8 Columns

The purpose of this section is to provide guidance on the analysis of columns. A column is a structural member whose cross-sectional dimensions are small with respect to its length. These members are primarily subjected to axial compressive stresses only, although they may sometimes have direct or induced bending stresses present.

A column is considered to have failed when it stops fulfilling the function for which it was designed. Failure, for columns, does not necessarily imply yielding of material, permanent damage or catastrophic failure. Column failure is more related to the stability of the member than it is to the stress or the cross-sectional area of the column; however, the geometry of the member is the primary predictor of what type of compression failure will occur. The geometrical characteristics of a column are summarized in a parameter known as the slenderness ratio, L′/ρ, or effective column length divided by the beam's applicable radius of gyration.

The effective length, L′, is a function of how the column is restrained at its ends and the effect this restraint has on the deflected shape of the column. For a pin ended column which can rotate freely at the ends, the effective length is the actual column length and the deflection is as shown in Figure 8.0.0-1(a). If, however, a compressive member is fixed against rotation at both ends that fixity will modify the deflected shape curve as shown in Figure 8.0.0-1(b) where at some point along its length there is a reversal in curvature, called the point of contraflexure. In the case of a fully fixed column, the points of contraflexure are at the quarter points of the column length. Since at the points of contraflexure, there is no curvature and no bending moment, it behaves as if it were a pin ended column. The effective length describes the length of the column which is behaving as a pin ended column due to the fixity provided by the supports. This is described in detail in Section 8.2.1.2.

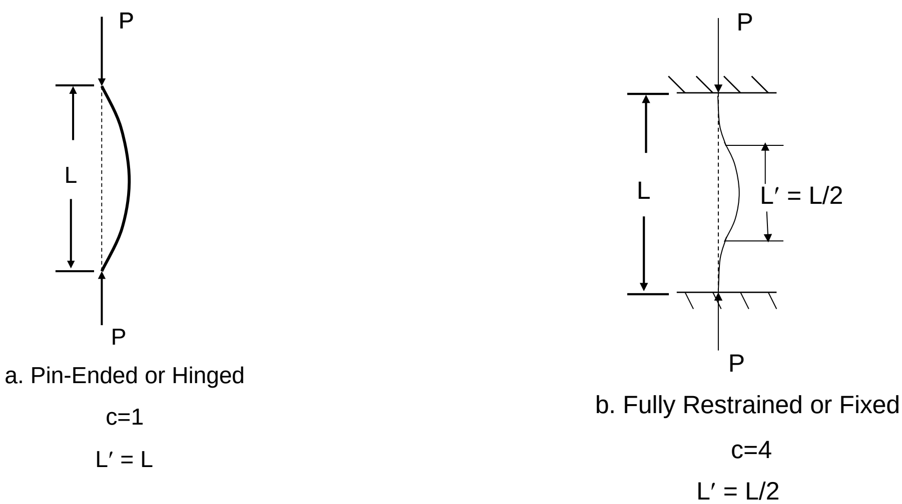

Figure 8.0.0-1 Column Deflected Shapes

The radius of gyration is given by

Equation 8.0.0-1

$$\rho = \sqrt{\frac{I_{min}}{A}}$$

where
I_{min} is the minimum moment of inertia of the section (in^4)
A is the area of the cross section (in^2)

In general, by determining the slenderness ratio of the column it is possible to predict the type of failure that column will experience. There are three ranges of slenderness ratio and types of failure – short, intermediate and long column. Additionally, any column with thin flanges may also fail due to local instability; *i.e.*, flange buckling or crippling. The four modes of failure are represented graphically in Figure 8.0.0-2 as a function of the slenderness ratio, L′/ρ.

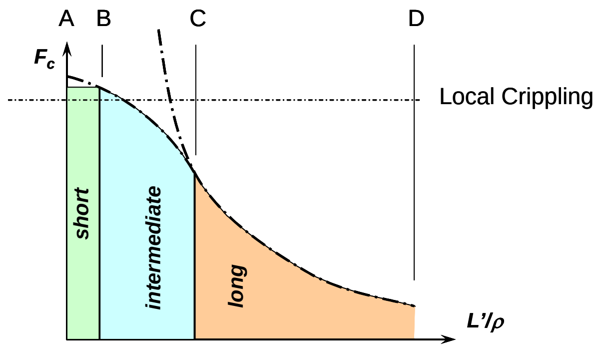

Figure 8.0.0-2 Column Stability Curve

The column fails by inelastic buckling from Points A to C, and elastic buckling from Points C to D. In the low L′/ρ, high stress region of the curve (Points A to B), thin-walled sections typically fail by local crippling or, for open sections, by torsional instability. In the absence of program guidance, it is recommended that the maximum allowable stress for these failure modes be limited to F_{cy}.

Some legacy Lockheed Martin programs limit the upper end of the column, crippling and torsional instability curves to F_{cmax}. Although this stress is a function of stiffness, not strength, the member is considered to have failed when its average stress reaches F_{cmax}. This value is defined as the minimum of the stress when the column has lost 95% of its stiffness or F_{tu}. See Section 3.3.3.3.2 for details on the calculation of F_{cmax}.

Analysts should adhere to specific program policy regarding the cutoff limit for allowable compression stress (F_{cy} or F_{cmax}).

Note that, when formulating the weighted-average crippling or buckling load for thin-walled sections, each individual element is limited to its applicable cutoff stress. Ensuring only that the weighted-average stress for the built-up section remains below the cutoff stress many lead to unconservative results.

Intermediate columns are members with an L′/ρ sufficiently small that the average stress on the cross-section reaches its elastic limit before the critical buckling load is reached. After small lateral deflections occur, the column will reach the conditions of instability associated with total collapse. The critical buckling stress can be determined by the use of the tangent modulus, E_{tan}, in Euler's equation. This will be discussed in detail in Section 8.2.

Long columns have high slenderness ratios and will fail before the average cross-section stress exceeds the elastic limit of the material. This type of column fails through the lack of stiffness instead of through the lack of strength. It should be noted that if the column is perfectly straight, the load is truly axial and the material perfectly homogenous, the column would remain straight under any value of load. If the column is slightly deflected laterally, the critical buckling load is that load which will hold the column in the slightly bent position. The significant characteristic of buckling is that the elastic deflections and stress are not proportional to the loads as buckling takes place; *i.e.*, a small increase in load can result in a significant increase in deflection. The slenderness ratio of long columns will be greater than the critical slenderness ratio, Point C of Figure 8.0.0-2. The critical buckling stress can be determined by the use of Euler's equation. This will be discussed in detail in Section 8.2.

Local crippling or secondary failure is any type of failure in which the cross sections are distorted in their own planes, but not translated or rotated. This type of failure is limited to thin-walled sections. This cutoff is shown notionally in Figure 8.0.0-2 as local crippling; however, its magnitude is a function of the geometry of the column's cross section. This type of failure will be discussed in detail in Section 8.4.

Another failure mode of thin walled open sections is lateral instability. This is a primary failure mode since the beam translates or rotates but there is no local distortion within the plane of the cross section. This failure mode is discussed in Section 8.5.

For any beam or column loaded in compression, lateral loads, or with applied moments the failure modes and analyses shown in Table 8.0.0-1 should be considered, as applicable.

Table 8.0.0-1 Summary of Analysis Requirements for Beams or Columns loaded in Compression

Failure Mode/Analysis	Applied Load	Applicability
Column Stability	Axial Compression	All Columns
Torsional Instability (Twisting Buckling)	Axial Compression	Thin Walled Open Sections
Lateral Instability (Sideways Buckling)	Transverse Loads or Moments	Tall Narrow Beams where lateral rigidity < flexural rigidity
Beam Column Analysis	Axial Compression PLUS Lateral Loads or Moment	All columns with both types of directly applied load shown or with directly applied axial load and an eccentricity that induces an additional bending load. Note: This is a Strength Check
Flange Buckling	Axial Compression and/or Bending resulting in a flange loaded in compression	Thin Walled Open Sections
Crippling	Axial Compression and/or Bending resulting in a flange loaded in compression	Thin Walled Sections (Open or Closed)
Fcy	Axial Compression and/or Bending	All Columns

Below is a summary of the organization of this section:

8.1 References and Nomenclature

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Table 8.1.0-1 Nomenclature

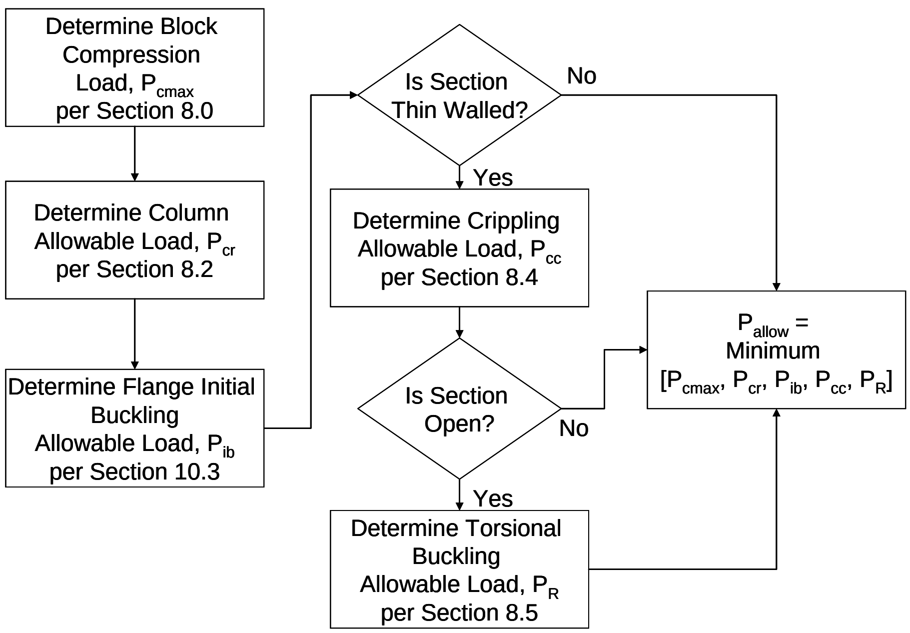
Symbol	Description	Units
A	Cross section area	in^2
A_1, A_2, A_3	Intermediate calculation constants. Reference Section 8.5.3	--
a	Length	in
a	Effective area	in^2
B	Ratio of support stiffness to column stiffness	--
B_o, B_s	Constants used in the calculation of St. Venant Torsion constant for sheet	--
b	Length	in
b	flange width	in
b	stiffener spacing	in
C_1, C_2	Beam column constants	--
C_w	Warping constant of the Section (given as Γ in some references) about the Section shear center	in^6
$C_{w,c}$	Warping constant of the Section (given as Γ in some references) about the Section Centroid	in^6
$C_{w\text{-}T}$	Warping constant of the stiffener and effective sheet about the center of torsion	in^6
c	Column end fixity coefficient	--
D	Diameter	in
D_o	Outer diameter	in
E	Modulus of Elasticity	psi
e	strain	in/in
e	Eccentricity	in
E_c	Compressive Young's Modulus	psi
E_{sec}	Secant Modulus	psi
E_{tan}	Tangent Modulus	psi
F	Stress	psi
f_o	Trapezoidal intercept stress	psi
$F_{0.05T}$	Stress at which a column has lost 95% of its initial stiffness	psi
F_{cc}	Crippling (or collapse) stress	psi
F_{cmax}	Material maximum allowable compression stress	psi
F_{col}	Column flexural buckling stress	psi
F_{cp}	Material compression proportional limit	psi
F_{cr}	Critical initial buckling stress	psi
F_{cy}	Material compression yield strength allowable	psi
F_{IR}	Inter-rivet Buckling Stress	psi
F_{tu}	Material Ultimate tensile strength allowable	psi
G_{tan}	Tangent Shear Modulus	psi
I	Moment of Inertia of the section	in^4
I_{max}, I_{min}	Moments of Inertia about the maximum and minimum principal axes, respectively	in^4
I_p	Polar moment of Inertia about the Section Centroid	in^4
$I_{p\text{-}sc}$	Polar moment of Inertia about the Section Shear Center	in^4
$I_{p\text{-}T}$	Polar moment of the stiffener and effective sheet about the center of torsion	in^4
I_{xy}	Product of Inertia	in^4
j	Ratio of beam stiffness to applied load	in
K	Spring stiffness	lb/in or in-lb/rad
K	St. Venant Torsional Constant (given as J in some references)	in^4

K	Cylinder Buckling Coefficient	
K_{cc}	Coefficient for Maximum Stress for a flat plate buckled in edge compression	--
k_c	Theoretical buckling constant	--
K_{ecc}	Reduction factor to account for unintentional cylinder imperfections in long cylinders	
k_x	Axial compression buckling coefficient of cylinder	
L	Length	in
ℓ	Length	in
L', L_e	Effective length	in
L'/ρ	Slenderness Ratio	--
M	Moment	in-lb
M_o	Moment resulting from direct application of transverse load	in-lb
m	Column buckling coefficient (beams with variable cross section)	--
$(mc/I)_b$	Bending modulus (Reference Section 6)	psi
M.S.	Margin of Safety	--
n	number of adjacent radii	--
n_c	Material Ramberg-Osgood number in compression	--
n_t	Material Ramberg-Osgood number in tension	--
NEF	No edges free (Crippling)	--
OEF	One edge free (Crippling)	--
P,p	Applied load	lb
P'	Applied axial load	lb
P_θ	Critical load for twisting failure of a column about the section shear center	lb
$P_{\theta,c}$	Critical load for twisting failure of a column calculated about the section centroid	lb
P_{cmax}	Column maximum allowable compression load	lb
P_{cr}	Column flexural buckling load	lb
P_{ib}	Column member initial buckling load	lb
P_{IR}	Inter-rivet Buckling Load	lb
P_{cc}	Column crippling load	lb
P_R	Column combined torsional, flexural instability load	lb
Q	Intermediate calculation constant. Reference Section 8.5.3	--
q	applied running shear load	lb/in
R	Mid-plane Radius	in
R	Warping moment of the section Reference Section 4.2.7	in^5
R	Intermediate calculation constant. Reference Section 8.5.3	--
R_{bend}	Bend radius	in
r	length fraction	--
r_o	outer radius of cylinder	in
s	spacing or distance	in
t	Thickness	in
V	Beam shear	lb
V_o	Beam Shear resulting from direct application of transverse load	lb
W	Applied point load	lb
w	Applied running load	lb/in
w	warping displacements	in^2
x	Arbitrary location on beam	in
y	Deflection or deformation	in
Z	Curvature parameter	
	Greek Symbols	

Symbol	Description	Units
α	Additional contribution of twist due to calculations made about the centroid rather than the shear center (Torsional Instability)	
β	Stiffness Parameter	1/in^2
δ	Deflection	in
ϕ	Slope	radians
ζ	correlation factor for cylinder buckling	
η	Plasticity reduction factor	--
η	Beam column magnification factor	--
ν	elastic Poisson’s ratio	--
ρ	Radius of gyration	in
ψ	Shell thickness parameter	
Subscripts and Superscripts		
Symbol	Description	
1,2, etc.	Defines which element is being considered	--
a,b,c, etc	Pertaining to Element a, b, c, etc.	--
allow	Allowable	--
appl	Applied	--
avg	Average	--
bulb	Flange bulb	--
c	Compression	--
E	Elastic	--
f	Flange	--
-i	An arbitrary element	--
ir	inter-rivet	--
larger	Pertaining to the larger feature	--
max	Maximum	--
min	Minimum	--
p	plastic	--
sh, sheet	Pertaining to the sheet	--
smaller	Pertaining to the smaller feature	--
stiffener	Pertaining to the stiffener	--
t	Tension	--
tot	Total	--
x or xx	in the y-z plane or about the x axis	--
y or yy	in the x-z plane or about the y axis	--

8.2 Basic Theory and Graphical Results

The stability allowable of a given member is the minimum of all possible failure modes which include the maximum block compression stress, column buckling , crippling, flange buckling, and, if applicable, torsional instability. Each of these failure modes, except for flange buckling, is discussed in Section 8. Flange buckling checks would be accomplished using analysis methods from Section 10. Additionally, a beam-column analysis would need to be performed, as appropriate, when lateral loading, eccentricity, or both are present. If there is no designed-in eccentricity, an assumed eccentricity may be appropriate. Refer to discussions in Sections 2.3.2.3.1 and 8.3.1.3 for further information. Figure 8.2.0-1 provides a flowchart for column analysis. A margin of safety can be written for each failure mode and the minimum margin selected, or the column allowables can be determined for each failure mode and the minimum selected for use in the margin of safety calculation.



In addition to above analysis, use Sections 2.3.2.3.1 and 8.3.1.3 to determine if a Beam Column analysis is required.

Figure 8.2.0-1 Column Analysis Flowchart

8.2.1 Constant Cross Section

The critical elastic buckling load, P_{cr}, of a constant cross-section column under axial compression is given by Euler's equation

$$P_{cr} = c\pi^2 E_c I/(L)^2 \qquad \textbf{Equation 8.2.1-1}$$

where
E_c is the Modulus of Elasticity of the material in compression (psi)
L is the column length (in)
I is the moment of inertia of the column cross section (in^4)
c is the end fixity coefficient

This mode of buckling is a bending of the column while the cross-section remains undistorted. This type of failure is called a primary instability mode.

In the equation above, the total length of the column, L, is used along with an end fixity coefficient. The end fixity coefficient is often used to reduce the actual, physical length of the column used in Equation 8.2.1-1 to an effective length, L′, to account for the effect of the end constraints on the deflection of the column. This is discussed in detail in Section 8.2.1.2 and Equation 8.2.1-9.

The critical, elastic buckling stress for a column is given by

$$F_{col} = \pi^2 E_c/(L'/\rho)^2$$ **Equation 8.2.1-2**

where
ρ is the radius of gyration given by Equation 8.0.0-1 (in)
L′ is the effective length of the column (in)

Here the transformation from load to stress introduced the radius of gyration. Euler assumed the stress-strain curve for the material is a straight line and, hence, that the stiffness, E, of the material remains constant as the load, P, increases to the Euler column load, P_E. This assumption limits the applicability of Equation 8.2.1-1 to elastic buckling, or buckling below the proportional limit of the material, and further limits the geometry to columns having a slenderness ratio above a certain value depending on the properties of the material, putting them into the long column category discussed in Section 8.0.

To determine the minimum elastic slenderness ratio, $(L'/\rho)_E$, Equation 8.2.1-2 is solved using a stress equal to the proportional limit of the material given by Equation 3.3.1-12, as follows

$$F_{cp} = F_{cy}(0.05)^{(1/n_c)}$$ Reference Equation 3.3.1-12

$$\left(\frac{L'}{\rho}\right)_E = \sqrt{\frac{\pi^2 E_c}{F_{cp}}}$$

where
F_{cp} is the material’s proportional limit stress (psi)
F_{cy} is the material’s compression yield stress (psi) from Reference 8-4 or other suitable source
n_c is the compression Ramberg-Osgood number from Reference 8-4 or other suitable source
E_c is the compressive elastic modulus (psi) from Reference 8-4 or other suitable source.

If the slenderness ratio of the column is greater than $(L'/\rho)_E$ then the material in the column has a constant stiffness, E, and the column behaves as an elastic column, hence this is referred to as the critical slenderness ratio.

When the slenderness ratio has a value such that the stress (P_E/A) is equal to the material’s elastic limit, any increase in load will cause a rapid decrease in stiffness in the column from the value of E to E_{tan}, the tangent modulus. The tangent modulus, as described in Section 3.3.2, is the slope of the stress strain curve at any location. In this case, it is the slope of the stress strain curve at the inelastic stress caused by increasing P slightly above the elastic limit. This decrease in stiffness permits the column to deflect nonlinearly and results in a large deflection with small increase in load.

The failure load and stress in the inelastic region are calculated from

$$P_{cr} = \pi^2 E_{tan} I/(L')^2$$ **Equation 8.2.1-3**

$$F_{col} = \pi^2 E_{tan}/(L'/\rho)^2$$ **Equation 8.2.1-4**

where
E_{tan} is the tangent modulus of the material in compression (psi)

The tangent modulus is often formulated as a reduction factor, η_{tan}, times the elastic modulus as shown below. In the elastic range the tangent modulus is equal to the elastic modulus and η_{tan} is equal to 1.0. The reduction factor is derived from Equation 3.3.1-8 and is also shown below.

$$E_{tan} = \eta_{tan} E_c$$ **Equation 8.2.1-5**

$$\eta_{\tan} = \frac{1}{1 + 0.002 \dfrac{E_c n_c}{F_{cy}} \left(\dfrac{F}{F_{cy}}\right)^{(n_c - 1)}}$$ **Equation 8.2.1-6**

where
n_c is the Ramberg-Osgood number in compression
F_{cy} is the allowable compression yield stress of the material (psi)
F is the stress at which the tangent modulus is being evaluated (psi)

Because the tangent modulus is a function of the stress and the stress is a function of the slenderness ratio of the column, the solution to Equation 8.2.1-4 is an iterative trial-and-error process. This is done by assuming a stress level, F, and calculating the tangent modulus using Equation 8.2.1-5 and 8.2.1-6. Then Equation 8.2.1-4 is solved for L'/ρ, using the tangent modulus and the assumed stress level, F. This is done for several different assumed stress ratios to bracket the L'/ρ of the column under analysis. Alternatively, material-specific column allowable curves can be generated, using the same approach, and used to determine the allowable for a specific slenderness ratio. This process is illustrated in the example problem of Section 8.2.1.1. Equation 8.2.1-3 can be used for the entire range of the stress-strain curve and a separate calculation using Equation 8.2.1-1 is not necessary. This is valid because the tangent modulus represents the slope of the stress-strain curve so that in the straight line portion prior to the proportional limit of the material, the tangent modulus and the elastic modulus have the same value.

A sample set of column curves is provided in Section 8.2.4 for a variety of materials. Column curves may be obtained for any material in the IDAT material library from the IDAT program SM83 under the metals tab.

Textbooks often refer to the Johnson curve for use in the intermediate length column region. This is not used in this manual. The formulation using the tangent modulus has proven, by testing of many different materials, to provide an overall better solution to the column buckling problem. For reference, the plot presented in association with the example in Section 8.2.1.1 also includes the Johnson curve.

If the column is buckling in the elastic range, the critical load can be determined directly from Equation 8.2.1-1 or Equation 8.2.1-3 with $\eta = 1.0$; however, if the buckling is inelastic, then it is more convenient to iterate on the column stress and once the allowable column stress is determined, the critical load in the column can be calculated from

$$P_{cr} = F_{col} A$$ **Equation 8.2.1-7**

where
F_{col} is the buckling stress obtained from iteration of Equation 8.2.1-3 (psi)
A is the gross cross sectional area (in^2)

In general, the gross area is used in compression calculations because filled fastener holes would transfer any compression loads. An exception would be the presence of lightening holes or other open holes. In that case the area used would be a net area and, in addition other analyses may need to be performed to ensure the presence of a large hole doesn't further reduce the load-carrying capability of the part in compression.

The margin of safety can be calculated from

$$M.S. = \frac{P_{cr}}{P} - 1$$ **Equation 8.2.1-8**

where
P_{cr} is the allowable column load determined from Equation 8.2.1-7 (lb)
P is the applied column compression load (lb)

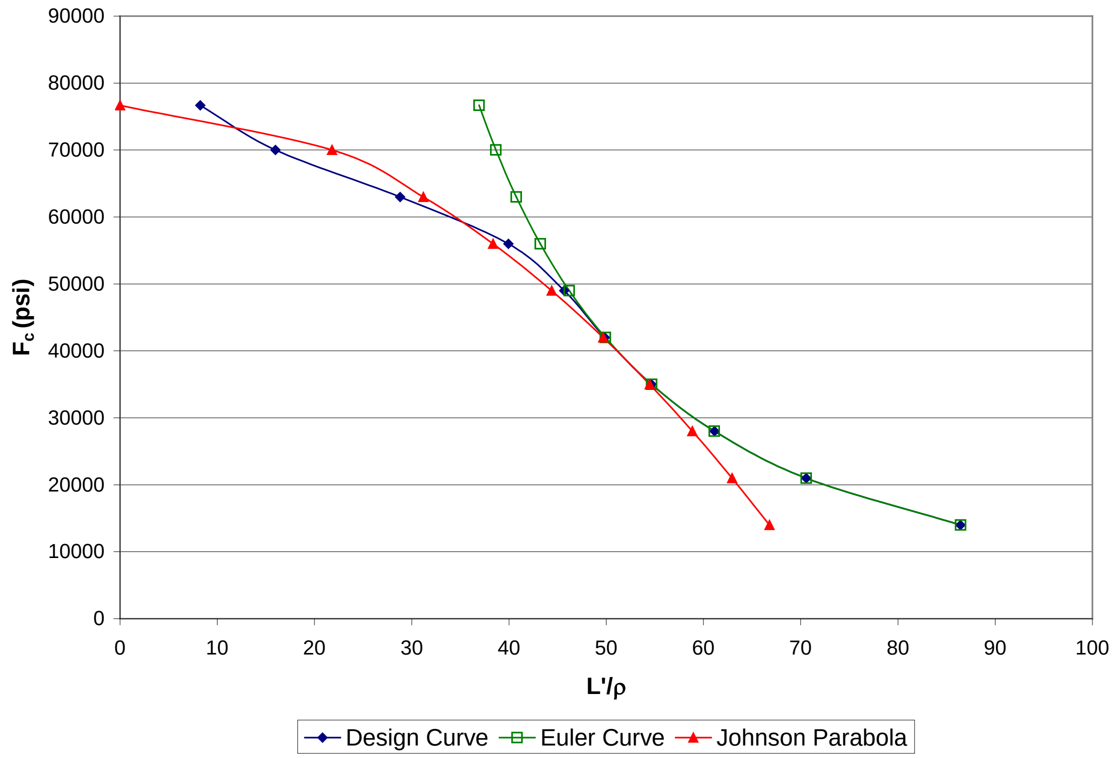
Always use the tangent modulus to determine the column buckling allowable.

Also note that in addition to column buckling, if the column is in the short or intermediate column range, other checks such as block compression, crippling and torsional instability may need to be performed. These will be discussed in later sections.

8.2.1.1 Example Problem – Constant Cross-Section Column

Given: 7075-T651 Plate 0.5-1.00 in thick. (Table 3.7.7.0(b_1), Reference 8-4) F_{tu} = 79000 psi F_{cy} = 70000 psi $E_c = 10.6x10^6$ psi n_c = 16 (Figure 3.7.7.1.6(h), Reference 8-4) Create a column buckling curve for $10 \le L'/\rho \le 80$				
Calculate $F_{0.05T}$ (Ref Eqn 8.0.0-3)	$F_{0.05T} = F_{cy}\left(\left[\frac{1}{0.05}-1\right]\frac{F_{cy}}{0.002E_c n_c}\right)^{\frac{1}{n_c-1}}$ $F_{0.05T} = 70000\left(\left[\frac{1}{0.05}-1\right]\frac{70000}{0.002(10.6x10^6)(16)}\right)^{\frac{1}{16-1}} = 76676$			$F_{0.05T}$= 76676 psi
Calculate F_{cmax}	F_{cmax}=Minimum[F_{tu},$F_{0.05T}$] = Minimum[79000,76676]			F_{cmax}=76676 psi
	F	η_t	E_{tan}	L'/ρ
		$\dfrac{1}{1+0.002\dfrac{E_c n_c}{F_{cy}}\left(\dfrac{F}{F_{cy}}\right)^{(n_c-1)}}$	$\eta_t E_c$	$\sqrt{\pi^2 E_{tan}/F}$
F_{cmax}	76676	0.0500	530000	8.26
F_{ty}	70000	0.1711	1813660	15.99
Even increments of stress	63000	0.5000	5300000	28.81
	56000	0.8543	9055580	39.94
	49000	0.9775	10361500	45.68
	42000	0.9977	10575620	49.85
	35000	0.9999	10598940	54.66
	28000	1.0000	10600000	61.13
	21000	1.0000	10600000	70.58
	14000	1.0000	10600000	86.44
Plot F vs. L'/ρ. Also shown on plot (but not calculated here) are the standard Euler and Johnson Curves.[1]				

[1] Johnson Curve: $F_c=F_{co}-F_{co}^2(L'/\rho)^2/(4\pi^2E_c)$



8.2.1.2 End Fixity

In the discussion of Section 8.2.1, the parameter L′ is used and described as the effective column length, but no information is provided on how to calculate its value. In the simplest form, the effective length is the actual length of the column which assumes that the column is hinged at both ends so that it can rotate freely. Such a condition occasionally exists in aircraft structure where a member is connected by a single bolt at each end. It also might exist where there is a shear clip that ties the column to adjacent structure, but in many cases the compression member is attached in such a way that the ends are restrained against rotation.

If a compressive member is rigidly fixed against rotation at both ends the deflection curve for elastic buckling will have the shape shown in the center illustration of Figure 8.2-1. At the quarter points of the fixed-fixed column, there will be a point of reverse curvature or points of contraflexure. At points of contraflexure there is no curvature and hence no bending moment. The portion of the column between points of contraflexure may therefore be treated as a pin-ended column and the length, L, is reduced to L′.

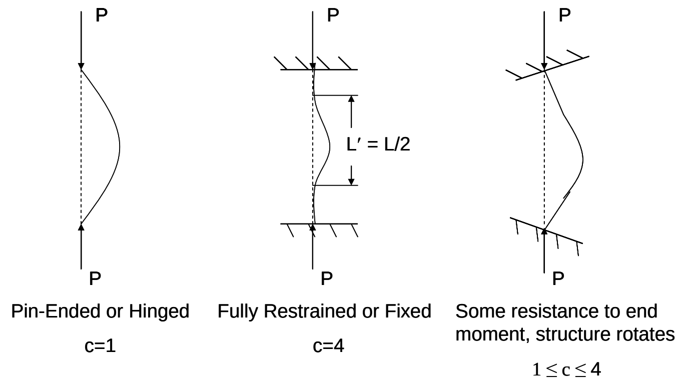


Figure 8.2.1-1 Illustration of End-Fixity

The end fixity term, c, is used to reduce the length of the beam to the effective length as follows

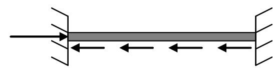
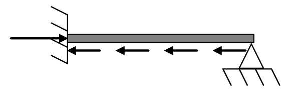
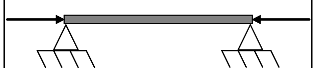
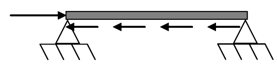
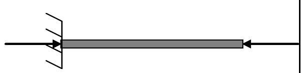
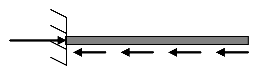
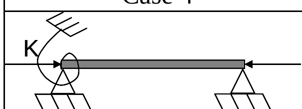
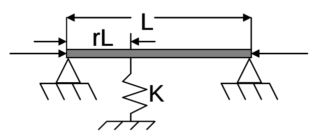
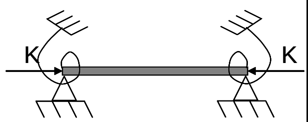
$$L' = \frac{L}{\sqrt{c}}$$ **Equation 8.2.1-9**

where
L is the total length of the column (in)
c is the end fixity coefficient

In the long column range, the end fixity coefficient is 4 for a fully restrained column. This indicates the fixed-end column resists 4 times the axial load of the pin-ended column. In the intermediate or short column range this relationship does not hold since the tangent modulus, E_{tan}, is also getting smaller.

Most practical columns have end conditions somewhere between hinged and fixed. The ends might be rigidly attached to a structure which deflects and permits the ends to rotate. The true end-fixity conditions can seldom be determined exactly and often conservative assumptions have to be made. However, most aerospace structural elements are in the short-to-intermediate column range where the end fixity effects on the critical loads are much smaller. Some common end fixities are provided in Table 8.2.1-1 for columns with constant cross section. In some cases the end fixities are applicable in both the elastic and plastic ranges; however, in other cases the end fixity coefficients are only intended for use in the elastic range or below the proportional limit of the material. The applicability of each case is noted in the table.

Table 8.2.1-1 End Fixity Coefficients for Constant Cross Section Columns

Type of Fixity	Applicability	c	Type of Fixity		c
 Case 1	Elastic and Plastic	4	 Case 7	Elastic Only	7.5
 Case 2	Elastic and Plastic	2.05	 Case 8	Elastic Only	6.08 (appx.)
 Case 3	Elastic and Plastic	1.0	 Case 9	Elastic Only	1.87
 Case 4	Elastic and Plastic	0.25	 Case 10	Elastic Only	0.794
 Case 5	Elastic and Plastic	Figure 8.2.1-2	 Case 11	Elastic and Plastic	Figures 8.2.1-4, 8.2.1-5 and 8.2.1-6
 Case 6	Elastic and Plastic	Figure 8.2.1-2			

The end fixity constraints for Cases 5, 6 and 11 are a function of the stiffness of the supports relative to the stiffness of the column and, for Case 11, the location of the support. Treatment of these cases is according to the following.

The spring constant, K, represents the restraint offered to the column by adjacent structure and the attachment scheme. This can be theoretically calculated, determined by an FEA approach, or measured in test, with some test verification being desirable. In its simplest form, the spring stiffness is the applied force divided by the deflection of the structure resulting from that force.

$$K = P/\delta \text{ or } K = M/\phi$$

where
K is the axial stiffness (lb/in) or rotational stiffness (in-lb/rad)
P is the applied force (lb)
δ is the deflection (in)
M is the applied moment (in-lb)
ϕ is the slope (radians)

Figure 8.2.1-2 provides curves to determine c for columns with intermediate fixity elastic restraint as shown in Cases 5 and 6 of Table 8.2.1-1. The end fixity curves are plotted in Figure 8.2.1-2 as a function of the variable, β. which represents the relative stiffness of the elastic end support, K, to the rotational stiffness of the column and is given by

$$\beta = \frac{KL}{EI}$$

where
K is the rotational stiffness of the elastic end support (in-lb/rad)
L is the length of the column (in)
E is E_c or E_{tan}, for elastic or plastic column buckling, respectively (psi)
I is the moment of inertia of the column (in^4)

Note if tangent modulus is used, since tangent modulus is a function of the stress level, an iterative solution will be required.

Figure 8.2.1-3 illustrates the end fixity rotational stiffness required to reduce the effective column length. This curve is from the same equation as is plotted in Figure 8.2.1-2 for Case 5, but as a function of the ratio of the effective length to the actual length. From an examination of the plot, it can be seen that very little restraint is required to reduced the effective column length from a pinned-pinned configuration; however, it is obvious that full fixity is quite difficult to achieve.

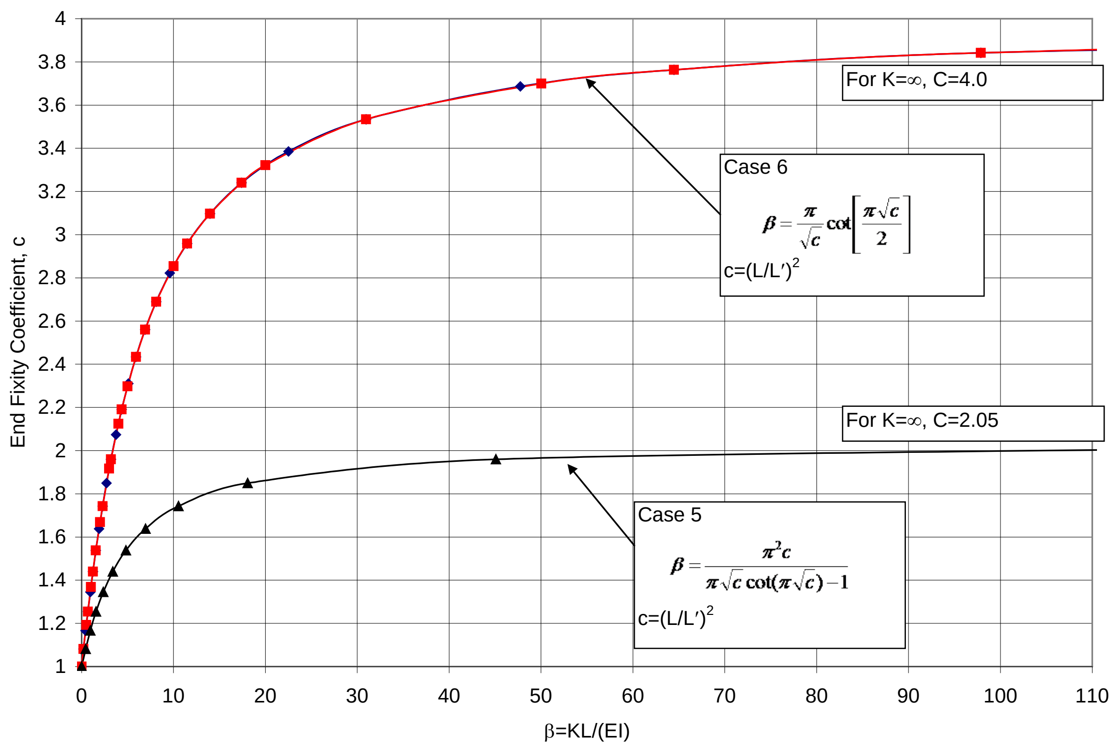


Figure 8.2.1-2 End Fixity Coefficient for Columns with End Supports Providing Elastic Moment Restraint

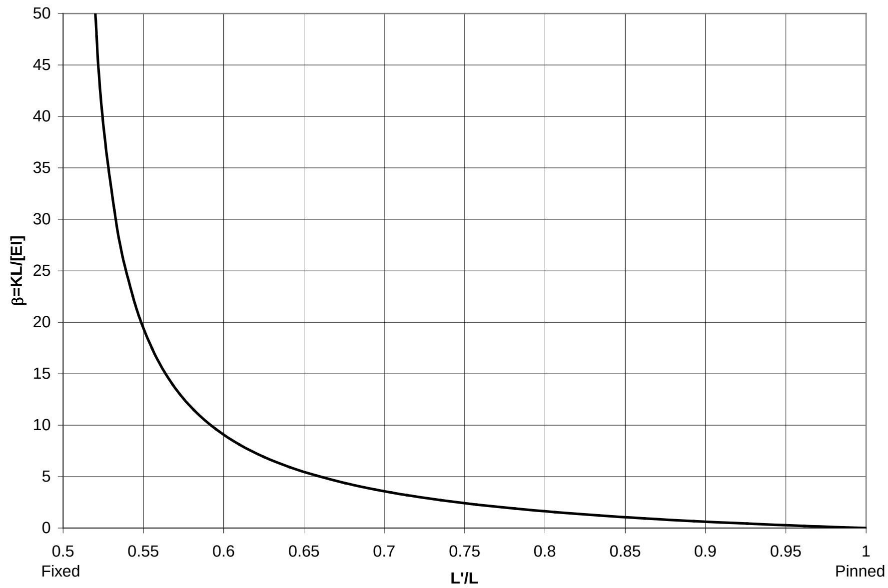


Figure 8.2.1-3 Effective Length Ratio vs. Stiffness

Figure 8.2.1-4 provides curves for the determination of c for elastic columns which are pinned-pinned but with an intermediate axial elastic support as shown in Table 8.2.1-1, Case 11. There are separate curves representing different stiffness values for the intermediate support. The variable B is used to represent the ratio of support stiffness to column stiffness and is given by

$$B = \frac{KL^3}{EI}$$

where
K is the stiffness of the intermediate support (lb/in)
L is the total length of the column (in)
E is the elastic modulus or the tangent modulus of the column, as appropriate (psi)
I is the moment of inertia of the column (in^4)

If the intermediate support has a relative stiffness approaching 0, the column behaves pinned-pinned. If the intermediate support has a relative stiffness approaching infinity and the support is very near one end (r = 0) the column behaves as a single pinned-fixed column. The spring stiffness of the intermediate support should be carefully determined and preferably verified either by experiment or by finite element model. In the absence of verification, a conservative value should be calculated.

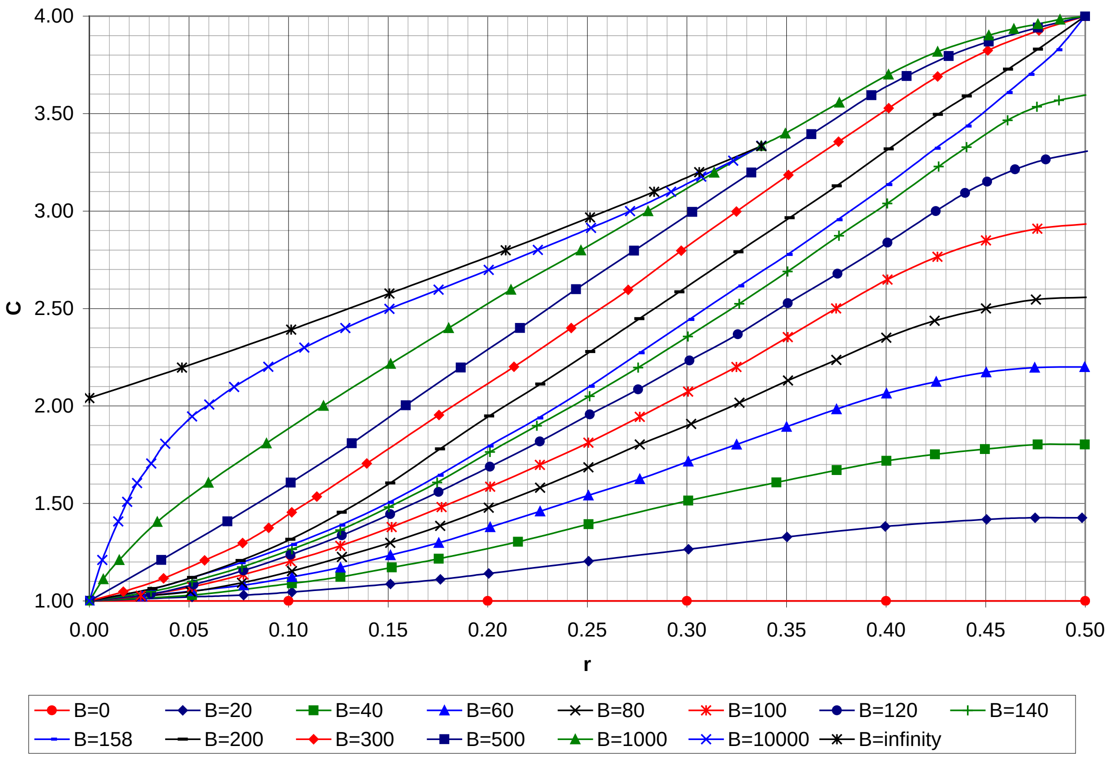


Figure 8.2.1-4 End Fixity Coefficient for Pinned-Pinned Columns with a Single Intermediate Elastic Support, Case 11, Table 8.2.1-1

While Figure 8.2.1-4 is useful in understanding and visualizing how the end fixity coefficient varies with the stiffness ratio, B and the location of the support, the same data are plotted using constant r curves in Figures 8.2.1-5 and 8.2.1-6. This form makes the iterative process required for the inelastic buckling region significantly easier. Example 8.2.1-5 will illustrate the technique for using these curves.

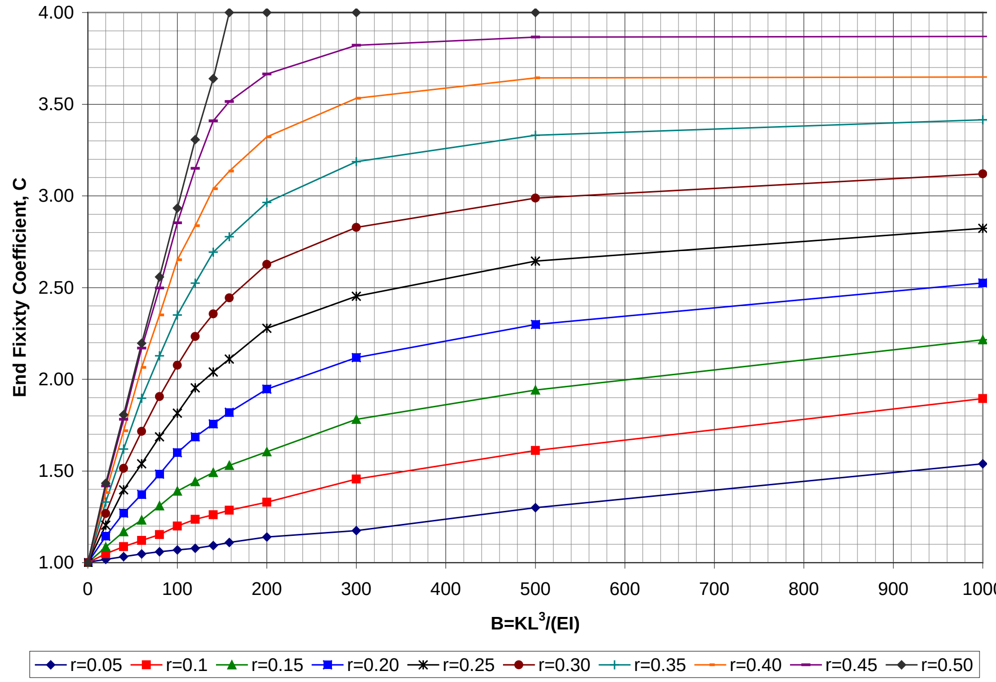


Figure 8.2.1-5 End Fixity Coefficient for Pinned-Pinned Columns with a Single Intermediate, Elastic Support, Case 11, Table 8.2.1-1 for B < 1000

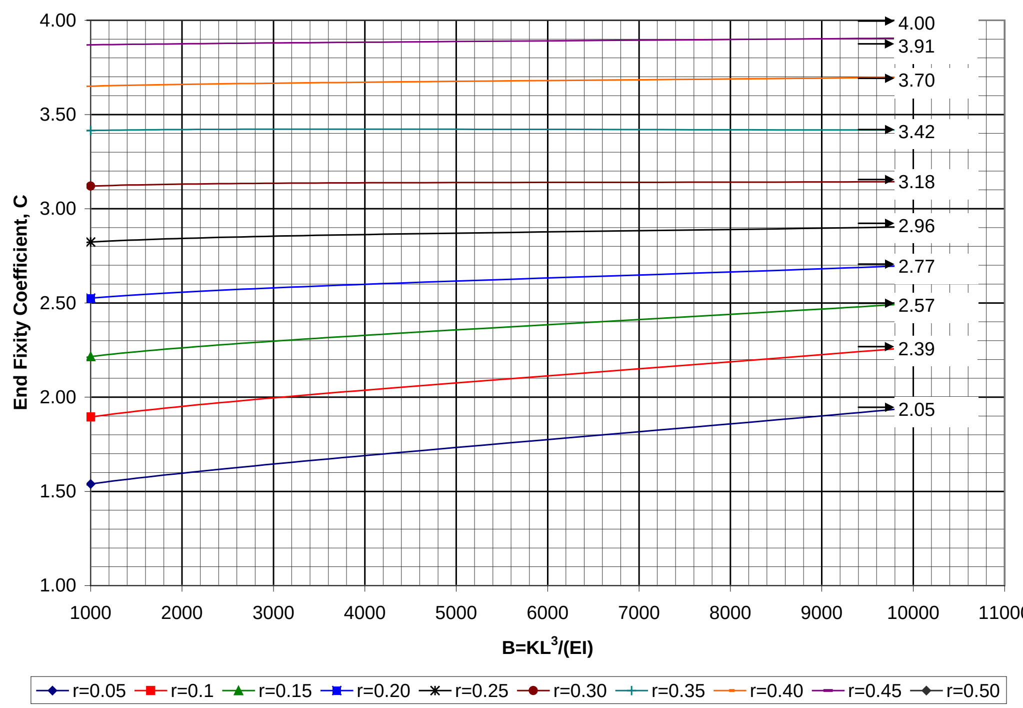


Figure 8.2.1-6 End Fixity Coefficient for Pinned-Pinned Columns with a Single Intermediate, Elastic Support, Case 11, Table 8.2.1-1 for B > 500

In some applications, columns are loaded both through shear and axial loading as illustrated in Figure 8.2.1-7. Figure 8.2.1-8 shows a plot of effective end fixity coefficients for columns loaded both axially and in shear as a function of the ratio of shear to applied axial load, P′. Table 8.2.1-3 provides curve fit equations for these curves.

These end fixity coefficients are only applicable in the elastic range of the material.

The column buckling load is determined using the elastic modulus and c determined from Table 8.2.1-3 for the appropriate loading and end constraints. This can be done either directly from Equation 8.2.1-1 or indirectly from the column buckling stress of Equation 8.2.1-2. Once the column buckling load has been determined, the margin of safety can be written using the total axial load reaction, P.

$$P = P' + qL \qquad \textbf{Equation 8.2.1-10}$$

$$M.S. = \frac{P_{cr}}{P} - 1 \qquad \textbf{Equation 8.2.1-11}$$

where
P′ is the applied axial load (lbs)
q is the applied running shear load (lb/in)
L is the total column length (in)
P_{cr} is the allowable column buckling load determined from Equation 8.2.1-1 or 8.2.1-2.

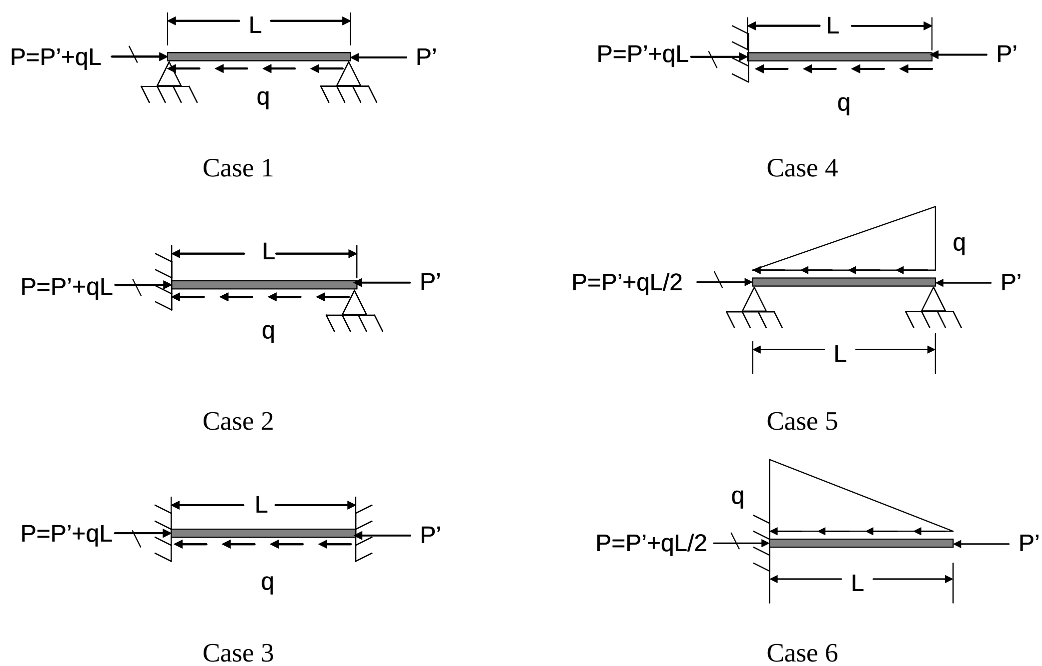


Figure 8.2.1-7 Combined Axial and Shear Loaded Column Geometry and Parameters

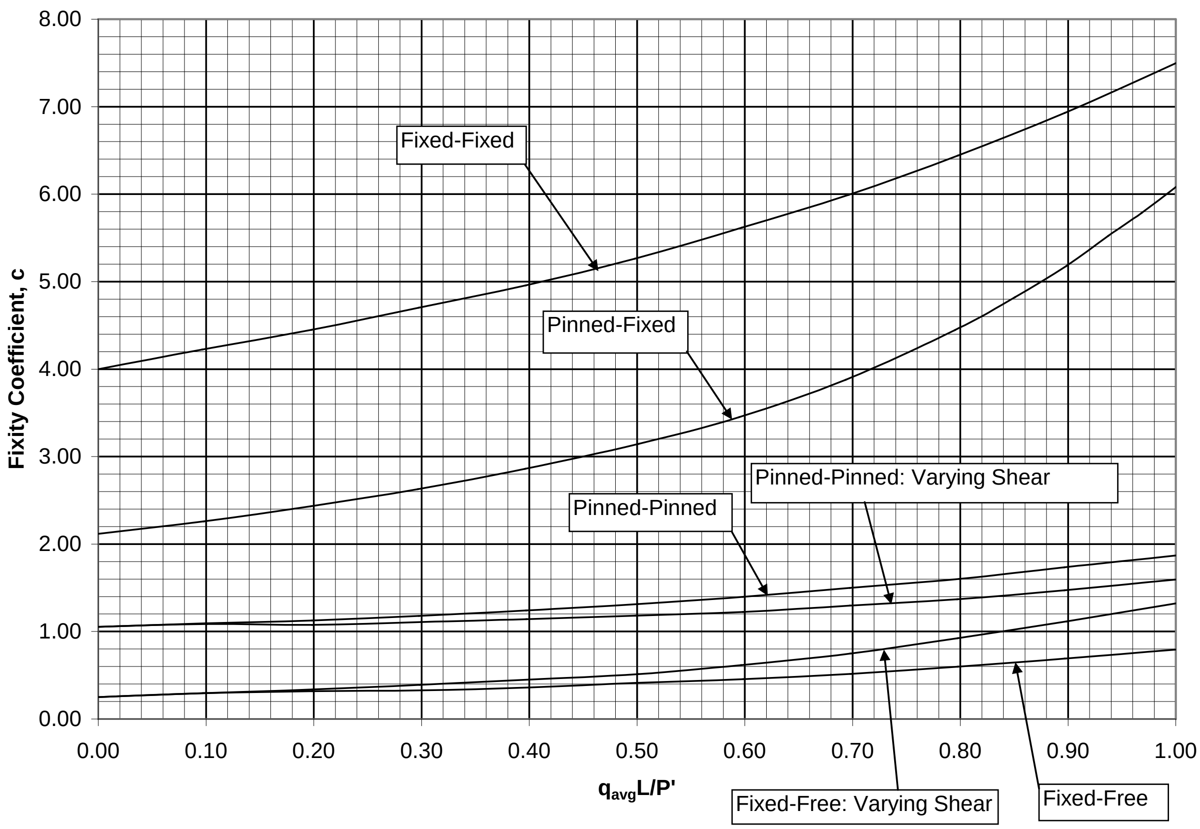


Figure 8.2.1-8 End Fixity Coefficients for Columns Loaded in Combined Shear and Axial Loading

Table 8.2.1-3 Fixity Coefficients for Shear Loaded Columns

$x = q_{avg} L/P'$ $0.0 \leq x \leq 1.0$	
where q_{avg} is the average shear running load (lb/in) L is the total column length (in) P′ is the applied axial load (lb)	
End Constraint	c
Pinned-Pinned, Case 1	$c = 0.134982x^3 + 0.393818x^2 + 0.288494x + 1.055461$
Fixed-Pinned, Case 2	$c = 3.690161x^3 - 1.668933x^2 + 1.958186x + 2.099683$
Fixed-Fixed, Case 3	$c = 1.074757x^3 + 0.282579x^2 + 2.137104x + 4.006007$
Fixed-Free, Case 4	$c = 0.393771x^3 - 0.108402x^2 + 0.255179x + 0.259000$
Pinned-Pinned, Variable Shear, Case 5	$c = 0.365486x^3 + 0.042399x^2 + 0.127590x + 1.058077$
Pinned-Free, Variable Shear, Case 6	$c = 0.686832x^3 + 0.036915x^2 + 0.352769x + 0.256120$

8.2.1.3 Example Problem – End Fixity – Constant Cross Section with Axial Load

Given a square beam of cross section 0.5 in. X 0.5 in, 10 inches long. It is made from 7075-T651 Plate 0.50-1.0 in thick (Reference Example 8.2.1.1)

Allowable curve of Example 8.2.1.1 has been extended to higher L′/ρ and plotted in more detail in Figure 8.2.1-9 for use in the solution of this problem.

Calculate the Margin of Safety if the column is loaded axially with an 8000 lb load.

Assume the following end fixities: 1) Pinned-Pinned 2)Fixed-Pinned 3)Fixed-Fixed 4)Fixed-Free

Compare the results

	Pinned-Pinned c=1.0	Fixed-Pinned c=2.05	Fixed-Fixed c=4.0	Fixed-Free c=0.25
Calculate Area A=wh	A=(0.5)(0.5) = 0.25 in^2	A=(0.5)(0.5) = 0.25 in^2	A=(0.5)(0.5) = 0.25 in^2	A=(0.5)(0.5) = 0.25 in^2
Calculate I $I=bh^3/12$	$I=(0.5)(0.5)^3/12$ =0.005208 in^4	$I=(0.5)(0.5)^3/12$ =0.005208 in^4	$I=(0.5)(0.5)^3/12$ =0.005208 in^4	$I=(0.5)(0.5)^3/12$ =0.005208 in^4
Calculate $\rho = (I/A)^{0.5}$	ρ =√(0.00502/0.25) =0.14434 in	ρ = √(0.00502/0.25) =0.14434 in	ρ = √(0.00502/0.25) =0.14434 in	ρ = √(0.00502/0.25) =0.14434 in
Calculate L′=L/√c	L′=10/√(1) = 10	L′=10/√(2.05) = 6.984	L′=10/√(4) = 5	L′=10/√(0.25) = 20
Calculate L′/ρ	L′/ρ=10/0.14434 =69.28	L′/ρ=6.984/0.14434 =48.39	L′/ρ=5/0.14434 =34.64	L′/ρ=20/0.14434 =138.56
Determine F_{col} – Figure 8.2-4	22050 psi	44000 psi	59600 psi	5800 psi
Calculate the allowable column load, $P_{cr}=F_{col}A$	P_{cr} = (22050)(0.25) = 5513 lb	P_{cr} = (44000)(0.25) = 11000 lb	P_{cr} = (59600)(0.25) = 14900 lb	P_{cr} = (5800)(0.25) = 1450 lb

Calculate the Margin of Safety M.S. = P_{cr}/P -1	MS = 5513/8000-1 = -0.31	MS = 11000/8000-1 = 0.38	MS = 14900/8000-1 = 0.86	MS = 1450/8000-1 = -0.82

Note that a comparison of the slenderness ratios for the fixed-fixed and fixed-free are a ratio of 4:1; however the column allowable for the fixed-fixed is approximately 10 times that of the fixed-free.

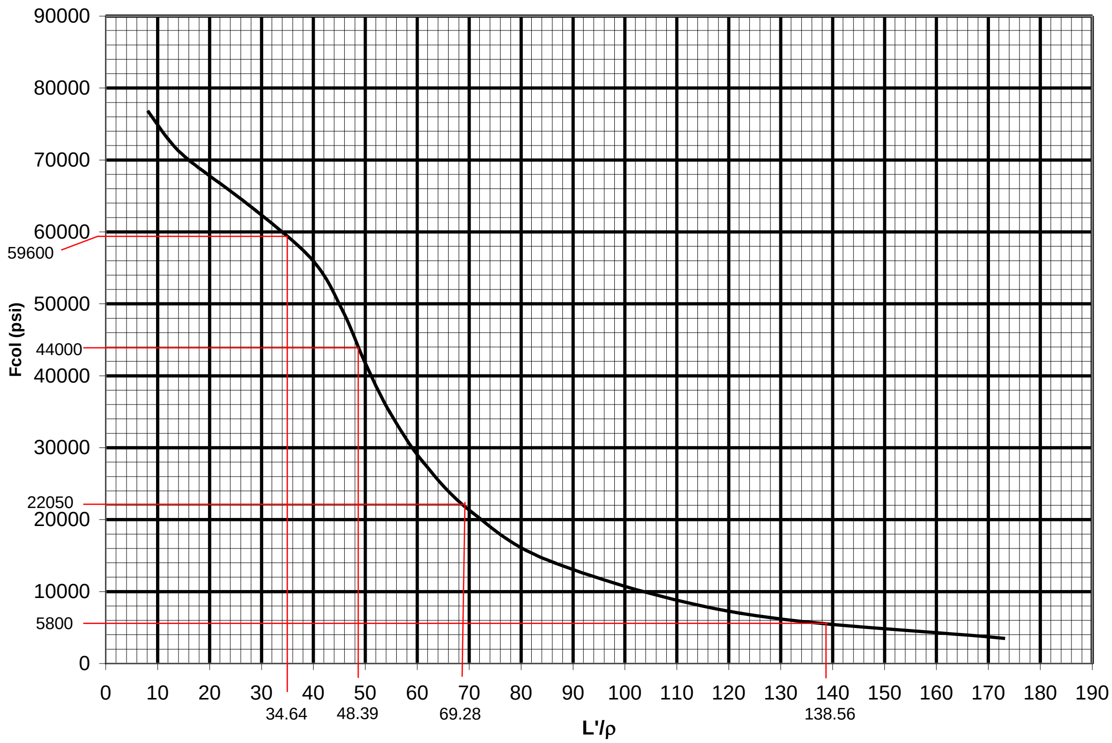


Figure 8.2.1-9 Sample Column Allowable Curve for Aluminum 7075-T651 Plate 0.5-1.0 in – Example 8.2.1.3 and 8.2.1.4

8.2.1.4 Example Problem – End Fixity – Constant Cross Section with Axial and Shear Load

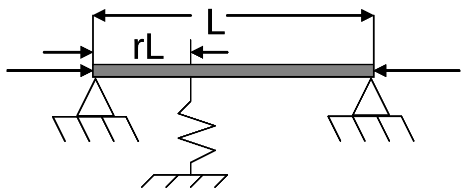
Given a square beam of cross section 0.5 in. X 0.5 in, 10 inches long. It is made from 7075-T651 Plate 0.50-1.0 in thick (Reference Example 8.2.1.1) Calculate the Margin of Safety if the column is loaded axially with an 100 lb/in shear load and a 7000 lb axial load that are reacted as an axial load at the end of the beam. Assume the following end fixities: 1) Pinned-Pinned 2)Fixed-Pinned 3)Fixed-Fixed 4)Fixed-Free Compare the results with Example 8.2.1.3				
These margins will only be valid if the material is below the proportional limit of the material Calculate the proportional limit stress: $F_{cp}=F_{cy}\ (0.05)^{(1/nc)} = (70000)(0.05)^{(1/16)} = 58048$ psi Calculate $(L'/\rho)_E = \sqrt{(\pi^2 E_c/F_{cp})} = \sqrt{[(\pi^2)(10.6\text{x}10^6)/58048]} = 42.45$; thus all $L'/\rho > 42.45$ will be valid solutions.				
	Pinned-Pinned	Fixed-Pinned	Fixed-Fixed	Fixed-Free
Calculate Area A=wh	A=0.5(0.5) = 0.25 in^2	A=0.5(0.5) = 0.25 in^2	A=0.5(0.5) = 0.25 in^2	A=0.5(0.5) = 0.25 in^2
Calculate I $I=bxh^3/12$	$I=0.5(0.5)^3/12$ =0.005208 in^4	$I=0.5(0.5)^3/12$ =0.005208 in^4	$I=0.5(0.5)^3/12$ =0.005208 in^4	$I=0.5(0.5)^3/12$ =0.005208 in^4
Calculate ρ = $(I/A)^{0.5}$	$\rho = \sqrt{(0.00502/0.25)}$ =0.14434 in	$\rho = \sqrt{(0.00502/0.25)}$ =0.14434 in	$\rho = \sqrt{(0.00502/0.25)}$ =0.14434 in	$\rho = \sqrt{(0.00502/0.25)}$ =0.14434 in
Calculate $q_{avg}L/P'$	$q_{avg}L/P'$ =100(10/7000) = 0.1429	$q_{avg}L/P'$ =100(10/7000) = 0.1429	$q_{avg}L/P'$ =100(10/7000) = 0.1429	$q_{avg}L/P'$ =100(10/7000) = 0.1429

Calculate c	$0.134982(0.1429)^3+$ $0.393818(0.1429)^2$ $+0.288494(0.1429)$ $+1.055461 = 1.105$	$3.690161(0.1429)^3-$ $1.668933(0.1429)^2$ $+1.958186(0.1429)$ $+2.099683 = 2.356$	$1.074757(0.1429)^3+$ $0.282579(0.1429)^2$ $+2.137104(0.1429)$ $+4.006007 = 4.320$	$0.393771(0.1429)^3-$ $0.108402(0.1429)^2$ $+0.255179(0.1429)$ $+0.259000 = 0.2944$
Calculate $L'=L/\sqrt{c}$	$L'=10/\sqrt{(1.105)}$ $= 9.513$	$L'=10/\sqrt{(2.356)}$ $= 6.515$	$L'=10/\sqrt{(4.320)}$ $= 4.811$	$L'=10/\sqrt{(0.2944)}$ $= 18.430$
Calculate L'/ρ	$L'/\rho=9.513/0.14434$ $=65.91$	$L'/\rho=6.515/0.14434$ $=45.14$	$L'/\rho=4.811/0.14434$ $=33.33$	$L'/\rho=18.430/0.14434$ $=127.68$
Is $L'/\rho > (L'/\rho)_E$ – if so, c is valid.	65.91>42.45 Valid	45.14>42.45 Valid/Borderline	33.33<42.45 Not valid	127.68>42.45 Valid
Calculate $P_{cr}=\pi^2E_cI/(L')^2$	$\pi^2(10.6x10^6)(0.005208)$ $/(9.513)^2$ 6020 lb	$\pi^2(10.6x10^6)(0.005208)$ $/(6.515)^2$ 12836 lb	--	$\pi^2(10.6x10^6)(0.005208)$ $/(18.430)^2$ 1604 lb
Calculate $P=P'+q_{avg}L$	P=7000+100(10) = 8000 lbs	P=7000+100(10) = 8000 lbs	--	P=7000+100(10) = 8000 lbs
Calculate the Margin of Safety $M.S. = P_{cr}/P - 1$	MS = 6020/8000-1 = -0.25	MS =12836/8000-1 = 0.60	--	MS = 1604/8000-1 = -0.80

Since the fixity coefficients for a combined shear and axial loading are valid only in the elastic range, beams with L'/ρ ratios <u>below</u> the critical slenderness ratio $(L'/\rho)_E$ cannot use the c values given in Table 8.2.1-2. This includes the fixed-fixed beam of Example 8.2.1.4.

8.2.1.5 Example Problem – End Fixity – Constant Cross Section with Intermediate Elastic Support

Given an I-Beam beam with 2 in. wide 0.25 in. thick caps and a total height of 3 in. with 0.10 in thick web. It is made from 7050-T76511 Extrusion (Material column curve given in Figure 8.2.1-10). The column is 45 inches long with an elastic support at 18 in. The stiffness of the elastic support, K= 5000 lb/in
Calculate the Margin of Safety if the column is loaded axially with a 70000 lb axial load.



Material Properties: F_{tu} = 79000 psi, F_{cy} = 71000 psi and $E_c = 10.7x10^6$ psi, n_c=22

Calculate basic section properties: I = 2.025 in^4
A = 1.25 in^2
ρ = √(I/A) = 1.273
r=18/45 = 0.40

Calculate $B=KL^3/(EI) = (5000)45^3/[(10.7x10^6)(2.025)] = 21.03$
From Figure 8.2.1-5 for r=0.40 c=1.4
L′ = L/√c = 45/√1.4 = 38.03
L′/ρ = 38.03/1.273 = 29.87
Enter Figure below with L′/ρ and determine F_{col} = 65000 psi. This is in the inelastic buckling range – calculate E_{tan}.

$$\eta_{\tan} = \frac{1}{1+0.002\dfrac{E_c n_c}{F_{cy}}\left(\dfrac{F}{F_{cy}}\right)^{(n_c-1)}} = \frac{1}{1+0.002\dfrac{10.7x10^6\,x22}{71000}\left(\dfrac{65000}{71000}\right)^{(22-1)}} = 0.491$$

$E_{tan} = \eta_{tan}E_c = 0.491(10.7x10^6) = 5.254x10^6$ psi

Recalculate B with E_{tan}; $B=KL^3/(EI) = (5000)45^3/[(5.254x10^6)(2.025)] = 42.82$
From Figure 8.2.1-5 for r = 0.40 c=1.78
L′ = L/√c = 45/√1.78 = 33.73
L′/ρ = 33.73/1.273 = 26.50
Enter Figure 8.2.1-9 with L′/ρ and determine F_{col} = 64000 psi. This is in the inelastic buckling range – calculate E_{tan}.

$$\eta_{\tan} = \frac{1}{1+0.002\dfrac{E_c n_c}{F_{cy}}\left(\dfrac{F}{F_{cy}}\right)^{(n_c-1)}} = \frac{1}{1+0.002\dfrac{10.7x10^6\,x22}{71000}\left(\dfrac{64000}{71000}\right)^{(22-1)}} = 0.571$$

$E_{tan} = \eta_{tan}E_c = 0.571(10.7x10^6) = 4.398x10^6$ psi
Calculate percent difference from previous allowable calculation Difference = [(65000-64000)/65000]x100 = 1.5% - Iterate

Recalculate B with new E_{tan}; $B=KL^3/(EI) = (5000)45^3/[(4.398x10^6)(2.025)] = 51.15$
From Figure 8.2.1-5 for r = 0.40 c=1.80
L′ = L/√c = 45/√1.80 = 33.54
L′/ρ = 33.54/1.273 = 26.35
Enter Figure 8.2.1-9 with L′/ρ and determine F_{col} = 63950 psi.
Calculate percent difference from previous allowable calculation Difference = [(63950-64000)/64000]x100 = 0.08% - Iteration complete

Calculate allowable load for column buckling
$P_{cr} = F_{col}(A) = 63950(1.25) = 79938$ lbs
M.S. = 79938/70000 – 1 = 0.14
Note other stability modes may be more critical and all should be checked per Figure 8.2.0-1.

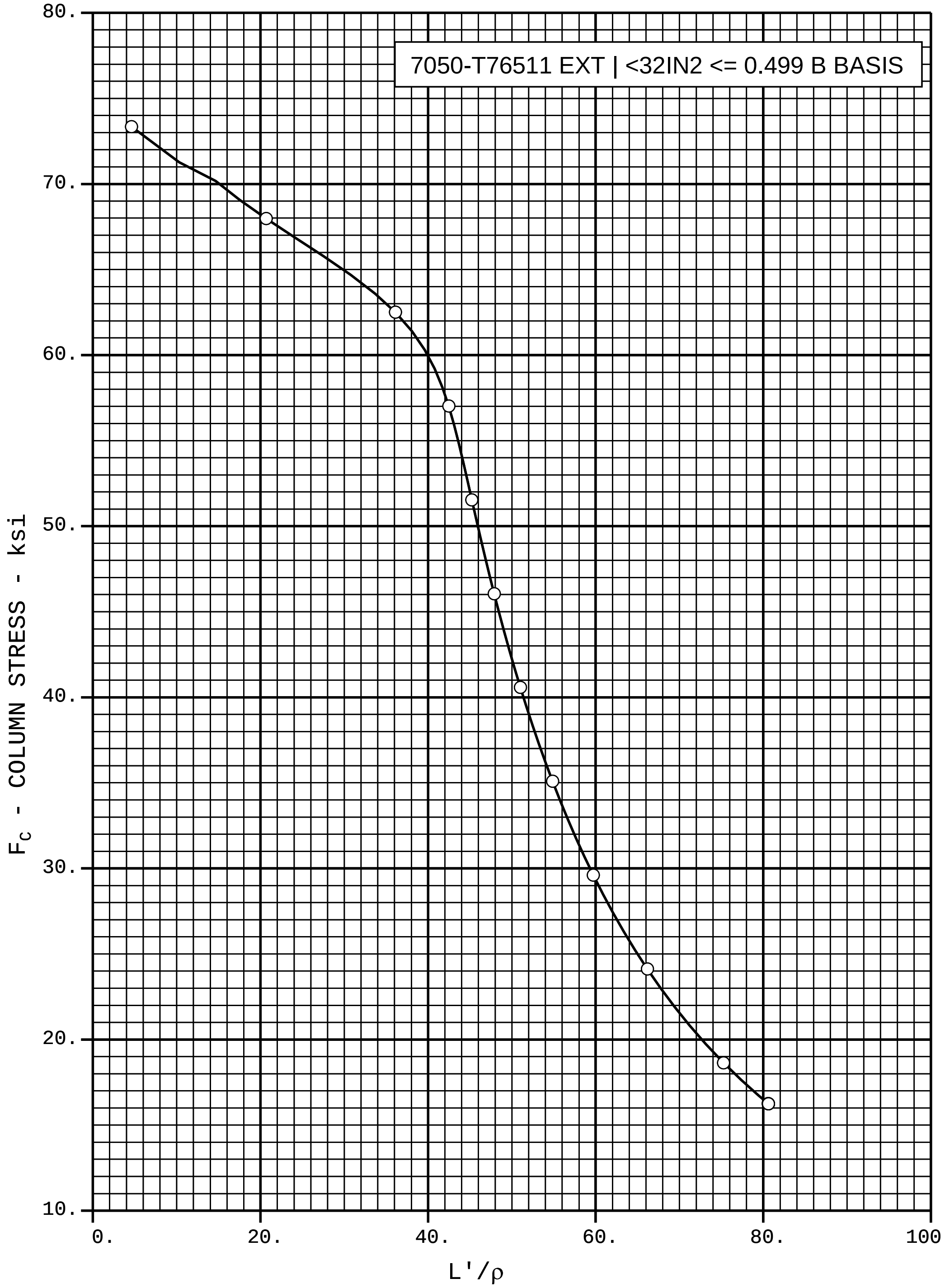


Figure 8.2.1-10 Column Curve for Example 8.2.1-5

8.2.2 Variable Cross Section

The equations discussed in Section 8.2.1 are valid for a straight column with constant cross section and bending rigidity along its length loaded in compression. The determination of the buckling load becomes a more difficult task when the bending rigidity varies along the length of the column.

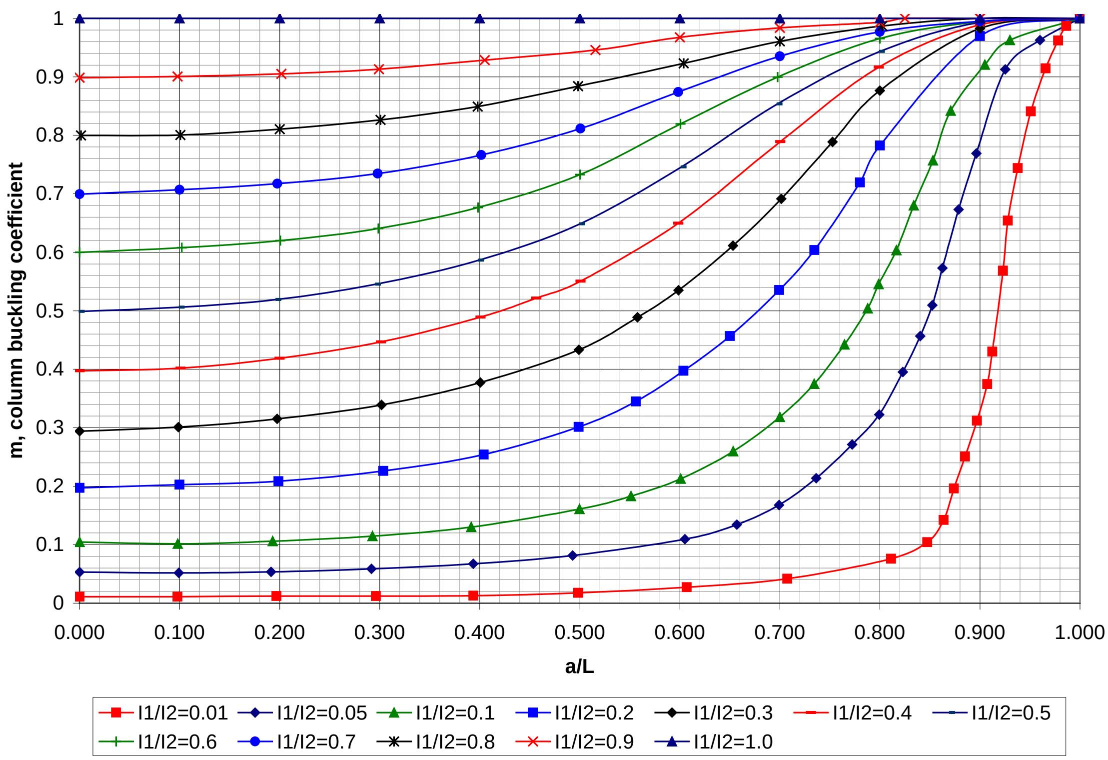
The general equation for the critical buckling load of variable cross section columns in the elastic range is

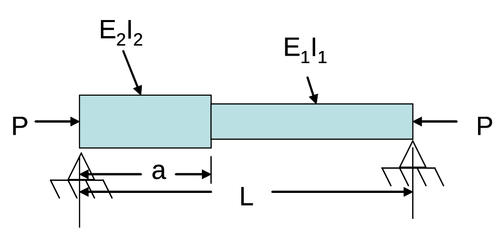
$$P_{cr} = (mc)\pi^2 E_2 I_2/(L)^2$$ **Equation 8.2.2-1**

where
m is the column buckling coefficient which is a function of the column geometry and bending rigidity
c is the end fixity coefficient from Section 8.2.1.2
E_2 is the Modulus of Elasticity of the portion of the column with the greater combined EI stiffness (psi)
I_2 is the moment of inertia of the portion of the column with the greater combined EI stiffness (in^4)

This equation is very similar to the Euler buckling equation given in Section 8.2.1. The column buckling coefficient is added but does not replace the end fixity coefficient, c. The column buckling coefficient accounts for the relative stiffness and geometry of the variable cross section. A limited number of geometries will be presented in this section. Reference 8-2, 8-5, 8-6 and 8-11 provide additional variable cross section geometries. The analyst should note that some of these sources provide curves with specific equations which may not specifically require either c or the π^2 term as they have been included in the plot values.

Figure 8.2.2-1 provides column buckling coefficients for a pinned-pinned single stepped column as a function of the larger stiffness beam length, a, divided by the total beam length, L, and the ratio of the relative stiffnesses. This configuration has a single step as shown in Figure 8.2.2-1. Although the ratio depicted is I_1/I_2, the equations are equally valid when the modulus is included as $E_1I_1/(E_2I_2)$.

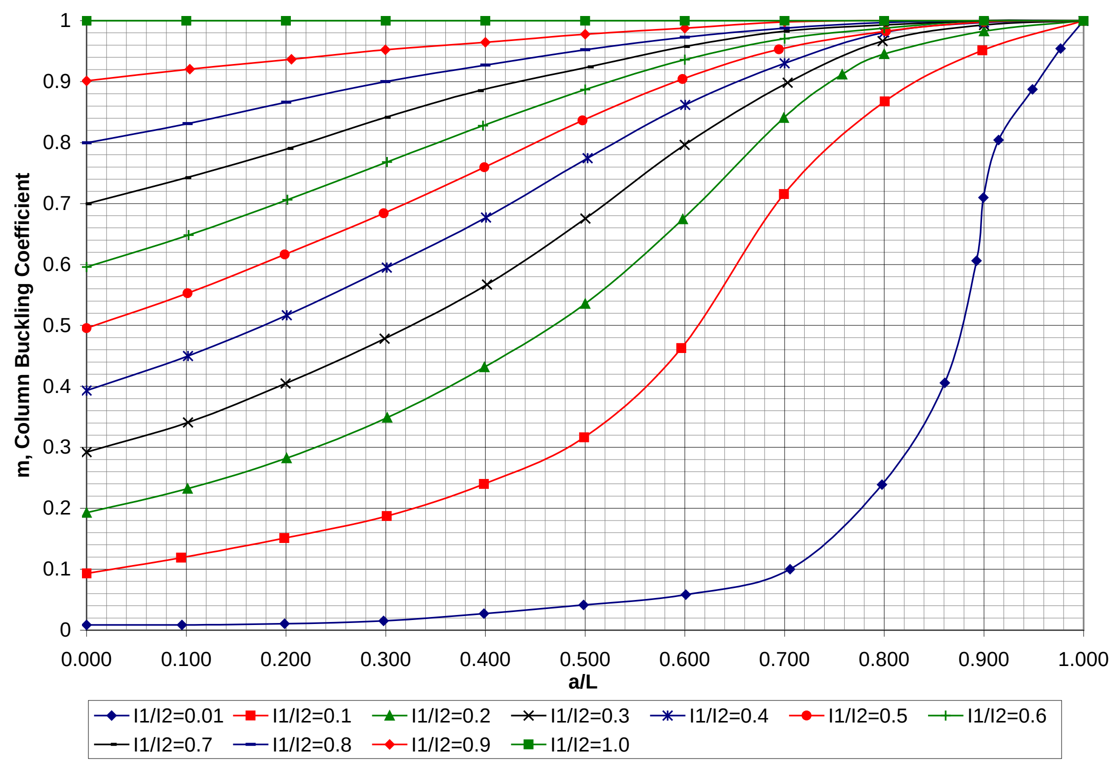


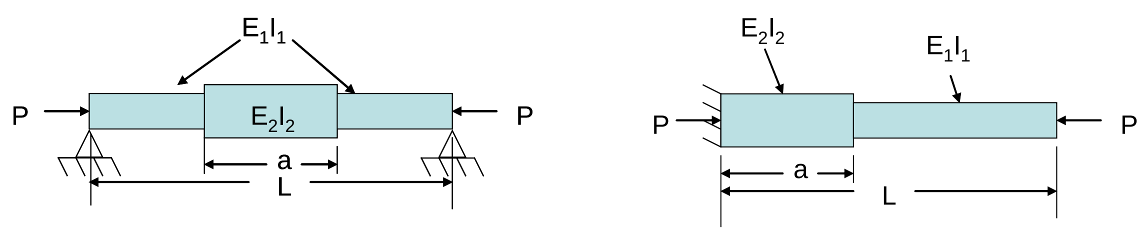


Configuration A: c=1.0

Figure 8.2.2-1 Column Buckling Coefficients for A Pinned-Pinned Stepped Column with No Transverse Axis of Symmetry

Figure 8.2.2-2 provides column buckling coefficients for a pinned-pinned single stepped column (Configuration B) or the fixed-free single step column (Configuration C) as a function of the larger stiffness beam length, a, divided by the total beam length, L, and the ratio of the relative stiffnesses. Configuration B has two steps with a smaller moment of inertia at both ends of the beam. Configuration C has a single step and is fixed at the larger stiffness end and free at the smaller stiffness end. Equation 8.2.2-1 is used with both of these configurations, with the appropriate values for the end fixity coefficients. As with Figure 8.2.2-1, the ratio depicted is I_1/I_2 but the equations are equally valid when the modulus is included as $E_1I_1/(E_2I_2)$.

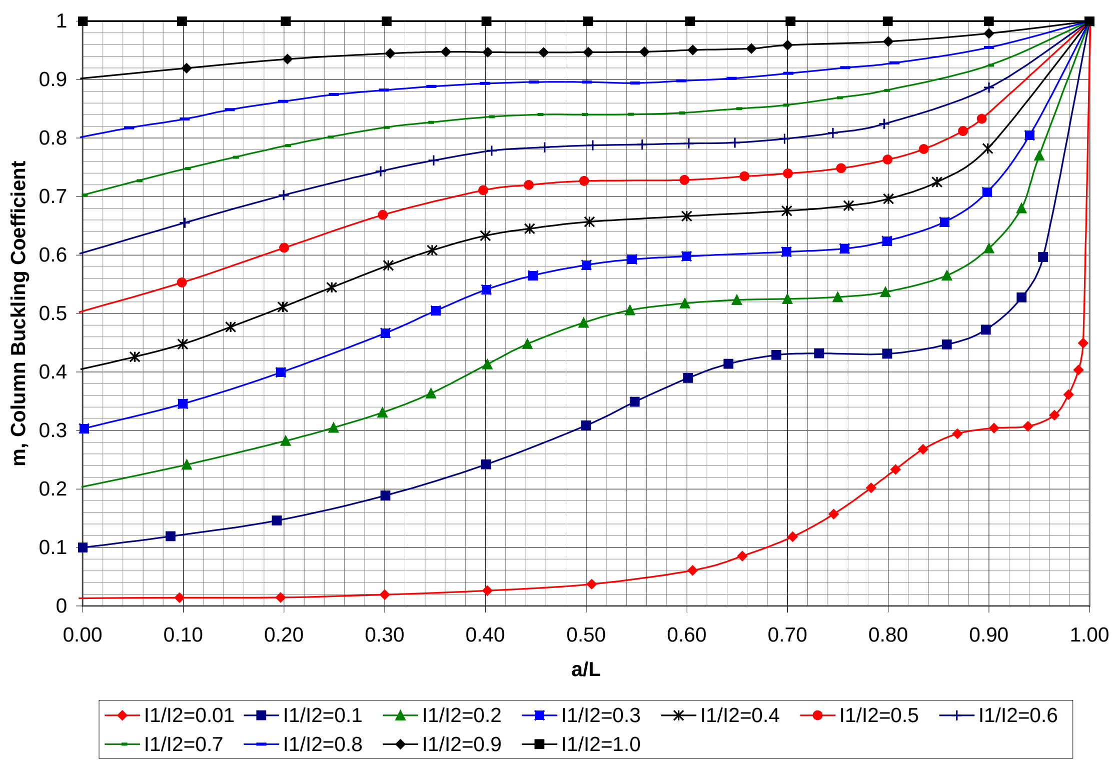


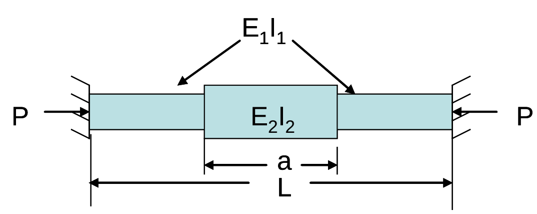


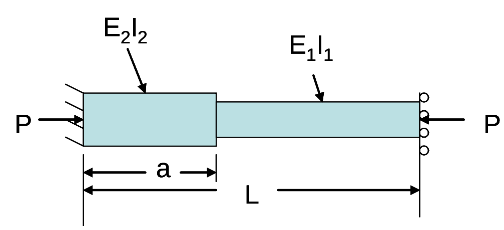
Configuration B : c=1.0 Configuration C: c=0.25

Figure 8.2.2-2 End Fixity Coefficients for A Pinned-Pinned Double Stepped Column or A Fixed-Free Single Stepped Column with No Transverse Axis of Symmetry

Figure 8.2.2-3 provides column buckling coefficients for the fixed-fixed double step column (Configuration D) or a fixed-guided single stepped column (Configuration E) as a function of the larger stiffness beam length, a, divided by the total beam length, L, and the ratio of the relative stiffnesses. Configuration D has two steps with a smaller moment of inertia at both ends of the beam. Configuration E has a single step and is fixed at the larger stiffness end and guided at the smaller stiffness end. Equation 8.2.2-1 is used with both of these configurations, with the appropriate values for the end fixity coefficients. As with Figure 8.2.2-1, the ratio depicted is I_1/I_2 but the equations are equally valid when the modulus is included as $E_1I_1/(E_2I_2)$.







Configuration D: c=4.0 Configuration E: c=1.0

Figure 8.2.2-3 Column Buckling Coefficients for A Fixed-Fixed Double Stepped Column or A Fixed-Guided Single Stepped Column with No Transverse Axis of Symmetry

Figure 8.2.2-4 used data from Figures 8.2.2-2 and 8.2.2-3 to provide a comparison of the values for m and mc for Beam configurations B and D for a/L=0.5.

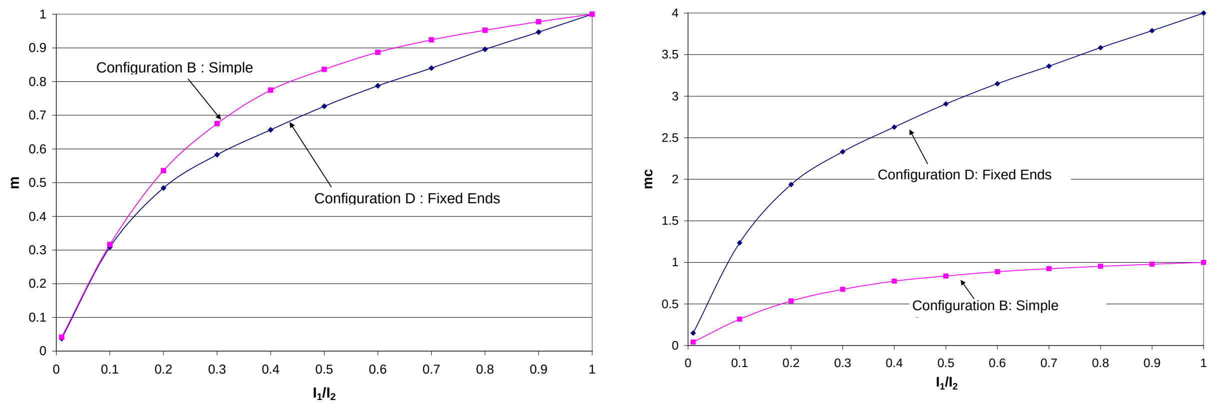


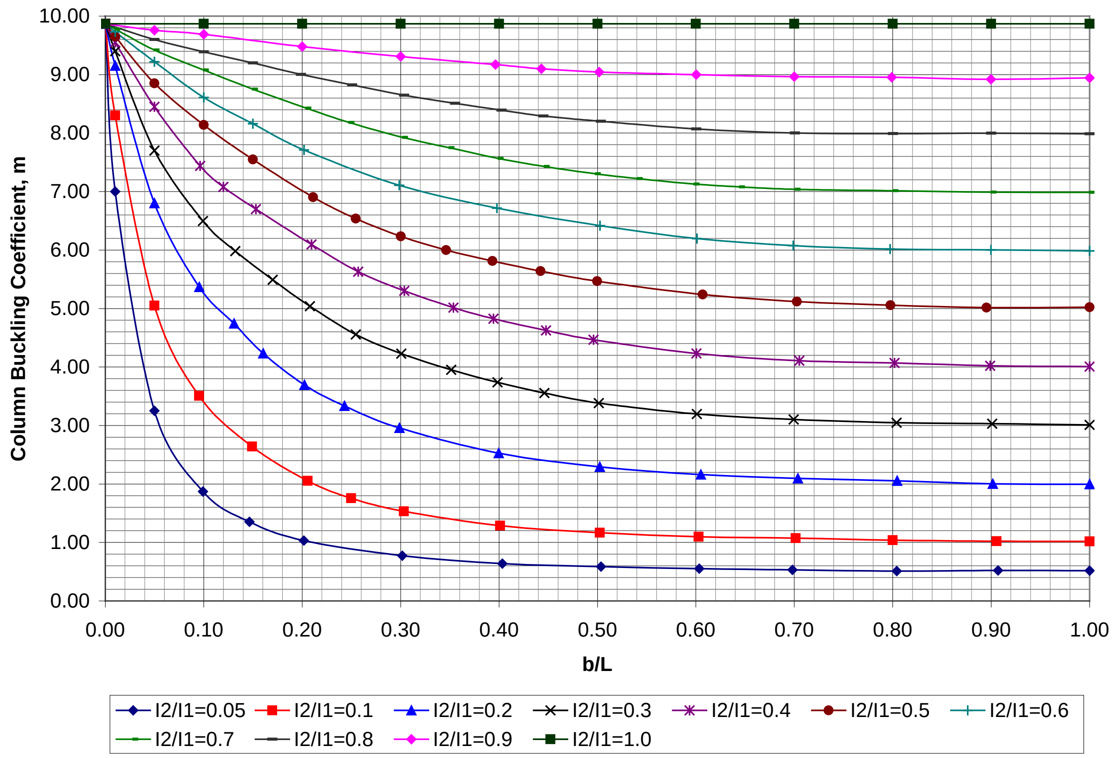
Figure 8.2.2-4 Comparison of Column Buckling Coefficients for A Double-Stepped Column

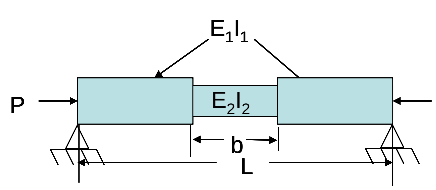
Figure 8.2.2-5 provides column buckling coefficients for the pinned-pinned double step column (Configuration F) and a fixed-simple single stepped column (Configuration G) as a function of the smaller stiffness beam length, b, divided by the total beam length, L, and the ratio of the relative stiffness. Configuration F has two steps with a smaller moment of inertia at the center of the beam. Configuration G has a single step and is fixed at the smaller stiffness end and free at the larger stiffness end. Equation 8.2.2-2 is used with both of these configurations, with the appropriate values for the end fixity coefficients.

$$P_{cr} = (mc)\pi^2 E_1 I_1/(L)^2 \qquad \textbf{Equation 8.2.2-2}$$

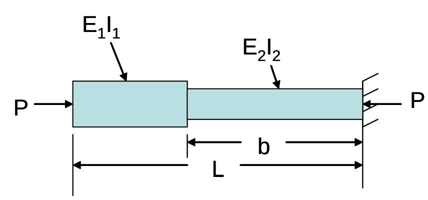
where
m is the column buckling coefficient which is a function of the column geometry and bending rigidity
c is the end fixity coefficient from Section 8.2.1.2
E_1 is the Modulus of Elasticity of the portion of the column with the smaller combined EI stiffness (psi)
I_1 is the moment of inertia of the portion of the column with the smaller combined EI stiffness (in^4)

As with Figure 8.2.2-1, the ratio depicted is I_1/I_2 but the equations are equally valid when the modulus is included as $E_1I_1/(E_2I_2)$.





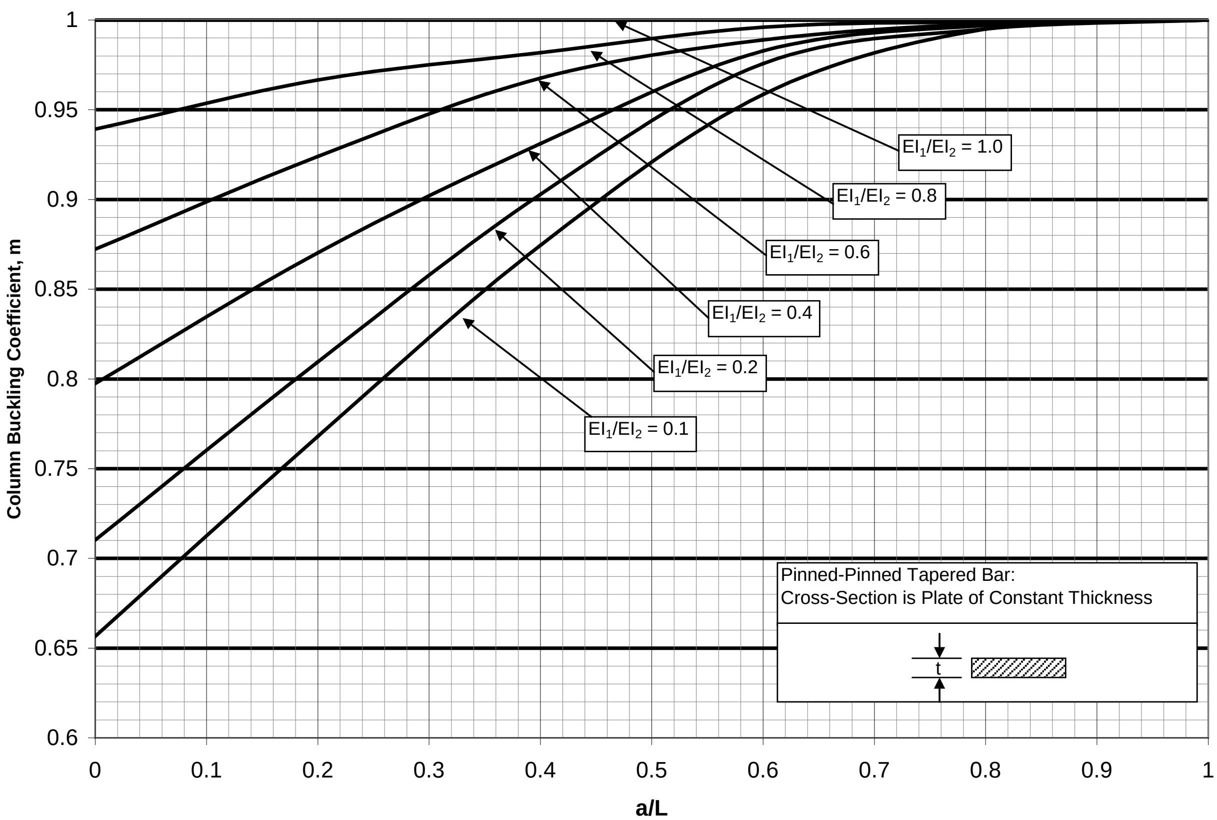
Configuration F: c=1.0

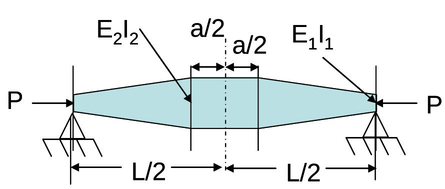


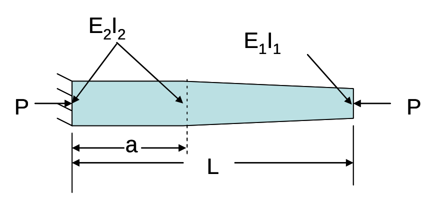
Configuration G: c=0.25

Figure 8.2.2-5 Column Buckling Coefficients for A Pinned-Pinned Double Stepped Column or A Fixed-Free Single Stepped Column with No Transverse Axis of Symmetry

Figure 8.2.2-6 provides column buckling coefficients for the pinned-pinned symmetric, linearly tapering constant thickness column (Configuration H) and the cantilevered, symmetric, linearly tapering constant thickness column (Configuration I) as a function of the tapered beam length, a, divided by the total beam length, L, and the ratio of the relative stiffness. Equation 8.2.2-1 is used with this configuration, with the appropriate end fixity coefficients.





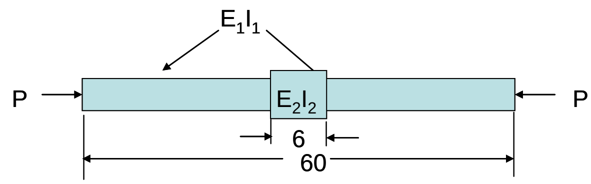


Configuration H: c=1.0 Configuration I: c=0.25

Figure 8.2.2-6 Column Buckling Coefficients for A Symmetric Linearly Tapering Column of Constant Thickness

8.2.2.1 Example Problem – Stepped Column

Given a beam shown below with the following properties:
$I_1 = 0.282$ in^4 $E_1 = 10.3 \times 10^6$ psi (7075-T6 Aluminum)
$I_2 = 0.400$ in^4 $E_2 = 29 \times 10^6$ psi (4130 150 HT Steel)
Find the critical elastic buckling load (P_{cr}) for Fixed-Fixed ends and Pinned-Pinned ends



	Pinned-Pinned	Fixed-Fixed
Calculate a/L	a/L=6/60=0.10	
Calculate E_1I_1/E_2I_2	$E_1I_1/E_2I_2=(10.3 \times 10^6)(0.282)/[(29 \times 10^6)(2.6)] = 0.250$	
Determine c –Table 8.2.1-1	Case 3: c=1.0	Case 1: c=4.0
Determine m	Figure 8.2.2-2 Configuration B: 0.287	Figure 8.2.2-3 Configuration D: 0.280
Calculate $P_{cr} = (mc)\pi^2 E_2I_2/(L)^2$	$P_{cr} = (0.287)(1.0)(\pi^2)(29 \times 10^6)(0.40)/(60)^2$ = 9127 lbs	$P_{cr} = (0.280)(4.0)(\pi^2)(29 \times 10^6)(0.40)/(60)^2$ = 35618 lbs

8.2.3 Column Buckling Material Allowable Curves

Material column curves can be generated by the IDAT program SM83 available on the metals tab for any material in the METDB database and any user defined material. In this section is a sampling of material column buckling curves for reference.

No	Material Alloy/Form	METDB No.	Thickness/ Grain	Basis	F_{tu} (ksi)	F_{cy} (ksi)	E_c (ksi)	n_c	F_{cmax} (ksi)	Figure
1	2024-T42 CLAD Sheet	8	0.010-0.062 L	B	59	35	10700	13	37.6	8.2.3-1
2	7075-T6 BARE Sheet	40	0.040-0.125 L	B	80	71	10500	13	80.0	8.2.3-1
3	7075-T6 CLAD Sheet	43	0.040-0.062 L	B	74	64	10500	13	72.5	8.2.3-1
4	2124-T8151 Plate	72	2.001-3.000 L	S	67	60	10900	16	64.9	8.2.3-2
5	7050-T7451 Plate	85	2.001-3.000 L	B	75	64	10600	19	68.1	8.2.3-2
6	7049-T73 Die Forging	134	2.001-3.000 L	B	72	63	10700	30	64.4	8.2.3-2
7	7075-T76511 Extrusion	249	<20 in^2 0.50-0.75 L	B	76	67	10700	25	69.5	8.2.3-2
8	Ti 6AL-4V Annealed Sheet	288	< 0.1857	B	139	138	16400	20	139.0	8.2.3-3
9	Ti 6AL-4V Annealed Plate	295	2.001 – 4.000	S	130	124	16400	20	130.0	8.2.3-3

10	Ti 6AL-4V Die Forging, A-B Annealed	311	2.001 – 4.000	S	130	123	16400	20	130.0	8.2.3-3
11	300M Steel 280 ksi Type A	355	0.42C	S	280	247	29000	13	280.0	8.2.3-4
12	Aermet 100 Bars, Forgings	358	<100 in^2	S	280	262	28000	11	280.0	8.2.3-4
13	PH13-8 Mo H1000 Bar	351	≤ 8D	B	208	211	29400	17	208	8.2.3-4

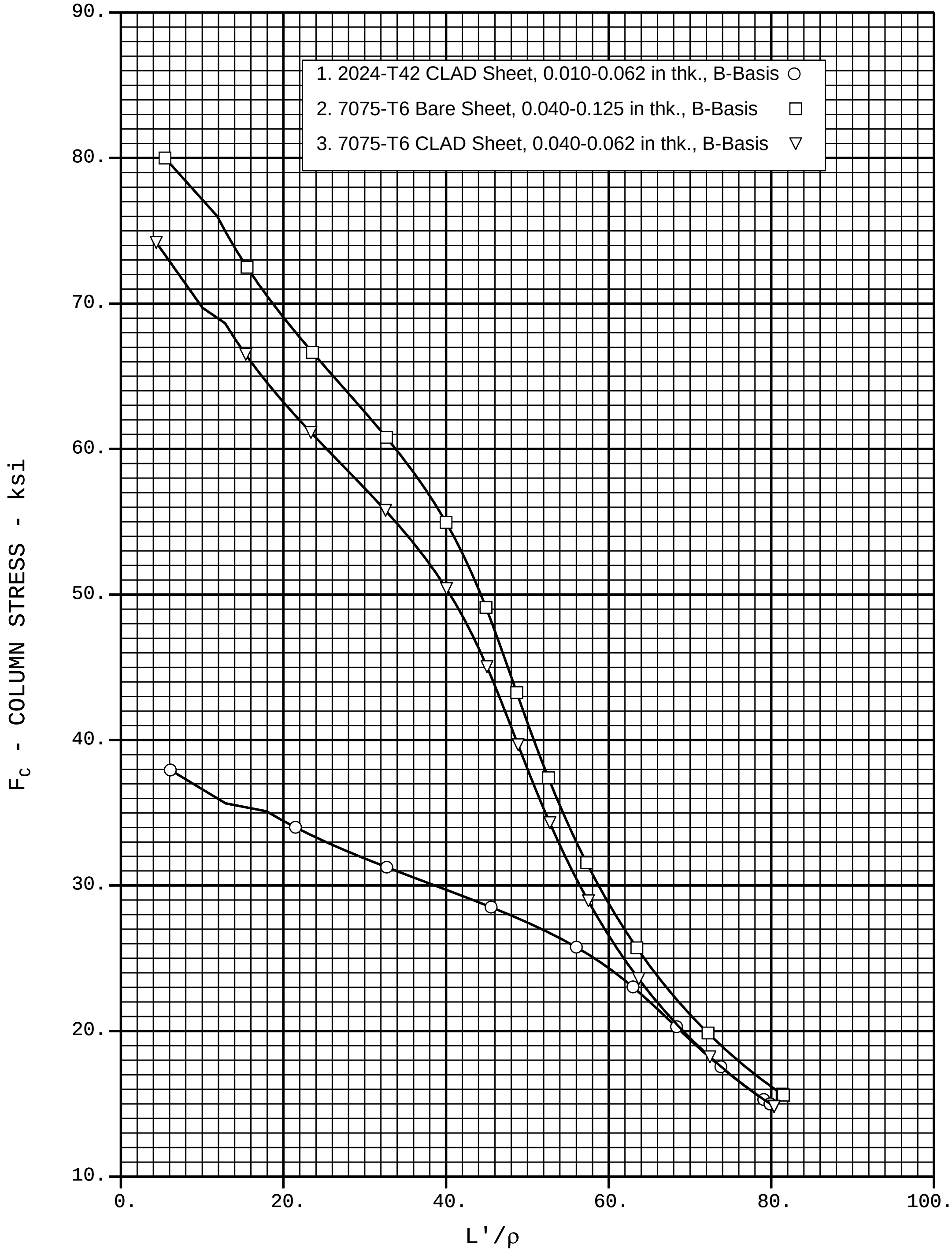


Figure 8.2.3-1 Sample of Column Buckling Allowable Curves for Aluminum Sheet Products

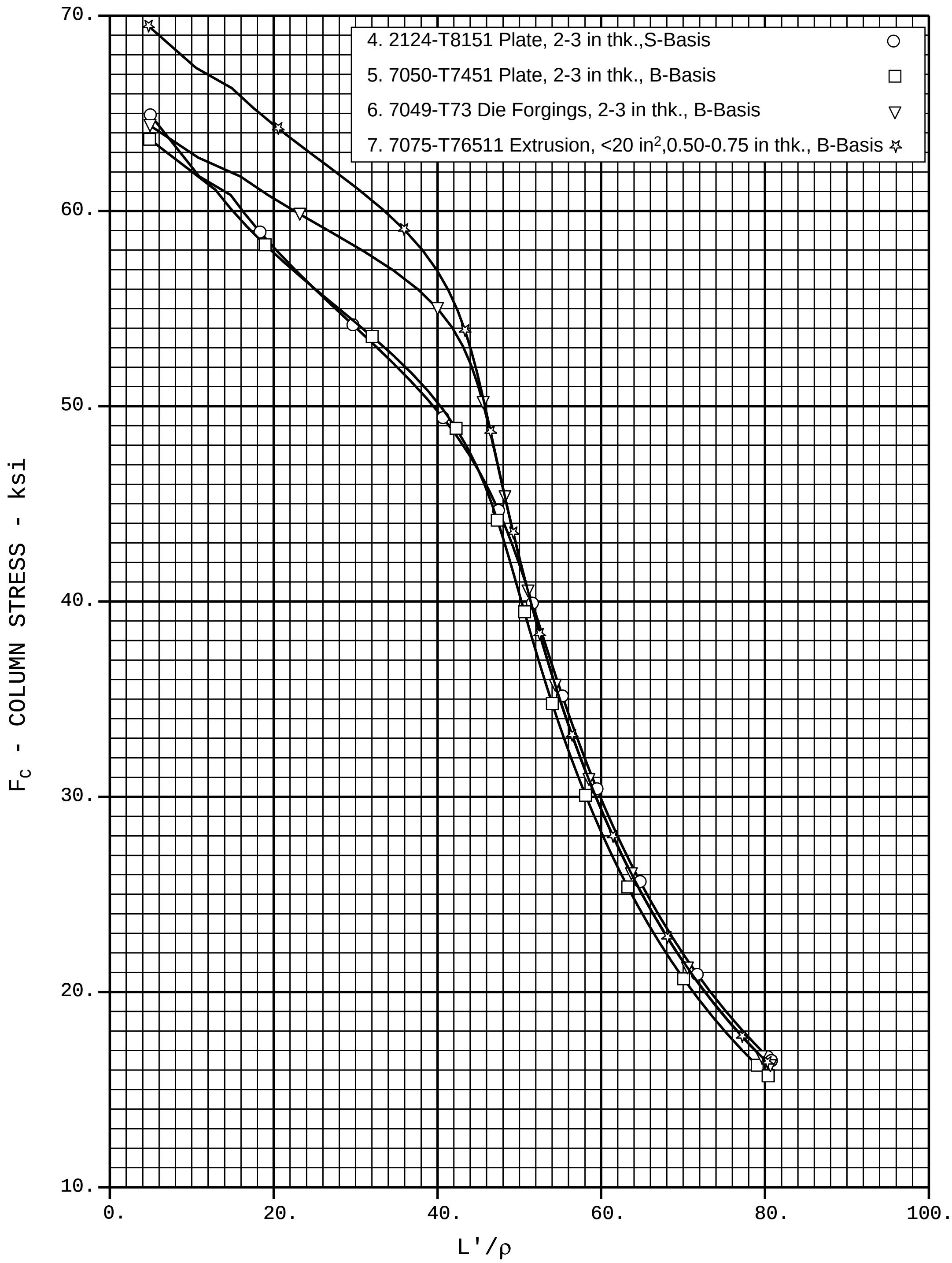


Figure 8.2.3-2 Sample of Column Buckling Allowable Curves for Aluminum Plate and Formed Products

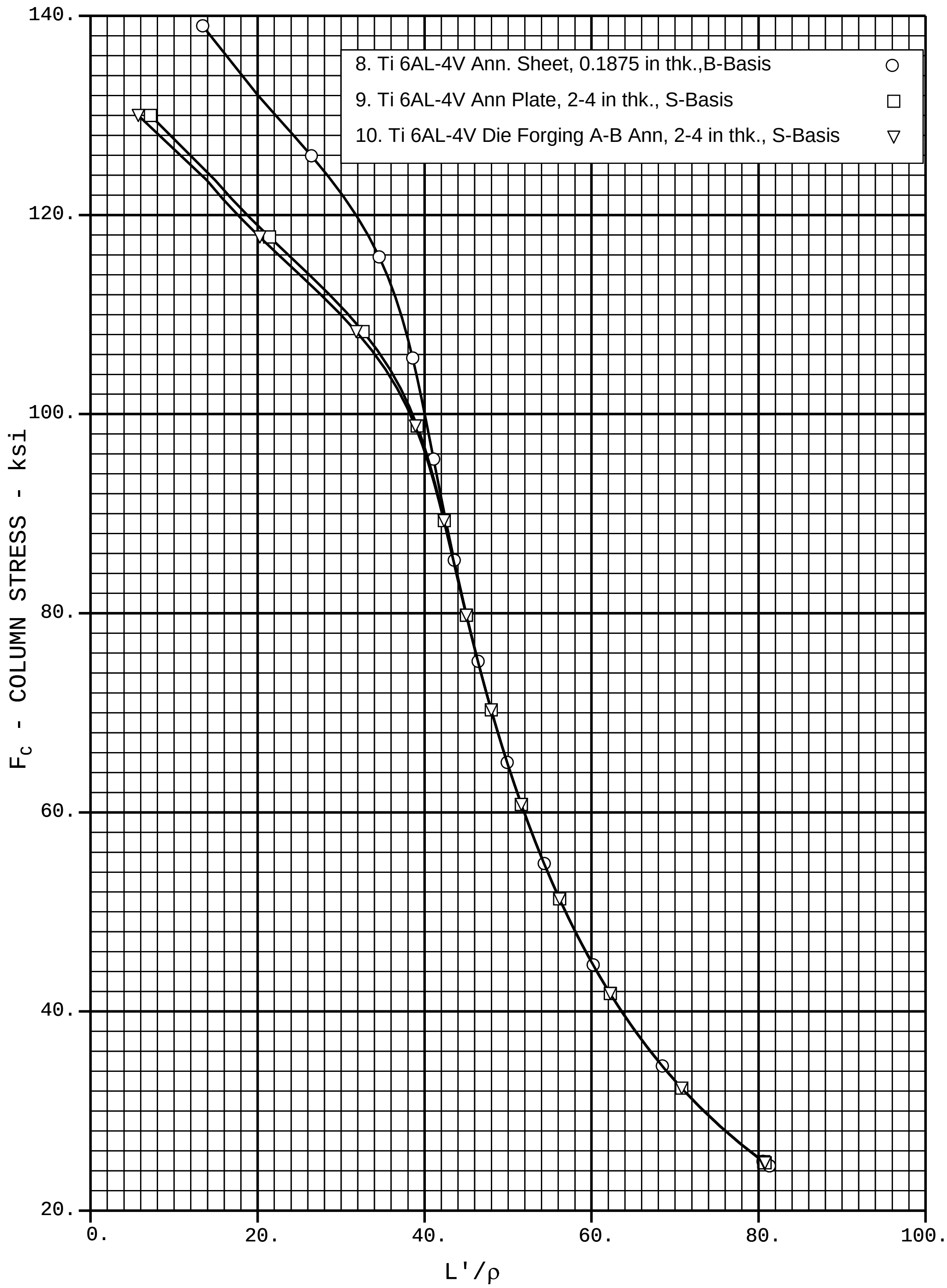


Figure 8.2.3-3 Sample of Column Buckling Allowable Curves for Titanium Products

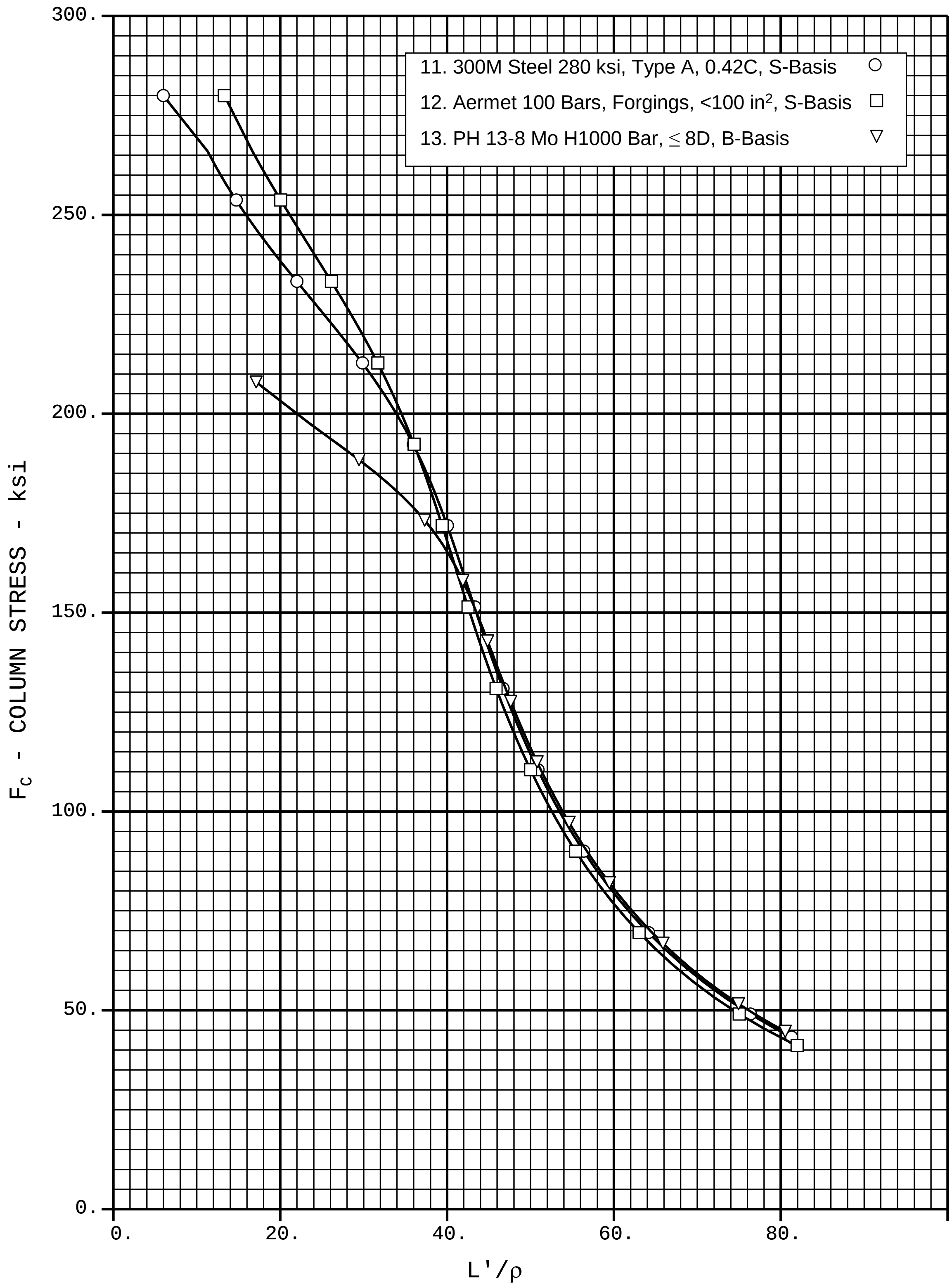


Figure 8.2.3-4 Sample of Column Buckling Curves for Steel Products

8.2.4 Special Case: Buckling of Cylinders

A cylinder subjected to axial compression can fail in either of two main ways: global buckling as a column or local buckling of the wall. In global column buckling, the cross section of the cylinder maintains its shape while the cylinder bends lengthwise along its axis as shown in Figure 8.2.4-1(a). In this case, standard column theory applies, and the methods of Section 8.2.1 can be used to compute the allowable loads. This form of buckling is likely to dominate when the length is relatively long, *i.e.*, L'/ρ is large, and the thickness is relatively large, *i.e.*, D_o/t is small.

(a) Global Buckling of Cylinder (b) Local Buckling of Wall[2]

Figure 8.2.4-1 Axial Compression Buckling of a Cylinder

Local buckling of the cylinder wall is a more complex problem. In this case, the cylinder axis tends to remain straight, but the walls develop undulating waves along both the circumference and the length as shown in Figure 8.2.4-1(b). Theoretical solutions for this mode are based on elastic shell theory but usually over-estimate test results, often by a large amount. Therefore, practical analysis methods require another approach, either by developing purely empirical curve fits to test data, or by developing correction factors that are applied to the theoretical solutions. The latter approach is presented here.

In local buckling analysis, the deviations between test and elastic theory are usually addressed by two different factors. The first is a plasticity correction factor that accounts for buckling in the inelastic range, which is usually dominant for thick walled cylinders, *i.e.*, low D_o/t. The second accounts for the sensitivity of the buckling load to slight imperfections in the initial shape of the shell. The correction for imperfections dominates when the wall is relatively thin, *i.e.*, high D_o/t.

When buckling loads are determined using a finite element model, the method used is usually a linear eigenvalue solution. In this case, the FEM will produce results that may closely match the theoretical elastic shell predictions, but may not be safe for design. The analysis methods presented here apply to unstiffened isotropic cylinders, and account for the relevant nonlinear effects, *i.e.*, plasticity and imperfection sensitivity. Most importantly, they have been correlated to test data to ensure safe designs. These are therefore the preferred methods for most practical applications. When special circumstances warrant a more complex analysis, *e.g.* the use of a finite element analysis, it should only be performed with close Subject Matter Expert guidance, and may require additional testing to substantiate the predictions.

The allowable buckling load is the minimum result of three analysis methods for any given combination of D_o/t and L'/ρ where D_o is the outer diameter of the cylinder, t is the wall thickness, L′ is the pin ended column length and ρ is the radius of gyration. The radius of gyration, for a cylinder is given as

$$\rho = 0.25\sqrt{D_o^2 + (D_o - 2t)^2}$$ **Equation 8.2.4-1**

Where
D_o is the outer diameter of the cylinder (in)
t is the cylinder wall thickness (in)

[2] From Reference 8-35, Kanemitsu (1939)

There are three analysis approaches used to describe the two failure modes. Mode 2 and Mode 3 are different empirical approaches used to describe the local buckling of the cylinder wall[3].

- Mode 1: Global Column Buckling as described in Section 8.2.1, Equations 8.2.1-4 through 8.2.1-6
- Mode 2: Method 1 - Local Shell Buckling from Reference 8-30, SP-8007
- Mode 3: Method 2 – Local Shell Buckling from Reference 8-31, Lockheed Missiles and Space Company Structural Methods Handbook, Section 6.1

Mode 1 Global Column Buckling

This method is the same as the general column buckling approach covered in PM4057 Section 8.2.1. Equation 8.2.4-1 can be used to calculate ρ.

The impact of potential eccentricities is larger for longer length cylinders as the combination of large deflections, dimensional tolerances and unintentional eccentricity may cause higher stresses than predicted by the Euler column. This is accounted for by a reduction in the allowable stress, F_{col}, for values of L'/ρ greater than 80.

This reduction and the resulting modified Euler buckling equations are given as

$$K_{ecc} = \sqrt{1 - \frac{L'/\rho - 80}{L'/\rho}} \qquad \text{For } L'/\rho > 80$$

$$K_{ecc} = 1.0 \qquad \text{For } L'/\rho \leq 80$$

$$F_{col} = K_{ecc} \cdot \frac{\pi^2 E}{\left(L'/\rho\right)^2}$$

Equation 8.2.4-2

Mode 2 Method 1 – Local Shell Buckling

Mode 2 buckling is based on classical theory with modifications resulting from testing. It is detailed in NASA SP-8007, Reference 8-36. Buckling analysis generally requires consideration of the end fixity in selection of buckling parameters. For cylinders this would require consideration of both tangential and normal displacements; however, it is nearly impossible to analytically characterize and justify end constraints of greater than simply supported. Assumptions of anything other than simply supported should be substantiated by testing. An empirical correlation factor, ζ, is included in the analysis to provide a better prediction of test results using theoretical equations.

The Mode 2 axial critical buckling stress is given by the following

$$F_{cr} = \frac{k_x \pi^2 E_c \eta t^2}{12(1 - \nu^2) L^2}$$

Equation 8.2.4-3

Where
k_x is the buckling coefficient of a cylinder subjected to axial compression
E_c is the compressive elastic modulus (psi) from Reference 8-4 or other suitable source
η is the plasticity correction factor
t is the wall thickness (in)
ν is the elastic Poisson's ratio
L is the length of the cylinder (in)

The plasticity correction factor for Equation 8.2.4-3, per Reference 8-30, is given as

$$\eta = \frac{\sqrt{E_{sec} E_{tan}}}{E_c}$$

Equation 8.2.4-4

Where
E_{tan} is the tangent modulus, given by Equation 3.3.1-3
E_{sec} is the secant modulus, given by Equation 3.3.1-4

[3] References 8-38, 8-39, and 8-40 provide much of the test data on this subject in the regions of interest.

The buckling coefficient, k_x, is dependent on tube geometry and is given by

$$k_x = k_{xo} + \frac{12}{\pi^4} \cdot \frac{\zeta^2 Z^2}{k_{xo}} \qquad \zeta Z < \frac{\pi^2 k_{xo}}{2\sqrt{3}}$$ **Equation 8.2.4-5**

$$k_x = \frac{4\sqrt{3}}{\pi^2} \zeta Z \qquad \zeta Z > \frac{\pi^2 k_{xo}}{2\sqrt{3}}$$ **Equation 8.2.4-6**

$$Z = \frac{L^2}{r_o t}\sqrt{1 - \nu^2}$$ Isotropic Cylinder

$$\zeta = 1 - 0.901\left(1 - e^{-\psi}\right)$$ Reference Equation 10.3.3-18

$$\psi = {}^{1}\!/_{16}\sqrt{\frac{r_o}{t}}$$ Reference Equation 10.3.3-19

Where
k_{xo} =1.0 It is the buckling coefficient of a compression cylinder when Z=0 (Reference 8-30)
Z is the non-dimensional curvature parameter
ζ is a correlation factor to account for the difference between classical theory and experimental instability loads
ψ is the shell thickness parameter
r_o is the outer radius of the cylinder (in)
t is the wall thickness (in)
ν is the elastic Poisson's ratio
L is the length of the cylinder (in)

For most practical tubes used in aircraft applications, Equation 8.2.4-6 for k_x is applicable. If it is substituted into Equation 8.2.4-3 and simplified, the following equation results

$$F_{cr} = \frac{\zeta E_c \eta}{\sqrt{3(1 - \nu^2)}} \cdot \frac{t}{r_o}$$ **Equation 8.2.4-7**

It is seen that this failure mode is not a function of tube length but a function of only the ratio of the tube radius to the tube wall thickness.

Mode 3 Method 2 Local Shell Buckling

Mode 3 is an alternative empirical approach to predicting local shell buckling based on the Lockheed Missiles and Space Company's Structural Methods Handbook, Reference 8-37. It looks at monocoque unstiffened circular cylinders. This approach is end-constraint-independent and relies on an empirically derived buckling coefficient, K.

The buckling load may be predicted from

$$F_{cr} = \frac{K E_c \eta_{sec}}{2\pi} \cdot \frac{t}{r_o}$$ **Equation 8.2.4-8**

Where
K is the buckling coefficient given in Table 8.2.4-1
r_o is the tube outer radius (in)
$\eta_{sec}=E_{sec}/E_c$, the plasticity correction factor based on the secant modulus

The buckling coefficient is a function of r_o/t ratio and is given in Table 8.2.4-1. For intermediate values linear interpolation may be used.

Table 8.2.4-1 Buckling Coefficient K versus r/t Ratio for $L/r_o \geq 2.0$

r_o/t	D_o/t	K
0	0	3.8
50	100	2.0
100	200	1.4
200	400	1.235
300	600	1.108
400	800	1.006
500	1000	0.925
600	1200	0.869
800	1600	0.778
1000	2000	0.717
1200	2400	0.676
1400	2800	0.639

This approach is also not a function of L'/ρ but only the radius, r_o, and the thickness, t. This results in Mode 2 or 3 typically providing a cutoff stress up to some value of L'/ρ which varies with material and D_o/t.

An example of this is provided in Figure 8.2.4-2. For this particular material, at $D_o/t = 100$, Mode 3 is critical up to $L'/\rho \sim 40$ and then Mode 1 is critical beyond that value so the allowable would follow the Mode 3 - Mode 1 line. Mode 2 is not critical for $D_o/t = 100$. For $D_o/t = 400$, Mode 3 is critical for all L'/ρ up to approximately $L'/\rho = 100$, while Mode 2 is not critical at all for this D_o/t so the allowable would again follow the Mode 3 – Mode 1 line.

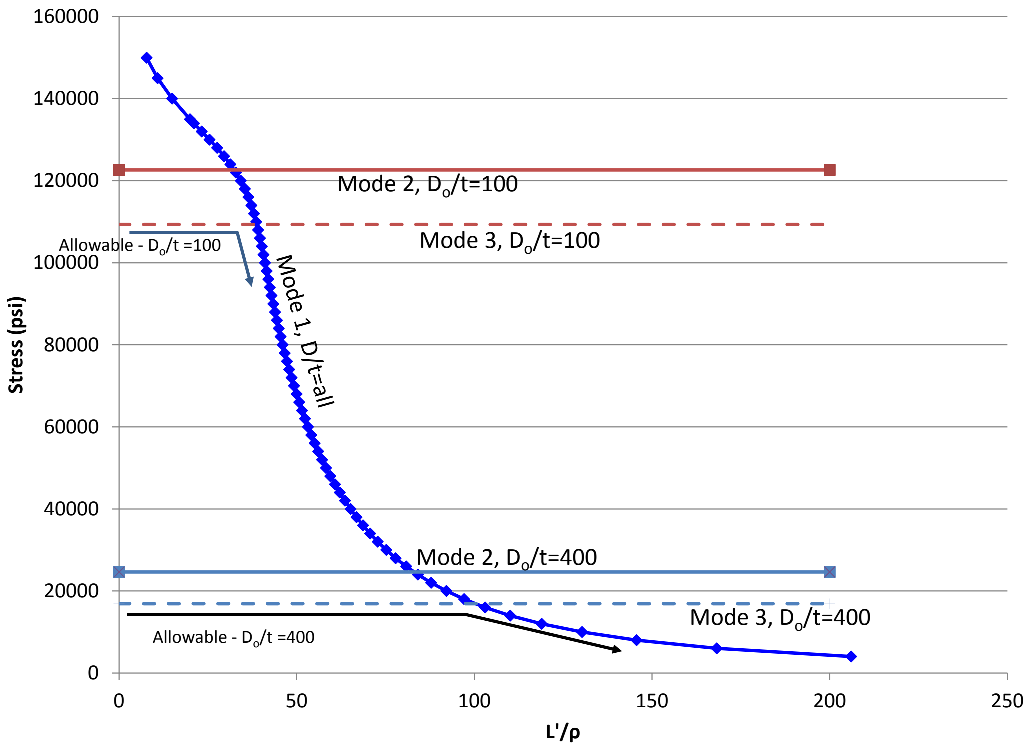


Figure 8.2.4-2 Example Plot of Failure Modes for Tube Axial Compression

IDAT/SM100 generates curves for the axial buckling analysis for cylinders described in this section. It analyzes for Modes 1-3. A sample set of curves is provided in Section 8.2.4.1, however they may be generated for any metallic material using IDAT. There are two output formats: text file and report ready. The text file contains tabular output for each of the D/t's calculated (D_o/t) and for a range of L/ρ provides the critical buckling stress, F_{co}, and the failure mode. This is shown in Figure 8.2.4-3 which is for a D_o/t of 150. For L/ρ up to 35.31, the failure is Mode 2 with a stress of 33.449 ksi. Beyond L/ρ=36.16, the failure mode is Mode 1 with a critical stress which varies with L/ρ. The report ready output provides the buckling curves.

```
SM100 - CYLINDERS IN COMPRESSION     DATE: 17-Dec-2015  VERSION V1.2

Material  1)  #377 L      2024-T42 DRAWN TUBE | .050-.259 | S BASIS | No Spec
FCY=  38.00 KSI   EC= 10700. KSI   RON= 15.0   F7C=  37.27 KSI   FCMAX=  40.27 KSI   MU=   0.33

                   L/D     L/RHO      FCO                 L/D     L/RHO      FCO
DO/T= 150.0       3.40      9.68    33.449  2            16.20     46.13    31.388  1
                  3.70     10.54    33.449  2            18.40     52.39    29.857  1
                  4.00     11.39    33.449  2            20.60     58.66    27.777  1
                  4.30     12.24    33.449  2            22.80     64.92    24.589  1
                  4.60     13.10    33.449  2            25.00     71.18    20.803  1
                  4.90     13.95    33.449  2            27.20     77.45    17.603  1
                  5.20     14.81    33.449  2            29.40     83.71    14.732  1
                  5.50     15.66    33.449  2            31.60     89.98    12.300  1
                  5.80     16.51    33.449  2            33.80     96.24    10.395  1
                  6.10     17.37    33.449  2            36.00    102.50     8.879  1
                  6.40     18.22    33.449  2            38.20    108.77     7.655  1
                  6.70     19.08    33.449  2            40.40    115.03     6.655  1
                  7.00     19.93    33.449  2            42.60    121.30     5.829  1
                  7.30     20.79    33.449  2            44.80    127.56     5.140  1
                  7.60     21.64    33.449  2            47.00    133.83     4.559  1
                  7.90     22.49    33.449  2            49.20    140.09     4.066  1
                  8.20     23.35    33.449  2            51.40    146.35     3.645  1
                  8.50     24.20    33.449  2            53.60    152.62     3.283  1
                  8.80     25.06    33.449  2            55.80    158.88     2.969  1
                  9.10     25.91    33.449  2            58.00    165.15     2.695  1
                  9.40     26.77    33.449  2            60.20    171.41     2.455  1
                  9.70     27.62    33.449  2            62.40    177.67     2.245  1
                 10.00     28.47    33.449  2            64.60    183.94     2.058  1
                 10.30     29.33    33.449  2            66.80    190.20     1.893  1
                 10.60     30.18    33.449  2            69.00    196.47     1.746  1
                 10.90     31.04    33.449  2            71.20    202.73     1.614  1
                 11.20     31.89    33.449  2            73.40    209.00     1.496  1
                 11.50     32.74    33.449  2            75.60    215.26     1.389  1
                 11.80     33.60    33.449  2            77.80    221.52     1.293  1
                 12.10     34.45    33.449  2            80.00    227.79     1.206  1
                 12.40     35.31    33.449  2            82.20    234.05     1.127  1
                 12.70     36.16    33.446  1            84.40    240.32     1.055  1
                 13.00     37.02    33.275  1            86.60    246.58     0.989  1
                 13.30     37.87    33.103  1            88.80    252.84     0.929  1
                 13.60     38.72    32.932  1            91.00    259.11     0.874  1
                 13.90     39.58    32.759  1            93.20    265.37     0.823  1
                 14.20     40.43    32.587  1            95.40    271.64     0.777  1
                 14.50     41.29    32.412  1            97.60    277.90     0.734  1
```

Figure 8.2.4-3 Sample Tabular Output from IDAT/SM100

8.2.4.1 Cylinder Buckling Allowable Material Curves

No	Material Alloy/Form	METDB No.	Thickness (in) / Carbon Content	Basis	F_{tu} (ksi)	F_{cy} (ksi)	E_c (ksi)	n_c	F_{cmax} (ksi)	Figure
1	2024-T42 Aluminum Drawn Tube	377	0.050-0.259	S	64	38	10700	15	40.3	8.2.4-4
2	6061-T6 Aluminum Tube	378	0.025-0.500	S	42	34	10100	20	34.9	8.2.4-5
3	7075-T6 Aluminum Bare Sheet	40	0.040-0.125	B	80	71	10500	13	80.0	8.2.4-6
4	Titanium 6AL-4V Annealed Sheet	288	≤0.1875	B	139	138	16400	20	139.0	8.2.4-7
5	300M Steel, 280 ksi Tube	355	0.42C	S	280	247	29000	13	280.0	8.2.4-8
6	Low alloy (4130) Steel Tube	391	0.035-0.187	S	95	75	29000	10	82.9	8.2.4-9

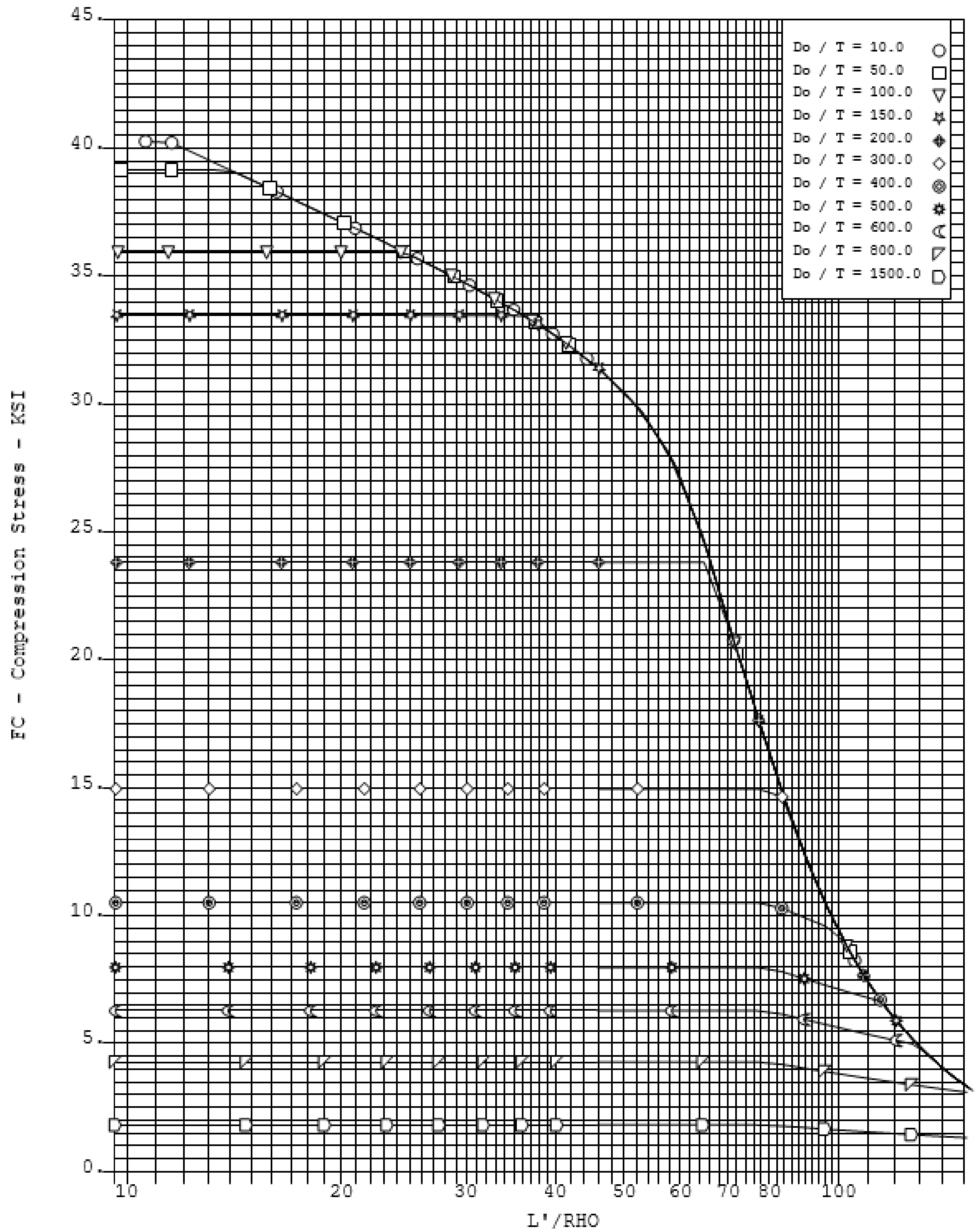


Figure 8.2.4-4 Sample IDAT/SM100 Cylinder Buckling Curve for 2024-T42 Aluminum Drawn Tube: METDB #377; 2024-T42 Aluminum Drawn Tube, t =0.050-0.259 in.; S-Basis; F_{tu}=64 ksi; F_{ty}=38 ksi; E_c=10,700 ksi; n_c=15, F_{cmax}=40.3 ksi

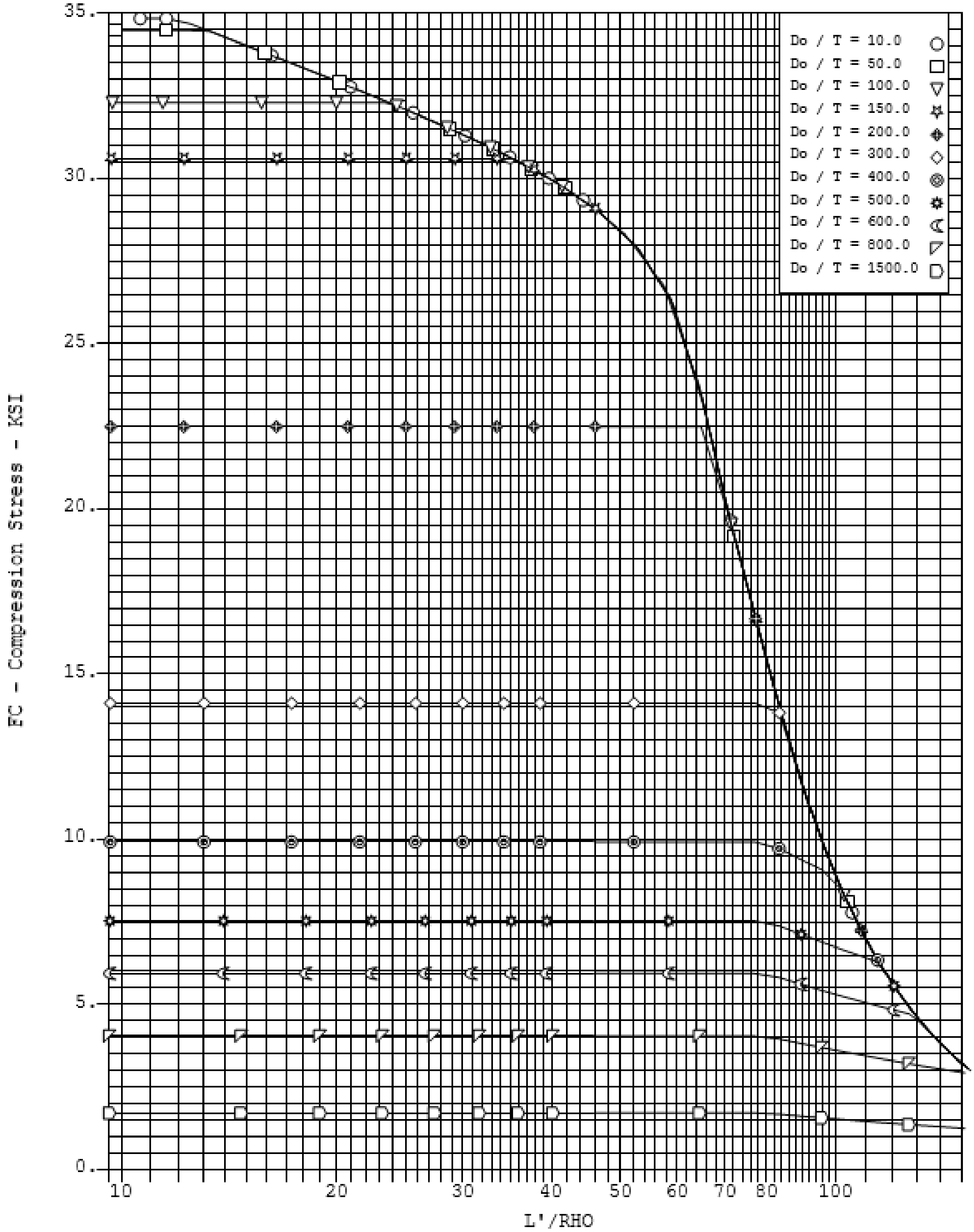


Figure 8.2.4-5 Sample IDAT/SM100 Cylinder Buckling Curve for 6061-T6 Aluminum Tube: METDB #378; 6061-T6 Aluminum Tube, t =0.025-0.500 in.; S-Basis; F_{tu}=42 ksi; F_{ty}=34 ksi; E_c=10,100 ksi; n_c=20, F_{cmax}=34.9 ksi

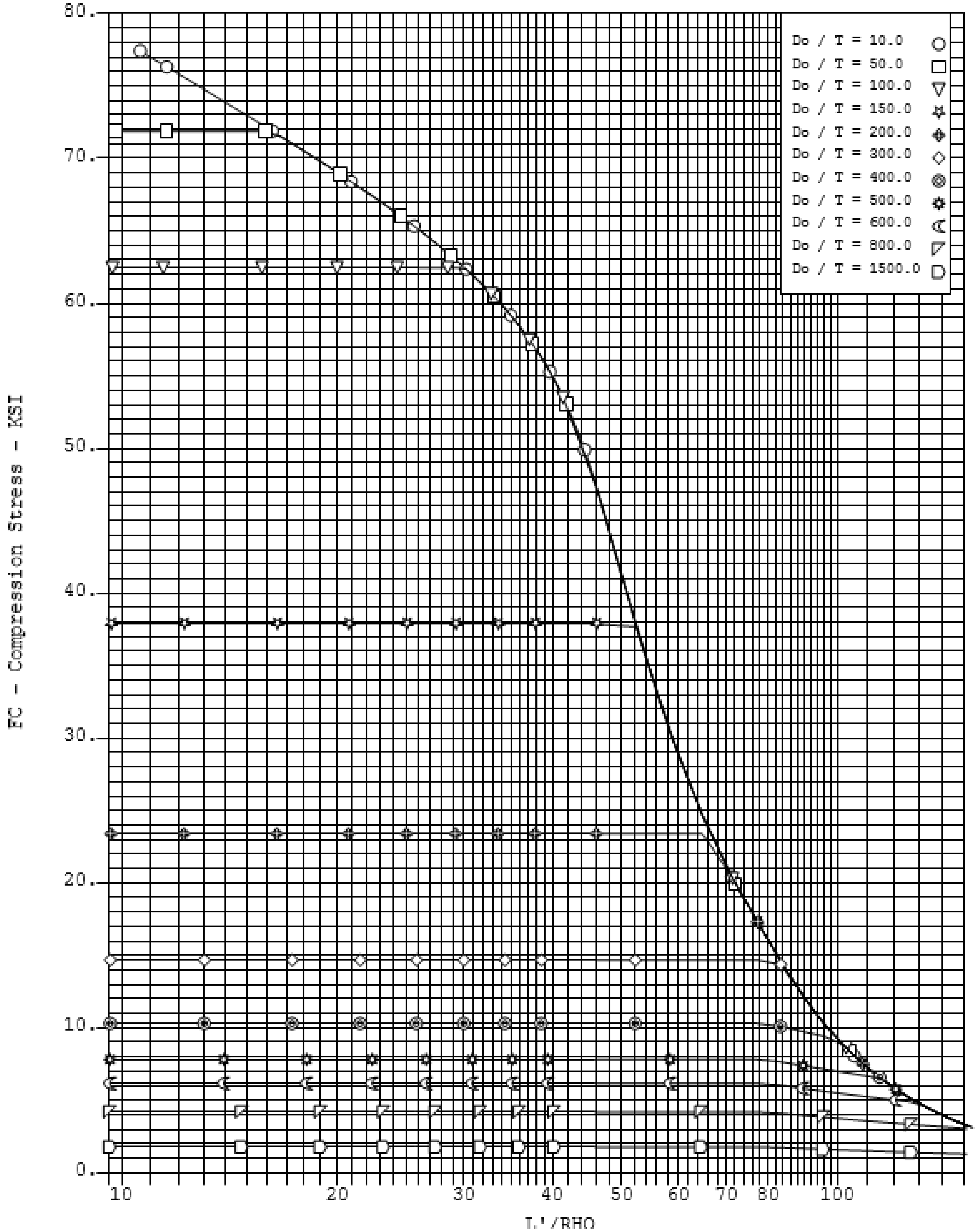


Figure 8.2.4-6 Sample IDAT/SM100 Cylinder Buckling Curve for 7075-T6 Aluminum Sheet: METDB #40; 7075-T6 Aluminum Bare Sheet, t =0.040-0.125 in.; B-Basis; F_{tu}=80 ksi; F_{ty}=71 ksi; E_c=10,500 ksi; n_c=13, F_{cmax}=80.0 ksi

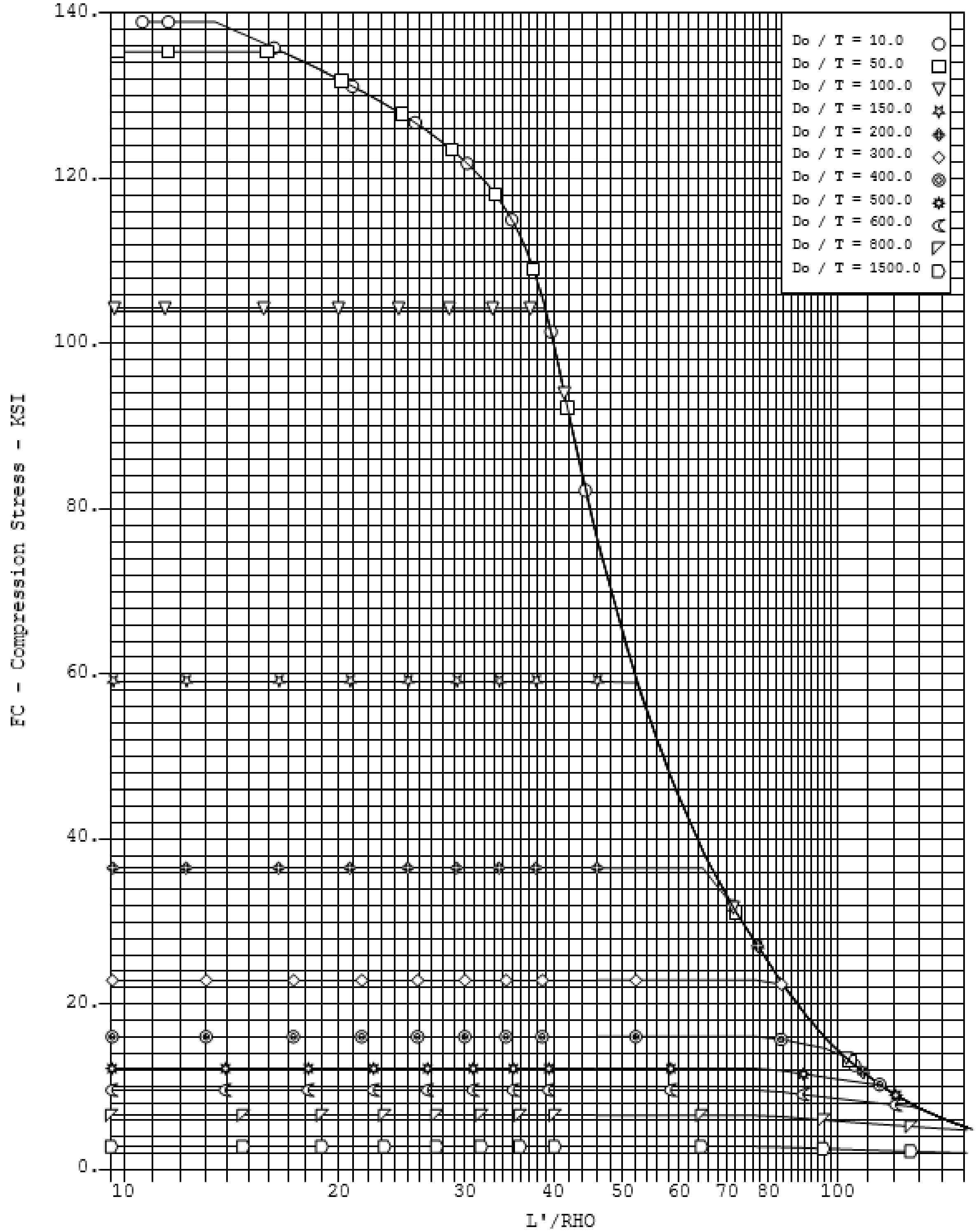


Figure 8.2.4-7 Sample IDAT/SM100 Cylinder Buckling Curve for Titanium 6AL-4V Annealed Sheet; METDB #288; Ti 6AL-4V Annealed Sheet, t ≤ 0.1875 in.; B-Basis; F_{tu}=139.0 ksi; F_{ty}=138.0 ksi; E_c=16,400 ksi; n_c=20, F_{cmax}=139.0 ksi

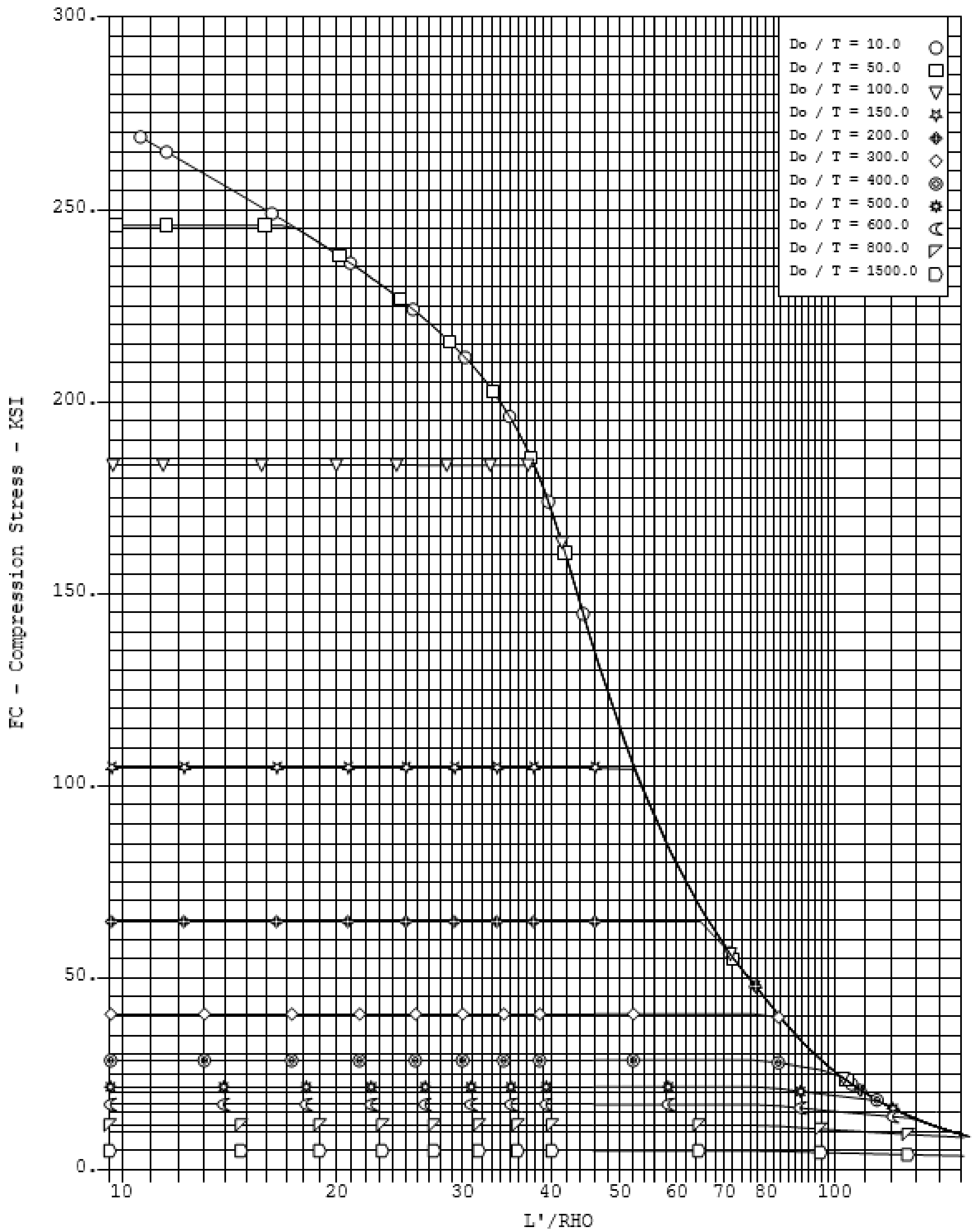


Figure 8.2.4-8 Sample IDAT/SM100 Cylinder Buckling Curve for 300M 280 ksi Steel Tube; METDB #355; 300M Steel Tube, 280 ksi, 42% Carbon.; S-Basis; F_{tu}=280.0 ksi; F_{ty}=247.0 ksi; E_c=29,000 ksi; n_c=13, F_{cmax}=280.0 ksi

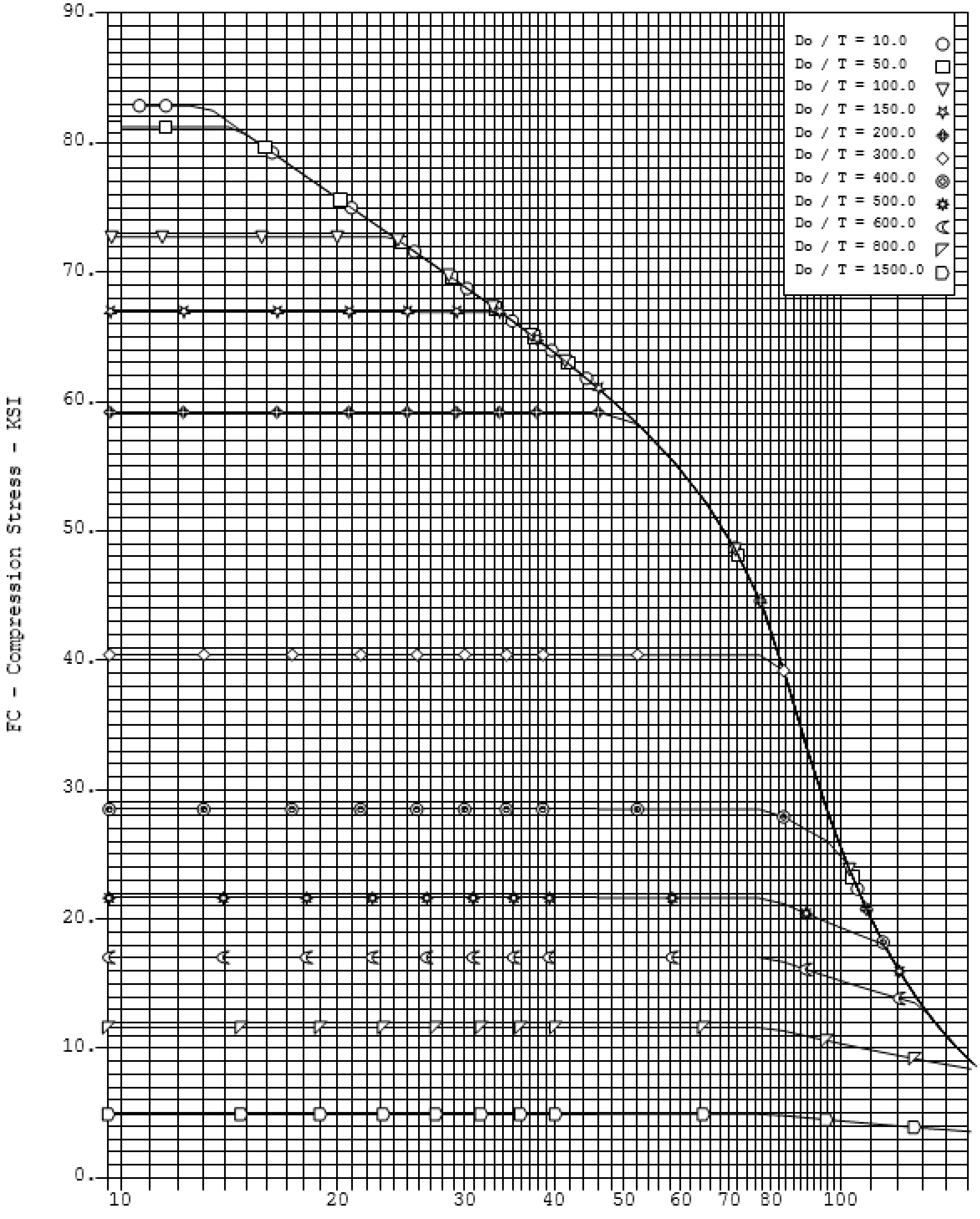


Figure 8.2.4-9 Sample IDAT/SM100 Cylinder Buckling Curve for Low Alloy (4130) 95 ksi Steel Tube; METDB #391; Low Alloy Steel Tube, 95 ksi; S-Basis;
F_{tu}=95.0 ksi; F_{ty}=75.0 ksi; E_c=29,000 ksi; n_c=10, F_{cmax}=82.9 ksi

8.3 Beam-Column Analysis

A beam-column is a structural member, either straight or having a small initial curvature, whose cross-sectional dimensions are small with respect to its length. It is subject to axial as well as transverse loading. The axial load can be either tension or compression; however, a beam-column tends to be more critical under compressive axial load.

The effect of the lateral load does not generally delay the buckling of the column and thus, for all practical cases, is assumed not to increase P_{cr}. As a result the beam-column analysis does not replace the column buckling analysis. Buckling is a bending phenomenon in which the bending rigidity, EI, resists the applied bending moment measured by an axial load. Beam-column analysis is a strength check of the beam due to applied loads plus amplified beam-column moments.

A column buckling analysis must still be performed even if a beam-column analysis is performed.

The critical column load is determined using the methods outlined in Section 8.2 and only the axial load. The critical column load is independent of both magnitude and location of any transverse loads and does not represent the maximum load the column can carry but the load under which the column is unstable without any transverse loads.

8.3.1 Beam-Column with Axial Compression

Beam-column action results in a magnification of the moments which result from eccentricities, transverse loads or applied moments which can make the stress level in the member more critical than would be calculated from a direct consideration of the applied loads.

The total beam-column moment and shear can be obtained classically in two ways. There is an approximate approach which works well with beams that deflect according to Equation 8.3.1-1. This method is described in Section 8.3.1.1.

$$y = a\sin\left(\pi x / L\right)$$

Equation 8.3.1-1

where
a is the magnitude of the maximum deflection (in)
x is a location on the beam, some arbitrary distance from the end (in)
y is the deflection of the beam at location x (in)
L is the total length of the beam (in)

A more exact solution can be obtained which is the result of the superposition of bending moment due to the effect of the combined action of one or more transverse loads and bending moments and the entire axial load. This method is discussed in Section 8.3.1.2 with standard solutions tabulated in Section 8.3.2. This method becomes inaccurate as the magnified beam-column moment is small, on the order of less than 110% of the direct moment. As the magnified moment increase, this method accurately predicts the beam column behavior.

A parameter which is used in beam-column calculations is a ratio of stiffness to applied load

$$j = \sqrt{\frac{EI}{P}}$$

Equation 8.3.1-2

where
E is the Modulus of Elasticity (psi)
I is the moment of inertia of the member about the plane of bending (in^4)
P is the applied load (lbs)

8.3.1.1 Approximate Beam-Column Solutions

An approximate solution for a beam-column magnification factor, η, can be obtained in a fairly simple manner by using the applied axial load and critical column load. It is applicable to beams which are pinned-pinned, have deflection characteristics described by Equation 8.3.1-1 and have a constant EI stiffness. This approximate solution is given by

Equation 8.3.1-3

$$\eta = \frac{1}{1 - \frac{P}{P_{cr}}}$$

where
P is the applied load (lbs)
P_{cr} is the allowable column buckling load per Section 8.2 about the plane of bending (lbs)

In Equation 8.3.1-3, the column buckling load is not necessarily the critical column buckling load, P_{col} about the minimum moment of inertia, although it may be. It is the column buckling load about the plane of bending.

Note the beam-column magnification factor, η, is not related to the tangent modulus ratio of Section 8.2. The beam-column magnification factor increases non-linearly as P approaches P_{cr}.

The magnified moment is then given by

Equation 8.3.1-4

$$M = \eta M_o$$

where
M_o is the moment resulting from direct application of transverse load (in-lbs)

This method gives an exact answer for a pinned-pinned member with a sinusoidally distributed lateral load. For a single span beam-column where L/j is less than or equal to 2.5 this approach results in an accuracy of within about 10%. As the L/j ratio increases, the error also increases. Error is also introduced where the applied loading includes concentrated moments, either directly applied or due to an axial load times an eccentricity. For continuous members, a single magnification factor for the entire beam should not be calculated because the inflection points shift with applied end load.

A better solution in the event there is an eccentricity or an applied moment is to assume the member is rigid and to calculate the bending moments resulting from all loads, moments and eccentricities. The deflection for this load state is determined either through a classical solution or from a finite element analysis using the calculated bending moments as applied loads.

Once the elastic deflections have been determined for all loads and moments, the magnification factor is used to magnify the sum of the elastic deflections and inherent eccentricities. Refer to Figure 8.3.1-1 for clarification. Deformations and eccentricities should be combined appropriately along the length of the beam.

For a simple beam which has a built-in sinusoidal bowing and is undergoing a uniform or point loading resulting in a sinusoidal variation, the total maximum deflection can then be calculated as

Equation 8.3.1-5

$$y_{tot} = (e + \Sigma \eta y_i)$$

where
e is the eccentricity (in)
y_i is the total maximum deflection for each of the external applied lateral loads and moments (in)
η is the beam-column magnification factor from Equation 8.3.1-3

The eccentricity should not be multiplied by the beam-column magnification factor since it is not a function of any external loading but rather a geometric feature of the beam.

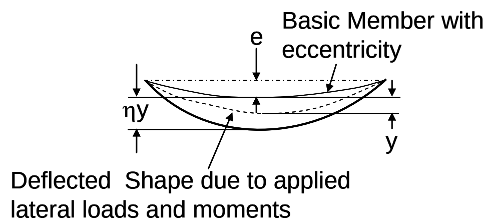
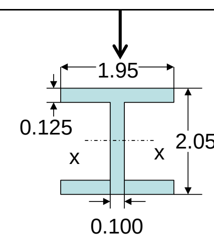
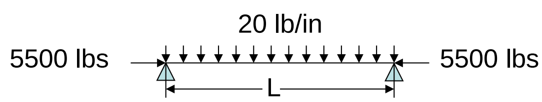


Figure 8.3.1-1 Beam-column Deflected Shape

From the final deflected shape, the maximum moments can be determined. For some beams, this can be done by classical equations. For many real aircraft beam geometries and loadings, this will need to be done by finite element analysis using enforced displacements. These models can be fairly simple beam or bar models.

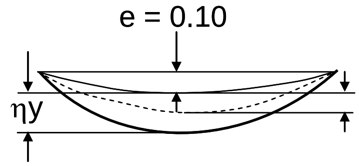
Once the magnified bending moment is calculated a combined margin of safety can be calculated using Section 6.3.6.2.

8.3.1.1.1 Example Problem – Beam-Column Analysis using Approximate Method

Given: A compression member 50 inches long Material is 7050-T7451 Aluminum $E = 10.6\text{x}10^6$ psi $F_{tu} = 75000$ psi $M_{allow} = 40857$ in-lb per Section 6.3.2 Section is an IBEAM shown c=1, thus L'=L=50 in. $P_{col} = \pi^2 E_c I/L^2 = \pi^2(10.6\text{x}10^6)(0.1546)/(50)^2 = 6470$ lbs	 $I_x = 0.5009\ in^4$ $I_y = 0.1546\ in^4$ $A = 0.6675\ in^2$ $k = 1.127$ $c_y = 1.025$
Calculate Column Buckling Margin of Safety: M.S. = $P_{cr}/P-1 = 6470/5500-1 = \underline{\underline{0.18}}$	
Calculate $J=(EI/P)^{0.5} =[(10.6\text{x}10^6)(0.5009)/(5500)]^{0.5} = 31.1$ $L/j = 50/31.1 = 1.61 < 2.5$, so reasonable accuracy would be expected.	
Beam is pinned-pinned so c=1.0, $L'=L/\sqrt{c} = 50/\sqrt{1.0} = 50$. Using Equation 8.0.0-1, calculate $\rho = \sqrt{(I_{Bending\text{-}Axis}/A)} = \sqrt{(0.5009/0.6675)} = 0.866$ Calculate $L'/\rho = 50/0.866 = 57.74$ – Column is a long slender column and will buckle elastically per Equation 8.2.1-1 Using Equation 8.2.1-1, calculate $P_{cr} = \pi^2 E_c I/L^2 = \pi^2(10.6\text{x}10^6)(0.5009)/(50)^2 = 20957$ lbs (about bending axis)	
Using Equation 8.3.1-3, calculate the beam-column magnification factor: $\eta = 1/(1-P/P_{cr}) = 1/(1-5500/20957)= 1.36$	
Calculate the elastic moment for a pin-ended uniformly loaded beam[1]: $M_o = wL^2/8 = (20)(50)^2/8= 6250$ in-lb	
Using Equation 8.3.1-4, calculate the beam-column moment: $M =\eta M_o = (1.36)(6250) = 8500$ in-lb	
Using Equation 6.3.6-9, $R_b=m_{appl}/M_{allow} = 8500/40857 = 0.208$ Using Equation 6.3.6-11, $R_c=P/P_{cr} = 5500/6470 = 0.850$ Using Equation 6.3.6-12, M.S.= $1/(R_b+R_c)-1 = 1/(0.208 +0.850)-1 = \underline{\underline{-0.05}}$	

[1] Reference 8-12, pg. 104

8.3.1.1.2 Example Problem – Beam-Column Analysis using Approximate Method with Initial Eccentricity

Given: Beam problem of Example 8.3.1.1.1, but the part has bowed during manufacture. The bowed shape is sinusoidal with the maximum eccentricity at mid-span is 0.10 in. How has this affected the margins of safety?	
L/j, c, L′, ρ, and L′/ρ are unaffected by the bow. P_{cr} is also unaffected so the column buckling margin remains unchanged. M.S. (Buckling) = <u>0.18</u>	
Using Equation 8.3.1-3, calculate the beam-column magnification factor: $\eta = 1/(1-P/P_{cr}) = 1/(1-5500/20957) = 1.36$	
Calculate the elastic moment for a pin-ended uniformly loaded beam[2]: $M_o = wL^2/8 = (20)(50)^2/8 = 6250$ in-lb	
Calculate the deflection for a pin-ended uniformly loaded beam: $y=-5wL^4/(384EI) = -5(20)(50)^4 / [384(10.6x10^6)(0.5009)] = 0.307$ in The bowed shape of the beam is sinusoidal. The deformed shape of the beam subject to a uniform loading is the same, thus the maximum deflections at the beam center can be combined directly. Using Equation 8.3.1-5, the total maximum deflection, $y_{tot} = e + \eta y = 0.10 + (1.36)(0.307) = 0.518$.	
Calculate a running load which would result in a total deformation of 0.518 in. Since $y=-5wL^4/(384EI)$, thus $w=384EIy/(5L^4) = [384(10.6x10^6)(0.5009)(0.518)]/[(5)(50)^4] = 33.8$ lb/in	
Calculate the beam-column moment for a pin-ended uniformly loaded beam: $M_{BC} = wL^2/8 = 33.8(50)^2/8 = 10563$ in-lb. This is the beam-column moment because it is based on the deflected shape which has had the magnification factor and includes the initial eccentricity. No further factoring is required.	
Using Equation 6.3.6-9, $R_b=m_{appl}/M_{allow} = 10563/40857 = 0.259$ Using Equation 6.3.6-11, $R_c=P/P_{cr} = 5500/6470 = 0.850$ Using Equation 6.3.6-12, $M.S.= 1/(R_b+R_c)-1 = 1/(0.259 +0.850)-1 =$ <u>-0.10</u>	

8.3.1.2 <u>Beam-Column</u> <u>Analysis</u> <u>Using</u> <u>Superposition</u>

For many compression beam-column loading scenarios found in aircraft design, the bending moment in a single span takes the form of

$$M=C_1 sin(x/j) + C_2 cos(x/j) + f(w) \qquad \textbf{Equation 8.3.1-6}$$

where
C_1, C_2 and f(w) are obtained from Table 8.3.2-1 and are dependent on the character of the transverse load
j is calculated from Equation 8.3.2-2
x is the location on the beam where the moment is to be obtained. See Table 8.3.2-1 for sign convention (in)

The general principle of superposition does not apply to beam-columns since the bending moments due to the axial and bending loads acting separately is not the same as the resulting bending moment when those same loads are acting simultaneously. However it is true that the bending moments resulting from a transverse load T_1 acting with an axial load A can be superimposed with the bending moments resulting from a transverse load T_2 acting with the <u>same</u> axial load A. Thus, the method of superposition can be applied as long as the bending moments calculated combine each of the transverse loads and moments with the entire axial load.

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[2] Reference 8-12, pg. 104

Section 8.3.2 provides some standard solutions for C_1, C_2 and f(w) for a single load input. To obtain a solution for a more complex load scenario, combine the solutions as required, always using the total axial load. Section 8.3.3 provides some example problems.

The following equations can be used to calculate shear, V, deflection, δ, and slope, θ, for a single span

$$V = \frac{C_1}{j}\cos\left(x/j \right) - \frac{C_2}{j}\sin\left(x/j \right) + f'(w) \qquad \textbf{Equation 8.3.1-7}$$

$$\delta = (M_o - M)/P \qquad \textbf{Equation 8.3.1-8}$$

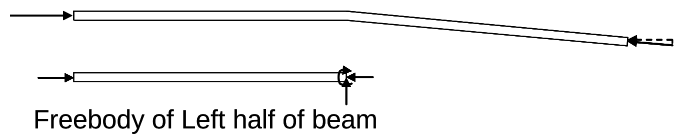
$$\theta = (V_o - V)/P \qquad \textbf{Equation 8.3.1-9}$$

where
f′(w) is the first derivative of f(w)
M_o is the bending moment of the beam with no beam-column magnification (in-lb)
M is the bending moment calculated from Equation 8.3.1-6 (in-lb)
V_o is the shear of the beam with no beam-column magnification (lb)
V is the shear calculated from Equation 8.3.1-7 (lb)
P is the beam axial load (lb)

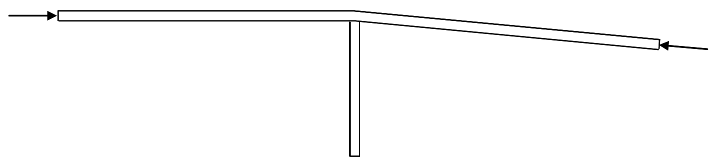
8.3.1.3 Eccentricity Considerations

Section 2.3.2.3.1 discusses guidelines for use in beam-column analysis of different types of structure due to unintentional eccentricities caused by adverse tolerance during manufacture. It is noted in Section 2.3.2.3.1 that the LM Aero legacy integrally-stiffened wing skin analysis programs assume an eccentricity of L/400. Reference 8-5 provides a discussion of research which has been conducted to determine reasonable initial deflections to account for unintentional eccentricities due to manufacturing, load application, material non-homogeneity, and, if applicable, initial curvature. From Reference 8-5, for all columns the recommended initial deflection is L/400.

Eccentricities can also be designed into a part because of space or other constraints. Every effort should be made to eliminate or minimize the amount of designed-in eccentricity. This includes straightening out load lines, removing kicks in a part and ensuring that the load line goes through the centroid of attachment fastener patterns. If any load line changes occur, providing an out of plane support is recommended. This is illustrated in Figure 8.3.1-2. The out-of-plane support eliminates the beam-column concerns since, in addition to reacting the out-of-plane load and moment due to the offset, it reduces the beam effective length by providing an intermediate support.



Not Recommended



Adding a lateral support reacts the out-of-plane load due to the change in load direction and the moment due to eccentricity of load and divides the beam into shorter spans while supporting it against out-of-plane deflection

Recommended

Figure 8.3.1-2 Supported Eccentric Load Path

8.3.2 Beam-Column Compression - Standard Solutions

Table 8.3.2-1 provides exact theoretical standard beam-column solutions for different loading scenarios. These may be superimposed to obtain a final beam-column solution per the method discussed in Section 8.3.1.2.

The variable j is given by Equation 8.3.1-2 and has a dimension of length (inches). All angles should be input in radians. It is recommended that all calculations using equations from Table 8.3.2-1 should be carried to at least four significant figures to maintain accuracy.

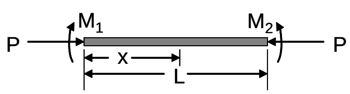
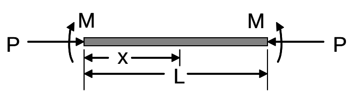
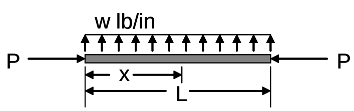
Cases 9 and 12 do not provide a maximum moment equation. Differential calculus must be used to find the location of maximum moment or moments at several locations on the span so that a smooth curve can be drawn through the plotted points to obtain the maximum value. That method is not provided in this manual and is found in Reference 8-5.

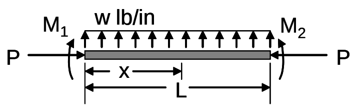
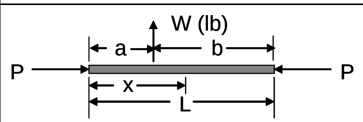
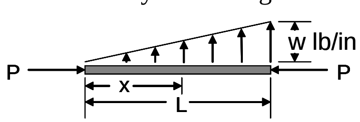
All points where concentrated loads or moments are acting should also be checked for maximum possible bending moments.

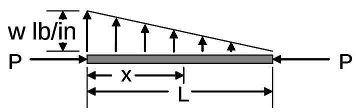
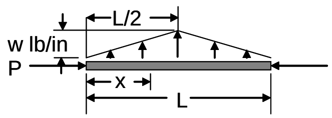
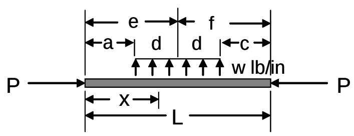
As noted previously, the beam may fail in column buckling before the stresses due to beam-column bending cause failure. Always calculate a margin of safety for column buckling in addition to any margins for beam-column effects.

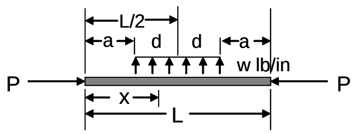
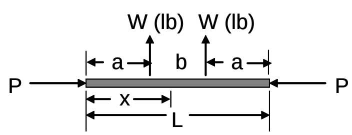
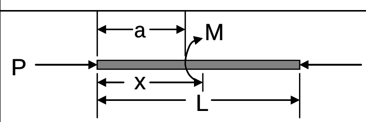
If a beam-column is of non-uniform cross-section, the equations describing M/EI become very complex and it is impossible to develop standard formulas similar to those included in Table 8.3.2-1. Analysis of non-uniform cross-sections is outside of the scope of this manual and often is best suited to a numerical analysis. References 8-7, 8-13 and 8-28 discuss analytical approaches to problems with non-uniform cross-sections. In addition, finite element solutions can be utilized. Refer to Section 8.6.3 for discussion of modeling techniques.

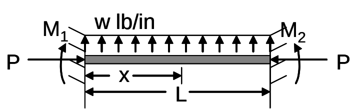
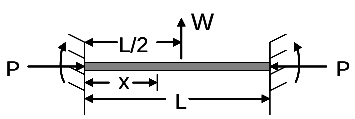
Table 8.3.2-1 Beam-Column Solutions with Axial Compression Loads for Columns with Uniform Cross-Section

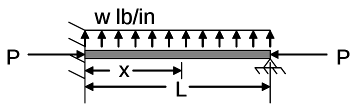
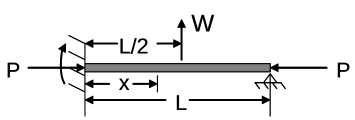
Case	Loading	C_1	C_2	f(w)	Maximum Moment
Sign Conventions all cases: Point load, W, or running load, w, is positive upward. M is positive when it produces compression in the upper fibers					
1 Pinned Pinned		$\dfrac{M_2 - M_1\cos(L/j)}{\sin(L/j)}$	M_1	*0*	$M_{max} = \dfrac{M_1}{\cos(x/j)}$ *where* $\tan(x/j) = \dfrac{M_2 - M_1\cos(L/j)}{M_1\sin(L/j)}$
2 Pinned Pinned	 Equal End Moments	$M_1\tan(L/2j)$	M_1	*0*	$M_{max} = \dfrac{M_1}{\cos(x/j)}$ *at midspan, L/2*
3 Pinned Pinned	 Uniform Load	$\dfrac{wj^2[\cos(L/j) - 1]}{\sin(L/j)}$	$-wj^2$	wj^2	$M_{max} = wj^2[1 - \sec(L/2j)]$ *at midspan, L/2*

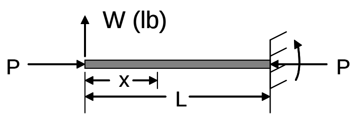
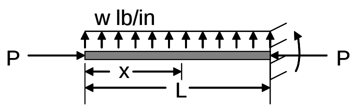
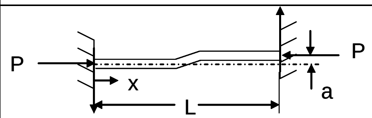
Case	Loading	C_1	C_2	f(w)	Maximum Moment
Sign Conventions all cases: Point load, W, or running load, w, is positive upward M is positive when it produces compression in the upper fibers					
4 Pinned Pinned		$$\frac{D_2 - D_1\cos(L/j)}{\sin(L/j)}$$ $D_1=M_1-wj^2$ $D_2=M_2-wj^2$	D_1	wj^2	$$M_{\max} = \frac{D_1}{\cos(x/j)} + wj^2$$ *where* $$\tan(x/j) = \frac{D_2 - D_1\cos(L/j)}{D_1\sin(L/j)}$$
5 Pinned Pinned		$$\frac{-Wj\sin(b/j)}{\sin(L/j)}$$ $x < a$	*0*	*0*	$$M_{\max} = \frac{C_1^2 + C_2^2}{C_2}\cos(x/j)$$
		$$\frac{Wj\sin(a/j)}{\tan(L/j)}$$ $x > a$	$-Wj\sin(a/j)$	*0*	*where* $$\tan(x/j) = \frac{C_1}{C_2}$$
6 Pinned Pinned	Uniformly Increasing Load 	$$\frac{-wj^2}{\sin(L/j)}$$	*0*	$$\frac{wj^2x}{L}$$	*Occurs at* *cos(x/j)=(j/L)sin(L/j)* *Solve for x/j and x, substitute into* Equation 8.3.1-6

Case	Loading	C_1	C_2	f(w)	Maximum Moment
Sign Conventions all cases: Point load, W, or running load, w, is positive upward M is positive when it produces compression in the upper fibers					
7 Pinned Pinned	 Uniformly Decreasing	$\frac{wj^2}{\tan(L/j)}$	$-wj^2$	$wj^2(1-\frac{x}{L})$	*Occurs at* $\cos(\frac{L-x}{j}) = \frac{j}{L}\sin(L/j)$ *Solve for x/j and x, substitute into* Equation 8.3.1-6
8 Pinned Pinned		$\frac{-2wj^3}{L\cos(L/2j)}$ $x < L/2$	0	$2wj^2(\frac{x}{L})$	$M_{max} = \frac{-2wj^3}{L}\tan(L/2j) + wj^2$ *at midspan, L/2*
		$\frac{2wj^3\cos(L/j)}{L\cos(L/2j)}$ $x > L/2$	$\frac{-4wj^3}{L}\sin(L/2j)$	$2wj^2(1-\frac{x}{L})$	
9 Pinned Pinned	 b=a+2d Partial Uniformly Distributed Load	$\frac{-2wj^2\sin(d/j)\sin(f/j)}{\sin L/j}$ $x < a$	0	0	
		$\frac{2wj^2\sin(d/j)\sin(e/j)}{\tan L/j} - wj^2\sin(b/j)$ $a < x < b$	$-wj^2\cos(a/j)$	wj^2	
		$\frac{-2wj^2\sin(d/j)\sin(e/j)}{\tan L/j}$ $b < x < L$	$-2wj^2\sin(d/j)\sin(e/j)$	0	

<table>
<tr><th>Case</th><th>Loading</th><th>C1</th><th>C2</th><th>f(w)</th><th>Maximum Moment</th></tr>
<tr><td colspan="6">Sign Conventions all cases:
Point load, W, or running load, w, is positive upward
M is positive when it produces compression in the upper fibers</td></tr>
<tr><td rowspan="3">10
Pinned
Pinned</td><td rowspan="3">
Symmetrical Partial Uniformly Distributed Load</td><td>$-wj^2 \sin(d/j)\sec(L/2j)$
$x < a$</td><td>0</td><td>0</td><td rowspan="3">$M_{max} = wj^2[1 - \frac{\cos(a/j)}{\cos(L/2j)}]$
at midspan, L/2</td></tr>
<tr><td>$-wj^2 \cos(a/j)\tan(L/2j)$
$a < x < (L-a)$</td><td>$-wj^2\cos(a/j)$</td><td>wj^2</td></tr>
<tr><td>$wj^2 \sin(d/j)\sec(L/2j)\cos(L/j)$
$(L-a) < x < L$</td><td>$-2wj^2 \sin(d/j)\sin(L/2j)$</td><td>0</td></tr>
<tr><td rowspan="3">11
Pinned
Pinned</td><td rowspan="3"></td><td>$-Wj\frac{\cos(b/2j)}{\cos(L/2j)}$
$x < a$
(continued next page)</td><td>0</td><td>0</td><td rowspan="3">$M_{max} = -Wj\frac{\sin(a/j)}{\cos(L/2j)}$
at midspan, L/2</td></tr>
<tr><td>$-Wj\frac{\sin(a/j)}{\tan(L/2j)}$
$a < x < (L-a)$</td><td>$-Wj\sin(a/j)$</td><td>0</td></tr>
<tr><td>$Wj\frac{\cos(b/2j)\cos(L/j)}{\cos(L/2j)}$
$(L-a) < x < L$</td><td>$-Wj\frac{\sin(L/j)\cos(b/2j)}{\cos(L/2j)}$</td><td>0</td></tr>
<tr><td>12
Pinned
Pinned</td><td></td><td>$-M\frac{\cos(b/j)}{\sin(L/j)}$
$x < a$</td><td>0</td><td>0</td><td></td></tr>
</table>

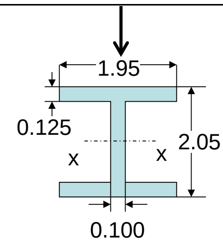
Case	Loading	C_1	C_2	f(w)	Maximum Moment
Sign Conventions all cases: Point load, W, or running load, w, is positive upward M is positive when it produces compression in the upper fibers					
	b=L-a Clockwise Couple or Applied Moment	$-M\dfrac{\cos(a/j)}{\tan(L/j)}$ *x > a*	*M cos(a/j)*	*0*	
13 Fixed Fixed	 Uniform Load	$-wj\dfrac{L}{2}$	$\dfrac{-wjL}{2\tan(L/2j)}$	wj^2	$M = wj^2[1-\dfrac{\frac{L}{2j}}{\tan(L/2j)}]$ *At x=0:* $M = -wj^2[\dfrac{\frac{L}{2j}}{\sin(L/2j)}-1]$ *At x=L/2*
14 Fixed FIxed	 Concentrated Load at Center	$-\dfrac{Wj}{2}$ *x < L/2*	$\dfrac{Wj[1-\cos(L/2j)]}{2\sin(L/2j)}$	*0*	$M_{\max} = \dfrac{Wj}{2}\left(\dfrac{1-\cos(L/2j)}{\sin(L/2j)}\right)$ *At x=0* $M = -M_{max}$ *At x=L/2*
		$\dfrac{Wj}{2}[2\cos(L/2j)-1]$ *x>L/2*	$\dfrac{Wj[\cos(L/j)-\cos(L/2j)]}{2\sin(L/2j)}$	*0*	

Case	Loading	C_1	C_2	f(w)	Maximum Moment
Sign Conventions all cases: Point load, W, or running load, w, is positive upward M is positive when it produces compression in the upper fibers					
15 Fixed Pinned	Uniform Load 	$C_1 = \frac{-\left(\tan(L/2j) - \frac{L}{2j}\right)}{\tan(L/j) - \frac{L}{j}}(wLj) - wj^2 \tan(L/2j)$	$C_2 = \frac{\tan(L/2j) - \frac{L}{2j}}{\tan(L/j) - \frac{L}{j}}(wLj \tan(L/j) - wj^2)$	wj^2	$M_{max} = wLj \tan(L/j) \frac{\tan(L/2j) - \frac{L}{2j}}{\tan(L/j) - \frac{L}{j}}$ *At x=0*
16 Fixed Pinned		$C_1 = -\frac{Wj}{2} \bullet \frac{j \tan(L/j) \sec(L/2j) - L}{j \tan(L/j) - L}$	$C_2 = \frac{WL}{2} \bullet \frac{j \tan(L/j)(\sec(L/2j) - 1)}{j \tan(L/j) - L}$ $x < L/2$	*0*	$\frac{WL}{2} \bullet \frac{j \tan(L/j)(\sec(L/2j) - 1)}{j \tan(L/j) - L}$ *at x=0*
		$C_1 = \frac{Wj}{2} \bullet \frac{L + 2j \sin(L/2j) - 2L \cos(L/2j)}{j \tan(L/j) - L}$	$C_2 = \frac{Wj}{2} \bullet \left(\frac{L \tan(L/j)\left[\sec(L/2j) - 1\right]}{j \tan(L/j) - 1} - 2\sin(L/2j) \right)$	*0*	

Case	Loading	C_1	C_2	f(w)	Maximum Moment
Sign Conventions all cases: Point load, W, or running load, w, is positive upward M is positive when it produces compression in the upper fibers					
		x > L/2			
17 Fixed Free					$M_{max} = Wj \tan L/j$ *at x=L*
18 Fixed Free					$M_{max} = wj^2[(1-\sec L/j) + \frac{L}{j}\tan L/j]$ *at x=L*
19 Fixed Fixed	 with Eccentricity				$M = \frac{aP}{2}\left(\frac{\tan L/2j}{\tan(L/2j) - \frac{L}{2j}}\right)$ $M_{max} = M$ *at x=0* $M_{max} = -M$ *at x=L*

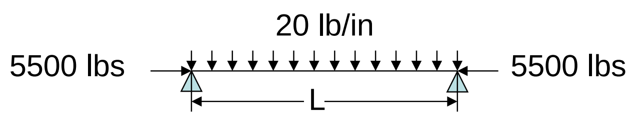
8.3.3 Example Problem – Beam-Column Compression

Given: Compression Member of Example 8.3.1.1.1
A compression member 50 inches long
Material is 7050-T7451 Aluminum E = $10.6x10^6$ psi
Ftu = 75000 psi
M_{allow} = 40857 in-lb per Section 6.3.2
Section is an IBEAM shown
c=1, thus L'=L=50 in.
$P_{col} = \pi^2 E_c I/L^2 = \pi^2(10.6x10^6)(0.1546)/(50)^2 =$ 6470 lbs



$I_x = 0.5009\ in^4$ I_y=0.1546 in^4
A = 0.6675 in^2 k = 1.127
c = 1.025

Calculate Column Buckling Margin of Safety:
M.S. = P_{cr}/P-1 = 6470/5500-1 = <u>0.18</u>



Calculate $j=(EI/P)^{0.5} = [(10.6x10^6)(0.5009)/5500)]^{0.5} = 31.1$
L/j = 50/31.1 = 1.61 < 2.5, so reasonable accuracy would be expected.

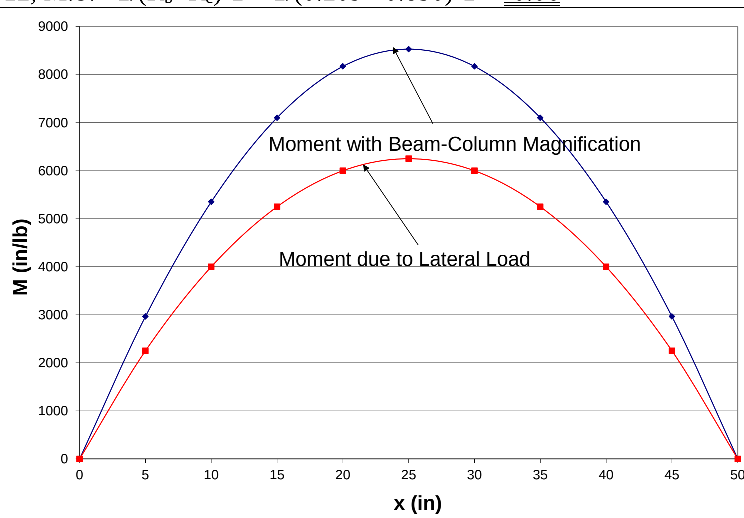
Beam is pinned-pinned with a uniformly distributed lateral load. Case 3 of Table 8.3.2-1 provides C_1, C_2 and f(w)
$C_1 = wj^2 [\cos(L/j)-1] / \sin(L/j) = (-20)(31.1)^2[\cos(50/31.1)-1]/\sin(50/31.1) = 20071.5419$
$C_2 = -wj^2 = -(-20)(31.1)^2 = 19344.2000$
$f(w) = wj^2 = (-20)(22.9)^2 = -19344.2000$

Equation 8.3.1-6

$$M=C_1\sin(x/j) + C_2\cos(x/j) + f(w) = (20071.5419)\sin(x/31.1) + (19344.2000)\cos(x/31.1) – 19344.2000$$

Maximum Moment occurs at x=L/2 = 50/2 = 25, so M= 8532 in-lb;
Positive moment indicates compression in the upper fibers.
This is 1.0% higher than was calculated using the approximate method.

Using Equation 6.3.6-9, $R_b = m_{appl}/M_{allow}$ = 8532/40857 = 0.209
Using Equation 6.3.6-11, $R_c = P/P_{cr}$ = 5500/6470= 0.850
Using Equation 6.3.6-12, M.S.= $1/(R_b+R_c)-1$ = 1/(0.209 +0.850)-1 = <u>-0.06</u>



Bending Moment as a Function of Beam Location

8.3.4 Beam-Column with Axial Tension

When the axial load applied to a beam is a tension load the effect is usually to reduce the primary bending moment rather than to increase it as a compressive axial load does. It is general practice to not take advantage of such relieving effects because in the complex load scenarios found in aircraft structure, the load or eccentricity which is providing the reduction may not occur as predicted. This could result in an understrength design. As result, the relieving beam-column effects of an axial tension load on the moment are ignored and the part is sized based on the primary bending moment combined, as appropriate, with the axial load and any other loading. If further information is required, References 8-2 and 8-13 provide standard solutions.

8.4 Crippling

Crippling failure or collapse failure is a failure mode which is unique to thin walled sections of short columns or laterally restrained compression members. It represents a local distortion of the cross section, as opposed to the overall distortion of the member seen in a buckling failure. Figure 8.4.0-1 illustrates how a member fails in crippling. In this mode of failure, at load P_2, the individual flanges of the cross-section buckle, but the corners and web of the cross-section continue to remain straight. If the flanges have different geometries, they may buckle at different load levels. At a different load, P_3, the web buckles, but again the corners remain straight. Finally as the load continues to increase to P_4, the corners reach material yield and the cross-section fails locally and takes on a permanent distortion which resembles a joggle, the member can carry no further load and collapses. At this point, it is said to have crippled. History has shown that this can be an explosive failure causing complete collapse of a structure when it occurs in a primary member. It is very important to recognize this failure mode as potentially catastrophic.

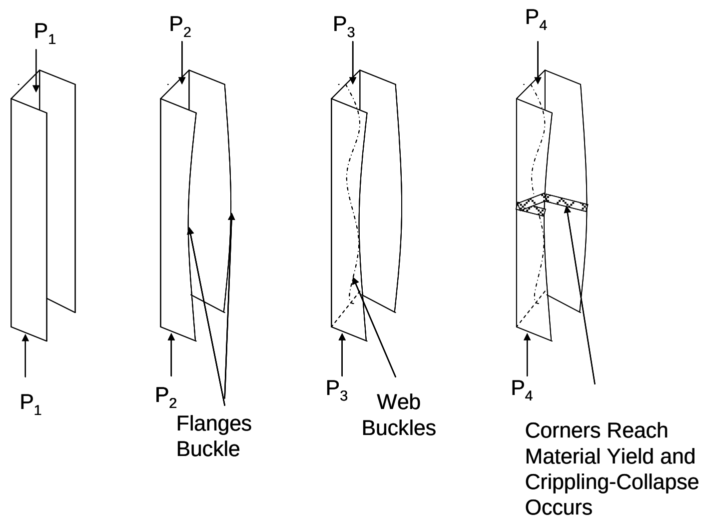


Figure 8.4.0-1 Crippling Failure of A Channel Section

This failure mode generally occurs in short columns which are too short to fail in column flexural buckling but which may be subject to torsional instability. Unlike other column failure modes, the techniques for determining crippling allowables are largely based on empirical data. The method outlined below was developed from and then used to predict failure for 323 specimens made from both sheet metal and extruded material. Reference 8-15[1] reports an average conservatism of 8.1 percent in these predictions.

This method breaks down the cross-section into flat-plate elements per guidelines presented in Section 8.4.1. A crippling allowable stress for each element is calculated. Then the weighted average of the individual flange element stresses is used to calculate a weighted average stress for the cross-section.

$$F_{cc} = \Sigma\,[F_{cc\text{-}i}\, b_i\, t_i] \,/\, \Sigma\,[b_i\, t_i] \quad \text{for } i=1, \text{no. of elements} \qquad \textbf{Equation 8.4.0-1}$$

Where
$F_{cc\text{-}i}$ is the crippling stress for an individual element of the cross-section(psi)
b_i is the effective width of an individual element calculated per the guidelines of Section 8.4.2(in)
t_i is the thickness of individual elements (in)

[1] Reference 8-14, LR 1793, reports the test results. Reference 8-15, LR9292, develops an analysis method for predicting the test results and shows the correlation.

The allowable crippling failure stress is the average stress of the cross-section, NOT the buckling failure stress of an individual element.

The allowable crippling load, P_{cc}, can be calculated as

$$P_{cc} = F_{cc}A$$ **Equation 8.4.0-2**

where
F_{cc} is the crippling stress calculated from Equation 8.4.0-1 (psi)
A is the actual area of the cross section (in^2), *i.e.*, <u>not</u> the area obtained in the calculation of the crippling stresses

A table similar to Table 8.4.0-1 is constructed to aid in the calculation process described in the previous paragraph. Each flat plate element is assigned a flat plate edge constraint which is either one-edge-free (OEF) or no-edge-free (NEF). An example of an OEF element would be the flange of a channel which is free along one edge and attached to the remainder of the channel along the other edge, while an example of a NEF element would be the channel's web, where both edges of the flat plate element are attached to adjacent flanges. The width, b, is taken per guidelines presented in Section 8.4.2.

Table 8.4.0-1 Sample Format for Calculation of Crippling Stress Allowable

Element No	b	t	b/t	bt	Edge – OEF or NEF	F_{cc-i} from appropriate material allowable curve	btF_{cc-i}
1							
2							
…							
n							
Summation				Σ			Σ
F_{cc}	$F_{cc}= \Sigma\, [F_{cc-i} b_i\, t_i\,]/ \Sigma\, [b_i\, t_i]$						
P_{cc}	$P_{cc} =F_{cc}A$						

The margin of safety is calculated from

$$M.S. = \frac{P_{cc}}{P} - 1$$ **Equation 8.4.0-3**

where
P is the axial applied load (lb)
P_{cc} is the crippling load allowable calculated from Equation 8.4.0-2 (lb)

8.4.1 Determination of Crippling Allowables

The allowable crippling curve is provided in terms of the ratio of flange width, b, and thickness, t and is defined by two regions of behavior. The curve is defined as the **<u>maximum</u>** of the two failure modes. One failure mode dominates in the low b/t, high stress region and the other in the high b/t, low stress region. The split between "low" and "high" b/t differs slightly with material and edge constraint but is clearly visible as a change in slope on the curve. This is illustrated for a one-edge-free (OEF) flange in Figure 8.4.1-1.

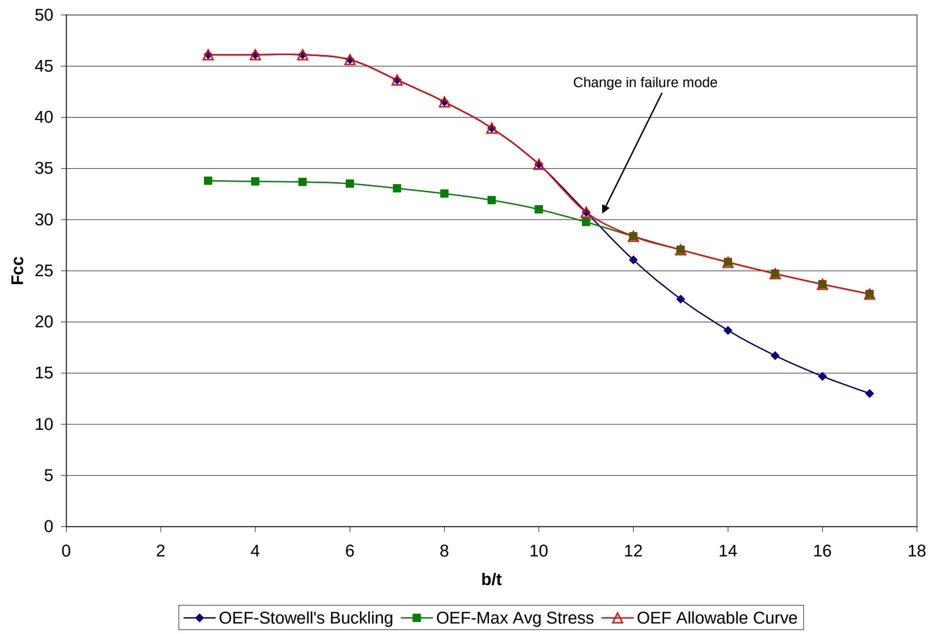


Figure 8.4.1-1 Typical One-Edge-Free Crippling Curve

In the high stress region the critical compressive buckling stress is given by the equation for a thin flat rectangular plate as

$$F_{cr} = \frac{k_c \eta \pi^2 E_c}{12(1-\nu_e^{\,2})}\left(\frac{t}{b}\right)^2 \qquad \textbf{Equation 8.4.1-1}$$

Where
k_c is the theoretical buckling constant which is a function of edge constraint and plate shape. For a long rectangular plate with one edge free, from Figures 10.3-13 and 10.3-14, k_c=0.429 and with no edge free, k_c=4.0[2,3]
η is the plasticity correction factor given by Equation 10.3-8, using Equations 8.4.1-2 and 8.4.1-3
E_c is the compressive modulus of elasticity for the material[4] (psi)
ν_e is the material's elastic Poisson's ratio
b is the flange or plate width (in)

Further discussion on this equation can be found in Section 10.3.1.2. Testing done by several researchers indicate that in the high stress region, the maximum load obtained by a flat plate after buckling is about 4% above the critical buckling stress[5], so this formulation is about 4% conservative.

[2] Reference 8-17, Gerard p.64
[3] This equation is sometimes reformulated as $F_{cr} = K\eta E_c(t/b)^2$, thus $K = k_c\pi^2/[12(1-\nu^2)]$ and K=3.62(NEF, ν=0.30) and 0.376(OEF, for ν=0.25) for a long rectangular plate.
[4] Note that for clad materials, per Section 3.3.2, the secondary modulus of the material is used.
[5] Reference 8-18

The plasticity correction factor[6] calculation varies with the edge constraints and is given by

$$\eta = \eta_{\sec}\frac{1-\nu_e^2}{1-\nu^2}$$ Reference Equation 10.3-8

$$\nu = \nu_p - \frac{E_{\sec}}{E_c}(\nu_p - \nu_e)$$ Reference Equation 10.3-7

$$\eta_{\sec} = \frac{E_{\sec}}{E_c}$$ **Equation 8.4.1-2 (One Edge Free)**

$$\eta_{\sec} = \frac{E_{\sec}}{E_c}\left(0.5 + 0.5\sqrt{\left(0.25 + 0.75\frac{E_{\tan}}{E_c}\right)}\right)$$ **Equation 8.4.1-3 (No Edge Free)**

Where
ν_e is the elastic Poisson's ratio
ν_p is the fully plastic value of the Poisson's ratio. For isotropic materials, $\nu_p = 0.5$
ν is the Poisson's ratio in the transition region between elastic and fully plastic, given by Equation 10.3-7
E_{sec} is the secant modulus given by Equation 3.3.1-4 or 3.3.1-9 (psi)
E_{tan} is the tangent modulus given by Equation 3.3.1-3 or 3.3.1-8 (psi)
E_c is the compressive modulus of elasticity (psi)

Further discussion of the tangent and secant modulus can be found in Section 3.2.2. Table 3.2-1 provides a summary of reduced modulus for buckling for different edge constraints. In the elastic region, E_{sec}=E_{tan}=E so that the plasticity correction factor, η is equal to 1.0. Because the reduced moduli represent a specific slope of the stress strain curve at a particular stress, η is a function of the stress state. As result, when the stresses are above the proportional limit, the solution to Equations 8.4.1-2 and 8.4.1-3 is iterative.

In the low-stress, high-b/t range the maximum stress differs considerably from the critical buckling stress of Equation 8.4.1-1 and the formulation used is based on Reference 8-18. The crippling stress in this region is given by

$$F_{cc}=K_{cc}F_{cr}^{0.25}F_{cy}^{0.75}$$ **Equation 8.4.1-4**

Where
K_{cc} is given by Figure 8.4-4 and is a function of b/t ratio and product form
F_{cr} is the critical buckling stress (psi)
F_{cy} is the compressive yield stress for the material (psi)

Figure 8.4.1-2 presents the curves for K_{cc} which are empirically derived.[7] Table 8.4.1-1 provides coefficients for an eighth-order polynomial curve fit of these curves for use in analysis.

[6] Reference 8-19
[7] Reference 8-20 and Reference 8-18

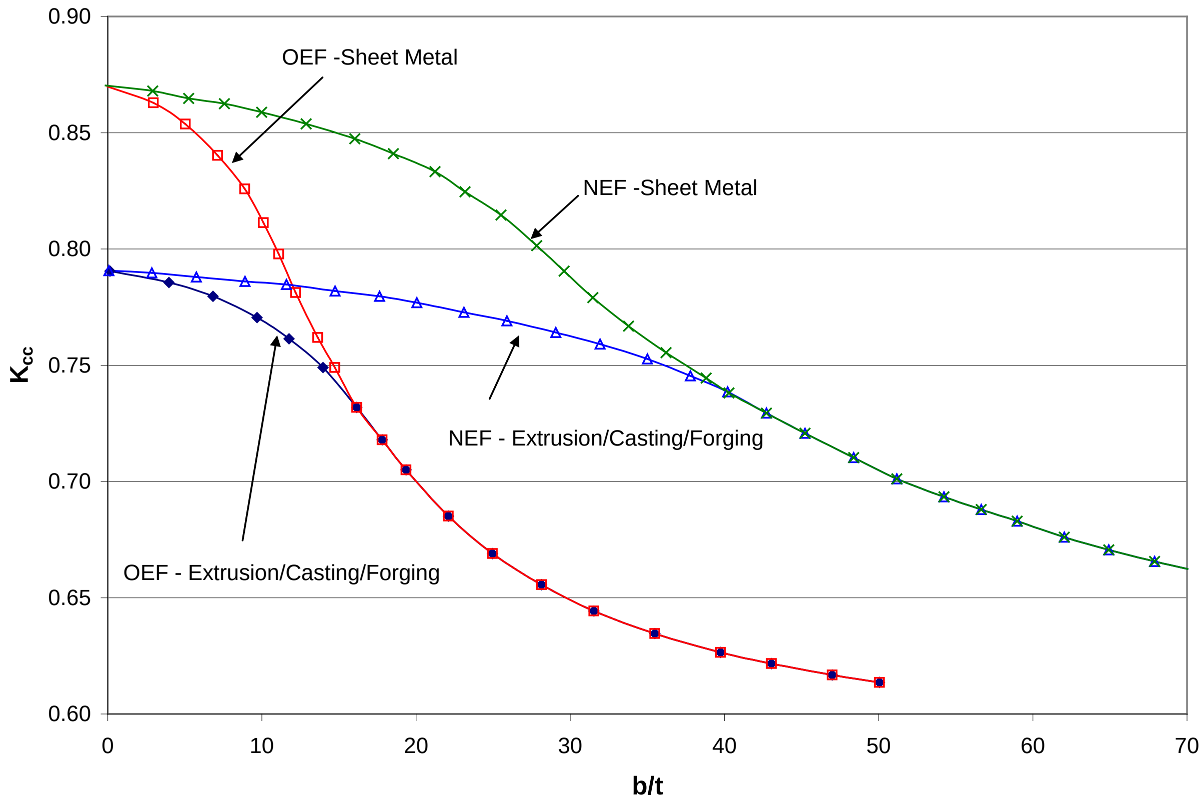


Figure 8.4.1-2 Coefficient K for Maximum Stress for a Plate Buckled in Edge Compression

Table 8.4.1-1 Polynomial Equations for K_{cc} used in Equation 8.4.1-4

Edge constraint/ Product Form	$K_{cc}=$
OEF –Extrusion, Casting, Forging	$-7.79413709\text{x}10^{-13}(b/t)^8 + 1.4440841\text{x}10^{-10}(b/t)^7 - 1.0243711\text{x}10^{-8}(b/t)^6 + 3.3436190\text{x}10^{-7}(b/t)^5 - 4.3432244\text{x}10^{-6}(b/t)^4 - 2.0683716\text{x}10^{-6}(b/t)^3 + 0.0002112(b/t)^2 - 0.0020979(b/t) + 0.7908436$
OEF – Sheet Metal	$1.9673403\text{x}10^{-12}(b/t)^8 - 4.2121913\text{x}10^{-10}(b/t)^7 + 3.6875234\text{x}10^{-8}(b/t)^6 - 1.6846383\text{x}10^{-6}(b/t)^5 + 4.2083595\text{x}10^{-5}(b/t)^4 - 5.3483378\text{x}10^{-4}(b/t)^3 + 0.0025632(b/t)^2 - 0.0066050(b/t) + 0.8698870$
NEF – Extrusion, Casting, Forging	$3.64222689\text{x}10^{-14}(b/t)^8 - 1.0398181\text{x}10^{-11}(b/t)^7 + 1.1882404\text{x}10^{-9}(b/t)^6 - 6.9244065\text{x}10^{-8}(b/t)^5 + 2.184868\text{x}10^{-6}(b/t)^4 - 3.7011575\text{x}10^{-5}(b/t)^3 + 0.0002895(b/t)^2 - 0.00130147(b/t) + 0.7910834$
NEF – Sheet metal	$-7.7289381\text{x}10^{-14}(b/t)^8 + 2.2306989\text{x}10^{-11}(b/t)^7 - 2.5899589\text{x}10^{-9}(b/t)^6 + 1.53220195\text{x}10^{-7}(b/t)^5 - 4.8015115\text{x}10^{-6}(b/t)^4 + 0.000075532(b/t)^3 - 0.00057804(b/t)^2 + 0.00054228(b/t) + 0.8702458$
Limits of Validity	OEF: $b/t \leq 50$ NEF: $b/t \leq 70$

Table 8.4.1-1 also provides a range of validity for each of the equations. This is a result of the curve fit process.

Section 8.4.3 provides a sampling of allowable crippling curves for typical aerospace materials. The analysis suite IDAT provides an automated means of generating these curves for any material in the program SM110. The program has two options: analysis of a standard shape or the generation of the curve. Part of the output generated with the curve option is a table of values for increments of b/t from 3 to 55. This can be used to interpolate intermediate values.

Occasionally a NEF element will have a b/t greater than 55, outside the IDAT range. In that event, Reference 8-20 derives a method whereby F_{cc} may be determined by calculating an effective OEF b/t per the following

$$\left(\frac{b}{t}\right)_{OEF} = \frac{\left(\frac{b}{t}\right)_{NEF}}{\sqrt{\frac{4.0}{0.58}}} = 0.3808\left(\frac{b}{t}\right)_{NEF} \qquad \text{Equation 8.4.1-5}$$

where
b is the flange width (in)
t is the flange thickness (in)
4 is the theoretical buckling coefficient, k_c, of a long plate simply supported on both edges
0.58 is the theoretical buckling coefficient, k_c, of a long plate simply supported on one edge[8]

This is effectively shifts the no-edge-free problem to a one-edge-free problem. Once the adjusted b/t_{OEF} has been determined, the equivalent F_{cc} for the NEF segment can be determined from the appropriate OEF material curve. After determining the crippling stress for the NEF segment, it is then used along with the actual b/t to determine the element's crippling load. This is illustrated in an example problem in Section 8.4.4.3.

8.4.2 Determination of Flat Plate Element Dimensions

The method for calculating the average allowable crippling stress, derived in Reference 8-15, requires that the cross-section of the member be divided up into flat plate elements. The method of dividing up the section differs depending on product form. If the part is manufactured from bent-up material, then Section 8.4.2.1 should be used. If the part is extruded, forged, cast or machined from any material product form, then Section 8.4.2.2 should be used.

8.4.2.1 Determination of Flat Plate Element Dimensions for Cross-Sections made from Bent-up Sheet Metal

Because the process of forming parts from sheet causes the material in the corners to be strengthened due to curvature and, in some alloys, work hardening, the crippling allowable is increased by decreasing effective flange length and, hence the b/t ratio. This is done by reducing the corner material used to the equivalent of 30 degrees for each flat flange. This is illustrated in Figure 8.4.2-1.
Also shown in Figure 8.4.2-1 is the difference between R and R_{bend}. For this particular approach to crippling analysis, the radius used in the calculations is the mid-plane radius or the radius to the centerline of the part thickness, not the bend radius. To calculate the mid-plane radius, R, when only the bend radius, R_{bend} is specified the following equation can be used.

$$R=R_{bend}+t/2$$

where
R_{bend} is the bend radius per the drawing (in)
t is the part thickness (in)

[8] Reference 8-20 (DOW TM15) and Reference 8-21, NACA-TN-734 for $\lambda/b = 2.5$.

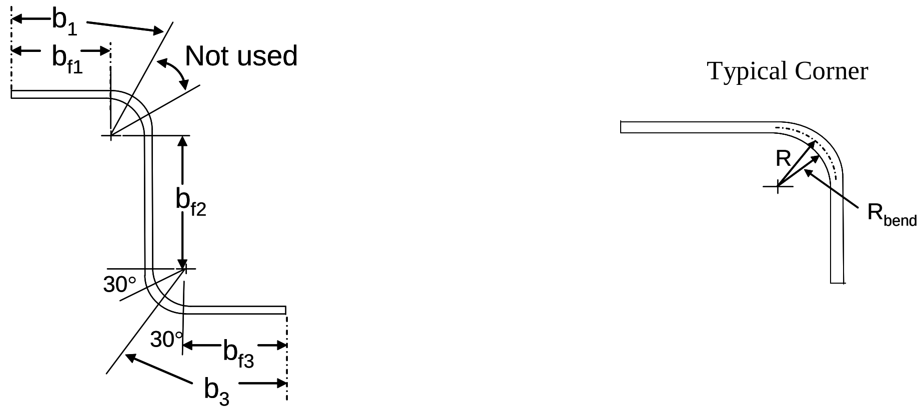


Figure 8.4.2-1 Segmentation of Formed Cross-Section

When calculating the flange lengths for the cross-section, Section 8.4.2.1.1 provides the general case while Section 8.4.2.1.2 shows the exception for handling angle sections and Section 8.4.2.1.3 discusses some special cases where the general rule must be modified, such as return flanges.

8.4.2.1.1 General Case

In general, for formed parts except for angles, to calculate the element length, b_i,

$$b_i=0.5235nR+b_{fi}$$ **Equation 8.4.2-1**

Where
n is the number of adjacent radii, *i.e.*, how many radii touch the flat plate element
R is the mid-plane radius (in)
0.5235 is the angle 30° in radians
b_{fi} is the length of the flat plate element (in)

8.4.2.1.2 Angle Sections

If the member is an angle, the one-edge-free geometry of each leg does not offer rotational support to the other leg. This is particularly true of narrow flanges. The flange lengths for angles are each calculated to include the material in the entire radius as follows

$$b_i=1.57R+b_{fi}$$ **Equation 8.4.2-2**

Once the two flange lengths, b_i, for an angle have been calculated in this manner, form the ratio of the larger numerical value for b_i divided by the smaller numerical value for b_i

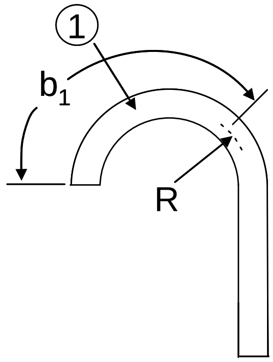
$$R_{flange}=b_{i\text{-}larger}/b_{i\text{-}smaller}$$

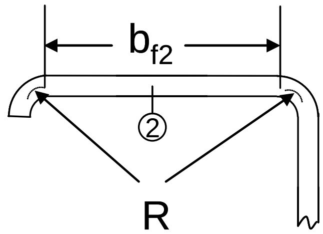
If R ≤ 1.5 – use the values calculated by Equation 8.4.2-2

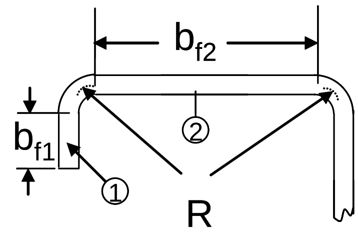
If R_{flange} > 1.5 – the general approach of Equation 8.4.2-1 can be used and b_i should be recalculated

8.4.2.1.3 Special Formed Stiffeners

If the member has multiple radii whose centers are on the same side of the part or very short flanges it requires special consideration. Some of these geometries are depicted in Figure 8.4.2-2 which also describes how the flange lengths should be calculated.







$b_1 = 2.10R$ Use OEF curve[9]	**Flange/Lip ineffective** *If* $[b_{f1}=0$ ***or*** $b_{f1} < R]$ *and* $b_{f2} < R$ $b_2=2.10R$ Neglect b_1, use OEF curve
	Flange/Lip partially effective *If* $[b_{f1}=0$ ***or*** $b_{f1} < R]$ *and* $b_{f2} \geq R$ $b_2=b_{f2}+2(0.5235)R$ (Determine $F_{cc\text{-}OEF}$ and $F_{cc\text{-}NEF}$ and average results) Neglect b_1
	Flange/Lip effective *If* $b_{f1} \geq R$ *and* $b_{f2} \geq 0$ $b_1=b_{f1}+0.5235R$ (OEF curve) $b_2=b_{f2}+2(0.5235)R$ (NEF curve)

Figure 8.4.2-2 Special Sheet Metal Flange Geometries

Examples 8.4.4.1 through 8.4.4.3 illustrate the benefit in section allowable that can be obtained by adding even a minimal lip or flange to a cross-section.

8.4.2.2 Determination of Flat Plate Element Dimensions for Cross-Sections made from Machining, Extrusions, Forgings or Castings

This approach of determining the flat plate elements applies to all product forms which are not formed from flat sheet material. This would include parts machined from plate, bar or forged stock or parts which have been net forged, extruded or cast.

It does not apply to cross-sections in which material has been removed from the corners, either through chamfering or through the machining of a radius as shown in Figure 8.4.2-3, even if a minimum thickness in the corner equal to the adjacent leg is maintained. The reason for this exclusion is found in the nature of how a part cripples. The corner material is the last to buckle/fail and if the material is reduced the analysis method is likely unconservative since there is no test data included or available for that type of configuration. This also cannot be analyzed using the approach described in Section 8.4.2.1 since this method is based on testing of bent up sheet metal in which the forming operation strengthened the corners. If compression loading is present it is recommended that testing be performed to validate analysis assumptions.

[9] This calculates b_1 as a curved section sweeping 120 degrees.

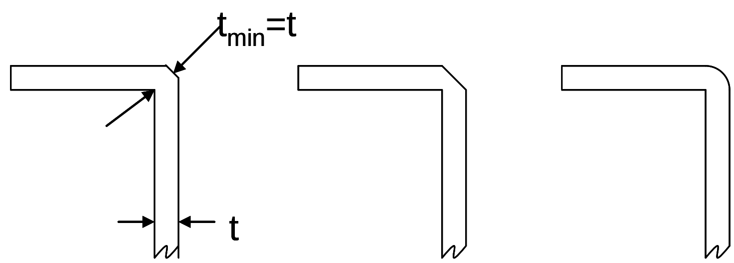


Figure 8.4.2-3 Cross-section Geometries Not Covered by Crippling Analysis Methodology

Figure 8.4.2-4A illustrates the basic approach to dividing the cross-section. For this product form, the flat plate widths include the radiused material, but not the material in the intersection between adjacent flanges. Figure 8.4.2-4B shows an example of a flanged cross-section.

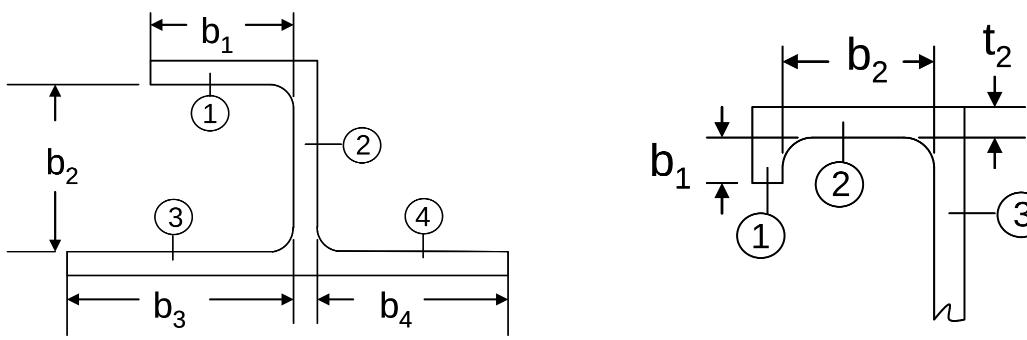


Figure 8.4.2-4 Segmentation of Non-Formed Cross-Section

There are times when a flange or lip may be too short to fully support the adjacent leg. An example of this is shown in Figure 8.4.2-4B in element 1. If the width of element 1, b_1, is greater than 3 times the thickness of the adjacent leg, t_2, the element is long enough to be considered a flange of width b_1. If, however, b_1 is less than or equal to 3 times the thickness, it is a lip and the following guidelines apply

If $t_2 < b_1 \leq 3t_2$, **Equation 8.4.2-3**

Neglect b_1

Calculate F_{cc2} as the average of $F_{cc\text{-}OEF}$ and $F_{cc\text{-}NEF}$ for Element 2, using b_2 as shown

The analyst should note that a lip still provides support to the adjacent leg, but it is not as much as would have resulted from a flange.

Another detail which can be present in extrusions and forgings is a bulb as shown in Figure 8.4.2-5. If the radius of the bulb is less than the thickness of the flange, it is ineffective in providing support and the adjacent flange should be analyzed as one-edge-free. If the radius of the bulb is equal to or greater than the adjacent leg's thickness then the crippling stress allowable for that leg is

$$F_{cc1} = Maximum[0.70F_{cc\text{-}NEF}, F_{cc\text{-}OEF}] \quad \text{if } R_{bulb} \geq t_1$$ **Equation 8.4.2-4**

Where

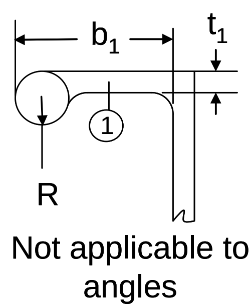
R_{bulb} is the radius of the bulb (in)

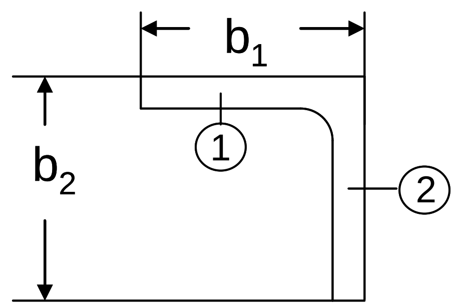
t_1 is the thickness of the flange adjacent to the bulb (in)

$F_{cc\text{-}NEF}$ is the crippling allowable for a no-edge-free condition using b_1/t_1 (psi)

$F_{cc\text{-}OEF}$ is the crippling allowable for a one-edge-free condition using b_1/t_1 (psi)

As with formed parts, angles must also be handled differently since both legs are one-edge-free and don't provide full support to the adjacent leg. The case of non-formed angles is shown in Figure 8.4.2-5B.





A- Bulbed End **B Angle**

Figure 8.4.2-5 Special Geometries for Non-formed Cross-Sections

Once the flange lengths for the angle have been determined as shown in Figure 8.4.2-5B, form the ratio of the larger numerical value for b_i divided by the smaller numerical value for b_i

$$R_{flange}=b_{i\text{-}larger}/b_{i\text{-}smaller}$$

If $R_{flange} \leq 1.5$ – use the b_i values determined as shown in Figure 8.4.2-5B

If $R_{flange} > 1.5$ – determine the flange lengths using the general formulation illustrated in Figure 8.4.2-4A (*i.e.*, don't include the corner material)

8.4.2.3 Determination of Flat Plate Element Dimensions for Built up Cross-Sections

In practical structures or structures which have been repaired in service, there are often cases when two or more parts are attached together by a row or rows of fasteners and the crippling allowable of this composite section is required. It is difficult to generalize an approach for this situation because there are many different ways to build up such a cross-section. The following method is a rational approach to determining a reasonable to conservative allowable for composite sections.[10] Once the analyst understands the approach, it is possible to extend it to other scenarios, using the same principles.

Composite sections, in general, fall into one of the following four categories shown in Figure 8.4.2-6. The first picture in each category depicts the idealized section. The section depicts a possible real configuration. Note that in Category I, both plate elements have support flanges. This Category is not applicable to an angle with an attached plate as the failure mode for it would involve torsional stability calculations. It will be addressed in Section 8.5.

[10] Reference 8-3, LTV Structures Manual

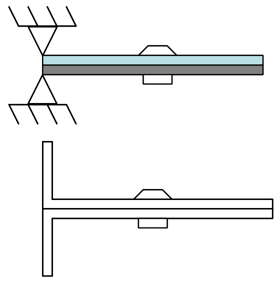
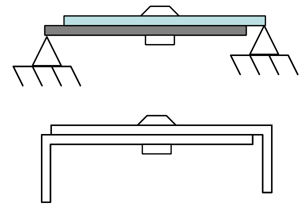
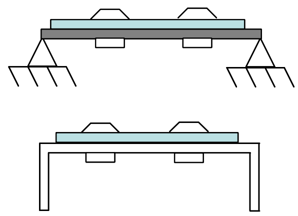
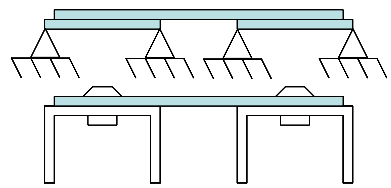
	
Category I	Category II
	
Category III	Category IV

Figure 8.4.2-6 Composite Section Summary

The complexity in determining the allowable crippling load of composite sections is in the determination of the crippling load of the members which are fastened together. Once that has been calculated, it can be used in conjunction with the approach defined for single piece sections to determine the allowable for the entire cross-section. Recall that at the beginning of this section it was noted that individual flanges of a section buckle, then the corners buckle and the section cripples or fails. So, in examining the elements in a composite section, the approach is to examine how individual elements can physically buckle and the effect of that element's buckling on other elements within the composite section. Thus, if an element can buckle without another element also buckling, then its buckling stress should be calculated in isolation. If an element cannot buckle without one or more attached elements also buckling, then the affected elements also should be considered.

The following items should be considered as each cross-section is examined:

- A flat plate under compressive load buckles as shown in Figure 8.4.2-7, depending on the support on the long edges.
- If a flat plate loses an edge support before it reaches its buckling stress, it will buckle at a lower stress, corresponding to one (or both) edge free conditions. In most cases this generally means it buckles immediately after the loss of support.
- The attaching rivets, fasteners or spotwelds should be spaced as closely as possible (4D ≤ spacing ≤ 6D) to keep the interrivet buckling stress greater than the elemental buckling stress and to make the elements act together as a single element. Smaller, more closely spaced rivets are more effective for this than larger more widely spaced rivets because the half-wavelength of the buckle is on the order of the width of the plate in the no edge free case.
    - If the rivet spacing exceeds 6D, F_{ir}, the inter-rivet buckling stress, should be calculated and the smaller of $F_{cc\text{-}element}$ and F_{ir} should be used.
    - Parts with rivet spacing greater than 8D shall be analyzed separately.
- Attached plates should be of similar thickness. Large differences in thickness could invalidate the reasoning behind this approach.

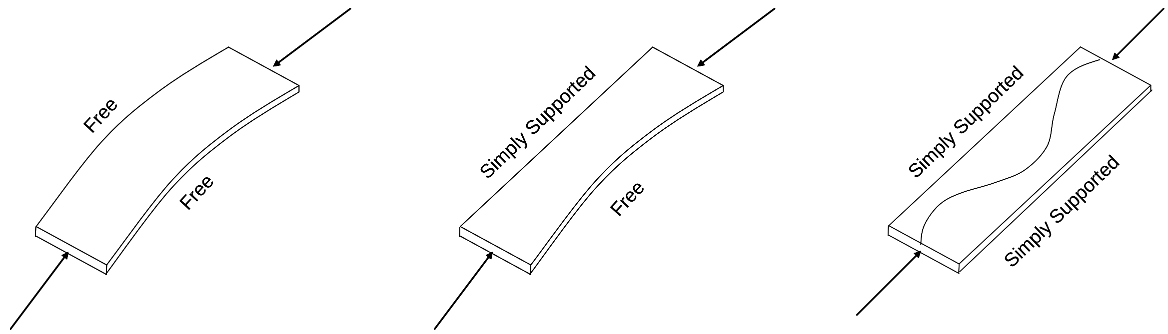


Figure 8.4.2-7 Plate Buckling Shapes

To determine the crippling load for Category I type sections, shown in Figure 8.4.2-8, where one edge is simply supported and the other edge is free, there are 3 possibilities: the entire combined thickness flange may buckle as a one edge free element, element b may buckle before the entire combined thickness flange buckles or both elements b and c buckle before the entire combined thickness flange.

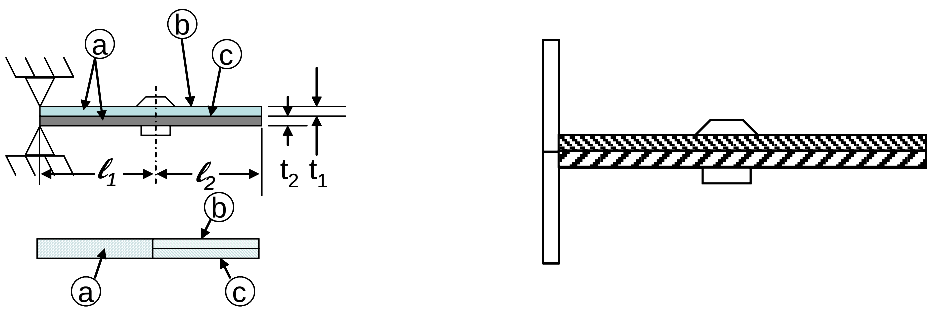


Figure 8.4.2-8 Category I One Edge Simply Supported, One Edge Free

Thus the following approach would be used. Calculate b/t for the single combined flange as shown in Equation 8.4.2-5 and determine F_{cc-a} for this value of b/t and an OEF edge condition

$$b/t_a = (\ell_1+\ell_2)/(t_1+t_2) \quad \textbf{Equation 8.4.2-5}$$

Where
ℓ_1,ℓ_2 are flange lengths shown in Figure 8.4.2-8 (in)
t_1, t_2 are flange thicknesses shown in Figure 8.4.2-8 (in)

Note the flange lengths are determined by the guidelines presented in Section 8.4.2.1 and 8.4.2.2, depending on the product form. In the machining depicted in Figure 8.4.2-8, the flange does not include material common with the vertical flange. If the part were sheet metal, a small portion of the radius would be included.

Calculate b/t for elements b and c as shown in Figure 8.4.2-8 and determine F_{cc-b} and F_{cc-c}, respectively for OEF edge condition.

$$b/t_b = \ell_2/t_1 \quad \textbf{Equation 8.4.2-6}$$
$$b/t_c = \ell_2/t_2$$

Then the buckling load of the composite section is

$$\begin{aligned}
&\text{If } F_{cc\text{-}a} \le F_{cc\text{-}b} \text{ and } F_{cc\text{-}a} \le F_{cc\text{-}c} : P_{cc} = F_{cc\text{-}a}(A_a + A_b + A_c)\\
&\text{If } F_{cc\text{-}a} > F_{cc\text{-}b} \text{ and } F_{cc\text{-}a} < F_{cc\text{-}c} : P_{cc} = F_{cc\text{-}a}(A_a + A_c) + F_{cc\text{-}b}(A_b)\\
&\text{If } F_{cc\text{-}a} < F_{cc\text{-}b} \text{ and } F_{cc\text{-}a} > F_{cc\text{-}c} : P_{cc} = F_{cc\text{-}a}(A_a + A_b) + F_{cc\text{-}c}(A_c)\\
&\text{If } F_{cc\text{-}a} > F_{cc\text{-}b} \text{ and } F_{cc\text{-}a} > F_{cc\text{-}c} : P_{cc} = F_{cc\text{-}a}(A_a) + F_{cc\text{-}b}(A_b) + F_{cc\text{-}c}(A_c)
\end{aligned}$$

Equation 8.4.2-7

Where

$F_{cc\text{-}a}$, $F_{cc\text{-}b}$ and $F_{cc\text{-}c}$ are the crippling stress found as described in discussion associated with Equations 8.4-11 and 8.4-12 (psi)

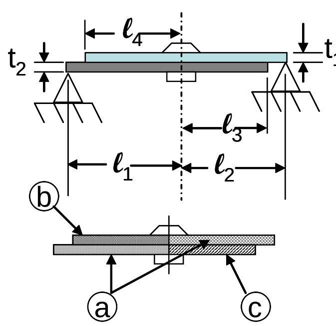
A_a is area of Element a shown in Figure 8.4.2-8 and given by $A_a=a(t_1+t_2)$ (in^2)

A_b is area of Element b shown in Figure 8.4.2-8 and given by $A_b=c(t_1)$ (in^2)

A_c is area of Element c shown in Figure 8.4.2-8 and given by $A_c=c(t_2)$ (in^2)

Once the crippling load for the multi-layer composite sub-section has been determined, it would be entered as a single element in Table 8.4.0-1 using P_{cc} from Equation 8.4.2-7 and the area as the summation of A_a, A_b and A_c. The remaining flanges and webs would be filled in and the calculation would proceed as described in Section 8.4. An example is provided in Section 8.4.4.6.

Category II, two one-edge free plates fastened together, also has 3 possible buckling scenarios. The upper and lower one edge free plates may have sufficient support for each other that they prevent each other from buckling and thus buckle together as a single NEF plate. Elements b and c, shown in Figure 8.4.2-9 will either buckle along with element a, or they may buckle independently. Once element a buckles, b and c lose their support and buckle immediately.



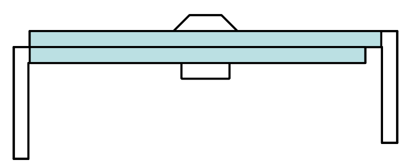


Figure 8.4.2-9 Category II Two One Edge Simply Supported, One Edge Free Plates

Thus the following approach would be used. Calculate b/t for the single combined flange as shown in Equation 8.4.2-8 and determine $F_{cc\text{-}a}$ for this value of b/t and an NEF edge condition

$$b/t_a = \ell_1/t_2+\ell_2/t_1$$

Equation 8.4.2-8

Where

ℓ_1, ℓ_2 are flange lengths shown in Figure 8.4.2-9 (in)

t_1, t_2 are flange thicknesses shown in Figure 8.4.2-9 (in)

Note the flange lengths are determined by the guidelines presented in Section 8.4.2.1 and 8.4.2.2, depending on the product form. In the machining depicted in Figure 8.4-12, the flange does not include material common with the vertical flange per Section 8.4.2.2. If the part were sheet metal, approximately 30 degrees of the radius would be included as discussed in Section 8.4.2.1.

Calculate b/t for elements b and c as shown in Equation 8.4.2-9 and determine F_{cc-b} and F_{cc-c}, respectively for OEF edge condition.

$$b/t_b = \ell_4/t_1$$
$$b/t_c = \ell_3/t_2$$ **Equation 8.4.2-9**

where

ℓ_3, ℓ_4 are flange lengths shown in Figure 8.4.2-9 (in)

t_1, t_2 are flange thicknesses shown in Figure 8.4.2-9 (in)

Then the buckling load of the composite section is given by Equation 8.4.2-10

$$\text{If } F_{cc-a} \le F_{cc-b} \text{ and } F_{cc-a} \le F_{cc-c} : P_{cc} = F_{cc-a}(A_a + A_b + A_c)$$
$$\text{If } F_{cc-a} > F_{cc-b} \text{ and } F_{cc-a} < F_{cc-c} : P_{cc} = F_{cc-a}(A_a + A_c) + F_{cc-b}(A_b)$$
$$\text{If } F_{cc-a} < F_{cc-b} \text{ and } F_{cc-a} > F_{cc-c} : P_{cc} = F_{cc-a}(A_a + A_b) + F_{cc-c}(A_c)$$
$$\text{If } F_{cc-a} > F_{cc-b} \text{ and } F_{cc-a} > F_{cc-c} : P_{cc} = F_{cc-a}(A_a) + F_{cc-b}(A_b) + F_{cc-c}(A_c)$$ **Equation 8.4.2-10**

Where

F_{cc-a}, F_{cc-b} and F_{cc-c} are the crippling stress found as described in discussion associated with Equations 8.4.2-6 and 8.4.2-7 (psi)

A_a is area of Element a shown in Figure 8.4.2-9 and given by $A_a= \ell_1 t_2+ \ell_2 t_1$ (in^2)

A_b is area of Element b shown in Figure 8.4.2-9 and given by $A_b= \ell_4 t_1$ (in^2)

A_c is area of Element c shown in Figure 8.4.2-9 and given by $A_c= \ell_3 t_2$ (in^2)

Category III, as depicted in Figure 8.4.2-10, consists of elements a and d which are no edge free elements and elements b and c are one edge free elements. This section can buckle in two general ways with some variations which depend on the buckling strength of the free edges. Fundamentally, elements b, c, and d can all be supported by element a in which case element d buckles before element a. This set of possible solutions is described by Equation 8.4.2-11. Alternately, element a can buckle before element d and element d provides support for element a. In this case a new combined element is formed. This is described by Equation 8.4.2-12. Note that in both Equation 8.4.2-11 or equation 8.4.2-12 if element b or c or both buckle at values greater than their supporting element, their buckling allowable is limited to that of the supporting element, because once the support is lost, they will buckle immediately.

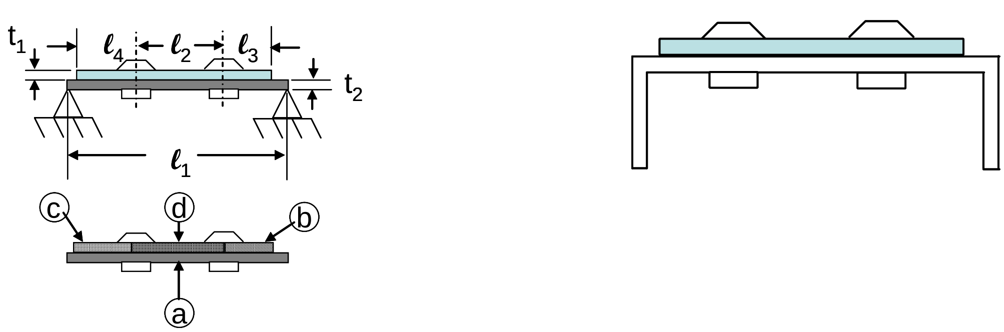


Figure 8.4.2-10 Category III Two No Edge Free and Two One Edge Free Elements

Thus, if $F_{cc\text{-}d} \le F_{cc\text{-}a}$ then

$$\begin{aligned}
&\text{If } F_{cc\text{-}a} > F_{cc\text{-}b} \text{ and } F_{cc\text{-}a} > F_{cc\text{-}c}: P_{cc} = F_{cc\text{-}a}(A_a) + F_{cc\text{-}b}(A_b) + F_{cc\text{-}c}(A_c) + F_{cc\text{-}d}(A_d)\\
&\text{If } F_{cc\text{-}a} < F_{cc\text{-}b} \text{ and } F_{cc\text{-}a} > F_{cc\text{-}c}: P_{cc} = F_{cc\text{-}a}(A_a + A_b) + F_{cc\text{-}c}(A_c) + F_{cc\text{-}d}(A_d)\\
&\text{If } F_{cc\text{-}a} > F_{cc\text{-}b} \text{ and } F_{cc\text{-}a} < F_{cc\text{-}c}: P_{cc} = F_{cc\text{-}a}(A_a + A_c) + F_{cc\text{-}b}(A_b) + F_{cc\text{-}d}(A_d)\\
&\text{If } F_{cc\text{-}a} < F_{cc\text{-}b} \text{ and } F_{cc\text{-}a} < F_{cc\text{-}c}: P_{cc} = F_{cc\text{-}a}(A_a + A_b + A_c) + F_{cc\text{-}d}(A_d)
\end{aligned}$$

Equation 8.4.2-11

Where

$F_{cc\text{-}a}$, $F_{cc\text{-}d}$ are the crippling stress allowable for a NEF plate of $b/t = \ell_1/t_2$ or ℓ_2/t_1, respectively.

$F_{cc\text{-}b}$, $F_{cc\text{-}c}$ are the crippling stress allowable for a OEF plate of $b/t = \ell_3/t_1$ or ℓ_4/t_1, respectively.

A_a is area of Element a shown in Figure 8.4.2-10 and given by $A_a = \ell_1 t_2$ (in^2)

A_b is area of Element b shown in Figure 8.4.2-10 and given by $A_b = \ell_3 t_1$ (in^2)

A_c is area of Element c shown in Figure 8.4.2-10 and given by $A_c = \ell_4 t_1$ (in^2)

A_d is area of Element d shown in Figure 8.4.2-10 and given by $A_d = \ell_2 t_1$ (in^2)

But, if $F_{cc\text{-}d} > F_{cc\text{-}a}$ then determine crippling stress allowable for element (a+d)

$$(b/t)_{a+d} = (\ell_1 - \ell_2)/t_2 + \ell_2/t_1 : \text{NEF Plate}$$

$$\text{Determine } F_{cc(a+d)}$$

$$\begin{aligned}
&\text{If } F_{cc(a+d)} > F_{cc\text{-}b} \text{ and } F_{cc(a+d)} > F_{cc\text{-}c}: P_{cc} = F_{cc(a+d)}(A_a + A_d) + F_{cc\text{-}b}(A_b) + F_{cc\text{-}c}(A_c)\\
&\text{If } F_{cc(a+d)} < F_{cc\text{-}b} \text{ and } F_{cc(a+d)} > F_{cc\text{-}c}: P_{cc} = F_{cc(a+d)}(A_a + A_d + A_b) + F_{cc\text{-}c}(A_c)\\
&\text{If } F_{cc(a+d)} > F_{cc\text{-}b} \text{ and } F_{cc(a+d)} < F_{cc\text{-}c}: P_{cc} = F_{cc(a+d)}(A_a + A_d + A_c) + F_{cc\text{-}b}(A_b)\\
&\text{If } F_{cc(a+d)} < F_{cc\text{-}b} \text{ and } F_{cc(a+d)} < F_{cc\text{-}c}: P_{cc} = F_{cc(a+d)}(A_a + A_b + A_c + A_d)
\end{aligned}$$

Equation 8.4.2-12

Category IV, depicted in Figure 8.4.2-11, is very similar to Category III in the rationale for determining the buckling load. Instead of a single support represented by element a, of Category III, the support has been split into two elements, a_1 and a_2 representing the two beams. Elements a_1, a_2 and d are NEF elements and elements b and c are OEF elements. Element d is supported by elements a_1 and a_2, so the two primary cases occur when either element a_1 or element a_2 or both buckle before element d or when element d buckles before either element a_1 or a_2.

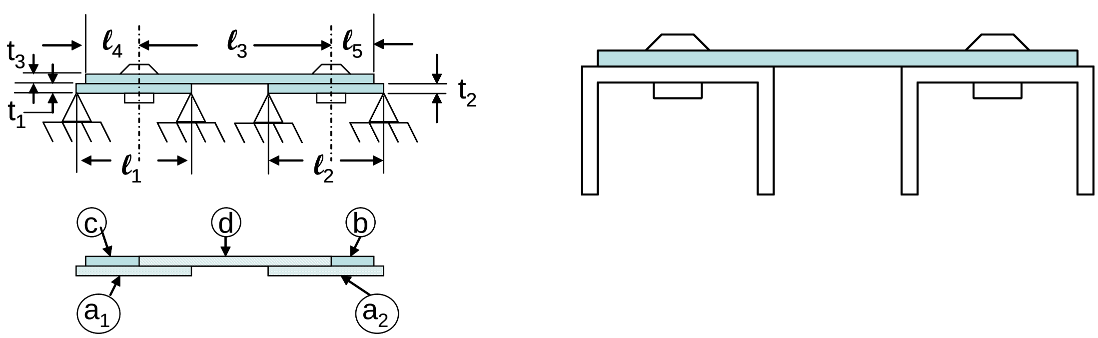


Figure 8.4.2-11 Category IV Three No Edge Free and Two One Edge Free Plates

The latter case is the more straightforward, so it will be addressed first. So, when $F_{cc\text{-}d} \le F_{cc\text{-}a1}$ or $F_{cc\text{-}d} \le F_{cc\text{-}a2}$ then

$$\begin{aligned}
&\textit{If } F_{cc\text{-}a1} > F_{cc\text{-}c} \textit{ and } F_{cc\text{-}a2} > F_{cc\text{-}b}\text{: } P_{cc} = F_{cc\text{-}a1}(A_{a1}) + F_{cc\text{-}a2}(A_{a2}) + F_{cc\text{-}b}(A_b) + F_{cc\text{-}c}(A_c) + F_{cc\text{-}d}(A_d)\\
&\textit{If } F_{cc\text{-}a1} < F_{cc\text{-}c} \textit{ and } F_{cc\text{-}a2} > F_{cc\text{-}b} : P_{cc} = F_{cc\text{-}a1}(A_{a1} + A_c) + F_{cc\text{-}a2}(A_{a2}) + F_{cc\text{-}b}(A_b) + F_{cc\text{-}d}(A_d)\\
&\textit{If } F_{cc\text{-}a1} > F_{cc\text{-}c} \textit{ and } F_{cc\text{-}a2} < F_{cc\text{-}b} : P_{cc} = F_{cc\text{-}a1}(A_{a1}) + F_{cc\text{-}c}(A_c) + F_{cc\text{-}a2}(A_{a2} + A_b) + F_{cc\text{-}d}(A_d)\\
&\textit{If } F_{cc\text{-}a1} < F_{cc\text{-}c} \textit{ and } F_{cc\text{-}a2} < F_{cc\text{-}b} : P_{cc} = F_{cc\text{-}a1}(A_{a1} + A_c) + F_{cc\text{-}a2}(A_{a2} + A_b) + F_{cc\text{-}d}(A_d)
\end{aligned}$$

Equation 8.4.2-13

Where

$F_{cc\text{-}a1}$, $F_{cc\text{-}a2}$, $F_{cc\text{-}d}$ are the crippling stress allowable for a NEF plate of b/t = ℓ_1/t_1, ℓ_2/t_2or ℓ_3/t_3, respectively.

$F_{cc\text{-}b}$, $F_{cc\text{-}c}$ are the crippling stress allowable for a OEF plate of b/t = ℓ_5/t_3 or ℓ_4/t_3, respectively.

A_{a1} is area of Element a_1 shown in Figure 8.4.2-11 and given by $A_{a1}= \ell_1 t_1$ (in^2)

A_{a2} is area of Element a_2 shown in Figure 8.4.2-11 and given by $A_{a2}= \ell_2 t_2$ (in^2)

A_b is area of Element b shown in Figure 8.4.2-11 and given by $A_b= \ell_5 t_3$ (in^2)

A_c is area of Element c shown in Figure 8.4.2-11 and given by $A_c= \ell_4 t_3$ (in^2)

A_d is area of Element d shown in Figure 8.4.2-11 and given by $A_d= \ell_3 t_3$ (in^2)

If either element a_1 or element a_2 buckles before element d, then element d loses its support and buckles also. In that event the crippling allowable for element d is the minimum of the crippling allowables for element d, a_1 or a_2. Thus, if $F_{cc\text{-}d} > F_{cc\text{-}a1}$ or $F_{cc\text{-}d} > F_{cc\text{-}a2}$ then

$$\begin{aligned}
&\textit{If } F_{cc\text{-}a1} > F_{cc\text{-}c} \textit{ and } F_{cc\text{-}a2} > F_{cc\text{-}b}\text{: } P_{cc} = F_{cc\text{-}a1}(A_{a1}) + F_{cc\text{-}a2}(A_{a2}) + F_{cc\text{-}b}(A_b) + F_{cc\text{-}c}(A_c) + \textit{Minimum } [F_{cc\text{-}a1}, F_{cc\text{-}a2}, \textit{ or } F_{cc\text{-}d}](A_d)\\
&\textit{If } F_{cc\text{-}a1} < F_{cc\text{-}c} \textit{ and } F_{cc\text{-}a2} > F_{cc\text{-}b} : P_{cc} = F_{cc\text{-}a1}(A_{a1} + A_c) + F_{cc\text{-}a2}(A_{a2}) + F_{cc\text{-}b}(A_b) + \textit{Minimum } [F_{cc\text{-}a1}, F_{cc\text{-}a2}, \textit{ or } F_{cc\text{-}d}](A_d)\\
&\textit{If } F_{cc\text{-}a1} > F_{cc\text{-}c} \textit{ and } F_{cc\text{-}a2} < F_{cc\text{-}b} : P_{cc} = F_{cc\text{-}a1}(A_{a1}) + F_{cc\text{-}c}(A_c) + F_{cc\text{-}a2}(A_{a2} + A_b) + \textit{Minimum } [F_{cc\text{-}a1}, F_{cc\text{-}a2}, \textit{ or } F_{cc\text{-}d}](A_d)\\
&\textit{If } F_{cc\text{-}a1} < F_{cc\text{-}c} \textit{ and } F_{cc\text{-}a2} < F_{cc\text{-}b} : P_{cc} = F_{cc\text{-}a1}(A_{a1} + A_c) + F_{cc\text{-}a2}(A_{a2} + A_b) + \textit{Minimum } [F_{cc\text{-}a1}, F_{cc\text{-}a2}, \textit{ or } F_{cc\text{-}d}](A_d)
\end{aligned}$$

Equation 8.4.2-14

8.4.2.4 Determination of Flat Plate Element Dimensions for Integrally Machined Webs and Skins

When stiffeners which are integrally machined on webs and skins undergo compression loading, they can experience crippling collapse failure in the same way as stand-alone stiffeners. Since crippling is the failure of the whole section a method for determining the crippling allowable must include more than the stiffener flange(s). Section 4.3 discusses effective width of web/skin and Table 4.3.1-2 provides appropriate assumptions for use in calculating the crippling allowable.

The load in the effective stiffener should include the appropriate additional load from the effective web/skin.

8.4.2.5 Crippling of Flanges in Bending

In practice, a crippling analysis may be required for a member whose loading is not simply axial compression, but may also include or be primarily a result of bending of the cross-section. Testing[11] performed on beams of various cross-sections indicated that beams can exhibit a crippling failure in bending and this failure mode can best be

[11] Reference 8-16, LR 11336.

predicted by calculating the modulus of rupture for the section where the outer fiber strain is limited to the average crippling strain allowable of the portion of the cross-section in compression. The approach for determining the allowable bending modulus of rupture and bending moment is described in Section 6.3.4 and an example problem is worked in Section 6.3.8.4, given a crippling stress allowable. An example of how to determine the crippling allowable is provided in Section 8.4.4.5. The IDAT program PLASBEND allows the user to enter a crippling stress cutoff and location and calculates a Margin of Safety using the approach outlined.

Note that in the case of a section loaded in bending, the crippling cutoff stress to be used for calculating the modulus of rupture is not the average of the entire cross-section but only the average of the portion of the section in compression. To determine the allowable crippling stress, the approach described by Equation 8.4.0.1 and Table 8.4.0-1 is used, but only the flanges and partial flanges on the compression side of the plastic neutral axis are included in the calculation. If a flange is cut by the neutral axis, the cut edge of the flange is considered to be supported. This is illustrated in Figure 8.4.2-13. The guidelines found in Sections 8.4.2.1 and 8.4.2.2 concerning flange and lip lengths also apply to any cut flanges.

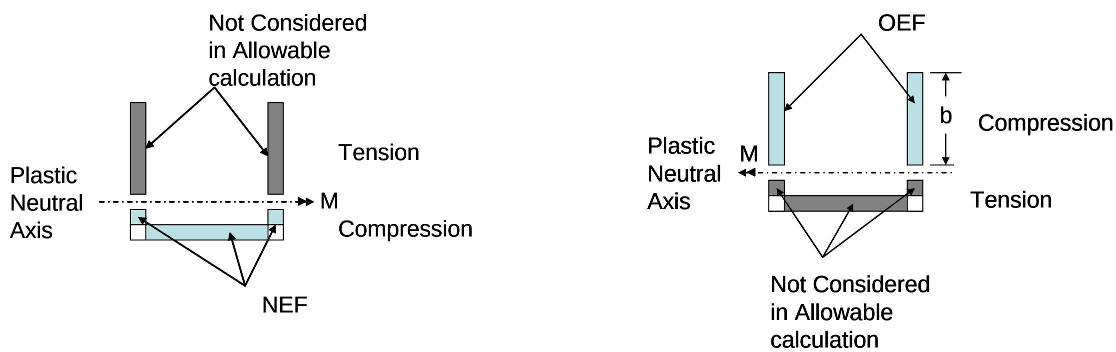


Figure 8.4.2-12 Correct Determination of Cross-Section for Use in Calculation of Crippling Allowable for a Section in Bending

Once the allowable bending moment based on the bending modulus limited by the crippling cutoff is obtained per Section 6.3.4, a margin of safety can be calculated from

$$M.S. = \frac{M_{allow}}{m_{appl}} - 1$$ **Reference Equation 6.3.2-3**

Where
M_{allow} is the allowable bending modulus of the section where the outer fiber of the compression side has been limited by the crippling stress based on the compression portion of the section in bending (in-lb)
m_{appl} is the applied bending moment on the section (in-lb)

If the section is loaded with both axial and bending loads, the allowable bending moment should be calculated as outlined above and a combined margin of safety calculated per Section 6.3.6

For tee or channel section loaded in <u>bending only</u>, for the case where the free flange is in compression, the above method may be modified in the following manner. When the allowable moment is calculated, it is the smaller of the moment resulting from the modulus of rupture or the calculated bending modulus when the crippling stress allowable is applied at two-thirds the distance from the free edge to the neutral axis or (2/3)b, where b is shown in Figure 8.4.2-13. If the tee or channel is also undergoing an axial compression load, the general method should be used. Example 8.4.4.5 illustrates this approach for a tee in bending

8.4.3 Material Crippling Curves

Material crippling curves can be generated by the IDAT program SM110 available on the metals tab for any material in the METDB database and any user defined material. This section includes a sampling of material crippling curves for reference.

No.	Material Alloy/Form	METDB No.	Thickness/ Grain	Basis	F_{tu} (ksi)	F_{cy} (ksi)	E_c (ksi)	n_c	F_{cmax} (ksi)	Figure
1	2024-T42 CLAD Sheet	8	0.010-0.062 L	B	59	35	10700	13	37.6	8.4.3-1
2	7075-T6 BARE Sheet	40	0.040-0.125 L	B	80	71	10500	13	80.0	8.4.3-1
3	7075-T6 CLAD Sheet	43	0.040-0.062 L	B	74	64	10500	13	72.5	8.4.3-1
4	2124-T8151 Plate	72	2.001-3.000 L	S	67	60	10900	16	64.9	8.4.3-2
5	7050-T7451 Plate	85	2.001-3.000 L	B	75	64	10600	19	68.1	8.4.3-2
6	7049-T73 Die Forging	134	2.001-3.000 L	B	72	63	10700	30	64.4	8.4.3-2
7	7075-T76511 Extrusion	249	<20 in^2 0.50-0.75 L	B	76	67	10700	25	69.5	8.4.3-2
8	Ti 6AL-4V Annealed Sheet	288	< 0.1857	B	139	138	16400	20	139.0	8.4.3-3
9	Ti 6AL-4V Annealed Plate	295	2.001 – 4.000	S	130	124	16400	20	130.0	8.4.3-3
10	Ti 6AL-4V Die Forging, A-B Annealed	311	2.001 – 4.000	S	130	123	16400	20	130.0	8.4.3-3
11	300M Steel 280 ksi Type A	355	0.42C	S	280	247	29000	13	280.0	8.4.3-4
12	Aermet 100 Bars, Forgings	358	<100 in^2	S	280	262	28000	11	280.0	8.4.3-4
13	PH13-8 Mo H1000 Bar	351	≤ 8D	B	208	211	29400	17	208	8.4.3-4

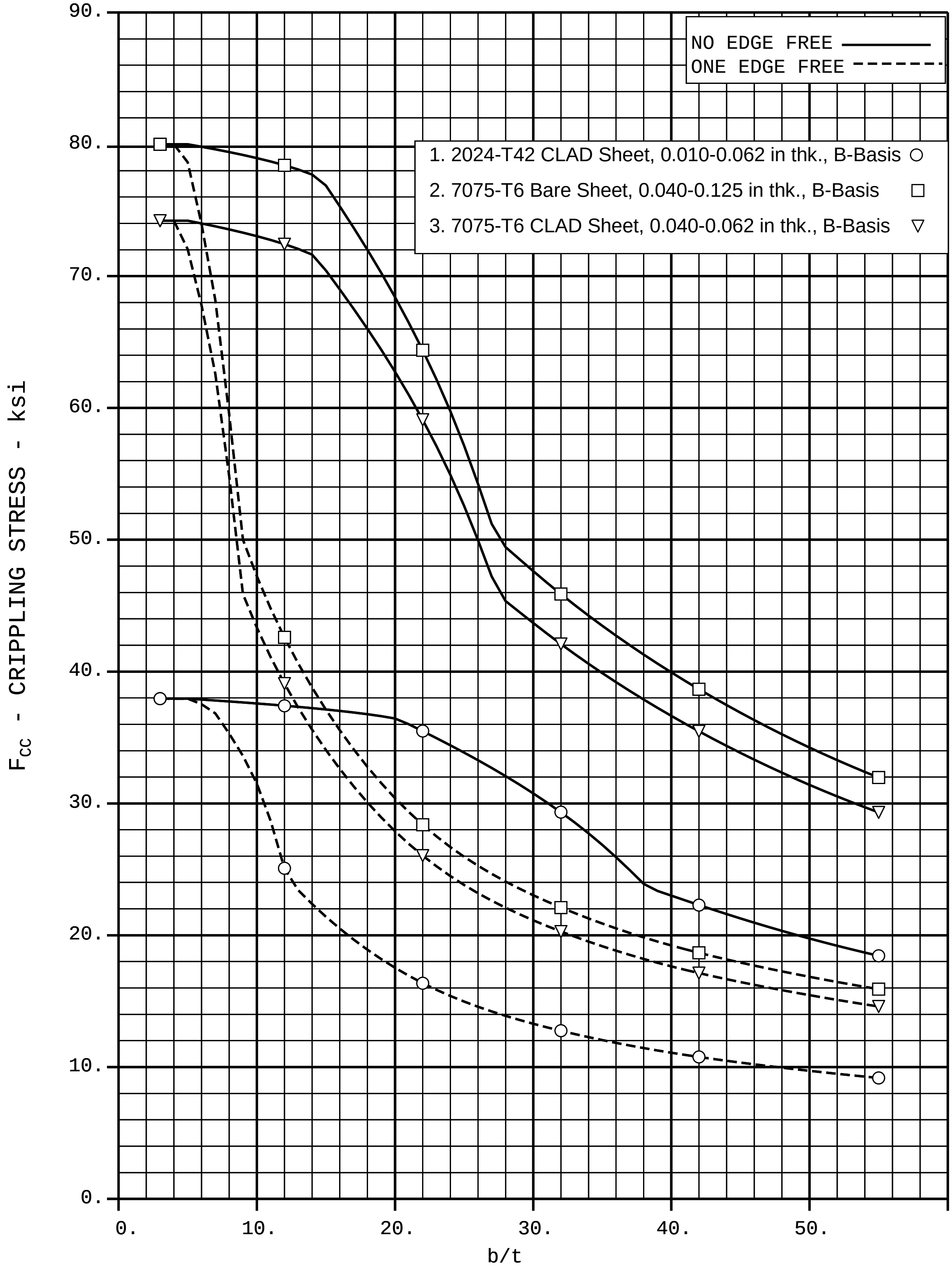


Figure 8.4.3-1 Sample of Crippling Allowables for Aluminum Sheet

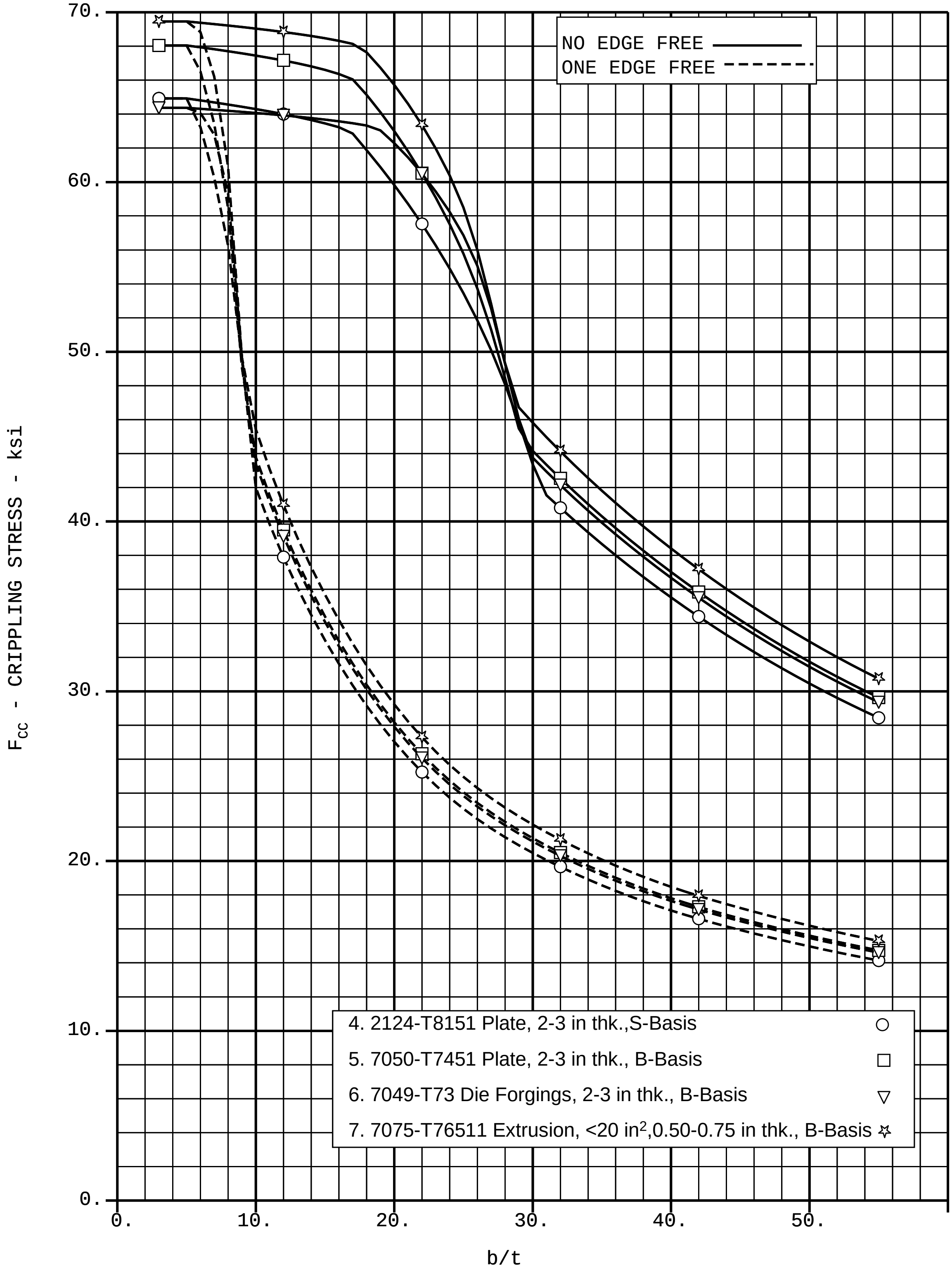


Figure 8.4.3-2 Sample of Crippling Allowables for Aluminum Plate, Forging and Extrusion

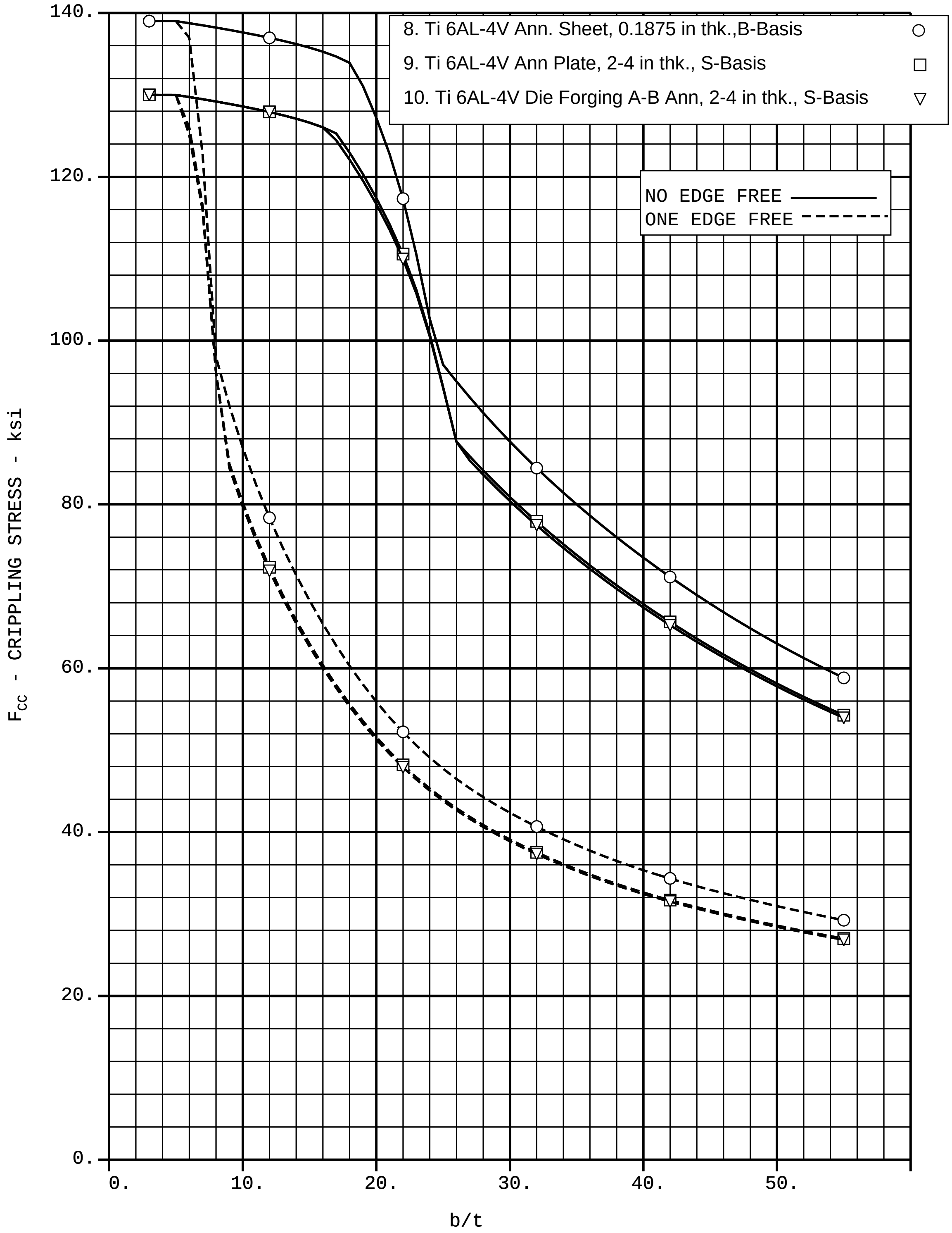


Figure 8.4.3-3 Sample of Crippling Allowables for Titanium 6AL-4V Sheet, Plate, and Forging

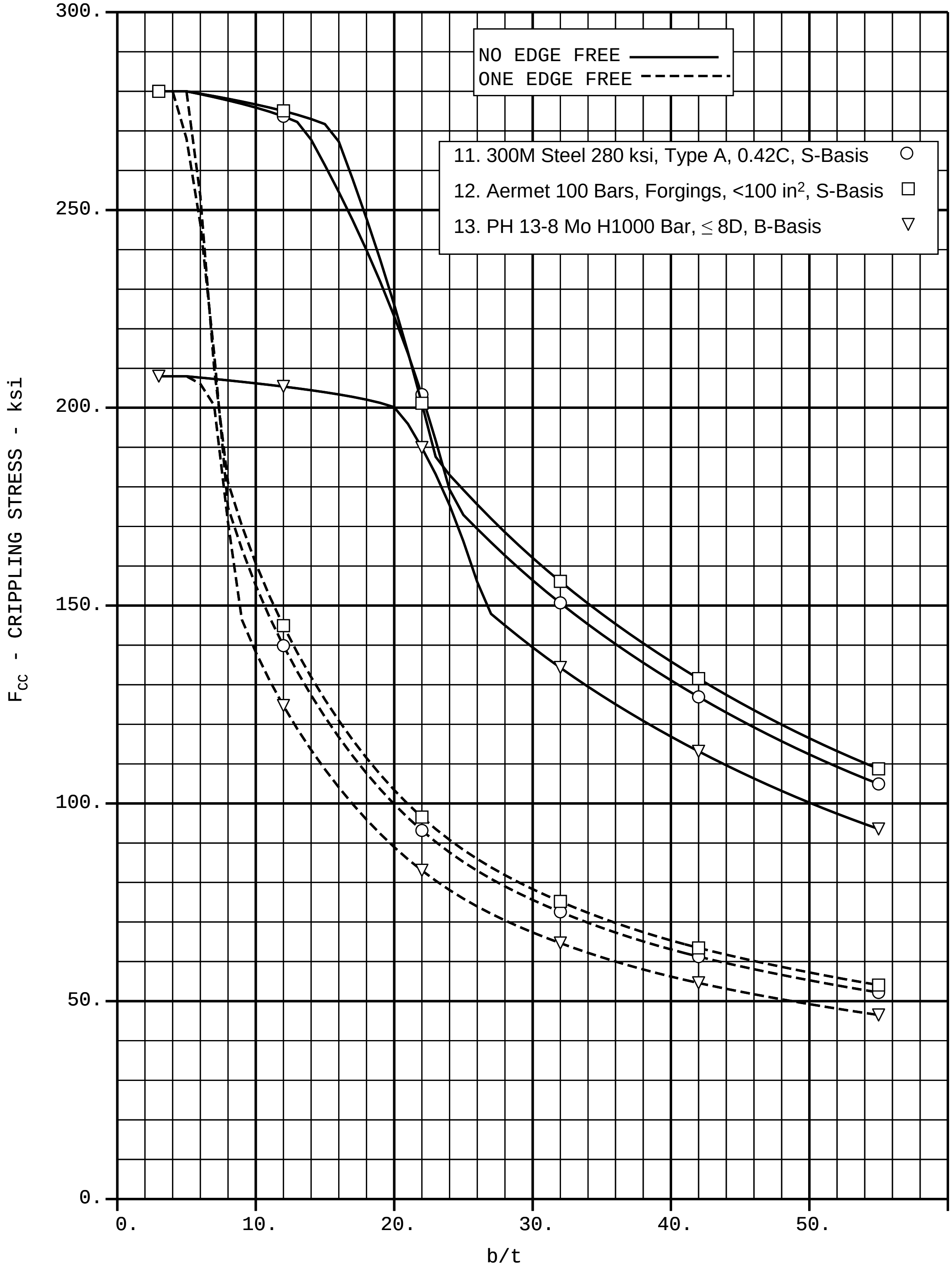


Figure 8.4.3-4 Sample of Crippling Allowables for Steel Products

8.4.4 Example Problems

This section provides several different sample problems to illustrate the use of the crippling analysis method.

8.4.4.1 Example Problem - Sheet Metal Zee

Determine the Margin of Safety in crippling for the Zee stiffener shown for an applied axial load of 7500 lb at the centroid of the section.

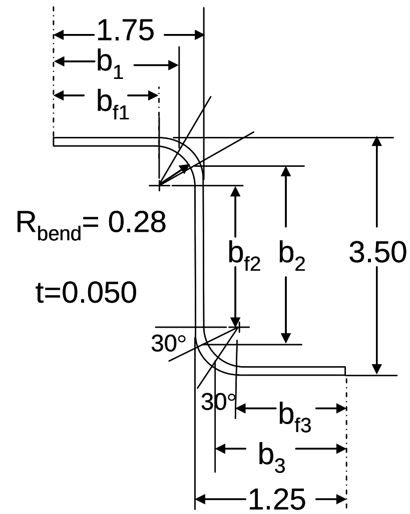
Material: 7075-T76 CLAD Aluminum 0.050 Sheet
F_{cy} = 55000 psi
E = $10.0\text{x}10^6$ psi
A=0.3069 in^2

Determine flange element widths, b_{fi}
b_{f1}=1.75-0.05-0.28=1.42
b_{f2}=3.50-2(0.05)-2(0.28)=2.84
b_{f3}=1.25-0.05-0.28=0.92

Calculate R = R_{bend}+t/2 = 0.28+0.050/2=0.305

Calculate flange widths, b_i
b_1=b_{f1}+0.5235R = 1.42+0.5235(0.305)=1.580 (OEF)
b_2=b_{f2}+0.5235nR =2.84+0.5235(2)(0.305)=3.159 (NEF)
b_3=b_{f3}+0.5235R = 0.92+0.5235(0.305)=1.080 (OEF)

Set up Table 8.4.0-1 and calculate the crippling allowable F_{cc}.



Not to Scale

Elem	b	t	bt	b/t	b/t_{OEF}	Ends	Fcc Fig. 8.4.4-1	btFcc
1	1.580	0.050	0.0790	31.6		OEF	18100	1430
2	3.159	0.050	0.1580	63.2[1]	24.1	OEF[1]	21700	3429
3	1.080	0.050	0.0540	21.6		OEF	23500	1269
Sum			0.291					6128

Calculate the average allowable crippling stress: $F_{cc}= \Sigma[F_{cc\text{-}i} b_i \; t_i \;]/ \Sigma \; [b_i \; t_i]$ = 6128/0.291=21058 psi

Calculate the allowable compression load on the section (use actual area), per Equation 8.4.0-3,
$P_{cc} = F_{cc}A$ = 21058(0.3069) = 6463 lbs
M.S. = P_{cc}/P-1 = 6463/7500-1 = -0.14
Possible options to resolving the negative margin could be to thicken the entire part or to add a lip or flange to one or both legs. The next example looks at adding a lip to leg 1.

Note:
1. A b/t of 63.2 for a NEF element exceeds the value of the curves. Use Equation 8.4.1-5 to obtain an effective b/t_{OEF} and use the OEF curve: $(b/t)_{OEF} = 0.3808(b/t)_{NEF}$ = 0.3808(63.2)=24.1

Figure 8.4.4-1 is the crippling allowable curve for the example problems of Section 8.4.4.1 through 8.4.4.3. Figure 8.4.4-2 is the crippling allowable curve for the Section 8.4.4.4 example.

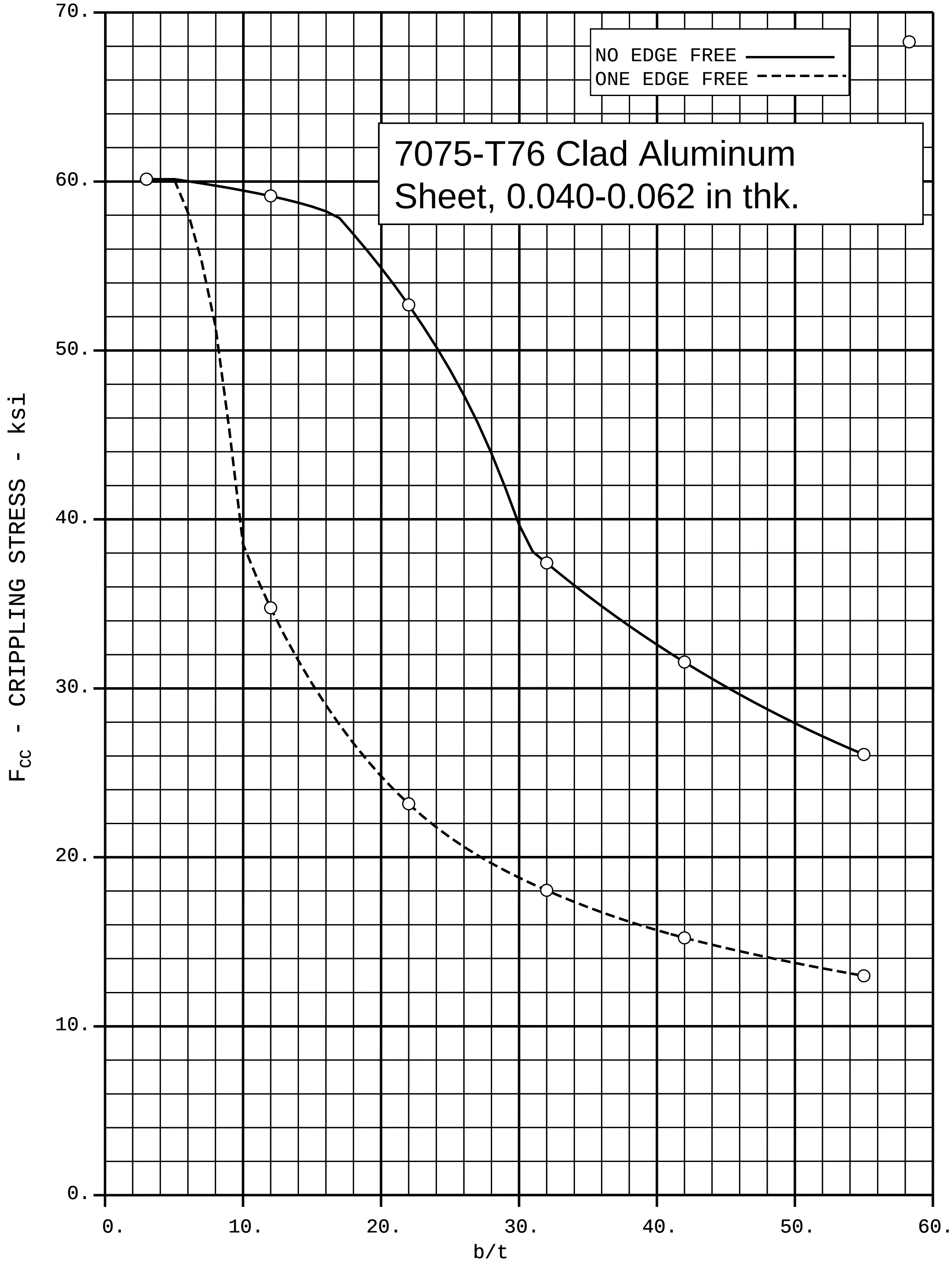


Figure 8.4.4-1 Crippling Allowable Curve 7075-T76 CLAD Aluminum Sheet

8.4.4.2 Example Problem - Sheet Metal Zee with Lip

Because of the negative margin discovered in the analysis of the Zee of Section 8.4.4.1, add a return lip to flange 1. Determine the Margin of Safety in crippling for the modified Zee stiffener shown for an applied axial load of 7500 lb at the centroid of the section.

Material: 7075-T76 CLAD Aluminum 0.050 Sheet
F_{cy} = 55000 psi
E = 10.0×10^6 psi
A=0.3434 in^2

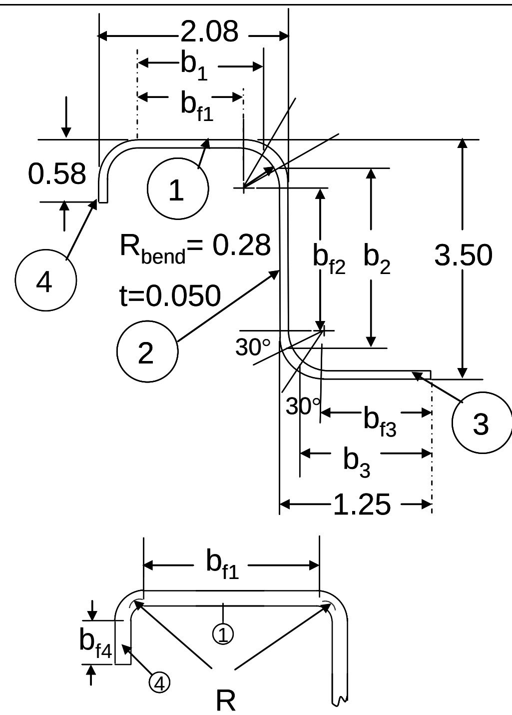
Determine flange element widths, b_{fi}
b_{f1}=2.08-2(0.05)-2(0.28)=1.42
b_{f2}=3.50-2(0.05)-2(0.28)=2.84
b_{f3}=1.25-0.05-0.28=0.92
b_{f4}=0.58-0.05-0.28=0.25

Calculate R = R_{bend}+t/2 = 0.28+0.050/2=0.305

Lip is b_{f4} and adjacent flange is b_{f1}:
b_{f1} = 1.42, b_{f4} = 0.25, R = 0.305
Per Figure 8.4.2-3 if b_{f4}<R and b_{f1}>R, neglect lip (b_{f4}) and b_1= b_{f1} + 2(0.5235)R – determine $F_{cc\text{-}OEF}$ and $F_{cc\text{-}NEF}$ and average.

Calculate flange widths, b_i
b_1=b_{f1}+0.5235R = 1.42+0.5235(2)(0.305)=1.739 (OEF and NEF)
b_2=b_{f2}+0.5235nR = 2.84+ 0.5235(2)(0.305)=3.159 (NEF)
b_3=b_{f3}+0.5235R = 0.92+0.5235(1)(0.305)=1.080 (OEF)

Set up Table 8.4.0-1 and calculate the crippling allowable F_{cc}.



Not to Scale

Elem	b	t	bt	b/t	b/t_{OEF}	Ends	Fcc Fig. 8.4.4-1	btFcc
1	1.739	0.050	0.0870	34.8		OEF	(17100+35800)/2	2301
2	3.159	0.050	0.1580	63.2[1]	24.1	OEF[1]	21700	3429
3	1.080	0.050	0.0540	21.6		OEF	23500	1269
Sum			0.2990					6999

Calculate the average allowable crippling stress: $F_{cc}= \Sigma[F_{cc\text{-}i} b_i\ t_i\ ]/ \Sigma\ [b_i\ t_i]$ = 6999/0.299=23408 psi

Calculate the allowable compression load on the section (use actual area), per Equation 8.4.0-3,
P = 23408(0.3434) = 8038 lb
M.S. =8038/7500-1 = <u>0.07</u>
Adding a lip resolved the negative margin – through the increase in actual area which resulted in a decrease in stress level and the averaging of the crippling stress allowable of element 1. For completeness, Section 8.4.4.3 extends the lip sufficiently to meet the criteria for a flange and recalculates the margins.

Note:
1. A b/t of 63.2 for a NEF element exceeds the value of the curves. Use Equation 8.4.1-5 to obtain an effective b/t_{OEF} and use the OEF curve: $(b/t)_{OEF} = 0.3808(b/t)_{NEF} = 0.3808(63.2)=24.1$

8.4.4.3 Example Problem - Sheet Metal Zee with added Flange

Although the negative margin discovered in the analysis of the Zee of Section 8.4.4.1 was resolved with the addition of a lip, investigate adding a minimum length return flange to flange 1. Determine the Margin of Safety in crippling for the modified Zee stiffener shown for an applied axial load of 7500 lb at the centroid of the section.

Material: 7075-T76 CLAD Aluminum 0.050 Sheet
F_{cy} = 55000 psi
E = $10.0\text{x}10^6$ psi
A=0.3484 in^2

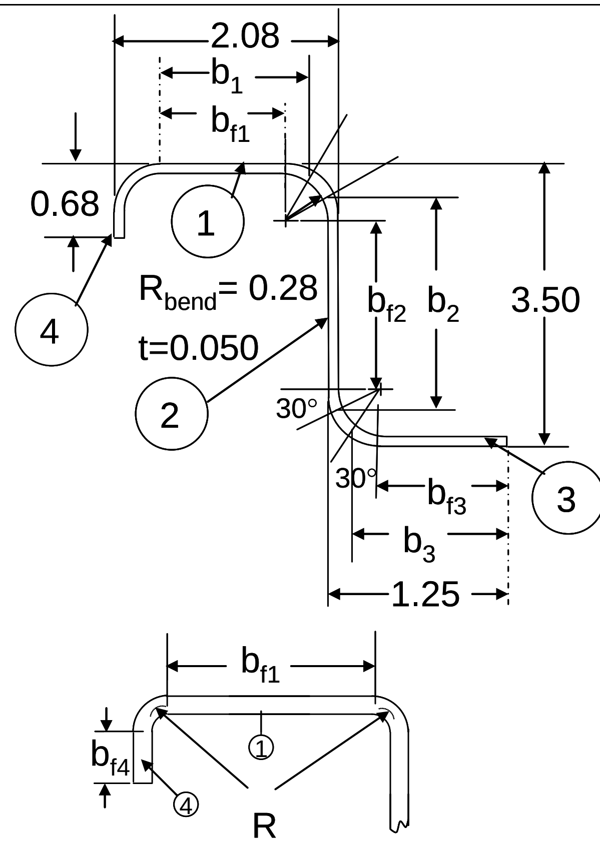
Determine flange element widths, b_{fi}
b_{f1}=2.08-2(0.05)-2(0.28)=1.42
b_{f2}=3.50-2(0.05)-2(0.28)=2.84
b_{f3}=1.25-0.05-0.28=0.92
b_{f4}=0.68-0.05-0.28=0.35

Calculate R = R_{bend}+t/2 = 0.28+0.050/2=0.305

Added flange is b_{f4} and adjacent flange is b_{f1}:
b_{f4} = 0.35, R = 0.305
Per Figure 8.4.2-3 if $b_{f4} \geq R$ and $b_{f1} > 0$
$b_1 = b_{f1} + 0.5235(2)R$ NEF
$b_4 = b_{f4} + 0.5235R$

Calculate flange widths, b_i
$b_1 = b_{f1}$+0.5235nR = 1.42+0.5235(2)(0.305)=1.739 (NEF)
$b_2 = b_{f2}$+0.5235nR = 2.84+0.5235(2)(0.305)=3.159 (NEF)
$b_3 = b_{f3}$+0.5235nR = 0.92+0.5235(1)(0.305)=1.080 (OEF)
$b_4 = b_{f4}$+0.5235nR = 0.35+0.5235(1)(0.305)=0.510 (OEF)

Set up Table 8.4.0-1 and calculate the crippling allowable F_{cc}.



Not to Scale

Elem	b	t	bt	b/t	b/t$_{OEF}$	Ends	Fcc Fig. 8.4.4-1	btFcc
1	1.739	0.050	0.0870	34.8		NEF	35800	3115
2	3.159	0.050	0.1580	63.2[1]	24.1	OEF[1]	21700	3429
3	1.080	0.050	0.0540	21.6		OEF	23500	1269
4	0.510	0.050	0.0255	10.2		OEF	38000	969
Sum			0.3245					8782

Calculate the average allowable crippling stress: $F_{cc} = \Sigma[F_{cc\text{-}i} b_i\, t_i] / \Sigma[b_i\, t_i]$ = 8782/0.3245=27063 psi

Calculate the allowable compression load on the section (use actual area), per Equation 8.4.0-3,
P = 27063(0.3484) = 9429 lb
M.S. = 9429/7500-1 = 0.26
The section crippling allowable with a very short flange is 30% greater than the same section with no return flange and 15% greater than the section with a return lip.

Note:
1. A b/t of 63.2 for a NEF element exceeds the value of the curves. Use Equation 8.4.1-5 to obtain an effective b/t$_{OEF}$ and use the OEF curve: $(b/t)_{OEF} = 0.3808(b/t)_{NEF}$ = 0.3808(63.2)=24.1

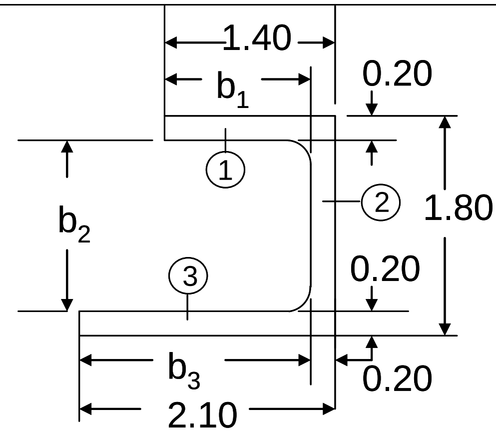
8.4.4.4 Example Problem - Extruded Section

Determine the section crippling allowable for the section below. Calculate the margin of safety for a load applied of 133950 lb at the centroid of the section.

Material: Titanium 6AL-4V STA Extrusion
F_{cy} = 157000 psi
E = 16.4×10^6 psi
A=1.0068 in^2

Determine flange element widths, b_i
b_1=1.40-0.20=1.20 (OEF)
b_2=1.80-0.20-0.20=1.40 (NEF)
b_3=2.10-0.20=1.90 (OEF)

Set up Table 8.4.0-1 and calculate the crippling allowable F_{cc}.



Not to Scale

Elem	b	t	bt	b/t	Ends	Fcc Fig. 8.4.4-2	btFcc
1	1.20	0.200	0.24	6	OEF	151000	36240
2	1.40	0.200	0.28	7	NEF	162900	45612
3	1.90	0.200	0.38	9.5	OEF	98500	37430
Sum			0.90				119282

Calculate the average allowable crippling stress: $F_{cc}= \Sigma[F_{cc-i} b_i\ t_i\ ]/ \Sigma\ [b_i\ t_i]$ = 119282/0.90=132536 psi

Calculate the allowable compression load on the section (use actual area), per Equation 8.4.0-3,
P = 132536(1.0068)= 133437 lb
M.S. = 133437/133950-1 = -0.004

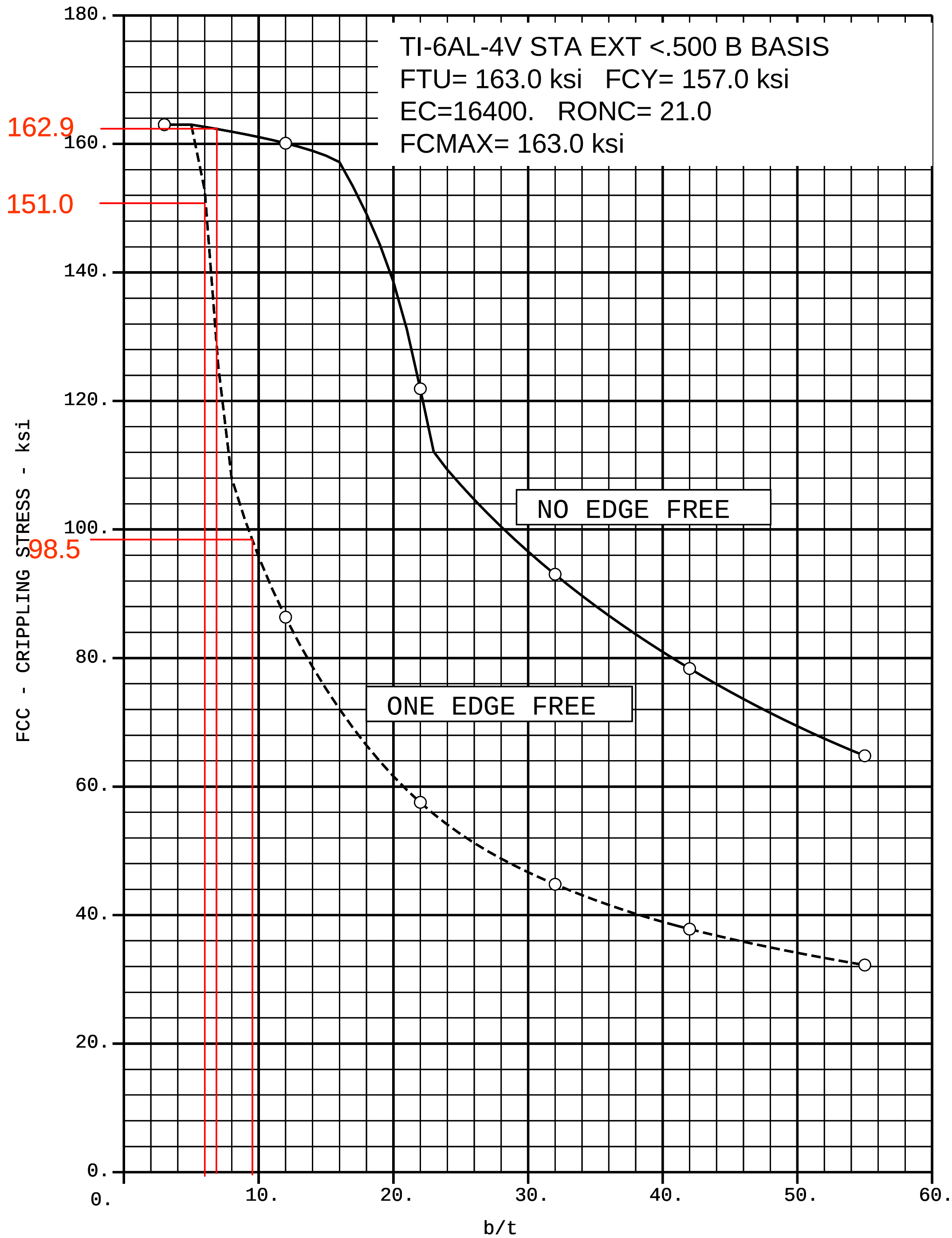


Figure 8.4.4-2 Crippling Allowable Curve Titanium 6AL-4V Extrusion

8.4.4.5 Crippling in Bending

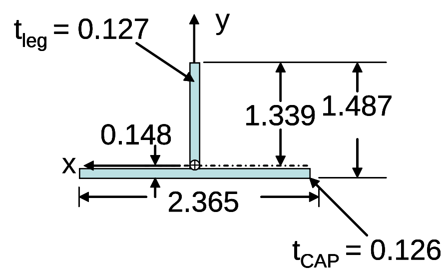
Determine the modulus of rupture for the section below for a section loaded in bending subject to crippling. Calculate the margin of safety for an applied moment of +4000 in-lb or -6000 in-lb.

Material: Aluminum 2024T42 Extrusion
METDB Material ID 194
Basis = S BASIS
$E = 10.8 \times 10^6$ psi
Ftu = 57000 psi
Fty = 38000 psi
e (in/in) = 0.120 in/in
Ramberg Osgood Number, n = 30.0

IDAT/PLASBEND has calculated

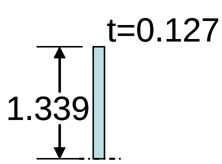
- The plastic neutral axis at 0.148 inches from the base of the cap
- The allowable bending moment based on the modulus of rupture as 6867 in-lb.
- $e_{ptu} = 0.05$ in/in
- $n_t = 7.939$
- The properties about the neutral axis are as shown in the figures below.

Sketch



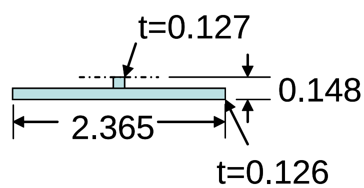
Area = 0.4708 in^2
I_{na} = 0.1041 in^4

Subsection 1 Compression Side



I_{NA1} = 0.1016 in^4 (for this subsection at the section NA)
A = 0.1700 in.^{2} Q = 0.1138 in^3 c = 1.339 in.

Subsection 2 Tension Side



I_{NA2} = 0.0026 in^4 (for this subsection at the section NA)
A = 0.3008 in.^{2} Q = 0.0255 in^3 c = 0.148 in.

Solution Step	Calculation	Result	Solution Step	Calculation	Result
Calculate k	$K_1 = Q_1/(I_{NA1}/c_{larger})$ = *0.11385/(0.1016/1.339)*	k_1=*1.50*	Calculate k	$K_2 = Q_2/(I_{NA2}/c_{smaller})$ = *0.0255/(0.0026/0.148)*	k_2=*1.45*

Part I – Applied moment results in compression in flange

- For an applied +4000 in-lb moment, the flange is in compression and the cap is in tension.
- Determine the crippling allowable for an OEF element where b=1.339 and t=0.127
  - Since this is a single element, a table is not required.
  - Calculate b/t = 1.339/0.127=10.54, OEF From Figure 8.4.4-3, F_{cc}=34500 psi
- Since this case is a Tee section with the free flange in compression, the modulus of rupture will be calculated with the crippling cutoff applied at (2/3)b for the flange: (2/3)1.339 = 0.893

Solution Step	Calculation	Result	
Calculate crippling strain Eqn 6.3.2-6	f_{cc} = 34500 psi $e = f/E + e_{ptu}(f/F_{tu})^{nt}$ = $34500/10.8 \times 10^6 + (0.0500)(34500/57000)^{7.939}$	e_{cc} = 0.004123 in/in	

Compression Side			Tension Side		
Solution Step	**Calculation**	**Result**	**Solution Step**	**Calculation**	**Result**
Calculate strain at outer compress'n fiber	$e_{outer-c} = e_{cc}(1.339/0.893) = 0.004123(3/2)$	$e_{outer-c} =$ 0.006185 in/in	Calculate strain at outer tension fiber	$e_{outer-t} = e_{cc}(0.148/0.893) = 0.004123(0.1657)$	$e_{outer-t} =$ 0.000683 in/in
Determine stress for given strain by iteration	$e = f/E + e_{ptu}(f/F_{tu})^{nt}$ Guess f= 41000 psi $e=41000/10.8x10^6 + 0.0500(41000/57000)^{7.939}$ $= 0.007450 > 0.006186$ Decrement, try 39600 $e =39600/10.8x10^6 + 0.0500(39600/57000)^{7.939}$ $= 0.006441 > 0.006186$ Reduce slightly: 39200 $e = 39200/10.8x10^6 + 0.0500(39200/57000)^{7.939}$ $= 0.006189 \sim 0.006185$	f =39200 psi	Determine stress for given strain by iteration	$e = f/E + e_{ptu}(f/F_{tu})^{nt}$ Guess f= 10000 psi $e=10000/10.8x10^6 + 0.050(10000/57000)^{7.939}$ $= 0.00926 > 0.000683$ Decrement, try $e = 7350/10.8x10^6 + 0.050(7350/57000)^{7.939}$ $= 0.000681 \sim 0.000683$	f = 7350 psi
Determine plastic strain per Eqn 6.3.2-4	$e_p = e - f/E = 0.006189 – 39200/10.8x10^6 = 0.00256$	$e_p =$ 0.00256 in/in	Determine plastic strain per Eqn 6.3.2-4	$e_p = e - f/E = 0.000681 – 7350/10.8x10^6 = 0.000001$	$e_p =$ 0.00000 in/in
Calculate f_o per Eqn 6.3.2-8	$f_o = 6f/e^2\ [f^2/(3E^2) + \{e_p(f/E)\}(n_t + 1) / (n_t + 2) + n_t(e_p^2) / (2n_t + 1)] – 2f =$ $6(39200/0.006185^2)x$ $\mathbf{[}39200^2/(3\ (10.8x10^6)^2)+ \{0.00256(39200/10.8x10^6)\}$ $x\ (7.939+1)/(7.939 + 2) + 7.939(0.00256)^2 / (2(7.939) + 1)\mathbf{]} - 2(39200)$	f_o = 18935 psi	Calculate f_o per Eqn 6.3.2-8	$f_o = 6f/e^2\ [f^2/(3E^2) + \{e_p(f/E)\}(n_t + 1) / (n_t + 2) + n_t(e_p^2) / (2n_t + 1)] – 2f =$ $6(7350/0.000683^2)x$ $\mathbf{[}7350^2/(3\ (10.8x10^6)^2) + \{0.00000(7350/10.8x10^6)\}$ $x\ (7.939+1)/(7.939 + 2) + 7.939(0.00000^2)/ (2(7.939)+1)\mathbf{]}\ -2(7350)$	$f_o = 0$
Calculate $(mc/I)_{b-larger-c}$	$(mc/I)_{b-larger-c} = f_{larger-c} + f_{o-larger-c}\ (k_{larger} -1)$ $= 39200 + 18935(1.5-1)$	$(mc/I)_{b-larger-c}=$ 48668 psi	Calculate $(mc/I)_{b-smaller-c}$	$(mc/I)_{b-smaller-c} = f_{smaller-c} + f_{o-smaller-c} * (k_{smaller} -1)$ $= 7350+ 0(1.45-1)$	$(mc/I)_{b-smaller-c} =$ 7350 psi
Calculate $M_{allow-larger-c}$	$M_{allow-larger-c} = [(mc/I)_{b-larger-c}]\ (I_{xx1}/c_{larger}) =$ *48668(0.1016 / 1.339)*	$M_{allow-larger-c}$ = *3693 in-lb*	Calculate $M_{allow-smaller-c}$	$M_{allow-smaller-c} = [(mc/I)_{b-smaller-c}]\ (I_{xx2}/c_{smaller})$ = *7350 (0.0026 / 0.148)*	$M_{allow-smaller-c}$= *129 in-lb*
Calculate M_{allow}	$M_{allow} = M_{allow-larger-c} + M_{allow-smaller-c}$ = *3693+ 129*				M_{allow} = *3822 in-lb*
Calculate M.S.	$M_{allow} / m_{appl} – 1$ = *3822 / 4000 – 1 =*				M.S.(bending-crippling) = ***-0.04***
If the approach taken had been to calculate the stress due to the bending moment and compare this directly with the crippling allowable an overly conservative negative margin would have resulted: f = 4000(1.339/0.1041) = 51540 psi; M.S. = (34500/51540) -1 = -0.33. Alternately, if the crippling cutoff had been ignored and only the modulus of rupture used, the margin would have been unconservative: M.S. = 6867/4000-1 = 0.72					

Part II – Applied moment results in compression in cap

- For an applied -6000 in-lb moment, the flange is in compression and the cap is in tension.
- In this case, the cap portion is made up of two one edge free elements (the cap) and a stub of the flange (0.148-0.126) = 0.022 in tall. Since this is less than the thickness, it will not be included in the calculation of F_{cc}, however because it is part of larger flange, which is loaded in tension, it still provides a support to the cap.
- Determine the crippling allowable for two OEF elements where b=(2.365-0.127)/2 = 1.119 and t=0.126
    - Since both elements have the same geometry, a table is not required.
    - Calculate b/t = 1.119/0.126=8.9, OEF From Figure 8.4.4-3, F_{cc}=37500 psi
- Since this case has the cap in compression, the crippling cutoff must be applied at the outer fiber of the cap, *i.e.*, c_2=0.148.

Tension Side			**Compression Side**		
Solution Step	**Calculation**	**Result**	**Solution Step**	**Calculation**	**Result**
			Calculate crippling strain Eqn 6.3.2-6	f_{cc} = 37500 psi $e=f/E+e_{ptu}(f/F_{tu})^{nt}$ = $37500/10.8\text{x}10^6$ + $0.0500(37500/57000)^{7.939}$	e_{cc} = 0.005272 in/in
Calculate strain at outer tension fiber	$e_{outer-t} = e_{cc}(1.339/0.148)$ = 0.005272(1.339/0.148)	$e_{outer-t}$ = 0.04770	Strain at outer compression fiber	$e_{outer-c} = e_{cc}$	$e_{outer-c}$ = 0.005272 in/in
Determine stress for given strain by iteration	$e=f/E+e_{ptu}(f/F_{tu})^{nt}$ Guess f= 55000 psi $e=55000/10.8\text{x}10^6$ + $0.0500(55000/57000)^{7.939}$ = 0.04274 < 0.04770 Increment, try 56000 $e=56000/10.8\text{x}10^6$ + $0.0500(56000/57000)^{7.939}$ = 0.04863 > 0.04770 Reduce slightly – 55850 $e=55850/10.8\text{x}10^6$ + $0.0500(55850/57000)^{7.939}$ = 0.04770 = 0.04770	f = 55850 psi	Stress at outer compression fiber = F_{cc}		f = 37500 psi
Determine plastic strain per Eqn 6.3.2-4	e_p= e-f/E = 0.04770 – $55850/10.8\text{x}10^6$ = 0.04253	e_p = 0.04253 in/in	Determine plastic strain per Eqn 6.3.2-4	e_p= e-f/E = 0.005272 – $37500/10.8\text{x}10^6$ = 0.001800	e_p = 0.001800 in/in
Calculate f_o per Eqn 6.3.2-8	$f_o = 6f/e^2\ [f^2/(3E^2)$ + $\{e_p(f/E)\}(n_t + 1) / (n_t + 2)$ + $n_t(e_p^2) / (2n_t + 1)]$ – 2f = $6(55850/0.004770^2)$ x $[55850^2/(3(10.8\text{x}10^6)^2)$+ $\{0.04253(55850/10.8\text{x}10^6)\}$ x (7.939+1) / (7.939 + 2) + $7.939(0.04253^2)/(2(7.939)$ + 1)] -2(55850)	f_o = 44052 psi	Calculate f_o per Eqn 6.3.2-8	$f_o = 6f/e^2\ [f^2/(3E^2)$ + $\{e_p(f/E)\}(n_t + 1) / (n_t + 2)$ + $n_t(e_p^2) / (2n_t + 1)]$ – 2f = $6(37500/0.005272^2)$ x $[37500^2/(3(10.8\text{x}10^6)^2)$+ {0.001800(37500 / $10.8\text{x}10^6)\}$ x (7.939+1) / (7.939 + 2) + $7.939(0.001800^2)$ / (2(7.939) + 1)] -2(37500)	f_o = 15375 psi
Calculate $(mc/I)_{b-larger-c}$	$(mc/I)_{b-larger-c} = f_{larger-c}$ + $f_{o-larger-c}$ x $(k_{larger}-1)$ = 55850 + 44052(1.5-1)	$(mc/I)_{b-larger-c}$ = 77876 psi	Calculate $(mc/I)_{b-smaller-c}$	$(mc/I)_{b-smaller-c} = f_{smaller-c}$ + $f_{o-smaller-c}$ x $(k_{smaller}-1)$ = 37500+15375(1.45-1)	$(mc/I)_{b-smaller-c}$= 44419 psi

Calculate $M_{allow\text{-}larger\text{-}c}$	$M_{allow\text{-}larger\text{-}c}$ = $(mc/I)_{b\text{-}larger\text{-}c}$x (I_{xx1}/c_{larger}) = 77876(0.1016)/ 1.339	$M_{allow\text{-}larger\text{-}c}$ = 5909 in-lb	Calculate $M_{allow\text{-}smaller\text{-}c}$	$M_{allow\text{-}smaller\text{-}c}$= $(mc/I)_{b\text{-}smaller\text{-}c}x(I_{xx2}/c_{smaller})$ = 44419(0.0026) / 0.148	$M_{allow\text{-}smaller\text{-}c}$= 780 in-lb
Calculate M_{allow}	M_{allow} =$M_{allow\text{-}larger\text{-}c}$+$M_{allow\text{-}smaller\text{-}c}$= 5909 + 780				M_{allow} = 6689 in-lb
Calculate M.S.	M_{allow} / m_{appl} – 1 = 6689 /6000 – 1 =				M.S.(bending-crippling) = +0.11
When the cap is in compression, the section can handle a much larger applied moment. The moment allowable, still limited by crippling ($M_{allow\text{-}crippling}$ = 6689 in-lb) is only slightly smaller than the modulus of rupture (M_{allow} = 6867 in-lb) for bending that puts the cap in compression.					

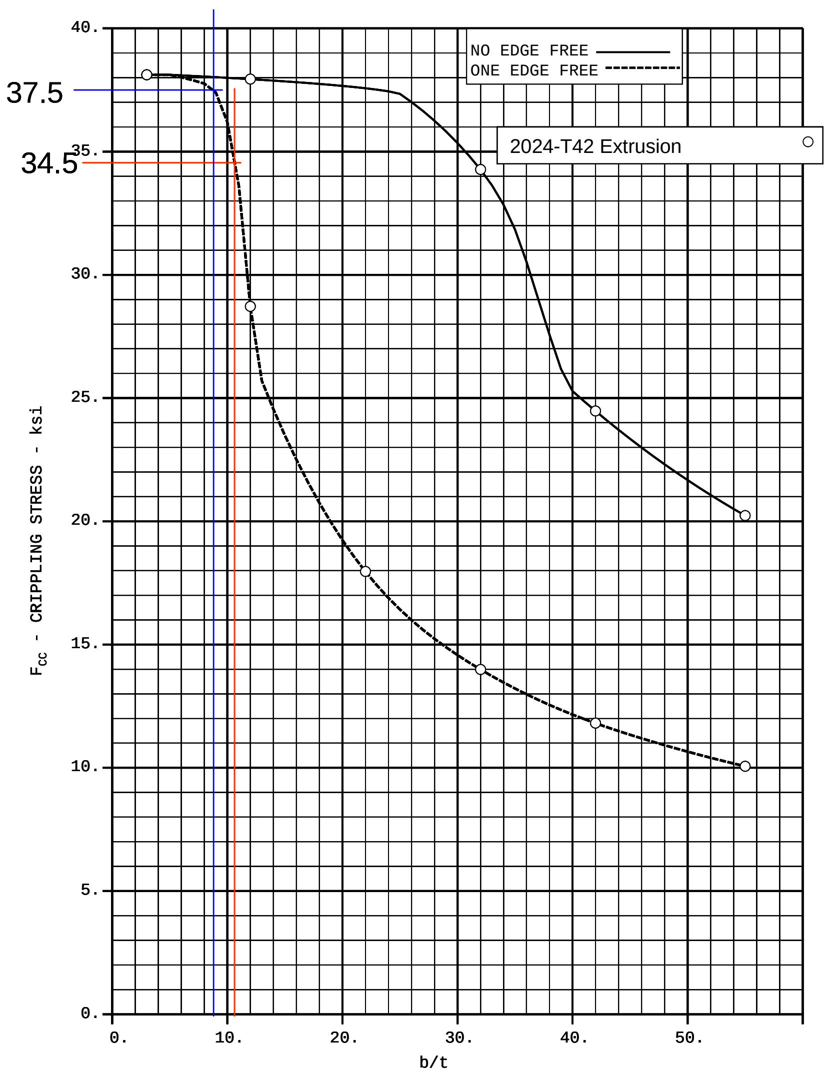
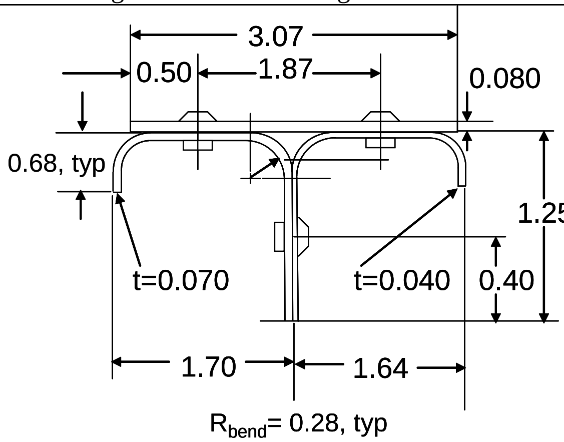
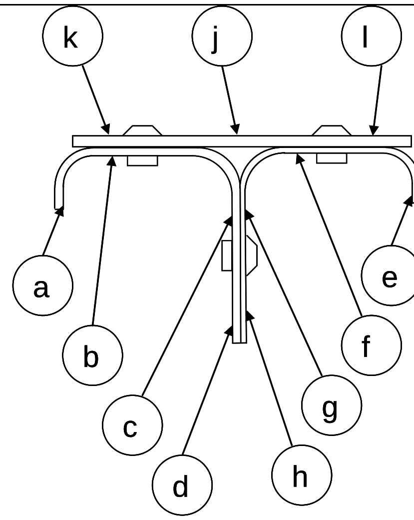


Figure 8.4.4-3 Crippling Allowable Curve 2024-T42 Aluminum Extrusion

8.4.4.6 Example Crippling of a Composite Section

Determine the Margin of Safety for the Composite Section shown below for an applied load of 25642 lb through the section centroid. All parts are made from 7075-T6 Bare Aluminum Sheet. Crippling curves for this material and thickness range can be found in Figure 8.4.4-4.





A = 0.600 in^2

Calculate $R_{0.070}$ = 0.28+0.070/2 = 0.315
$R_{0.040}$ = 0.28+0.040/2 = 0.300

Determine the length of the flats for each element:		Calculate the Element Lengths		
a	0.68-0.070-0.28 = 0.33 in.	a OEF	Figure 8.4.2-2 a>R, b>0	$b_a = a+0.5235R_{0.070}$ = 0.33+0.5235(0.315) = 0.495
b	1.70-2(0.070)-2(0.28) = 1.0 in.	b NEF	Figure 8.4.2-2 a>R, b>0	$b_b = b+2(0.5235)R_{0.070}$ = 1.0+2(0.5235)(0.315) = 1.330
c	1.25-0.070-0.28-0.40=0.50 in.	c NEF	Equation 8.4.2-1, n=1	$b_i=0.5235nR+b_{fi}$ = 0.5235(0.315)+0.50 = 0.665
d	0.40 in.	d OEF	--	0.40
e	0.68-0.040-0.28 = 0.36 in.	e	Figure 8.4.2-2 e>R, f>0	$b_e = e+0.5235R_{0.040}$ = 0.36+0.5235(0.300) = 0.517
f	1.64-2(0.040)-2(0.28) = 1.0 in.	f	Figure 8.4.2-2 e>R, f>0	$b_f = b+2(0.5235)R_{0.040}$ = 1.0+2(0.5235)0.300 = 1.314
g	1.25-0.040-0.28-0.40=0.53 in.	g	Equation 8.4.2-1, n=1	$b_i=0.5235nR+b_{fi}$ = 0.5235(0.300)+0.53 = 0.687
h	0.40 in.	h	--	0.40
j	1.87 in.	j	--	1.87
k	0.50 in.	k	--	0.50
l	3.07-1.87-0.50 = 0.70 in.	l	--	0.70

There are two composite sections in this built up section. The first, a Category IV section, is made up of the flat plate (elements j, k, and l) and the flat portion of the stiffener caps (elements b and f). The second, a Category I section is made up of the vertical legs of the two stiffeners, (elements c, d, g, and h). The crippling strength of each of these sections must be calculated before the total section crippling allowable can be calculated.

Category IV section:

Element	b	t	b/t	bt	Edge	F_{cc}
j	1.870	0.080	1.870/0.080=23.38	0.1496	NEF	61900
b	1.330	0.070	1.330/0.070=19.00	0.0931	NEF	70000
f	1.314	0.040	1.315/0.040=32.85	0.0526	NEF	45200
k	0.50	0.080	0.50/0.08=6.25	0.0400	OEF	71900
l	0.70	0.080	0.70/0.08=8.75	0.0560	OEF	51800
			Summation	0.3913		

There are two cases for Category IV composite sections – the buckling stress for the flat plate element j is less than the buckling stresses for the two supports (elements b and f) indicating they adequately support element j. Alternately one or both of the supports has a buckling stress below that of element j, in which case the buckling of element j will occur almost immediately and is thus limited to the buckling stress of the “weakest” support.

In this example, the buckling stress of element f is below that of element j, so the latter scenario is the case. Thus, equation 8.4.2-12 applies. Elements k and l must also be examined relative to elements b and f, respectively. Element k has a higher buckling stresses than element b and element l has a buckling stress higher than element f.

So $F_{cc\text{-}j} > F_{cc\text{-}f}$ and $F_{cc\text{-}b} < F_{cc\text{-}k}$ and $F_{cc\text{-}f} < F_{cc\text{-}l}$

$P_{cc} = F_{cc\text{-}b}(A_b + A_k) + F_{cc\text{-}f}(A_f + A_l) + \text{Minimum}[F_{cc\text{-}b}, F_{cc\text{-}f}, F_{cc\text{-}j}] x A_j$
$= 70000(0.0931 + 0.0400) + 45200(0.0526+0.0560) + 45200(0.1496)$
$= 20987.64$ lbs

Category I section: Per Equation 8.4.2-3, an effective b/t is calculated for the total combined flange and the F_{cc} stress is found for an OEF condition. Since b for the two legs is slightly different due to the thickness difference of the parts, use the larger value and the F_{cc} curve for the thinner material. Additionally, F_{cc} is found for each of the OEF flanges d and h

Element	b	t	b/t	bt	Edge	F_{cc}
c	0.665	0.070	(0.687+0.40)/(0.070+0.040) = 9.88	0.0740	OEF	48000
g	0.687	0.040				
d	0.40	0.070	0.40/0.070=5.71	0.0280	OEF	73900
h	0.40	0.040	0.40/0.040=10	0.0160	OEF	47200
			Summation	0.1180		

So $F_{cc\text{-}c+g} > F_{cc\text{-}h}$ (combined element c+g adequately supports element h) and $F_{cc\text{-}c+g} < F_{cc\text{-}d}$ (combined element c+g buckles before element d, so element buckling of element d is limited by combined element c+g).

$P_{cc} = F_{cc\text{-}c+g}(A_{c+g} + A_d) + F_{cc\text{-}h}A_h$
$= 48000(0.0740+0.0280) + 47200(0.0160)$
$= 5651.2$ lbs

Set up Table 8.4.0-1 and calculate the allowable crippling load for the entire section

Element	b	t	b/t	bt	Edge	Fcc	btFcc
a	0.495	0.070	7.07	0.0347	OEF	68700	2383.89
e	0.517	0.040	12.92	0.0207	OEF	41100	850.77
b, f, j, k and l	--	--	--	0.3913	--	--	20987.64
c, g, d and h	--	--	--	0.1180	--	--	5651.2
Summation				0.5647			29873.5

$F_{cc} = \Sigma btF_{cc}/\Sigma bt = 29873.5/0.5647 = 52902$ psi
$P_{cc} = F_{cc}A = 52902(0.600) = 31741$ lb
M.S. 31741/25642 -1 = 0.24 (Crippling)

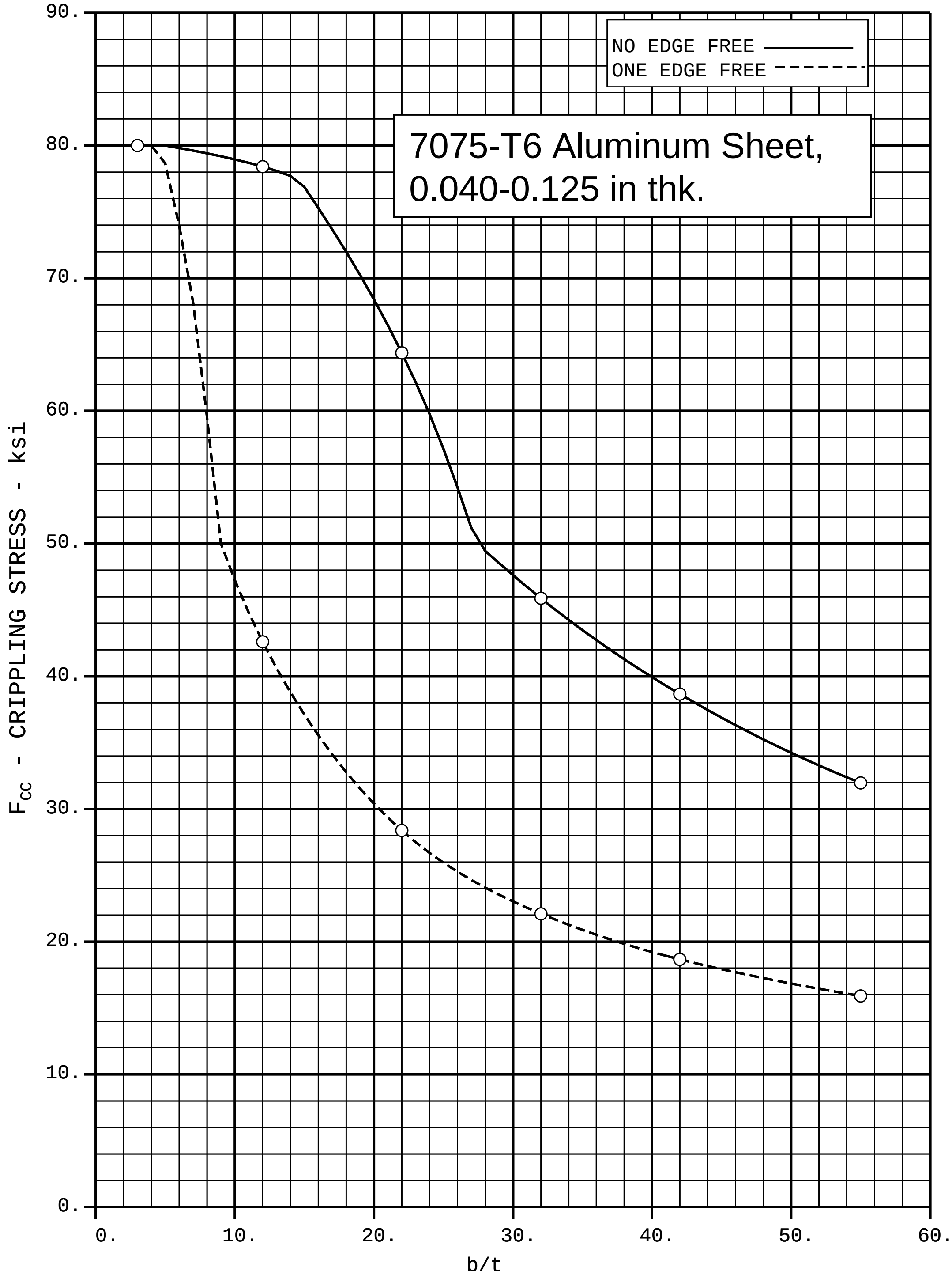


Figure 8.4.4-4 Crippling Curves for 7075-T6 Bare Aluminum Sheet

8.5 Torsional Instability Analysis

When a thin-walled section is loaded in pure axial compression through the section centroid, the column can assume a bent shape, a twisted shape or it can both twist and bend. The bent-shape mode is flexural buckling, covered in Section 8.2. The latter two modes are called torsional instability or torsional buckling and they occur even though there is no applied twisting or torsion load. This failure mode can be the critical failure mode for short, open columns with wide flanges or thin cross-sections.

The fundamental equation which governs this failure mode is a cubic equation which is developed in Reference 8-22. The three resulting solutions correspond to different axes of twist. In many cases the axis of twist for one of the solutions is very close to the shear center of the section, resulting in a torsion instability failure mode, while for the other two solutions it can be located on the maximum and minimum principal axes where it approaches the Euler column failure modes about these axes. The type of torsional buckling exhibited is a function of the type of symmetry present in the cross-section. A cross-section which is symmetric in both axes can twist without bending, while unsymmetrical sections and sections with a single plane of symmetry will both twist and bend. Reference 8-25 provides a significant body of empirical data which confirms the analytical approach described in Reference 8-22 and this section.

As with flexural buckling, torsional instability is a primary failure mode. Primary failure modes occur when the cross-section is translated or rotated or both but the cross-section is not distorted in its own plane. When the cross-section is distorted it is considered a secondary or local failure. An example of a secondary failure mode is crippling which distorts the cross-section. Figure 8.5.0-1 illustrates the twisting present in a beam undergoing torsional instability.

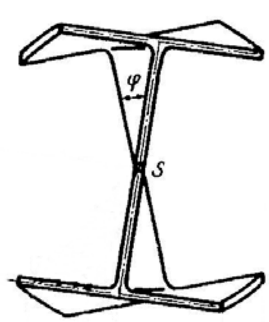


Figure 8.5.0-1 I-BEAM Exhibiting Twist Due to Torsional Instability

Section 8.5.1 describes the analysis of cross-sections which are symmetric about both axes. This is the simplest solution since there is no coupling between modes. Section 8.5.2 describes the analysis for cross-sections which are symmetric about a single axis and Section 8.5.3 describes the analysis for cross-sections which have no symmetry.

As with flexural buckling, end constraint of the flanges factors into the predicted torsional buckling load. If the ends are free to rotate and warp, the length used in the calculation is the full length of the column. If the ends are restrained from rotation and are not free to warp, the length used in the calculation is ½ the length of the column. So

If ends of flange are free to warp and twist: $L' = L$ **Equation 8.5.0-1**

Else if ends of flange are restrained against warp and twist: $L' = \frac{1}{2}L$

where

L is the length of the column from support to support (in)

Twisting can be calculated about either the shear center or the centroid which, for all but sections symmetric about both axes, are not coincident. Section 4.2.7 provides the shear center location and warping constants about the shear center for some common aircraft cross-section shapes. Equations for other sections can be found in References 8-23 and 8-24. Section 4.2.8 provides a means of calculation of the warping constant and warping moments for sections which do not have exact equations readily available. Later in this section the approach for the use of the warping constant and warping moments will be discussed in more detail.

8.5.1 Sections Symmetric about Both Axes

Columns with doubly symmetric cross-sections such as symmetric I-beams, H-beams and cruciforms exhibit only two buckling modes: a pure bending or Euler buckling mode about the minor principal axis and a pure twisting mode in which the beam twists and the flanges warp. Columns with point symmetry, such as Zee beams behave in the same manner. The critical load is the smaller of the two values.

The pure bending mode is the subject of Section 8.2, Equation 8.2.1-3 where the value for I is taken as I_{min}.

$$P_{cr} = \pi^2 E_{tan} I_{min}/(L')^2 \qquad \text{Reference Equation 8.2.1-3}$$

where
I_{min} is the minimum principal moment of inertia (in^4)

The more general form of the flexural buckling equation utilizing the tangent modulus is shown because in the elastic range the tangent modulus is equal to the modulus of elasticity and the equation results in the same value as the Euler buckling equation.

The critical load for pure twisting failure is given by

$$P_\theta = \frac{A}{I_p}\left(G_{\tan} K + \frac{\pi^2 E_{\tan} C_w}{(L')^2}\right) \qquad \textbf{Equation 8.5.1-1}$$

where
A is the cross-section area (in^2)
I_p is the polar moment of inertia about the centroid (in^4). Refer to Section 4.2.1.
G_{tan} is the tangent shear modulus of elasticity (psi) given by Equation 3.2.3-3.
K is the torsional constant of the cross-section (in^4). Refer to Section 4.2.2. In some literature, this is given the symbol J.
E_{tan} is the tangent modulus (psi) given by Equation 3.3.1-3 or 3.3.1-8.
C_w is the warping constant of the section about the shear center (in^6). Refer to Section 4.2.7. In some literature, this is given the symbol Γ.
L′ is the length of the column per Equation 8.5.0-1 (in)

The first term, $G_{tan}K/I_p$, represents that part of the critical compressive stress caused by the resistance of the column to pure twisting. The second term, $\pi^2 E_{tan} C_w/[(L')^2 I_p]$, is that part of the critical compressive stress caused by the resistance of the column to bending. Examining Equation 8.5.1-1 it can be seen that if the warping constant, C_w, is zero, the torsional buckling load is not a function of length as that term is zero; however the section can still be critical in torsional buckling.

Because the tangent modulus and tangent shear modulus are a function of the stress and the stress is a function of the geometry of the column, the solution to Equation 8.5.1-1 is an iterative trial-and-error process. This is done by assuming a stress level, F, and calculating the tangent and tangent shear moduli. Then Equation 8.5.1-1 is solved for L′, using the two moduli, the assumed stress level, F, and the specific geometry of the section in question. Unlike basic column-bending buckling, which can be solved for the general case as a function of only L′/ρ, the solution to

Equation 8.5.1-1 must be done for a specific cross-section. In this way a material and geometry specific allowable torsional stability curve can be generated. If the values obtained are plotted against L′/ρ then the column buckling curve can be shown on the same plot. This process is illustrated in the example problem of Section 8.5.1.1.

The margin of safety for symmetric sections and sections with point symmetry is calculated as

$$M.S. = \frac{\min[P_{cr}, P_\theta]}{P} - 1 \qquad \textbf{Equation 8.5.1-2}$$

where
P_{cr} is the critical allowable load due to flexural buckling (lb)
P_θ is the critical allowable load due to twisting (lb)
P is the applied axial compression load (lb)

With doubly symmetric sections there is no coupling of modes and the shear center is at the section centroid. A notional plot of F_{cr} versus L′/ρ, shown in Figure 8.5.1-1, illustrates the relationship between the criticality of torsional stability and bending stability. For short columns, torsional stability dominates. At higher column lengths, the classic bending-buckling mode dominates. Where the crossover between the two modes occurs is a function of cross-section geometry and material.

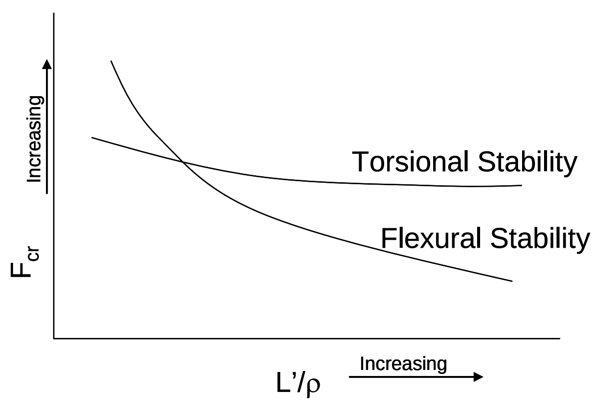
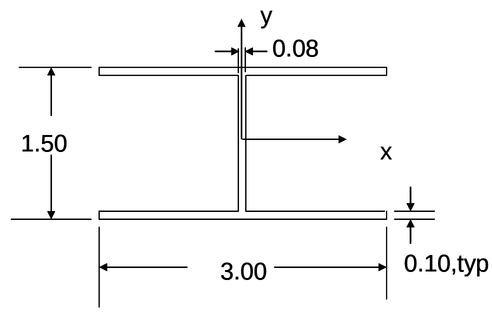


Figure 8.5.1-1 F_{cr} versus L’/ρ

8.5.1.1 Example Problem – Torsional Stability of Symmetric Section

Create a column buckling curve for 7076-T651 plate and, for the I-Beam geometry shown below, create a plot showing the torsional stability and column buckling critical values.

Given: 7075-T651 Plate 1.00-2.00 in. stk. (Table 3.7.7.0(b_1), Reference 8-4)
F_{tu} = 78000 psi
F_{cy} = 68000 psi
$E_c = 10.6 \times 10^6$ psi
n_c = 16
ν = 0.33
Create a torsional stability buckling curve for
$10 \le L'/\rho \le 80$ and the geometry shown



Section Properties for this geometry:
$A = 0.704\ in^2$
$I_x = 0.3091\ in^4$
$I_y = 0.4501\ in^4$

Reference Section 4 for Equations
$I_p = I_x + I_y = 0.3091 + 0.4501 = 0.7592\ in^4$
$\rho_x = \sqrt{(I_x/A)} = \sqrt{(0.3091/0.704)} = 0.6626$ in
$\rho_y = \sqrt{(I_y/A)} = \sqrt{(0.4501/0.704)} = 0.7996$ in
$K = ⅓\Sigma b_n t_n^3 = ⅓\{(2)(3.0)(0.10)^3 + [1.50-(2)(0.10)](0.08)^3\} = 0.0022\ in^4$
$C_w = (H-t)^2 I_y/4 = (1.5-0.10)^2(0.4501)/4 = 0.2205\ in^6$

Calculate $F_{0.05T}$

$$F_{0.05T} = F_{cy}[(\frac{1}{0.05}-1)\frac{F_{cy}}{0.002E_c n_c}]^{\frac{1}{n_c-1}}$$

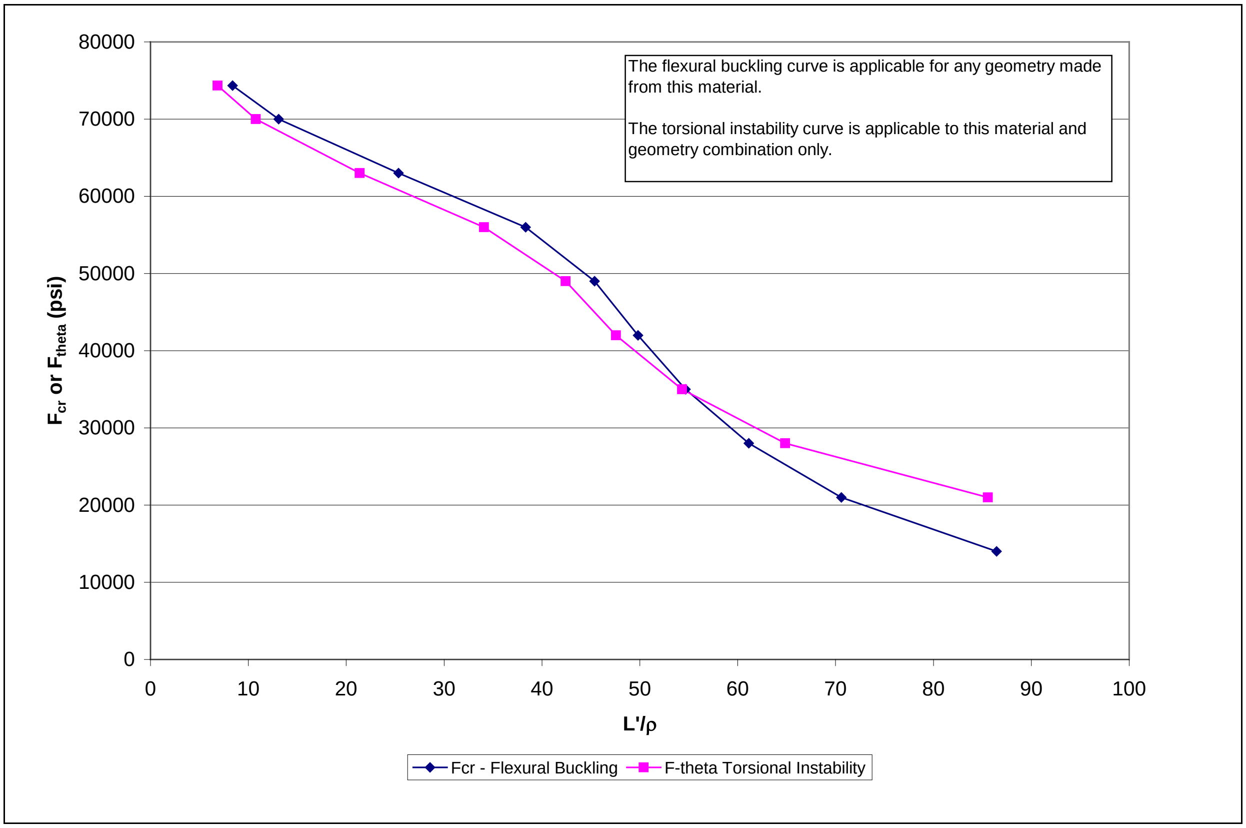
$$F_{0.05T} = 68000\left[\left(\frac{1}{0.05}-1\right)\frac{68000}{(0.002)(10.6x10^6)(16)}\right]^{\frac{1}{16-1}}$$

$F_{0.05T}$= 74341 psi

Calculate F_{cmax}: F_{cmax}=Minimum[F_{tu},$F_{0.05T}$] = Minimum[78000,74341] — F_{cmax}=74341 psi

	F	η_t	E_{tan}	G_{tan}	L′	L′/ρ_{min}
		$1/[1+(0.002E_c n_c/F_{cy}) x(F/F_{cy})^{(n_c-1)}]$	$\eta_t E_c$	$E_{tan}/[2(1+\nu)]$	$[(\pi^2 E_{tan} C_w)/(FI_p - G_{tan}K)]^{0.5}$	L′/0.6626
F_{cmax}	74341	0.0500	530000	199248	4.538	6.849
F_{ty}	70000	0.1711	1813660	681690	7.129	10.759
Even increments of stress per Example 8.2.1.1)	63000	0.5000	5300000	1994786	14.167	21.381
	56000	0.8543	9055580	3404515	22.572	34.067
	49000	0.9775	10361500	3895349	28.107	42.420
	42000	0.9977	10575620	3975904	31.509	47.554
	35000	0.9999	10598940	3984373	35.988	54.313
	28000	1.0000	10600000	3984942	42.975	64.857
	21000	1.0000	10600000	3984962	56.697	85.567
	14000	1.0000	10600000	3984962	111.309	167.989

Plot F vs. L′/ρ. Also shown on plot (calculated in Section 8.2.1.1) is F_{cr}.



8.5.2 Sections Symmetric about a Single Axis

For columns which have a single axis of symmetry such as symmetric channel, angle and tee sections there are two modes of buckling. As with the doubly symmetric sections, one mode is the pure bending or critical flexural buckling mode described in Section 8.2. This can be the classical elastic Euler buckling or it can be inelastic buckling. The critical flexural mode will occur in the plane of symmetry; *i.e.*, the plane containing the axis of symmetry. For the purposes of this discussion, it is assumed the x-z plane is always the plane of symmetry, thus the critical flexural buckling mode will occur about the y-y axis as illustrated in Figure 8.5.2-1. Pure flexural buckling can ONLY occur in the plane of symmetry.[1]

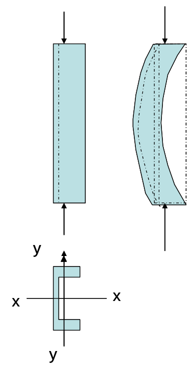
The critical flexural buckling load is given by Equation 8.2.1-3, using $I = I_y$.

$$P_{yy} = P_{cr} = \pi^2 E_{tan} I_y/(L')^2$$

Equation 8.5.2-1
Reference Equation 8.2.1-3

where
I_y is the moment of inertia in the x-z plane or about the centroidal y axis(in^4)
E_{tan} is the tangent modulus of the material (psi)
L′ is the effective length of the column per Equation 8.2.1-9 (in)

[1] Reference 8-22.



X-Z is the Plane of Symmetry.
The critical bending buckling mode occurs in the plane of symmetry or about the y-y axis.
Figure 8.5.2-1 Flexural Buckling For Sections with a Single Plane of Symmetry

The remaining modes are coupled bending-twisting modes in the non-symmetric plane. For this category of section, the twisting calculation can either be done about the shear center, which simplifies the calculation because the warping moments are zero about the shear center, or it can be done about the centroid, which requires the calculation of x and y warping moments. Both approaches are described in this section.

Warping moments are geometric properties of the cross-section which are used for the calculation of twist. They are similar in concept to static moments and moments of inertia. Refer to Section 4.2.9.

If the warping constant can be calculated about the shear center for the section under analysis, then the allowable critical combined mode buckling load is calculated from the critical buckling load for bending in the non-symmetric plane and the critical twisting load. The critical buckling load for bending in the non-symmetric plane (or about the symmetric axis) is given by Equation 8.2.1-3, using I_x.

$$P_{xx} = P_{cr} = \pi^2 E_{tan} I_x / (L')^2$$

Equation 8.5.2-2
Reference Equation 8.2.1-3

where
I_x is the moment of inertia in the y-z plane or about the centroidal x axis (in^4)

The critical twisting load is calculated from Equation 8.5.1-1, where instead of using the polar moment of inertia about the centroid, the polar moment of inertia about the shear center is used.

Equation 8.5.2-3

$$P_\theta = \frac{A}{I_{p-sc}}\left(G_{\tan} K + \frac{\pi^2 E_{\tan} C_w}{(L')^2}\right)$$

where
I_{p-sc} is the polar moment of inertia about the shear center (in^4). Reference Section 4.2.1
C_w is the warping constant about the shear center of the section (in^6). Refer to Section 4.2.7.

The combination of these loads results in a quadratic equation which results in two roots. The lower of these is the critical allowable load. The two solutions are given by

Equation 8.5.2-4

$$P_R = \frac{I_{p-sc}}{I_p}\left(\frac{P_{xx}+P_\theta}{2}\right) \pm \tfrac{1}{2}\sqrt{\left(\frac{I_{p-sc}}{I_p}(P_{xx}+P_\theta)\right)^2 - 4\frac{I_{p-sc}}{I_p}(P_{xx}P_\theta)}$$

where
P_{xx} is the allowable load for flexural stability given by Equation 8.5.2-2 (lbs)
P_θ is the allowable load for twisting about the shear center given by Equation 8.5.2-3 (lbs)
I_{p-sc} is the polar moment of inertia about the shear center (in^4). Reference Section 4.2.1
I_p is the polar moment of inertia about the section centroid (in^4)

In general, P_R resulting from the difference equation is the minimum load. As was discussed in Section 8.5.1, the solution to the allowable torsional buckling curve requires an iterative approach since both E_{tan} and G_{tan} are functions of stress. With an unsymmetric section, this is complicated by the coupled bending-twisting mode making it no longer possible to solve the bending buckling or only the twisting buckling equations independently. The iteration must occur on the combined mode for a given cross-section given by Equation 8.5.2-4. Table 8.5.2-1 provides an outline of the steps necessary for the iterative solution of the combined mode to determine the torsional buckling allowable curve for a specific section.

Table 8.5.2-1 Iteration Steps for Solution of Equation 8.5.2-4

1. Calculate the geometric values for the cross-section A, I_x, I_p, I_{p-sc}, K, C_w
2. Assume a buckling load, P_R
3. Calculate the buckling stress P_R/A
4. Calculate η_{tan}, E_{tan} and G_{tan} for this stress level (Equation 3.3.1-8 or 3.3.1-3 and Equation 3.2.3-3)
5. Assume L′
6. Using η_{tan} and L′, calculate P_{xx} and P_θ (Equations 8.5.2-2 and 8.5.2-3)
7. Calculate $P_{R\text{-}calc}$ (Equation 8.5.2-4)
8. Compare $P_{R\text{-}calc}$ to P_R. If not equal, increment L′. Repeat steps 6 -8 until match is achieved.
9. Increment P_R. Repeat steps 3-8 for new P_R
10. Plot Buckling stress vs. L′ or L′/ρ.

Figure 8.5.2-2 shows the critical load curves for the individual bending buckling modes, the torsional buckling mode and the combined bending-torsion buckling mode for a tee section made from 7075-T651 aluminum plate. The iterative solution outlined in Table 8.5.2-1 is represented by the bending-torsion curve with the open diamond symbol. The two allowable curves for this section are the critical bending buckling curve (F_{yy}) represented by the closed diamond symbols and the combined bending-torsion buckling curve represented by the open diamond symbols. Note that the bending-torsion curve follows the torsion buckling curve in the short length range and the F_{xx} bending buckling curve in the long length range and transitions between them in the intermediate range. It is applicable only to the material and section analyzed, and the lower of the two curves at a given L′ would be the allowable.

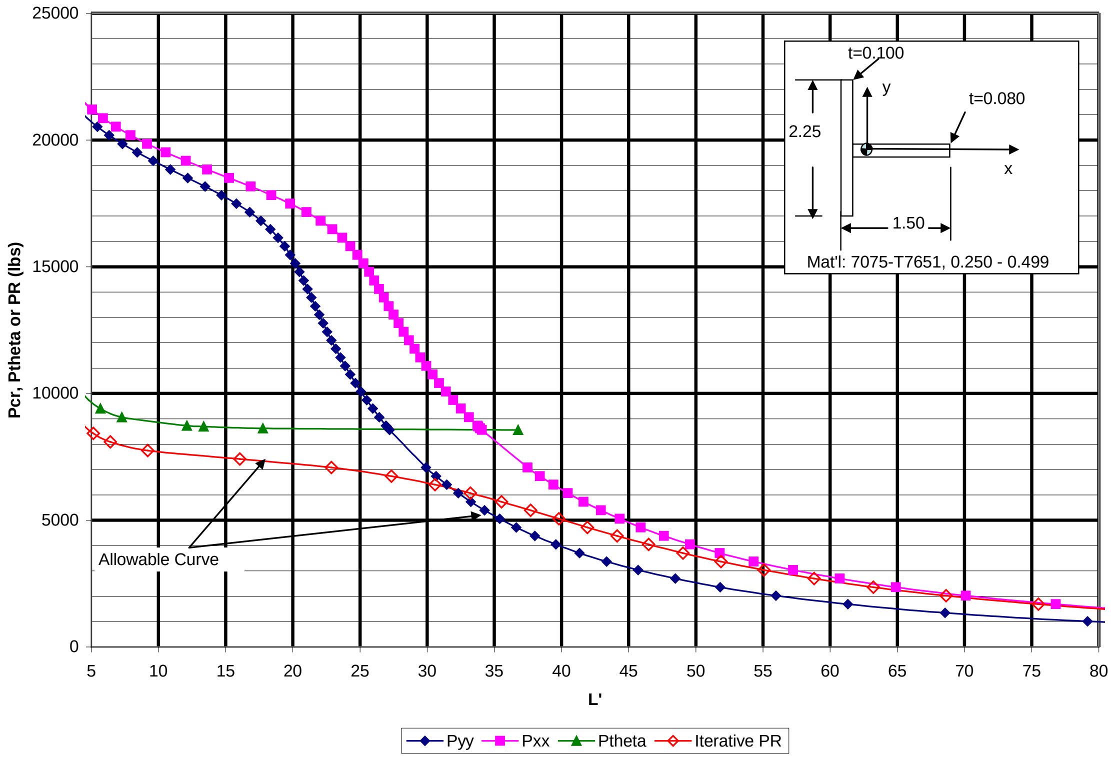


Figure 8.5.2-2 Column Buckling Curves

The margin of safety is calculated as

$$M.S. = \frac{\min[P_{cr}, P_R]}{P} - 1$$ **Equation 8.5.2-5**

where
P_{cr} is the allowable load for bending buckling stability about the un-symmetric axis (P_{yy} per Equation 8.5.2-1) (lb)
P_R is the critical allowable load due to bending and twisting from Equation 8.5.2-4 or Equation 8.5.2-7 (lb)
P is the applied axial compression load (lb)

If the warping constant about the shear center is not available, an alternative approach which does not require the calculation of the shear center is available. As described in Section 4.2.7, the warping constant about the centroid can be evaluated, saving the time to calculate the shear center for the section; however, this does require the calculation of warping moments. Warping moments (in^5) are geometric properties of the cross-section and are similar in concept to static moment and moment of inertia and are used to calculate the twisting moments.

The calculation of the allowable bending load about the symmetric axis is done as in Equation 8.5.2-2. The calculation of the critical twisting load is done similarly to Equation 8.5.2-3 except the polar moment of inertia and the warping constant are calculated about the centroid rather than about the shear center

$$P_{\theta,c} = \frac{A}{I_p}\left(G_{\tan}K + \frac{\pi^2 E_{\tan} C_{w,c}}{(L')^2}\right) \qquad \textbf{Equation 8.5.2-6}$$

where
I_p is the polar moment of inertia about the section centroid (in^4)
$C_{w,c}$ is the warping constant about the centroid of the section (in^6)[2]. Refer to Section 4.2.9

The critical allowable load[3] is then calculated from

$$P_R = \frac{P_{xx} + P_{\theta,c}}{2} - \sqrt{\left(\frac{P_{\theta,c} - P_{xx}}{2}\right)^2 + (A\alpha_{xx})^2} \qquad \textbf{Equation 8.5.2-7}$$

where
A is the cross-section area (in^2)
α_{xx} is given by Equation 8.5.2-8 (psi)
P_{xx} is the allowable bending load about the symmetric axis and is given by Equation 8.5.2-2 (lb)
$P_{\theta,c}$ is the critical twisting load about the section centroid and is given by Equation 8.5.2-6 (lb)

The additional contribution due to the calculation of twist resulting from using the warping constant about the centroid rather than the shear center is given by

$$\alpha_{xx} = \frac{\pi^2 E_{\tan} R_{xx}}{(L')^2 \sqrt{A I_p}} \qquad \textbf{Equation 8.5.2-8}$$

where
R_{xx} is the warping moment about the x axis (in^5). Reference Section 4.2.9

The margin of safety is calculated per Equation 8.5.2-5.

8.5.3 Sections with No Symmetry

For sections with no symmetry, there is no possibility of either pure bending or pure twisting and what occurs is a combination of bending and twisting about each of the principal axes. This leads to a cubic equation with 3 solutions, the smallest of which is the critical load for the section. This section will present an approach to the solution of an unsymmetrical section which does not require the calculation of the shear center of the section. The derivation of the equations for sections with no symmetry is shown in detail in Reference 8-22.

[2] Note that for straight line elements whose flanges all meet at a single point (*e.g.*, tees and L's) the warping constant C_w is very small and is often assumed, for engineering purposes, to be zero. This is illustrated in Section 8.5.3.1.
[3] Reference 8-22.

The general cubic equation for the critical load about any convenient pair of rectangular x-y axes is given by

$$P_R^3 - P_R^2(P_{\theta,c}+P_{xx}+P_{yy}) + P_R(P_{\theta,c}P_{xx} + P_{\theta,c}P_{yy}+P_{xx}P_{yy}-A^2\alpha_{xy}^2-A^2\alpha_{xx}^2-A^2\alpha_{yy}^2)$$
$$- (P_{\theta,c}P_{xx}P_{yy} + 2A^3\alpha_{xy}\alpha_{xx}\alpha_{yy} - P_{\theta,c}A^2\alpha_{xy}^2 - P_{xx}A^2\alpha_{yy}^2 - P_{yy}A^2\alpha_{xx}^2) = 0$$ **Equation 8.5.3-1**

where
P_{xx} is the flexural buckling load about a convenient centroidal x axis and is given by Equation 8.2.1-3 using I_x (lbs)
P_{yy} is the flexural buckling load about a convenient centroidal y axis and is given by Equation 8.2.1-3 using I_y (lbs)
$P_{\theta,c}$ is the critical twisting load about the section centroid and is given by Equation 8.5.2-6 (lbs)
α_{xx} is given by Equation 8.5.2-8 (psi)
α_{yy} is given by Equation 8.5.3-2 (psi)
α_{xy} is given by Equation 8.5.3-3 (psi)

The additional contributions to twist due to the calculation of the warping constant about the centroid rather than the shear center are given by

$$\alpha_{yy} = \frac{\pi^2 E_{\tan} R_{yy}}{(L')^2\sqrt{AI_p}}$$ **Equation 8.5.3-2**

$$\alpha_{xy} \frac{\pi^2 E_{\tan} I_{xy}}{A(L')^2}$$ **Equation 8.5.3-3**

where
R_{yy} is the warping moment about the y axis (in^5). Reference Section 4.2.9
I_{xy} is the product of inertia given by Equation 4.2-10 (in^4)
E_{tan} is the tangent modulus of the material (psi)
L′ is per Equation 8.5.0-1 (in)
A is the area of the section (in^2)
I_p is the polar moment of inertia about the centroidal axis (in^4)

The positive, real roots of Equation 8.5.3-1 can be obtained as follows.

$$P_R^3 - A_1P_R^2 + A_2P_R - A_3 = 0$$
$$Q = (A_1^2-3A_2)/9$$
$$R=(2A_1^3-9A_1A_2+27A_3)/54$$ **Equation 8.5.3-4**

Where
Q, R are convenient intermediate variables
$A_1=-(P_{\theta,c}+P_{xx}+P_{yy})$
$A_2=(P_{\theta,c}P_{xx} + P_{\theta,c}P_{yy}+P_{xx}P_{yy}-A^2\alpha_{xy}^2-A^2\alpha_{xx}^2-A^2\alpha_{yy}^2)$
$A_3=-(P_{\theta,c}P_{xx}P_{yy} + 2A^3\alpha_{xy}\alpha_{xx}\alpha_{yy} - P_{\theta,c}A^2\alpha_{xy}^2 - P_{xx}A^2\alpha_{yy}^2 - P_{yy}A^2\alpha_{xx}^2)$

The solution to any cubic equation may have 1 positive real root and 2 imaginary roots or 3 positive real roots. If there are 3 real roots, two may be equal.

If Q^3-$R^2 \geq 0$, then Equation 8.5.3-1 has 3 real roots which can be obtained as follows

$$\theta = \cos^{-1}\left(\frac{R}{\sqrt{Q^3}}\right)$$

$$P_{R1} = -2\sqrt{Q}\cos\left(\theta/3\right) - \frac{A_1}{3}$$

$$P_{R2} = -2\sqrt{Q}\cos\left((\theta + 2\pi)/3\right) - \frac{A_1}{3}$$

Equation 8.5.3-5

$$P_{R3} = -2\sqrt{Q}\cos\left((\theta + 4\pi)/3\right) - \frac{A_1}{3}$$

$$P_R = minimum[P_{R1}, P_{R2}, P_{R3}]$$

If Q^3-$R^2 < 0$, then Equation 8.5.3-1 has 1 real root and two imaginary roots. The real root can be obtained from

$$P_R = -sign(R)\left[\sqrt[3]{\sqrt{R^2 - Q^3} + |R|} + \frac{Q}{\sqrt[3]{\sqrt{R^2 - Q^3} + |R|}}\right] - \frac{A_1}{3}$$

Equation 8.5.3-6

where
sign(R) is the sign of the number R calculated above

When the stresses are in the inelastic range such that E_{tan} and G_{tan} vary with stress, an iterative solution is required with multiple dependencies and the solution to the cubic equation can only be done reliably by computer program. The way to determine if the use of tangent modulus is required is to solve the Equations 8.5.3-1 through 8.5.3-6 using the Modulus of Elasticity. Once the critical buckling load is determined, calculate the critical buckling stress. If it is below the proportional limit, then the buckling is elastic and the solution is valid. If the critical buckling stress is above the proportional limit, the critical buckling load will need to be determined using the tangent modulus.

Also note that the critical buckling load determined by Equation 8.5.3-5 or 8.5.3-6 is for a load applied at the centroid of the section. Equation 8.5.3-8 will provide an Equation which will result in the critical buckling load applied at the section's shear center. Example problems provided in Sections 8.5.3.1 and 8.5.3.2 will illustrate the difference in critical load for these two approaches.

The margin of safety is then calculated from

Equation 8.5.3-7

$$M.S. = \frac{P_R}{P} - 1$$

where
P_R is the critical allowable load due to bending and twisting from Equation 8.5.3-5 or 8.5.3-6 (lb)
P is the applied axial compression load (lb)

Note that for this margin of safety calculation, only P_R is considered. This is because the critical buckling load for an unsymmetrical section which satisfies Equation 8.5.3-5 or 8.5.3-6 is **always** smaller than the P_{xx} or P_{yy} critical bending buckling loads, so comparing the result obtained for P_R to the minimum of P_{xx} and P_{yy} is a good check of the solution. The difference between P_R and minimum[P_{xx} or P_{yy}] can be large for a short column and becomes smaller as the column gets longer.

If the warping constant for an unsymmetric section is available about the shear center, then a simpler formulation than Equation 8.5.3-1 is available.

$$\frac{I_p}{I_{p-sc}}P_R^3 - [P_\theta + P_{xx} + P_{yy} - \frac{A}{I_{p-sc}}(P_{xx}y_o^2 + P_{yy}x_o^2)]P_R^2 + (P_\theta P_{xx} + P_\theta P_{yy} + P_{xx}P_{yy})P_R - P_\theta P_{xx}P_{yy} = 0$$

Equation 8.5.3-8

where
P_{xx} is the flexural buckling load about the x principal axis and is given by Equation 8.2.1-3 using $I_{x\text{-principal}}$ (lbs)
P_{yy} is the flexural buckling load about the y principal axis and is given by Equation 8.2.1-3 using $I_{y\text{-principal}}$ (lbs)
P_θ is the critical twisting load about the shear center and is given by Equation 8.5.2-3 (lbs)
I_p is the polar moment of inertia about the centroidal axis (in^4)
I_{p-sc} is the polar moment of inertia about the shear center (in^4). Reference Section 4.2.1
A is the section area (in^2)
x_o, y_o are the distances along the x and y axes from the centroid to the shear center (in)

Note that this simplified approach uses the principal moments of inertia rather than any convenient centroidal moment of inertia used in Equation 8.5.3-1.

Dividing Equation 8.5.3-8 by (I_p/I_{p-sc}) and rewriting into a format consistent with Equation 8.5.3-4

$$P_R^3 - A_1P_R^2 + A_2P_R - A_3 = 0$$
$$Q = (A_1^2 - 3A_2)/9$$
$$R = (2A_1^3 - 9A_1A_2 + 27A_3)/54$$

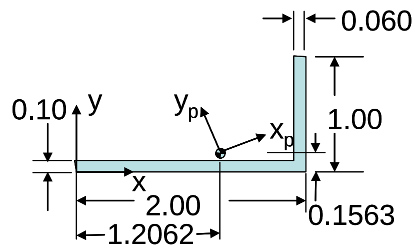
Equation 8.5.3-9

Where Q, R are convenient intermediate variables
$A_1 = -[P_\theta + P_{xx} + P_{yy} - (A/I_{p-sc})(P_{xx}y_o^2 + P_{yy}x_o^2)] / (I_p/I_{p-sc})$
$A_2 = (P_\theta P_{xx} + P_\theta P_{yy} + P_{xx}P_{yy}) / (I_p/I_{p-sc})$
$A_3 = -(P_\theta P_{xx}P_{yy}) / (I_p/I_{p-sc})$

Equation 8.5.3-9 can be solved using the approach outlined in Equation 8.5.3-5 and 8.5.3-6 and then the margin of safety calculated per Equation 8.5.3-7.

Again note the method described by Equations 8.5.3-8 and 8.5.3-9 is only valid when the warping constant about the shear center is known. The most practical application of this is for unsymmetric sections comprised solely of straight elements that ALL meet at a single, common point, such as a tee or angle with no symmetry. For these elements, within engineering accuracy, the shear center is at the point where the elements intersect and the warping constant is zero.

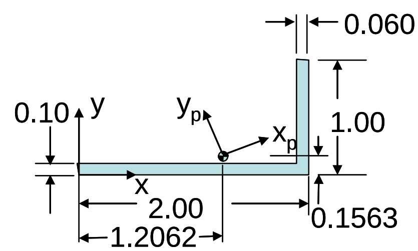
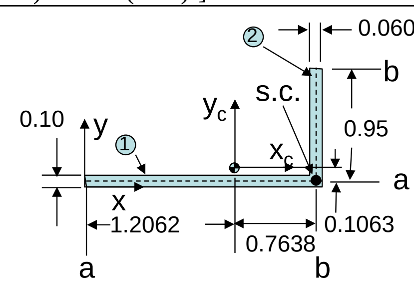
8.5.3.1 Example – Section with No Symmetry C_w about Shear Center

Determine P_R for the unequal leg angle shown. The column is 20 in long and simply supported at both ends. The material is 7075-T651 Plate 0.5-1.00 in stk. thick. (Table 3.7.7.0(b_1), Reference 8-4) F_{tu} = 79000 psi F_{cy} = 70000 psi $E_c = 10.6x10^6$ psi n_c = 16 ν = 0.33 $G=E_c/[2\ (1+\mu)]= (10.6x10^6)/[2\ (1+0.33)] = 3.985x10^6$ psi	
Summary of section properties: A=0.254 in^2 $I_x = 0.0144\ in^4$ ybar = 0.1563 in $I_y = 0.1067\ in^4$ xbar = 1.2062 in	 $I_{xy} = 0.0207\ in^4$ $I_{yp} = I_{max} = 0.1111\ in^4$ $I_{xp} = I_{min} = 0.0100\ in^4$
Calculate the warping constant about the shear center (Section 4.2.7) $C_w=(h'^3t^3_{leg1} + w'^3t^3_{leg2})/36$ h′ = 2-0.06 = 1.94 in w′ = 1.0-0.10 = 0.90 in $C_w = [(1.94)^3(0.10)^3 + (0.90)^3(0.06)^3]/36 = 0.0002\ in^6$ x_{sc}= 1.97 in; y_{sc}=0.05 in	Calculate the St. Venant torsion constant $K=1/3\Sigma bt^3 = 1/3[2.0(0.10)^3 + 0.90(0.06^3)]$ $K = 0.0007\ in^4$
Calculate the polar moment of inertia: About the centroid: $I_p=I_x+I_y = 0.0144 + 0.1067 = 0.1211\ in^4$ About the shear center: $I_{p\text{-}sc} = I_{xx} + Ay_o^2 + I_{yy} + Ax_o^2$ $y_o = y_{bar}-y_{sc} = 0.1563 – 0.05 = 0.1063$ in $x_o = x_{bar}-x_{sc} = 1.2062 – 1.97 = -0.7638$ in $I_{p\text{-}sc} = 0.0144 + (0.254)(0.1063)^2 + 0.1067+ (0.254)(-0.7638)^2 = 0.2722\ in^4$	
Calculate the flexural buckling load about the principal axes (assume solution is in the elastic range): $P_{cr\text{-}xprincipal} = \pi^2E_{tan}I/L'^2 = \pi^2(10.6x10^6)(0.0100)/20^2 =$ 2615.45 lb $P_{cr\text{-}yprincipal} = \pi^2E_{tan}I/L'^2 = \pi^2(10.6x10^6)(0.1111)/20^2 =$ 29057.60 lb	Calculate the twisting buckling load about the shear center (assume the solution is in the elastic range): $P_\theta = (A/I_{p\text{-}sc})(G_{tan}K + \pi^2E_{tan}C_w/L'^2)$ $=(0.254/0.2722)[(3.985x10^6)(0.0007) + \pi^2(10.6x10^6)(0.0002)/20^2]$ P_θ = (0.9331) (2789.50 + 52.31) = 2651.70 lb

Using Equation 8.5.3-8 (since the warping constant about the shear center is known): $(I_p/I_{p\text{-}sc})P_R^3 - P_R^2[P_\theta+P_{xx} +P_{yy} – (A/I_{p\text{-}sc})(P_{xx}y_o^2+P_{yy}x_o^2)] + P_R(P_\theta P_{xx} + P_\theta P_{yy}+P_{xx}P_{yy}) - P_\theta P_{xx}P_{yy} = 0$ $A= I_p/I_{p\text{-}sc} = 0.1211/0.2722 = 0.445$ $B= P_\theta+ P_{xx} +P_{yy} – (A/I_{p\text{-}sc})(P_{xx}y_o^2+P_{yy}x_o^2) = 2651.70+2615.45+29057.60 - (0.254/0.2722)[(2615.45)(0.1063)^2+(29057.60)(-0.7638)^2] = 18478.70$ $C=(P_\theta P_{xx} + P_\theta P_{yy}+P_{xx}P_{yy}) = (2651.70)(2615.45) + (2651.70)(29057.60) + (2615.45)(29057.60) = 1.5999x10^8$ $D =(P_\theta P_{xx}P_{yy}) = (2651.70)(29057.60)(2615.45) = 2.1053x10^{11}$
Determine the coefficients of $P_R^3 - A_1P_R^2 + A_2P_R – A_3= 0$: $A_1=-B/A = -18478.70/ 0.445 = -41525.17$ $A_2=C/A = 1.5999x10^8/0.445= 3.5953x10^8$ $A_3=-D/A = -2.1053*10^{11}/0.445 = -4.731*10^{11}$
Calculate Q, R $Q = (A_1^2-3A_2)/9 = [(41525.17)^2-3(3.5953x10^8)]/9 = 7.1750x10^7$ $R=(2A_1^3-9A_1A_2+27A_3)/54 = [2(-41525.17)^3 – 9(-41525.17)(3.5953x10^8)+27(-4.731x10^{11})]/54 =$ $-4.0027x10^{11}$

Is $Q^3-R^2 \geq 0$: $(7.1750x10^7)^3-(-4.0027x10^{11})^2 = 2.0916x10^{23} > 0$, 3 real roots which can be obtained as follows
$\theta = \arccos(R/Q^{3/2}) = \arccos[(-4.0027x10^{11})/(7.1750x10^7)^{3/2}] = 2.290$ radians $P_{R1}=-2Q^{1/2}\cos(\theta/3)-A_1/3 = -2(7.1750x10^8)^{1/2}\cos(2.290/3) - (-41525.17)/3 = 1601$ lb $P_{R2}=-2Q^{1/2}\cos[(\theta+2\pi)/3]-A_1/3 = -2(7.1750x10^8)^{1/2}\cos[(2.290+2\pi)/3] - (-41525.17)/3 = 30105$ lb $P_{R3}=-2Q^{1/2}\cos[(\theta+4\pi)/3]-A_1/3 = -2(7.1750x10^8)^{1/2}\cos[(2.290+4\pi)/3] - (-41525.17)/3 = 9819$ lb
P_R = minimum[P_{R1},P_{R2}, P_{R3}] = minimum[1601, 30105, 9819] = 1601 lb
Note the combined bending twisting failure mode at 1601 lb is less than the critical flexural buckling load of 2615 lb by a significant amount. Also note that the critical stress, F_{cr} = 1601/0.254 = 6303 psi is within the elastic range and the use of E rather than E_{tan} is valid.

8.5.3.2 Example – Section with No Symmetry C_w about Centroid

Determine P_R for the unequal leg angle of Section 8.5.3.1 using the method of solution about the centroid. The column is 20 in long and simply supported at both ends. The material is 7075-T651 Plate 0.5-1.00 in stk. thick. (Table 3.7.7.0(b_1), Reference 8-4) $E_c = 10.6x10^6$ psi $n_c = 16$ $\nu = 0.33$ $G=E_c/[2(1+\mu)]= (10.6x10^6)/[(2(1+0.33)] = 3.985x10^6$ psi	
Summary of section properties: A=0.254 in^2 $I_x = 0.0144$ in^4 ybar = 0.1563 in $I_y = 0.1067$ in^4 xbar = 1.2062 in	$I_{xy} = 0.0207$ in^4 $I_{yp} = I_{max} = 0.1111$ in^4 $I_{xp} = I_{min} = 0.0100$ in^4
Calculate the St. Venant torsion constant $K=1/3\Sigma bt^3 = 1/3[2.0(0.10)^3 + 0.90(0.06)^3] = 0.0007$ in^4	
Calculate the warping constant about the centroid (Section 4.2.9) x_{sc}= 1.97 in; y_{sc}=0.05 in	

Elem	r_i	L_i	r_iL_i	w_{i-a}	w_{i-b}	$w_{i-a}+w_{i-b}$	$L_i/2$	t_i	L_it_i
1	0.1063	1.97	0.2094	0	0.2094	0.2094	0.985	0.10	.197
2	0.7638	0.95	0.7256	0.2094	0.9350	1.1444	0.475	0.06	.057

Elem	$(t_iL_i/2)(w_{i-a}+w_{i-b})$	$L_it_i(w_{i-a}^2+w_{i-a}w_{i-b}+w_{i-b}^2)/3$	$C_{w-c} =\Sigma L_it_i(w_{i-a}^2+w_{i-a}w_{i-b}+w_{i-b}^2)/3 - (1/A)[\Sigma(t_iL_i/2)(w_{i-a}+w_{i-b})]^2$
1	0.0206	0.0029	
2	0.0326	0.0212	$C_{w-c} = 0.0241 - (1/0.254)(0.0532)^2 = 0.0130$ in^6
Σ	0.0532	0.0241	

Calculate R_x

Elem	y_{i-a}	y_{i-b}	$2y_{i-a}+y_{i-b}$	$y_{i-a}+2y_{i-b}$	w_{i-a}	w_{i-b}	$w_{i-a}(2y_{i-a}+y_{i-b})$	$w_{i-b}(y_{i-a}+2y_{i-b})$
1	-0.1063	-0.1063	-0.3189	-0.3189	0	0.2094	0	-0.0668
2	-0.1063	0.8437	0.6311	1.5811	0.2094	0.9350	0.1322	1.4783

Elem	$w_{i-a}(2y_{i-a}+y_{i-b}) + w_{i-b}(y_{i-a}+2y_{i-b})$	L_i	t_i	$L_it_i/6$	ΔR_x	$R_x = \Sigma\Delta R_x$ = -0.00219 + 0.01530
1	-0.0668	1.97	0.10	0.0328	-0.00219	

2	1.6105	0.95	0.06	0.0095	0.01530	= 0.01311 in^5

Calculate R_y

Elem	$x_{i\text{-}a}$	$x_{i\text{-}b}$	$2x_{i\text{-}a}+x_{i\text{-}b}$	$x_{i\text{-}a}+2x_{i\text{-}b}$	$w_{i\text{-}a}$	$w_{i\text{-}b}$	$w_{i\text{-}a}(2x_{i\text{-}a}+x_{i\text{-}b})$	$w_{i\text{-}b}(x_{i\text{-}a}+2x_{i\text{-}b})$
1	-1.2062	0.7638	-1.6486	0.3214	0	0.2094	0	0.0673
2	0.7638	0.7638	2.2914	2.2914	0.2094	0.9350	0.4798	2.1425

Elem	$w_{i\text{-}a}(2x_{i\text{-}a}+x_{i\text{-}b}) + w_{i\text{-}b}(x_{i\text{-}a}+2x_{i\text{-}b})$	L_i	t_i	$L_it_i/6$	ΔR_y	$R_y = \Sigma\Delta R_y$ = 0.00221+0.02491 = 0.02712 in^5
1	0.0673	1.97	0.10	0.0328	0.00221	
2	2.6223	0.95	0.06	0.0095	0.02491	

Calculate the polar moment of inertia:
About the centroid: $I_p=I_x+I_y = 0.0144 + 0.1067 = 0.1211\ in^4$

Calculate $\alpha_{xx}=\pi^2E_{tan}R_{xx} / [L'^{\,2}(AI_p)^{0.5}] = \pi^2(10.6x10^6)(0.01311)/\{(20)^2 [(0.254)(0.1211)]^{0.5}\} = 19550.57$

Calculate $\alpha_{yy}=\pi^2E_{tan}R_{yy} / [L'^{\,2}(AI_p)^{0.5}] = \pi^2(10.6x10^6)(0.02712)/\{(20)^2[(0.254^*)(0.1211)]^{0.5}\} = 40443.28$

Calculate $\alpha_{xy}=\pi^2E_{tan}I_{xy} / [AL'^{\,2}] = \pi^2(10.6x10^6)(0.0207)/[(0.254)(20)^2] = 21314.85$

Calculate the flexural buckling load about the centroidal axes (assume solution is in the elastic range):
$P_{cr\text{-}x} = \pi^2E_{tan}I/L'^2 = \pi^2(10.6x10^6)(0.0144)/20^2 =$
3766.24 lb
$P_{cr\text{-}y}= \pi^2E_{tan}I/L'^2 = \pi^2(10.6x10^6)(0.1067)/20^2 =$
27906.80 lb

Calculate the twisting buckling load about the shear center (assume the solution is in the elastic range):
$P_{\theta,c} = A/I_p(G_{tan}K + \pi^2E_{tan}C_w/(L')^2)$
$=(0.254/0.1211)[(3.985x10^6)(0.0007) + \pi^2(10.6x10^6)(0.0130)/(20)^2]$
$P_{\theta,c} =2.0974(2789.50 + 3400.08) = 12961.05$ lb

Using Equation 8.5.3-1:
$P_R^3 - P_R^2(P_{\theta,c}+P_{xx} +P_{yy}) + P_R(P_{\theta,c}P_{xx} + P_{\theta,c}P_{yy}+P_{xx}P_{yy}-A^2\alpha_{xy}^2-A^2\alpha_{xx}^2-A^2\alpha_{yy}^2)$
$- (P_{\theta,c}P_{xx}P_{yy} + 2A^3\alpha_{xy}\alpha_{xx}\alpha_{yy} - P_{\theta,c}A^2\alpha_{xy}^2 - P_{xx}A^2\alpha_{yy}^2 - P_{yy}A^2\alpha_{xx}^2) = 0$
$A_1 = -(P_{\theta,c}+ P_{xx} +P_{yy}) = 3766.24+27906.80+12961.05 = 44634.09$
$A_2=(P_{\theta,c}P_{xx} + P_{\theta,c}P_{yy}+P_{xx}P_{yy}-A^2\alpha_{xy}^2-A^2\alpha_{xx}^2-A^2\alpha_{yy}^2) = (12961.05)(3766.24) + (12961.05)(27906.80) + (3766.24)(27906.80) – (0.254)^2(21314.85)^2– (0.254)^2(19550.57)^2– (0.254)^2(40443.28)^2 = 3.5612x10^8$
$A_3 =-(P_{\theta,c}P_{xx}P_{yy} + 2A^3\alpha_{xy}\alpha_{xx}\alpha_{yy} - P_{\theta,c}A^2\alpha_{xy}^2 - P_{xx}A^2\alpha_{yy}^2 - P_{yy}A^2\alpha_{xx}^2) = -[(12961.05)(3766.24)(27906.80) + 2(0.254)^3(21314.85)(19550.57)(40443.28)-(12961.05)(0.254)^2(21314.85)^2-(3766.24)(0.254)^2(40443.28)^2 -(27906.80)(0.254)^2(19550.57)^2] =-4.4910x10^{11}$

Summarizing the coefficients of $P_R^3 - A_1P_R^2 + A_2P_R – A_3= 0$:
$A_1= -44634.09$
$A_2= 3.5612x10^8$
$A_3= -4.4910x10^{11}$

Calculate Q, R
$Q = (A_1^2-3A_2)/9 = [(-44634.09)^2-3(3.5612x10^8)]/9 = 1.0265x10^8$
$R=(2A_1^3-9A_1A_2+27A_3)/54 = [2(-44634.09)^3 – 9(-44634.09)(3.5612x10^8)+27(-4.4910x10^{11})]/54 =$
$-8.6871x10^{11}$
Is $Q^3-R^2 \geq 0$: $(1.025x10^8)^3-(-8.6871x10^{11})^2 = 3.2697x10^{23} > 0$, 3 real roots which can be obtained as follows

$\theta = \arccos(R/Q^{3/2}) = \arccos[-8.6871x10^{11}/(1.0265x10^8)^{3/2}]= \arccos (-.83529) = 2.559$ radians
$P_{R1}=-2Q^{1/2}\cos(\theta/3)-A_1/3 = -2(1.0265x10^8)^{1/2}\cos(2.559/3) - (-44634.09.13)/3 = 1550$ lb
$P_{R2}=-2Q^{1/2}\cos[(\theta+2\pi)/3]-A_1/3 = -2(1.0265x10^8)^{1/2}\cos[(2.559+2\pi)/3]- (-44634.09)/3 = 8323$ lb
$P_{R3}=-2Q^{1/2}\cos[(\theta+4\pi)/3]-A_1/3 = -2(1.0265x10^8)^{1/2}\cos[(2.5559 +4\pi)/3] - (-44634.09)/3 = 34760$ lb

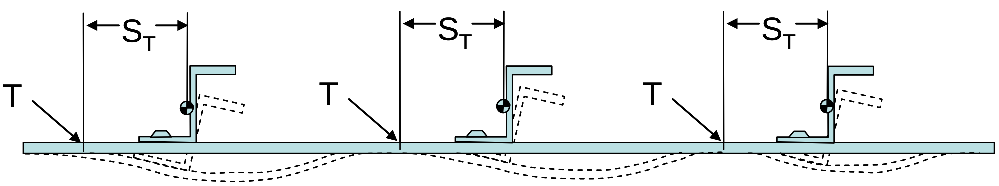
$P_R = \text{minimum}[P_{R1},P_{R2}, P_{R3}] = \text{minimum}[1550, 8323, 34760] = 1550$ lb

Note the combined bending-twisting failure mode at 1550 lb is less than the critical flexural buckling load about the minor principal axis of 2615 lb (from Section 8.5.3.1) by a significant amount. Also note that the critical stress, Fcr = 1550/0.254 = 6102 psi is within the elastic range and the use of E rather than E_{tan} is valid.

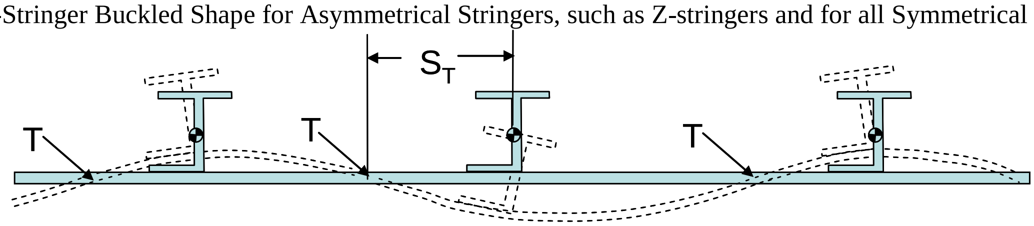
The solution of 1550 lb in this example and 1601 lb in Example 8.5.3.2 are, within engineering accuracy, the same load, thus it is seen that calculating the critical load about the shear center and about the centroid results in the same critical buckling load.

8.5.4 Torsional Buckling of Restrained Columns

When open columns are attached to skins of some minimum thickness, the natural buckling mode of a column is restrained and it is no longer free to buckle in the plane of the sheet due to the presence of the sheet. It can still buckle in the plane normal to the sheet and to twist. The twisting or rolling of the stiffeners can either occur in the same direction or in alternate directions, depending on the geometric configuration of the stringer. This is illustrated in Figure 8.5.4-1. The twisting is accompanied by flexural buckling normal to the skin. This twisting-buckling can be considered as a twisting only about some center of rotation, T, which lies in the plane of the skin and the distance between the center of rotation and the centroid of the section is given as S_T.



a) Skin-Stringer Buckled Shape for Asymmetrical Stringers, such as Z-stringers and for all Symmetrical Sections



b) Skin-Stringer Buckled Shape for Most Unsymmetrical Stringers

Figure 8.5.4-1 Skin-Stringer Buckled Shapes

The critical compressive stress for torsional buckling of an open section stringer attached to a sheet can be computed from[4]

$$P_{cr} = (A+a)\left(\frac{(G_{\tan}K)_{stiffener} + (G_{\tan}K)_{sheet}}{I_{p-T}} + \frac{\pi^2 E_{\tan} C_{w-T}}{I_{p-T} L^2}\right) \qquad \textbf{Equation 8.5.4-1}$$

where
A is the area of the stiffener (in^2)
a is the effective sheet area given by Equation 8.5.4-2 or other appropriate calculation (in^2)
G_{tan} is the tangent shear modulus of the stiffener or sheet, as appropriate(psi)
E_{tan} is the tangent modulus of the stiffener or sheet, as appropriate (psi)
K is the St. Venant torsion constant given in Section 4.2.2 for the stiffener and by Equation 8.5.4-3 for the sheet (in^4)
I_{p-T} is the polar moment of inertia of the stiffener and effective sheet area about the center of torsion, T, given by Equation 8.5.4-6 (in^4)
C_{w-T} is the warping constant of the stringer and effective sheet area about the center of torsion, T, given by Equation 8.5.4-7.
L is the effective length of the stiffener-sheet panel (in) as discussed below

The effective length of the stiffener-sheet panel, L, is given as the rib spacing where the stiffeners are continuous across supporting ribs. Where there is a single span and the ends are simply supported and the flanges are unrestrained, L is the panel length.

[4] Reference 8-26

To calculate the sheet effective area, the most straightforward approach is given by

$$a = bt_{sh} \text{ (unbuckled sheet)}$$
$$a = bt_{sh}/2 \text{ (buckled sheet) for } b/t_{sh} \leq 80$$ **Equation 8.5.4-2**

where
b is the stringer spacing (in)
t_{sh} is the thickness of the sheet (in)

This effective area is assumed to be acting at the fastener line and is based on the premise that the effective width for an unbuckled skin panel is the stiffener spacing. For a buckled panel the effective width is reduced to half the stiffener spacing. For stiffener spacing to sheet thickness ratios less than 80 this provides a reasonable to conservative estimate. For a less conservative approach or for cases where b/t_{sh} is greater than 80 and the panel is buckled, Reference 8-1, SM23 should be used.

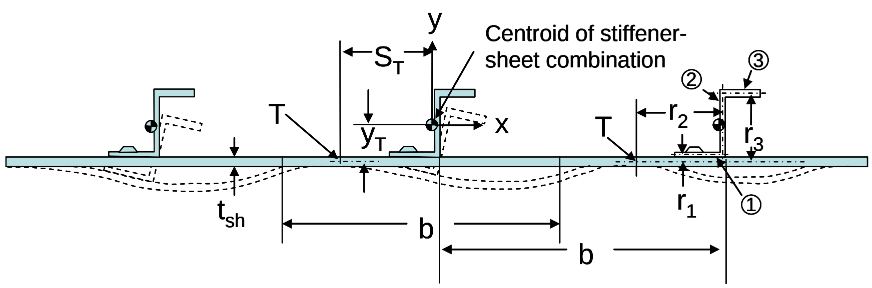


Figure 8.5.4-2 Geometry Definition

The St. Venant torsion constant for the stiffness of the sheet material is given by

$$K_{sheet} = B_o + B_s S_T^2$$ **Equation 8.5.4-3**

where
B_o and B_s are constants defined by Equations 8.5.4-4 and 8.5.4-5
S_T is the distance from the center of torsion, T, and the centroid of the striffener/skin cross-section (in)

Reference 8-24 derives and defines the constants B_o and B_s, for deflections of the type shown in Figure 8.5.4-1a, as

$$B_o = bt_{sh}^3[0.0952 + 0.289(L/b)^2]$$
$$B_s = 2.35(bt_{sh}^3/L^2)$$ **Equation 8.5.4-4**

where
b is the stiffener spacing
t_{sh} is the sheet thickness (in)
L is the effective length of the stiffener-sheet panel (in). See discussion in conjunction with Equation 8.5.4-1 (in)

For deflections of the type shown in Figure 8.5.4-1b, B_o and B_s are given by

$$B_o = bt_{sh}^3[0.238 + 0.119(L/b)^2 + 0.119(b/L)^2]$$
$$B_s = (bt_{sh}^3/L^2)[1.175 + 2.35(L/b)^2 + 1.175(L/b)^4]$$ **Equation 8.5.4-5**

The appropriate equation to use for the calculation of B_o and B_s is the one which results in the lowest critical load. There are some general rules which aid the analyst in determining this, but often some trial calculations using both are required to determine the appropriate set. Equation 8.5.4-4 applies for most Z and C sections which are asymmetrical and I and U sections which are symmetrical. Equation 8.5.4-5 often applies for unsymmetrical sections.

The polar moment of inertia of the stiffener-sheet combination is given by

$$I_{p\text{-}T} = I_p + (A+a)(y_T^2 + S_T^2)$$ **Equation 8.5.4-6**

where
I_p is the polar moment of inertia of the stiffener-sheet combined section about the centroid of the combined section (in^4)
y_T is the distance between the centerline of the sheet and the location of the combined section centroid (in)
S_T is the distance between the center of rotation and the centroid of the combined section (in)

The warping constant $C_{w\text{-}T}$, about the center of rotation is calculated in the same fashion as the warping constant about the centroid given by Equation 4.2.7-3 and Table 4.2.7-1 with two deviations. The distance r_i is the normal distance from the center of rotation, T, to the midline of the element, as shown in Figure 8.5.4-2 and the area used in the calculation is the area of the combined stiffener-skin section.

Thus, Equation 4.2.7-3 can be re-written as

$$C_{w,c} = \sum_i \frac{L_i t_i (w_{i-a}^2 + w_{i-a} w_{i-b} + w_{i-b}^2)}{3} - \frac{1}{A+a} \sum_i \left(\frac{L_i t_i (w_{i-a} + w_{i-b})}{2} \right)^2$$ **Equation 8.5.4-7**

where
L_i is the length of element i (in)
$w_{i\text{-a}}$, $w_{i\text{-b}}$ are the warping displacement of element i at ends a and b, respectively (in^2) given by $w = r_i L_i$ where
r_i is the distance from the center of rotation to the midline of the element as shown in Figure 8.5.4-2
t_i is the thickness of element i (in)

Equations 8.5.4-1 through 8.5.4-7 are dependent on the location of the center of rotation, T, and the distance from it to the centroid of the stiffener/skin element. In the general case, the location of the center of rotation is an unknown which must be found by iteration. Figure 8.5.4-3 provides a flow chart with the sequence of steps which must be followed to determine the center of rotation. The actual center of rotation is the one that results in the smallest critical buckling load. This is determined by repeating the calculations with several assumed values for the location of T and plotting P_{cr} vs S_T to determine the minimum value. Even if the assumed center of rotation is on the wrong side of the centroid, creating the plot will result in finding the correct location.

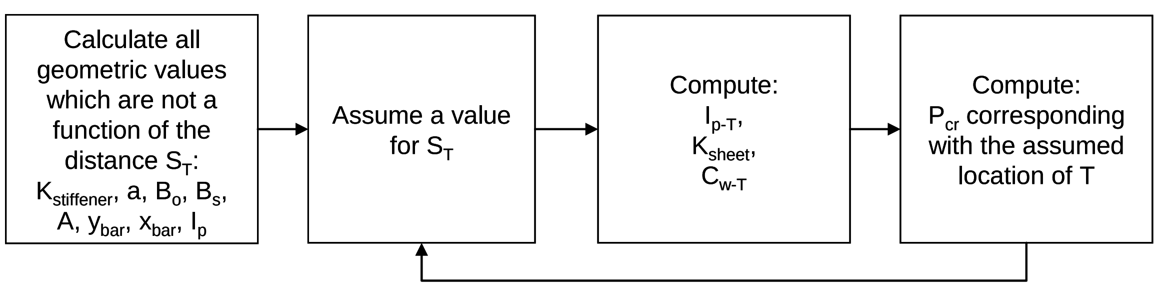


Figure 8.5.4-3 Flowchart For Calculation of Critical Torsional Buckling Load for Stiffened Panel

The margin of safety is calculated as

$$M.S. = \frac{\min(P_{cr})}{P} - 1$$ **Equation 8.5.4-8**

If the resulting critical stress, $P_{cr}/(A+a)$, exceeds the proportional limit of the material or if the local buckling of any element occurs at an axial load lower than P_{cr}, then the tangent modulus and the tangent shear modulus must be used and the solution becomes increasingly complex and a torsional buckling curve for the stiffener/sheet combination must be constructed. This approach is best suited to computer applications. See Section 8.7.2 for further information.

If the stiffener (including attachment points) is symmetric about the y-y axis, *i.e.*, normal to the sheet, then the calculations become simplified. For this case, the critical load will either be at torsional buckling load when $S_T = 0$ or a flexural buckling load when $S_T = \pm\infty$. If $S_T = 0$ or the center of rotation is at the centerline of the stiffener, the torsional buckling load, $P_{cr,1}$ is calculated from Equation 8.5.4-1 with $S_T = 0$. If $S_T = \pm\infty$, the critical load is a flexural buckling-only load given by

$$P_{cr,2} = \pi^2 E_{tan} I_x / L^2$$ **Equation 8.5.4-9**

where
I_x is the moment of inertia of the combined stiffener-sheet section about the centroidal x axis (in^4)
L is the effective length of the stiffener-sheet panel (in). See discussion in conjunction with Equation 8.5.4-1 (in)

And the critical buckling load is the minimum of ($P_{cr,1}$, $P_{cr,2}$) so the margin of safety can be calculated from Equation 8.5.2-5.

8.6 Inter-rivet Buckling

Fasteners are intended to transfer load and keep the structure in intimate contact. It is only when they remain in intimate contact that the concept of effective width can be used in calculating the section properties of stringer/skin or stiffener/web for column buckling. If the skin buckles between fasteners then there is a loss of stiffness in the skin and section properties calculated using the skin are unconservative.

8.6.1 Calculation of Inter-rivet Buckling Allowable Stress

Inter-rivet buckling is a specialized form of column buckling analysis which should be performed where there are two pieces of structure joined by fasteners to ensure that each piece of structure is thick and stiff enough to not buckle between fasteners. This is illustrated in Figure 8.6.1-1

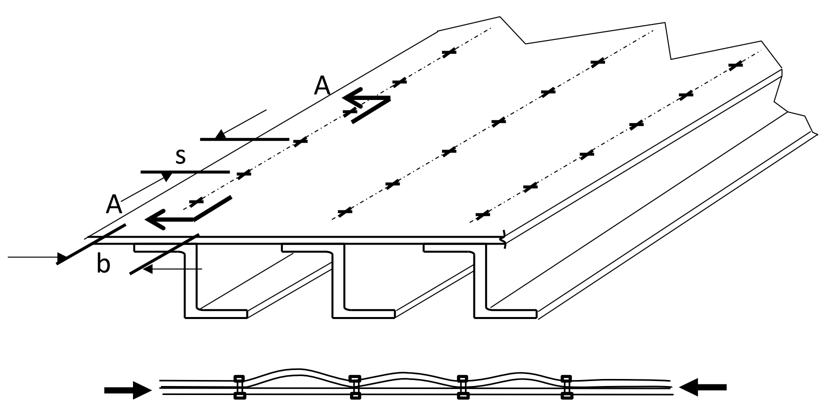


Figure 8.6.1-1 Illustration of Inter-rivet Bucking

Inter-rivet buckling is predicted using the column buckling allowable curves of Section 8.2.3. The small segment of skin between fasteners is assumed to be a column with end restraints provided by the fasteners. The spacing to thickness ratio is then used to formulate the L'/ρ necessary to use the column buckling curves. This approach has been correlated to the panel tests of References 8-30 through 8-34 to determine what end fixity to assume.

Equation 8.2.1-4 is solved for $(L'/\rho)^2$ in terms of F_{col} as

$$\left(\frac{L'}{\rho}\right)^2 = \frac{\pi^2 E_{tan}}{F_{col}}$$ **Equation 8.6.1-1**

where
L' is the effective length of the column (in), L/√c, where c is the end fixity coefficient
ρ is the radius of gyration (in), √(I/A)
E_{tan} is the tangent modulus of the material in compression (psi)
F_{col} is the column buckling stress (psi)

The inter-rivet buckling load is given by Equation 8.2.1-1 which has been modified to include the plasticity correction and the fastener spacing, *s* has been substituted for the column length, *L*

$$P_{IR} = \frac{c\pi^2 E_{tan} I}{s^2}$$

where
s is the fastener spacing (in)
c is the end fixity coefficient
I is the moment of inertia of the column cross section (in^4)

The inter-rivet buckling stress is then

$$F_{IR} = \frac{c\pi^2 E_{tan}}{s^2} \cdot \frac{I}{A}$$ **Equation 8.6.1-2**

where
A is the area of the column cross section (in^2)

For the flat plate column of width *b* and thickness *t*, *I/A* can be determined as

$$\frac{I}{A} = \frac{bt^3}{12} \cdot \frac{1}{bt} = \frac{t^2}{12}$$ **Equation 8.6.1-3**

where
b is the width of the flat plate column-element (in)
t is the thickness of the skin or stiffener, whichever is smaller (in)

Substituting Equation 8.6.1-3 into the inter-rivet buckling equation 8.6.1-2,

$$F_{IR} = \frac{c\pi^2 E_{tan}}{12\left(\frac{s}{t}\right)^2}$$

Set $B = c\pi^2/12$, substitute into the above equation

$$F_{IR} = \frac{B E_{tan}}{\left(\frac{s}{t}\right)^2}$$ **Equation 8.6.1-4**

The inter-rivet buckling stress of Equation 8.6.1-4 can be substituted as F_{col} into Equation 8.6.1-1 and the relationship between L'/ρ and s/t can be determined

$$\left(\frac{L'}{\rho}\right)^2 = \frac{\pi^2 E_{tan}}{\frac{BE_{tan}}{\left(\frac{s}{t}\right)^2}} = \frac{\pi^2}{B}\left(\frac{s}{t}\right)^2$$

$$\frac{L'}{\rho} = \frac{\pi}{\sqrt{B}}\left(\frac{s}{t}\right) \qquad \textbf{Equation 8.6.1-5}$$

where
$B = c\pi^2/12$
c is the end fixity coefficient of the column

Table 8.6.1-1 gives values of c, B and $\pi/\sqrt{B}$ with recommendations for use.

Table 8.6.1-1 Equations for Inter-rivet Buckling for use in Determining Allowable from Column Curves

Fastener Type and Diameter	Fastener Spacing	End Fixity Coefficient, c	B	$\pi/\sqrt{B}$	Equation	
Protruding Head and Countersunk Fasteners and Rivets, D≥5/32 in,	s≤8D	4	3.290	1.732	$\frac{L'}{\rho} = 1.732\left(\frac{s}{t}\right)$	**Equation 8.6.1-6**
Protruding Head and Countersunk Fasteners and Rivets, D≥5/32 in	10D>s>8D	1	0.822	3.465	$\frac{L'}{\rho} = 3.465\left(\frac{s}{t}\right)$	**Equation 8.6.1-7**
Blind Fasteners, D≥5/32 in	All	1	0.822	3.465		
Protruding Head and Countersunk Fasteners and Rivets, D < 5/32 in	All	1	0.822	3.465		

For spot welds, use $0.75F_{col}$ as determined using Equation 8.6.1-6.

After determining the appropriate L'/ρ, the inter-rivet buckling allowable stress is equal to the allowable column stress as determined from the curves in Section 8.2.3. If the joint is made of two different materials and 2 different thicknesses, the column curve selected should be consistent with the material of the thickness selected for the calculation of *s/t*. If it is not apparent which of the two layers will be critical, the inter-rivet buckling calculation will need to be performed on each one. The allowable inter-rivet buckling is F_{col}

$$F_{IR} = F_{col} \qquad \textbf{Equation 8.6.1-8}$$

where
F_{col} is the allowable column buckling stress based on L'/ρ as calculated from Equation 8.6.1-6 or 8.6.1-7 (psi)

The applied stress is the stress in the joint layer between fasteners which will generally be the maximum between the end two fasteners in the fastener pattern. The margin of safety is given as

$$M.S. = \frac{F_{IR}}{f} - 1 \qquad \textbf{Equation 8.6.1-9}$$

where
F_{IR} is the inter-rivet buckling allowable stress (psi)
f is the maximum applied gross area stress in the fastener line, between fasteners (psi)

8.7 Unix/PC-Based Calculations

Many of the equations which predict bending buckling, crippling, beam-column solutions and torsional buckling require iterative solutions which lend themselves to computer programs and the use of finite element analysis. This section will discuss the tools currently available.

8.7.1 Integrated Detail Analysis Tool (IDAT)

IDAT, the integrated detail analysis tool suite has programs which can calculate and provide curves for most of the compression stability beam problems. IDAT/SM83 can generate generic column curves for use in flexural buckling column failures as a function of L'/ρ. IDAT/SM110 can generate crippling allowable curves for use in the crippling analysis of Section 8.4. In addition SMM110 has an analysis tab which allows for the crippling analysis of a specific cross section. IDAT/SM83 can also generate the combined torsional-bending instability allowable curves of Section 8.5 if a specific cross-section is input. IDAT/SM97 will analyze specific beam-column problems.

8.7.2 Legacy Integrally Stiffened Buckling Programs

The integrally stiffened wing skin analysis of transport aircraft is ideally suited for computer application. There are several reports which detail the methodology and offer computer programs available for this type of analysis which include various features of compression buckling, torsional buckling, crippling, and panel buckling. The reports are summarized in Table 8.6.2-1. Some of these methods recommend assumed eccentricities for beam column analysis and assumed column fixities. For access to these reports and recommendations on applicability, please contact the 6E5 methods group or your Core Engineering Manager.

Table 8.7.2-1 Reports Detailing Methods for Sizing of Stiffened Panels.

Document Number	Title
SMN198	Tee Flanged Integrally Stiffened Wing Covers
LR11916	Optimum Design of Integrally Stiffened Panels in Compression
LR14724	General Instability of Long Orthotropic Elastic Plates Supported by Flexible Rib Caps with Rigid Posts under Combined Longitudinal Compression and Shear
LR17684	Local Instability of Tee Flanged Integrally Stiffened Sheet with Intermediate Ribs
LR18473	Buckling of Orthotropic Plates Under Biaxial Compression and Shear
LR22808	Margins of Safety for Unflanged Integrally Stiffened Sheet Wing Surfaces
LR23385	Margins of Safety for Extruded I Stiffened Wing Surfaces

8.7.3 Finite Element Analysis

A finite element approach is often a good way to investigate the compression stability of aircraft structure. This section provides some recommendations for analysis based on studies conducted at Lockheed. Experience has shown that the FEA code ABAQUS, using a modified Riks algorithm is best suited to the solution of stability problems. Solutions must be run nonlinearly (material and geometry) and the mesh density is an important parameter. NASTRAN Solution 105 is also available for obtaining eigenvalue buckling solutions while Solution 106 can be used for a non-linear beam column analysis.

If the mesh is not sufficiently dense then critical failure modes can be missed. If the mesh is too fine the problem will run for an excessive period of time and may not converge. To address the question of how fine the mesh should be, the analyst should perform a series of test problems in which the mesh is made increasingly finer and the problem solved. When subsequent solutions are within 1-2% of each other the proposed mesh is sufficient. The test problems should be constructed to model the type of structure being analyzed but not be the full depth and breadth of the problem. As an example, for an integrally stiffened panel the test problem might consist of a single stringer or a single span of a single stringer. When the model is fine enough to predict the mode shape correctly, then the same density of mesh could be used for the entire panel.

Another crucial aspect of using an FEA solution in stability analysis is the boundary constraints. If the model is over-constrained then the buckling solution may be unconservative and the actual structure could fail below predictions. As has been discussed in Section 8.2, actual aircraft structure very rarely approaches 100% fixity.

For a non-linear solution, non-linear material properties must be supplied. Section 3.3.1.7 discusses the approach for the generation of the stress-strain curve necessary for the FEA code. Different codes require different inputs and it is incumbent on the analyst to know what input is required.

It is recommended that where test data of a similar configuration and constrained panel can be obtained, it be used for model correlation prior to use of the model or modeling techniques be used for design of aircraft.