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# Bose-Einstein Condensation in a Harmonic Potential

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## Abstract

We examine several features of Bose-Einstein condensation (BEC) in an external harmonic potential well. In the thermodynamic limit, there is a phase transition to a spatial Bose-Einstein condensed state for dimension  $D \geq 2$ . The thermodynamic limit requires maintaining constant average density by weakening the potential while increasing the particle number  $N$  to infinity, while of course in real experiments the potential is fixed and  $N$  stays finite. For such finite ideal harmonic systems we show that a BEC still occurs, although without a true phase transition, below a certain “pseudo-critical” temperature, even for  $D=1$ . We study the momentum-space condensate fraction and find that it vanishes as  $1/\sqrt{N}$  in any number of dimensions in the thermodynamic limit. In  $D \leq 2$  the lack of a momentum condensation is in accord with the Hohenberg theorem, but must be reconciled with the existence of a spatial BEC in  $D=2$ . For finite systems we derive the  $N$ -dependence of the spatial and momentum condensate fractions and the transition temperatures, features that may be experimentally testable. We show that the  $N$ -dependence of the 2D ideal-gas transition temperature for a finite system cannot persist in the interacting case because it violates a theorem due to Chester, Penrose, and Onsager.

## I. Introduction

Bose-Einstein condensation (BEC) has been observed recently in several laboratories<sup>1,2</sup> using magnetic traps to confine and cool alkali atoms. Related experiments on lithium have been reported by third group<sup>3</sup>. The number of atoms involved ranged from a few thousand to a few million, in potential wells that were to a good approximation anisotropic harmonic oscillator potentials. Such a spatial BEC gives rise to several theoretical questions.

First, did these recent experiments observe a true phase transition to the BEC state? The answer is, obviously, no, because a true phase transition, with nonanalytic thermodynamic functions, requires taking the number of particles and the volume to infinity while keeping the density constant. Of course, no real system ever has such properties, but in most homogeneous macroscopic systems the thermodynamic limit is a good approximation to the experimental situation in which boundaries seem to play a relatively unimportant role. In the magnetic traps, not only is the number of particles quite small, compared to the usual case, but the “boundary,” formed by a quadratic potential well, extends literally throughout the whole system. In order to take the thermodynamic limit in such a system it is necessary to weaken the potential so that, as the number of particles increases, the average density remains constant. This is well-defined mathematically, but is of course physically unrealizable. On the other hand, taking the box size to infinity in the homogeneous case is also unrealized experimentally. One can argue that the situation there is *not* qualitatively different from a gas in a harmonic trap. For the ideal gas all that matters is the density of states, and the thermodynamic limit simply carries that to a continuum in each case.

The above discussion leads one to ask if the experiments then observed a

real Bose-Einstein condensation. The answer seems surely positive that they did see a macroscopic number of particles occupying the lowest harmonic oscillator state. Moreover the transition occurs quite abruptly as temperature is lowered. This result is in accordance with the findings of several authors<sup>4–11</sup> that in finite homogeneous Bose systems there is an accumulation point or “pseudo-critical” temperature where the increase in the chemical potential slows and the number of particles in the ground state begins increasing rapidly. Many different definitions of this accumulation point have been offered for an ideal gas – all of which approach the true phase transition point in the thermodynamic limit.

Thirdly, can a pseudo-transition occur in cases where there is no real phase transition in the thermodynamic limit? It can. One can show<sup>12</sup> that there is a real transition in the harmonic potential for an ideal gas in any dimension greater than or equal to two. However, although there is no real transition in one dimension (1D), there is a pseudo-transition<sup>12,13</sup> that occurs at temperature that would go to zero as  $1/\ln N$  in the thermodynamic limit.

The results quoted in the last paragraph bring up yet another question. How can there be a transition in 2D in the harmonic potential when the well-known Hohenberg theorem<sup>14</sup> says that there is no BEC transition in that number of dimensions? What the Hohenberg theorem actually says is that there can be no BEC into the  $k = 0$  state where  $k$  is wave number. Despite the restriction to  $k$ -states, this theorem would seem to be relevant to the case of a BEC into a harmonic oscillator ground state, because there is yet another theorem (CPO theorem), due to Chester<sup>15,16</sup>, based on a lemma of Penrose and Onsager<sup>17</sup>, that notes that there can be no BEC into *any* single-particle state unless one occurs into a  $k = 0$  state. So it seems that the 2D ideal gas transition ought not to be allowed! This situation has

arisen before and the answer found<sup>18,19</sup>: The CPO theorem requires that the density be finite everywhere. In the thermodynamic limit, the density of an ideal gas becomes infinite at the origin in the harmonic oscillator problem, which negates the validity of the CPO theorem. So there can be and is a BEC into the harmonic oscillator ground state in 2D in the thermodynamic limit.

But the Hohenberg theorem does *not* depend on the finiteness of the density for *its* validity. So it must still be valid to say that, in the thermodynamic limit, there is *no* BEC into the  $k = 0$  state for the 2D oscillator problem, while there *is* one for the lowest oscillator state; that is, there is a spatial condensation but not a momentum condensation. We see then that we must define two condensate numbers:  $n_0$ , the number of particles in the lowest oscillator state, and  $f_0$ , the number of particles in the zero-momentum state. If there are  $N$  particles in the system, then for the 2D ideal gas at non-zero temperature in the thermodynamic limit,  $f_0/N$  approaches zero, while  $n_0/N$  remains a finite fraction. Basically these two quantities are Fourier transforms of one another, and, despite the presence of the harmonic potential, it is still possible to have particles in the lowest  $k$  state. This peculiar relationship between them has not been noted previously.

Our calculations are for the ideal gas. In the thermodynamic limit the density becomes infinite at the origin in a harmonic potential. If a hard-core interparticle repulsion were included, the density would not become infinite, the CPO theorem would apply and the 2D BEC in a harmonic potential would disappear in the thermodynamic limit. It is easy to extend the ideal gas calculations to include a mean-field interaction<sup>22,23</sup>, but to the author's knowledge no one has yet done the 2D calculation for a system with hard-

core interactions in a harmonic potential.<sup>†</sup> Because the actual experiments are done with just a few particles, the ideal gas should become a fair approximation to the real experiments, and the density divergence of the ideal gas should never become an issue. Thus our calculations of  $n_0$ ,  $f_0$ , and the transition temperature might be relevant to experiment.<sup>‡</sup> However, the CPO theorem tells us that the pseudo-transition temperature would vanish were we to take the thermodynamic limit. Thus the  $N$ -dependence of the pseudo-transition temperature must be *different* from that of the ideal gas, meaning it must depend more weakly on particle number than for the ideal gas. This is in contrast to the 3D case where the ideal gas transition temperature seems to describe experiment quite well<sup>26</sup>. A full hard-core calculation of the transition temperature in 2D would of course be useful.

In Sec. II, we review the treatment of the Bose condensation in the thermodynamic limit. We look at the finite case in more detail in Sec. III. Sec. IV examines the momentum distribution and its relation to the distribution in the harmonic states. Further discussion occurs in Sec. V. An Appendix gives some mathematical details and makes connection with previous work.

## II. Harmonic Bose Systems in the Thermodynamic Limit

Bose-Einstein condensation in an external harmonic potential has been considered previously<sup>12–13,19–25,25–29</sup>. We re-examine and extend the analysis here. Consider the two-dimensional system of  $N$  noninteracting Bosons with the particles contained in an isotropic two-dimensional harmonic potential

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<sup>0 †</sup>A 3D calculation of the transition temperature for a gas in a harmonic potential with hard core interactions has been done recently by path-integral Monte Carlo methods<sup>25</sup>.

<sup>0 ‡</sup>As pointed out recently in Ref. 13, experiments on systems with reduced dimensionality (1D or 2D) are entirely possible in magnetic traps.

given by

$$U(\mathbf{r}) = \frac{1}{2}U_0 \left(\frac{r}{R}\right)^2 \quad (1)$$

where  $r^2 = x^2 + y^2$  and  $R$  is a range parameter for the potential. It is simple to generalize to any number of dimensions greater than two. The energy levels are given

$$E_{m_x, m_y} = \hbar\omega(m_x + m_y + 1) \quad (2)$$

with  $m_x, m_y = 0, 1, 2, \dots$ . The angular frequency is

$$\omega = \sqrt{\frac{U_0}{R^2 m}} \quad (3)$$

where  $m$  is the mass of a particle. In the grand ensemble at temperature  $T$ , the total number of particles  $N$  satisfies the equation

$$N = \sum_{m_x, m_y=0}^{\infty} \frac{1}{e^{\beta[\hbar\omega(m_x+m_y+1)-\mu]} - 1} \quad (4)$$

where  $\mu$  is the chemical potential,  $\beta = 1/k_B T$ , and  $k_B$  is the Boltzmann constant.

If one wishes to take the thermodynamic limit, then one must increase the “volume” of the system while increasing the number of particles with the average density kept constant. The average density is proportional to  $\rho \equiv N/R^2$ , where  $R$  is the range parameter in the harmonic potential of Eq. (1). To take the thermodynamic limit requires that  $R$  increase, that is, that the potential weaken, as  $N$  increases. Such a situation has been considered previously<sup>18,19,30</sup>, but might seem peculiar when considering real experiments in a potential fixed by external magnets. We follow this procedure, because it is the only way to reach the true nonanalytic behavior characteristic of phase transitions. One does not feel so uncomfortable with

the thermodynamic limit in the case of a homogeneous particles-in-a-box situation because there it seems that all that the thermodynamic limit does is to remove boundary effects, which are mostly negligible anyway in the real bulk experiments. Making such a qualitative distinction between the two cases seems unwarranted, however.

If  $\rho$  is kept constant we require, in 2D, that

$$\omega \propto \frac{1}{R} \propto \frac{1}{\sqrt{N}} \quad (5)$$

In  $D$  dimensions,  $\rho \propto \frac{N}{R^D}$  and  $\omega \propto \frac{1}{N^{1/D}}$ .

Introduce

$$T_0^{(2)} \equiv \frac{\hbar}{k_B} \sqrt{\frac{U_0}{m}} \sqrt{\rho} \quad (6)$$

and

$$\alpha = -\beta\mu + \hbar\omega. \quad (7)$$

Then

$$\frac{\hbar\omega}{k_B T} = \frac{T_0^{(2)}}{T\sqrt{N}} \quad (8)$$

and Eq. (4) can be written as

$$N = \sum_{m_x, m_y} \frac{1}{e^{\frac{T_0^{(2)}}{T\sqrt{N}}(m_x+m_y)+\alpha} - 1}. \quad (9)$$

To simplify, one can change the sums over  $m_x$  and  $m_y$  to sums over  $p = m_x + m_y$  and  $l = m_x$ . Then, since  $l$  no longer appears in the summand, one can do that sum to give

$$N = \sum_{p=0}^{\infty} \frac{p+1}{e^{\frac{T_0^{(2)}}{T\sqrt{N}}p+\alpha} - 1}. \quad (10)$$

One can use this formula to do numerical calculations of  $\alpha$  for given finite  $N$ . We will discuss this procedure in the next section. Here we want to take



the thermodynamic limit and it is useful to separate off the ground state by writing

$$\begin{aligned} N &= n_0 + \sum_{p=1}^{\infty} \frac{p+1}{e^{\frac{T_0^{(2)}}{T\sqrt{N}}p+\alpha} - 1} \\ &= n_0 + \sum_{p=0}^{\infty} \frac{p+2}{e^{\frac{T_0^{(2)}}{T\sqrt{N}}p+\alpha'} - 1}. \end{aligned} \quad (11)$$

where

$$n_0 = \frac{1}{e^{\alpha} - 1}. \quad (12)$$

In the last form we have taken  $p \rightarrow p+1$  and  $\alpha' = \alpha + \frac{T_0^{(2)}}{T\sqrt{N}}$  to reset the sum from  $p=0$ .

When  $N$  gets large, the states become very closely spaced and we can replace the sum by an integral (see Appendix) to good approximation to become

$$N = n_0 + \int_0^{\infty} dp \frac{p+2}{e^{\frac{T_0^{(2)}}{T\sqrt{N}}p+\alpha'} - 1} \quad (13)$$

Changing variables to  $u = pT_0^{(2)}/\sqrt{N}T$ , we find

$$\begin{aligned} N &= n_0 + N \left( \frac{T}{T_0^{(2)}} \right)^2 \int_0^{\infty} du \frac{u}{e^{u+\alpha'} - 1} + 2\sqrt{N} \left( \frac{T}{T_0^{(2)}} \right) \int_0^{\infty} du \frac{1}{e^{u+\alpha'} - 1} \\ &= n_0 + N \left( \frac{T}{T_0^{(2)}} \right)^2 F_2(\alpha') + 2\sqrt{N} \left( \frac{T}{T_0^{(2)}} \right) F_1(\alpha') \end{aligned} \quad (14)$$

where

$$F_{\sigma}(\alpha) \equiv \frac{1}{\Gamma(\sigma)} \int_0^{\infty} du \frac{u^{\sigma-1}}{e^{u+\alpha} - 1} = \sum_{p=1}^{\infty} \frac{e^{-\alpha p}}{p^{\sigma}} \quad (15)$$

are the Bose integrals<sup>31</sup>;  $\Gamma(\sigma)$  is the Gamma function. The term in  $F_2$  is of order  $N$  while the term in  $F_1$  is of order  $\sqrt{N}$  and can be neglected in the thermodynamic limit. We will however use it in the next section to estimate

the error made by the thermodynamic limit form when  $N$  is finite. Since  $F_2$  has a finite limit for small  $\alpha$ , there is a BEC for temperatures lower than some critical temperature.  $F_2(\alpha)$  behaves for small  $\alpha$  as<sup>31</sup>

$$F_2(\alpha) = \zeta(2) - \alpha(1 - \ln \alpha) + \dots \quad (16)$$

where  $\zeta(\sigma) = \sum_{p=1}^{\infty} 1/p^\sigma$  is the Riemann  $\zeta$ -function. We have  $\zeta(2) = \pi^2/6$ . The 2D Bose condensation temperatures  $T_c^{(2)}$  is given by the equation

$$N = N \frac{T_c}{T_0^{(2)}} F_2(0). \quad (17)$$

or

$$T_c^{(2)} = T_0^{(2)} \sqrt{\frac{6}{\pi^2}} \approx 0.78 T_0^{(2)}. \quad (18)$$

The occupation of the lowest oscillator state is, for  $T \leq T_c^{(2)}$ ,

$$\frac{n_0}{N} = 1 - \left( \frac{T}{T_c^{(2)}} \right)^2 \quad \text{2D} \quad (19)$$

and zero above  $T_c^{(2)}$ . Unlike the 2D homogeneous system the 2D oscillator has a BEC phase transition in the thermodynamic limit.

Recently the authors of Ref. 29 have claimed that the harmonic potential is fundamentally different from a homogeneous system and that a true phase transition cannot occur in the harmonic potential. We see that this is not correct, although the physical conditions for its occurrence (weakening the potential) may seem a bit strange. However, the same arguments could be applied to the system of particles in a box. Real experiments do not occur in an infinitely large box and experimentalists do not observe actual phase transitions with truly discontinuous functions. Further, all the particles know is the energy levels and their spacing; in each case the limiting process is just

changing the spacing. We feel that the two cases are not fundamentally different.

In 3D we have

$$N = \sum_{m_x, m_y, m_z=0}^{\infty} \frac{1}{e^{\frac{T_0^{(3)}}{TN^{1/3}}(m_x+m_y+m_z)+\alpha} - 1} \quad (20)$$

where  $T_0^{(3)}$  is given by

$$T_0^{(3)} \equiv \frac{\hbar}{k_B} \sqrt{\frac{U_0}{m}} \rho^{1/3} \quad (21)$$

with  $\rho \equiv N/R^3$  and

$$\alpha = -\beta\mu + 3T_0^{(3)}/(2TN^{1/3}). \quad (22)$$

One can again reduce the sum of Eq. (20) to one variable over  $p = m_x + m_y + m_z$  to give to

$$N = n_0 + \sum_{p=0}^{\infty} \frac{\frac{1}{2}p^2 + \frac{5}{2}p + 3}{e^{\frac{T_0^{(3)}}{TN^{1/3}}p+\alpha'} - 1} \quad (23)$$

with  $\alpha' = \alpha + T_0/(TN^{1/3})$ . We have again separated off the  $n_0$  term and have reset the sum from  $p = 0$ . Changing the sum to an integral (see Appendix) gives

$$N = n_0 + N \left( \frac{T}{T_0^{(3)}} \right)^3 F_3(\alpha') + \frac{5}{2} N^{2/3} \left( \frac{T}{T_0^{(3)}} \right)^2 F_2(\alpha') + 3N^{1/3} \left( \frac{T}{T_0^{(3)}} \right) F_1(\alpha') \quad (24)$$

In the thermodynamic limit the terms in  $F_2$  and  $F_1$  are negligible, as is the difference between  $\alpha'$  and  $\alpha$ , and the condensate fraction is given by

$$\frac{n_0}{N} = 1 - \left( \frac{T}{T_c^{(3)}} \right)^3 \quad \text{3D} \quad (25)$$

where  $T_c^{(3)} = T_0^{(3)}/(\zeta(3))^{1/3}$ .

Eq. (24) is very similar to one given recently in Ref. 24, but differs in the coefficients of the various terms and in the argument being  $\alpha'$  rather than  $\alpha$ . We show in the Appendix that the two equations are the same. Our equation also has the same form as that appearing in Ref. 27.

It is similarly easy to show that the D-dimensional harmonic oscillator system has a BEC, for  $D \geq 2$ , at temperature

$$T_c^{(D)} = T_0^{(D)} \zeta(D)^{-1/D} \quad (26)$$

with  $T_0^{(D)}$  given by

$$T_0^{(D)} \equiv \frac{\hbar}{k_B} \sqrt{\frac{U_0}{m}} \rho^{1/D} \quad (27)$$

and  $\rho \equiv N/R^D$ . In D-dimensions ( $D \geq 2$ ) the condensate fraction is given by

$$\frac{n_0}{N} = 1 - \left( \frac{T}{T_c^{(D)}} \right)^D \quad \text{D dimensions.} \quad (28)$$

As  $D \rightarrow \infty$ ,  $T_c^{(D)} \rightarrow T_0^{(D)}$  and  $n_0/N$  becomes a step function in  $T$ .

The 1D system is a special case for which there is no condensation in the thermodynamic limit. The same procedure as used above leads, in the continuum limit, to

$$N = N \left( \frac{T}{T_0^{(1)}} \right) \int_0^\infty \frac{1}{e^{u+\alpha} - 1} = N \left( \frac{T}{T_0^{(1)}} \right) F_1(\alpha). \quad (29)$$

with  $T_0^{(1)}$  given by Eq. (27) with  $D=1$ .  $F_1(\alpha)$  does not approach a finite limit as  $\alpha \rightarrow 0$ , but is given exactly by

$$F_1(\alpha) = -\ln(1 - e^{-\alpha}) \quad (30)$$

which approaches  $-\ln \alpha$  as  $\alpha \rightarrow 0$ . However, Refs. 12 and 13 have pointed out that, in the finite system, the 1D system *does* have a BEC below a certain pseudo-critical temperature. We will discuss this case below.

### III. Finite Systems

To consider finite harmonic systems<sup>12–13,25,27–29</sup> most easily in, say, 2D, we can just compute  $\alpha$  via Eq. (10) by iteration and then compute the number of particles in the lowest state from Eq. (12). Fig. 1 illustrates the 2D situation. The dotted and dashed lines are the exact results for  $N = 10$  to  $10^4$  with the thermodynamic limit (Eq. (19)) shown as the solid line.

One notes that the exact result is smaller than the value given by the infinite  $N$  limit. A better approximation than Eq. (19) is the form of Eq. (14); the  $F_1$  term in Eq. (14) aids in giving an estimate of the difference due to finite  $N$ . In 3D a better analytic approximation is given by Eq. (24)<sup>24,27–29</sup>. Since, for small temperature,  $\alpha = O(1/N)$ , the  $F_\sigma$  behave as

$$\begin{aligned} F_1(\alpha) &\rightarrow -\ln(\alpha) \\ F_2(\alpha) &\rightarrow \zeta(2) - \alpha(1 - \ln(\alpha)) \end{aligned} \quad (31)$$

and we can show that the condensate fraction is roughly

$$\frac{n_0}{N} \approx 1 - \left( \frac{T}{T_c^{(2)}} \right)^2 - \frac{T}{T_0^{(2)}} \frac{\ln(N)}{\sqrt{N}} \quad (32)$$

The correction is negative so that the value of  $n_0$  valid for finite  $N$  is smaller than the infinite limit value, at least when  $\alpha$  is of order  $1/N$  as we see in the figure.

For a finite harmonic system it is not particularly appropriate to express the pseudo-transition temperature in terms of the density. The potential is fixed and scaling it in terms of  $N$  with fixed density makes little sense. We now express it instead in terms of a fixed frequency. The effective or pseudo-transition temperature is only a bit less (less by terms of order  $\ln N/\sqrt{N}$ ) than the infinite-limit transition temperature and can be taken as roughly

the same. By Eqs. (6) and (18) the pseudo-transition temperature is given by

$$T_c^{(2)} = \sqrt{\frac{N}{\zeta(2)}} \frac{\hbar\omega}{k_B} \quad (33)$$

A similar analysis for three dimensions leads to a pseudo-transition temperature a bit less than

$$T_c^{(3)} = \left( \frac{N}{\zeta(3)} \right)^{1/3} \frac{\hbar\omega}{k_B} \quad (34)$$

A pseudo-transition in one-dimension is a special case<sup>12,13</sup> because there is no real phase transition in 1D. Pseudo-transition temperatures can be defined in many ways. Ref. 7 defines six different ways in the homogeneous case that all lead to the same  $T_c$  for infinite  $N$  in cases where there is a real phase transition. We compare just two approaches. First we follow<sup>12</sup> the analysis used in the higher dimensional cases: Write

$$N = \sum_{p=0}^{\infty} \frac{1}{e^{\frac{T_0^{(1)}}{TN} p + \alpha} - 1} \quad (35)$$

where  $T_0^{(1)}$  is given by

$$T_0^{(1)} \equiv \frac{\hbar}{k_B} \sqrt{\frac{U_0}{m}} \frac{N}{R} \quad (36)$$

and  $\alpha = -\beta\mu + T_0^{(1)}/(2TN)$ . Separate off  $n_0$  as above to get

$$N = n_0 + \sum_{p=0}^{\infty} \frac{1}{e^{\frac{T_0^{(1)}}{TN} p + \alpha'} - 1} \quad (37)$$

in which

$$\alpha' = \alpha + \frac{T_0^{(1)}}{TN} \quad (38)$$

and the sum has been reset from  $p = 0$ . The continuum limit becomes

$$N = n_0 + N \frac{T}{T_0^{(1)}} F_1(\alpha'). \quad (39)$$

The assumption that  $n_0$  is of order  $N$  implies, from Eq. (12), that  $\alpha$  is of order  $1/N$ . We have, for small  $\alpha'$ ,

$$N = n_0 - \frac{T}{T_0^{(1)}} \ln(\alpha') \quad (40)$$

Taking  $\alpha'$  of order  $1/N$  and assuming  $N$  large give

$$N = n_0 + \frac{T}{T_0^{(1)}} N \ln N \quad (41)$$

or

$$\frac{n_0}{N} = 1 - \frac{T}{T_0^{(1)} / \ln N} \quad (42)$$

This equation has just the form shown in Eq. (28) with an  $N$ -dependent transition temperature. (But, of course, in the case of a finite  $N$  and fixed harmonic potential *all* pseudo-transition temperatures are  $N$ -dependent.) Here we have

$$T_c^{(1)} = \frac{T_0^{(1)}}{\ln N} = \frac{N}{\ln N} \frac{\hbar\omega}{k_B} \quad (43)$$

in which the first form is appropriate to the case where the density  $N/R$  is being kept constant (weakening potential) and shows the transition temperature decreasing to zero as the number of particles increases so that there is no true phase transition. However, the second form shows that for constant potential as in the real experiments the temperature for which a substantial BEC occurs actually gets larger as  $N$  increases.

An alternative derivation<sup>7,13</sup> comes easily from noting that in a BEC one can write

$$N = n_0(\alpha) + N'(\alpha) \quad (44)$$

where  $N'$  is the number of particles not in the ground state. At a real BEC transition temperature  $\alpha$  is very small, but  $n_0$  is also still microscopic. Thus

at the transition temperature, we have approximately

$$N'(0) = N \quad (45)$$

In the case of the 1D system this would imply, from Eqs. (40) and (38) that  $T_c^{(1)}$  is given by

$$-\frac{T_c^{(1)}}{T_0} \ln \left( \frac{T_0^{(1)}}{T_c^{(1)} N} \right) = 1 \quad (46)$$

or to a good approximation for large  $N$ ,

$$T_c^{(1)} = \frac{T_0^{(1)}}{\ln N} \quad (47)$$

in agreement with the first estimate.

In Fig. 2, we plot the exact quantum result  $n_0/N$  from Eq. (35) as a function of  $T/T_0^{(1)}$ . Such a graph is appropriate to the case of taking the thermodynamic limit for which the density (and therefore  $T_0^{(1)}$ ) is held constant. One sees that then the transition temperature gets smaller as  $N$  increases, indicating that there is no real transition in this system. Fig. 3, on the other hand, plots  $n_0/N$  as a function of  $T \ln N/T_0^{(1)}$ . This method of graphing<sup>13</sup> includes all the  $N$  dependence that occurs in  $T_c^{(1)}$  and is more appropriate for the fixed potential case. One sees the asymptotic approach to an exact straight line for large  $N$  as given by Eq. (42).

#### IV. The Hohenberg and CPO Theorems

The Hohenberg theorem<sup>14</sup> states that there is no condensation into the  $k = 0$  state in the infinite system in two dimensions or less. The CPO theorem<sup>15–17</sup> says that there can be no condensation into any single particle-state unless there is one into the  $k = 0$  state. However, the CPO theorem relies on the assumption that the density is everywhere finite. This is not



the case in a harmonic potential, where we will show that the density becomes infinite at the origin in the thermodynamic limit. We first review the Hohenberg theorem as presented by Chester<sup>15</sup>.

Chester has shown how to apply the Hohenberg theorem to a finite, inhomogeneous system. To quote Chester<sup>15</sup>: “... [F]or any system whatsoever, we can always ask about the mean number of particles with momentum zero. This follows from the general principles of quantum mechanics and we are in no way limited, in asking this type of question, to systems that are homogeneous.” We consider the following picture. Assume that the harmonic potential extends out to radius  $R$ , but that there is a hard wall at  $r = R$ . It is possible, of course, to find the energy eigenvalues for such a mixed potential system, but we can use the eigenvalues of the harmonic potential if we just assume that  $k_B T$  is small enough that particles rarely ever actually reach the hard wall. That is, we assume that  $R$  is sufficiently large that

$$k_B T < V(R) \quad (48)$$

The Bogoliubov inequality is

$$\left\langle \frac{1}{2} [A, A^\dagger] \right\rangle \geq \frac{k_B T |\langle [C, A] \rangle|^2}{\langle [[C, H], C^\dagger] \rangle} \quad (49)$$

in which Chester takes

$$A = a_0 a_k^\dagger \quad (50)$$

and

$$C = \sum_q a_q^\dagger a_{q+k} \quad (51)$$

where  $a_k^\dagger$  is the creation operator for the state with wave number  $\mathbf{k}$ . To quote Chester once again<sup>15</sup>: “Since we are allowed to use any set of complete

functions for our second quantization formalism we choose a continuum of plane waves.” Direct calculation of the commutators leads to

$$f_0 f_k + \frac{1}{2}(f_0 + f_k) \geq \frac{(f_0 - f_k)^2 k_B T m}{N(\hbar k)^2} \quad (52)$$

where  $f_k = \langle a_k^\dagger a_k \rangle$  is the number of particles in the momentum state  $\mathbf{k}$ . This inequality is summed over  $k$  values from some minimum  $k_0$  to maximum  $k_c$ . The first term on the left can have the sum changed to *all*  $k$  values, which just increases the inequality and gives  $Nf_0$ . The second term becomes  $Nf_0\Gamma$ , where  $\Gamma$  is of order unity. The third term is of order  $N$  and is negligible if  $f_0$  is assumed to be of order  $N$ . On the right side we can drop  $f_k$  as negligible compared to  $f_0$ . Thus we have

$$Nf_0 \left(1 + \frac{\Gamma}{N}\right) \geq \frac{f_0^2 m k_B T}{N \hbar^2} \sum_{k=k_0}^{k_c} \frac{1}{k^2} \quad (53)$$

And so in 2D, we have (assuming that dropping the  $\Gamma$  term does not seriously affect the inequality)

$$N \geq \frac{f_0 m k_B T A}{h^2} \ln \frac{k_c}{k_0} \quad (54)$$

where  $A = \pi R^2$  is the area of the system. We take  $k_0 \propto 1/R$  and  $k_c \propto 1/a$ , where  $a = 1/\sqrt{\rho} = \sqrt{R^2/N}$ . Then our result for the 2D case is

$$\frac{f_0}{N} \leq \frac{\hbar^2}{m a^2 k_B T} \frac{1}{\ln N} \quad (55)$$

Thus the Hohenberg theorem states that, in the thermodynamic limit, the condensate fraction in the  $k = 0$  momentum state must go to zero at least as fast as  $1/\ln N$  for any non-zero temperature. We will see that it actually goes to zero faster than this, namely as  $1/\sqrt{N}$ .

The CPO theorem goes as follows: Let  $\sigma_1(\mathbf{r}, \mathbf{r}')$  be the single-particle reduced density matrix in the position representation. Penrose and Onsager's

lemma states that if the quantity

$$W = \frac{1}{V} \int d\mathbf{r} \int d\mathbf{r}' |\sigma_1(\mathbf{r}, \mathbf{r}')|, \quad (56)$$

(where  $V$  is the volume) is not proportional to  $N$ , then there is no BEC into *any* eigenstate of the single-particle density matrix. However, the number of particles in the zero momentum state is given by

$$\begin{aligned} f_0 &= \int d\mathbf{r} \int d\mathbf{r}' \langle \mathbf{k} = 0 | \mathbf{r} \rangle \langle \mathbf{r} | \sigma_1 | \mathbf{r}' \rangle \langle \mathbf{r}' | \mathbf{k} = 0 \rangle \\ &= \frac{1}{V} \int d\mathbf{r} \int d\mathbf{r}' \sigma_1(\mathbf{r}, \mathbf{r}'). \end{aligned} \quad (57)$$

Chester proves that a Bose system has a positive-semidefinite single-particle density matrix, so that, if  $f_0/N$  approaches zero as  $N \rightarrow \infty$ , then  $W/N$  must also go to zero, eliminating the possibility of BEC into any single-particle state.

The weak link in the theorem's proof for our purposes is that the Penrose-Onsager lemma requires the average density at each point be finite. That is not the case in the harmonic potential as we will show. Thus the fact that  $f_0/N \rightarrow 0$  does *not* imply that  $n_0/N \rightarrow 0$ . In the thermodynamic limit it is possible, for a 2D harmonic potential, to have a condensation into the lowest harmonic oscillator state while still satisfying the Hohenberg theorem that there be no BEC into the zero momentum state in the thermodynamic limit. We will indeed show that  $f_0/N$  approaches zero as  $1/\sqrt{N}$ .

A direct computation of  $f_k$  is straightforward. We have

$$f_k = \sum_m \langle \mathbf{k} | m \rangle \langle m | \sigma_1 | m \rangle \langle m | \mathbf{k} \rangle \quad (58)$$

where the  $|m\rangle$  represent the eigenstates of  $\sigma_1$ , in this case the harmonic oscillator eigenstates. The states  $\langle \mathbf{k} | m \rangle$  are the Fourier transforms of the

harmonic oscillator states. These latter functions are, just like the states in configuration space, Hermite functions times a Gaussian. In 1D

$$\langle k|m \rangle = \frac{C_m \sqrt{2\pi}}{\xi \sqrt{L}} H_m(k/\xi) e^{-k^2/2\xi^2} \quad (59)$$

in which  $L$  is a box length,

$$\xi = \sqrt{\frac{m\omega}{\hbar}}, \quad (60)$$

and

$$C_m = \sqrt{\frac{\xi}{\sqrt{2} 2^m m!}} \quad (61)$$

is the usual wave function normalization factor in configuration space.

The density matrix is given in 2D by

$$\sigma_{m_x m_y} \equiv \langle m_x m_y | \sigma_1 | m_x m_y \rangle = \frac{1}{e^{\beta[\hbar\omega(m_x+m_y+1)-\mu]} - 1} \quad (62)$$

The result of the indicated calculations is

$$f_k = \frac{(2\pi)^2}{A\xi^4} e^{-(\kappa_x^2 + \kappa_y^2)} \sum_{m_x, m_y=0}^{\infty} \sigma_{m_x m_y} C_{m_x}^2 C_{m_y}^2 [H_{m_x}(\kappa_x) H_{m_y}(\kappa_y)]^2 \quad (63)$$

where  $\kappa_i \equiv k_i/\xi$ , and  $A$  is the area. Using the relations derived in Sec. II we can rewrite the last equation as

$$f_k = \frac{4}{\sqrt{N} k_B T_0} \frac{\hbar^2}{m a^2} e^{-(\kappa_x^2 + \kappa_y^2)} \times \sum_{m_x, m_y=0}^{\infty} \sigma_{m_x m_y} \frac{1}{2^{m_x} 2^{m_y} m_x! m_y!} [H_{m_x}(\kappa_x) H_{m_y}(\kappa_y)]^2 \quad (64)$$

where  $a \equiv \sqrt{1/\rho}$ .

The value of the zero momentum term is

$$f_0 = \frac{4}{\sqrt{N} k_B T_0} \frac{\hbar^2}{m a^2} \sum_{m_x, m_y=0,2,\dots}^{\infty} \sigma_{m_x m_y} \left[ \frac{m_x! m_y!}{2^{m_x} 2^{m_y} [(m_x/2)!(m_y/2)!]^2} \right] \quad (65)$$

We can find the density at position  $\mathbf{r}$  by a similar analysis:

$$\rho(\mathbf{r}) = \langle \mathbf{r} | \sigma_1 | \mathbf{r} \rangle = e^{-\xi^2 r^2} \sum_{m_x, m_y=0}^{\infty} \sigma_{m_x m_y} C_{m_x}^2 C_{m_y}^2 [H_{m_x}(\xi x) H_{m_y}(\xi y)]^2 \quad (66)$$

Because of the relation between harmonic oscillator wave functions and their Fourier transforms, we see that there is a close relation between  $f_0$  and  $\rho(0)$ . This is given by

$$f_0 = 4\pi \left( \frac{\hbar^2}{ma^2 k_B T_0} \right)^2 \rho(0). \quad (67)$$

We can find WKB approximations for  $f_k$  and  $\rho(\mathbf{r})$  by using the Boltzmann distribution function for the number of particles of wave number  $\mathbf{k}$  at position  $\mathbf{r}$ , given in 2D by

$$f_k^{WKB}(\mathbf{r}) = \frac{1}{A} \frac{1}{\exp \beta \left[ \frac{\hbar^2 k^2}{2m} + U(\mathbf{r}) - \mu \right] - 1}. \quad (68)$$

If we integrate over  $\mathbf{r}$  and sum over  $\mathbf{k}$  we should get the number of particles in the system. However, in 2D we actually get just the  $F_2(\alpha)$  term in Eq. (14). This is  $N'$ , the number of particles in excited states. We need to treat the condensate separately.

If we integrate  $f_k^{WKB}(\mathbf{r})$  just over  $\mathbf{r}$  we get  $f_k^{WKB}$ , the wave-number distribution, and if we just sum over  $\mathbf{k}$  we get  $\rho(\mathbf{r})$ , the local density. The results of these integrations are

$$f_k^{WKB} = -\frac{2\hbar^2 k_B T}{ma^2 (k_B T_0)^2} \ln \left( 1 - \exp[-\beta(\hbar k)^2/2m - \alpha] \right) \quad (69)$$

and

$$\rho(\mathbf{r}) = -\frac{2\pi m k_B T}{h^2} \ln \left( 1 - \exp[-\beta U(\mathbf{r}) - \alpha] \right) \quad (70)$$

where  $\alpha = -\beta\mu$ .

These formulas are valid only above any condensation or pseudo-critical temperature. We could add  $\delta$ -function terms to account for the  $n_0$  term, but we do not bother here. Were we to take  $\mathbf{k} = 0$  and consider low temperatures such that  $\alpha \propto 1/N$ , the first of these functions would depend on  $-\ln \alpha \propto \ln N$ . A similar behavior would occur for the second function for  $\mathbf{r} = 0$ . We will see next, however, that the true quantum behavior gives a stronger divergence with  $N$ .

In Fig. 4 we show plots of the quantum  $f_k$ , as a function of  $k$  as computed from Eq. (64). The lower temperature  $0.5T_0$  is below the transition temperature  $T_c = 0.78T_0$  and the higher temperature is above  $T_c$ . Also shown are the WKB approximations at the same temperatures from Eq. (69) and Eq. (70). We see that the WKB approximation is quite good if we are above  $T_c$ , but very poor below as expected.

One important feature of  $f_k$  shown is that  $f_0$  is continuous with  $f_k$  for  $k \neq 0$ . This is quite different from homogeneous Bose condensed systems for which the  $k = 0$  term is much larger than the  $k \neq 0$  terms. The pseudo-condensate in momentum space (terms not included in the WKB expression) *is spread over many  $k$ -values*. However, note that the  $k$ -scale factor determining the width of the spread in  $k$ -space of the functions in the Fig. 4 is

$$\xi = \sqrt{\frac{mk_B T_0}{\hbar^2}} \frac{1}{N^{1/4}} \quad (71)$$

which goes to zero as  $N \rightarrow \infty$  in the thermodynamic limit.

In Fig. 5, we show the  $k = 0$  momentum occupation number as a function of temperature for  $N = 1000$ . Clearly there is a pseudo-condensation at the same temperature  $T_c = 0.78T_0$  as the spatial condensation into the oscillator ground state.

The  $N$ -dependence of  $f_0$  is shown in Fig. 6. For the two lowest temperatures the behavior is found numerically to be  $f_0 \propto \sqrt{N}$ . The highest temperature shown is just below the transition temperature and has not yet settled into its asymptotic form. With this result we will have

$$\frac{f_0}{N} \propto \frac{1}{\sqrt{N}} \rightarrow 0 \quad (72)$$

as  $N \rightarrow \infty$ . This supports our claim that the  $k = 0$  condensate fractions vanishes in the thermodynamic limit.

We can verify this numerical result analytically. The first term in the sum in Eq. (65) is just  $\sigma_{00} = n_0$  so that the first term is of order  $\sqrt{N}$  when  $n_0$  is of order  $N$ . The other terms in the series are positive so that the whole sum is at least of order  $\sqrt{N}$ . The quantity in square brackets in Eq. (65) is always less than 1. This implies that

$$f_0 < \frac{4\hbar^2}{ma^2k_BT_0} \frac{1}{\sqrt{N}} \sum_{m_x, m_y} \sigma_{m_x m_y} \quad (73)$$

Since the sum on the right is just  $N$ , we have

$$f_0 < \frac{4\hbar^2}{ma^2k_BT_0} \sqrt{N} \quad (74)$$

This result implies that  $f_0/N$  approaches zero with  $N$  as  $1/\sqrt{N}$  as our numerical treatment showed. Actually we can do somewhat better in our analysis. The quantity in square brackets in Eq. (65) can be shown (via the Sterling's approximation) to obey

$$\left[ \frac{m_x! m_y!}{2^{m_x} 2^{m_y} [(m_x/2)! (m_y/2)!]^2} \right] \approx \frac{2}{\pi \sqrt{m_x m_y}} \quad (75)$$

for large  $m_x$  and  $m_y$ . If we separate off the  $m_x, m_y = 0$  term, use Eq. (75) for *all*  $m_x, m_y$ , and then replace the resulting sum by an integral, we find

$$f_0 \approx \frac{4\hbar^2}{ma^2k_BT_0} \frac{n_0}{\sqrt{N}} + f_0^{WKB} \quad (76)$$

where  $f_0^{WKB}$  is gotten from Eq. (69). The WKB term in Eq. (76) diverges as  $\ln N$ . In this approximation the  $\sqrt{N}$  behavior comes from just the term in  $n_0/\sqrt{N}$ . We see from Eq. (76), that for low  $T$ ,  $f_0 \propto \sqrt{N}/a^2 T_0 \propto \sqrt{N\rho}$ . Thus  $f_0/N$  goes to zero in the thermodynamic limit as  $1/\sqrt{N}$ . However, when the potential remains fixed, so that the density increases with  $N$ , then  $f_0 \propto N$ .

Because of the relation Eq. (67), we see that the density at the origin also diverges as  $\sqrt{N}$  in the thermodynamic limit.

It is possible to extend our discussion to arbitrary dimension. We find that in any dimension  $f_0/N$  tends to zero as  $1/\sqrt{N}$  so that in the thermodynamic limit there is no BEC into the  $k = 0$  state. While the Hohenberg theorem forbids a  $k = 0$  condensate only for  $D \leq 2$ , there is, on the other hand, there is no theorem that *requires* a BEC into  $k = 0$  for  $D > 2$ .

We should again point out that our results apply only to the ideal gas. For a real gas with hard-core interactions there can be no divergence in the density. If the particles have a hard-core interaction, the density will stay finite everywhere and the connection between the CPO and Hohenberg theorem will be re-established. This implies that the 2D BEC into the lowest oscillator state will disappear for interacting particles in the thermodynamic limit. Thus the CPO theorem implies that for interacting finite 2D systems the dependence of the transition temperature on particle number must change from  $T_c^{(2)} \propto \sqrt{\rho} \propto \sqrt{N}$  as shown for an ideal gas to something more weakly dependent on  $N$ . An example would be the form  $T_c \propto \sqrt{N}/\ln N$  analogous to the 1D gas, but the determination of the exact form, of course, requires explicit hard-core calculations. Curiously, the 3D, and even the 1D, dependences are not required to change.



## V. Discussion

We have examined several features of BEC in harmonic potentials for the ideal gas. We have shown that, in the thermodynamic limit, there is no condensation into the zero-momentum state in any number of dimensions. This result occurs in spite of the fact that there is a BEC into the lowest oscillator state for dimension  $D \geq 2$ .

For 2D there is a peculiarity in finding a condensate in the lowest oscillator state in the thermodynamic limit given the usual connection, via the CPO theorem, between the  $k = 0$  condensation and the condensation into any single-particle state. The fact that the Hohenberg theorem requires that there be no real condensation via a phase transition into the  $k = 0$  state in 2D would normally rule out the possibility of BEC into any other state. However, we have seen that the density diverges at the origin in the oscillator case (in the thermodynamic limit), which invalidates the CPO proof. Thus while there is no BEC into the  $k = 0$  momentum state in 2D in the thermodynamic limit, there *is* one into the  $m = 0$  oscillator state.

The thermodynamic limit is an approximation to reality that might seem less reasonable in a harmonic trap than in homogeneous systems. The numbers of particles in the initial real experiments were quite small – only a few thousand to a few million. Further, the mode by which one must take the thermodynamic limit, weakening the potential while increasing the particle number in such a way that the average density remains constant, might seem qualitatively different from enlarging the size of a box of particles at constant density. In the box case it would seem that the enlargement is merely a change in the boundary conditions, which affects the minority of particles at the walls. On the other hand, varying the size of the experimen-

tal harmonic trap affects every particle. However, looked in another way we can say that all that the particles in either case know is the energy levels and their spacing. In both the box and the harmonic trap the thermodynamic limit treats the transition to the case where the spacing becomes infinitesimally small. The process is thus completely analogous in the two cases, simply with differing densities of states.

The experimental situation, whether it be with a box or a harmonic magnetic trap, usually has a fixed potential (which of course can be altered) with possibly varying numbers of particles; obviously the thermodynamic limit is never approached. Thus it becomes more useful to look at finite systems and how their properties depend on the number of particles  $N$ . Although finite systems do not have actual phase transitions, they have pseudo-condensations in which a substantial fraction of the  $N$  particles fall into the lowest oscillator state below some temperature whose value depends on  $N$ . Real experiments involve such pseudo-transitions and we have shown that they can occur in any number of dimensions, even in 1D. Experimentalists using magnetic traps may have the capability of manipulating the traps to simulate 2D or even 1D.<sup>13</sup>

The finiteness of the systems in real experiments may be precisely what makes our results relevant. Because there are so few particles in the real systems, they do indeed behave much like ideal gases. Our finite-size ideal gas model has a density that is well-behaved at the origin without the unrealistic divergent behavior shown in the thermodynamic limit. In our finite-sized model, we can investigate how such systems behave as  $N$  varies. There are pseudo-condensations into both the lowest oscillator state and into the  $k = 0$  state, and the relevant variables have predicted dependences on  $N$ . The 1D case is interesting because of the behavior of the transition temperature;

the case of 2D is particularly interesting from the point of view of the CPO theorem. It is possible that these  $N$  dependences could be measured experimentally and the predictions demonstrated. What needs to be done in further research is to investigate the effect of hard-cores to check that, in the low density limit, the results we find persist.

The CPO theorem can predict that the persistence of ideal gas behavior cannot be complete. If the particles have a hard-core interaction, the density will stay finite everywhere even in the thermodynamic limit and the connection between the CPO and Hohenberg theorem will be re-established. This implies that the 2D BEC into the lowest oscillator state will disappear for interacting particles in the thermodynamic limit. The theorem thus implies that for interacting finite 2D systems the dependence of  $T_c^{(2)}$  on particle number must change from that shown by an ideal gas in such a way that it can vanish in the thermodynamic limit.

We note that it is possible that the prediction for  $f_0$  could be checked rather directly in experiments using a common method of measuring the condensate. The magnetic trap is removed and the particle cloud expands. Its “shadow” is then examined after it has blown up to a size large enough to be easily visible. The spatial distribution after removal of the trap is determined by the velocity distribution. The particles staying at the center of the cloud are those with zero momentum.

## Appendix

Eqs. (14) and (24) give corrections to the large- $N$  limit of the relation giving  $n_0(T, N)$ . It is useful to consider a more rigorous derivation than we have given above and to make connection with previous work on the subject.

In particular, Eq. (24) looks different from the equivalent formula of Ref. 24 [Eq. (4) in the first paper of the reference], in having for example a  $\frac{5}{2}$  in the  $F_2$  term instead of a  $\frac{3}{2}$ . We will see however that our formula agrees completely with the results of Ref. 24.

We use the Euler-Maclaurin summation formula<sup>32</sup> to change our sums to integrals. There is more than one form of this formula. Eq. (3.6.28) of Ref. 32 gives

$$\begin{aligned} \sum_{k=1}^{n-1} f(k) &= \int_0^n f(k) dk - \frac{1}{2}[f(0) + f(n)] + \frac{1}{12}[f'(n) - f'(0)] \\ &\quad - \frac{1}{720}[f'''(n) - f'''(0)] + \dots \end{aligned} \quad (77)$$

Note that the sum starts at  $k = 1$ . Eq. (20) can be written as

$$N = n_0 + \sum_{p=1}^{\infty} \frac{\frac{1}{2}p^2 + \frac{3}{2}p + 1}{e^{bp+\alpha'} - 1} \quad (78)$$

where

$$b = \frac{T_0^{(3)}}{TN^{1/3}}. \quad (79)$$

Use of Eq. (77) leads to

$$N = n_0 + \frac{1}{b^3}F_3(\alpha) + \frac{3}{2}\frac{1}{b^2}F_2(\alpha) + \dots \quad (80)$$

exactly as in Ref. 24. However, the corrections to the Euler-Maclaurin integral form a divergent series. It is easy to see that the various derivative terms are in powers of  $n_0 = 1/[\exp(\alpha) - 1]$ , which for low  $T$  is of order  $N$ .

To find a correction series that converges we use a slightly different form of the Euler-Maclaurin formula<sup>27</sup>; Eq. (23.1.30) of Ref. 32 is

$$\begin{aligned} \sum_{k=0}^n f(k) &= \int_0^n f(k) dk + \frac{1}{2}[f(0) + f(n)] + \frac{1}{12}[f'(n) - f'(0)] \\ &\quad - \frac{1}{720}[f'''(n) - f'''(0)] + \dots \end{aligned} \quad (81)$$

This form, in which the sum starts at  $k = 0$ , can be used directly on Eq. (23).

Using it we get

$$\begin{aligned} N &= n_0 + \frac{1}{b^3}F_3(\alpha') + \frac{5}{2}\frac{1}{b^2}F_2(\alpha') + 3\frac{1}{b}F_1(\alpha') \\ &+ \frac{3}{2}\frac{1}{e^{\alpha'} - 1} - \frac{1}{12}\left[\frac{5/2}{e^{\alpha'} - 1} - \frac{3}{(e^{\alpha'} - 1)^2}\right] + \dots \end{aligned} \quad (82)$$

Because  $\alpha' = \alpha + b$  appears in the arguments instead of  $\alpha$  the series of corrections is well-behaved as we now see.

First, we consider the case when  $T > T_c$  so that  $\alpha$  is of order unity while  $b$  remains of order  $N^{-1/3}$ . We can simply Taylor expand around  $\alpha' = \alpha$ . We make use of the easily proved relation

$$\frac{d}{d\alpha}F_\sigma(\alpha) = -F_{\sigma-1}(\alpha) \quad (83)$$

for  $\sigma > 1$ . After a bit of algebra we find to the same order

$$\begin{aligned} N &= n_0 + \frac{1}{b^3}F_3(\alpha) + \frac{3}{2}\frac{1}{b^2}F_2(\alpha) + \frac{1}{b}F_1(\alpha) \\ &= +\frac{31}{24}n_0 + \frac{1}{12}n_0^2\left[3 - \frac{31}{2}b\right] - \frac{1}{2}bn_0^3 + \dots \end{aligned} \quad (84)$$

Since we are above  $T_c$ ,  $n_0$  is of order unity;  $b$  is of order  $N^{-1/3}$  so that the entire series in various powers of  $n_0$  and  $b$  is negligible. We have then

$$N \approx \frac{1}{b^3}F_3(\alpha) + \frac{3}{2}\frac{1}{b^2}F_2(\alpha) + \frac{1}{b}F_1(\alpha) \quad (85)$$

in agreement with Refs. 24. Ref. 27 has given a similar analysis.

Finally, below  $T_c$ ,  $\alpha$  is of order  $1/N$ , which is much smaller than  $b$ , so that  $\alpha' \approx b$  and

$$\frac{1}{e^{\alpha'} - 1} \approx 1/b = O(N^{1/3}). \quad (86)$$

The full Taylor expansion then gives

$$N = n_0 + N\frac{1}{b^3}F_3(0) + \frac{3}{2}\frac{1}{b^2}F_2(0) + O\left(\frac{1}{b}\right) \quad (87)$$

The first three terms correspond to the formula of Ref. 24, which is therefore valid below the transition as well as above.

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### Figure Captions

Figure 1: Spatial condensate fraction  $n_0/N$  versus temperature  $(T/T_0)$  in 2D. The dotted and dashed lines are for finite  $N$  values, generated from Eq. (10); the solid curve is the thermodynamic limit, Eq. (19).  $T_0$  is given by Eq. (6).

Figure 2: Spatial condensate fraction  $n_0/N$  versus  $T/T_0$  in 1D from Eq. (35). This curve is appropriate for considering the thermodynamic limit for which the average density and thus  $T_0$  are held constant by weakening the potential as  $N$  increases. The pseudo-critical temperature then decreases with  $N$  as  $1/\ln N$ . In 1D  $T_0$  is given by Eq. (27) with  $D=1$ .

Figure 3: Spatial condensate fraction  $n_0/N$  versus  $(T/T_0) \ln N$  in 1D from Eq. (35). This plot removes the  $N$  dependence from the approximate linear expression, Eq. (42). As  $N$  increases the exact expressions approach the linear limit. When the harmonic potential is fixed while  $N$  increases, as in real experiments,  $T_0/\ln N \propto N/\ln N$ ; that is, the pseudo-critical temperature increases with  $N$ .

Figure 4: Reduced momentum distribution function  $f_k^*$  versus reduced momentum component  $k_x/\xi$  in 2D. Here  $f_k^* \equiv f_k[ma^2k_BT_0/\hbar^2]$  and we have taken  $k_y = k_x$ . The upper solid curve is the exact quantum results from Eq. (64) for a temperature above  $T_c$  and the lower one for a temperature below  $T_c$ . The dotted curves are the WKB approximations at the same temperatures as the solid curves. The WKB expression is a good approximation only if  $T > T_c$ .

Figure 5: Reduced zero-momentum distribution  $f_0^*$  versus  $T/T_0$  in 2D from Eq. (65) in 2D for  $N = 1000$ . Here  $f_k^* \equiv f_k[ma^2k_BT_0/\hbar^2]$ . A pseudo-condensate in momentum space clearly occurs for  $T < T_c \approx 0.78T_0$ .

Figure 6: Reduced zero-momentum distribution  $f_0^*$  versus  $N$  in 2D [Eq. (65)] for several temperatures below  $T_c$ . These are log-log plots, with slopes of the upper two curves numerically very close to 0.5, indicating that  $f_0 \propto \sqrt{N}$ . The lowest curve, for  $T$  just below  $T_c$ , has not yet reached its asymptotic  $N$  dependence.

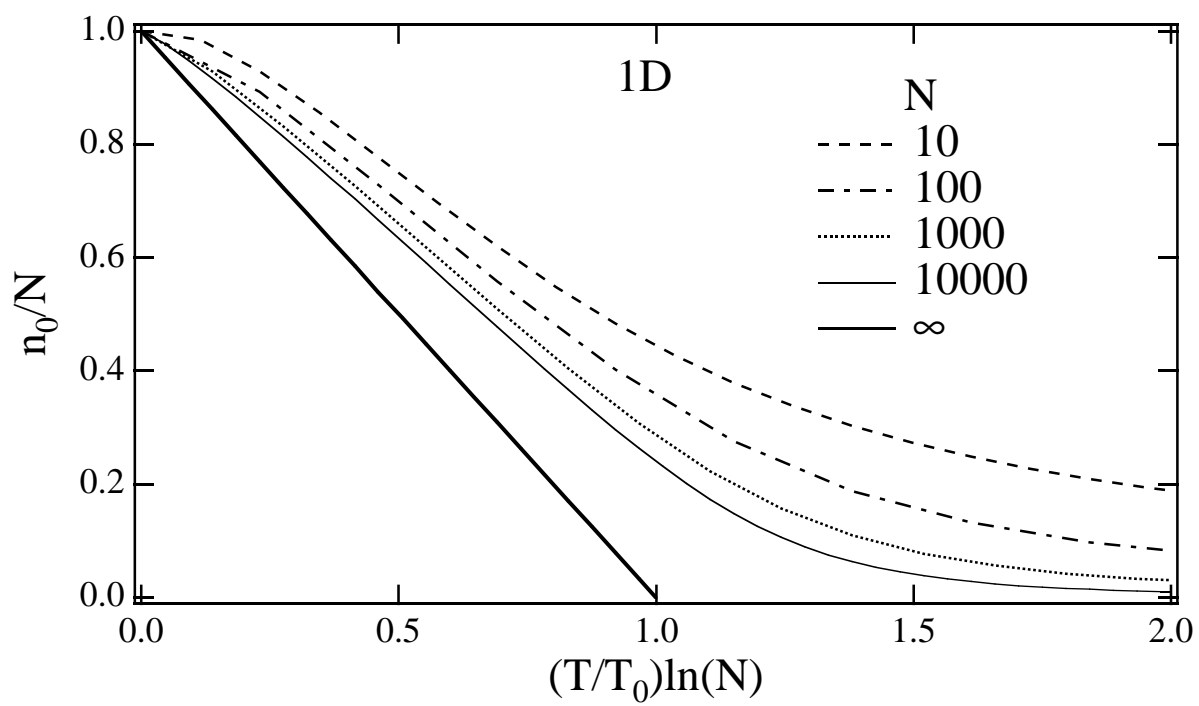


Fig. 3

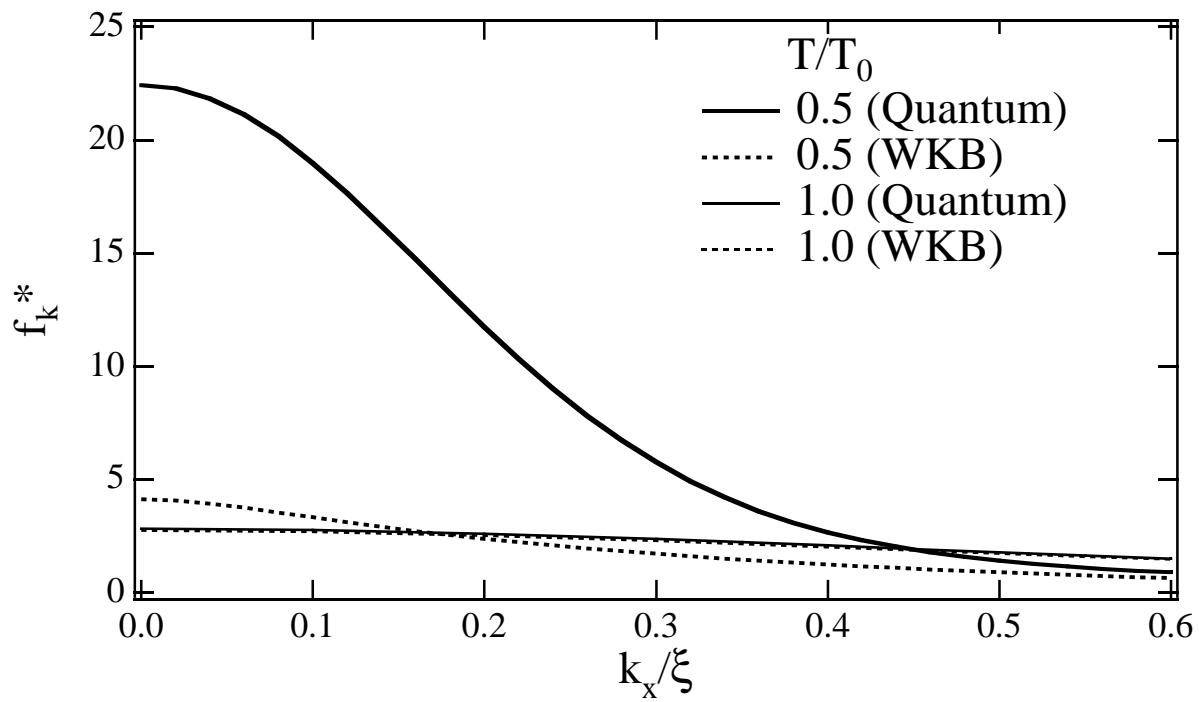


Fig. 4

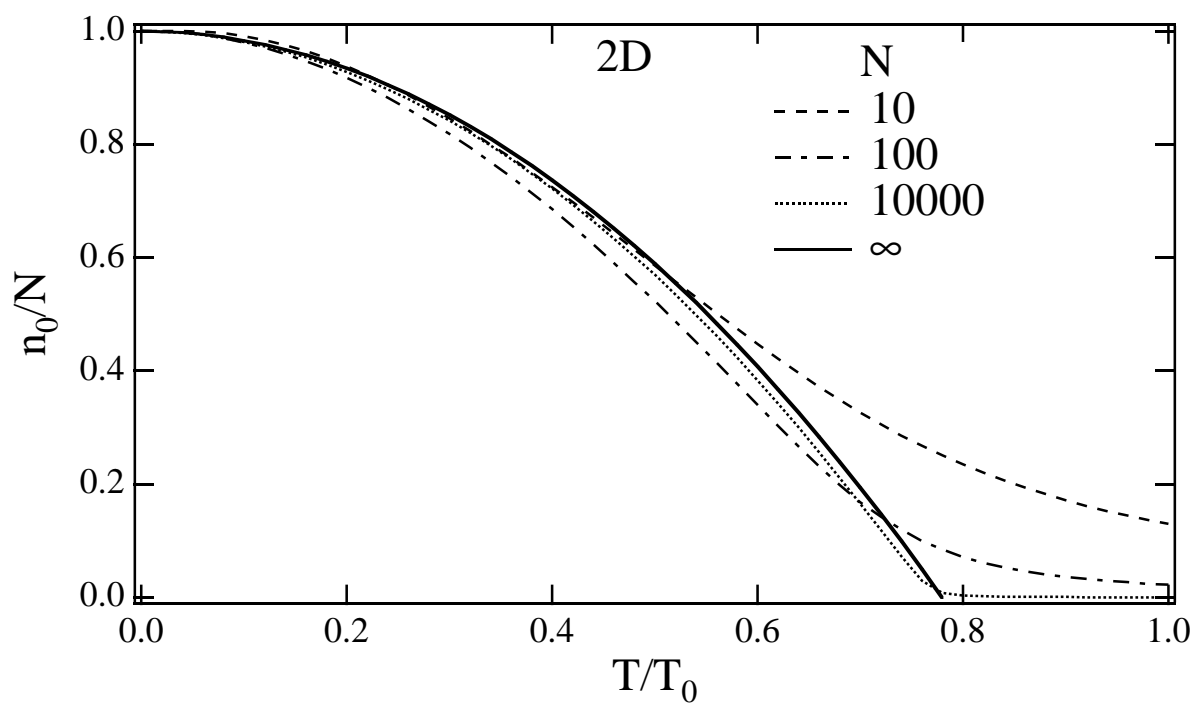


Fig. 1

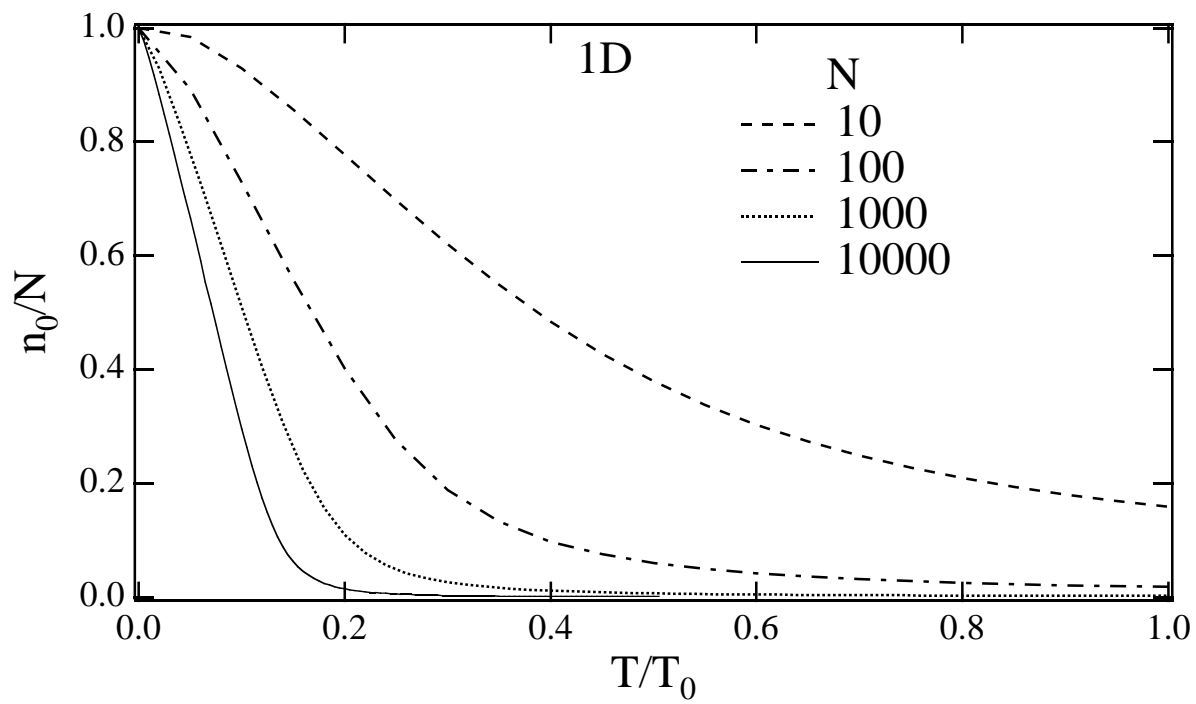


Fig. 2

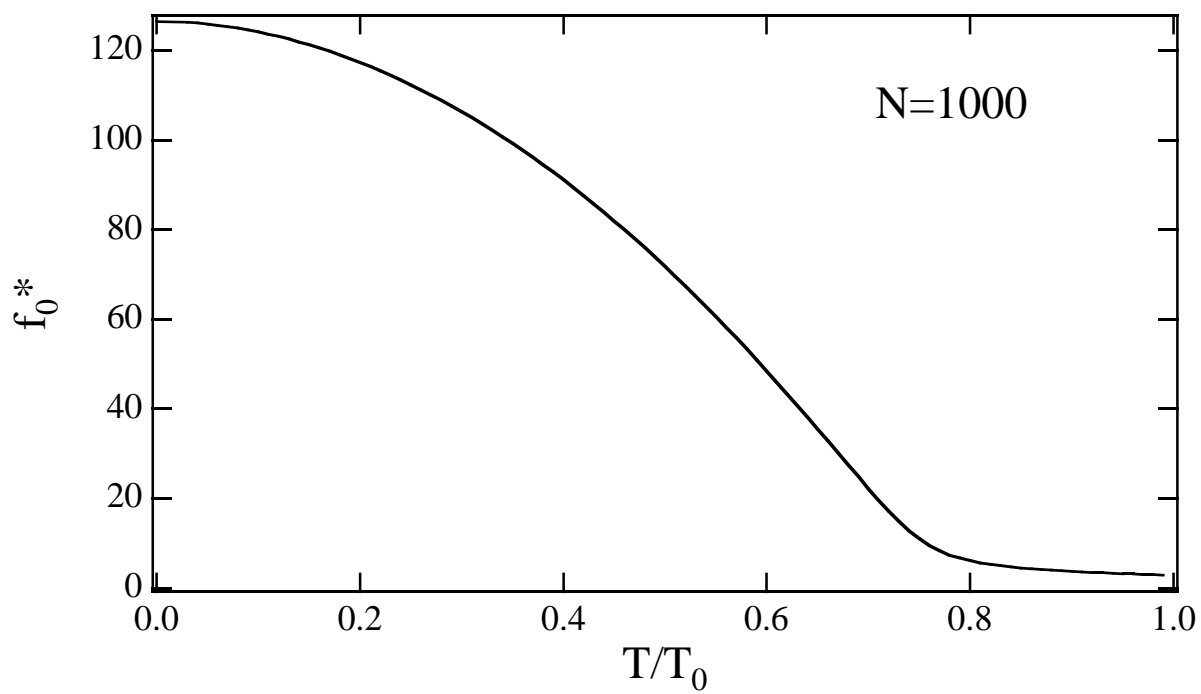


Fig. 5

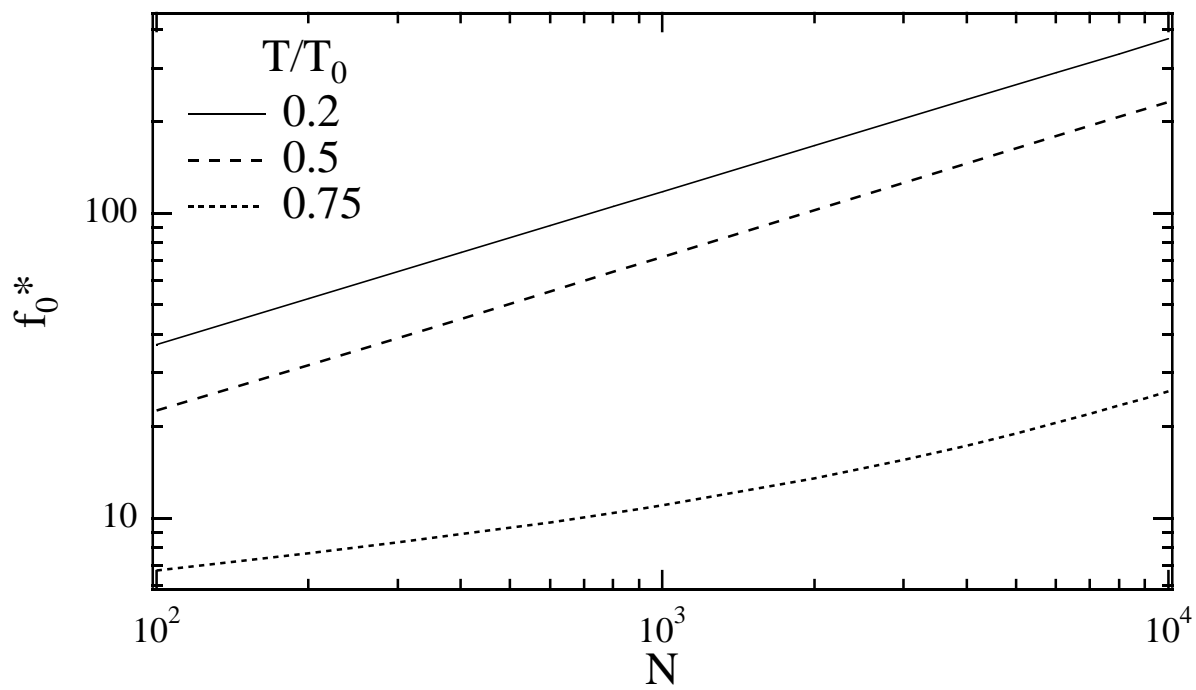


Fig. 6