13. The Recursion Theorem

The following theorem, due to Kleene, is a powerful tool in computability theory. Its proof uses the S^n_m theorem, and involves a dazzling use of diagonalisation.

Theorem 13.1 (Recursion Theorem, RT). *Let g be an $(m+1)$-ary computable function. Then there is some e such that for all $\vec{x}$,*

$$\varphi_e^{(m)}(\vec{x}) \simeq g(\vec{x}, e).$$

Proof: There is some d such that for all $v, \vec{x}$

$$\begin{aligned} g(\vec{x}, S^1_m(v,v)) &\simeq \varphi_d^{(m+1)}(\vec{x}, v), \\ &\simeq \varphi^{(m)}_{S^1_m(d,v)}(\vec{x}) \qquad \text{(by } S^n_m\text{)} \end{aligned}$$

Putting $v = d$ and $e = S^1_m(d,d)$, we obtain

$$g(\vec{x}, e) \simeq \varphi_e^{(m)}(\vec{x}). \;\; \square$$

Corollary 13.2 (Fixed Point Theorem, FPT). *Let f be a total computable unary function. Then for any $m > 0$ there is some e such that*

$$\varphi_e^{(m)} = \varphi_{f(e)}^{(m)}.$$

Proof: Let

$$g(\vec{x}, z) \simeq \varphi^{(m)}_{f(z)}(\vec{x}).$$

Then g is computable by the UFT. Therefore by the RT there exists e such that for all x,

$$\varphi_e^{(m)}(\vec{x}) \simeq g(\vec{x}, e) \simeq \varphi^{(m)}_{f(e)}(\vec{x}). \;\; \square$$

Discussion of FPT.

Define the '$\sim$' relation on $\mathbb{N}$ by

$$x \sim y \;=_{df}\; \varphi_x = \varphi_y.$$

Then the FPT states (taking $m = 1$):

$$\forall f \in \mathrm{TCOMP}^{(1)}\, \exists e\, (f(e) \sim e),$$

i.e., any total computable function has an ***"extensional fixed point"***.

Note: Obviously, we cannot replace '$\sim$' by '$=$' in general.

Examples: $f(x) = x + 1$, $\quad f(x) = 2^x$, $\ldots$

More Examples:

1. There is some e such that for all x, $\varphi_e(x) = e$, i.e. there is a program which gives its own gn as output!

   This is the basic idea behind "self-reproducing programs" and viruses.

   Proof: Let $g(x, z) = z$. Clearly, $g \in$ TCOMP. By the RT there is some e such that for all x,

   $$\varphi_e(x) \simeq g(x, e) = e. \;\; \square$$

2. More generally: Take *any* total computable unary function f, for example $f(x) = x^x$. Then there is some e such that for all x,

   $$\varphi_e(x) = f(e) = e^e.$$

Exercise: Prove the result stated in Example 2.