{\pi}\, r^{n+1}} z(z^2 - 1)^n, \qquad n = 0, 1, \ldots \tag{7.16}$$

are also complete and orthonormal for $L^2(\mathrm{OC})$.

If $f(z)$ is expanded in a series of q_n's,

$$f(z) = \sum_{n=0}^{\infty} c_n q_n(z), \qquad c_n = \iint_{\mathrm{OC}} f\overline{q_n}\, dx\, dy, \tag{7.17}$$

the change of variable (7.15) leads to the identification

$$c_n = \frac{1}{n!} \left[\frac{f(\sqrt{1 + rw})}{\sqrt{1 + rw}} \right]_{w=0}^{(n)}. \tag{7.18}$$

Each c_n is therefore an n-th-order differential operator on f at $z = 1$.

The two sets of polynomials are, of course, related by a unitary transformation and hence also the corresponding Fourier coefficient functionals.

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