Homework 2 review

1) Cor. 1.5 (c), the $\Rightarrow$ direction.

i.e., $f : A \to B$ is bijective $\Rightarrow$ f has a 2-sided inverse.

Pf. Assume f is bijective.
By Cor. 1.5(a), (b), we know,
since f is injective and surjective,

$\exists\ g_1, g_2 : B \to A$ s.t.

g_1 is a left inverse of f

and g_2 is a right inverse of f.

Prove $g_1 = g_2$.

Recall

$g_1 \circ f = 1_A$

$f \circ g_2 = 1_B$.

$$
\begin{aligned}
g_1 &= g_1 \circ 1_B && \langle \text{identity} \rangle \\
&= g_1 \circ (f \circ g_2) && \langle g_2 \text{ is right inv.} \rangle \\
&= (g_1 \circ f) \circ g_2 && \langle \circ \text{ is associative} \rangle \\
&= 1_A \circ g_2 = g_2 && \langle g_1 \text{ is left inv.} \rangle
\end{aligned}
$$

2) (1) Prove a right inv. need not be a left inv., and v.v.

Pf/ Let $A = \{0, 1\}$ and $B = \{\bullet\}$

Define $f : A \to B$ as

$$f(0) = \bullet$$
$$f(1) = \bullet$$

Define $g : B \to A$ as

$$g(\bullet) = 0$$

Claim: g is a right inv. of f, and so f is left inverse of g.

But: g is not a left inverse of f.

$$\cancel{(f \circ g)} \quad (g \circ f)(1) = 0 \neq 1,$$

f is not a right inverse of g.

$\square$

2) (2) A left inverse need not be unique.

Pf/ Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as

$$f(n) = n.$$

Define $g_1 : \mathbb{N} \to \mathbb{Z}$ as

$$g_1(0) = 0$$
$$g_1(n+1) = 1$$

and $g_2 : \mathbb{N} \to \mathbb{Z}$ as

$$g_2(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

It is clear that g_1, g_2 are both left inverses of f, but $g_1 \neq g_2$. $\square$

3) Prove (2) $\Rightarrow$ (3) in Theorem 1.9.

$$\mathbb{N} \hookrightarrow A \Rightarrow A \approx \text{some proper subset of } A.$$

Pf/ Assume $f: \mathbb{N} \hookrightarrow A$.

Let $A = \{a_0, a_1, a_2, \ldots\} \cup A'$

s.t. $a_n = f(n)$, and

$A' = A - \text{ran}(f)$.

Let $B = \{a_1, a_2, \ldots\} \cup A'$

Define ~~$g: B \to A$~~

$g: A \to B$ as

$$g(x) = \begin{cases} f(x+1) & \text{if } x \notin A' \\ x \end{cases}.$$

4) Prove: A ~~countable~~

subset of a countable set is countable.

i.e.,

Given A is countable and $B \subseteq A$,

B is countable.

By Thm 1.11(a), A is countable $\Leftrightarrow$ ~~[illegible]~~

$\exists f: A \hookrightarrow \mathbb{N}$.

Let ~~[illegible]~~

$g: B \to A$ s.t.

$g(x) = x$.

Clearly, $g: B \hookrightarrow A$.

Then $(g \circ f): B \hookrightarrow \mathbb{N}$.

Therefore, by 1.1~~[illegible]~~, B is countable.

Hwk 3 hints

1) Chase definitions!

2) $PRED(\mathbb{N}) = \{ p \mid p : \mathbb{N} \to \mathbf{2} \}.$

$TFN(\mathbb{N}) = \{ f \mid f : \mathbb{N} \to \mathbb{N} \}.$

3) Please include comments!

Note: you can use earlier programs as macros in later programs!

Macro usage:

Assuming we named program (1) ZERO(x) use it as a macro like so.

$Z \leftarrow ZERO(x)$