Lemma 1.2 (a) proof

(a) f, g total $\Rightarrow g \circ f$ total.

Chase definitions.

Def. of total: $f : A \to B$ is <u>total</u> if $\operatorname{dom}(f) = A$.

Goal: ~~dom~~ Assume: $\cdot \operatorname{dom}(f) = A$

$\cdot \operatorname{dom}(g) = B$

Prove: $\operatorname{dom}(g \circ f) = A$

~~i.e. dom~~

Chase the definition again.

Recall: $\operatorname{dom}(f) = \{ x \in A \mid \exists y \in B : (x,y) \in f \}$

Proof: $\operatorname{dom}(g \circ f) = \{ x \in A \mid \exists z \in C : (x, z) \in g \circ f \}$

$= \{ x \in A \mid \exists z \in C : \exists y \in B : f(x) = y \wedge g(y) = z \}$

Difficulty we have to massage this set comprehension to be A.

Try another way.

Lemma 1.2 (a) ~~informally~~

not so informal in the end

Assume f, g total.

For any $x \in A$:

$\exists\, y_1 \in B : f(x) = y_1.$ $\langle f \text{ total} \rangle$

For any $y_2 \in B$:

$\exists\, z \in C : g(y_2) = z$ $\langle g \text{ total} \rangle$

<u>So</u> for any $x \in A$,

$\exists\, z \in C : g(f(x)) = z$

Lemma 1.2 (b) starting point

for calculational proof

f, g 1-1

$\equiv \langle$ definition of 1-1 $\rangle$

$$(\forall x, y \in dom(f) : (f(x) = f(y) \Rightarrow x = y))$$
$$\wedge (\forall x, y \in dom(g) : (g(x) = g(y) \Rightarrow x = y))$$

Ah! Too many quantifiers!
Try another way.

Lemma 1.2 (b) starting point (easier)

Goal: $g \circ f$ 1-1 $\equiv \forall x, y \in dom(g \circ f) : (g \circ f(x) = g \circ f(y) \Rightarrow x = y)$

For any $x, y \in A$:

$(g \circ f)(x) = (g \circ f)(y)$

$\equiv \langle ? \rangle$

...

$\Rightarrow \langle ? \rangle$

Some of the justifications include "f is 1-1" and "g is 1-1".

...

$\equiv \langle ? \rangle$

$x = y$

You can present less formal proofs to me.
But calculational proofs help you catch mistakes. If you can't justify a step, it might be a mistake.

Example of common mistakes: Invalid assumptions w/ infinite sets

"Assume A is infinite.
Let $A = \{x_0, x_1, x_2, \ldots\}$".

Can't always list (count) elements this way.

Cor. 1.3(b)

Goal: $A \approx B \Rightarrow B \approx A$.

Hint: use Cor. 1.5(c).

$f: A \to B$ is bijective $\Leftrightarrow$ f has a 2-sided inverse $g: B \to A$

Proof: Assume $A \approx B$.

i.e., $\exists\, f: A \to B \bullet f$ is bijective.

$\exists f [f: A \approx B]$

We prove: $B \approx A$.

i.e. $\exists\, g [g: B \approx A]$.

Let g be the 2-sided inverse of f provided by Cor. 1.5(c).

In general, avoid using higher-numbered facts (from later in the notes) in earlier proofs (of lower number facts).

This could lead to circular proofs, which would be invalid.

Only do this when it's recommended.