5. Some Techniques for Defining PR Functions

5.1 Explicit Definability

This is a convenient method for showing that certain functions are PR.

Given a sequence $\vec{g} \equiv g_1, \ldots, g_m$ of functions of arity $r_1, \ldots, r_m$, and a sequence $\vec{x} \equiv x_1, \ldots, x_n$ of *indeterminates* or *variables*, the class $\boldsymbol{Expr}(\vec{g}, \vec{x})$ of ***expressions in*** $\vec{g}$, $\vec{x}$ is defined inductively by:

1. $x_i \in \boldsymbol{Expr}(\vec{g}, \vec{x}) \quad (i = 1, \ldots, n)$,
2. $\bar{0} \in \boldsymbol{Expr}(\vec{g}, \vec{x})$, where $\bar{0}$ a symbol for the number 0,
3. If $E \in \boldsymbol{Expr}(\vec{g}, \vec{x})$, then so is $\overline{\boldsymbol{S}}(E)$,
   where $\overline{\boldsymbol{S}}$ is a symbol for the successor function $\boldsymbol{S}$,
4. If $E_1, \ldots, E_{r_i} \in \boldsymbol{Expr}(\vec{g}, \vec{x})$, so is $\bar{g}_i(E_1, \ldots, E_{r_i})$ $(i = 1, \ldots, m)$,
   where $\bar{g}_i$ is a symbol for the function g_i.

(More on inductive definitions can be found in [`Kle52`], §55.)

Since each expression in $\vec{g}$, $\vec{x}$ represents an *explicit definition* of an n-ary function, we say that f is ***explicitly definable from*** $\vec{g}$ iff $f = \lambda\vec{x} \cdot E$ for some $E \in \boldsymbol{Expr}(\vec{g}, \vec{x})$.

Examples. (1) $E \equiv \boldsymbol{exp}(y, 2) \dot{-} \boldsymbol{sqrt}(x + y)$.
Then $E \in \boldsymbol{Expr}\,(\boldsymbol{exp},\ \boldsymbol{sqrt},\ +,\ \dot{-},\ x,\ y)$.

Note that E ***defines a function relative to a list of variables*** $\vec{x}$, where $\boldsymbol{var}(E) \subseteq \vec{x}$.

In this case, take $\vec{x} \equiv (x, y)$.

Then $f = \lambda x, y \cdot E = \lambda x, y \cdot \big(\boldsymbol{exp}(y, 2) \dot{-} \boldsymbol{sqrt}(x + y)\big)$,

i.e., $f(x, y) = \boldsymbol{exp}(y, 2) \dot{-} \boldsymbol{sqrt}(x + y)$,

(2) The constant function $\boldsymbol{C}^n_k = \lambda\vec{x} \cdot k$ is explicitly defined from $\langle\rangle$ by the *numeral* $\bar{k} =_{df} \underbrace{\overline{\boldsymbol{S}}(\cdots\overline{\boldsymbol{S}}}_{k \text{ times}}(\bar{0})\cdots)$.

Note. In general we will not distinguish between functions and their symbols, or between numbers and their numerals.

Definition. $\mathrm{ED}(\vec{g})$ is the class of functions explicitly definable from $\vec{g}$.

Theorem 5.1. $\mathrm{ED}(\vec{g}) \subseteq \mathrm{PR}(\vec{g})$.
Hence if $\vec{g} \in \mathrm{PR}$, *then* $\mathrm{ED}(\vec{g}) \subseteq \mathrm{PR}$.

Proof: We must show:

$$f \textit{ expl. def. by } E \textit{ from } \vec{g} \implies f \in \mathrm{PR}(\vec{g})$$

by

- ***structural induction*** on E, or
- ***CVI*** on ***compl***(E).

(*Details in class.*) $\square$

Corollary 5.2. *In particular, we can define new PR functions from old by:*

(a) *permuting arguments, e.g.* $f(x, y) = g(y, x)$

(b) *using dummy arguments, e.g.* $f(x, y, z) = g(x, y)$

(c) *identifying arguments, e.g.* $f(x) = g(x, x)$

(d) *substituting numerals for arguments, e.g.* $f(x) = g(\bar{2}, x)$

(e) *any combination of the above.*

Proof: (a) $f \in \mathrm{PR}(\vec{g})$ since

$$f(x, y) = g(\boldsymbol{U}_2^2(x, y), \boldsymbol{U}_1^2(x, y)).$$

(b)-(e) Similarly. $\square$

EXAMPLE:

If $f(x, y, z) = g(x, h(z, k(x)), \bar{2})$, then f is explicitly definable from g, h, k. Putting $\vec{x} \equiv (x_1, x_2, x_3)$,

$$f(\vec{x}) = g(\boldsymbol{U}_1^3(\vec{x}), h(\boldsymbol{U}_3^3(\vec{x}), k(\boldsymbol{U}_1^3(\vec{x})), \boldsymbol{C}_2^3(\vec{x}))),$$

which suggests a PR-derivation of f from g, h, k.

- So from now on, we will freely use *explicit definitions*, as well as *infix* and *postfix* notation, to show that functions are PR.

More examples of PR functions:

- **Exponential** $\boldsymbol{exp}(x, y) = x^y$

  Defined by *primitive recursion on the second argument*:

$$\begin{cases} \boldsymbol{exp}(x, 0) &= 1 \\ \boldsymbol{exp}(x, y+1) &= \boldsymbol{exp}(x, y) * x. \end{cases}$$

- **Predecessor** $\boldsymbol{pd}(x) = \begin{cases} x - 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

  By primitive recursion:

$$\begin{cases} \boldsymbol{pd}(0) &= 0 \\ \boldsymbol{pd}(x+1) &= x. \end{cases}$$

- **Monus** $x \dot{-} y = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$

  By primitive recursion on the second argument:

$$\begin{cases} x \dot{-} 0 &= x \\ x \dot{-} (y+1) &= \boldsymbol{pd}(x \dot{-} y). \end{cases}$$

- **Absolute difference** $\lambda x, y \cdot |x - y|$

  By *explicit definition* from ***monus*** and ***sum*** which are both PR:

$$|x - y| = (x \dot{-} y) + (y \dot{-} x).$$

- **Zero predicate (characteristic function of 0)**

$$\boldsymbol{zero}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

By primitive recursion:

$$\begin{cases} \boldsymbol{zero}(0) &=& 1 \\ \boldsymbol{zero}(x+1) &=& 0 \end{cases}$$

or by explicit definition from $\boldsymbol{monus}$: $\boldsymbol{zero}(x) = 1 \dot{-} x$.

- **Characteristic function of positive integers**

$$\boldsymbol{pos}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

By primitive recursion:

$$\begin{cases} \boldsymbol{pos}(0) &=& 0 \\ \boldsymbol{pos}(x+1) &=& 1. \end{cases}$$

- **Equality predicate (char. fn. of equality)**

$$\boldsymbol{eq}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

By explicit definition:

$$\boldsymbol{eq}(x, y) = \boldsymbol{zero}(|x - y|).$$

- **Less-than-or-equal predicate**

$$\boldsymbol{leq}(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

By explicit definition:

$$\boldsymbol{leq}(x, y) = \boldsymbol{zero}(x \dot{-} y)$$

Theorem 5.3. *Let P and Q be n-ary predicates. Define the predicates*

$$\begin{aligned} R_1(\vec{x}) &\Leftrightarrow \neg P(\vec{x}), \\ R_2(\vec{x}) &\Leftrightarrow P(\vec{x}) \wedge Q(\vec{x}), \text{ and} \\ R_3(\vec{x}) &\Leftrightarrow P(\vec{x}) \vee Q(\vec{x}). \end{aligned}$$

Then $R_1 \in \mathrm{PR}(P)$ and R_2, $R_3 \in \mathrm{PR}(P, Q)$. More informally, the predicate $\neg P$ is PR in P, and the predicates $P \wedge Q$, and $P \vee Q$ are PR in P, Q. Hence if P, $Q \in \mathrm{PR}$, then so are $\neg P$, $P \wedge Q$, $P \vee Q$.

Proof: By explicit definition:

$$\begin{aligned} R_1(\vec{x}) &= \boldsymbol{zero}(P(\vec{x})), \\ R_2(\vec{x}) &= P(\vec{x}) * Q(\vec{x}), \text{ and} \\ R_3(\vec{x}) &= \boldsymbol{pos}(P(\vec{x}) + Q(\vec{x})). \end{aligned}$$

Alternatively for R_3, by De Morgan's law, $P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$. $\square$

Hence

- **Less predicate** $\lambda x, y \cdot x < y$

  is PR, since $x < y \Leftrightarrow \neg(y \leq x)$.

Lemma 5.4 (Definition by cases). *Define f by*

$$f(\vec{x}) \simeq \begin{cases} g(\vec{x}) & \text{if } P(\vec{x}) \\ h(\vec{x}) & \text{otherwise.} \end{cases}$$

Then $f \in \mathrm{PR}(g, h, P)$. Hence if $g, h, P \in \mathrm{PR}$, then so is f.

Proof: $f(\vec{x}) \simeq g(\vec{x}) * P(\vec{x}) + h(\vec{x}) * \boldsymbol{zero}(P(\vec{x}))$. $\square$

Lemma 5.5. *Let P be an n-ary predicate, and $f_1, \ldots, f_n$ m-ary functions. Define the predicate*

$$Q(\vec{x}) \;=\; P(f_1(\vec{x}), \ldots, f_n(\vec{x})).$$

Then $Q \in \mathrm{PR}(P, f_1, \ldots, f_n)$. Hence if $P, f_1, \ldots, f_n \in \mathrm{PR}$, then so is Q.

Proof: By composition. $\square$

Corollary 5.6. *Given m-ary functions f_1, f_2, define the predicate*

$$Q(\vec{x}) \;=\; (f_1(\vec{x}) = f_2(\vec{x})).$$

Then $Q \in \mathrm{PR}(f_1, f_2)$. Hence if $f_1, f_2 \in \mathrm{PR}$, then so is Q.

Note: In Lemma 5.5 and Cor. 5.6, if the f's are total, then Q is a predicate.

EXERCISES:

1. Does the converse of Theorem 5.1 hold, i.e., $\mathrm{PR}(\vec{g}) \;\subseteq\; \mathrm{ED}(\vec{g})$? If so, prove it. If not, state a modified result which *is* true, and prove it.

2. **(Generalised definition by cases)** Let, for some $n \geq 2$, $g_1, \ldots, g_n$ be functions and $P_1, \ldots, P_{n-1}$ predicates. For the function f, as defined below, show that $f \in \mathrm{PR}(g_1, \ldots, g_n, P_1, \ldots, P_{n-1})$. Hence if $\vec{g}, \vec{P} \in \mathrm{PR}$, then so is f. (*Hint*: Induction on n with basis $n = 2$).

$$f(\vec{x}) \simeq \begin{cases} g_1(\vec{x}) & \text{if } P_1(\vec{x}) \\ g_2(\vec{x}) & \text{if } \neg P_1(\vec{x}) \wedge P_2(\vec{x}) \\ g_3(\vec{x}) & \text{if } \neg P_1(\vec{x}) \wedge \neg P_2(\vec{x}) \wedge P_3(\vec{x}) \\ \quad\vdots & \\ g_{n-1}(\vec{x}) & \text{if } \neg P_1(\vec{x}) \wedge \cdots \wedge \neg P_{n-2}(\vec{x}) \wedge P_{n-1}(\vec{x}) \\ g_n(\vec{x}) & \text{otherwise.} \end{cases}$$

5.2 Finite sums and products.

Theorem 5.7. *Let f be an $(n{+}1)$-ary function. Let*

$$g(y,\vec{x}) = \sum_{z<y} f(z,\vec{x}), \qquad h(y,\vec{x}) = \prod_{z<y} f(z,\vec{x}).$$

Then $g, h \in \mathrm{PR}(f)$. Hence if $f \in \mathrm{PR}$, then so are g, h.

Proof: Define g, h by *primitive recursion* on y:

$$\begin{cases} g(0,\vec{x}) &=& 0 \\ g(y+1,\vec{x}) &=& g(y,\vec{x}) + f(y,\vec{x}), \end{cases}$$

and

$$\begin{cases} h(0,\vec{x}) &=& 1 \\ h(y+1,\vec{x}) &=& h(y,\vec{x}) * f(y,\vec{x}). \ \square \end{cases}$$

Corollary 5.8. *Let*

$$g'(y,\vec{x}) = \sum_{z=0}^{y} f(z,\vec{x}), \qquad h'(y,\vec{x}) = \prod_{z=0}^{y} f(z,\vec{x}).$$

Then $g', h' \in \mathrm{PR}(f)$.

Proof: $g'(y,\vec{x}) = g(y+1,\vec{x})$, and $h'(y,\vec{x}) = h(y+1,\vec{x})$. $\square$

Corollary 5.9. *Let*

$$g''(y,\vec{x}) = \sum_{z=1}^{y} f(z,\vec{x}), \qquad h''(y,\vec{x}) = \prod_{z=1}^{y} f(z,\vec{x}).$$

Then $g'', h'' \in \mathrm{PR}(f)$.

Exercise: Prove Corollary 5.9.

5.3 Bounded quantification.

Theorem 5.10. *Let P be an $(n+1)$-ary predicate. Let*

$$\begin{aligned} Q(y,\vec{x}) &= (\exists z < y)P(z,\vec{x}) \\ \text{and} \quad R(y,\vec{x}) &= (\forall z < y)P(z,\vec{x}). \end{aligned}$$

Then $Q, R \in \mathrm{PR}(P)$. Hence if $P \in \mathrm{PR}$, then so are Q and R.

Proof:

$$\begin{aligned} R(y,\vec{x}) &= \textstyle\prod_{z<y} P(z,\vec{x}) \\ \text{and} \quad Q(y,\vec{x}) &= \boldsymbol{pos}(\textstyle\sum_{z<y} P(z,\vec{x})), \end{aligned}$$

or alternatively, $Q(y,\vec{x}) \Leftrightarrow \neg((\forall z < y)\neg P(z,\vec{x}))$. $\square$

Corollary 5.11. *Let*

$$\begin{aligned} Q'(y,\vec{x}) &= (\exists z \le y)P(z,\vec{x}) \\ \text{and} \quad R'(y,\vec{x}) &= (\forall z \le y)P(z,\vec{x}). \end{aligned}$$

Then $Q', R' \in \mathrm{PR}(P)$. Hence if $P \in \mathrm{PR}$, then so are Q' and R'.

Corollary 5.12. *Let*

$$\begin{aligned} Q''(y,\vec{x}) &\simeq (\exists z < f(y,\vec{x}))P(z,\vec{x}) \\ \text{and} \quad R''(y,\vec{x}) &\simeq (\forall z < f(y,\vec{x}))P(z,\vec{x}). \end{aligned}$$

Then $Q'', R'' \in \mathrm{PR}(f, P)$. Hence if $f, P \in \mathrm{PR}$, then so are Q'' and R''.

Intuitively, ***bounded quantification*** is ***effective*** in P since there are only finitely many cases to check, whereas ***unbounded quantification*** is ***not*** (in general).

EXERCISE: Prove Corollaries 5.11 and 5.12.

5.4 Bounded minimalisation (μ).

Theorem 5.13. *Let P be an $(n{+}1)$-ary predicate. Let*

$$f(y,\vec{x}) \;=\; (\mu z < y)P(z,\vec{x}),$$

i.e., "the least $z < y$ such that $P(z,\vec{x})$ holds, if such z exists; y otherwise." Then $f \in \mathrm{PR}(P)$. Hence if $P \in \mathrm{PR}$, then so is f.

Proof: Put

$$g(y,\vec{x}) \;=\; \sum_{z<y}\prod_{t<z} \boldsymbol{zero}(P(t,\vec{x})). \tag{1}$$

Clearly, $g \in \mathrm{PR}(P)$. We will show that $f = g$. There are two cases:

- Case 1: There exists $t < y$ such that $P(t,\vec{x})$ is true, i.e. $P(t,\vec{x}) = 1$. Let t_0 be the least such t. Then, for any $t < t_0$, $P(t,\vec{x}) = 0$ so that $\boldsymbol{zero}(P(t,\vec{x})) = 1$, and $\boldsymbol{zero}(P(t_0,\vec{x})) = 0$. So for all z,

  $$\prod_{t<z} \boldsymbol{zero}(P(t,\vec{x})) \;=\; \begin{cases} 1 & \text{if } z < t_0 \\ 0 & \text{if } z \geq t_0 \end{cases}$$

  Therefore,

  $$\sum_{z<y}\prod_{t<z} \boldsymbol{zero}(P(t,\vec{x})) \;=\; \underbrace{1+\cdots+1}_{t_0 \text{ times}}+0+0+\cdots \;=\; t_0 \tag{2}$$

- Case 2: For all $t < y$, $P(t,\vec{x})$ is false, i.e. $P(t,\vec{x}) = 0$. Clearly, $\boldsymbol{zero}(P(t,\vec{x})) = 1$. So for all $z < y$,

  $$\prod_{t<z} \boldsymbol{zero}(P(t,\vec{x})) \;=\; 1.$$

  Therefore,

  $$\sum_{z<y}\prod_{t<z} \boldsymbol{zero}(P(t,\vec{x})) \;=\; \underbrace{1+\cdots+1}_{y \text{ times}} \;=\; y. \tag{3}$$

From (1), (2) and (3):

$$g(y, \vec{x}) \;=\; \begin{cases} \text{"least } z < y \text{ such that } P(z, \vec{x}) \\ \qquad \text{if such } z \text{ exists"} \;\; \text{(Case 1)} \\ y \quad \text{otherwise} \;\; \text{(Case 2).} \end{cases}$$

Hence $f = g \in \mathrm{PR}(P)$. $\square$

NOTE:

The condition that there is no $z < y$ s.t. $P(z, \vec{x})$ holds is an "*error case*" for $f(y, \vec{x})$. In this case, we set $f(\vec{x}) = y$ to indicate *without ambiguity* that no such z was found.

Corollary 5.14. *If $f(y, \vec{x}) = (\mu z \leq y) P(z, \vec{x})$, then $f \in \mathrm{PR}(P)$.*

Corollary 5.15. *If $f(y, \vec{x}) \simeq (\mu z < g(y, \vec{x})) P(z, \vec{x})$, then $f \in \mathrm{PR}(g, P)$.*

Proofs: Like Cors 5.11, 5.12 (p. 5-8). $\square$

5.5 A note on unbounded minimalisation

Let P be an $(n{+}1)$-ary predicate, and f an n-ary function defined by

$$f(\vec{x}) \;\simeq\; \mu y P(\vec{x}, y) \;\simeq\; \begin{cases} \text{the least } y \text{ s.t. } P(\vec{x}, y) \text{ holds, if such } y \text{ exists} \\ \uparrow \quad \text{otherwise} \end{cases} \tag{1}$$

Intuitively, $f \in \mathrm{EFF}(P)$ since the following algorithm, which uses an oracle for P, computes f:

"Test $P(\vec{x}, 0)$, $P(\vec{x}, 1)$, $P(\vec{x}, 2), \ldots$
until y is found such that $P(\vec{x}, y)$.
Then halt, with output y."

EXERCISE. Is f as defined in (1) in $\mathrm{PR}(P)$?

NOTES:

1. The n-ary function

$$g(\vec{x}) = \begin{cases} \mu y P(\vec{x}, y) & \text{if } \exists y P(\vec{x}, y) \\ 0 & \text{otherwise} \end{cases}$$

is total, but *not* (in general) effective in P.

2. In (1), $f \in \mathcal{G}$-COMP(P). Hence if $P \in \mathcal{G}$-COMP, then so is f. You may try to prove this now, or wait for Lemma 11.1.

5.7 More examples of PR functions and predicates

- **Integer division**

$$\begin{aligned} x \; \boldsymbol{div} \; y &= \lfloor x/y \rfloor \\ &= \mu z[z * y \leq x \wedge (z+1) * y > x] \\ &= (\mu z \leq x)[(z+1) * y > x]. \end{aligned}$$

- **Remainder**

$$x \; \boldsymbol{mod} \; y = x \mathbin{\dot{-}} (x \; \boldsymbol{div} \; y) * y.$$

- **Divisibility predicate**

$$y|x \iff x \; \boldsymbol{mod} \; y = 0,$$

or alternatively,

$$y|x \iff \exists z(x = y * z) \iff (\exists z \leq x)(x = y * z).$$

- **Primality predicate**

$$\begin{aligned} \boldsymbol{prime}(x) &\iff x > 1 \wedge \neg \exists y \, [1 < y \wedge y < x \wedge y|x] \\ &\iff x > 1 \wedge \neg(\exists y < x)[1 < y \wedge y|x]. \end{aligned}$$

- **Prime number sequence**

  Let p_n denote the n-th prime, with $p_0 = 0$. Is $\lambda n \cdot p_n \in$ PR?
  The PR definition

$$\begin{cases} p_0 & = & 0 \\ p_{n+1} & = & \mu y \, [\boldsymbol{prime}(y) \wedge y > p_n] \end{cases}$$

  is problematic as it stands, since (i) μ is unbounded, and (ii) it assumes the existence of a prime $> p_n$, or equivalently, the existence of infinitely many primes. Euclid comes to the rescue!

Theorem 5.16 (Euclid). *There are infinitely many primes.*
More precisely,

$$\forall x \exists p \, [\boldsymbol{prime}(p) \wedge x < p \leq (x! + 1)].$$

Proof: Let $y = x! + 1$.
For $2 \leq k \leq x$, $k|x!$, and so $y \;\boldsymbol{mod}\; k = 1$, and so $k \not| y$.
But y has at least one prime factor p. So $x < p \leq y$. $\square$

We can now re-define the prime number sequence as:

$$\begin{cases} p_0 & = & 0 \\ p_{n+1} & = & (\mu y \leq (p_n! + 1))[\boldsymbol{prime}(y) \wedge y > p_n] \end{cases}$$

which is PR, by Corollary 5.15.

EXERCISES:

1. Show that the following functions and predicates are PR:
   (a) $\boldsymbol{even}(x)$ (x is even)
   (b) $\boldsymbol{max}(x, y)$
   (c) $\boldsymbol{perfsq}(x)$ (x is a perfect square)
   (d) $\boldsymbol{sqrt}(x)$ (integral square root of x)
   (e) $\boldsymbol{gcd}(x, y)$.

2. For any unary function f, define

$$g(n, x) \simeq f^n(x)$$

(the n-th iterated composition of f). Is $g \in \mathrm{PR}(f)$?

3. (a) Show that every finite subset of $\mathbb{N}$ is PR.
(A set or relation is said to be PR if its char. pred. is PR.)

(b) Is every co-finite subset of $\mathbb{N}$ PR?
(A set is co-finite if its complement is finite.)

4. Prove the following function is PR:

$$\boldsymbol{d}(n, k) = (k+1)\text{st digit in decimal expansion of } 1/n,$$

e.g., $1/7 = 0.142\ldots$, so $\boldsymbol{d}(7,0) = 1$, $\boldsymbol{d}(7,1) = 4$, $\boldsymbol{d}(7,2) = 2$.