10 Plates and Shells

The purpose of this section is to provide guidance on the structural analysis of plates and shells. Both plates and shells are structural members whose thickness is small as compared to their length or width. Plates may be flat or curved and carry pressure loading as a combination of internal moments and in-plane loads while shells have at least one plane of curvature and carry pressure loads entirely as in-plane loads. These may be hoop or longitudinal depending on the edge restraints. Both plates and shells have the ability to carry additional in-plane aircraft loads. As pressures increase on a plate, it assumes an increasingly curved shape due to deformation and may start to act as a membrane-shell.

According to Reference 10-4, the thickness of a plate is not more than one-quarter of its least edge dimension. The middle surface of a plate is defined as midway between the top and bottom surfaces. Plates are classified as flat or curved depending upon whether the middle surface lies in a plane or in a curve. A flat plate is a plate whose middle surface lies in a plane whereas a curved plate has a curvature in its middle surface. Plates are structural members designed to carry aircraft in-plane loads as well as pressure loads or a combination of both. A plate is considered as a two-dimensional counterpart of a beam except that a plate bends in all planes normal to the surface of the plate whereas a beam bends in only one plane. Since the plate thickness is small as compared to the other dimensions, its bending behavior strongly depends upon its thickness as compared to the other dimensions. Based on the thickness, plates are classified into four groups:

- **Thick Plate** – A thick plate has considerable thickness and transverse shearing stresses are significant, similar to short deep beams. Thick plate theory based on three-dimensional elasticity is used to describe its behavior. Consequently, the analysis becomes complicated such that closed-form solutions exist for only a few cases.
- **Thin Plate** – A thin plate supports the transverse load entirely by developing bending stresses. The maximum transverse deflection w_{max} is small in comparison to its thickness ($w_{max} < t / 2$). Small deflection theory is used to describe the bending behavior of the plate and it is assumed that the middle surface of the plate does not undergo any deformation. Sometimes the thin plates are also referred as medium thick plates.
- **Very Thin Plates** – The maximum transverse deflection, w_{max}, of a very thin plate is very large as compared to its thickness ($w_{max} > t / 2$). The very thin plate resists the transverse load by bending in conjunction with direct tension stresses caused by stretching of its middle surface. The stretching and bending of the middle surface results in a set of non-linear equations and the solution of the problem becomes more complicated.
- **Membrane** – The membrane is very thin, thus has a negligible flexural rigidity, and consequently cannot carry any transverse load by bending action. A plate which has a very small thickness and carries the transverse load primarily by stretching its middle surface is classified as a membrane. The maximum transverse deflection is very large as compared to its thickness and is non-linear. The resulting membrane stresses are also non-linear.

Some examples of plates under transverse pressure in aircraft structures are:

- Fuel tank walls are loaded by static head and inertia fuel pressure
- All outside skins are loaded by aerodynamic pressure
- Wheel wells walls must be designed for tire burst pressure for transport aircraft
- Gun bays are loaded by blast pressure
- Sonic fatigue on flat panels is produced by acoustic pressure
- Engine stall causes hammershock pressure on walls of engine inlet duct
- Cockpit floors are loaded by cockpit pressure
- Fuselage skins are loaded by cabin pressure
- Cabin floors must be designed for instantaneous decompression
- Pressure bulkheads are loaded by cabin or cockpit pressure

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10.1 References

10-1 anon., *Lockheed Martin Engineering Stress Memo Manual*, Lockheed Martin Aeronautical Systems, Marietta, GA (October 1998 Release; April 2002 Revision)

10-2 anon., *Structures Analysis Manual, Volume 1 and Volume 2*, General Dynamics Convair and Space Structures Division 1988)

10-3 anon., "Metallic Materials and Elements for Aerospace Vehicles Structures," *MIL-HDBK-5*[1], Battelle Memorial Inst., Secretariat (2001)

10-4 Roark, R. J. and Young, W. C., "Formulas for Stress and Strain," McGraw-Hill Book Company Inc., New York, NY, Fifth Edition, 1982

10-5 Timoshenko, S. and Woinowsky-Krieger, S., "Theory of Plates and Shells," McGraw-Hill Book Company, New York, NY, Second Edition, 1959

10-6 Roark, R. J. and Young, W. C., "Formulas for Stress and Strain," McGraw-Hill Book Company Inc., New York, NY, Third Edition, 1954

10-7 *Astronautic Structures Manual*, George C. Marshall Space Flight Center, Marshall Space Flight Center, Alabama, NASA TM X-73305, August 1975

10-8 *Stress Analysis Manual*, Air Force Flight Dynamics Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, Technical Report AFFDL-TR-69-42, February 1970

10-9 Griffel, W., "Handbook of Formulas for Stress and Strain," Frederick Ungar Publishing Company, 1966

10-10 Timoshenko, S. P. and Gere, J. M., "Theory of Elastic Stability," McGraw-Hill Book Company, New York, NY, Second Edition, 1961

10-11 Brush, D. O. and Almroth, B. O., "Buckling of Bars, Plates, and Shells", McGraw-Hill Book Company, Inc., New York, 1975

10-12 Bleich, F., "Buckling Strength of Metal Structures", McGraw-Hill Book Company, Inc., New York, 1952

10-13 Ramberg, W. and Osgood, W. R., "Description of Stress-Strain Curves by Three Parameters", NACA-TN 902, July 1943

10-14 Hill, H.,"Determination of Stress-Strain Relations from Offset Yield Strength Values", NACA-TN 927, February 1944

10-15 Gerard, G. and Wildhorn, S., "A Study of Poisson's ratio in the Yield Region", NACA-TN 2561, January 1952

10-16 Stowell, E. Z. and Pride, R. A., "The Effect of Compressibility of the Material on Plastic Buckling Stresses", Reader's Forum, Journal Aeronautical Science, Vol. 18, No. 11, November 1951

10-17 Batdorf, S. B. and Stein, M, "Critical Combinations of Shear and Direct Stress for Simply Supported Rectangular Flat Plates", NACA-TN 1223, March 1947

10-18 Batdorf, S.B. and Houbolt, J. C., "Critical Combinations of Shear and Transverse Direct Stress for an Infinitely Long Plate with Edges Elastically Restrained Against Rotation", NACA-TR 847, 1946

10-19 Schuette, E. H. and McCulloch, J. C., "Charts for the Minimum-Weight Design of Multiweb in Bending", NACA-TN 1323, June 1947

10-20 Johnson, Jr. A. E. and Buchert, K. P., "Critical Combinations of Bending, Shear, and Transverse Compressive Stresses for Buckling of Infinitely Long Flat Plates", NACA-TN 2536, December 1951

10-21 Noel, R. G., "Elastic Stability of Simply Supported Flat Rectangular Plates Under Critical Combination of Longitudinal Bending, Longitudinal Compression, and Lateral Compression", Journal of the Aeronautical Sciences, December 1952, pp. 829-834

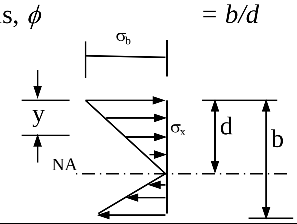
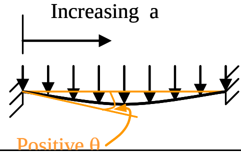
10-22 Grossman, N., "Elastic Stability of Simply supported Flat Rectangular Plated Under Critical Combination of Transverse Compression and Longitudinal Bending", Journal of the Aeronautical Sciences, Volume 16, 1949, pp. 272-276

10-23 Stowell, E.Z. and Schwartz, E. B., "Critical Stress for an Infinitely Long Flat Plate with Elastically Restrained Edges under Combined Shear and Direct Stress", NACA-ARR 3K13, November 1943

[1] In 2003, MIL-HDBK-5 was superseded by the Metallic Materials Properties Development and Standardization (MMPDS) Handbook, Battelle Memorial Institute, Secretariat (2003)

10-24 Peters, R. W., "Buckling of Long Square Tubes in Combined Compression and Torsion and Comparison with Flat-Plate Buckling Theories", NACA-TN 3184, May 1954
10-25 Donnell, L. H., "Stability of Thin-Walled Tubes under Torsion", NACA-TR 479, 1933
10-26 Batdorf, S. B., "A Simplified Method of Elastic Stability for Thin Cylindrical Shells I – Donnell's Equation", NACA-TN 1341, June 1947
10-27 Gerard, G. and Becker, H., "Handbook of Structural Stability Part III – Buckling of Curved Plates and Shells", NACA-TN 3783, August 1957
10-28 Donnell, L. H., "A New Theory for the Buckling of Thin Cylinders under Axial Compression and Bending", Trans. A.S.M.E., Volume 56, No. 11, November 1934, pp. 795-806
10-29 Donnell, L. H., and Wan, C.C., "Effect of Imperfections on Buckling of Thin Cylinders and Columns under Axial Compression", Journal of Applied Mechanics, Volume 17, No. 1, March 1950, pp. 73-82
10-30 "Buckling of Thin-Walled Circular Cylinders", NASA SP-8007, August 1968
10-31 Batdorf, S. B., Stein, M. and Schildcrout, M., "Critical Shear Stress of Curved Rectangular Panels", NACA-TN 1348, May 1947
10-32 Batdorf, S. B., Schildcrout M., and Stein, M., "Critical Combinations of Shear and Longitudinal Direct Stress for Long Plates with Transverse Curvature", NACA-TN 1347, June 1947
10-33 Schildcrout, M. and Stein, M., "Critical Combinations of Shear and Direct Axial Stress for Curved Rectangular Panels", NACA-TN 1938, August 1949
10-34 Fung, Y. C. and Kaplan, A., "Buckling of Low Arches or Curved Beams", NACA-TN 2840, November 1952
10-35 Kuhn, P., Peterson, J. P. and Levin, L. R., "A Summary of Diagonal Tension Part I – Methods of Analysis", NACA-TN 2661, May 1952
10-36 Kuhn, P., Peterson, J.P. and Levin, L. R., "A Summary of Diagonal Tension Part II – Experimental Evidence", NACA-TN 2662, May 1952
10-37 Bruhn, E. F., "Analysis and Design of Flight Vehicle Structures", Jacobs Publishing, Inc., Carmel, Indiana, June 1973
10-38 "Buckling of Thin-Walled Truncated Cones", NASA SP-8019, September 1968
10-39 "Buckling of Thin-Walled Doubly Curved Shells", NASA SP-8032, August 1969
10-40 Peery, D. J., "Post-Buckling Strength of A Pressurized Cylinder", GD Report No. GD/A-DDG 64-024, October, 1964
10-41 Wang, Chi-Teh, "Bending of Rectangular Plates with Large Deflections," NACA-TN-1462, April 1948
10-42 Levy, Samuel and Greenman, Samuel, "Bending with Large Deflection of a Clamped Rectangular Plate with Length to Width Ratio of 1.5 under Normal Pressure,' NACA-TN-853, July 1942

10.1.1 List of Symbols and Nomenclature

Symbol	Description	Units
α	ratio of plate dimensions; $\alpha = b/a$	
δ	deflection	in
Δ	plate geometric parameter	
$\Delta m_{abruptness}$	snap-through instability slope Abruptness parameter	
ε	strain	in/in
ε_r	rotational edge restraint coefficient	
ϕ	ratio of plate loaded-edge length to distance between compression edge and neutral axis, ϕ = b/d 	
η	plasticity correction factor	
η_s	Stowell modulus	
θ	slope of the plate measured from horizontal. It is positive when the deflection increases as dimension a, b or r increases 	radians
θ	angle bounded by the curved plate’s cylindrical generators	radians
ζ	correlation factor for the curved plate	
σ	internal stress in the plate. Positive stress indicates tension on the bottom fibers. Subscripts a, b or r denote stress direction along dimension a, b or r  Positive Stress Convention	psi
σ_b	maximum compressive bending stress	psi
τ_{xz}	shear stress parallel to Z-Axis on an edge perpendicular to X-Axis	psi
τ_{yz}	shear stress parallel to Z-Axis on an edge perpendicular to Y-Axis	psi

ν	Poisson's ratio	
Symbol	**Description**	**Units**
ν_e	elastic Poisson's ratio	
ν_p	plastic Poisson's ratio	
ξ	circumferential displacement in the Y-direction	in
χ	axial displacement	in
ψ	sub-correlation factor for the curved plate	
Ω	compressive buckling constant for long cylinders	
a	long plate dimension	in
b	short plate dimension or stiffener spacing	in
c	plate dimension	in
C	plate end fixity coefficient	
d	distance between compression edge and neutral axis, see picture on page 10-3 for ϕ	in
D	plate flexural rigidity	lb-in
dx	dimension of the plate differential element in the X-direction	in
dy	dimension of the plate differential element in the Y-direction	in
E	Young's modulus of the material	psi or ksi
E_c	compression modulus of elasticity	psi or ksi
E_{sec}	secant modulus	psi or ksi
E_{tan}	tangent modulus	psi or ksi
f	stress	psi or ksi
f_{cr}	initial buckling stress	psi or ksi
f_{cre}	initial elastic buckling stress	psi or ksi
f_{cx}	applied compressive stress in X-direction	psi or ksi
f_{ty}	applied tensile stress in X-direction	psi or ksi
f_{crx}	initial buckling stress in X-direction	psi or ksi
f_{scr}	initial shear buckling stress	psi or ksi
$f_{0.7}$	secant intercept at 0.7E of the stress strain curve	psi or ksi
$f_{0.85}$	secant intercept at 0.85E of the stress strain curve	psi or ksi
F_{cp}	proportional limit stress in compression	psi or ksi
F_{cy}	material compressive yield stress	psi or ksi
F_{su}	material shear ultimate stress	psi or ksi
F_{tu}	material ultimate tensile stress	psi or ksi

F_{cmax}	maximum allowable compressive buckling stress	psi or ksi
F_{smax}	maximum allowable shear buckling stress	psi or ksi
Symbol	**Description**	**Units**
g	unit running applied load	lb/in
I	moment of inertia	in^4
IDAT	Integrated Detail Analysis Tools	
k	buckling coefficient	
k_b	bending buckling coefficient	
k_c	uniaxial compression buckling coefficient	
k_{cu}	buckling coefficient in axial compression for the curved plate	
k_{pl}	buckling coefficient of the flat plate in axial compression	
k_s	shear buckling coefficient	
k_{su}	shear buckling coefficient for the curved plate	
K	effective buckling coefficient	
L	longitudinal grain direction	
Ln	Natural Logarithm	
m	reciprocal of Poisson’s ratio; *i.e.*, m = 1/ν	
m_{AB}	Incremental slope of line between points A and B	
m_{secant}	slope of line from origin to point A	
M_x	bending moment on an edge parallel to Y-Axis	in-lb/in
M_y	bending moment on an edge parallel to X-Axis	in-lb/in
M_{xy}	twisting moment on an edge parallel to Y-Axis	in-lb/in
M_{yx}	twisting moment on an edge parallel to X-Axis	in-lb/in
M_r	radial bending moment	in-lb/in
M_t	tangential bending moment	in-lb/in
MS	margin of safety	
n	Ramberg-Osgood shape parameter. Subscript c is for compression and t is for tension	
N	generic in-plane running load	lb/in
N_x	in-plane normal load parallel to X-axis	lb/in
N_y	in-plane normal load parallel to Y-axis	lb/in
N_{xy}	in-plane shear load	lb/in
NA	Neutral axis	
p	pressure	psi

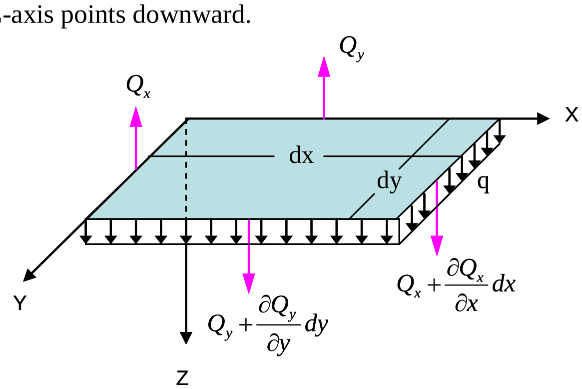
p_{cr}	critical pressure, *i.e.*, transverse pressure at which snap-through will occur	psi
P	total applied load	lb
Symbol	**Description**	**Units**
P_{all}	allowable load	lb
q	unit uniformly distributed applied load	psi
Q_x	shear force on an edge parallel to Y-Axis	lb/in
Q_y	shear force on an edge parallel to X-Axis	lb/in
r	plate dimension; radius of curvature of the curved plate	in
R	reaction force normal to the plate surface exerted by the boundary support on the edge of the plate	lb/in
R_b, R_s, R_c	stress ratios in bending, shear an compression respectively	
R_{cr}	loading parameter in calculation of snap-through instability	
$R_{SnapThrough}$	ratio of the load level of predicted snap-through to the criteria load level	
R_x, R_y	stress ratios along X and Y axis respectively	
R_{1Allow}, R_{2Allow}	allowable stress ratios in 1 and 2 directions respectively	
r_0	radius of the concentrated load circular area	in
r_0'	the equivalent radius of contact for a load concentrated on a very small area	in
t	thickness of the plate	in
T	transverse grain direction	
U	uneven factor relating to the initial imperfections of the cylinder	
V	shear force	lb
w	total plate deflection (initial plus that due to loads) measured from the original position. Positive deflection is vertically downward. Subscript 0 is for initial deflection and 1 is for deflection due to loads	in
w	applied transverse running load	lb
w_{cr}	critical running load, *i.e.*, running load at which snap-through instability will occur	lb
y	distance measured from the compression edge	in
Z_b	curved plate geometric parameter related to radius and thickness of the plate	
%DLL	Percent of Design Limit Load	
%DUL	Percent of Design Ultimate Load	
Subscripts		
$_{A}$, $_{B}$, $_{C}$	Points on the Deflection – Load Level Plot	
$_{bend}$		
$_{bending\text{-}x}$, $_{bending\text{-}y}$	Denoting bending stress in the x or y direction	

ctr, ctr-Ledge, ctr-Sedge	Denoting center of plate, center of long edge or center of short edge	
membrane-x, membrane-y	Denoting membrane stress in the x or y direction	
x,y	Denoting stress, load or moment in x and y directions	
Edge Conditions		
SS	simply supported edge	
FR	free edge	
/////	fixed edge	

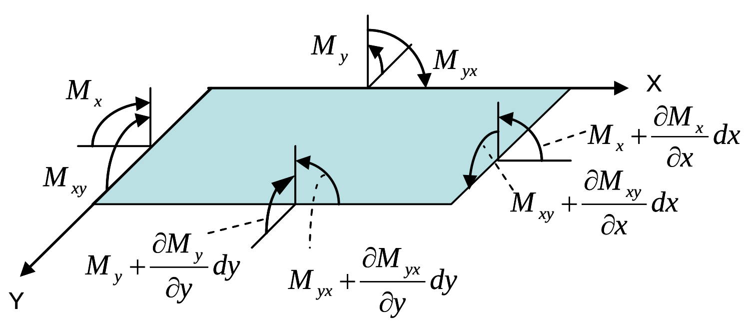
10.2 Plate Bending Theory

This section presents the theoretical background of bending of transversely loaded thin plates in an abbreviated form because the derivation is long and it is covered in an excellent manner in many books such as Reference 10-5. Therefore, only key equations are presented here.

The derivation is done by taking a small element of the plate with the sides parallel to its edges. Figure 10.2.0-1 shows such a small plate element with lateral load, q (lb/in^2), shears and bending moments. The coordinate system is a right-hand system and +Z-axis points downward.



(a)



(b)

Figure 10.2.0-1: Moments and Shears on a Differential Element
Note all shear forces are in (lb/in) and bending moments are in (in-lb/in)

It is assumed that the load acting on the plate is normal to its surface and that the deflections are small as compared to its thickness. This means that small deflection theory can be used. It is also assumed that the edges at the boundary are free to move in the plane of the plate; thus, the reactive forces at the edges are normal to the plate.

In applying small deflection theory for plate bending, the following additional assumptions are made:

1. There is no in-plane deformation in the middle surface of the plate.
2. Points in the plate lying initially on a normal-to-the-middle plane of the plate remain on the normal-to-the-middle surface of the plate after bending. This assumption is same as in the case of beam bending (one-dimensional structural member).
3. The normal stresses σ_z in the direction transverse to the plate can be disregarded. Implication of this assumption is that when applying the equilibrium condition in the Z direction, the transverse load is carried by the shearing forces in the X and Y direction as shown in Equation 10.2.0-1. If the effect of the normal stress σ_z needs to be considered, thick plate theory has to be used and the reader is referred to Reference 10-5.

The plate element, unlike a beam element, has twisting moments in addition to the bending moments. Since there is no deformation of the middle surface of the plate, all shear forces are vertical. The moments and shear forces are functions of x-y coordinates and small change in their magnitude is also included in the analysis.

The shear forces Q_x and Q_y are related to the shear stresses τ_{xz} and τ_{yz} as follows:

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\, dz \qquad Q_y = \int_{-h/2}^{h/2} \tau_{yz}\, dz$$ **Equation 10.2.0-1**

Now the equilibrium equations can be written by noting that the normal stress resultant σ_z can be neglected. The equilibrium equations are:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$$ **Equation 10.2.0-2**

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0$$ **Equation 10.2.0-3**

$$\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0$$ **Equation 10.2.0-4**

By using Equations 10.2.0-3 and 10.2.0-4, Equation 10.2.0-2 can be written as:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} = -q$$ **Equation 10.2.0-5**

Noting that $M_{yx} = - M_{xy}$ because $\tau_{xy} = - \tau_{yx}$, so the above equation can be written as

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = -q$$

Equation 10.2.0-6

Equation 10.2.0-6 is a second order differential equation in terms of M_x, M_y and M_{xy}. In order to solve this equation it is preferable to reduce it in terms of one variable. It is done by expressing this equation in terms of the deflected surface, w, of the plate. If the deflected surface is known, bending moments M_x, M_y and M_{xy} can be expressed in terms of the deflection of the plate by using the curvature of the middle surface of the plate due to the loads. The resulting equations are analogous to well-known beam bending equations.

The bending moments M_x and M_y are shown below in terms of the curvature of deflected surface

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

Equation 10.2.0-7

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

Equation 10.2.0-8

Where,
D is the flexural rigidity (lb-in) of the plate given by Equation 10.2.0-9 and it is similar to the beam bending stiffness EI.

The flexural rigidity of the plate is defined as:

$$D = \frac{Et^3}{12\left(1-\nu^2\right)}$$

Equation 10.2.0-9

Where,
E is the Young's modulus of the material (psi)
t is the thickness of the plate (in)
ν is the Poisson's ratio

The twist of the element, $\delta^2 w/\delta x \delta y$ or $\delta^2 w/\delta y \delta x$ is the change in x-direction slope per unit distance in the y-direction or vice versa. The twisting moment is related to the curvature of the deflected surface as defined by the Equation 10.2.0-10:

$$M_{xy} = -M_{yx} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

Equation 10.2.0-10

A summary of above plate equations and corresponding beam equations for comparison are presented in Table 10.2.0-1.

Table 10.2.0-1: Comparison of Plate versus Beam Equations

Class	Item	Plate Theory	Beam Theory
Geometry	Coordinates	x,y	x
	Deflection	w	y
	Deformations	$\dfrac{\partial^2 w}{\partial x^2};\ \dfrac{\partial^2 w}{\partial y^2};\ \dfrac{\partial^2 w}{\partial x\,\partial y}$	$\dfrac{d^2 y}{dx^2}$
Structural Characteristics	Bending Stiffness	$D = \dfrac{Et^3}{12\left(1-\nu^2\right)}$	EI
Loadings	Couples	M_x, M_y, M_{xy}	M
	Shears	Q_x, Q_y	V
	Lateral	q	q
Hooke's Law	Moment	$M_x = -D\left(\dfrac{\partial^2 w}{\partial x^2} + \nu\dfrac{\partial^2 w}{\partial y^2}\right)$	$M = EI\dfrac{d^2 y}{dx^2}$
	Distortion	$M_y = -D\left(\dfrac{\partial^2 w}{\partial y^2} + \nu\dfrac{\partial^2 w}{\partial x^2}\right)$	
	Relation	$M_{xy} = D(1-\nu)\dfrac{\partial^2 w}{\partial x \partial y}$	
Equilibrium	Moment Equations	$Q_x = \dfrac{\partial M_{yx}}{\partial y} + \dfrac{\partial M_x}{\partial x}$ $Q_y = \dfrac{\partial M_y}{\partial y} - \dfrac{\partial M_{xy}}{\partial x}$	$V = \dfrac{dM}{dx}$
	Force Equation	$q = -\left(\dfrac{\partial Q_x}{\partial x} + \dfrac{\partial Q_y}{\partial y}\right)$	$q = \dfrac{dV}{dx}$

Using Equations 10.2.0-7 and 10.2.0-8, Equation 10.2.0-6 becomes

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$ **Equation 10.2.0-11**

The plate bending problem is thus reduced to solving a 4th-order differential equation. Thus, a deflected surface is sought for a given lateral loading condition which satisfies Equation 10.2.0-11 and the specified boundary conditions as defined in section 10.2.1. Once the deflected surface is known, internal bending moments can be calculated from Equations 10.2.0-7, 10.2.0-8 and 10.2.0-10 and shears can be calculated from Equations 10.2.0-3 and 10.2.0-4. With known bending moments and shears, the stresses can be calculated.

It was assumed in the development of plate bending equations that the middle surface does not stretch and thus there are no membrane stresses. However, in a great majority of the plate bending problems the middle surface undergoes some stretching. Therefore, it is the amount of stretching of the middle surface that restricts the applicability of this theory; it is assumed that the plate bending formulas apply accurately if the deflection is less than half its thickness.

10.2.1 Edge Restraint

Edge restraints provided by adjoining structural elements are used as boundary conditions for solving the differential Equation 10.2.0-11. The extent of edge restraint provided has a marked effect on the flexural and buckling load carrying capacity of the plate element. The degree of edge restraint provided is usually approximated from the following idealized edge conditions.

(a) Elastically Restrained Edge: This is an edge about which the plate is elastically restrained from rotating freely. The deflection in the transverse direction along the edges is zero. The degree of restraint is defined by the rotational restraint coefficient, ε_r, which is proportional to the ratio of the stiffness of the restraining element to that of the plate.

(b) Simply Supported Edge ($\varepsilon_r = 0$): This is an edge condition where the plate is free to rotate about the centerline of the edge; *i.e.*, no restraining bending moments about the edge. The deflection in the transverse direction along the edge is zero. In general, torsionally weak edge elements such as open sections in compression members will act as simply supported edges.

(c) Fixed or Built-In Edge ($\varepsilon_r = \infty$): This is an edge about which the plate cannot rotate. The deflection in the transverse direction along the edge is zero. In general, torsionally stiff edge members such as closed sections and heavy flanges will provide almost-clamped edges. The tangent plane to the deflected surface along this edge coincides with the initial position of the plate.

(d) Free Edge ($\varepsilon_r = 0$): This is an edge that is entirely free to rotate and to deflect transversely.

10.2.2 Flat Plates Under Transverse Loads

This section includes solutions for stresses and deflections of flat plates having different geometric configurations, loading cases and boundary conditions. Solutions are presented for the most commonly encountered cases in aircraft design. Other cases may be found in References 10-2 and 10-4 to 10-9.

- Rectangular plates of uniform thickness having simple support and fixed boundary conditions and subjected to different transverse loads are presented in Table 10.2.2-1.

- Rectangular plates of uniform thickness that have mixed simple support or fixed or free edges subjected to uniformly distributed load, linearly distributed transverse load, and edge moments are presented in Table 10.2.2-2.

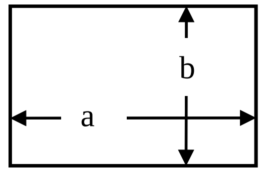
- Circular plates of constant thickness having simple support and fixed boundary conditions and subjected to different transverse loads are presented in Table 10.2.2-3.

- Annular circular plates of constant thickness having simple support and fixed boundary conditions and subjected to different transverse loads are presented in Table 10.2.2-4.

All solutions are based on small deflection theory; *i.e.*, maximum deflection is less than half of plate thickness.

Table 10.2.2-1: Formulas for Constant Thickness, Straight Edges Rectangular Plates with all Edges Either Simply Supported or Fixed

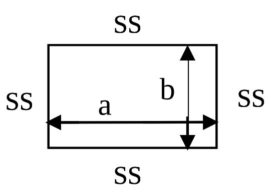
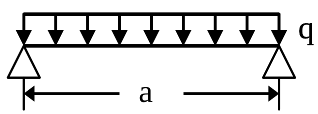
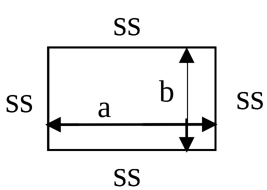
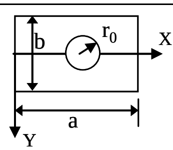
$$\frac{b}{a} = \alpha$$

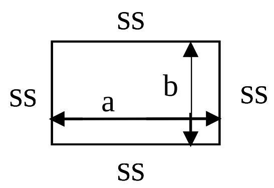
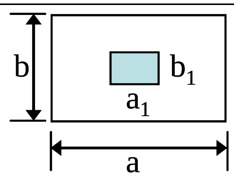


$$r_0' = (1.6\, r_0^2 + t^2)^{0.5} - 0.675\, t \text{ if } r_0 < 0.5t$$
$$r_0' = r_0 \text{ if } r_0 > 0.5t$$

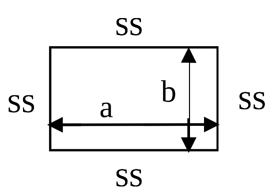
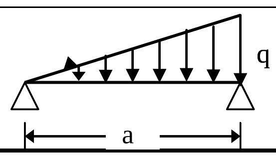
Equation 10.2.2-1

where
r_0' is the equivalent radius of contact (in)
t is the thickness of the plate (in)
r_0 is the radius of the load circular area (in)

	Case	Loading	Deflection and Stresses
1	All edges simply supported 	Uniform load over entire surface 	At center for $\nu = 0.3$ $\sigma_a = \frac{qb^2(0.225 + 0.382\alpha^2 - 0.320\alpha^3)}{t^2}$ $\sigma_a = \frac{0.75\, qb^2}{t^2(1+1.61\alpha^3)} = Max\,\sigma$ $Max\; w = \frac{0.1422\, qb^4}{E\, t^3(1+2.21\alpha^3)}$
2	All edges simply supported 	Uniform load over concentric circular area of radius r_0 	At center $\sigma_b = \frac{3P}{2\pi m t^2}\left[(m+1)Ln\frac{b}{2r_0} + m(1+k)\right]$ $, where,\; k = \left(\frac{0.914}{1+1.6\alpha^5}\right) - 0.6$ $Max\; w = \frac{0.203\, Pb^2(m^2-1)}{m^2 E\, t^3(1+0.462\alpha^4)}$ $, where\; P = \pi r_0^2 q$

	Case	Loading	Deflection and Stresses
3	All edges simply supported 	Uniform load over central rectangular area shown shaded 	At center, $Max\ \sigma = \beta \frac{P}{t^2}$ and $P = a_1\ b_1\ q$ Where β may be found from following table by interpolation for ν = 0.3

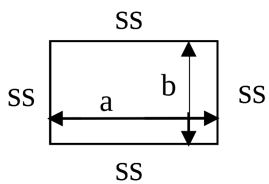
a_1/b \ b_1/b	a =b						a = 1.4 b						a= 2 b					
	0	0.2	0.4	0.6	0.8	1.0	0	0.2	0.4	0.8	1.2	1.4	0	0.4	0.8	1.2	1.6	2.0
0		1.82	1.38	1.12	0.93	0.76		2.0	1.55	1.12	0.84	0.75		1.64	1.20	0.97	0.78	0.64
0.2	1.82	1.28	1.08	0.90	0.76	0.63	1.78	1.43	1.23	0.95	0.74	0.64	1.73	1.31	1.03	0.84	0.68	0.57
0.4	1.39	1.07	0.84	0.72	0.62	0.52	1.39	1.13	1.00	0.80	0.62	0.55	1.32	1.08	0.88	0.74	0.60	0.50
0.6	1.12	0.90	0.72	0.60	0.52	0.43	1.10	0.91	0.82	0.68	0.53	0.47	1.04	0.90	0.76	0.64	0.54	0.44
0.8	0.92	0.76	0.62	0.51	0.42	0.36	0.90	0.76	0.68	0.57	0.45	0.40	0.87	0.76	0.63	0.54	0.44	0.38
1.0	0.76	0.63	0.52	0.42	0.35	0.30	0.75	0.62	0.57	0.47	0.38	0.33	0.71	0.61	0.53	0.45	0.38	0.30

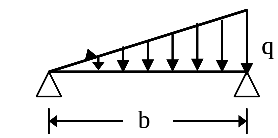
	Case	Loading	Deflection and Stresses	
4	All edges simply supported 	Distributed load varying linearly along length 	$Max\ \sigma = \beta \frac{qb^2}{t^2}$	$Max\ w = \delta \frac{qb^4}{E\ t^3}$

For ν = 0.3, β and δ may be found from following table by interpolation

a/b	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**
β	0.16	0.26	0.34	0.38	0.43	0.47	0.49
δ	0.022	0.043	0.060	0.070	0.078	0.086	0.091

	Case	Loading	Deflection and Stresses
5	All edges simply supported	Distributed load varying linearly along width	



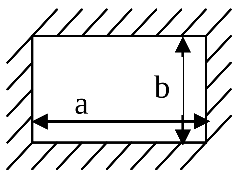


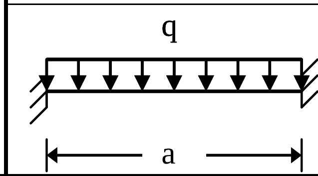
$$Max\sigma = \beta \frac{qb^2}{t^2} \qquad Max\ w = \delta \frac{qb^4}{E\ t^3}$$

For ν = 0.3, β and δ may be found from following table by interpolation

a/b	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**
β	0.16	0.26	0.32	0.35	0.37	0.38	0.38
δ	0.022	0.042	0.056	0.063	0.067	0.069	0.070

	Case	Loading	Deflection and Stresses
6	All edges clamped	Uniform load over entire surface	





At center of long edges

$$Max\sigma = -\beta_1 \frac{qb^2}{t^2}$$

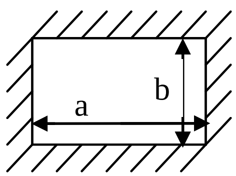
At center

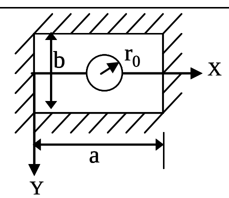
$$\sigma = \beta_2 \frac{qb^2}{t^2} \qquad Max\ w = \delta \frac{qb^4}{E\ t^3}$$

For ν = 0.3, β_1, β_2 and δ may be found from following table by interpolation

a/b	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**	∞
β_1	0.3078	0.3834	0.4356	0.4680	0.4872	0.4974	0.5000
β_2	0.1386	0.1794	0.2094	0.2286	0.2406	0.2472	0.2500
δ	0.0138	0.0188	0.0226	0.0251	0.0267	0.0277	0.0284

	Case	Loading	Deflection and Stresses
7	All edges clamped	Uniform load over small concentric circle of r_0	





At center

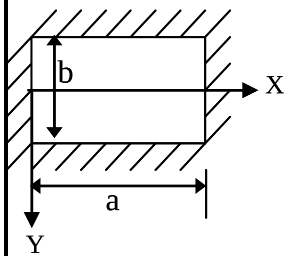
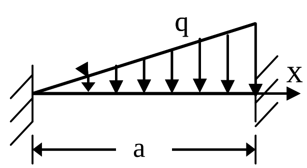
$$Max\sigma = \frac{3P}{2\pi t^2}\left[(1+\nu) Ln \frac{2b}{\pi r_0'} + \beta_1\right] \qquad Max\ w = \delta \frac{Pb^2}{E\ t^3}$$

At center of long edges

$$\sigma = -\frac{\beta_2 P}{t^2}$$

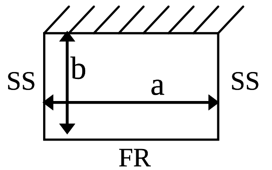
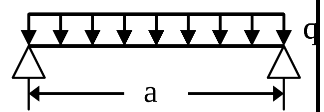
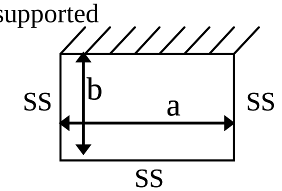
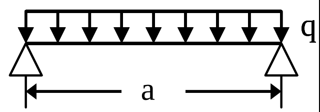
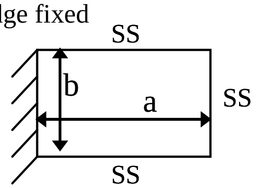
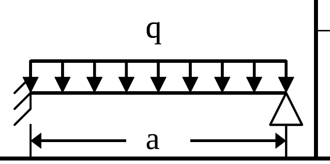
For ν = 0.3, β_1, β_2 and δ may be found from following table by interpolation

a/b	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**	∞
β_1	- 0.238	- 0.078	0.011	0.053	0.068	0.067	0.067
β_2	0.7542	0.8940	0.9624	0.9906	1.0000	1.004	1.008
δ	0.0611	0.0706	0.0754	0.0777	0.0786	0.0788	0.0791

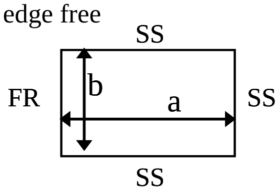
<table>
<tr><th></th><th>Case</th><th>Loading</th><th colspan="2">Deflection and Stresses</th></tr>
<tr><td rowspan="4">8</td><td rowspan="4">All edges clamped

</td><td rowspan="3">Distributed load varying linearly along length a</td><td>At Y = ± b/2, X = 0.55a
$$Max\,\sigma_b = -\beta_1 \frac{qa^2}{t^2}$$</td><td>At Y = 0, X = 0.6a
$$\sigma_b = \beta_2 \frac{qa^2}{t^2}$$</td></tr>
<tr><td>At Y = 0, X = a
$$\sigma_a = -\beta_3 \frac{qa^2}{t^2}$$</td><td>At Y = 0, X = 0
$$\sigma_a = -\beta_4 \frac{qa^2}{t^2}$$</td></tr>
<tr><td>At Y = 0, X = 0.6a
$$\sigma_a = \beta_5 \frac{qa^2}{t^2}$$</td><td>$$Max\ w = \delta \frac{qb^4}{E\,t^3}$$</td></tr>
<tr><td>
</td><td colspan="2">For ν = 0.3, β_1, β_2, β_3, β_4, β_5 and δ may be found from following table by interpolation</td></tr>
</table>

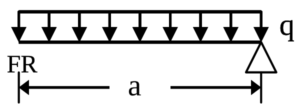
a/b	**0.6**	**0.8**	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**
β_1	0.1308	0.1434	0.1686	0.1800	0.1842	0.1872	0.1902	0.1908
β_2	0.0636	0.0688	0.0762	0.0715	0.0612	0.0509	0.0415	0.0356
β_3	0.0832	0.1778	0.2365	0.2561	0.3004	0.3092	0.3100	0.3000
β_4	0.0206	0.0497	0.0898	0.1249	0.1482	0.1615	0.1680	0.1709
β_5	0.0410	0.0633	0.0869	0.1038	0.1128	0.1255	0.1157	0.1148
δ	0.0016	0.0047	0.0074	0.0097	0.0113	0.0126	0.0133	0.0136

Table 10.2.2-2: Formulas for Constant Thickness, Straight Edges Rectangular Plates with Mixed Boundary Conditions

<table>
<tr><th></th><th>Case</th><th>Loading</th><th colspan="2">Deflection and stresses</th></tr>
<tr><td rowspan="2">1</td><td rowspan="2">One long edge fixed, other free. Short edges simply supported
</td><td>Uniform load over entire surface</td><td colspan="2">For ν = 0.3 at center of fixed edge
$$\sigma_b = -\frac{3qb^2}{t^2(1+3.2\alpha^3)} = Max\,\sigma$$</td></tr>
<tr><td></td><td>For ν = 0.3 at center of free edge
$$\sigma_a = \frac{0.75qb^2}{t^2\left(1+\frac{0.285}{\alpha^4}\right)}$$</td><td>$$Max\ w = \frac{1.37qb^4}{E\,t^3(1+10\alpha^3)}$$</td></tr>
<tr><td rowspan="2">2</td><td rowspan="2">One long edge fixed, other three edges simply supported
</td><td>Uniform load over entire surface</td><td>$$Max\ \sigma = \beta\frac{qb^2}{t^2}$$</td><td>$$Max\ w = \delta\frac{qb^4}{E\,t^3}$$</td></tr>
<tr><td></td><td colspan="2">For ν = 0.3, β and δ may be found from following table by interpolation
<table>
<tr><th>a/b</th><th>1.0</th><th>1.5</th><th>2.0</th><th>2.5</th><th>3.0</th><th>3.5</th><th>4.0</th></tr>
<tr><td>β</td><td>0.50</td><td>0.66</td><td>0.73</td><td>0.74</td><td>0.74</td><td>0.75</td><td>0.75</td></tr>
<tr><td>δ</td><td>0.030</td><td>0.046</td><td>0.054</td><td>0.056</td><td>0.057</td><td>0.058</td><td>0.058</td></tr>
</table></td></tr>
<tr><td rowspan="2">3</td><td rowspan="2">Three edges simply supported and one short edge fixed
</td><td>Uniform load over entire surface</td><td>$$Max\,\sigma = \beta\frac{qb^2}{t^2}$$</td><td>$$Max\ w = \delta\frac{qb^4}{E\,t^3}$$</td></tr>
<tr><td></td><td colspan="2">For ν = 0.3, β and δ may be found from following table by interpolation
<table>
<tr><th>a/b</th><th>1.0</th><th>1.5</th><th>2.0</th><th>2.5</th><th>3.0</th><th>3.5</th><th>4.0</th></tr>
<tr><td>β</td><td>0.50</td><td>0.67</td><td>0.73</td><td>0.74</td><td>0.75</td><td>0.75</td><td>0.75</td></tr>
<tr><td>δ</td><td>0.030</td><td>0.071</td><td>0.101</td><td>0.122</td><td>0.132</td><td>0.137</td><td>0.139</td></tr>
</table></td></tr>
</table>

	Case	Loading	Deflection and Stresses
4	Three edges simply supported and one short edge free	Uniform load over entire surface	$Max\,\sigma = \beta \frac{qb^2}{t^2}$ $Max\ w = \delta \frac{qb^4}{E\,t^3}$

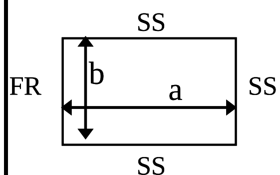


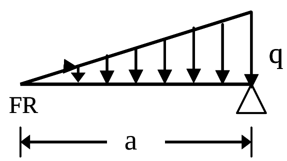


For ν = 0.3, β and δ may be found from following table by interpolation

a/b	**1.0**	**1.5**	**2.0**	**4.0**
β	0.67	0.77	0.79	0.80
δ	0.140	0.160	0.165	0.167

	Case	Loading	Deflection and Stresses
5	Three edges simply supported and one short edge free	Distributed load varying linearly along length	$Max\,\sigma = \beta \frac{qb^2}{t^2}$ $Max\ w = \delta \frac{qb^4}{E\,t^3}$

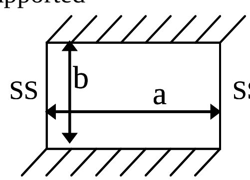


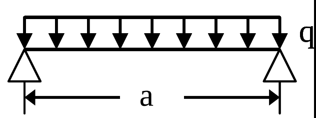


For ν = 0.3, β and δ may be found from following table by interpolation

a/b	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**
β	0.20	0.28	0.32	0.35	0.36	0.37	0.37
δ	0.040	0.050	0.058	0.064	0.067	0.069	0.070

	Case	Loading	Deflection and Stresses
6	Two long edges fixed and short edges simply supported	Uniformly distributed load over entire surface	$Max\,\sigma = -\beta \frac{qb^2}{t^2}$ $Max\ w = \delta \frac{qb^4}{E\,t^3}$

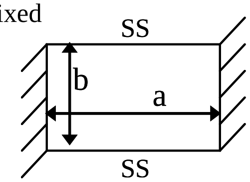


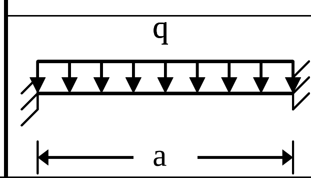


For ν = 0.3, β and δ may be found from following table by interpolation

a/b	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**	∞
β	0.4182	0.4626	0.4860	0.4968	0.4971	0.4973	0.50
δ	0.0210	0.0243	0.0262	0.0273	0.0280	0.0283	0.0285

	Case	Loading	Deflection and Stresses
7	Two long edges simply supported and short edges fixed	Uniformly distributed load over entire surface	$Max\,\sigma = -\beta \frac{qb^2}{t^2}$ $Max\ w = \delta \frac{qb^4}{E\,t^3}$

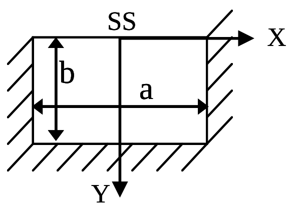


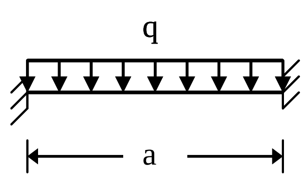


For ν = 0.3, β and δ may be found from following table by interpolation

a/b	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**	∞
β	0.4182	0.5208	0.5988	0.6540	0.6912	0.7146	0.7500
δ	0.0210	0.0349	0.0502	0.0658	0.0800	0.0922	

	Case	Loading	Deflection and Stresses	
8	Three edges fixed and long edge simply supported	Uniform load over entire surface	At X =0, Y = b $\sigma_b = -\beta_1 \frac{qb^2}{t^2}$	$R = \gamma_1 qb$
			At X =0, Y= 0.4b $\sigma_b = \beta_2 \frac{qb^2}{t^2}$	$\sigma_a = \beta_3 \frac{qb^2}{t^2}$
			At X =0, Y = 0 $R = \gamma_2 qb$	
			At X = ± a/2, Y= 0.4b $\sigma_a = -\beta_4 \frac{qb^2}{t^2}$	$R = \gamma_3 qb$

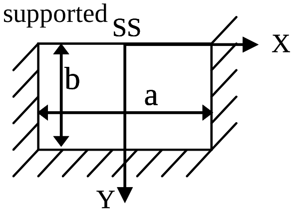


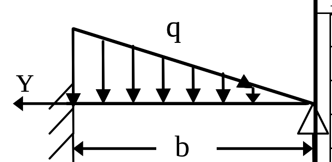


For ν = 0.2, β_1, β_2, β_3, β_4, Υ_1, Υ_2 and Υ_3 may be found from following table by interpolation

a/b	**0.25**	**0.50**	**0.75**	**1.0**	**1.5**	**2.0**	**3.0**
β_1	0.020	0.081	0.173	0.307	0.539	0.657	0.718
β_2	0.004	0.018	0.062	0.134	0.284	0.370	0.422
β_3	0.016	0.061	0.118	0.158	0.164	0.135	0.097
β_4	0.031	0.121	0.242	0.343	0.417	0.398	0.318
Υ_1	0.115	0.230	0.343	0.453	0.584	0.622	0.625
Υ_2	0.123	0.181	0.253	0.319	0.387	0.397	0.386
Υ_3	0.125	0.256	0.382	0.471	0.547	0.549	0.530

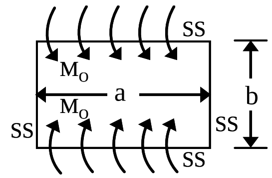
	Case	Loading	Deflection and Stresses	
9	Three edges fixed and the long edge simply supported	Uniformly decreasing from fixed edge to simply supported edge	At X = 0, Y = b $\sigma_b = -\beta_1 \frac{qb^2}{t^2}$	$R = \gamma_1 qb$
			At X = ± a/2, Y = 0.6b $Max\, \sigma_a = -\beta_2 \frac{qb^2}{t^2}$	$R = \gamma_2 qb$





For ν = 0.2, β_1, β_2, Υ_1 and Υ_2 may be found from following table by interpolation

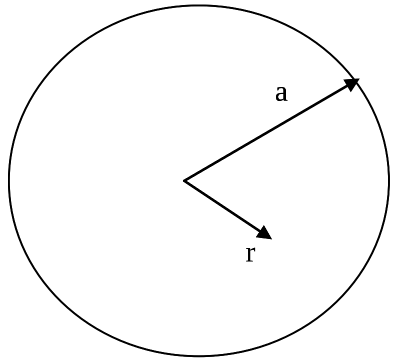
a/b	**0.25**	**0.50**	**0.75**	**1.0**	**1.5**	**2.0**	**3.0**
β_1	0.018	0.064	0.120	0.192	0.303	0.356	0.382
β_2	0.019	0.068	0.124	0.161	0.181	0.168	0.132
Υ_1	0.106	0.195	0.265	0.323	0.383	0.399	0.400
Υ_2	0.075	0.152	0.212	0.245	0.262	0.258	0.250

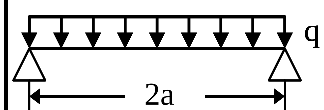
<table>
<tr><th></th><th>Case</th><th>Loading</th><th colspan="3">Deflection and Stresses</th></tr>
<tr><td rowspan="4">10</td><td rowspan="4">All four edges simply supported</td><td rowspan="4">Uniformly distributed symmetric edge moment on two edges

</td><td colspan="3">At Center</td></tr>
<tr><td>$M_a = \beta M_o$</td><td>$M_b = \beta_1 M_o$</td><td>$w = \delta \dfrac{M_o b^2}{D}$</td></tr>
<tr><td colspan="3">For ν = 0.3, β, β_1 and δ can be found from the following table:</td></tr>
<tr><td colspan="3">

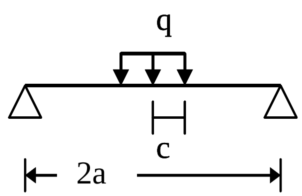
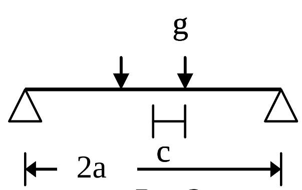
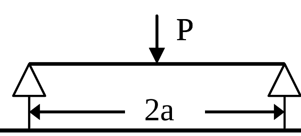
a/b	δ	β	β_1
0	0.1250	0.300	1.000
0.50	0.0964	0.387	0.770
0.75	0.0620	0.424	0.476
1.00	0.0368	0.394	0.256
1.50	0.0280	0.264	0.046
2.00	0.0174	0.153	-0.010

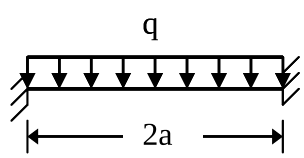
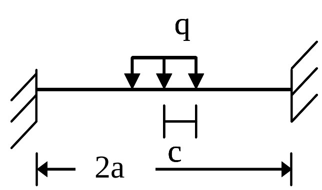
</td></tr>
</table>

Table 10.2.2-3: Formulas for Constant Thickness Circular Plates



	Case	Loading	Deflection and Bending Moments	
1	Simply supported edges	Uniform load over entire surface	At center (for r=0) $w_{\max} = \frac{(5+\nu)}{64(1+\nu)} \frac{qa^4}{D}$	$(M_r)_{\max} = (M_t)_{\max} = \frac{(3+\nu)}{16} qa^2$
			At any other point $w = \frac{q\,(a^2-r^2)}{64D}\left(\frac{5+\nu}{1+\nu}\,a^2 - r^2\right)$	
			At any other point $M_r = \frac{q}{16}(3+\nu)\,(a^2-r^2)$	$M_t = \frac{q}{16}[a^2(3+\nu) - r^2(1+3\nu)]$

	Case	Loading	Deflection and Bending Moments
2	Simply supported edges	Uniform load over concentric circular area of radius, c .	At center (for r =0) $$w_{\max} = \frac{P}{16\pi D}\left[\frac{3+\nu}{1+\nu}a^2 + c^2 Ln\frac{c}{a} - \frac{7+3\nu}{4(1+\nu)}c^2\right]$$
			For r >c $$w = \frac{P}{16\pi D}\left[\frac{3+\nu}{1+\nu}(a^2 - r^2) + 2r^2 Ln\frac{r}{a} + c^2\left\{Ln\frac{r}{a} - \frac{1-\nu}{2(1+\nu)}\frac{a^2-r^2}{a^2}\right\}\right]$$
		P = $\pi c^2 q$	At center (for r= 0) $$M_{\max} = \frac{P}{4\pi}\left[(1+\nu)Ln\frac{a}{c} + 1 - \frac{(1-\nu)}{4a^2}c^2\right]$$
		For r >c $$M_r = \frac{(1+\nu)P}{4\pi}Ln\frac{a}{r} + \frac{(1-\nu)Pc^2}{16\pi}\left(\frac{1}{r^2} - \frac{1}{a^2}\right)$$	$$M_t = \frac{P}{4\pi}\left[(1+\nu)Ln\frac{a}{r} + 1 - \nu\right] - \frac{(1-\nu)Pc^2}{16\pi}\left(\frac{1}{r^2} + \frac{1}{a^2}\right)$$
3	Simply supported edges	Uniform load over concentric circular ring of radius, c	At center (for r = 0) $$w = \frac{P}{8\pi D}\left[c^2 Ln\frac{c}{a} + (a^2 - c^2)\frac{3+\nu}{2(1+\nu)}\right]$$
			For r >c $$w = \frac{P}{8\pi D}\left[(a^2 - r^2)\left(1 + \frac{1}{2}\frac{1-\nu}{1+\nu}\frac{a^2-c^2}{a^2}\right) + (c^2 + r^2)Ln\frac{r}{a}\right]$$
		P = $2\pi cg$	At center (for r=0) $$M_{\max} = \frac{(1-\nu)P(a^2 - c^2)}{8\pi a^2} - \frac{(1+\nu)PLn\frac{b}{a}}{4\pi}$$
4	Simply supported edges	Concentrated load at the center	At center $$Max\ w = \frac{(3+\nu)Pa^2}{16\pi(1+\nu)D}$$ At any other point $$w = \frac{P}{16\pi D}\left[\frac{(3+\nu)}{(1+\nu)}(a^2 - r^2) + 2r^2 Ln\frac{r}{a}\right]$$
			At any point $$M_r = \frac{P}{4\pi}(1+\nu)Ln\frac{a}{r}$$ $$M_t = \frac{P}{4\pi}\left[(1+\nu)Ln\frac{a}{r} + 1 - \nu\right]$$

	Case	Loading	Deflection and Bending Moments	
5	Fixed edges	Uniform load over entire surface	At center $w_{\max} = \frac{qa^4}{64D}$	$(M_r)_{r=0} = (M_t)_{r=0} = \frac{qa^2}{16}(1+\nu)$
			At any other point $w = \frac{q}{64D}(a^2 - r^2)^2$	
			Any where $M_r = \frac{q}{16}\left[a^2(1+\nu) - r^2(3+\nu)\right]$	$M_t = \frac{q}{16}\left[a^2(1+\nu) - r^2(1+3\nu)\right]$
			At the edge (for r=a) $(M_r)_{r=a} = -\frac{qa^2}{8}$	$(M_t)_{r=a} = -\frac{\nu qa^2}{8}$
6	Fixed edges	Uniform load over concentric circular area of radius, c .	At center (for r =0) $w_{\max} = \frac{P}{64\pi D}\left[4a^2 - 4c^2 Ln\frac{a}{c} - 3c^2\right]$	
		 $P = \pi c^2 q$	For r >c $w = \frac{P}{64\pi D}\left[4a^2 - (8r^2 + 4c^2)Ln\frac{a}{r} - \frac{2r^2c^2}{a^2} - 4r^2 + 2c^2\right]$	
			At center (for r= 0) $M_{\max} = \frac{P}{4\pi}\left[(1+\nu)Ln\frac{a}{c} + \frac{(1+\nu)}{4a^2}c^2\right]$ $when\ c < 0.588a$	
		For r >c $M_r = \frac{P}{4\pi}\left[(1+\nu)Ln\frac{a}{r} + (1+\nu)\frac{c^2}{4a^2} + (1-\nu)\frac{c^2}{4r^2} - 1\right]$		$M_t = \frac{P}{4\pi}\left[(1+\nu)Ln\frac{a}{r} + (1+\nu)\frac{c^2}{4a^2} - (1-\nu)\frac{c^2}{4r^2} - \nu\right]$

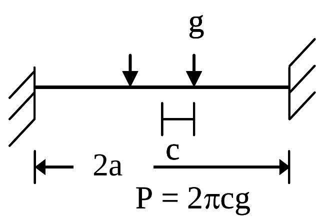
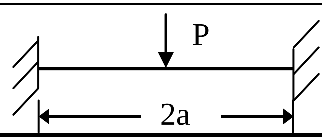
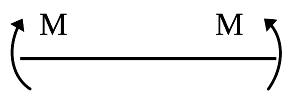
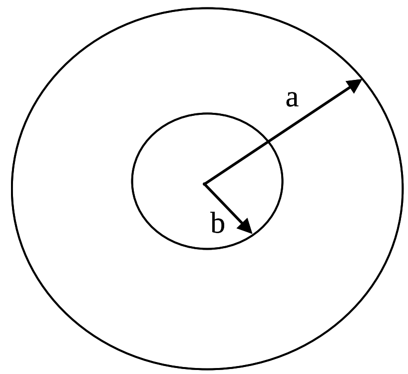
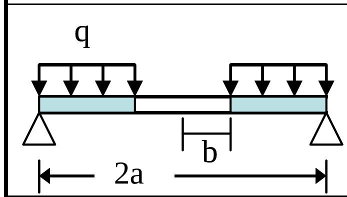
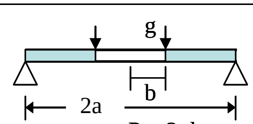
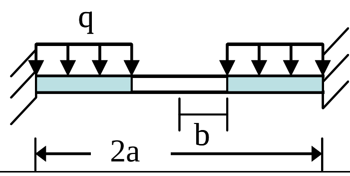
	Case	Loading	Deflection and Bending Moments	
7	Fixed edges	Uniform load over concentric circular ring of radius, c	At center (for r = 0) $w = \frac{P}{8\pi D}\left[c^2 Ln\frac{c}{a} + \frac{(a^2-c^2)}{2}\right]$	
			For r >c $w = \frac{P}{8\pi D}\left[(c^2+r^2)Ln\frac{r}{a} + \frac{(a^2+c^2)(a^2-r^2)}{2a^2}\right]$	
			At the edges (for r=a) $M = -\frac{P(a^2-c^2)}{4\pi a^2}$	
8	Fixed edges	Concentrated load at the center	At center $Max\ w = \frac{Pa^2}{16\pi D}$	At any other point $w = \frac{\Pr^2}{8\pi D}Ln\frac{r}{a} + \frac{P}{16\pi D}(a^2-r^2)$
			At any point $M_r = \frac{P}{4\pi}\left[(1+\nu)Ln\frac{a}{r} - 1\right]$	$M_t = \frac{P}{4\pi}\left[(1+\nu)Ln\frac{a}{r} - \nu\right]$
9	No Support	Uniform edge moment	At center $Max\ w = \frac{6(m-1)Ma^2}{Emt^3}$	At any other point $w = \frac{6(m-1)M(a^2-r^2)}{Emt^3}$
			At any point $M = M$	At edge $\theta = \frac{12(m-1)Ma}{Emt^3}$

Table 10.2.2-4: Formulas for Constant Thickness Annular Circular Plates



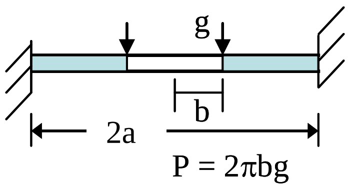
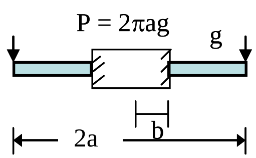
	Case	Loading	Deflection and Bending Moments
1	Simply supported outer edges	Uniform load over entire actual surface	At inner edge $Max\ M = M_t = \frac{q}{8m(a^2-b^2)}\left[a^4(3m+1)+b^4(m-1)-4ma^2b^2-4(m+1)a^2b^2 Ln\frac{a}{b}\right]$
			$When\ b\ is\ very\ small, Max\ M = M_t = \frac{qa^2(3m+1)}{8m}$
		$w_{\max} = \frac{q}{8D}\left[\frac{a^4(5m+1)}{8(m+1)}+\frac{b^4(7m+3)}{8(m+1)}-\frac{a^2b^2(3m+1)}{2(m+1)}+\frac{a^2b^2(3m+1)}{2(m-1)}Ln\frac{a}{b}-\frac{2a^2b^4(m+1)}{(a^2-b^2)(m-1)}\left(Ln\frac{a}{b}\right)^2\right]$	
2	Simply supported outer edges	Uniform distributed load along inner edge.	$Max\ M = M_t = \frac{P}{4\pi m}\left[\frac{2a^2(m+1)}{a^2-b^2}Ln\frac{a}{b}+(m-1)\right]$
			$w_{\max} = \frac{P}{16\pi D}\left[\frac{(a^2-b^2)(3m+1)}{(m+1)}+\frac{4a^2b^2(m+1)}{(m-1)(a^2-b^2)}\left(Ln\frac{a}{b}\right)^2\right]$

	Case	Loading	Deflection and Bending Moments
3	Fixed outer edges	Uniform load over entire actual surface	At outer edge $$Max\ M = M_r = \frac{q}{8}\left[a^2 - 2b^2 + \frac{b^4(m-1) - 4b^4(m+1)Ln\frac{a}{b} + a^2b^2(m+1)}{a^2(m-1)+b^2(m+1)}\right]$$
			At inner edge $$Max\ M = M_t = \frac{qa^2(m^2-1)}{8m}\left[\frac{a^4 - b^4 - 4a^2b^2Ln\frac{a}{b}}{a^2(m-1)+b^2(m+1)}\right]$$

Maximum deflection

$$w_{max} = \frac{q}{64D} \bullet$$

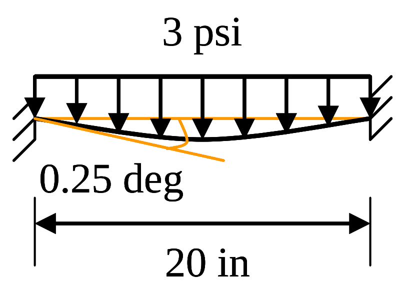
$$\left[a^4 + 5b^4 - 6a^2b^2 + 8b^4Ln\frac{a}{b} - \frac{\left\{8b^6(m+1) - 4a^2b^4(3m+1) - 4a^4b^2(m+1)\right\}Ln\frac{a}{b} + 16a^2b^4(m+1)\left(Ln\frac{a}{b}\right)^2 - 4a^2b^4 + 2a^4b^2(m+1) - 2b^6(m-1)}{a^2(m-1)+b^2(m+1)}\right]$$

	Case	Loading	Deflection and Bending Moments
4	Fixed outer edges	Uniform distributed load along inner edge.	At outer edge $Max\ M = M_r = \frac{P}{4\pi}\left[1-\frac{2mb^2-2b^2(m+1)Ln\frac{a}{b}}{a^2(m-1)+b^2(m+1)}\right] \quad when\ \frac{a}{b}<2.4$
			At inner edge $Max\ M = M_t = \frac{P}{4\pi m}\left[1+\frac{ma^2(m-1)-mb^2(m+1)-2(m^2-1)a^2Ln\frac{a}{b}}{a^2(m-1)+b^2(m+1)}\right] \quad when\ \frac{a}{b}>2.4$
			$w_{\max} = \frac{P}{16\pi D}\left[a^2-b^2+\frac{2mb^2(a^2-b^2)-8ma^2b^2Ln\frac{a}{b}+4a^2b^2(m+1)\left(Ln\frac{a}{b}\right)^2}{a^2(m-1)+b^2(m+1)}\right]$
5	Inner edge fixed support	Uniform distributed load along outer edge.	At inner edge $Max\ M = M_r = \frac{P}{4\pi}\left[\frac{2a^2(m+1)Ln\frac{a}{b}+a^2(m-1)-b^2(m-1)}{a^2(m+1)+b^2(m-1)}\right]$
			
		At outer edge $w_{\max} = \frac{P}{16\pi D}\left[\frac{a^4(3m+1)-b^4(m-1)-2a^2b^2(m+1)-8ma^2b^2Ln\frac{a}{b}-4a^2b^2(m+1)\left(Ln\frac{a}{b}\right)^2}{a^2(m+1)+b^2(m-1)}\right]$	

10.2.2.1 Example

Given: A circular steel plate is fully supported in vertical direction but partially supported in horizontal direction such that when uniformly distributed load of q= 3 psi is applied it results in a slope of 0.25 degrees at the edge.
t = 0.20 in
a = 10 in (radius of plate)
$E = 30x10^6$ psi
$\nu = 0.30$
$\theta = (0.25^0)(\pi/180) = 4.3633x10^{-3}$ radians

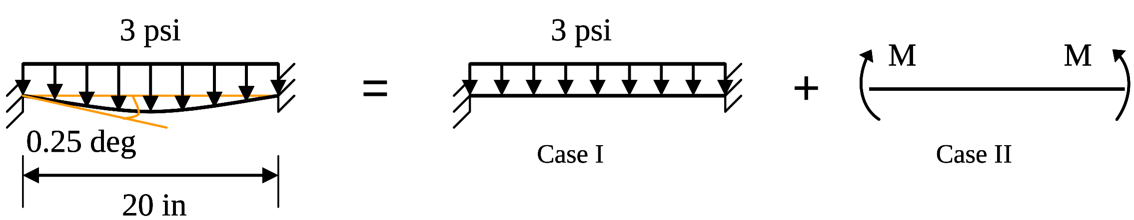
Calculate the stress and deflection at the center and the stress at the edge of the plate.



Discussion: The support offers partial fixity at the edges with respect to the rotation. This partial fixity problem will be solved by using superposition. It is solved by combining the solutions of two cases:

- Case I – the stresses and deflections are computed by assuming that the edges provide complete fixity. Table 10.2.2-3 Case 5
- Case II - the stresses and deflections are computed by applying a moment that produces a rotation of 0.25 degrees. Table 10.2.2-3 Case 9

After having obtained the solutions for both the cases, the stresses and deflections are combined appropriately to obtain the solution of the plate problem in hand.



Equation 10.2.0-9, $D = Et^3/[12(1-\nu^2)] = (30.0x10^6)(0.2)^3 / [12(1 - 0.3^2)] = 21978$ lb-in
Stress due to a moment is given by $\sigma = Mc/I$, where for a rectangular section $c = t/2$,
b = width and $I = bt^3/12$. Substituting, $\sigma = Mc/I = M(t/2)/[bt^3/12] = 6M/bt^2$
And for a unit width rectangular section, b = 1, thus $\sigma = 6M/t^2$

Case I: Fixed Edges – Reference Table 10.2.2-3 Case 5

Calculation	Equation	Result
Calculate maximum radial moment at edge	$M_r = -qa^2/8 = -(3.0)(10^2) / 8 = -37.5$ in-lb/in	$M_r = -37.5$ in-lb/in
Calculate maximum radial stress at edge	$\sigma_r = 6M_r/t^2 = 6(37.5) / 0.2^2 = \pm 5625$ psi	$\sigma_r = \pm 5625$ psi Upper surface is in tension
Calculate deflection at center	$w = qa^4/(64D)$ $= 3.0(10^4) / [(64)(21978)] = 0.02133$ in	w = 0.02133 in

Calculate maximum radial moment at center	$M_r = qa^2(1+\nu)/16$ $= 3.0(10^2)(1+0.3) / 16 = 24.375$ in-lb/in	$M_r = 24.375$ in-lb/in
Calculate maximum radial stress at center	$\sigma_r = 6M_r/t^2 = 6(24.375) / 0.2^2 = 3656$ psi	$\sigma_r = \pm 3656$ psi Upper surface is in compression
Case II: Free Edges with an Equivalent Moment – Reference Table 10.2.2-3 Case 9		
Determine equation for M, given edge rotation	$\theta = 12M(m-1)a/(Emt^3)$, thus $M = Emt^3\theta / [12(m-1)a]$ where $m = 1/\nu$	
Calculate m	$m = 1/\nu = 1/0.3 = 3.333$	$m = 3.333$
Calculate M	$M = (30x10^6)(3.333)(0.20^3)(4.3633x10^{-3}) / [12(3.333-1)(10)] = 12.467$ in-lb/in	$M = 12.467$ in-lb/in
Calculate stress at edge, M_r = M	$\sigma_r = 6M_r/t^2 = 6(12.467) / 0.2^2 = 1870$ psi	$\sigma_r = \pm 1870$ psi Upper surface is in compression
Calculate stress at center, M_r=M	$\sigma_r = 6M_r/t^2 = 6(12.467) / 0.2^2 = 1870$ psi	$\sigma_r = \pm 1870$ psi Upper surface is in compression
Calculate deflection at center for M	$w = 6(m-1)Ma^2/(Emt^3)$ $= 6(3.333-1)(12.467)(10^2)$ $/[(30x10^6)(3.333)(0.2^3)] = 0.02182$ in.	$w = 0.02182$ in
Superimpose results		
Total maximum tensile stress at edge	$\sigma_r = 5625-1870 = 3755$ psi	$\sigma_r = 3755$ psi (Upper surface)
Total maximum tensile stress at center	$\sigma_r = 3656+1870 = 5526$ psi	$\sigma_r = 5526$ psi (Lower surface)
Total deflection at center	$w = 0.02133 + 0.02182 = 0.04315$ in	$w = 0.0432$ in

10.2.3 Bending of Plates with a Small Initial Curvature

This section presents the procedure for the analysis of plates having a small initial curvature such that, at any point of the middle surface, there is an initial deflection, w_0, which is small in comparison to the thickness of the plate. The transverse loads on such a plate will produce additional deflection, w_1, of the middle surface. If deflections are small, small deflection theory can be used and furthermore, the principle of superposition can be used such that the total deflection is a sum of deflections w_0 and w_1. For a plate which does not have in-plane loads like a flat plate, the deflection w_1 is calculated by using the same Equation 10.2.0-11**.** Since the moments depend not on the total curvature but only on change in curvature, only the deflection w_1 should be used.

$$\frac{\partial^4 w_1}{\partial x^4} + 2\frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_1}{\partial y^4} = \frac{q}{D}$$ **Equation 10.2.3-1**

Thus, a deflected surface is sought for a given lateral loading condition which satisfies the above equation and the specified boundary conditions. The total deflection will be sum of w_0 and w_1. Once the deflected surface is known, internal bending moments and shears can be calculated as shown in section 10.2. With known bending moments and shears, the stresses can be calculated. It should be noted that only deflection, w_1, caused by the applied loads not the total deflection should be used in computing shear and bending moments and thus induced stresses will correspond only to the applied loads.

10.3 Plate Buckling

In section 10.2, the bending of thin plates subjected to transverse loads was discussed. Aircraft structures made up of skins and webs of stringers, longerons and bulkheads are also designed for in-plane loads in addition to or independent of transverse loads. If these in-plane loads are compressive or shearing in nature then the aircraft structures must be checked for stability similar to column stability (section 8) for carrying the loads safely and efficiently.

10.3.1 Plate Buckling Theory

Plate buckling is a much-studied subject and has been covered in great detail in technical literature. Many excellent books are available on this subject, namely References 10-10 to 10-12; therefore only the pertinent equations are presented in this section.

Figure 10.3.1-1 shows a rectangular plate subjected to in-plane loads, N_x, N_y and N_{xy}. The in-plane loads can all act together or separately. Usually, solutions are developed for each kind of load separately and if more than one kind of load is present interaction equations are used to determine the margin of safety.

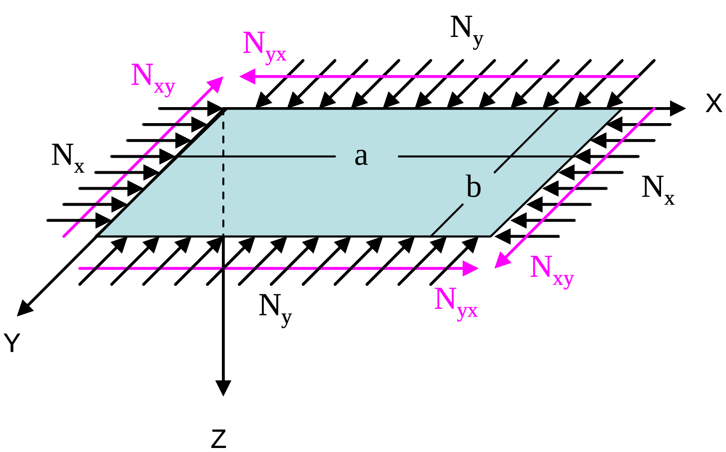


Figure 10.3.1-1: Plate Subjected to In-Plane Loads

Note all the loads are in (lb/in).
The coordinate system is right-hand system with Z-axis pointing downward.

Several ways are available to determine the critical values of the forces acting in the middle plane of a plate by assuming that from the beginning the plate has some (1) initial curvature or (2) some lateral loading or (3) that the plate buckles slightly under the action of forces applied in its middle plane. Then, the magnitudes that the forces must have in order to keep the plate in such a slightly buckled shape are calculated. By assuming that the plate buckles slightly under the action of in-plane loads, the differential equation of the deflected surface can be developed similarly as in section 10.2 by including the in-plane loads but removing the lateral load effects. If there are no body forces (*e.g.*, inertia forces), by following the same procedure as in section 10.2 the equation for the buckled plate can be developed in terms of the deflection of the middle surface, w, of the plate and is given by Equation 10.3.1-1.

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y}\right)$$ **Equation 10.3.1-1**

Where,
D is the flexural rigidity (lb-in) given by Equation 10.2.0-9
w is the deflection of the plate (in)
N_x is the compressive load (lb/in) along dimension, a
N_y is the compressive load (lb/in) along dimension, b
N_{xy} is the shear load (lb/in)

The solution of the above differential equation of equilibrium provides the initial buckling load of a flat rectangular plate. If the load is increased further it leads to a rapid increase in the deflection perpendicular to the plane of the plate and eventual failure. The mathematical solution requires that the equilibrium and the boundary conditions be satisfied. This is accomplished by integration of the differential equation, which provides the exact solution, or by some mathematical methods which may not completely satisfy boundary and equilibrium conditions but can provide an approximate solution. Energy methods are such methods and are such that one has to start with an assumed solution, *i.e.*, a deflection function which may not completely satisfy the boundary and equilibrium conditions. The reason for using such approximate methods is that the exact solution of the differential equation is available only for a few cases. Even though energy methods may not completely satisfy every condition, they provide solutions which are usually very accurate.

The initial buckling load or stress is normally calculated by solving Equation 10.3.1-1 for one directional load at a time*; i.e.*, compressive load N_x or N_y, or shear load N_{xy}. If a plate has multiple loads, for example compressive load and shear load, the buckling load for compression and shear are calculated separately and then the margin of safety or ratio to requirement is calculated through the use of interaction curves or equations.

10.3.1.1 Assumptions

It is assumed that the plate is perfectly elastic under the action of external forces within the elastic limit of the material. The plate material is assumed to be homogeneous and continuously distributed over its entire volume. It is further assumed that the plate material is isotropic*; i.e.*, that the elastic properties are the same in all directions. Basic assumptions are listed below:

a) Perfectly Elastic
b) Homogeneous
c) Continuous
d) Isotropic
e) Constant Thickness

With these assumptions, the general solution of the Equation 10.3.1-1 for a single load is given by the following equation:

$$f_{cre} = k\frac{\pi^2 D}{t b^2}$$ **Equation 10.3.1-2**

Where,
f_{cre} is the initial elastic buckling stress
k is the non-dimensional buckling coefficient for the configuration (loading and boundary conditions) under consideration
t is the thickness of the plate (in)
b is the length of the loaded edge unless otherwise noted (in)
D is the flexural rigidity (lb-in) defined by Equation 10.2.0-9

$$D = \frac{E_c t^3}{12\left(1-\nu_e^2\right)}$$

E_c is the compression modulus of the material
ν_e is the elastic Poisson's ratio of the material

Using Equation 10.2.0-9 for D, Equation 10.3.1-2 becomes

$$f_{cre} = k\frac{\pi^2 E_c}{12(1-\nu_e^2)}\left(\frac{t}{b}\right)^2$$ **Equation 10.3.1-3**

Define $(b/t)_e$ as follows:

$$\left(\frac{b}{t}\right)_e = \left(\frac{b}{t\sqrt{K}}\right)$$ **Equation 10.3.1-4**

and effective buckling coefficient, K is defined as

$$K = \frac{k\pi^2}{12\left(1-\nu_e^2\right)}$$ **Equation 10.3.1-5**

Using Equation 10.3.1-4, buckling Equation 10.3.1-3 can be written as:

$$f_{cre} = \frac{E_c}{\left(\frac{b}{t}\right)_e^2}$$ **Equation 10.3.1-6**

Equation 10.3.1-6 is the plate buckling equation in the elastic region and is modified in the inelastic region by using plasticity correction factors.

10.3.1.2 Plasticity Correction

Efficiently designed plates often buckle at a stress above the proportional limit of the material. If the buckling stress is above the proportional limit, the stress provided by Equation 10.3.1-6 will be unconservative. Such situations are handled by introducing a plasticity correction factor, η, into Equation 10.3.1-6.

In order to calculate the factor η, the stress-strain curve of the material between the proportional limit and yield stress is often modeled mathematically. One of the most used methods is the Ramberg-Osgood three-parameter method (Reference 10-13). Section 3.3.1.1 of this manual provides the discussion and other mathematical formulations of the stress-strain curve. Repeated below is the conventional Ramberg-Osgood formulation provided by Equations 3.3.1-1 and 3.3.1-2. However, the formulation given by Hill (Reference 10-14) could also be used for defining the strain and shape parameter as defined by Equations 3.3.1-5 and 3.3.1-6 of this manual.

The strain is defined as:

$$\varepsilon = \left(\frac{f}{E_c}\right)\left[1+\left(\frac{3}{7}\right)\left(\frac{f}{f_{0.7}}\right)^{n-1}\right]$$

Reference Equation 3.3.1-1

And the shape parameter

$$n = 1+\frac{\log(17/7)}{\log(f_{0.7}/f_{0.85})}$$

Reference Equation 3.3.1-2

The stresses denoted by $f_{0.7}$ and $f_{0.85}$ are experimentally determined for $0.7E_c$ and $0.85E_c$ secant intercepts of the stress-strain curve. It is to be noted that these secant intercepts may be determined for either typical (mean value) or statistically reduced (A- or B-basis) stress-strain curves and are usually determined separately for tension and compression. The shape parameter n is determined separately for compression and tension and typical data is used. The subscripts c and t are used with n to denote compression and tension shape parameters.

The tangent modulus, given by Equation 3.3.1-3, using compression shape parameter n_c is

$$E_{\tan} = \frac{E_c}{\left(1+\frac{3}{7}n_c\left(\frac{f}{f_{0.7}}\right)^{n_c-1}\right)}$$

Reference Equation 3.3.1-3

The secant modulus, given by Equation 3.3.1-4, using compression shape parameter n_c is

$$E_{\sec} = \frac{E_c}{\left(1+\frac{3}{7}\left(\frac{f}{f_{0.7}}\right)^{n_c-1}\right)}$$

Reference Equation 3.3.1-4

For inelastic buckling, it is expedient to use the inelastic Poisson's ratio of the material since it appears in the buckling stress Equation 10.3.1-6. For most engineering materials (Reference 10-3), the elastic Poisson's ratio ranges between 0.25 and 0.35. Based on the incompressibility of the isotropic materials, the theoretical Poisson's ratio in the plastic region is 0.5. The transition from the elastic to the plastic value is most pronounced in the yield region of the stress-strain curve. Gerard and Wilhorn (Reference 10-15) have shown that Poisson's ratio is seriously affected by the anisotropy of the material. For materials which can be considered orthotropic (*e.g.*, having the same properties in y and z directions if loaded along the x-axis), the following equation is used to compute the Poisson's ratio in the yield region:

$$\nu = \nu_p - \left(E_{sec} / E_c\right)\left(\nu_p - \nu_e\right)$$ **Equation 10.3.1-7**

Where,
ν is the Poisson's ratio in the transition region
ν_p is the fully plastic value of the Poisson's ratio. For isotropic materials $\nu_p = 0.5$
ν_e is the elastic Poisson's ratio

For simplicity of calculations, all effects of exceeding the proportional limit are generally incorporated in a single coefficient referred to as the plasticity correction factor η, which is a ratio of plastic buckling stress and the elastic buckling stress. Stowell and Pride (Reference 10-16) defines η_s for simply supported long plates as per Equation 10.3.1-8 and 10.3.1-9, which take into account the effect of the inelastic Poisson's ratio.

$$\eta = \eta_s \frac{\left(1 - \nu_e^2\right)}{\left(1 - \nu^2\right)}$$ **Equation 10.3.1-8**

and

$$\eta_s = \frac{E_{sec}}{E_c}\left[0.5 + 0.5\left(0.25 + 0.75\left(\frac{E_{tan}}{E_{sec}}\right)\right)^{0.5}\right]$$ **Equation 10.3.1-9**

Where,
η_s is the Stowell modulus as defined by Equation 3.2.2-3 divided by E_c

Using Equation 10.3.1-8, the buckling stress Equation 10.3.1-6 becomes

$$f_{cr} = \frac{\eta E_c}{\left(\frac{b}{t}\right)_e^2}$$ **Equation 10.3.1-10**

The above equation is perfectly general since $\eta = 1$ provides the buckling stress for the elastic case. Thus, it is not necessary to distinguish between the elastic and plastic buckling stress equations since the same equation is used for both cases. The values of k and ν_e represent elastic values and η incorporates all the corrections needed for inelastic behavior.

10.3.1.3 Buckling Criteria

Most metallic aerospace structural components have very small thicknesses as compared to their length and width. The members are usually built up from thin plates in the case of transport aircraft but usually machined for modern fighter aircraft. These structures typically include stiffeners with cross sections that are small compared to their lengths. When loaded in compression, shear, or combined loading that includes compression and shear components, such structures become subject to instability or collapse. Modes of instability can be

- global, where the overall structure of plates and stiffeners collapses
- plate, where a plate buckles locally between stiffeners
- crippling, where a stiffener cross-section fails (see section 8.4)
- or a complex interaction between local buckling and overall deformation of plates and stiffeners.

Table 10.3.1-1 provides recommended buckling criteria that have been successfully applied to metallic structures on programs at Lockheed Martin Aeronautics. Subject to the limitations in this section, structural panels and webs may be allowed to buckle where affordable weight savings can be demonstrated and where durability and damage tolerance requirements allow. The criteria presented are the minimum limits allowed. In general:

- Global stability is required
- No oil canning is permitted (phenomenon is a function of thickness and size but not load)

Programs that plan to allow buckling of metallic structure at lower load levels than Table 10.3.1-1 should coordinate with the certifying authority to ensure concurrence with defined buckling criteria and perform appropriate component and full scale testing to ensure structural integrity.

Where buckling is allowed below ultimate load, a redistribution analysis shall be performed and adjacent structure shall have positive margins for the additional load. Where buckling is allowed below limit load, durability and damage tolerance analysis, as required by structural classification, shall include the effects of the redistributed load, and, for the buckled panel, any increased loads due to diagonal tension effects.

Table 10.3.1-1: Metallic Structure Buckling Criteria

Type of Structure[1]		**Buckling Allowed at Percent Limit Load(Shear plus Compression)[9]**
Mold Line Skins	Wing, vertical and horizontal tails	120[2]
	Fuselage—metallic	37.5[2]
	Fuselage—Syncore stiffened	150
	Control surface	150
Periphery Webs[4,5] (incl. interfaces and end bays)	With holes	150
	Without holes	37.5[6]
Non-Periphery Webs[4,5]	With holes	150
	Without holes	37.5
	With reinforced holes[7]	37.5
	With attached subsystem fitting	150
	Fuel Boundary[3]	37.5[3]
Keel beam, spar web, or longeron end bays	with or without holes	150
Access Panels		150
Duct		150
Core Stiffened Panel (including honeycomb & Syncore)		150
Small Radius of Curvature ($r \le 30$; eccentricity, $e < t$)[8]		120

[1] Where more than one classification applies use the most conservative criterion
[2] Buckled panel deflection and waviness must comply with program external features design standard
[3] A Fuel boundary is defined as a panel or web adjacent to fuel sealant or sealing grooves, see Figure 10.3.1-2. These criteria cover a structural basis only. Fuel boundaries shall comply with program fuel seal design guide
[4] See Figure 10.3.1-3 for definition
[5] Webs adjacent to boundary flanges that rely on in-plane web support for curvature effects should not buckle, see Figure 10.3.1-4
[6] Periphery webs without holes may buckle only when web stability is not needed for OML flange strength, stability, or stiffness requirements
[7] Reinforced hole has local pad up and no system attachments
[8] No compression buckling (loads normal to curvature, see Figure 10.3.1-5) with $r \leq 30$ in. or where panel eccentricity exceeds panel thickness, t, based on fatigue experience
[9] Buckling under specific load conditions may be considered on an exception basis. These conditions may include: crash, jacking, spin, taxi, emergency arrestment, launch, tie down, and ground handling. Additionally, buckling under any ground handling conditions must not cause functional impairment.

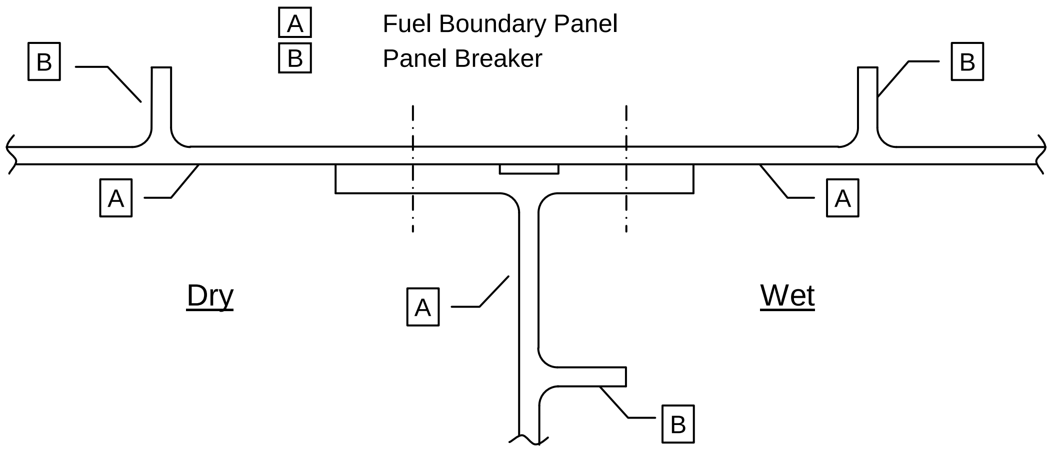


Figure 10.3.1-2: Fuel Boundary Definition

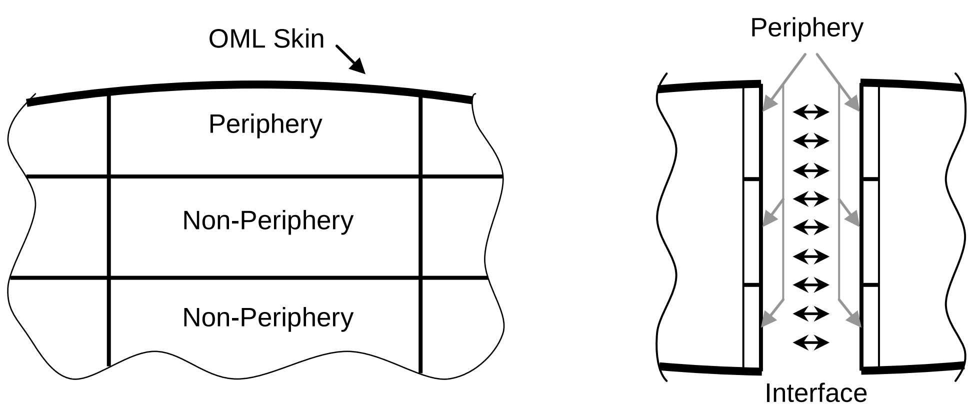


Figure 10.3.1-3: Definition of Periphery and Non-Periphery Webs

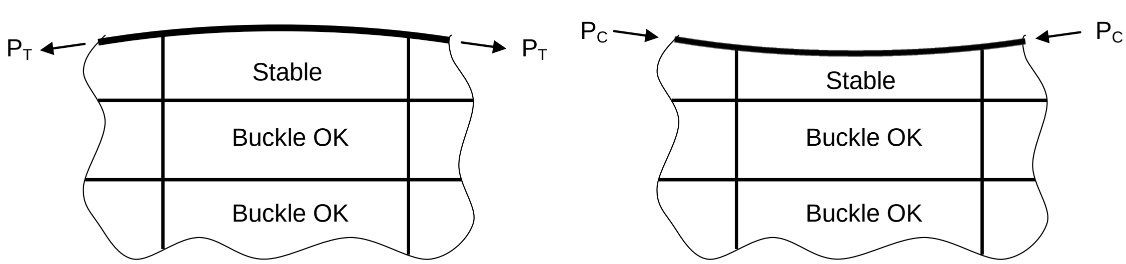


Figure 10.3.1-4: Definition of Webs Adjacent to Curved Boundary Flanges

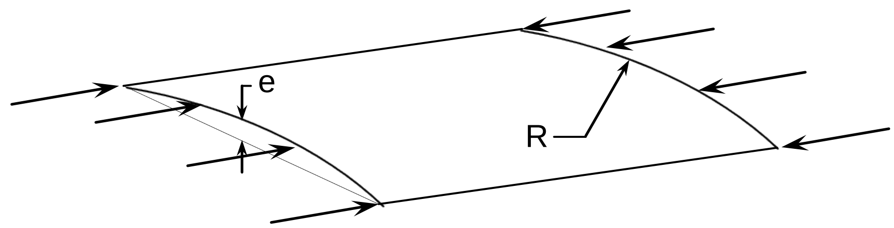


Figure 10.3.1-5: Definition of Compression Loads Normal to Curvature

10.3.1.4 Buckling Considerations

Metallic skins, webs, and covers must be checked for stability failure in accordance with the criteria stated in section 10.3.1.3. The items to consider when conducting buckling analysis include:

- Loads
  - Allowable buckling load level—see Section 10.3.1.3
  - Pressure loading effects
  - Applied internal loads to analyze for buckling
  - Methods to derive buckling applied loads from internal loads (averaging methods, etc.)
  - Combined loading effects
- Geometry:
  - Panel thickness
  - Panel size and shape
  - Cutouts and pad-ups
  - Edge fixity
  - Panel stiffening (including minimum stiffener sizing for buckling panel breakers)
- Material Properties:
  - Buckling cut-off stress values
  - Elevated temperature material modulus
- Post-buckling—see section 10.6
- Analysis Theory and Methodology—see sections 10.3.1 to 10.3.1.3

10.3.1.5 Margins of Safety and Interaction Curves

Section 2.5 of this manual discusses margins of safety calculations when the structure is designed to carry a single load or multiple loads. Likewise, in plates the margin of safety calculation is done the same way whether it is a single load or combination of loads. For a single load, margin of safety is calculated using Equation 2.5.0-1, which is repeated here.

$$M.S. = \frac{P_{all}}{P} - 1$$

Reference Equation 2.5.0-1

where,

P_{all} is allowable load (lb)
P is applied load (lb)

In the event the stress is linear up to failure, *i.e.*, no plasticity or non-linear effects, the margin of safety can be calculated using the allowable stress and applied stress.

For plates that are subjected to combined loading, it is general engineering practice (as noted in section 2.5) to use interaction curves or equations which make use of load ratios. Interaction curves or equations provide a convenient approach to calculate a margin of safety for an otherwise complex problem. For a certain class of problems as discussed in section 2.5, closed-form solutions exist for margin of safety calculations but, in general, the interaction curves are generated empirically.

Section 2.5.2 provides interaction equations for several different classes of load combinations and section 2.5.3 provides interaction curves.

Sections 10.3.2.9 to 10.3.2.20 discuss plate buckling interaction equations for combined loadings.

10.3.1.6 Initial Buckling Stress Curves

Two methods discussed in this section to compute the initial buckling stress of a given plate of a given configuration are: (1) by solving the nonlinear Equation 10.3.1-10 iteratively or (2) by using allowable stress curves generated by using program SM33 of the IDAT suite of programs. It is to be noted that Equation 10.3.1-10 entails determination of the Stowell modulus η_s, which is a function of the tangent and the secant moduli. These moduli are a non-linear function of stress and the stress must be known before the moduli can be computed. Usually the starting values of the stresses are estimated and the correct values are determined by iteration. Figure 10.3.1-6 provides such a flow chart for a sequence of steps which must be followed to determine the initial buckling stress.

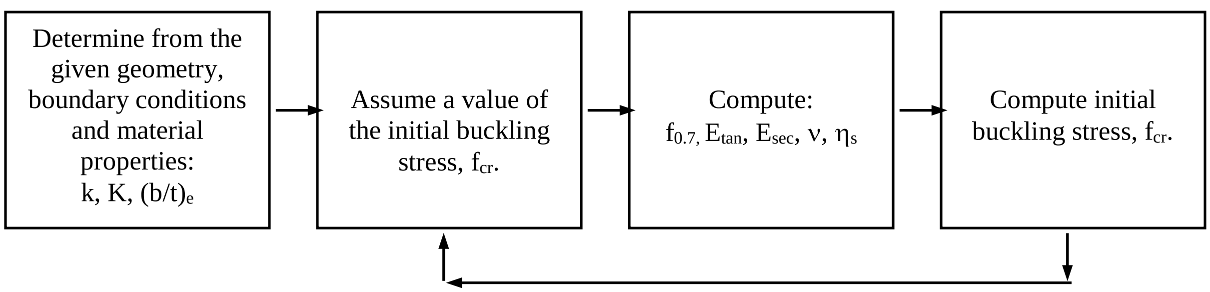


Figure 10.3.1-6: Flowchart for Calculation of the Initial Buckling Stress of a Flat Plate

On the other hand, program SM33 of the IDAT suite provides a convenient way of calculating the initial buckling stress by generating curves for compression and shear loading for any material in its material database, METDB, or for any user-defined material. The curves provided here are examples for some aerospace structural materials shown in Table 10.3.1-2. These curves, presented in Figure 10.3.1-7 to 10.3.1-12, provide allowable initial buckling stress for compression and shear for any $(b/t)_e$ as per Equation 10.3.1-10.

Table 10.3.1-2: List of Materials for Plate Buckling Curves

Material Alloy/Form	METDB No	Thick./ Grain	Basis	F_{tu} (ksi)	F_{cy} (ksi)	F_{su} (ksi)	E_c (ksi)	n_c	F_{cmax} (ksi)	Fsmax	Fig.
2024-T351 Bare Plate	68	0.500 – 1.000 L,T	R.T. B	66.0	41.0	33.0	10900	9.0	48.7	33.0	For compression see Figure 10.3.1-7 and for shear see Figure 10.3.1-10
2124-T8151 Plate	75	5.001-6.000 L,T	R.T. S	65.0	54.0	38.0	10900	16.0	58.0	38.0	
7050-T7651 Plate	95	2.501-3.000 L,T	R.T. S	76.0	64.0	38.0	10800	18.0	68.4	38.0	
7050-T7451 Plate	87	3.001-4.000 L,T	R.T. B	74.0	63.0	37.5	10600	19.0	66.9	37.5	
7075-T6 Bare Sheet	40	0.040-0.125 L,T	R.T. B	80.0	71.0	40.0	10500	12.0	80.0	40.0	
300M-0.40C Forging	356	0.40C L,T	R.T. S	270.0	236.0	135.0	29000	13.0	270.0	135.0	For compression see Figure 10.3.1-8 and for shear see Figure 10.3.1-11
301-Stainless sheet 1/2 hardness	320	>0.16 L,T	R.T. B	151.0	69.0	76.0	26000	9.0	78.5	73.4	
Ti 6Al-4V Annealed Plate	294	0.1875 – 2.000 L,T	R.T. B	135.0	129.0	69.0	16400	20.0	135.0	69.0	For compression see Figure 10.3.1-9 and for shear see Figure 10.3.1-12
Ti 6Al-4V A-B Annealed Die Forging	312	4.001 – 6.000 L,T	R.T. S	130.0	123.0	78.0	16400	20.0	130.0	78.0	

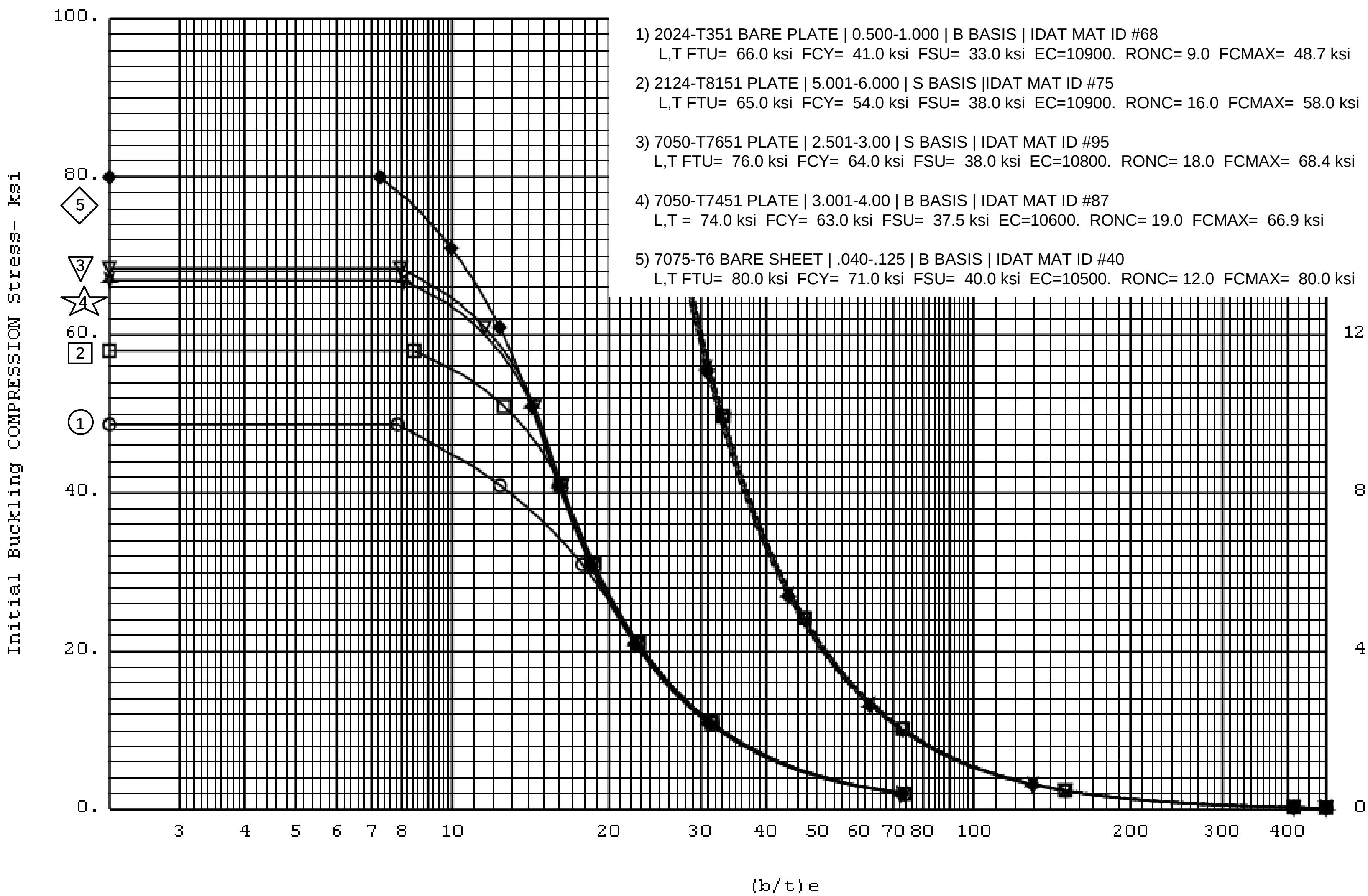


Figure 10.3.1-7: Initial Buckling Compression Stress for Aluminum Plates

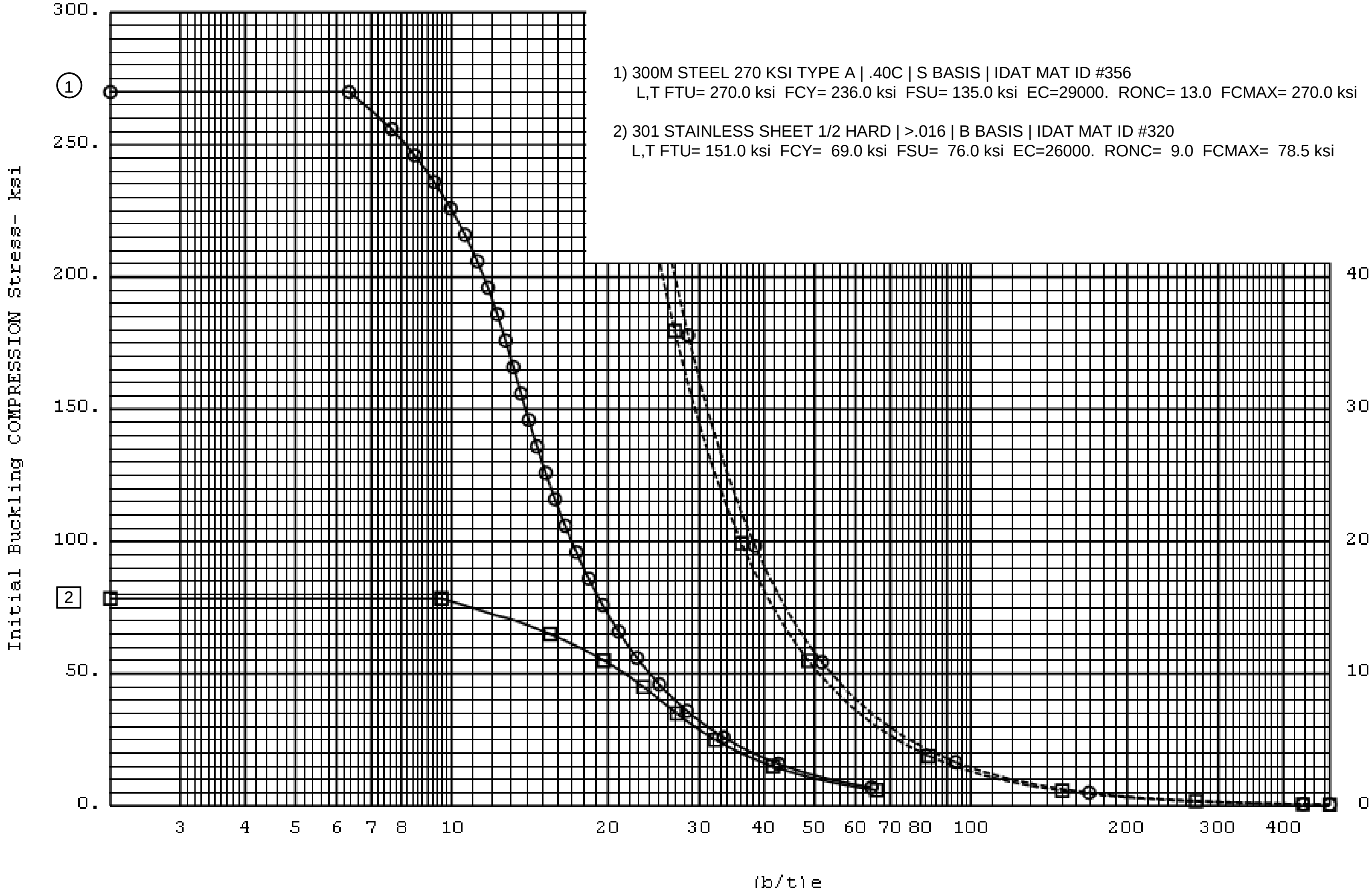


Figure 10.3.1-8: Initial Buckling Compression Stress for Steel Plates

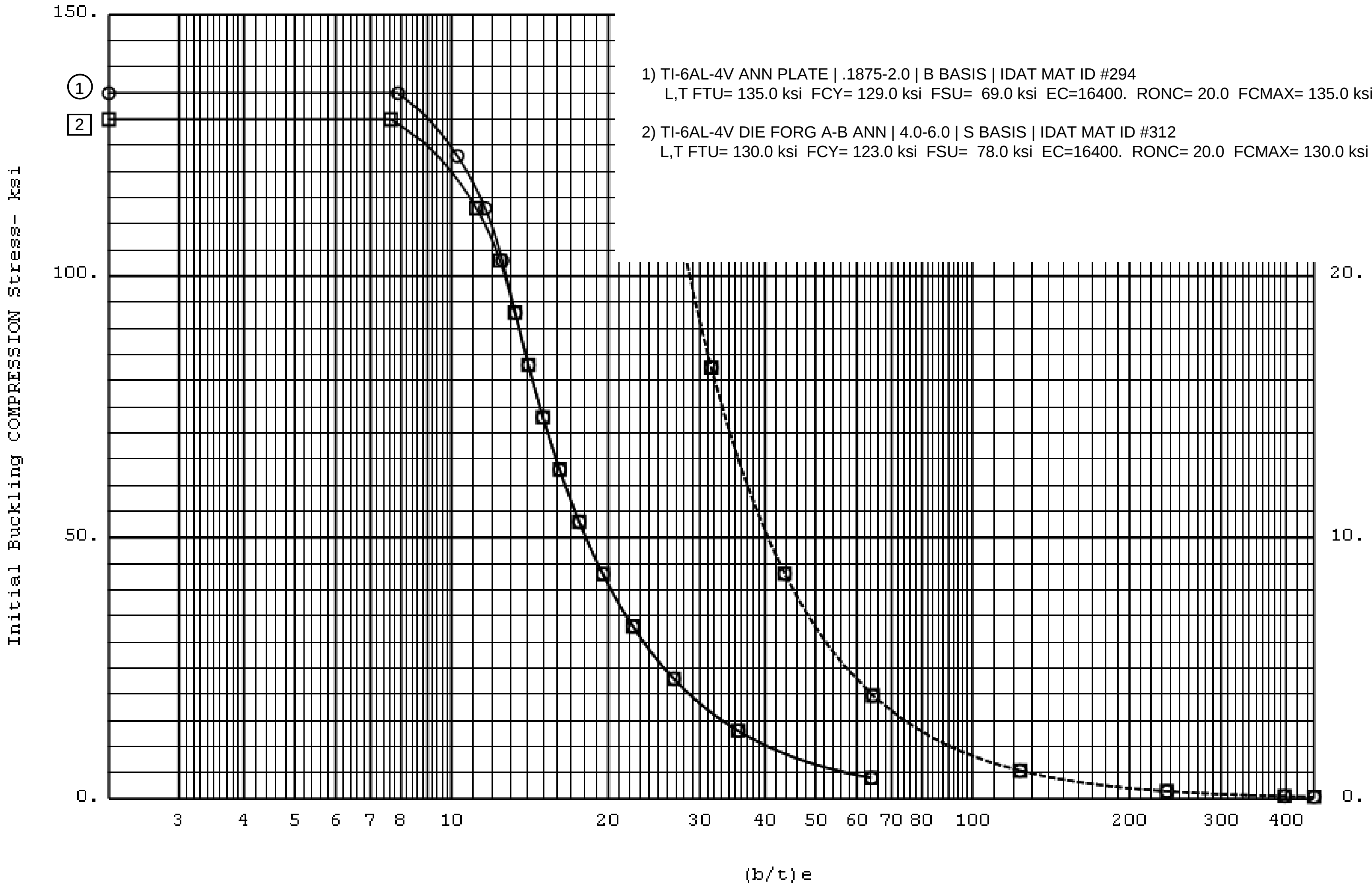


Figure 10.3.1-9: Initial Buckling Compression Stress for Titanium Plates

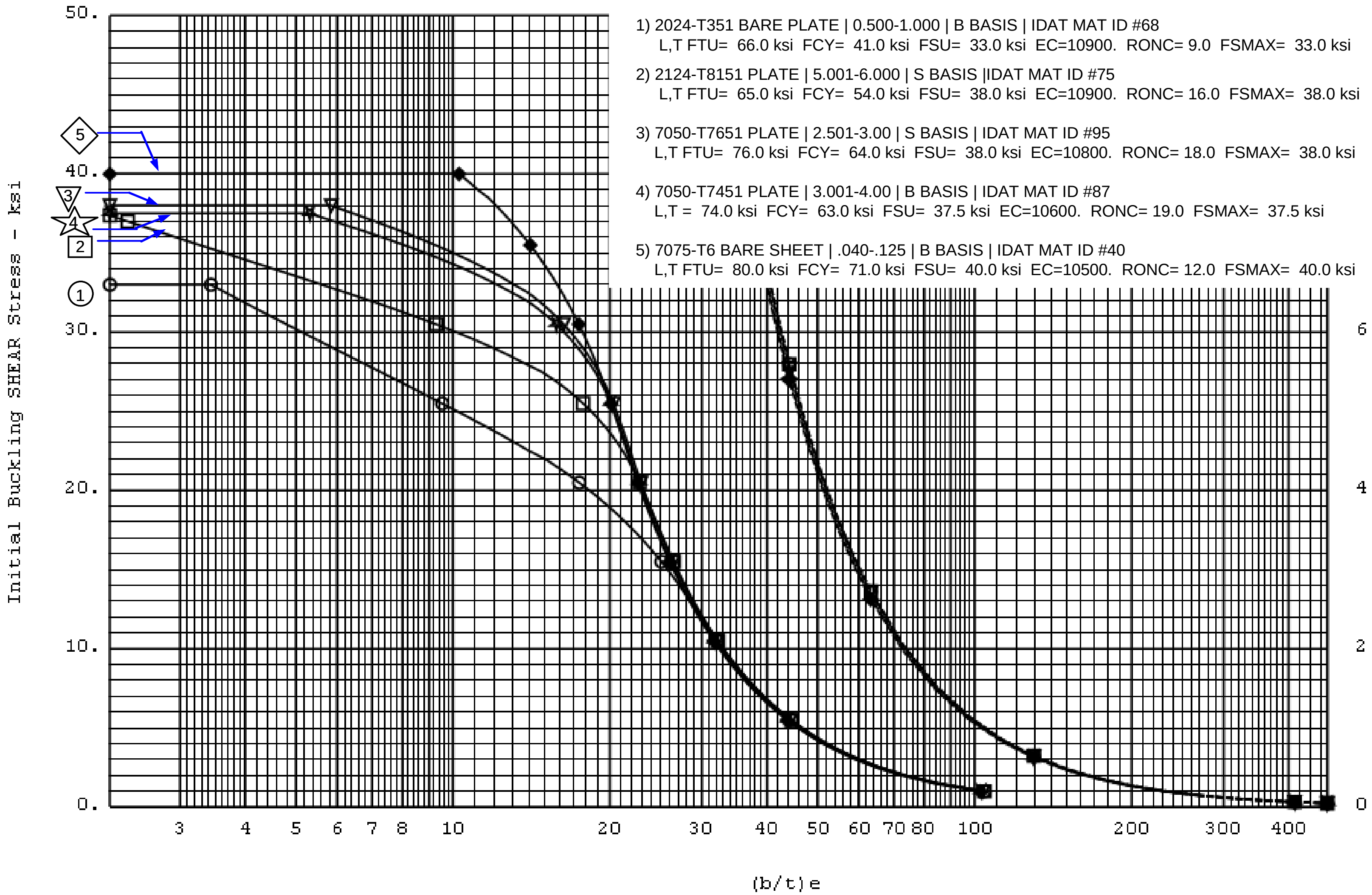


Figure 10.3.1-10: Initial Buckling Shear Stress for Aluminum Plates

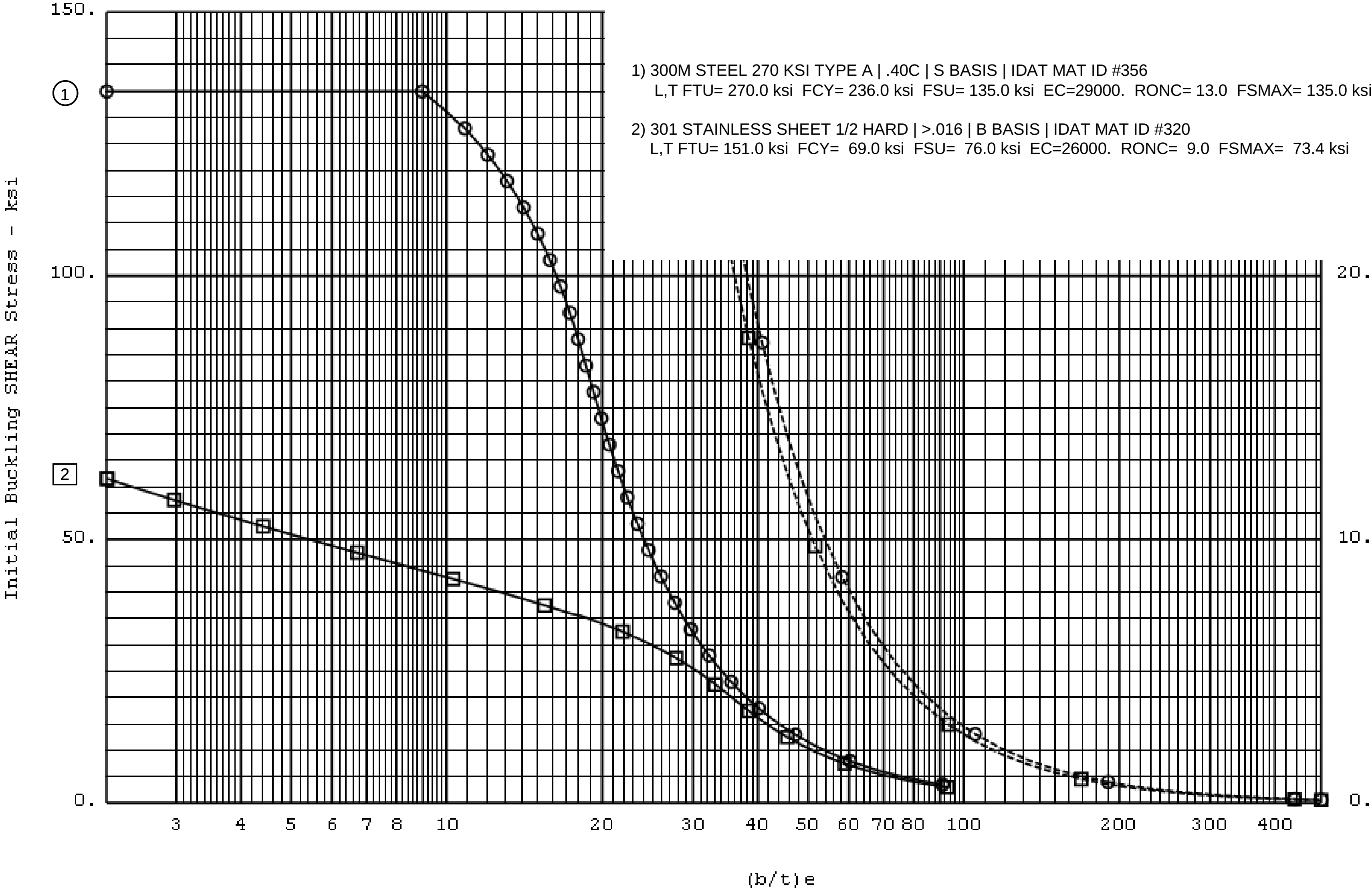


Figure 10.3.1-11: Initial Buckling Shear Stress for Steel Plates

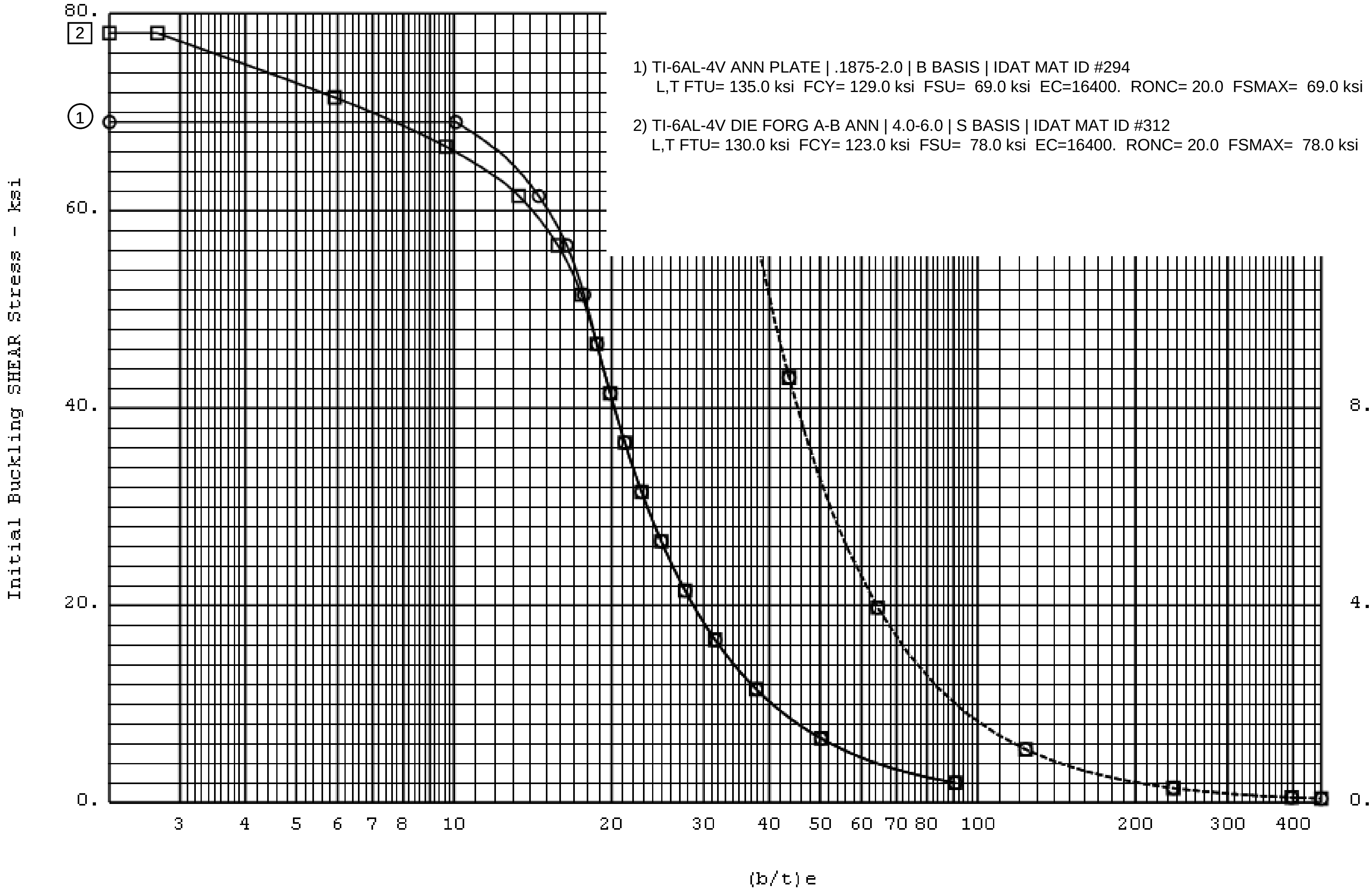


Figure 10.3.1-12: Initial Buckling Shear Stress for Titanium Plates

10.3.1.7 Plate Buckling Allowable Stress

The procedure involves determining the value of k, the buckling coefficient defined in Equation 10.3.1-2, for a given a/b ratio and loading. It is to be noted that dimension b is the loaded edge unless indicated otherwise. For a given k, Poisson's ratio ν_e and thickness t, $(b/t)_e$ can be computed as per Equation 10.3.1-4. The initial buckling stress can then be read from an appropriate graph for a computed $(b/t)_e$. The procedure outlined below offers the step-by-step process to obtain the initial buckling stress for a given flat plate.

1. From the plate dimensions, a and b, determine k for a loading and boundary conditions. Note dimension b is the loaded edge unless indicated otherwise
2. Compute K as per Equation 10.3.1-5 for a given k and Poisson's ratio ν_e
3. Compute $(b/t)_e$ as per Equation 10.3.1-4
4. Compute the initial buckling stress as discussed in section 10.3.1.6 either by solving Equation 10.3.1-10 or by using curves generated by running program SM33 of IDAT. If using curves, read the initial buckling stress f_{cr} for $(b/t)_e$ computed in step 3. For a compression loading and materials listed in Table 10.3.1-2, use Figure 10.3.1-7 to Figure 10.3.1-9 . For a shear loading and materials listed in Table 10.3.1-2 use Figure 10.3.1-10 to Figure 10.3.1-12. The buckling stress so obtained has all the material corrections required in the yield region if the stress is more than the proportional limit of the material

10.3.2 Buckling of Flat Rectangular Plates

Webs of structural members that are modeled as flat rectangular plates for analysis may be subjected to a single load or a combination of loads and may have different edge support conditions. For such plates the buckling coefficient, k will depend upon the type of loading and support boundary conditions. Buckling coefficient, k curves are presented in this section for different loading and boundary conditions. With known k, the initial buckling stress is calculated by using the procedure of section 10.3.1.7.

10.3.2.1 Uniaxial Compression

For rectangular plates that carry uniaxial compressive load, the buckling coefficient, k is defined as k_c by Equation 10.3.2-1

$$k = k_c$$ **Equation 10.3.2-1**

For uniaxial compression, use the buckling coefficient as defined by Equation 10.3.2-1 in computing the effective buckling coefficient, K, defined by Equation 10.3.1-5.

Curves of the buckling coefficient, k_c, for plates that have loaded edges that are simply supported but the other edges have different support conditions, are presented in Figure 10.3.2-1. The asymptotic values for each plate configuration are also depicted in the figure.

Figure 10.3.2-2 presents curves of the buckling coefficient, k_c, for plates that have loaded edges that are clamped but the other edges have different support conditions. The asymptotic values for each plate case are also depicted in the figure.

Figure 10.3.2-3 presents curves of the buckling coefficient, k_c, for wide plates with different support conditions for the loaded edges and unloaded edges. For wide plates, $(b/t)_e$ defined in Equation 10.3.1-4 is modified as per Equation 10.3.2-2 to reflect the correct plate dimension to use for computing the initial buckling stress.

$$\left(\frac{b}{t}\right)_e = \left(\frac{a}{t\sqrt{K}}\right) \qquad \textbf{Equation 10.3.2-2}$$

This modified value of $(b/t)_e$ is used in Equation 10.3.1-10 to compute the initial buckling stress.

Figure 10.3.2-4 presents buckling coefficients, k_c, for flat rectangular plates with one loaded edge free and the other edges simply supported.

Figure 10.3.2-5 presents buckling coefficients, k_c, for flat rectangular plates with one loaded edge fixed and the other loaded edge free. The remaining edges are simply supported.

For flat rectangular plate columns having simply supported and clamped ends, Figure 10.3.2-6, k_c is determined as per Equation 10.3.2-3.

$$k_c = Ck' \qquad \textbf{Equation 10.3.2-3}$$

Where,
k_c is the buckling coefficient for uniaxial compression
k' is the buckling coefficient shown in Figure 10.3.2-6
C is the fixity coefficient. For a simply supported plate column, C= 1 and for a fixed end plate column, C = 4

For plate columns, the effective buckling coefficient, K, is calculated as per Equation 10.3.2-4.

$$K = \frac{k_c \pi^2}{12\left(1-\nu_e^2\right)}\left(\frac{b}{a}\right)^2 \qquad \textbf{Equation 10.3.2-4}$$

With known k and K, the initial buckling stress is calculated by using the procedure of section 10.3.1.7.

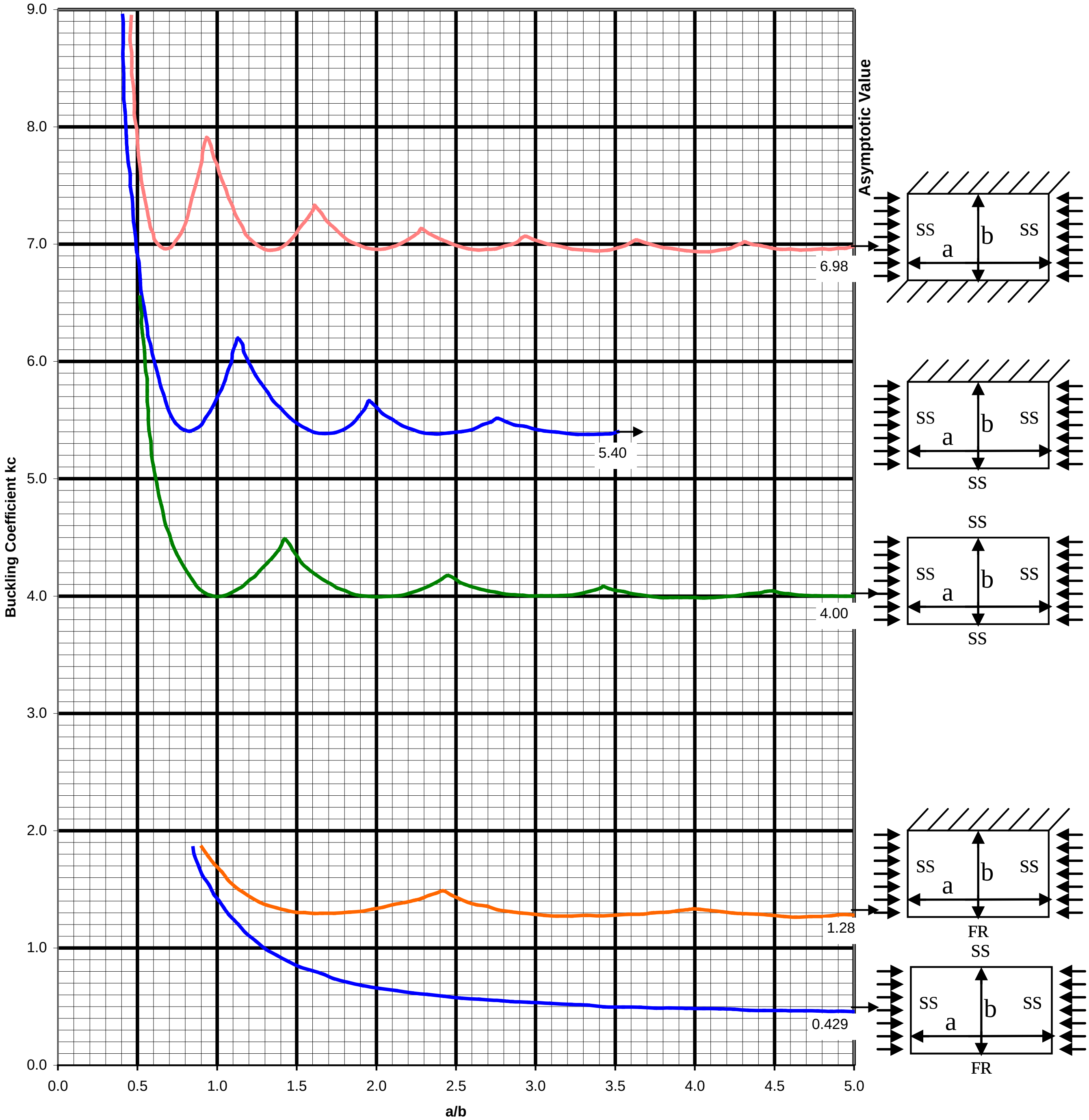


Figure 10.3.2-1: Uniaxial Compression Buckling Coefficients for Flat Rectangular Plates with Simply Supported Loaded Edges

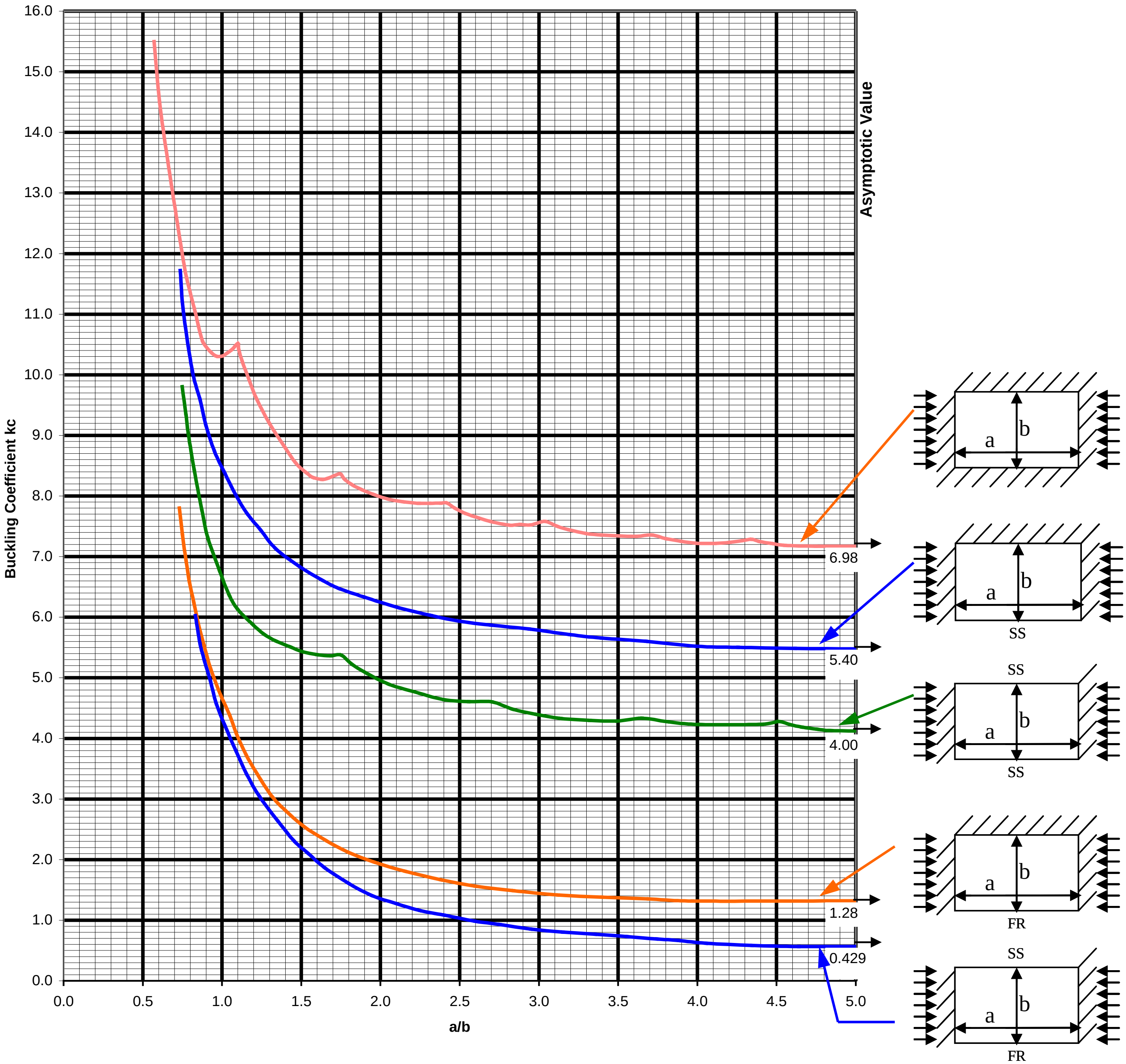


Figure 10.3.2-2: Uniaxial Compression Buckling Coefficients for Flat Rectangular Plates with Clamped Loaded Edges

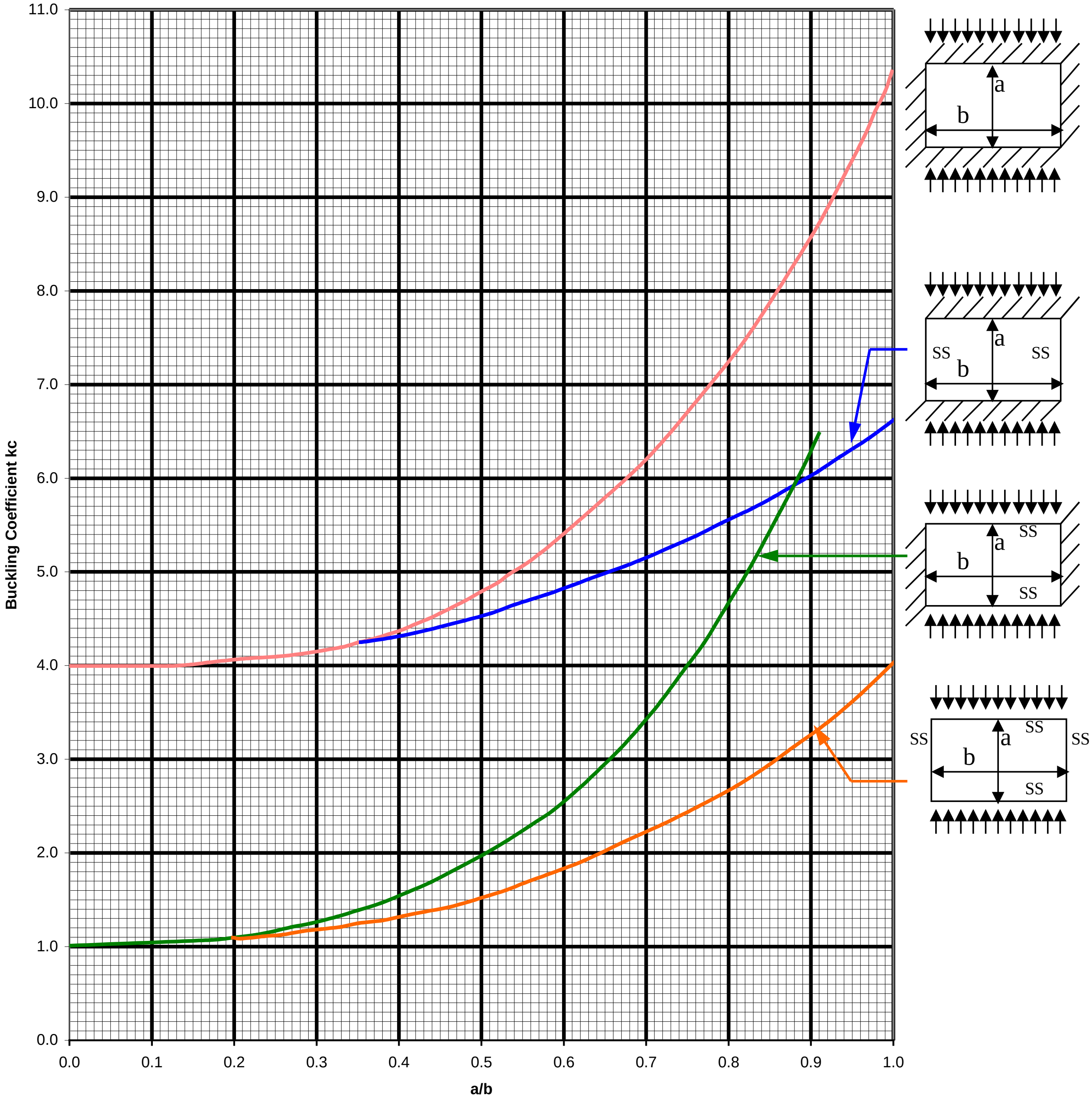


Figure 10.3.2-3: Uniaxial Compression Buckling Coefficients for Wide Flat Rectangular Plates

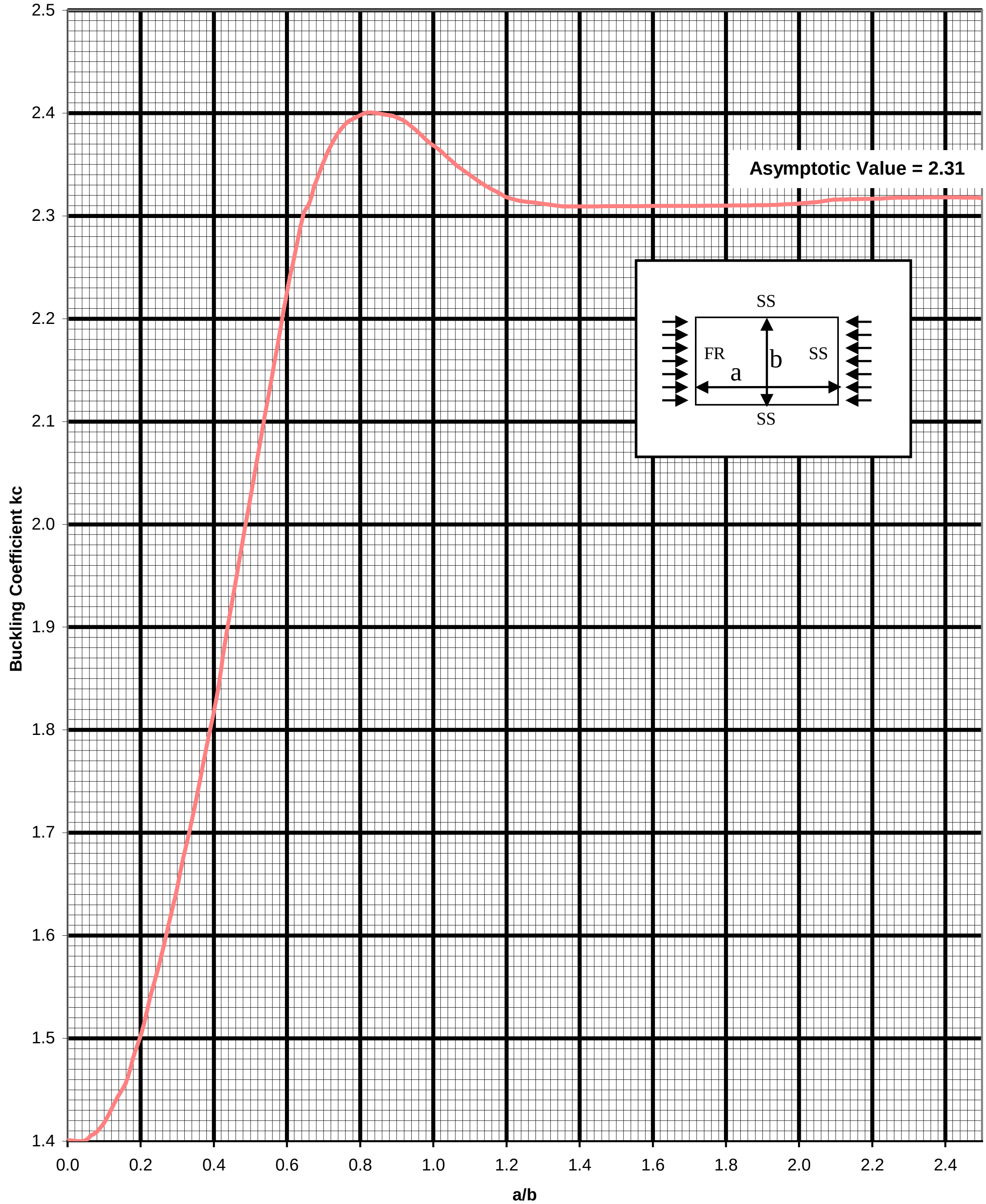


Figure 10.3.2-4: Uniaxial Compression Buckling Coefficient for Flat Rectangular Plates with One Loaded Edge Free and Other Edges Simply Supported

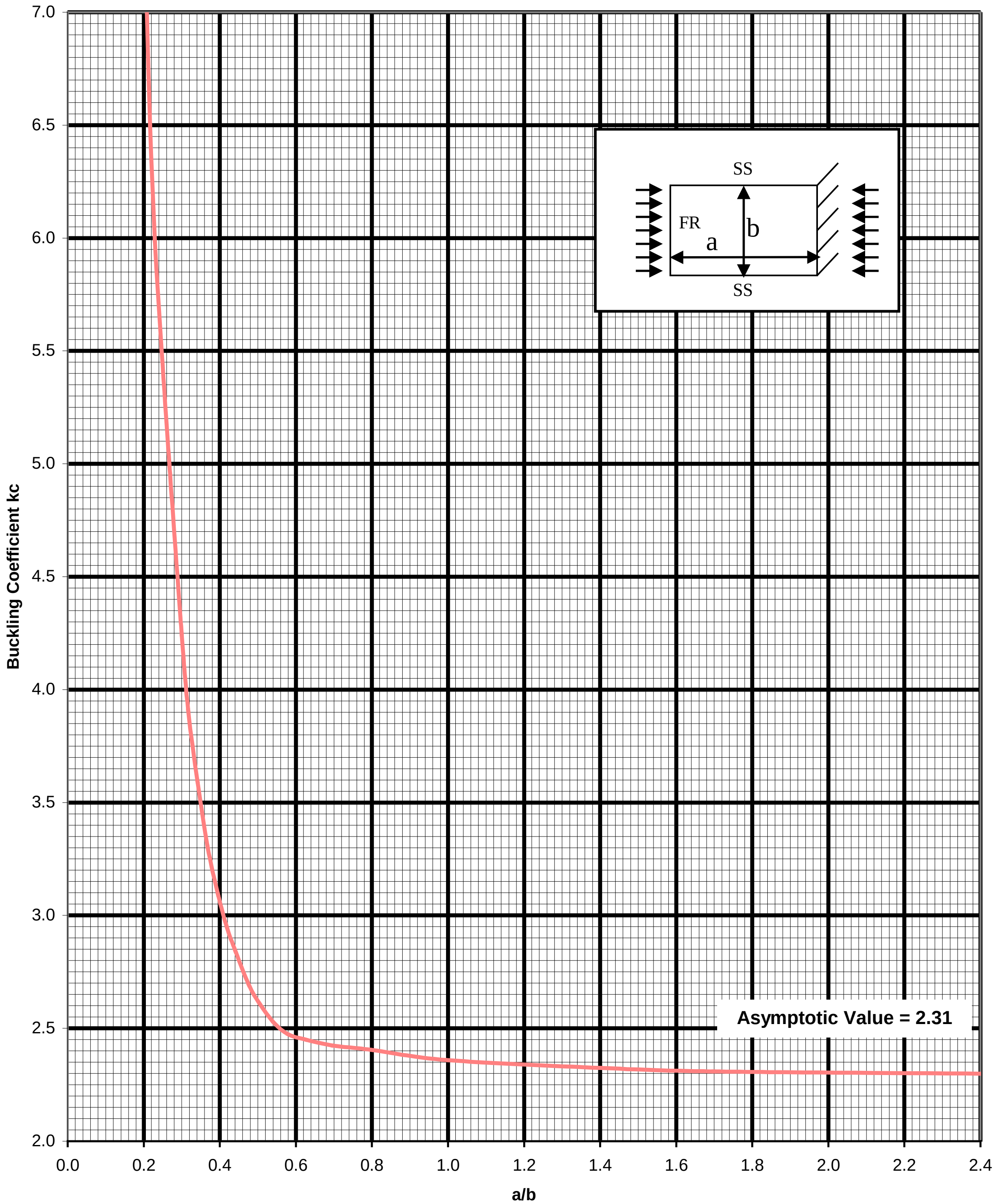


Figure 10.3.2-5: Uniaxial Compression Buckling Coefficient for Flat Rectangular Plate with One Loaded Edge Free and Other Loaded Edge Fixed

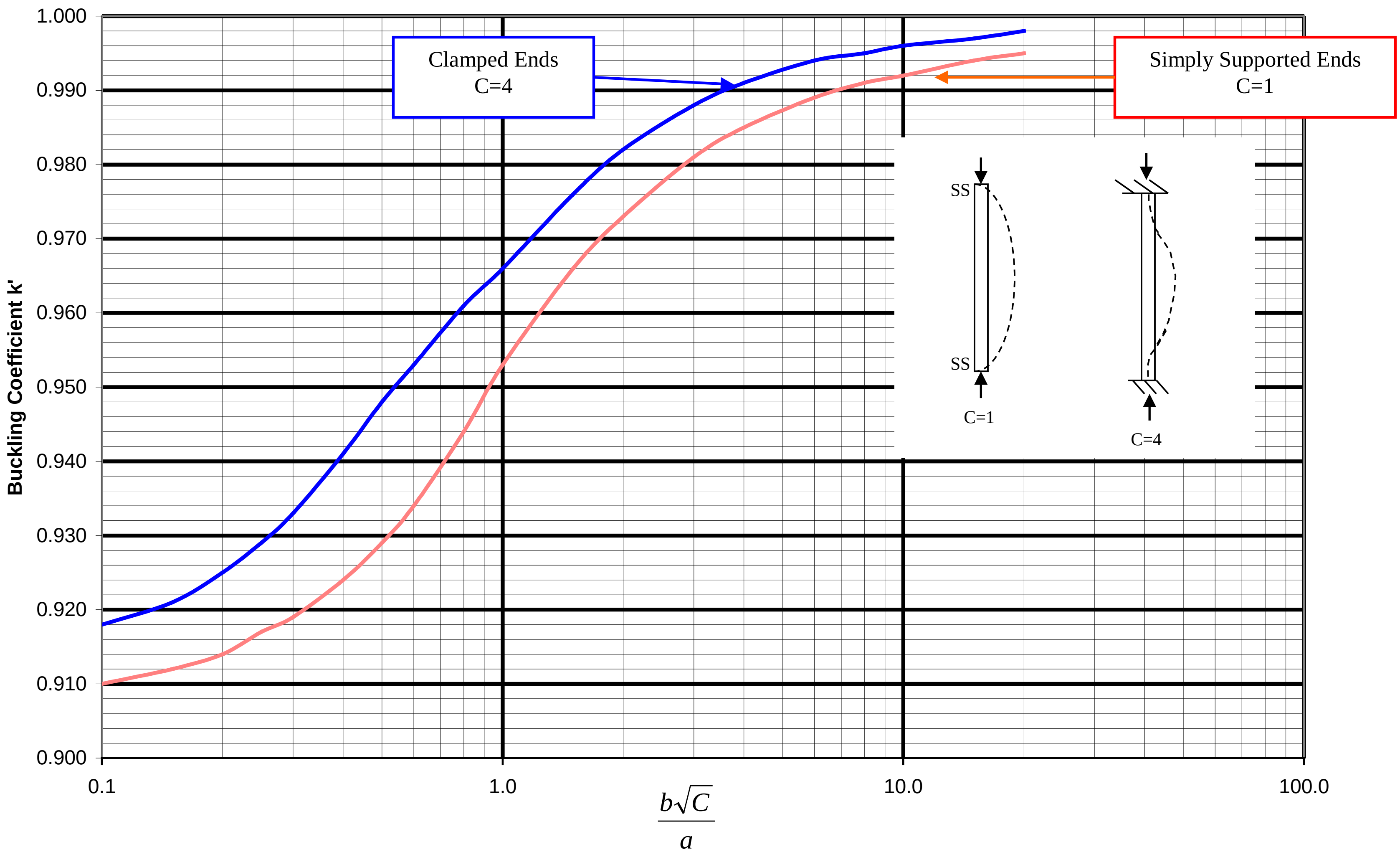


Figure 10.3.2-6: Uniaxial Compression Buckling Coefficients for Flat Rectangular Plate Columns

10.3.2.2 Shear

For flat rectangular plates that carry shear load, the buckling coefficient, k, is defined as k_s by Equation 10.3.2-5.

$$k = k_s$$ **Equation 10.3.2-5**

For shear, use the buckling coefficient as defined by Equation 10.3.2-5 in computing the effective buckling coefficient, K, defined by Equation 10.3.1-5. With known k and K, the initial buckling stress is calculated by using the procedure of section 10.3.1.7 either by solving Equation 10.3.1-10 or using the shear curves generated by program SM33 of IDAT as discussed in section 10.3.1.6.

Figure 10.3.2-7 presents curves of the shear buckling coefficient, k_s, for flat rectangular plates that have different support conditions.

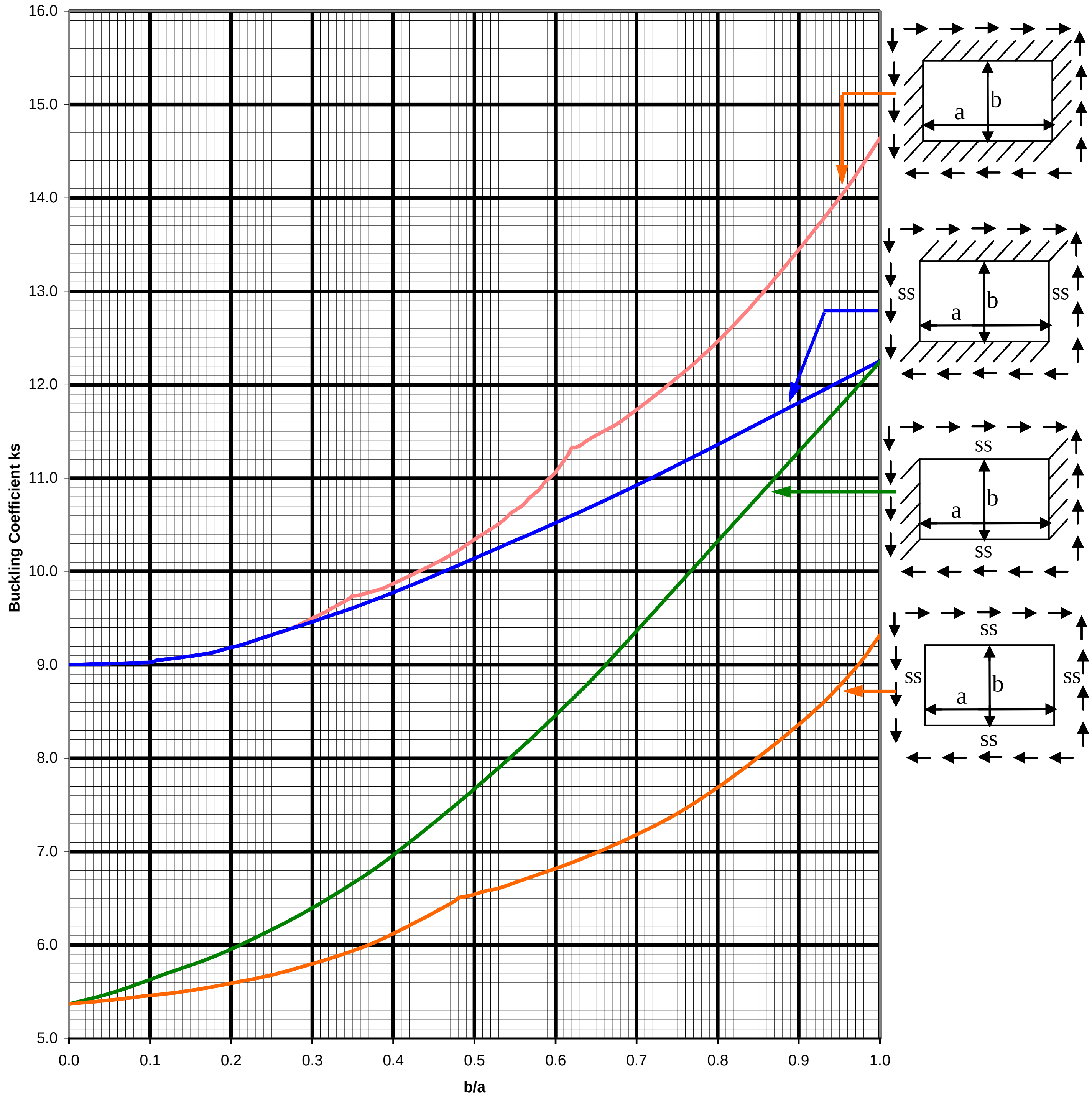


Figure 10.3.2-7: Shear Buckling Coefficients for Flat Rectangular Plates

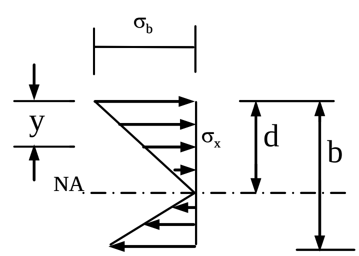
10.3.2.3 Bending

For flat rectangular plates that are subjected to in-plane bending, the buckling coefficient, k, is defined as k_b by Equation 10.3.2-6.

$$k = k_b$$ **Equation 10.3.2-6**

For bending, use the buckling coefficient as defined by Equation 10.3.2-6 in computing the effective buckling coefficient, K, defined by Equation 10.3.1-5. With known k and K, the initial buckling stress is calculated by using the procedure of section 10.3.1.7 either by solving Equation 10.3.1-10 or by using compression curves generated by program SM33 of IDAT as discussed in section 10.3.1.6.

Figures 10.3.2-8 to 10.3.2-13 present bending buckling coefficients for flat rectangular plates having different edge conditions. The edge conditions for each case are depicted in the figures. The plate stress distribution is given by Equation 10.3.2-7 and is shown below.



$$\sigma_x = \sigma_b \left[1 - \phi \frac{y}{b} \right]$$ **Equation 10.3.2-7**

Where,
σ_x is the stress at distance y from the edge (psi)
σ_b is the maximum compressive stress at the edge (psi)
y is the distance from the compressive edge (in)
b is the length of loaded edge (in)
d is the distance from the compression edge to the neutral axis
$\phi = b/d$, Note when $\phi = 2$, the plate is under pure bending
$\phi = 0$, the plate is under uniform compression
$0 < \phi < 2$, the plate is under combined bending and compression

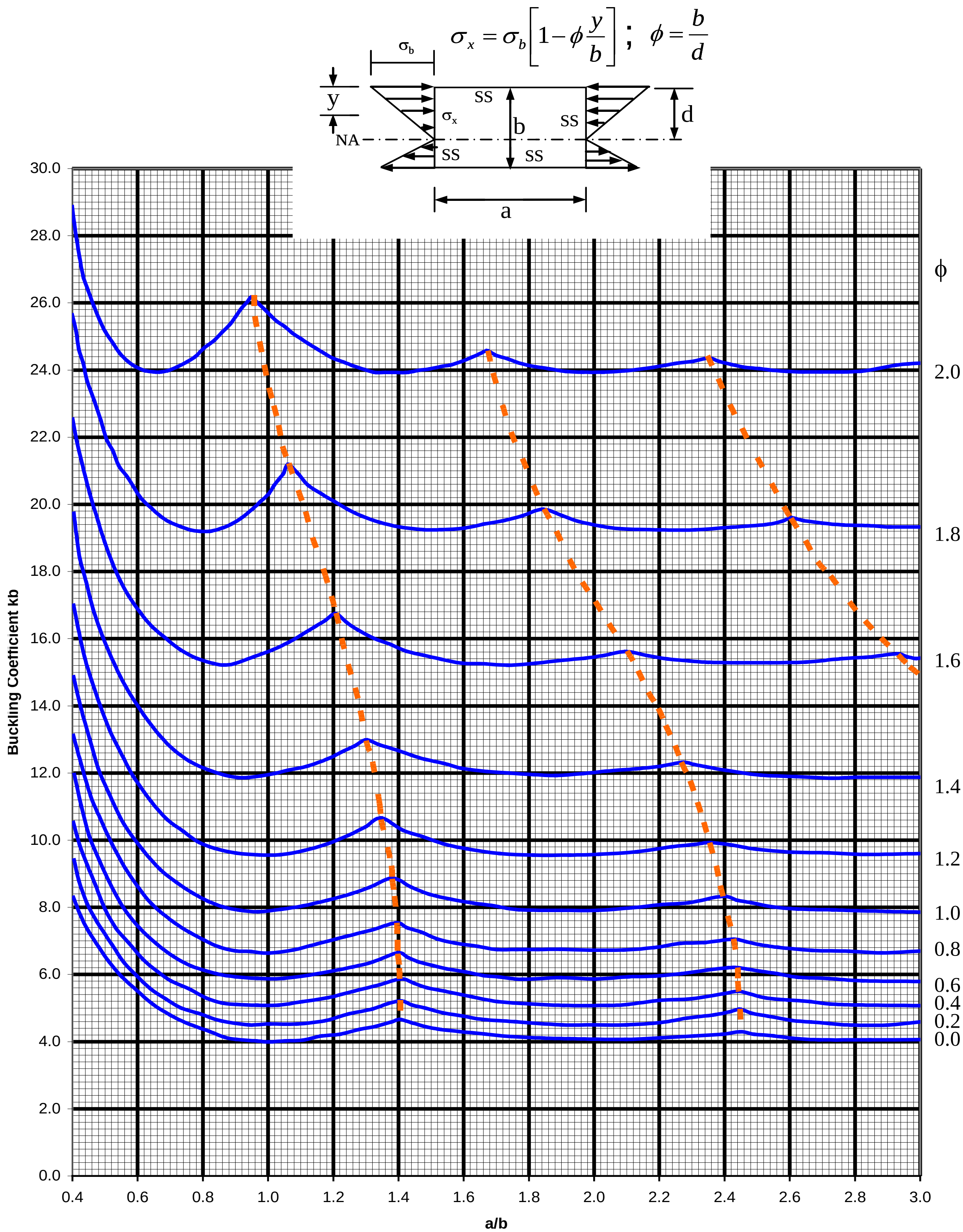


Figure 10.3.2-8: Bending Buckling Coefficients for Flat Rectangular Plates with all Edges Simply Supported for a/b $\leq$ 3.0

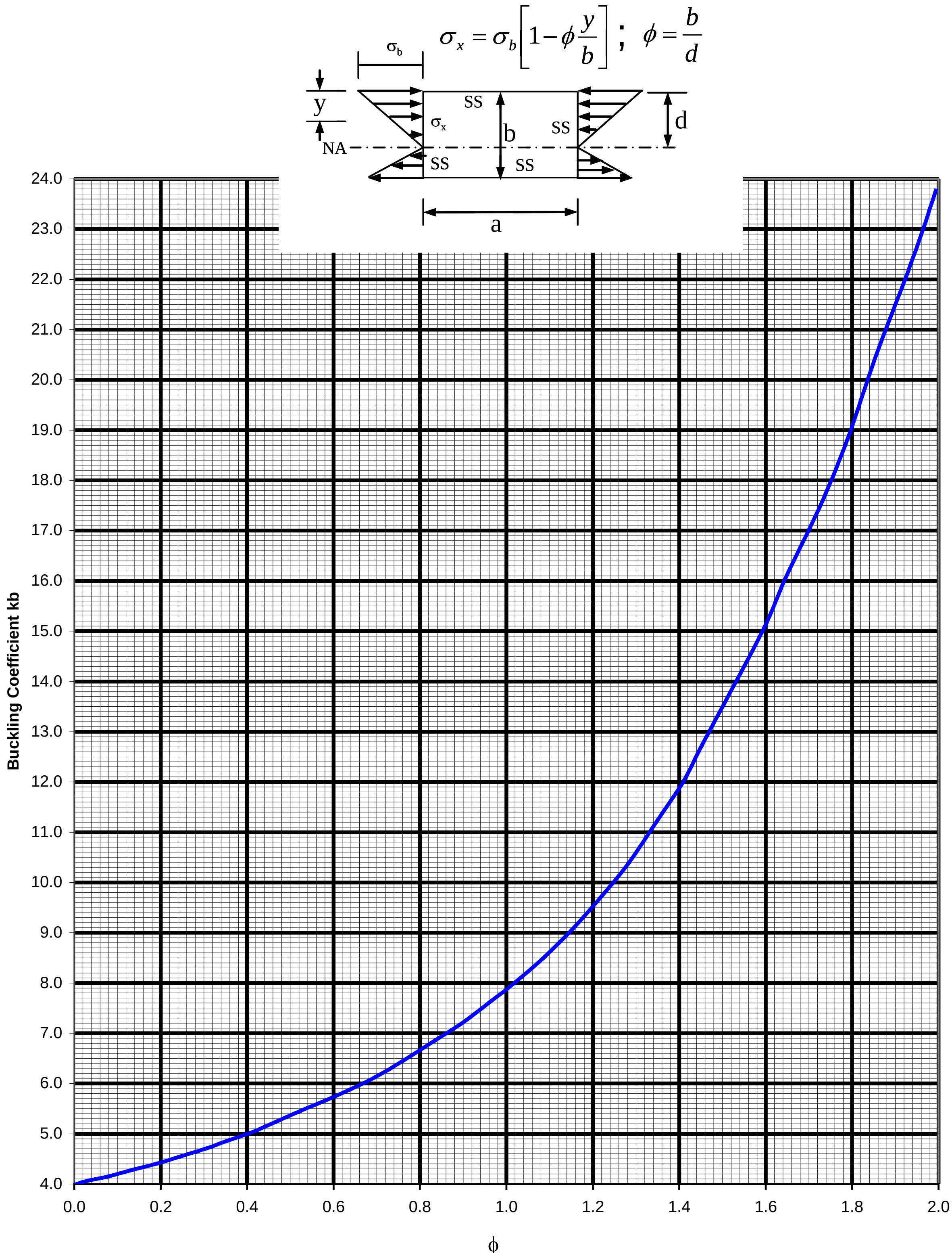


Figure 10.3.2-9: Bending Buckling Coefficient for Flat Rectangular Plate with all Simply Supported Edges for a/b ≥ 4.0

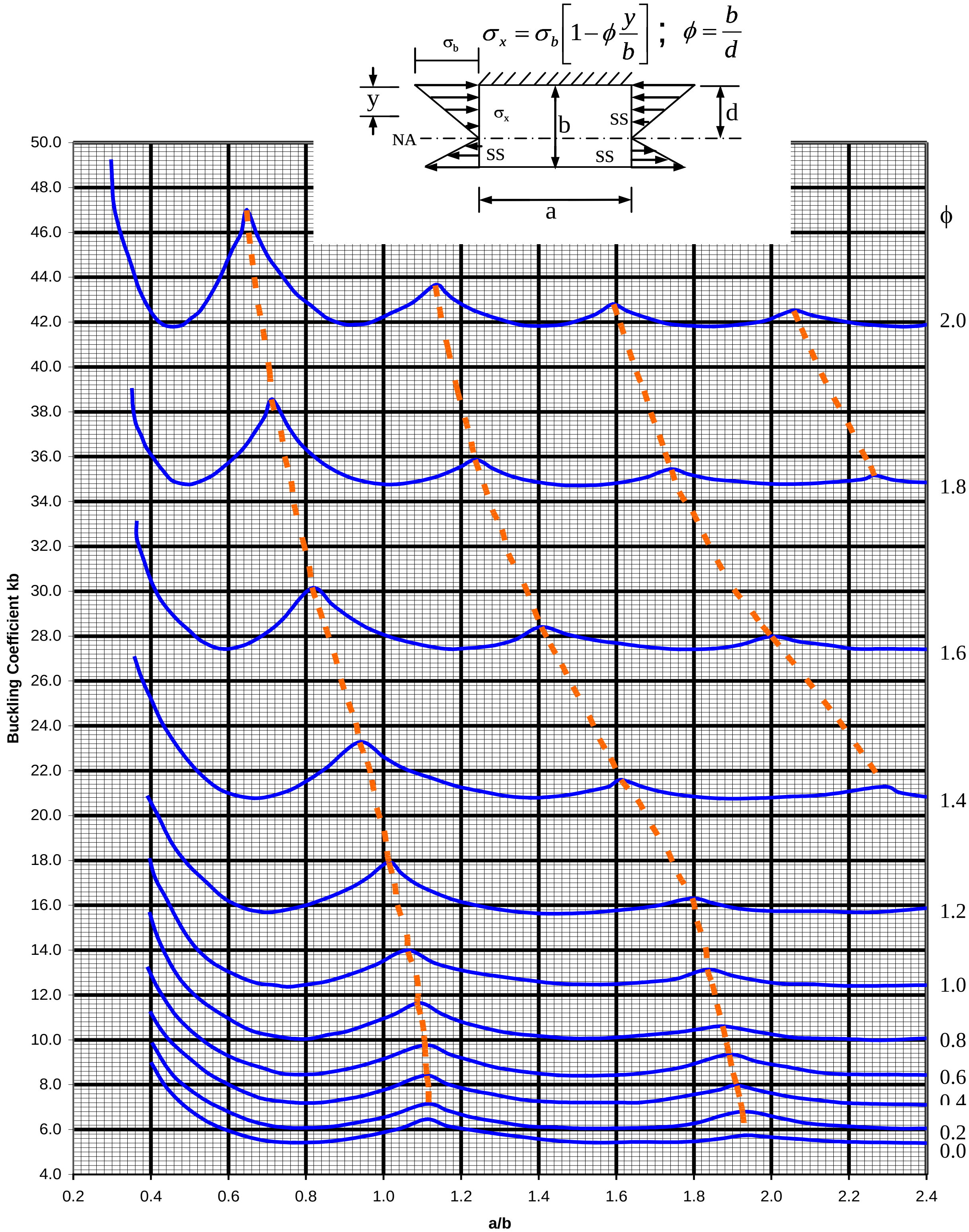


Figure 10.3.2-10: Bending Buckling Coefficients for Flat Rectangular Plates Having Compression Edge Fixed and other Edges Simply Supported for a/b ≤ 2.4

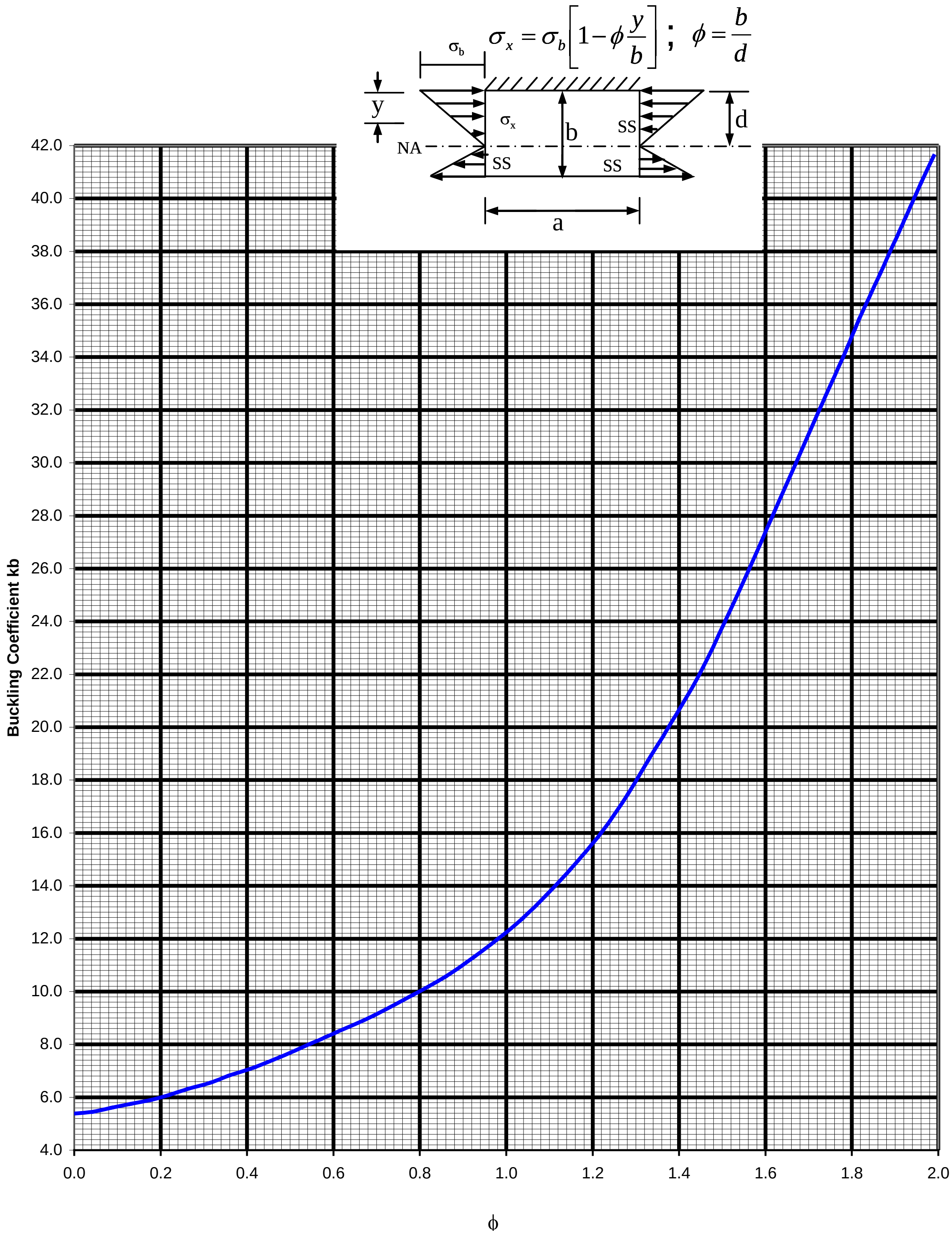


Figure 10.3.2-11: Bending Buckling Coefficient for Flat Rectangular Plate Having Compression Edge Fixed and other Edges Simply Supported for a/b ≥ 2.4

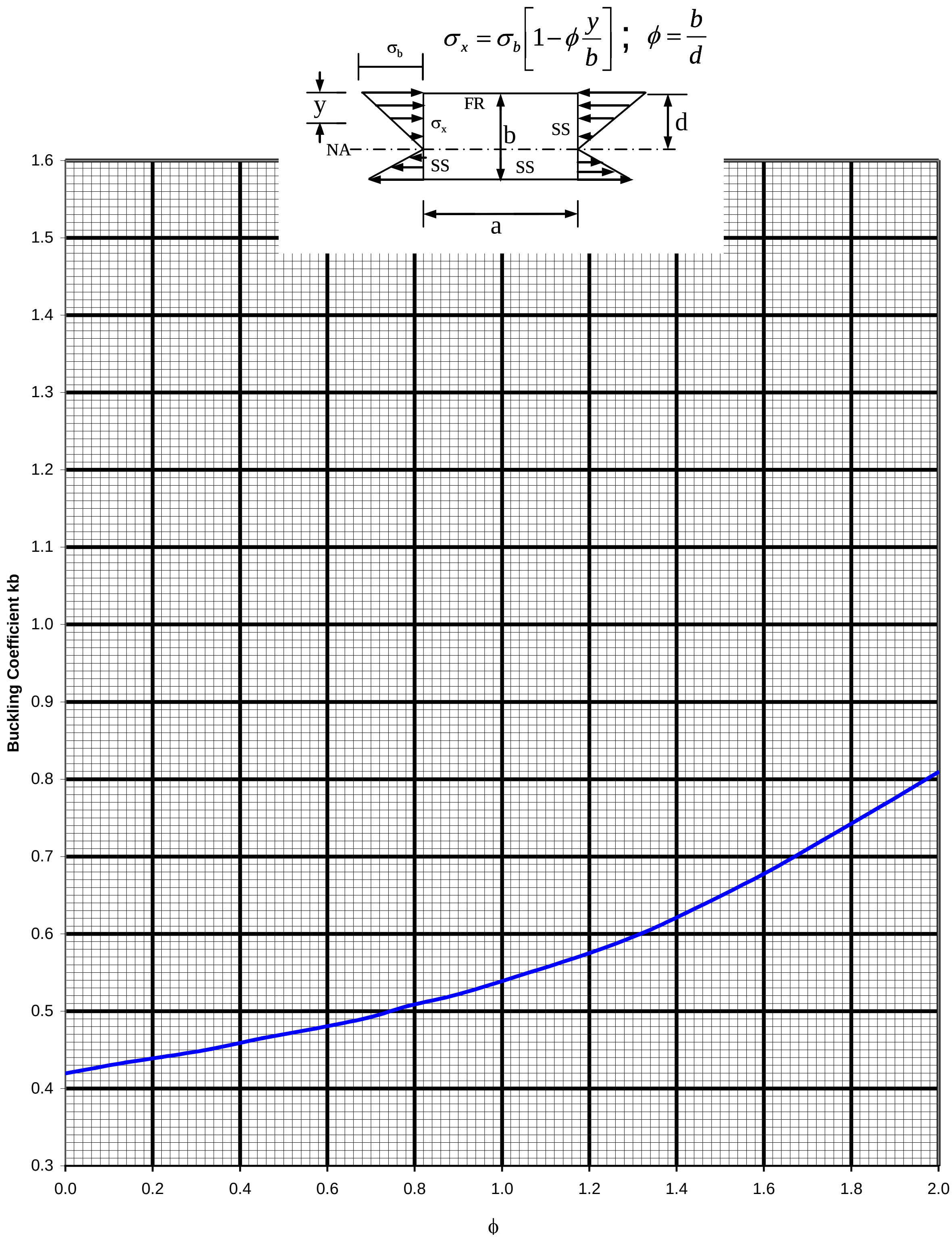


Figure 10.3.2-12: Bending Buckling Coefficient for Flat Rectangular Plate Having Compression Edge Free and other Edges Simply Supported for a/b ≥ 4.0

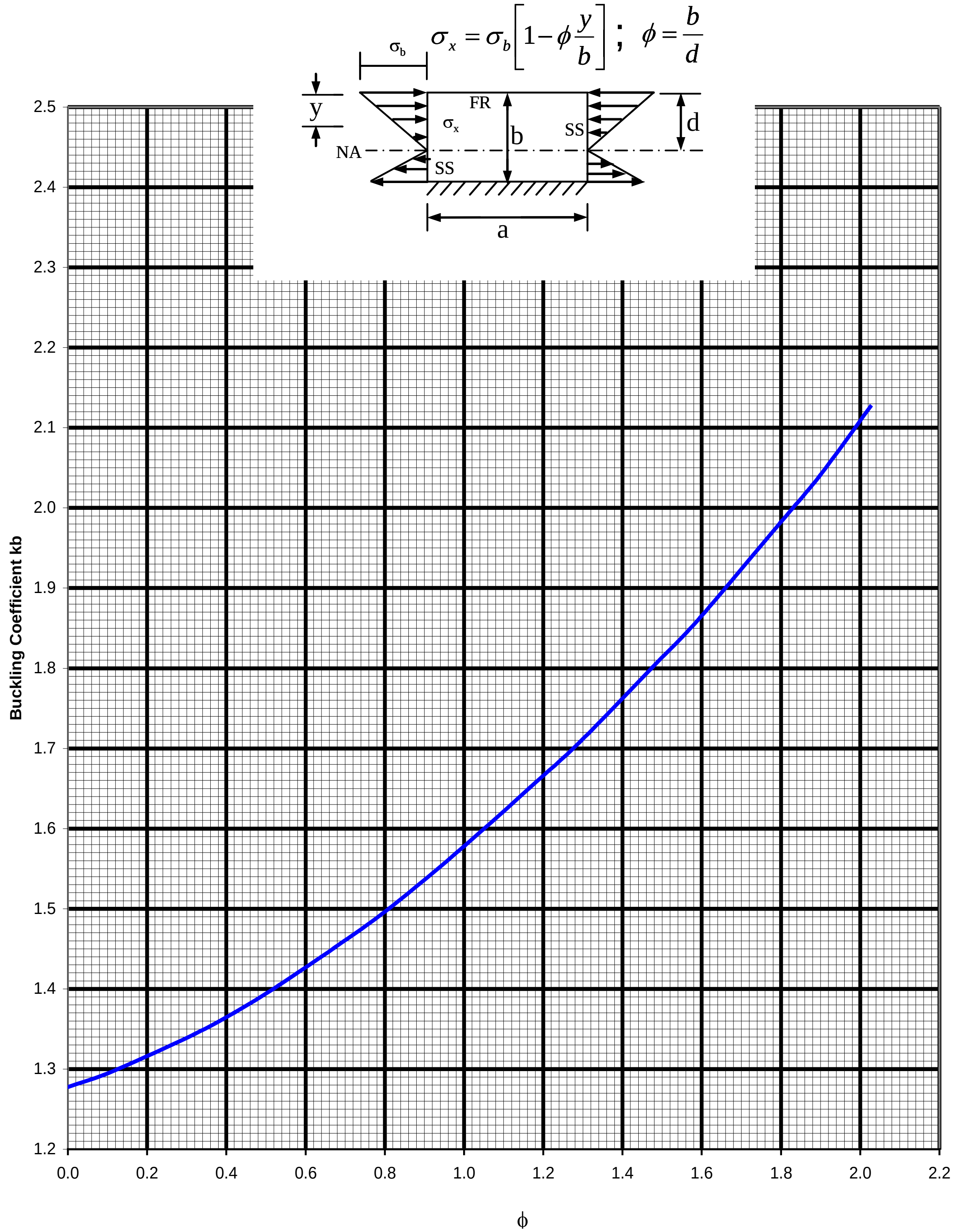
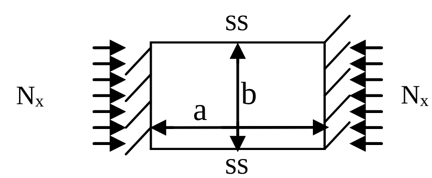


Figure 10.3.2-13: Bending Buckling Coefficient for Flat Rectangular Plate Having Compression Edge Free, Tension Edge Fixed and Loaded Edges Simply Supported for a/b ≥ 2.0

10.3.2.4 Example - Web Longeron Compression Buckling Check

Given: A 7075-T6 (bare sheet) web of a longeron is analyzed as a flat rectangular plate. Two of the edges where compressive load is applied are considered as fixed. Other edges are considered as simply supported. The dimensions of the web model are:
$t = 0.125$ in, $a = 9.0$ in, $b = 3.0$ in
$N_x = 7200$ lb/in

Calculate the margin of safety in buckling.



Solution: The margin of safety can be calculated either by using the allowable stress chart generated by using SM33 program of IDAT or solving the Equation 10.3.1-10 iteratively. Here the solution is presented by using allowable initial buckling stress chart of Figure 10.3.1-7.

Material properties for 7075-T6 (bare sheet, 0.04 – 0.125) are obtained from Table 10.3.1-2 (developed from METDB) or can be obtained directly from the METDB database of IDAT.

$F_{tu} = 80$ ksi, $F_{cy} = 71.0$ ksi, $F_{su} = 40.0$ ksi, $E_c = 10500$ ksi, $n_c = 12.0$, $\nu_e = 0.33$ (Poisson's ratio obtained from METDB of IDAT)

Solution: Using the Allowable Stress Chart

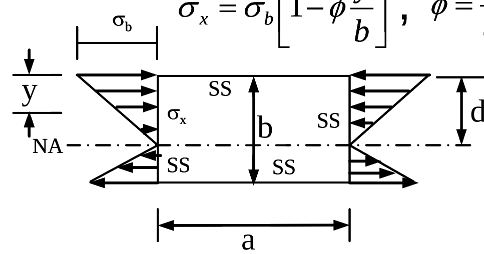
Calculation	**Equation/ Figure**	**Result**
Obtain buckling coefficient, k_c from the uniaxial compression chart	From Figure 10.3.2-2 For $a/b = 9.0/3.0 = 3.0$ $k_c = 4.4$	$k_c = 4.4$
Buckling coefficient, k	$k = k_c$ Equation 10.3.2-1 $k = 4.4$	$k = 4.4$
Calculate effective buckling coefficient, K	$K = k\,\pi^2 / [12(1-\nu_e^2)]$ Equation 10.3.1-5 $= (4.4)(\pi^2) / [12(1 - 0.33^2)] = 4.0611$	$K = 4.0611$
Calculate $(b/t)_e$	$(b/t)_e = b/(t\sqrt{K})$ Equation 10.3.1-4 $= 3.0 / [0.125(\sqrt{4.0611})] = 11.909$	$(b/t)_e = 11.9$
Read initial buckling stress	From Figure 10.3.1-7 for $(b/t)_e = 11.9$ $f_{cr} = 63$ ksi	$f_{cr} = 63$ ksi
Allowable load	$P_{all} = f_{cr}\,(bt)$ $= 63[(3.0)(0.125)](1000) = 23625$ lb	$P_{all} = 23625$ lb
Applied load	$P = N_x\,b$ $= 7200(3.0) = 21600$ lb	$P = 21600$ lb
Margin of Safety	$MS = P_{all} / P - 1$ Equation 2.5.0-1 $= 23625 / 21600 - 1 = +0.09$	$MS = +0.09$
Discussion: If the initial buckling stress is calculated by assuming that the stress is linear up to buckling, *i.e.*, no plasticity is involved, then the margin of safety will be unconservative.		
Linear initial buckling stress	From Equation 10.3.1-3 $f_{cre} = k\,\pi^2 / [12(1-\nu_e^2)]\,(E_c)\,(t/b)^2$ $f_{cre} = 4.4(\pi^2)/[12(1-0.33^2)]\,(10.5\text{x}10^3)(0.125/3.0)^2$ $f_{cre} = 74$ ksi	$f_{cre} = 74$ ksi
Margin of Safety	$MS = 74(1000)/(7200/0.125) - 1 = +0.28$	$MS = +0.28$ (unconservative)
Note: The linear initial buckling stress, 74 ksi, is greater than the compressive proportional limit and the yield stresses, which implies that the plate is in the plastic state, thus the 0.28 margin of safety based on the linear initial buckling stress is unconservative. The actual margin of safety, which accounts for plasticity effects, is 0.09.		

10.3.2.5 Example - Bending Buckling Check

Given: A 7075-T6 (bare sheet) liner is provided to shield the heat producing equipment. The liner is modeled as a flat rectangular plate for analysis purposes. All the edges are considered as simply supported. The dimensions of the plate model are:
t = 0.080 in, a= 9.0 in, b= 3.0 in, d = 2 in
σ_b = 65 ksi

Calculate the margin of safety in bending buckling.

$$\sigma_x = \sigma_b\left[1-\phi\frac{y}{b}\right]; \quad \phi=\frac{b}{d}$$



Solution: The plate is subjected to in-plane bending. As discussed in section 10.3.2.3 the buckling of the plate can be analyzed either by using the allowable stress chart generated by using SM33 program of IDAT or solving the Equation 10.3.1-10 iteratively. Since all the edges are simply supported, the buckling coefficient is determined from **Figure 10.3.2-8**.

Material properties for 7075-T6 (bare sheet, 0.04 - 0.125) are obtained from Table 10.3.1-2 (developed from METDB) or can be obtained directly from the METDB database of IDAT.

F_{tu} = 80 ksi, F_{cy} = 71.0 ksi, F_{su} = 40.0 ksi, E_c = 10500 ksi, n_c = 12.0, ν_e = 0.33 (Poisson's ratio obtained from METDB of IDAT)

Solution: Using the Allowable Stress Chart

Calculation	**Equation/ Figure**	**Result**
Obtain buckling coefficient, k_b from the bending chart	From **Figure 10.3.2-8** For a/b = 9.0/3.0 = 3.0 and ϕ = 3/2 = 1.5 k_b = 13.60 (interpolated between ϕ = 1.4 and 1.6)	k_b = 13.60
Buckling coefficient, k	$k = k_b$ Equation 10.3.2-6 k = 13.60	k = 13.60
Calculate effective buckling coefficient, K	$K = k\,\pi^2 / [12(1-\nu_e^2)]$ Equation 10.3.1-5 $= 13.60(\pi^2) / [12(1 - 0.33^2)] = 12.5525$	K = 12.5525
Calculate $(b/t)_e$	$(b/t)_e = b/(t\sqrt{K})$ Equation 10.3.1-4 $= 3.0 / [0.08(\sqrt{12.5525})] = 10.58$	$(b/t)_e$ = 10.60
Read initial buckling stress	From Figure 10.3.1-7 for $(b/t)_e$ = 10.60 f_{cr} = 68 ksi	f_{cr} = 68 ksi
Allowable maximum load intensity	$P_{all} = f_{cr}\,t$ = (68)(0.080)(1000) = 5440 lb/in	P_{all} = 5440 lb/in
Applied load intensity	$P = \sigma_b\,t$ = (65)(0.08)(1000) = 5200 lb/in	P = 5200 lb/in
Margin of Safety	$MS = P_{all} / P - 1$ Equation 2.5.0-1 = 5440 / 5200 – 1 = + 0.05	MS = + 0.05

Discussion: Assume that the plate is in pure bending, *i.e.*, ϕ = 2 (or d = b/2), then the buckling coefficient, k_b, is 24.2 from **Figure 10.3.2-8**.

Buckling coefficient, k	$k = k_b$ Equation 10.3.2-6 k = 24.20	k = 24.20
Calculate effective buckling coefficient, K	$K = k\,\pi^2 / [12(1-\nu_e^2)]$ Equation 10.3.1-5 $= 24.20(\pi^2) / [12(1 - 0.33^2)] = 22.3361$	K = 22.3361
Calculate $(b/t)_e$	$(b/t)_e = b/(t\sqrt{K})$ Equation 10.3.1-4 $= 3.0 / [0.08(\sqrt{22.3361})] = 7.9346$	$(b/t)_e$ = 7.9

Calculation	Equation/ Figure	Result
Read initial buckling stress	From Figure 10.3.1-7 for $(b/t)_e$ = 7.9 f_{cr} = 78 ksi	f_{cr} = 78 ksi
Allowable maximum load intensity	$P_{all} = f_{cr}\ t$ = (78)(0.080)(1000) = 6240 lb/in	P_{all} = 6240 lb/in
Applied load intensity	$P = \sigma_b\ t$ = (65)(0.08)(1000) = 5200 lb/in	P = 5200 lb/in
Margin of Safety	$MS = P_{all} / P - 1$ Equation 2.5.0-1 = 6240 / 5200 – 1 = + 0.20	MS = + 0.20

10.3.2.6 Special Case, Flat Panel in Compression with a Single Row of Attachments between Sheet and Stiffener

Figure 10.3.2-14 presents the curve for the buckling coefficient, k_c, for flat panels that are joined to stringers with a single row of attachments. These flat panels are modeled as flat rectangular plates for analysis and the panels are subjected to uniaxial compressive load that is parallel to the stiffeners.

The buckling coefficient, k, is defined as k_c as per Equation 10.3.2-1. Equation 10.3.1-5 is used to calculate the effective buckling coefficient, K, for a given value of k. It is to be noted that $(b/t)_e$ as defined in Equation 10.3.1-4 is computed by taking b as the distance between the stiffeners and then the initial buckling stress is calculated by using the procedure of section 10.3.1.7.

10.3.2.7 Special Case, Flat Panel in Shear with a Single Row of Attachments between Sheet and Stiffener

Figure 10.3.2-15 presents the buckling coefficient, k_s curves for flat panels for different (b/t) ratios that are joined to stringers with a single row of attachments. These flat panels are modeled as flat rectangular plates for analysis and are subjected to shear load.

The buckling coefficient k is defined as k_s as per Equation 10.3.2-5. Equation 10.3.1-5 is used to calculate the effective buckling coefficient, K for a given value of k. Plate dimension b is the short side of the panel that is used for computing $(b/t)_e$ as defined by Equation 10.3.1-4. With known k and K, the initial buckling stress is calculated by using the procedure of section 10.3.1.7 either by solving Equation 10.3.1-10 or using the shear curves generated by program SM33 of IDAT as discussed in section 10.3.1.6.

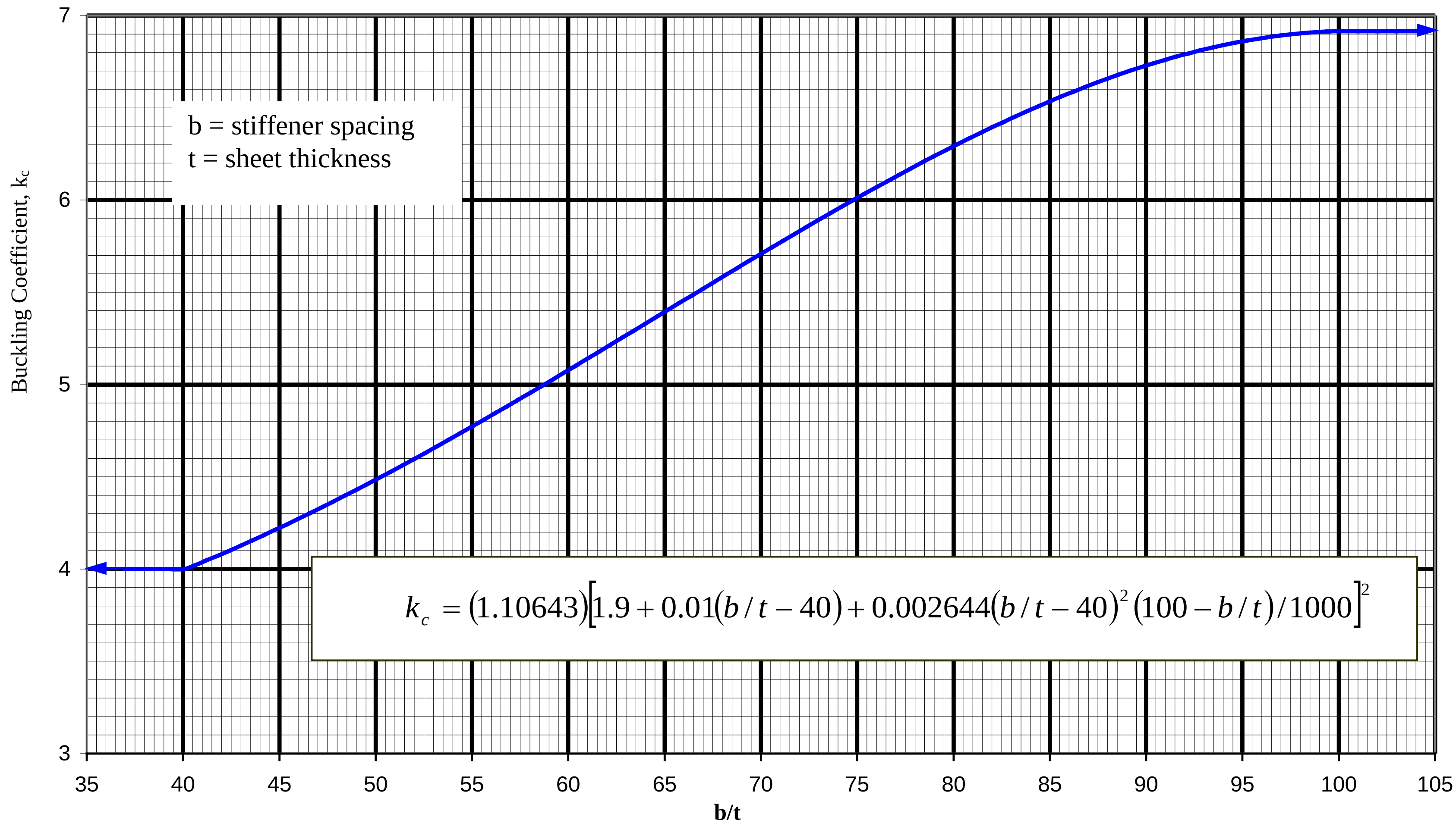


Figure 10.3.2-14: Special Case, Buckling Coefficient for Flat Panel in Compression with a Single Row of Attachments Between Sheet and Stiffener, Load Parallel to Stiffeners

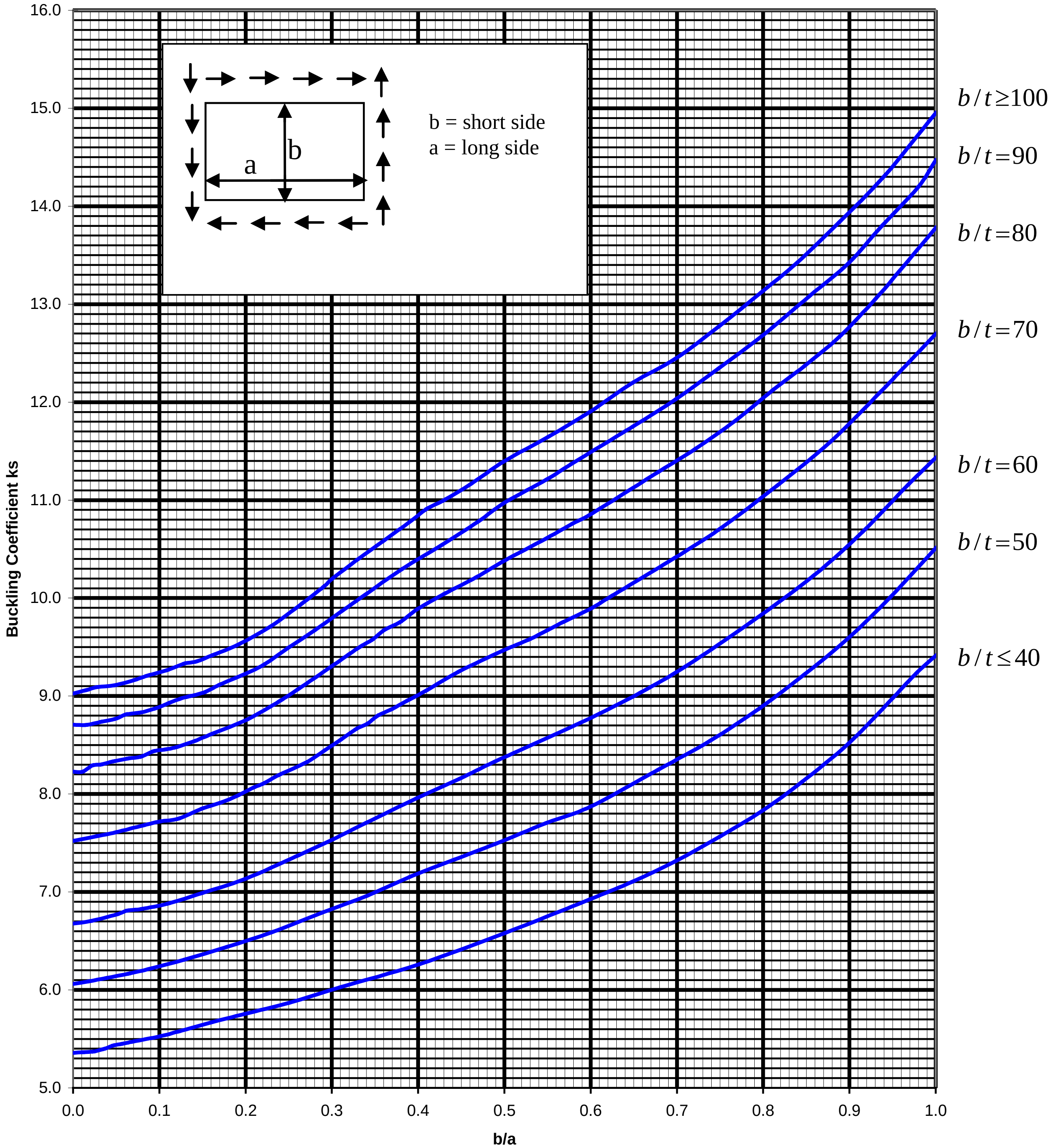
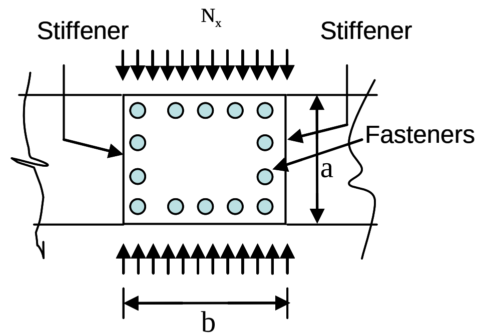


Figure 10.3.2-15: Special Case, Buckling Coefficient for Flat Panel in Shear with a Single Row of Attachments Between Sheet and Stiffener

10.3.2.8 Example – Panel in Between the Stiffeners

Given: A 7075-T6 (bare sheet) sheet is attached to stiffeners with a single row of fasteners. The sheet is to be checked for buckling due to in-plane compressive load as shown. The dimensions of the sheet are:
t = 0.125 in, a= 8.0 in, b= 10.0 in,
N_x = 1100 lb/in

Calculate the margin of safety in buckling.



Solution: The plate is subjected to compressive in-plane load, which is parallel to the stiffeners. This plate is analyzed using the special chart depicted in **Figure 10.3.2-14**. The chart provides the buckling coefficient for a given b/t ratio. It is to be noted that dimension b is the distance between the stiffeners.

Material properties for 7075-T6 (bare sheet, 0.04 - 0.125) are obtained from Table 10.3.1-2 (developed from METDB) or can be obtained directly from the METDB database of IDAT.

F_{tu} = 80 ksi, F_{cy} = 71.0 ksi, F_{su} = 40.0 ksi, E_c = 10500 ksi, n_c = 12.0, ν_e = 0.33 (Poisson's ratio obtained from METDB of IDAT)

Solution: Using the Allowable Stress Chart

Calculation	**Equation/ Figure**	**Result**
Obtain buckling coefficient, k_c from the special case chart	From **Figure 10.3.2-14** For b/t = 10.0/0.125 = 80.0 k_c = 6.3	k_c = 6.3
Buckling coefficient, k	$k = k_c$ Equation 10.3.2-1 k = 6.3	k = 6.3
Calculate effective buckling coefficient, K	$K = k\,\pi^2 / [12(1-\nu_e^2)]$ Equation 10.3.1-5 $= 6.3\,(\pi^2) / [12(1 - 0.33^2)] = 5.8148$	K = 5.8148
Calculate $(b/t)_e$	$(b/t)_e = b/(t\,\sqrt{K})$ Equation 10.3.1-4 $= 10.0 / [0.125(\sqrt{5.8148})] = 33.175$	$(b/t)_e$ = 33.2
Read initial buckling stress	From Figure 10.3.1-7 for $(b/t)_e$ = 33.2 f_{cr} = 9.4 ksi	f_{cr} = 9.4 ksi
Allowable load	$P_{all} = f_{cr}\,b\,t$ = (9.4)(10.0)(0.125)(1000) = 11750 lb	P_{all} = 11750 lb
Applied load	$P = N_x\,b$ = (1100)(10.0) = 11000 lb	P = 11000 lb
Margin of Safety	$MS = P_{all} / P - 1$ Equation 2.5.0-1 = 11750 / 11000 – 1 = + 0.07	MS = + 0.07

10.3.2.9 Shear and Transverse Compression or Tension

Interaction curves for simply supported flat rectangular plates of finite length loaded in combined shear and transverse compression or tension are presented in **Figure 10.3.2-16**. These curves were developed using energy methods by Batdorf and Stein (Reference 10-17) for different aspect ratios for simply supported plates. Interaction curves are for elastic stresses and the curves in the plastic regime are not available. However, the general practice in the aerospace industry is to use the elastic interaction curves for stresses in the plastic region. Note: if the aspect ratio, a/b, is less than 1, the curve for a/b = 1 is valid for all edge restraints ($0 < \varepsilon_r < \infty$) defined in Section 10.2.1.

Figure 10.3.2-17 presents interaction curves for infinitely long rectangular plates having simple supports on all four edges and clamped on all four edges. Interaction curves were developed from closed form solutions by Batdorf and Houbolt (Reference 10-18). These curves are for elastic stresses and the curves for plastic stresses are not available. In the absence of plastic interaction curves, the recommendation is to use elastic interaction curves.

Interaction curves presented in Sections 10.3.2.9 to 10.3.2.20 can be interpolated for a/b values other than those presented in the figures. Additionally, the curves and the equations are for initial plate buckling.

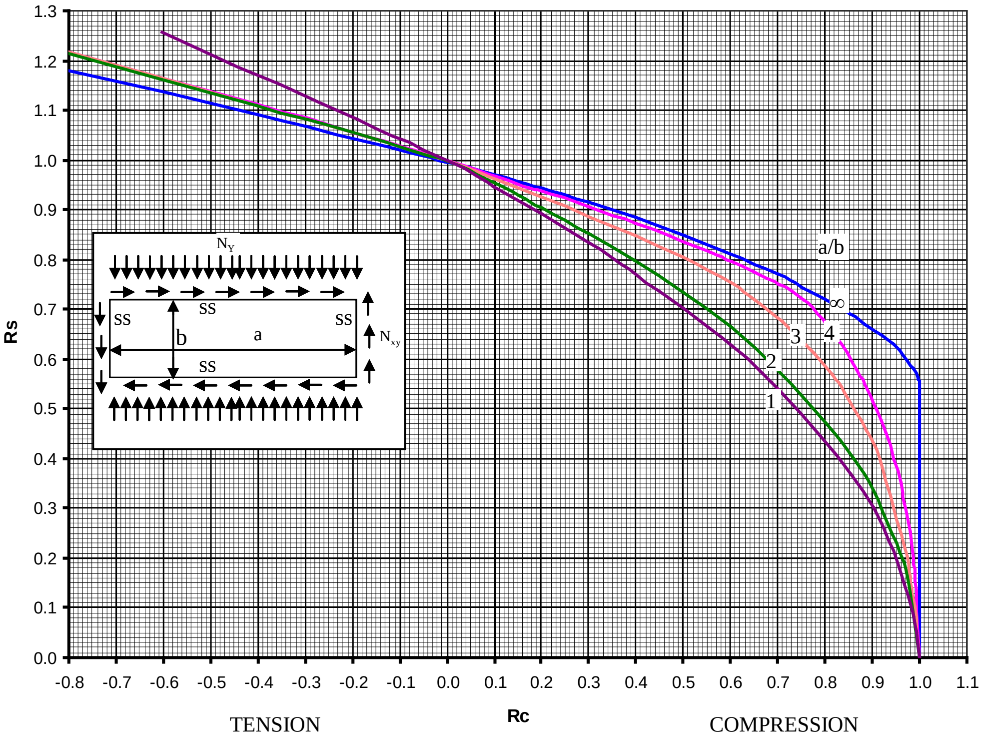


Figure 10.3.2-16: Interaction Curves for Simply Supported Plates of Finite Length Loaded in Transverse Compression or Tension and Shear

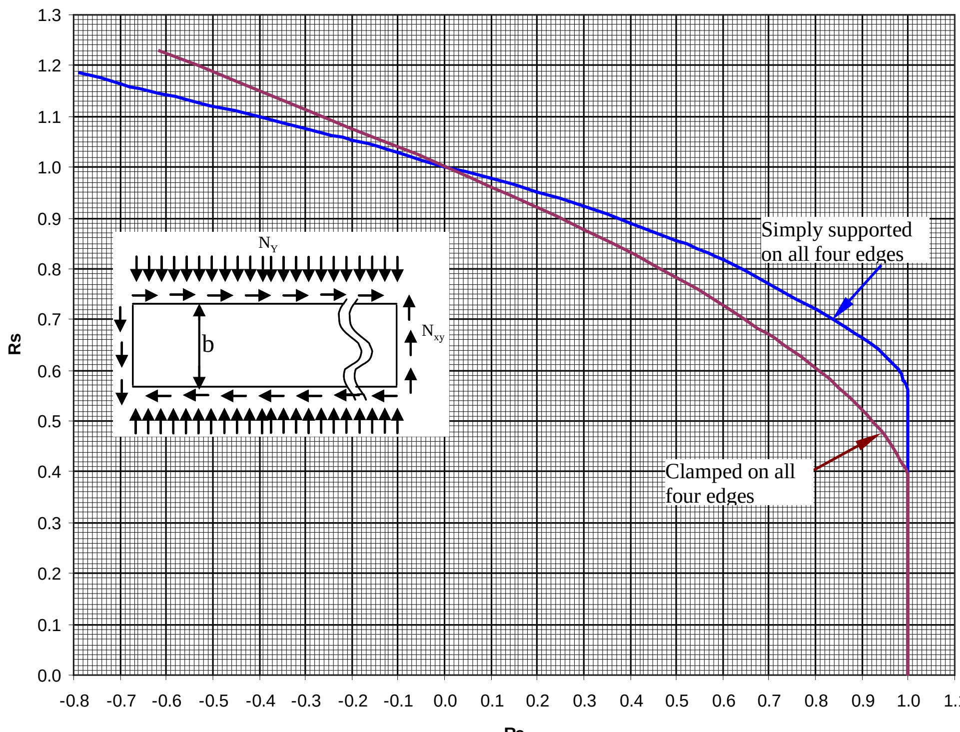


Figure 10.3.2-17: Interaction Curves for Infinitely Long Plate Loaded in Transverse Compression or Tension and Shear

10.3.2.10 Biaxial Compression or Compression and Tension

Figure 10.3.2-18 presents interaction curves for flat rectangular plates having all edges simply supported under biaxial compression or compression and tension loads. Figure 10.3.2-19 presents interaction curves for flat rectangular plates having all edges fixed under biaxial compression or compression and tension loads. These curves are obtained from Volume 1 of Reference 10-2. Interaction curves are for elastic stability and the same curves are recommended for the stresses in the plastic regime.

For the case when N_y is tension and $b < a$, the tension load is on the edge having length a. In order to use the curves, compute ratio a/b and reverse the coordinate axes by taking R_x as ordinate and R_y as abscissa. The reason for this digression is that the curves are generated on the assumption that ordinate of the curves represents compression and abscissa represents compression and tension. By reversing the coordinate axes, the correct R's will be obtained. This is illustrated by an Example 10.3.2.11.

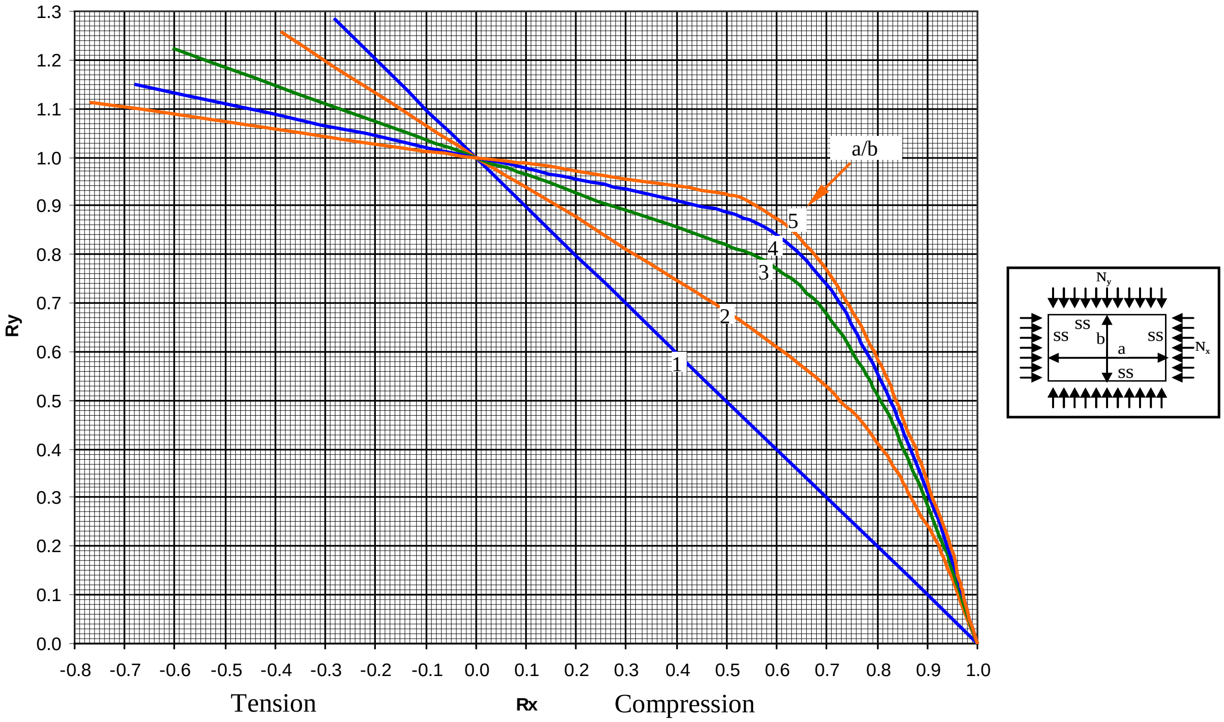


Figure 10.3.2-18: Interaction curves for Simply Supported Plates Subjected to Biaxial Compression or Compression and Tension

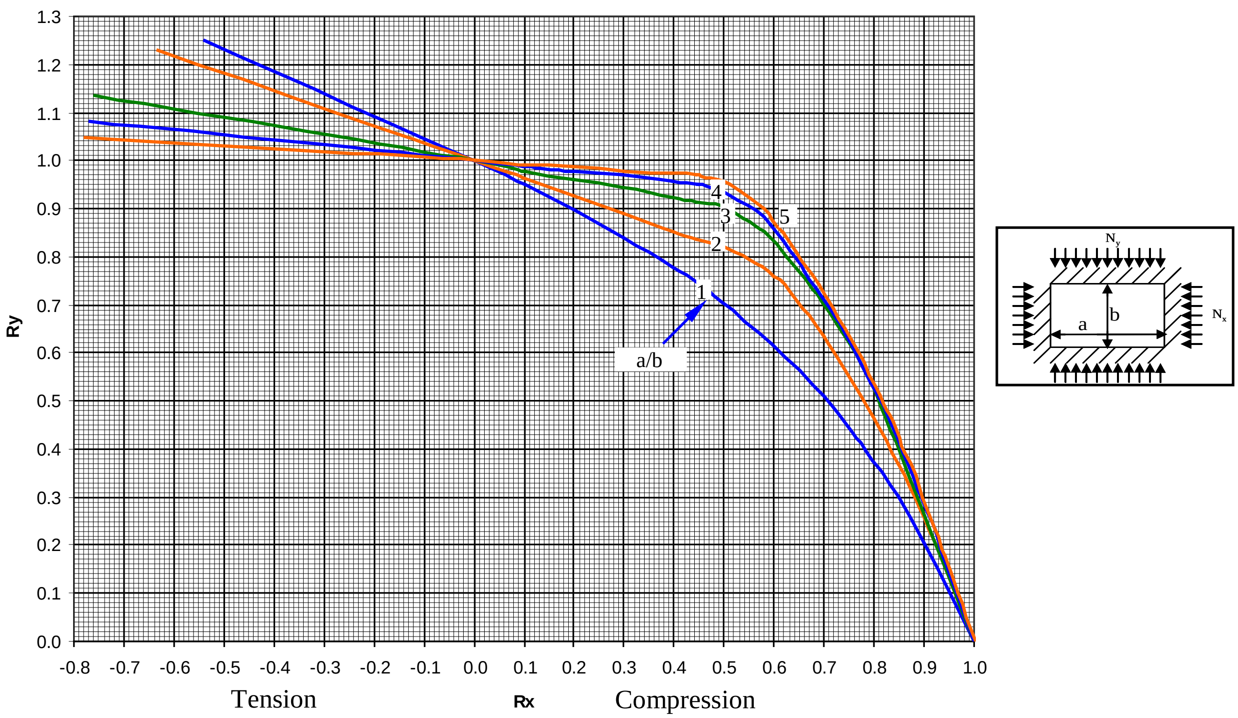
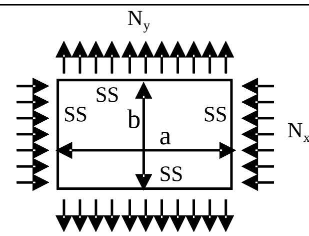


Figure 10.3.2-19: Interaction Curves for Flat Plates with Fixed Edges Subjected to Biaxial Compression or Compression and Tension

10.3.2.11 Example – Buckling of a Plate Subjected to Compression and Tension

Given: A 7075-T6 (bare sheet) panel is subjected to a combination of compression and tension. It is assumed that the edges are all simply supported. The dimensions of the panel are:
t = 0.125 in, a = 4.5 in, b = 3.0 in
N_x = 6000 lb/in compression
N_y = 1200 lb/in tension
All loads are ultimate. Calculate the margin of safety in buckling.



Solution: Material properties for 7075-T6 (bare sheet, 0.04 – 0.125) are obtained from Table 10.3.1-2 (developed from METDB) or can be obtained directly from the METDB database of IDAT.

F_{tu} = 80 ksi, F_{cy} = 71.0 ksi, F_{su} = 40.0 ksi, E_c = 10500 ksi, n_c = 12.0, ν_e = 0.33 (Poisson's ratio obtained from METDB of IDAT)

Solution:

Calculation	Equation/ Figure	Result
Compute the compression stress ratio R_x in X-direction		
Applied compressive stress, f_{cx}, in ksi due to N_x.	$f_{cx} = \frac{6000}{(0.125)(1000)} = 48.0$	f_{cx} = 48.0 ksi
Obtain compression buckling coefficient, k_c from the uniaxial compression chart	From Figure 10.3.2-1 For a/b = 4.5 / 3 = 1.5 k_c = 4.30	Taking the limiting value, k_c = 4.0 which is conservative
Buckling coefficient, k	$k = k_c$ Equation 10.3.2-1 k = 4.0	k = 4.0
Calculate effective buckling coefficient, K	$K = k\,\pi^2 / [12\,(1 - \nu_e^2)]$ Equation 10.3.1-5 $= (4.0)(\pi^2) / [12\,(1 - 0.33^2)] = 3.6919$	K = 3.692
Calculate $(b/t)_e$	$(b/t)_e = b/(t\,\sqrt{K})$ Equation 10.3.1-4 $= 3.0 / [(0.125)\,(\sqrt{3.692})] = 12.49$	$(b/t)_e$ = 12.5
Read initial buckling stress in compression	From Figure 10.3.1-7, curve 5, for $(b/t)_e$ = 12.5 f_{crx} = 59.5 ksi	f_{crx} = 59.5 ksi
Compressive stress ratio in the X-direction, R_x	$R_x = f_{cx} / f_{crx}$ = 48.0 / 59.5 = 0.806	R_x = 0.81
Compute the compression stress ratio R_y in Y-direction		
Applied tensile stress, f_{ty}, in ksi due to N_y	$f_{ty} = \frac{-1200}{(0.125)(1000)} = -9.60$	f_{ty} = - 9.6 ksi
Ultimate stress	F_{tu} = 80 ksi	F_{tu} = 80.0 ksi
Tensile stress ratio in the Y-direction, R_y	$R_y = f_{ty} / F_{tu}$ = - 9.60 / 80 = - 0.12	R_y = - 0.12
Coordinate axes are reversed since N_y is tensile	Abscissa R_x = - 0.12 and ordinate R_y = 0.81 a/b = 4.5/3.0 = 1.5	R_x = - 0.12 R_y = 0.81 a/b= 1.5
Margin of Safety is computed using Interaction curves. Interaction curve for a/b = 1.5 is interpolated between a/b values of 1 and 2	From Figure 10.3.2-20 and Equation 2.5.1-1 $MS = \frac{OB}{OA} - 1 = \frac{R_{1\,Allow}}{R_1} - 1 = \frac{R_{2\,Allow}}{R_2} - 1$	MS = 0.41

	Refer to the Figure below to see how the numbers are obtained $MS = \frac{(-0.17)}{(-0.12)} - 1 = 0.416 \cong \frac{1.14}{0.81} - 1 = 0.407$	

Note: On some programs tension load may not be considered; the analyst is advised to consult the appropriate design criteria for the applicable guidelines. In instances where the tension load is not considered, the margin of safety will be computed based on the compressive load. The margin of safety in such a case will be: MS = 59.5 / 48.0 -1 = 0.24

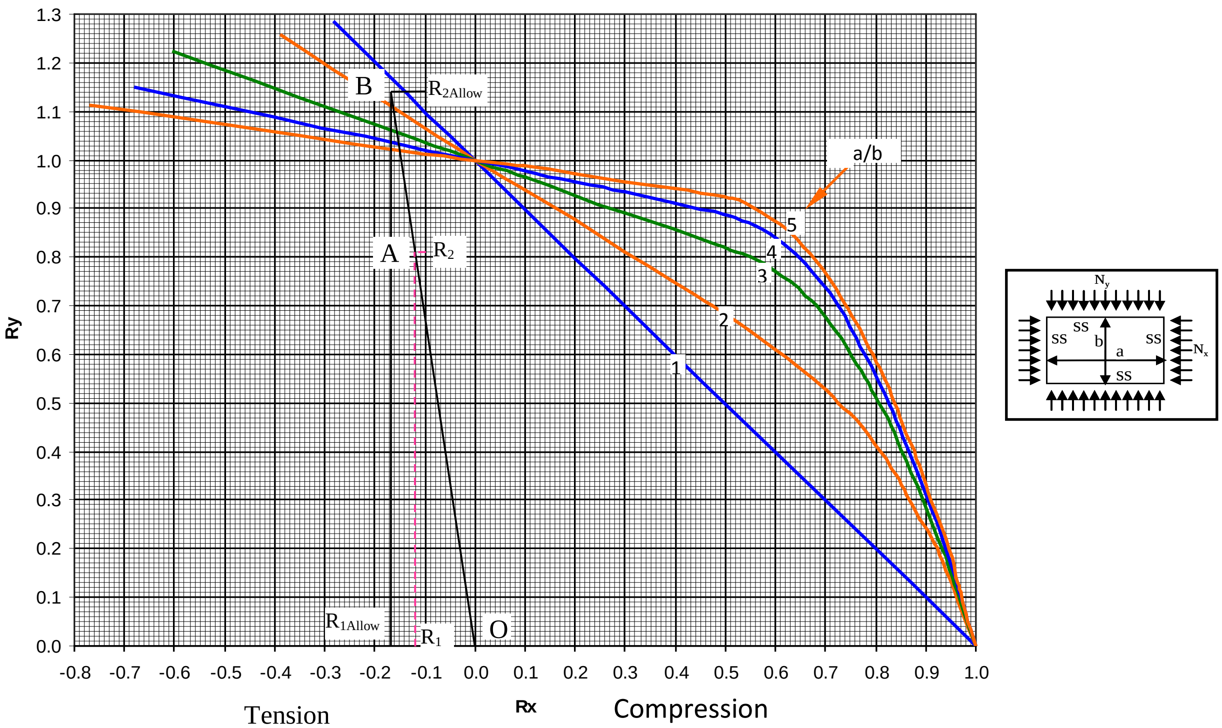


Figure 10.3.2-20: To Illustrate the use of the Figure 10.3.2-18

10.3.2.12 Biaxial Compression and Shear

Interaction curves for simply supported flat rectangular plates subjected to combined shear and biaxial compression having different aspect ratios a/b, are presented in **Figure 10.3.2-21** through Figure 10.3.2-25. These interaction curves were developed from theoretical considerations and were obtained from Reference 10-2. In general, there are two possible buckling modes, symmetrical and antisymmetrical. Interaction curves presented correspond to the mode which gives the lower combination of these stress ratios. Interaction curves are for elastic stability and the same curves are recommended for the stresses in the plastic regime.

R_s represents the shear stress ratio and if $R_s = 0$, the result is identical with Section 10.3.2.10; *i.e*., biaxial compression. If R_x or R_y is equal to zero, the results are identical with Section 10.3.2.9; *i.e*., combined shear and transverse compression.

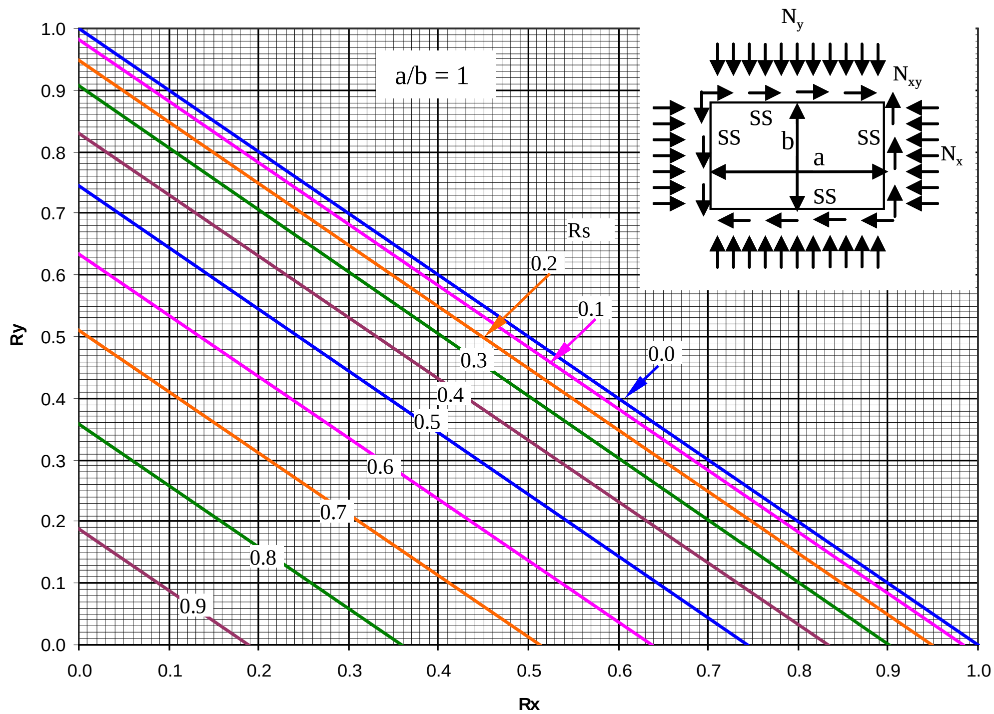


Figure 10.3.2-21: Interaction Curves for Simply Supported Flat Rectangular Plates under Biaxial Compression and Shear for a/b =1.0

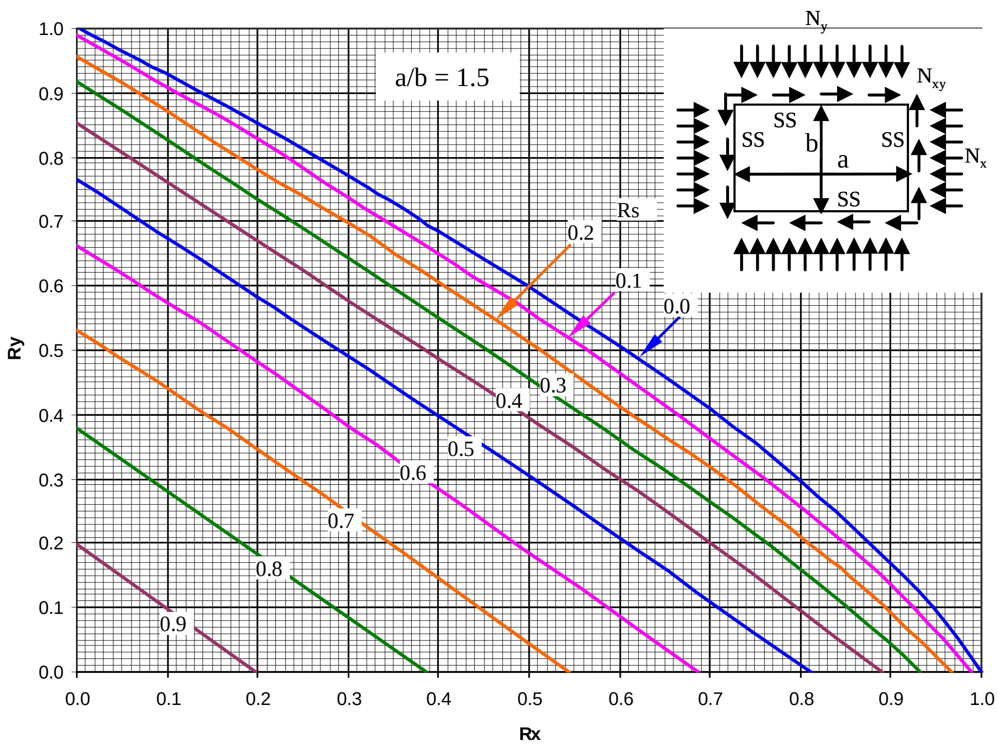


Figure 10.3.2-22: Interaction Curves for Simply Supported Flat Rectangular Plates under Biaxial Compression and Shear for a/b = 1.5

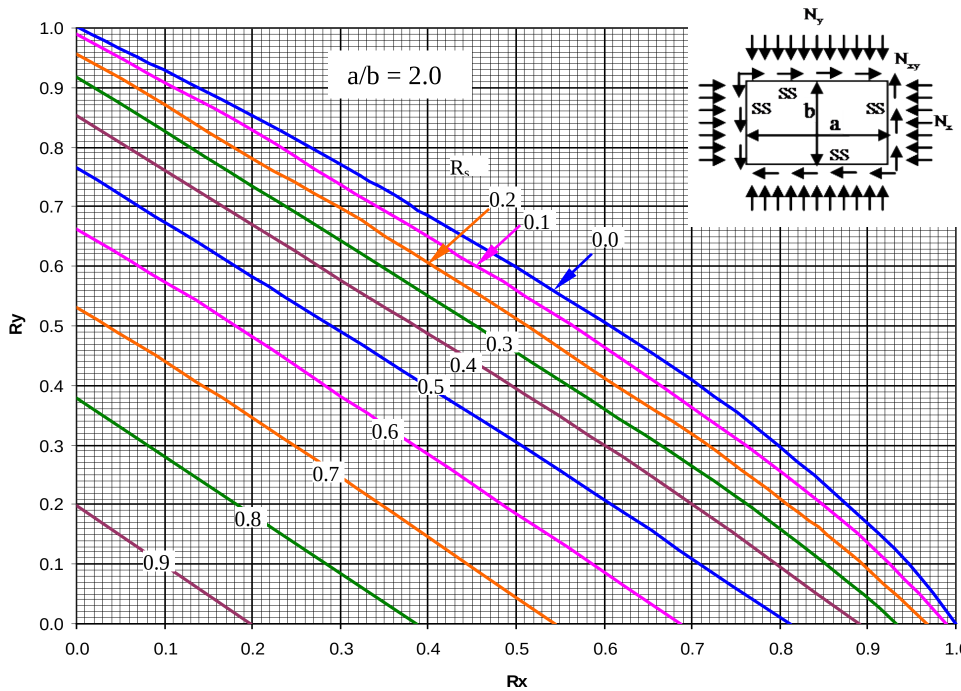


Figure 10.3.2-23: Interaction Curves for Simply Supported Flat Rectangular Plates under Biaxial Compression and Shear for a/b =2.0

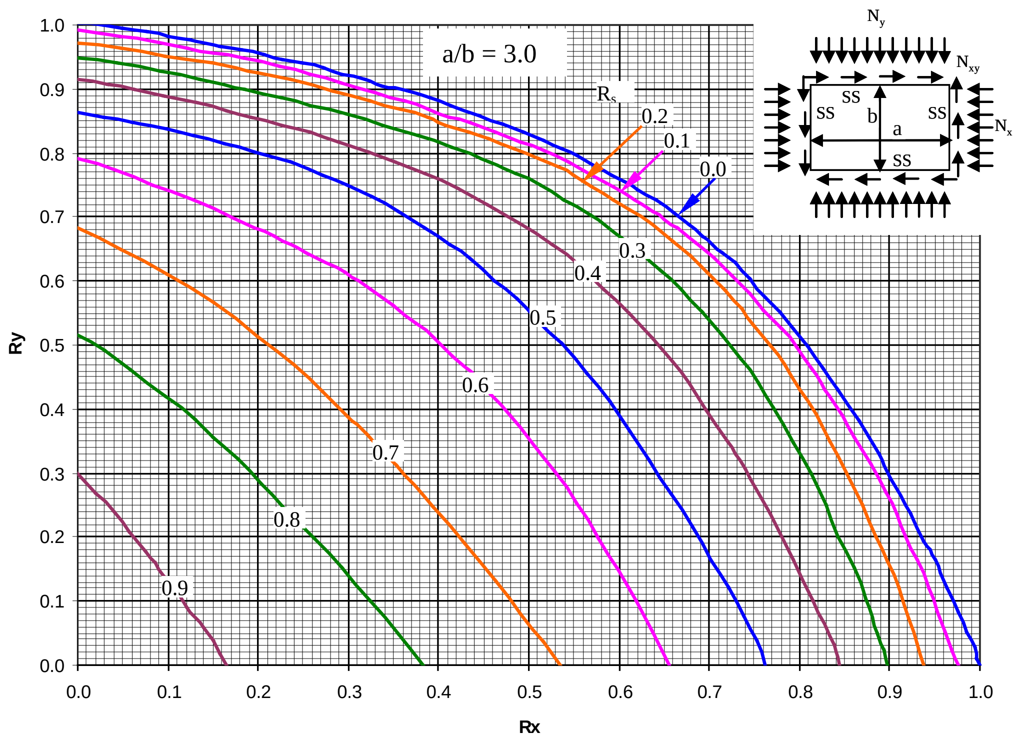


Figure 10.3.2-24: Interaction Curves for Simply Supported Flat Rectangular Plates under Biaxial Compression and Shear for a/b =3.0

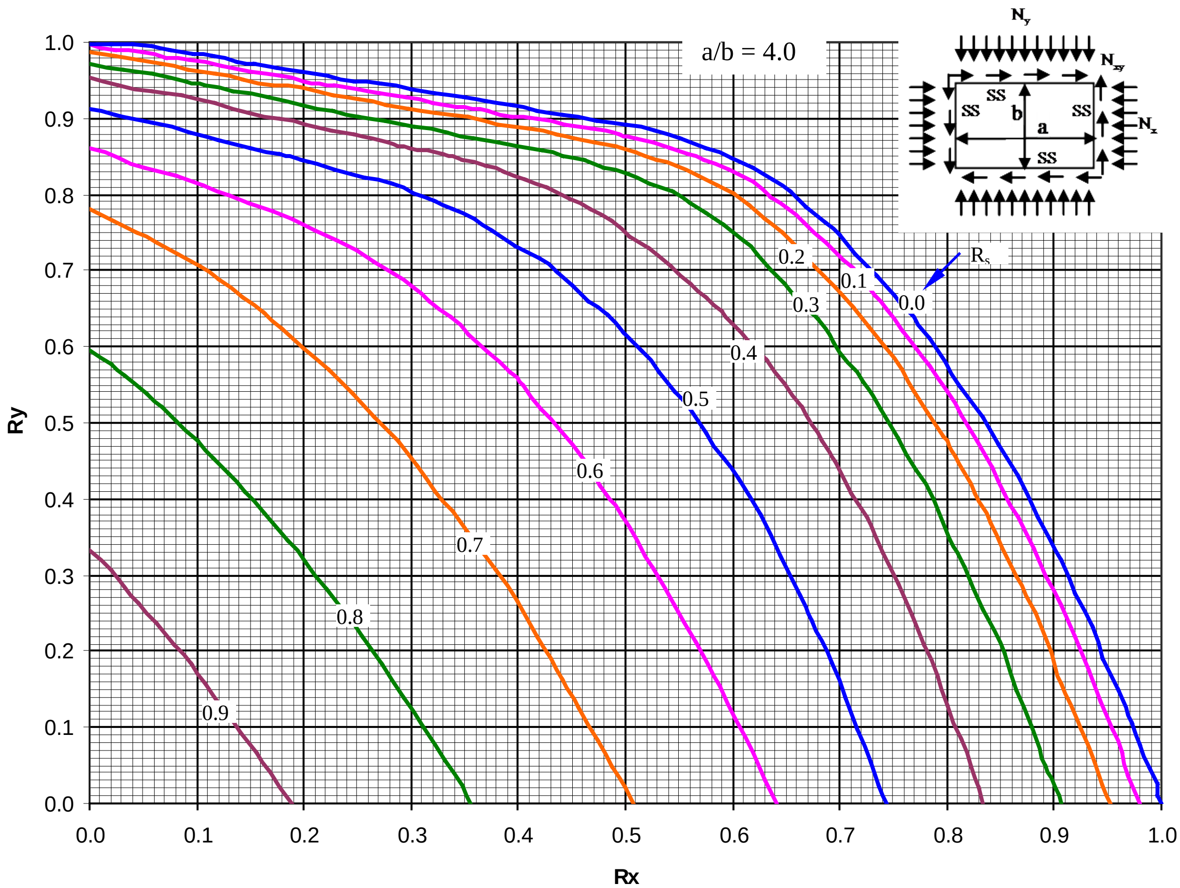


Figure 10.3.2-25: Interaction Curves for Simply Supported Flat Rectangular Plates under Biaxial Compression and Shear for a/b =4.0

10.3.2.13 Bending and Shear

The interaction curve for simply supported rectangular plates under combined bending and shear is depicted in Figure 10.3.2-26 and is obtained from Reference 10-19. This curve is a plot of Equation 10.3.2-8 and is valid for any plate aspect ratio, a/b.

$$R_b^2 + R_s^2 = 1 \qquad \textbf{Equation 10.3.2-8}$$

Where,
R_b is the stress ratio for bending
R_s is the stress ratio for shear

The bending stress distribution is per Equation 10.3.2-7 and the bending stress allowable is determined per Section 10.3.2.3 which is used to calculate the bending stress ratio, R_b. The neutral axis for bending is assumed halfway between the upper and lower edges. The shear stress allowable is determined per Section 10.3.2.2 which is used to calculate shear stress ratio, R_s. The interaction equation is for elastic stability and the same equation is recommended for stresses in the plastic regime.

The margin of safety is calculated per Case b from Table 2.5.1-1 of Section 2.

$$MS = \frac{1}{\sqrt{R_b^2 + R_s^2}} - 1 \qquad \textbf{Equation 10.3.2-9}$$

Where,
$R_b = R_1$ of equation in Table 2.5.1-1, Case b
$R_s = R_2$ of equation in Table 2.5.1-1, Case b

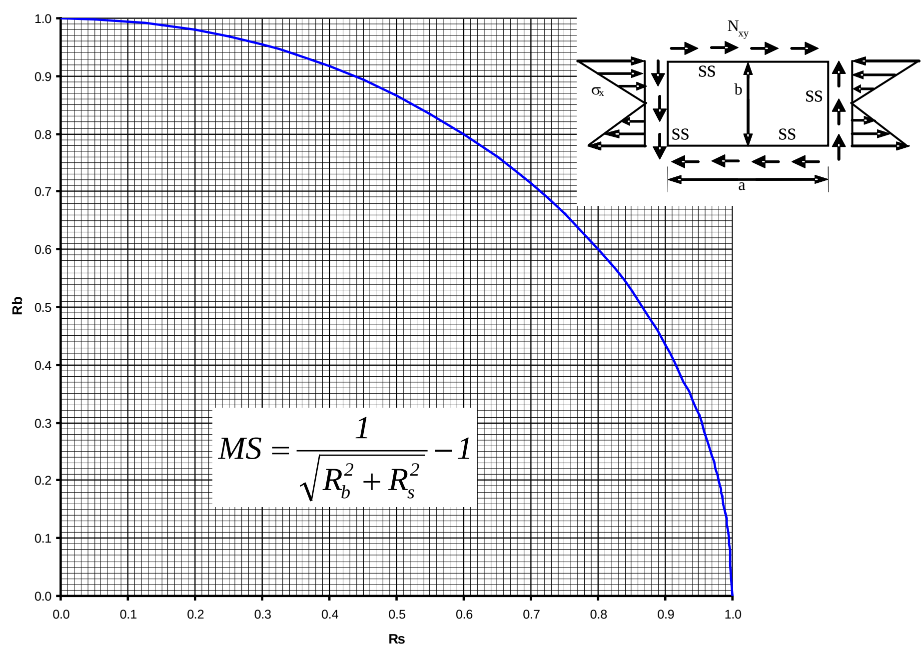


Figure 10.3.2-26: Interaction Curves for Simply Supported Flat Rectangular Plates under Bending and Shear

10.3.2.14 Bending, Shear and Transverse Compression

Interaction curves for infinitely long flat rectangular plates subjected to combined in-plane bending, shear and transverse compression are presented in Figure 10.3.2-27 and Figure 10.3.2-28. The curves in Figure 10.3.2-27 are for a plate whose compression edge defined by bending is clamped and tension edge defined by bending is simply supported. The curves in Figure 10.3.2-28 are for simply supported flat rectangular plates.

Interaction curves were derived based on theoretical considerations and were developed by Johnson *et al* (Reference 10-20) by assuming that the flat rectangular plate is elastic and infinitely long, and that the bending moment, shear, and transverse compression are constant along the length of the plate. The neutral axis for bending stress is assumed halfway between the upper and lower edges. The interaction surface is a three dimensional surface representing the interaction between bending, shear and transverse compression. For R_c = 1.0 curve, the plate acts as an Euler column and it can carry appreciable bending and shear stresses. The interaction curves are for elastic stability and the same curves are recommended for stresses in the plastic regime.

Interaction curves in Figure 10.3.2-27 need special attention because of bulging of the curves. The bulging of the curves is due to the fact that the tension on the simply supported edge due to the bending increases the buckling strength of the plate. This increase of strength necessitates that the analyst must be cautious in selecting the critical bending to use in calculation of the margin of safety. A look at the curves indicates that the maximum bending load case may not be critical as compared to the lesser bending load case.

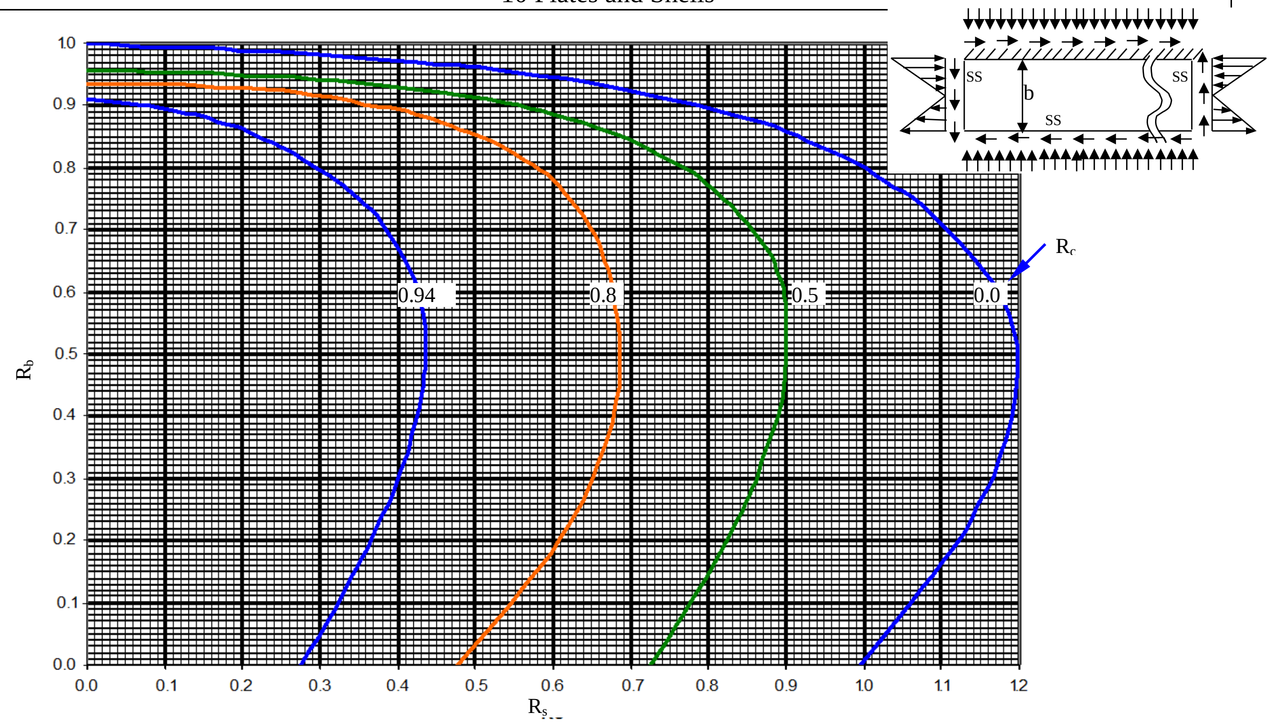


Figure 10.3.2-27: Compression Edge Clamped Infinitely Long Rectangular Plate under Bending, Shear and Transverse Compression

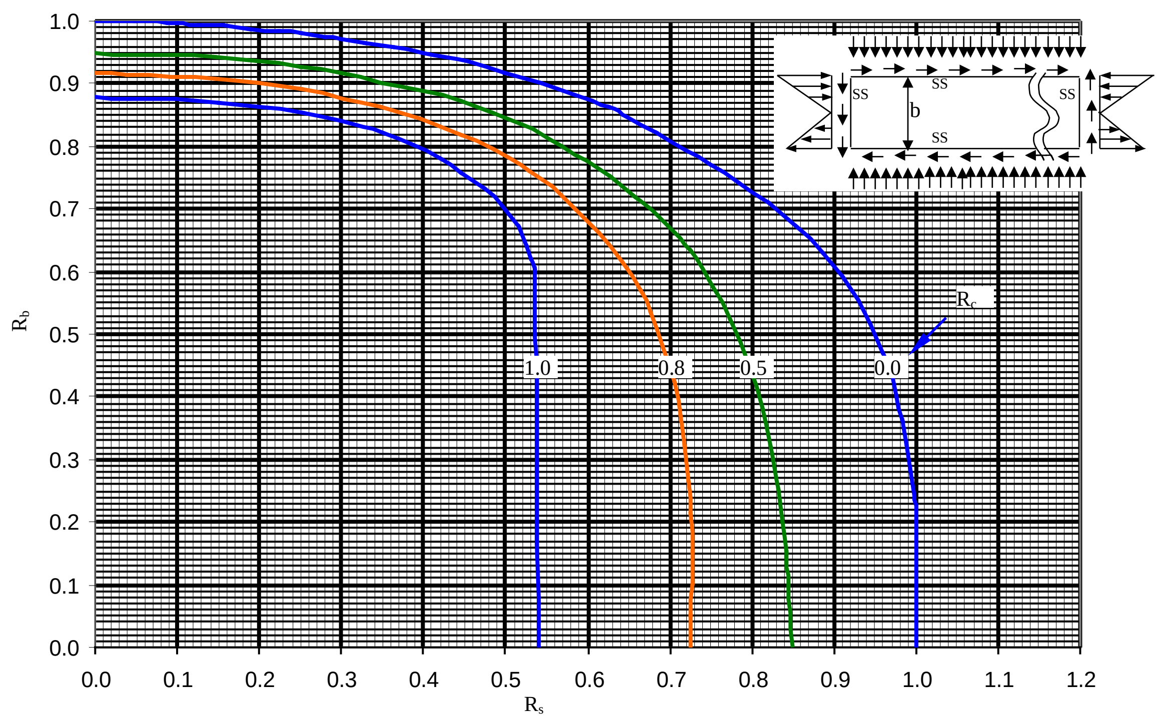


Figure 10.3.2-28: Simply Supported Infinitely Long Rectangular Plates under Bending, Shear and Transverse Compression

10.3.2.15 Bending and Biaxial Compression

Interaction curves for simply supported flat rectangular plates under combined longitudinal bending and biaxial compression are presented in **Figure 10.3.2-29** through **Figure 10.3.2-35** for various aspect ratios, a/b, ranging from 0.8 to ∞. The neutral axis for bending stress is assumed halfway between the upper and lower edges. These curves were developed based on theoretical considerations by Noel (Reference 10-21). The interaction curves are for elastic stability and the same curves are recommended for stresses in the plastic regime.

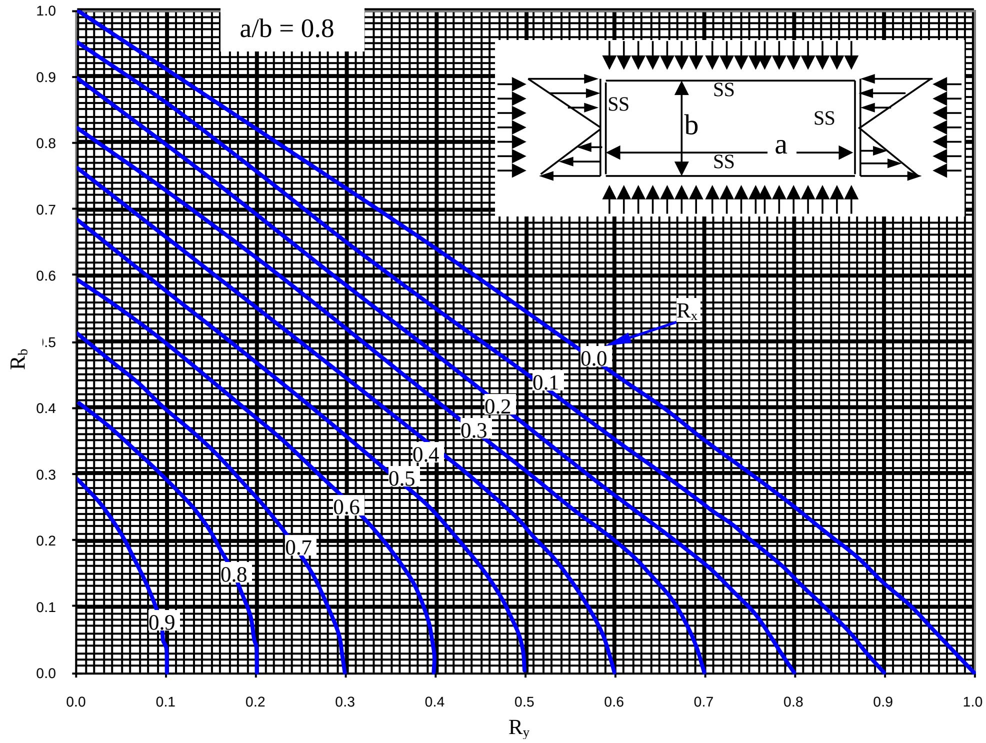


Figure 10.3.2-29: Simply Supported Rectangular Plates under Bending, and Biaxial Compression, a/b=0.8

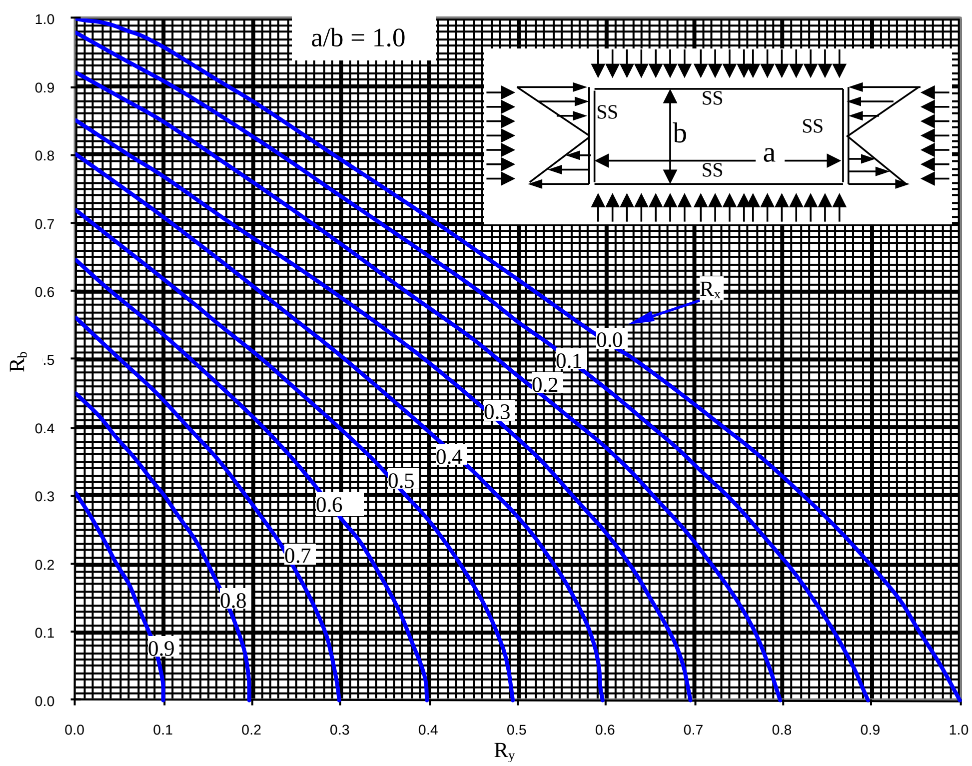


Figure 10.3.2-30: Simply Supported Rectangular Plates under Bending, and Biaxial Compression, a/b=1.0

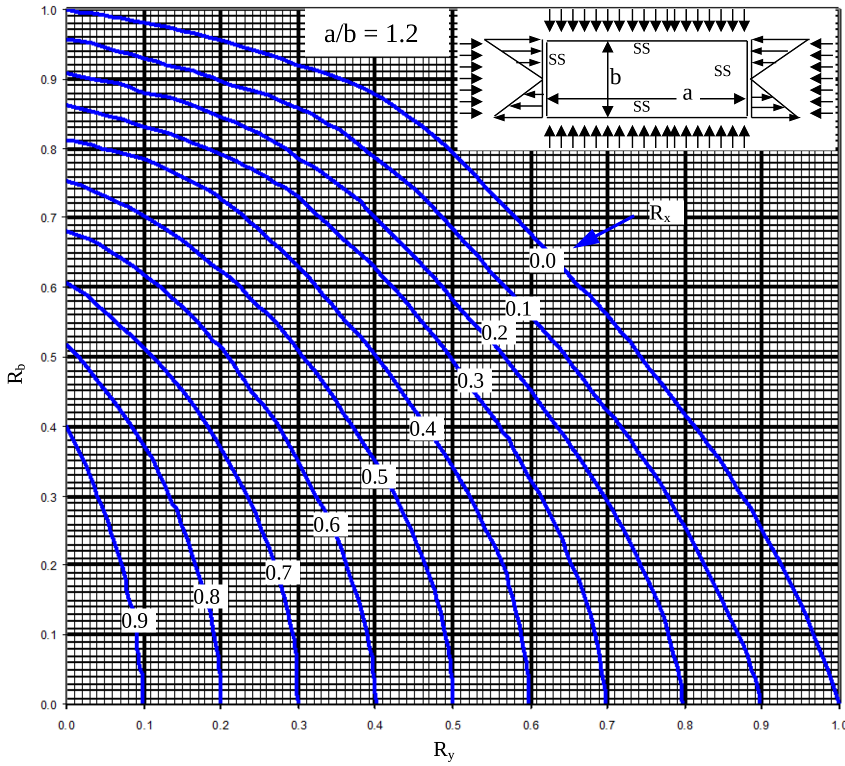


Figure 10.3.2-31: Simply Supported Rectangular Plates under Bending, and Biaxial Compression, a/b=1.2

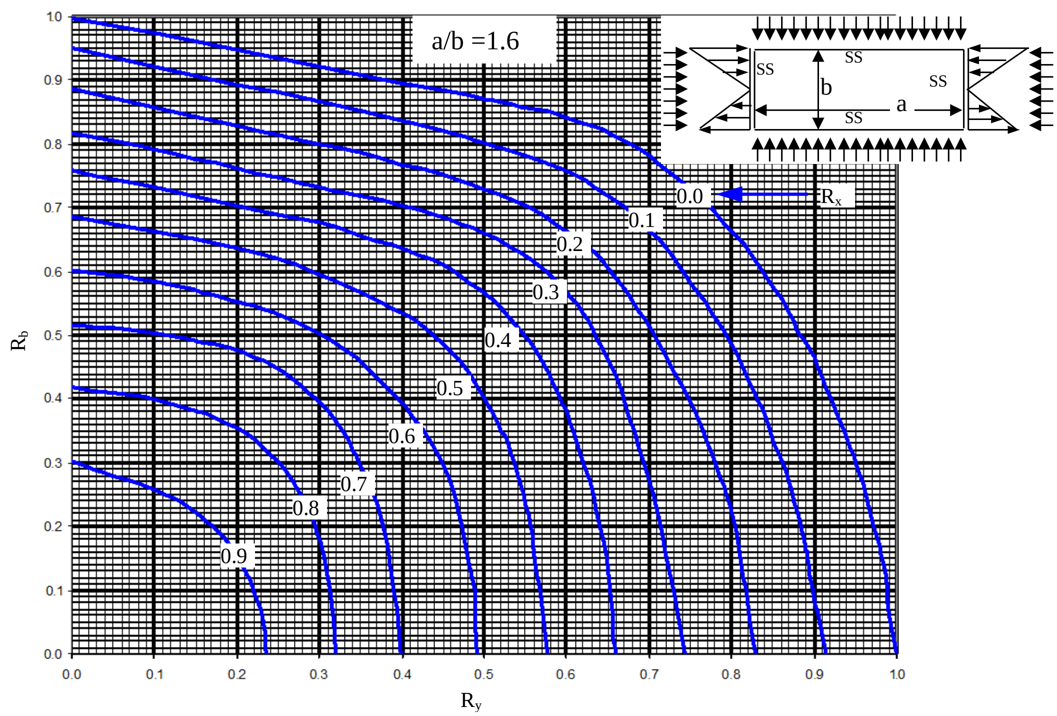


Figure 10.3.2-32: Simply Supported Rectangular Plates under Bending, and Biaxial Compression, a/b=1.6

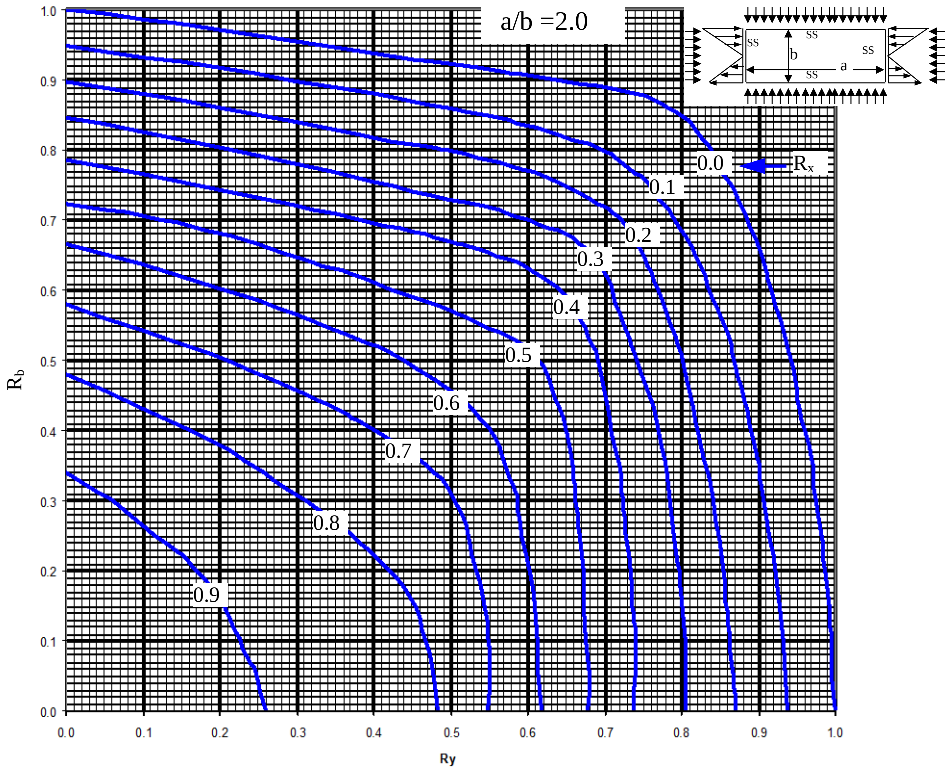


Figure 10.3.2-33: Simply Supported Rectangular Plates under Bending, and Biaxial Compression, a/b=2.0

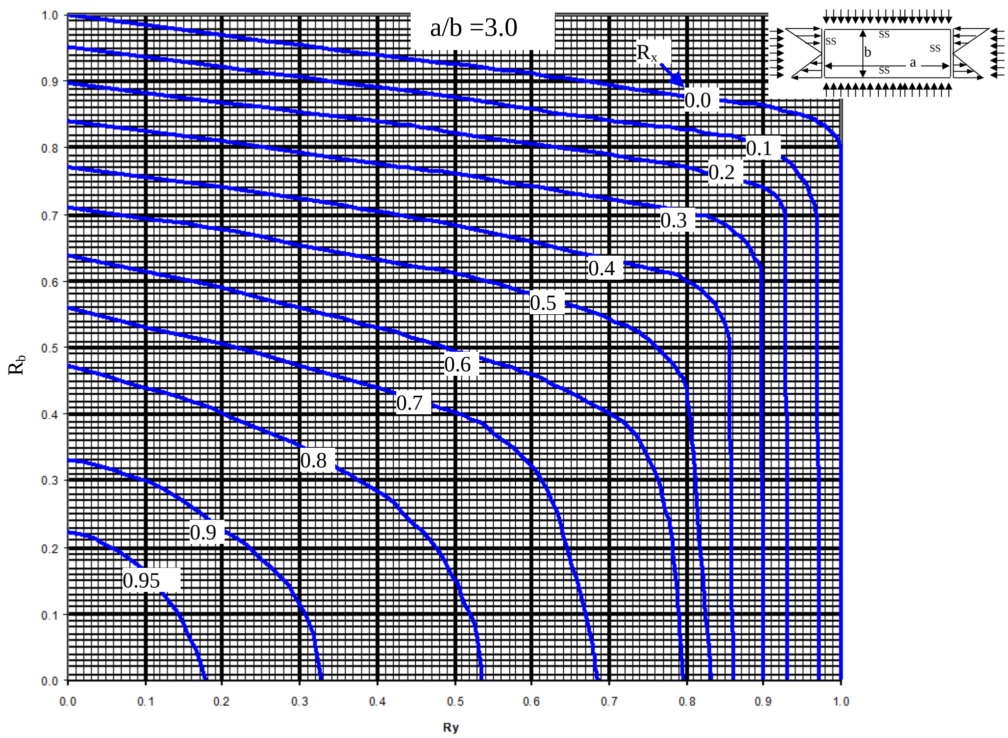


Figure 10.3.2-34: Simply Supported Rectangular Plates under Bending, and Biaxial Compression, a/b=3.0

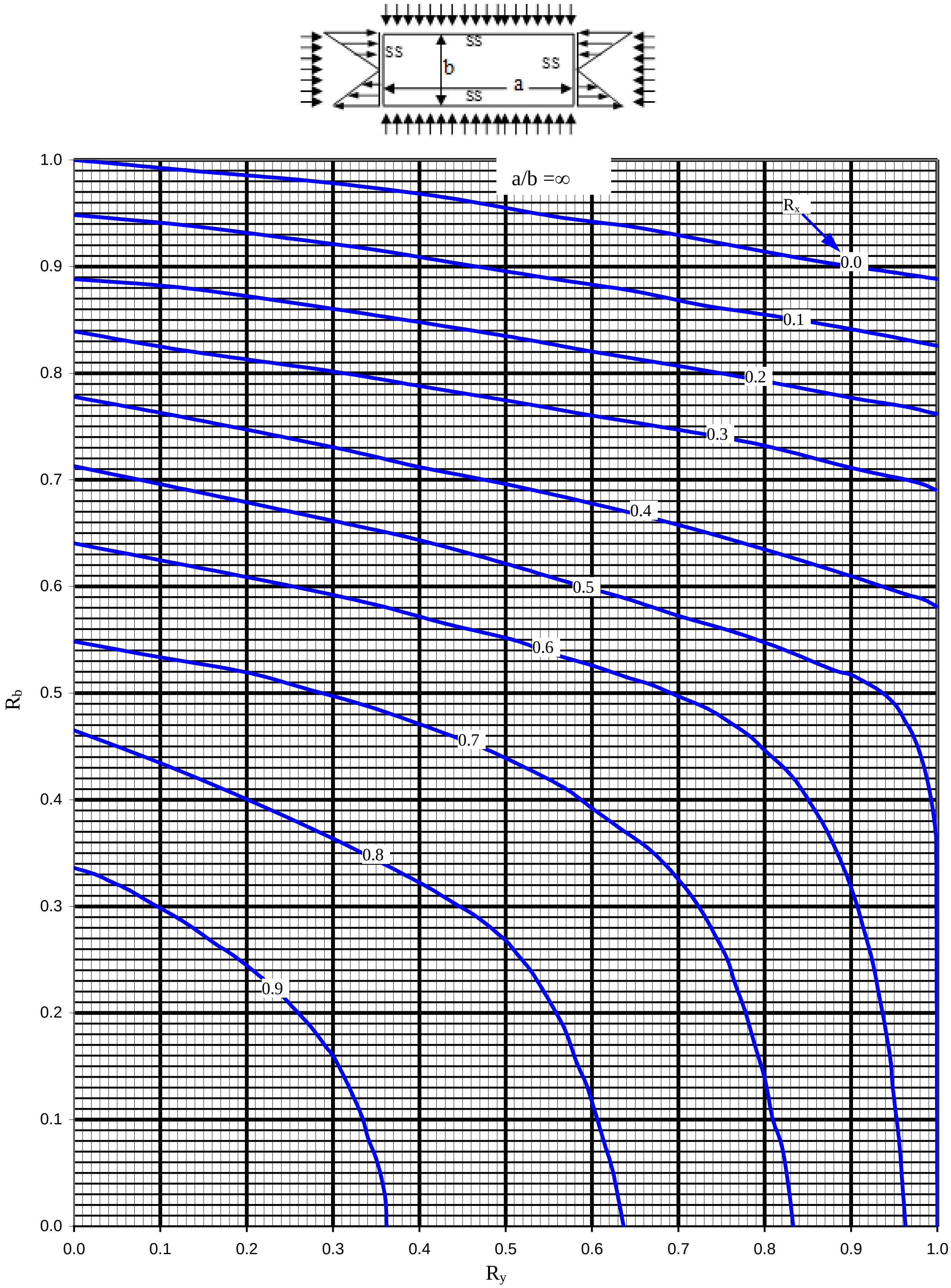


Figure 10.3.2-35: Simply Supported Rectangular Plates under Bending, and Biaxial Compression, a/b=∞

10.3.2.16 Bending and Transverse Compression

The interaction curve for simply supported flat rectangular plates under combined longitudinal bending and transverse compression is presented in Figure 10.3.2-36 for various aspect ratios, a/b, ranging from 0.8 to ∞. The neutral axis for bending stress is assumed halfway between the upper and lower edges. These curves were developed based on theoretical considerations by Grossman (Reference 10-22). The interaction curves are for elastic stability and the same curves are recommended for stresses in the plastic regime.

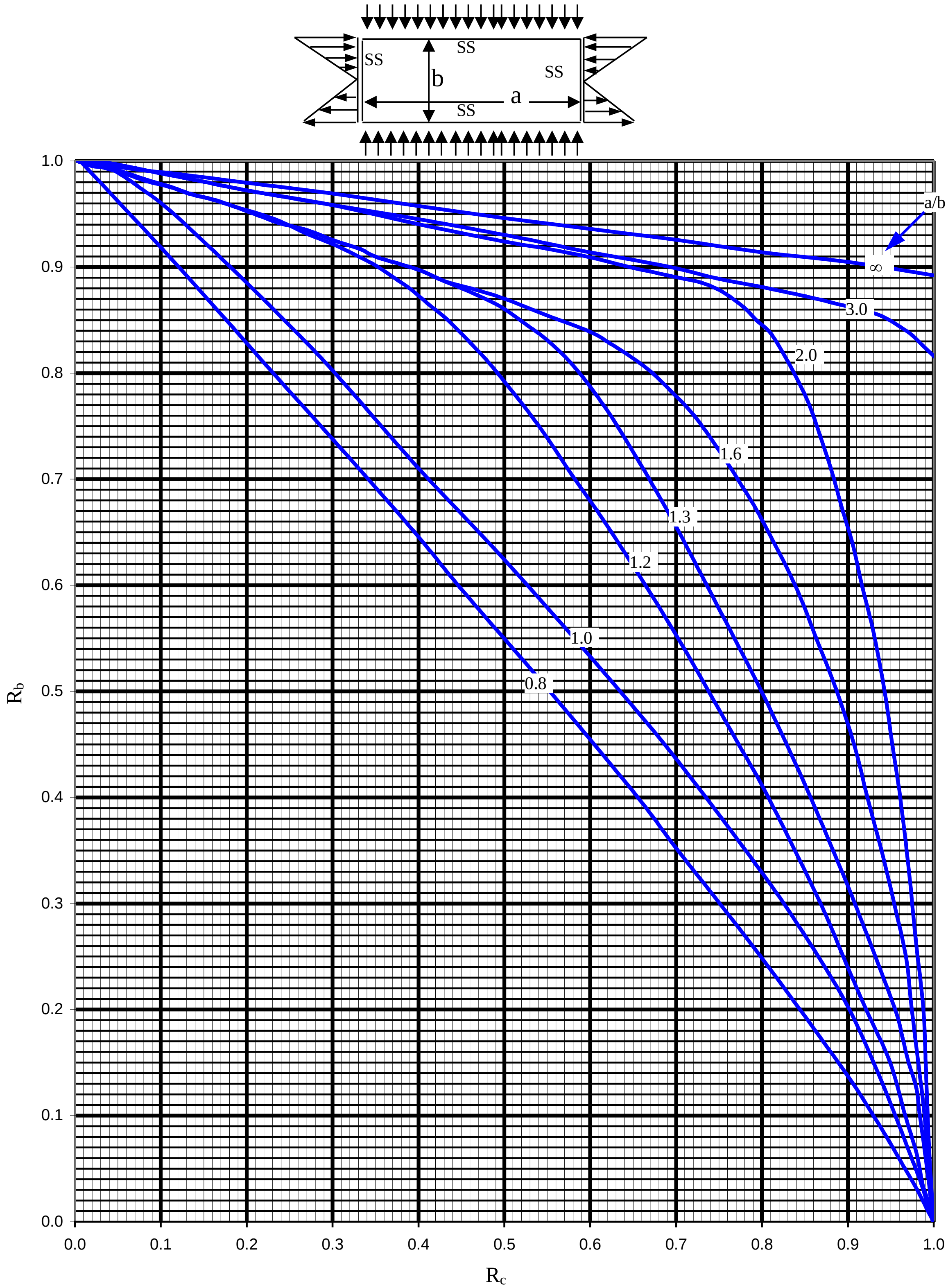


Figure 10.3.2-36: Simply Supported Rectangular Plates under Bending, and Transverse Compression

10.3.2.17 Shear and Longitudinal Compression or Tension

For this case of combined loading, the interaction equations are different for elastic and plastic buckling. The interaction equation for elastic buckling of an infinitely long rectangular plate subjected to shear and longitudinal compression as shown in Figure 10.3.2-37 was developed from theoretical consideration by Stowell (Reference 10-23).

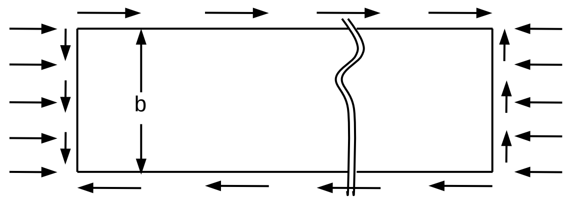


Figure 10.3.2-37: Infinitely Long Rectangular Plate under Combined Shear and Longitudinal Compression

The interaction relationship between the elastic stress ratios, shown in Equation 10.3.2-10, is valid for simply supported and fixed long edges.

Equation 10.3.2-10

$$R_c + R_s^2 = 1$$

Where,
R_c is the elastic compression or tension stress ratio
R_s is the elastic shear stress ratio

The margin of safety is calculated per Case c, Table 2.5.1-1 of Section 2.

Equation 10.3.2-11

$$MS = \frac{2}{R_c + \sqrt{R_c^2 + 4R_s^2}} - 1$$

Where,
$R_c = R_1$ of equation in Table 2.5.1-1, Case c
$R_s = R_2$ of equation in Table 2.5.1-1, Case c

The interaction equation for inelastic buckling of an infinitely long rectangular plate subjected to shear and compression as shown in Figure 10.3.2-37 was developed from empirical consideration by Peters (Reference 10-24). The interaction relationship between the inelastic stress ratios is given by Equation 10.3.2-12 and is valid for simply supported and fixed long edges.

Equation 10.3.2-12

$$R_c^2 + R_s^2 = 1$$

Where,
R_c is the inelastic compression stress ratio
R_s is the inelastic shear stress ratio

The margin of safety is calculated per Case b, Table 2.5.1-1 of Section 2.

$$MS = \frac{1}{\sqrt{R_c^2 + R_s^2}} - 1$$ **Equation 10.3.2-13**

Where,
$R_c = R_1$ of equation in Table 2.5.1-1, Case b
$R_s = R_2$ of equation in Table 2.5.1-1, Case b

10.3.2.18 Bending, Shear and Compression or Tension

Interaction curves for simply supported flat plates under combined longitudinal bending, shear and longitudinal compression or tension shown in Figure 10.3.2-38 are presented in Figure 2.5.3-3, which is repeated here. **Figure 10.3.2-39** is an expanded form of the interaction curves of Figure 2.5.3-3 using a shear stress ratio, R_s and the bending stress ratio, R_b and a fixed compression or tension stress ratio, R_c. The neutral axis for bending stress is assumed halfway between the upper and lower edges. These curves are plotted from Equation 10.3.2-14 provided by Volume I of Reference 10-2. The interaction equation is for elastic stability and the same equation is recommended for stresses in the plastic regime.

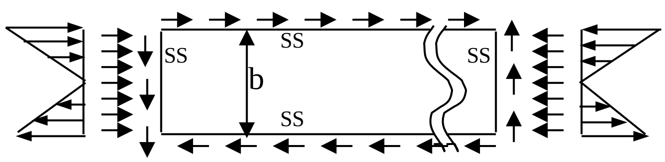


Figure 10.3.2-38: Infinitely Long Rectangular Plate under Combined Bending, Shear and Longitudinal Compression or Tension

Equation 10.3.2-14

$$R_s^2 + R_b^2 + R_c = 1$$

Where,
R_s is the shear stress ratio
R_b is the bending stress ratio
R_c is the compression or tension stress ratio

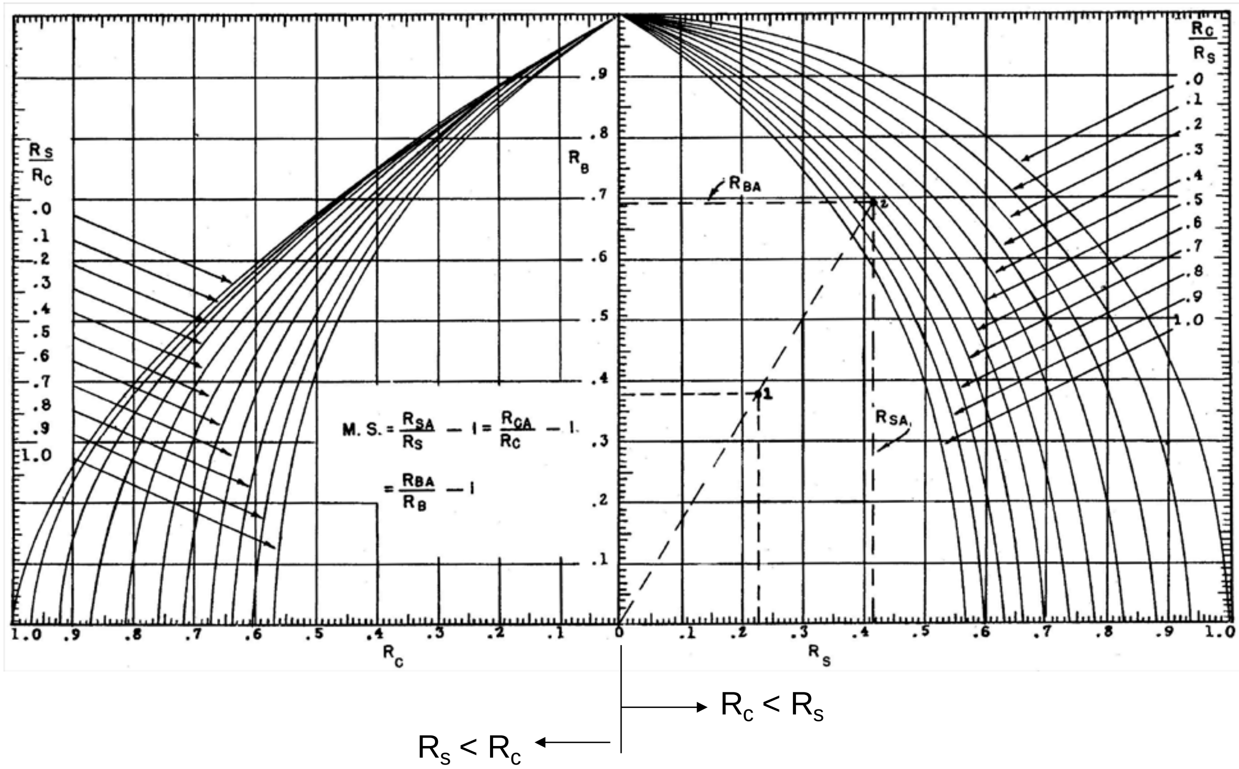
The margin of safety is calculated per Case e, Table 2.5.1-1.

Equation 10.3.2-15

$$MS = \frac{2}{R_c + \sqrt{R_c^2 + 4\left(R_s^2 + R_b^2\right)}} - 1$$

Where,
$R_c = R_1$ of equation in Table 2.5.1-1, Case e
$R_s = R_2$ of equation in Table 2.5.1-1, Case e
$R_b = R_3$ of equation in Table 2.5.1-1, Case e

The margin of safety can also be calculated using interaction curves shown in Figure 2.5.3-3. Section 2.5.3 explains how to use these curves to compute the margin of safety.



Reference Figure 2.5.3-3: Shear, Compression and In-plane Bending Interaction Curve

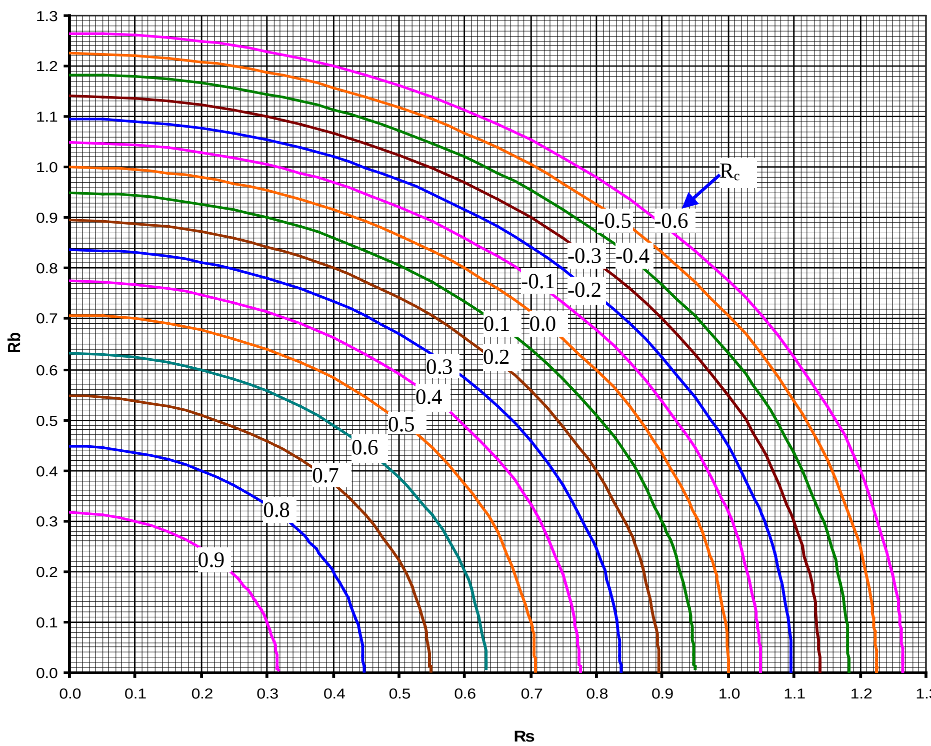
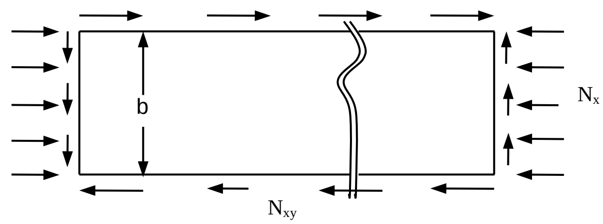


Figure 10.3.2-39: Expanded Interaction Curves for Combined Bending and Shear for a Given Value of Compressive Stress Ratio, R_c

10.3.2.19 Example – Buckling of a Plate Subjected to Shear and Compression

Given: A 7075-T6 (bare sheet) skin panel is subjected to a combination of shear and longitudinal compression loads. It is assumed that the edges are all simply supported and the plate is considered infinitely long. The dimensions of the panel are:
$t = 0.125$ in, $b = 3.0$ in
$N_x = 7000$ lb/in
$N_{xy} = 1500$ lb/in
Calculate the margin of safety in buckling.



Solution: Material properties for 7075-T6 (bare sheet, 0.04 – 0.125) are obtained from Table 10.3.1-2 (developed from METDB) or can be obtained directly from the METDB database of IDAT.

$F_{tu} = 80$ ksi, $F_{cy} = 71.0$ ksi, $F_{su} = 40.0$ ksi, $E_c = 10500$ ksi, $n_c = 12.0$, $\nu_e = 0.33$ (Poisson's ratio obtained from METDB of IDAT)

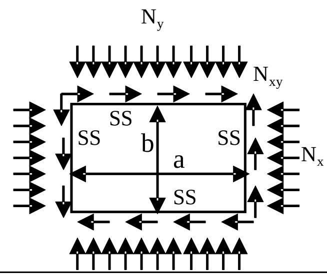
Solution:

Calculation	**Equation/ Figure**	**Result**
Calculate F_{cp}, proportional limit in compression	$F_{cp} = F_{cy}(0.05)^{\left(\frac{1}{n_c}\right)}$ Equation 3.3.1-12 $F_{cp} = 71.0(0.05)^{\left(\frac{1}{12}\right)} = 55.3145$	$F_{cp} = 55.315$ ksi
Applied compressive stress, f_c, in ksi	$f_c = \dfrac{7000}{(0.125)(1000)} = 56.0$	$f_c = 56.0$ ksi
Since applied stress, f_c is greater than the proportional limit, the plate is subjected to inelastic buckling		
Obtain compression buckling coefficient, k_c, from the uniaxial compression chart	From Figure 10.3.2-1 For a/b = high $k_c = 4.0$	$k_c = 4.0$
Buckling coefficient, k	$k = k_c$ Equation 10.3.2-1 $k = 4.0$	$k = 4.0$
Calculate effective buckling coefficient, K	$K = k\,\pi^2 / [12\,(1 - \nu_e^2)]$ Equation 10.3.1-5 $= (4.0)(\pi^2) / [12\,(1 – 0.33^2)] = 3.6919$	$K = 3.692$
Calculate $(b/t)_e$	$(b/t)_e = b/(t\,\sqrt{K})$ Equation 10.3.1-4 $= 3.0 / [(0.125)\,(\sqrt{3.692})] = 12.49$	$(b/t)_e = 12.5$
Read initial buckling stress in compression	From Figure 10.3.1-7, curve 5, for $(b/t)_e = 12.5$ $f_{cr} = 59.5$ ksi	$f_{cr} = 59.5$ ksi
Compressive stress ratio, R_c	$R_c = f_c / f_{cr}$ $= 56.0 / 59.5 = 0.9412$	$R_c = 0.941$
Applied shear stress, f_s, in ksi	$f_s = N_{xy} / [(t)]$ $= 1500 / [(0.125)\,(1000)] = 12.0$	$f_s = 12.0$ ksi
Obtain shear buckling coefficient, k_s, from the shear chart	From **Figure 10.3.2-7**, For $b/a \approx 0$ $k_s = 5.4$	$k_s = 5.4$
Buckling coefficient, k	$k = k_s$ Equation 10.3.2-5 $k = 5.4$	$k = 5.4$
Calculate effective buckling coefficient, K	$K = k\,\pi^2 / [12\,(1 - \nu_e^2)]$ Equation 10.3.1-5 $= (5.4)(\pi^2) / [12\,(1 – 0.33^2)] = 4.984$	$K = 4.984$
Calculate $(b/t)_e$	$(b/t)_e = b/(t\,\sqrt{K})$ Equation 10.3.1-4 $= 3.0 / [(0.125)\,(\sqrt{4.984})] = 10.75$	$(b/t)_e = 10.8$
Read initial buckling stress in shear	From Figure 10.3.1-10, curve 5, for $(b/t)_e = 10.8$ $f_{scr} = 39.4$ ksi	$f_{scr} = 39.4$ ksi

Shear stress ratio, R_s	$R_s = f_s / f_{scr}$ $= 12.0 / 39.4 = 0.3046$	$R_s = 0.305$
Margin of Safety	$MS = \frac{1}{\sqrt{R_c^2 + R_s^2}} - 1$ Equation 10.3.2-13 $MS = \frac{1}{\sqrt{0.941^2 + 0.305^2}} - 1 = 0.01$	MS = 0.01

10.3.2.20 Example – Buckling of a Plate Subjected to Biaxial Compression and Shear

Given: A 7075-T6 (bare sheet) web is subjected to a combination of biaxial compression and shear. It is assumed that the edges are all simply supported. The dimensions of the panel are:
t = 0.125 in, a = 4.5 in, b = 3.0 in
N_x = 5000 lb/in
N_y = 1200 lb/in
N_{xy} = 1500 lb/in
Calculate the margin of safety in buckling.



Solution: Material properties for 7075-T6 (bare sheet, 0.04 – 0.125) are obtained from Table 10.3.1-2 (developed from METDB) or can be obtained directly from the METDB database of IDAT.

F_{tu} = 80 ksi, F_{cy} = 71.0 ksi, F_{su} = 40.0 ksi, E_c = 10500 ksi, n_c = 12.0, ν_e = 0.33 (Poisson's ratio obtained from METDB of IDAT)

Solution:

Calculation	**Equation/ Figure**	**Result**
Compute the shear stress ratio R_s		
Applied shear stress, f_s, in ksi	$f_s = N_{xy} / [(t)]$ $= 1500 / [(0.125)(1000)] = 12.0$	f_s = 12.0 ksi
Obtain shear buckling coefficient, k_s, from the shear chart	From Figure 10.3.2-7, For b/a = 3.0 / 4.5 = 0.67 k_s= 7.05	k_s = 7.05
Buckling coefficient, k	$k = k_s$ Equation 10.3.2-5 k = 7.05	k = 7.05
Calculate effective buckling coefficient, K	$K = k\,\pi^2 / [12(1 - \nu_e^2)]$ Equation 10.3.1-5 $= (7.05)(\pi^2) / [12(1 - 0.33^2)] = 6.507$	K = 6.507
Calculate $(b/t)_e$	$(b/t)_e = b/(t\sqrt{K})$ Equation 10.3.1-4 $= 3.0 / [(0.125)(\sqrt{6.507})] = 9.409$	$(b/t)_e$ = 9.4
Read initial buckling stress in shear	From Figure 10.3.1-10, curve 5, for $(b/t)_e$ = 9.4 f_{scr} = 40 ksi	f_{scr} = 40.0 ksi
Shear stress ratio, R_s	$R_s = f_s / f_{scr}$ $= 12.0 / 40.0 = 0.3$	R_s = 0.30
Compute the compression stress ratio R_x in X-direction		
Applied compressive stress, f_{cx}, in ksi due to N_x	$f_{cx} = \frac{5000}{(0.125)(1000)} = 40.0$	f_{cx} = 40.0 ksi
Obtain compression buckling coefficient, k_c, from the uniaxial compression chart	From Figure 10.3.2-1 For a/b = 4.5 / 3 = 1.5 k_c = 4.30	Taking the limiting value, k_c = 4.0, which is conservative

Buckling coefficient, k	$k = k_c$ Equation 10.3.2-1 $k = 4.0$	$k = 4.0$
Calculate effective buckling coefficient, K	$K = k\,\pi^2 / [12\,(1 - \nu_e^2)]$ Equation 10.3.1-5 $= (4.0)(\pi^2) / [12\,(1 - 0.33^2)] = 3.6919$	$K = 3.692$
Calculate $(b/t)_e$	$(b/t)_e = b/(t\,\sqrt{K})$ Equation 10.3.1-4 $= 3.0 / [(0.125)(\sqrt{3.692})] = 12.49$	$(b/t)_e = 12.5$
Read initial buckling stress in compression	From Figure 10.3.1-7, curve 5, for $(b/t)_e = 12.5$ $f_{crx} = 59.5$ ksi	$f_{crx} = 59.5$ ksi
Compressive stress ratio in the X-direction, R_x	$R_x = f_{cx} / f_{crx}$ $= 40.0 / 59.5 = 0.6723$	$R_x = 0.67$
Compute the compression stress ratio R_y in Y-direction		
Applied compressive stress, f_{cy}, in ksi due to N_y	$f_{cy} = \frac{1200}{(0.125)(1000)} = 9.6$	$f_{cy} = 9.6$ ksi
Obtain compression buckling coefficient, k_c, from the uniaxial compression chart of wide plates	From Figure 10.3.2-3 For a/b = 3 / 4.5 = 0.67 $k_c = 2.1$	$k_c = 2.1$
Buckling coefficient, k	$k = k_c$ Equation 10.3.2-1 $k = 2.1$	$k = 2.1$
Calculate effective buckling coefficient, K	$K = k\,\pi^2 / [12\,(1 - \nu_e^2)]$ Equation 10.3.1-5 $= (2.1)(\pi^2) / [12\,(1 - 0.33^2)] = 1.9383$	$K = 1.938$
Calculate $(b/t)_e$	$(b/t)_e = a/(t\,\sqrt{K})$ Equation 10.3.2-2 $= 3.0 / [(0.125)(\sqrt{1.938})] = 17.240$	$(b/t)_e = 17.2$
Read initial buckling stress in compression	From Figure 10.3.1-7, curve 5, for $(b/t)_e = 17.2$ $f_{cry} = 35.5$ ksi	$f_{cry} = 35.5$ ksi
Compressive stress ratio in the Y-direction, R_y	$R_y = f_{cy} / f_{cry}$ $= 9.60 / 35.5 = 0.270$	$R_y = 0.270$
Margin of Safety is computed using Interaction curves	From Figure 10.3.2-40 and Equation 2.5.1-1 $MS = \frac{OB}{OA} - 1 = \frac{R_{1Allow}}{R_1} - 1 = \frac{R_{2Allow}}{R_2} - 1$ Refer next page to see how the numbers are obtained $MS = \frac{0.689}{0.67} - 1 = 0.028 \cong \frac{0.278}{0.27} - 1 = 0.029$	MS = 0.03

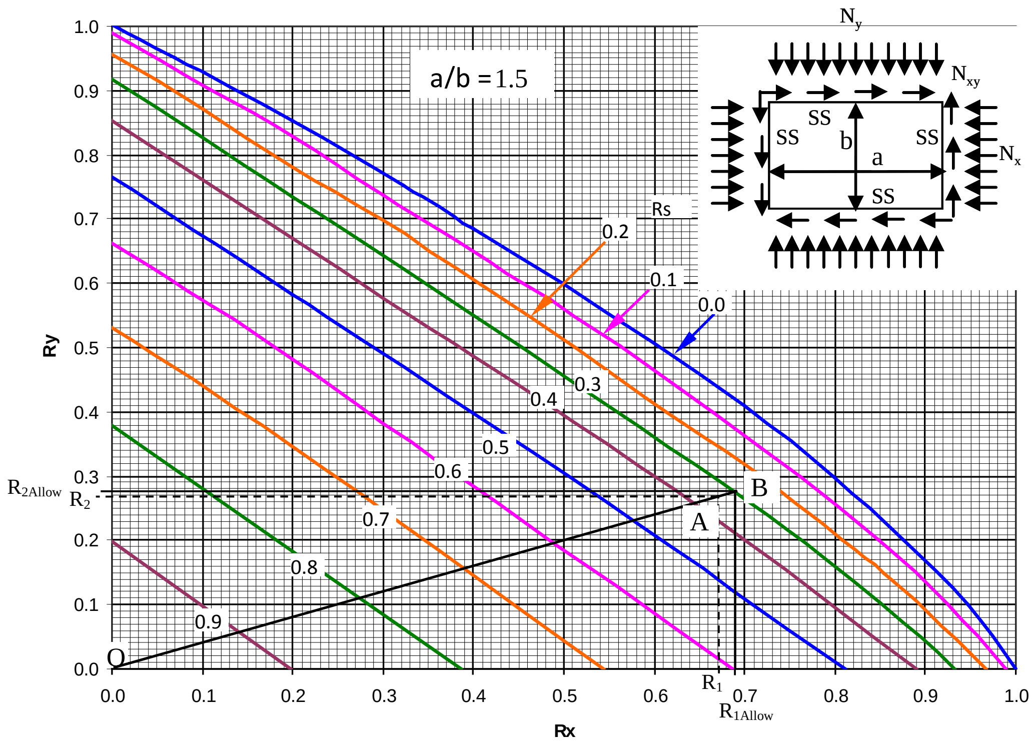


Figure 10.3.2-40: To Illustrate the use of the Figure 10.3.2.-21

10.3.3 Buckling of Curved Plates

The purpose of this section is to provide guidance on the structural analysis of curved plates/sheets. A curved plate is a structural member whose middle surface has a curvature unlike a flat plate whose middle surface is planar. An example of the curved sheet panel is fuselage skin. For an unstiffened curved sheet – *i.e.*, no longitudinal stiffeners – under the action of loads, the failure of the sheet occurs when the sheet buckles. If the curved sheet has longitudinal stiffeners, the buckling of the sheet does not constitute the failure of the composite panel made up of sheet and stiffeners. The ultimate strength of the composite panel is far greater than the initial buckling of the sheet alone. In aircraft design, it is usually required to know the magnitude of loads that would cause the buckling of the curved plate acting alone. This section provides the necessary design aids for determining the buckling of an unstiffened plate under the action of loads acting individually or in combination. Unstiffened curved plates are covered in Sections 10.3.3.1 to 10.3.3.3.

Two kinds of loads are considered for curved plate buckling: (1) axial compression and (2) shear. Curved plate buckling under axial compression is discussed in Section 10.3.3.1 and curved plate buckling under shear is discussed in Section 10.3.3.2. The combined axial and shear load buckling is described in Section 10.3.3.3.

The equations describing the behavior of curved plates under axial compression loads are very complex and involve complicated mathematical analysis. An introduction to the theory is provided in this section, to allow the analyst to understand the basis and limitations of the method; however, for the purpose of performing the analysis, design curves are provided which are easy to use and independent of the material used for the construction of the curved plate. The design curves for axially loaded curved plates are provided in Section 10.3.3.1.4.

The behavior of curved plates loaded in shear more closely parallels flat plate behavior which utilizes a shear buckling coefficient. Section 10.3.3.2 provides the equations and buckling coefficients for curved plates loaded in shear.

10.3.3.1 Curved Plate under Axial Compression

The behavior of a curved plate under compression as shown in Figure 10.3.3-1 is similar to that of a circular cylinder under compression since a curved plate is essentially a section of a cylinder. Long curved plates of ample curvature and width display the same characteristics as long cylinders. Both buckle at stresses considerably below the predictions of linear theory. Thus, there is a considerable disagreement between theory and the test data. Consequently, considerable reliance is placed on semi empirical methods using theory as a guide. Geometric parameters – *viz.*, radius of curvature, r, and thickness of the plate, t – are employed to enhance the correlation between test and theory by grouping the data for a fixed value of r/t.

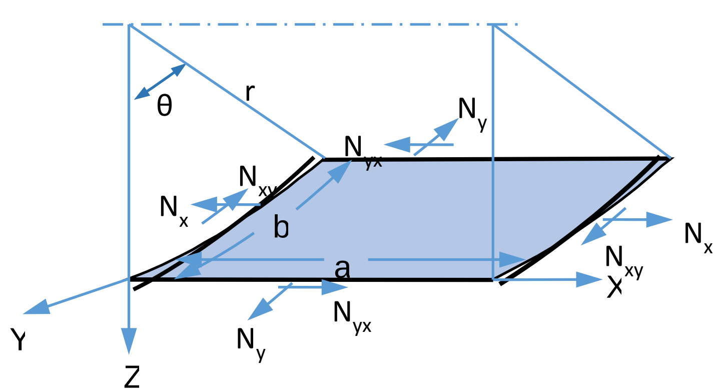


Figure 10.3.3-1: A Curved Plate Definition

To account for the geometry effects in curved plates, a parameter Z_b is defined by Equation 10.3.3-1.

$$Z_b = \left(\frac{b^2}{rt}\right)\left(1-\nu_e^2\right)^{0.5} \qquad \textbf{Equation 10.3.3-1}$$

Where,
b is the width of the plate measured along the arc
t is the thickness of the plate
r is the radius of the curvature of the plate
ν_e is the elastic Poisson's ratio

Buckling behavior of axially compressed curved plates are classified into four ranges of behavior similar to axially compressed circular cylinders. For small values of Z_b, the buckling behavior of curved plates approaches that of flat plates with the characteristic sinusoidal buckle shape. At large values of Z_b, it is observed that the behavior of curved plates is same as that of long axially loaded cylinders with the characteristic diamond pattern buckle shape. For the intermediate values of Z_b, defined here as the transition range, the behavior of curved plates is represented by a transition curve that is fitted between the flat plate behavior and that of long plate. In the transition range, it appears that there is an interaction between diamond and sinusoidal buckles. For very large values of Z_b, the behavior of curved plate is that of Euler column buckling.

10.3.3.1.1 Curved Plate under Axial Compression - Theory

A curved plate loaded in axial compression buckles in the same manner as a cylinder when the plate curvature is significant. Curvature is defined as the reciprocal of the radius of the plate. When curvature of a plate is very small it buckles essentially as a flat plate. Between these two limits, there is a transition from one type of behavior to the other.

Figure 10.3.3-2 shows a cylindrical element of radius r subjected to axial load N_x, circumferential load N_y and shear load N_{xy}. The figure also shows the displacement components: axial displacement χ, circumferential displacement ξ, and radial displacement w. The loads can all act together or separately. Usually, solutions are developed for each kind of load separately and if more than one kind of load is present interaction equations/curves are used to determine the margin of safety.

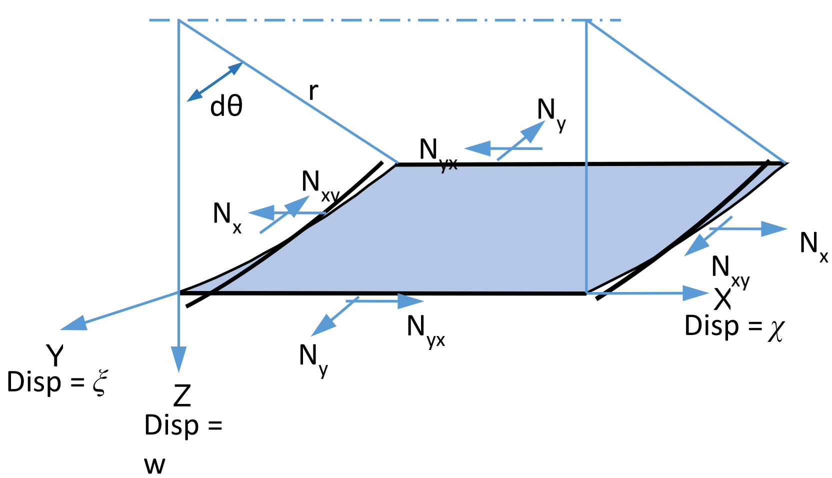


Figure 10.3.3-2: An Element of Cylindrical (or Curved Plate) Shell

In small deflection stability theory, the deflections are assumed infinitesimal. Thus, the strains are linear functions of the displacements and therefore stresses are also linear functions of displacements. In the case of curved plates/cylinders, the geometry of the stressed cross-sections of the plates/cylinders due to loads is very complex. It is possible to obtain different strain displacement equations depending upon the inclusion of minor terms.

The mathematical formulation of stability theory for axially compressed cylinders is very complex and thus only pertinent equations will be shown. Similar to flat plate buckling theory presented in Section 10.3.1, it is desired to express the curved plate buckling behavior in terms of the radial deflection, w, of the middle surface of the plate. Such an equation was developed by Donnell (Reference 10-25), Equation 10.3.3-2. This is an 8th order partial differential equation and expresses the relationship between applied loads, radial displacement, w, flexural rigidity, D of the plate, radius r of the plate, thickness of the plate t, and angle θ bounded by the plate's cylindrical generators. It is to be noted that by letting 1/r = 0, and replacing r ∂θ by ∂y the equation reduces to the governing equation for the flat plate, Equation 10.3.1-1.

$$D\nabla^8 w + \frac{E_c t}{r^2}\frac{\partial^4 w}{\partial x^4} + \nabla^4\left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{r^2 \partial\theta^2} + 2N_{xy}\frac{\partial^2 w}{r\partial x \partial\theta}\right) = 0$$

Equation 10.3.3-2

Where,
D is the flexural rigidity (lb-in) given by Equation 10.2.0-9
E_c is the compression modulus of the plate material (psi)
w is the radial deflection of the plate (in)
θ is the angle bounded by the plate's cylindrical generators (radians)
N_x is the axial compressive load (lb/in)
N_y is the circumferential compressive load (lb/in)
N_{xy} is the shear load (lb/in)

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\theta^2}\right), \quad \nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\theta^2}\right),$$

$$\nabla^8 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\theta^2}\right)$$

The relationship for other displacements χ and ξ are given by

$$\nabla^4 \chi = -\nu \frac{\partial^3 w}{r\partial x^3} + \frac{\partial^3 w}{r^3 \partial x \partial\theta^2}$$

Equation 10.3.3-3

Where,
ν is the Poisson's ratio of the plate material
χ is the axial displacement of the plate (in)

$$\nabla^4 \xi = -(2+\nu)\frac{\partial^3 w}{r^2 \partial x^2 \partial\theta} - \frac{\partial^3 w}{r^4 \partial\theta^3}$$

Equation 10.3.3-4

Where,
ξ is the displacement in the Y-direction (in)

The solution of the equilibrium Equation 10.3.3-2 provides the initial buckling load of a curved plate. The mathematical solution requires that the equilibrium and the boundary conditions be satisfied. A curved plate has four edges similar to a flat plate where boundary conditions can be applied. However, a cylinder has only two boundaries, *i.e.*, two ends, instead of four boundaries for the curved plate. Thus, for the cylinder, two additional boundary conditions are provided by the fact that the displacements are cyclic functions of the angle θ with a cyclic length of (2πr).

Similar to the flat plate buckling solution, energy methods are employed to solve the governing differential equation. Batdorf, Reference 10-26, used such a method to provide a compressive buckling solution for axially compressed cylinders. As alluded to earlier, there is a considerable disagreement between theory and the test data. Thus, considerable reliance is placed on semi-empirical methods using theory as a guide.

It is assumed that the plate is perfectly elastic under the action of external forces within the elastic limit of the material. The plate material is assumed to be homogeneous and continuously distributed over its entire volume. It is further assumed that the plate material is isotropic; *i.e.*, that the elastic properties are the same in all directions. Basic assumptions are listed below:

f) Perfectly Elastic
g) Homogeneous
h) Continuous
i) Isotropic
j) Constant Thickness

With these assumptions, the general solution of the Equation 10.3.3-2 for compressive load N_x is given by the following equation, Reference 10-27:

$$f_{cre} = \frac{\pi^2 E_c}{12(1-\nu_e^2)}\left(\frac{t}{b}\right)^2 k_{cu} \qquad \textbf{Equation 10.3.3-5}$$

Where,
f_{cre} is the initial elastic buckling stress (psi)
k_{cu} is the non-dimensional buckling coefficient in axial compression for the curved plate
E_c is the compression modulus of the material (psi)
ν_e is the elastic Poisson's ratio of the material

Define $(b/t)_e$ as follows:

$$\left(\frac{b}{t}\right)_e^2 = \frac{12\left(1-\nu_e^2\right)}{\pi^2}\left(\frac{b}{t}\right)^2\left(\frac{1}{k_{cu}}\right) \qquad \textbf{Equation 10.3.3-6}$$

Using Equation 10.3.3-6, buckling Equation 10.3.3-5 is written as:

$$f_{cre} = \frac{E_c}{\left(\frac{b}{t}\right)_e^2} \qquad \textbf{Equation 10.3.3-7}$$

Equation 10.3.3-7 is the plate buckling equation in the elastic region and is modified in the inelastic region by using plasticity correction factors.

10.3.3.1.1.1 Curved Plate under Axial Compression - Plasticity Correction

Plasticity correction discussed in Section 10.3.1.2 for flat rectangular plate is also applicable to the curved plate. For simplicity of calculations, all effects of exceeding the proportional limit are generally incorporated in a single coefficient referred to as the plasticity correction factor η, which is a ratio of plastic buckling stress and the elastic buckling stress. Following Stowell modulus definition in Equation 10.3.1-9 and plasticity correction factor η defined in Equation 10.3.1-8 the initial buckling stress Equation 10.3.3-7 becomes

$$f_{cr} = \frac{\eta\, E_c}{\left(\frac{b}{t}\right)_e^2}$$ **Equation 10.3.3-8**

The above equation is perfectly general since η = 1 provides the buckling stress for the elastic case. Thus, it is not necessary to distinguish between the elastic and plastic buckling stress equations since the same equation is used for both cases. The values of k_{cu} and ν_e represent elastic values and η incorporates all the corrections needed for inelastic behavior.

This is the same equation as given in Equation 10.3.1-10 for flat plates and the solution procedure will be the same for curved plates. For curved plates, $(b/t)_e$ will be developed using curvature effects as shown in Sections 10.3.3.1.2 to 10.3.3.1.3. As noted before, a curved plate loaded in axial compression buckles in the same manner as a cylinder and its buckling behavior is classified into four ranges. For small values of Z_b, the curved plate buckles as a flat plate. Flat plate initial buckling is covered earlier in Section 10.3.1, where analytical methods are discussed and equations are provided to compute the initial buckling stress in elastic as well as in plastic regimes. Solutions are provided when the loads act separately and if more than one kind of load is acting, interaction equations/curves are provided. For very high values of Z_b, the curved plate buckles as an Euler column which is covered in Section 8.

For intermediate values of Z_b, the curved plate buckling is in transition from flat plate buckling to long cylinder buckling. Transition range and long range buckling behavior will be covered here. Design curves are provided in Section 10.3.3.1.4 such that analyst can determine allowable stresses by using only geometric properties of the plate. The theoretical background is provided in Sections 10.3.3.1.2 and 10.3.3.1.3 for the analyst to understand the behavior of curved plates and the development of the design curves.

10.3.3.1.2 Curved Plate under Axial Compression in the Transition Range

In the transition range, the curved plate buckling coefficient in compression, k_{cu}, is defined as k_{cut} by Equation 10.3.3-9.

$$k_{cu} = k_{cut}$$ **Equation 10.3.3-9**

The coefficient k_{cut} in the transition range is defined empirically in Reference 10-27 as follows:

$$k_{cut} = k_{pl} + \left[\frac{\left(0.581\,\Omega\, Z_b\right)^2}{k_{pl}}\right]$$ **Equation 10.3.3-10**

Where,
k_{cut} is the curved plate buckling coefficient in the transition range
k_{pl} is the buckling coefficient of the flat plate in axial compression, Refer Section 10.3.2.1
Z_b is the plate parameter defined by Equation 10.3.3-1
Ω is a compressive buckling coefficient constant for long cylinders

As noted before, a thin long cylinder under axial compression buckles at stresses considerably below the prediction of small deflection theory. Donnell, Reference 10-28 proposed that this disparity could be explained by using large deflection theory of shells which he developed in conjunction with the initial imperfection of the cylinder. Later research by Donnell and Wan, Reference 10-29 showed that the sensitivity of axially compressed cylinders to the initial imperfection was due to the fact that these imperfections usually were of the same size as the relatively small buckles generated at critical load. They defined, theoretically, the relationship between Ω and r/t in terms of an unevenness factor U, related to the initial imperfections. Unevenness factor, U is determined by using energy minimization principle together with the trial and error procedure where a certain value of U is assumed first and then the total energy of the system is minimized. The value of U is then used to obtain Ω for a given value of r/t, Reference 10-27. The design curves are generated by using the value of U equal to 0.00025.

Note the solution is developed only for simply supported curved plates since simply supported boundary conditions provide a lower bound for the initial buckling stress. Solutions for other boundary conditions are not available because of complexity and non-availability of the test data to validate the predictions.

Two cases are investigated based on the geometry of the plate: aspect ratio a/b < 1 and aspect ratio a/b > 1.

10.3.3.1.2.1 Curved Plate under Axial Compression in the Transition Range for Aspect Ratio a/b < 1

For aspect ratio a/b < 1 and assuming that the plate buckles in one half-wave, the flat plate buckling coefficient, k_{pl}, for simply support boundary conditions is defined by Reference 10-10

$$k_{pl} = \left(\frac{b}{a} + \frac{a}{b}\right)^2 = 4\left(\frac{1}{\Delta^2}\right)$$
$$and$$
$$\Delta = \frac{2ab}{a^2 + b^2}$$

Equation 10.3.3-11

Where,
Δ is the plate geometric parameter

Knowing k_{pl} from the above equation, the buckling coefficient k_{cu} is computed from Equation 10.3.3-9 by using Equation 10.3.3-10 for a given value of r, t, and Ω. Substituting Equation 10.3.3-9 into Equation 10.3.3-6, $(b/t)_e$ is computed as follows:

$$\left(\frac{b}{t}\right)_e^2 = \frac{12\left(1-\nu_e^2\right)}{\pi^2}\left\{\frac{1}{\left(\frac{b\Delta}{2r}\right)^2\left[\left(\frac{2r}{b\Delta}\right)^4\left(\frac{t}{r}\right)^2 + 0.581^2\,\Omega^2\left(1-\nu_e^2\right)\right]}\right\}$$

Equation 10.3.3-12

Valid for r/t ≤ 1500

For different values of r/t, and corresponding values of Ω, and for a given value of Poisson's ratio, $(b/t)_e$ versus (bΔ/2r) curves can be plotted. It is to be noted that the Poisson's ratio ranges from 0.31 to 0.33 for the commonly used materials for the construction of curved plates. Calculations indicate that for a small change in the Poisson's ratio of 0.02, *i.e.,* 0.33 to .31, the effect on $(b/t)_e$ is less than 0.72%. The maximum value of r/t is restricted to 1500 because test data to validate Equation 10.3.3-12 does not exist beyond that value. Thus, design curves $(b/t)_e$ versus

($b\Delta/2r$) are generated by using a Poisson's ratio of 0.33 for all different values of r/t and such curves will be valid for all the materials.

10.3.3.1.2.2 Curved Plate under Axial Compression Cut-off Line for Transition Range Behavior to Long Plate Behavior

There is a boundary between the transition range and long curved plate buckling behavior. This is referred to as a cut-off line. The transition range buckling behavior is represented by a fitted curve between the flat plate behavior and the long-range behavior. For a particular value of r/t, the curve for $(b/t)_e$ as a function of ($b\Delta/2r$) becomes tangent to the line representing a long curved plate. Thus for a range of r/t, a curve passing through all such tangent points is called the cut-off line. For the points falling to the right side of the cut-off line, the behavior of the plate is governed by long curved plate buckling.

The curved plate buckling coefficient in compression, k_{cu} at the tangent point is defined as k_{cul} by Equation 10.3.3-13.

$$k_{cu} = k_{cul}$$ **Equation 10.3.3-13**

The coefficient, k_{cul} at the tangent point is defined by Reference 10-27.

$$k_{cul} = 1.162\,\Omega\,Z_b$$ **Equation 10.3.3-14**

Knowing k_{cul} from the above equation for a given value of r, t, and Ω, the buckling coefficient k_{cu} is computed from Equation 10.3.3-13. Substituting Equation 10.3.3-13 into Equation 10.3.3-6, $(b/t)_e$ is computed as follows:

$$\left(\frac{b}{t}\right)_e^2 = \frac{12\left(1-\nu_e^2\right)^{0.5}}{\pi^2}\left\{\frac{1}{1.162\,\Omega\left(\frac{t}{r}\right)}\right\}$$ **Equation 10.3.3-15**

For different values of r/t and corresponding values of Ω, and for a given value of Poisson's ratio, $(b/t)_e$ can be calculated from the above equation at the tangent point. Thus, a cut-off line is established by using a Poisson's ratio of 0.33.

10.3.3.1.2.3 Curved Plate under Axial Compression in the Transition Range for Aspect Ratio a/b > 1

For aspect ratio a/b > 1, the flat plate buckling coefficient depends upon the number of half-wave buckle shapes and can be obtained from Figure 10.3.2-1 for a simply supported flat plate. Conservatively taking the limiting value of k_{pl} as defined by the following equation for simply supported edges:

$$k_{pl} = 4$$ **Equation 10.3.3-16**

This assumption is equivalent to setting Δ = 1 in Equation 10.3.3-11. Knowing k_{pl}, the buckling coefficient k_{cu} is computed from Equation 10.3.3-9 by using Equation 10.3.3-10. Equation 10.3.3-12, developed for a/b < 1, can also be used for a/b > 1 by setting $\Delta = 1$. Thus, the same design curves generated for a/b < 1 can be used for a/b > 1 if $\Delta = 1$ is substituted in $(b\Delta/2r)$.

10.3.3.1.3 Curved Plate under Axial Compression in the Long Curved Plate Region

The compression buckling behavior of a long curved plate is the same as that of a long cylinder. The $(b/t)_e$ of a long curved plate is computed using the method described in Reference 10-28 which was developed based on the correlation between theory and test data. Using the correlation factor ζ between the test data and theory, $(b/t)_e$ is defined as:

$$\left(\frac{b}{t}\right)_e^2 = \left(\frac{r}{t}\right)\left(\frac{1}{\zeta}\right)\sqrt{3\left(1-\nu_e^2\right)}$$ **Equation 10.3.3-17**

The correlation factor ζ is defined as:

$$\zeta = 1 - 0.901\left(1 - e^{-\psi}\right)$$ **Equation 10.3.3-18**

Sub-correlation factor ψ is defined in terms of the ratio r/t as:

$$\psi = \left(\frac{1}{16}\right)\sqrt{\left(\frac{r}{t}\right)} \qquad \left(for\ \frac{r}{t} \le 1500\right)$$ **Equation 10.3.3-19**

It is to be noted that the maximum value of r/t is restricted to 1500 since test data to validate Equation 10.3.3-17 does not exist beyond that value.

10.3.3.1.4 Compression Design Curves

Design curves presented in Figures 10.3.3-3 to 10.3.3-5 are for curved plates buckling in the transition range and long range. Design curves are valid for r/t < 1500 because test data do not exist beyond 1500 for validating the equations. Design curves provide a convenient way to obtain $(b/t)_e$ for a given curved plate. It is not required to compute the buckling coefficients and $(b/t)_e$ from the equations presented earlier in the discussion. As noted earlier, only geometric properties are needed to use the design curves. The required geometric properties are: (1) dimensions a and b of the plate, (2) the radius of curvature, r of the plate, and (3) the thickness, t of the plate. The technical discussion in Sections 10.3.3.1.2 and 10.3.3.1.3 was provided to give the analyst an understanding of the design curves.

Figure 10.3.3-3 provides the relationship between a/b and Δ represented by the Equation 10.3.3-11 for the curved plate having an aspect ratio, a/b < 1. Figure 10.3.3-4 shows the design curves for a simply supported curved plate in the transition range. Design curves $(b/t)_e$ versus $(b\Delta/2r)$ are generated by using Equation 10.3.3-12 for a Poisson's ratio of 0.33 for r/t between 100 and 1500. These curves are valid for all materials since a small change in Poisson's ratio has a negligible effect as noted earlier. Figure 10.3.3-4 also shows a cut-off line separating the long plate buckling from the transition range buckling. A curved plate is said to exhibit long range plate behavior if a point corresponding to a combination of $(b\Delta/2r)$ and r/t falls on the right side of the cut-off line. Figure 10.3.3-5 provides the curve between $(b/t)_e$ and r/t for a long plate.

To perform an analysis of a curved plate in axial compression using the design curves:

1. From the geometric properties of the curved plate, compute the aspect ratio, a/b and r/t.
2. If a/b < 1, obtain Δ from Figure 10.3.3-3. For a/b > 1, assume $\Delta = 1$.
3. Compute $(b\Delta/2r)$ and knowing r/t, ascertain if the plate exhibits transition or long range plate buckling from Figure 10.3.3-4.

   a. For transition-range, obtain $(b/t)_e$ from Figure 10.3.3-4.
   b. For long-range obtain $(b/t)_e$ from Figure 10.3.3-5.
4. Use $(b/t)_e$ to determine the initial buckling stress per Section 10.3.1.6.

Note that in case of a flat rectangular plate, variables k and K are obtained first before computing $(b/t)_e$, unlike a curved plate, in which $(b/t)_e$ is directly obtained from the design curves. The allowable initial compression buckling stress can be calculated by solving Equation 10.3.3-8 iteratively using the procedure described in Section 10.3.1.6 or it may also be read directly from a plot of $(b/t)_e$ versus allowable initial buckling compression stress for a particular material. The allowable curves are generated by using program SM33 of the IDAT suite of programs. Sample curves are shown in Figures 10.3.1-7 to 10.3.1-9 for the materials shown in Table 10.3.1-2.

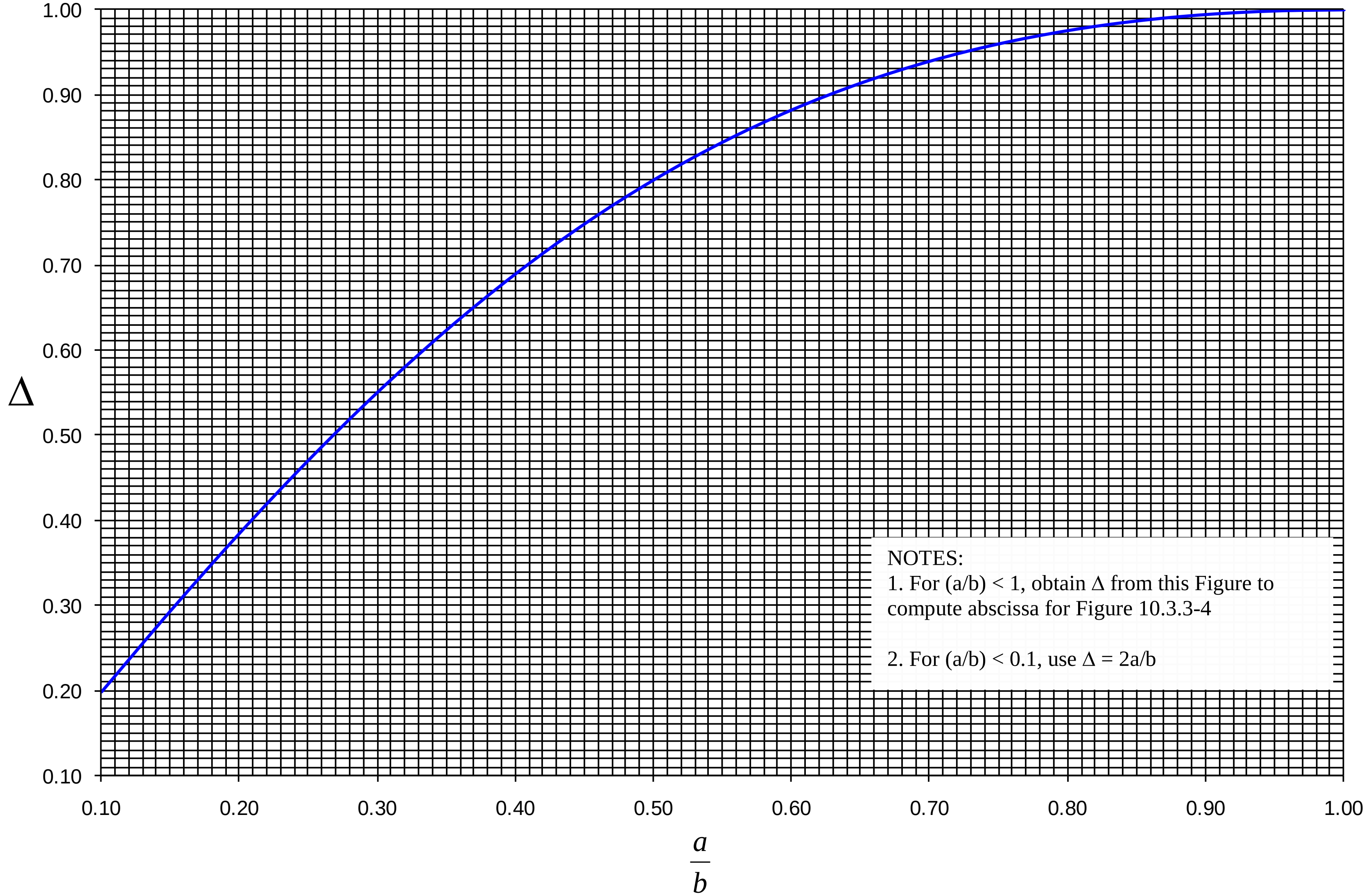


Figure 10.3.3-3: Parameter Δ for Figure 10.3.3-4 for (a/b) < 1 for Curved Plates

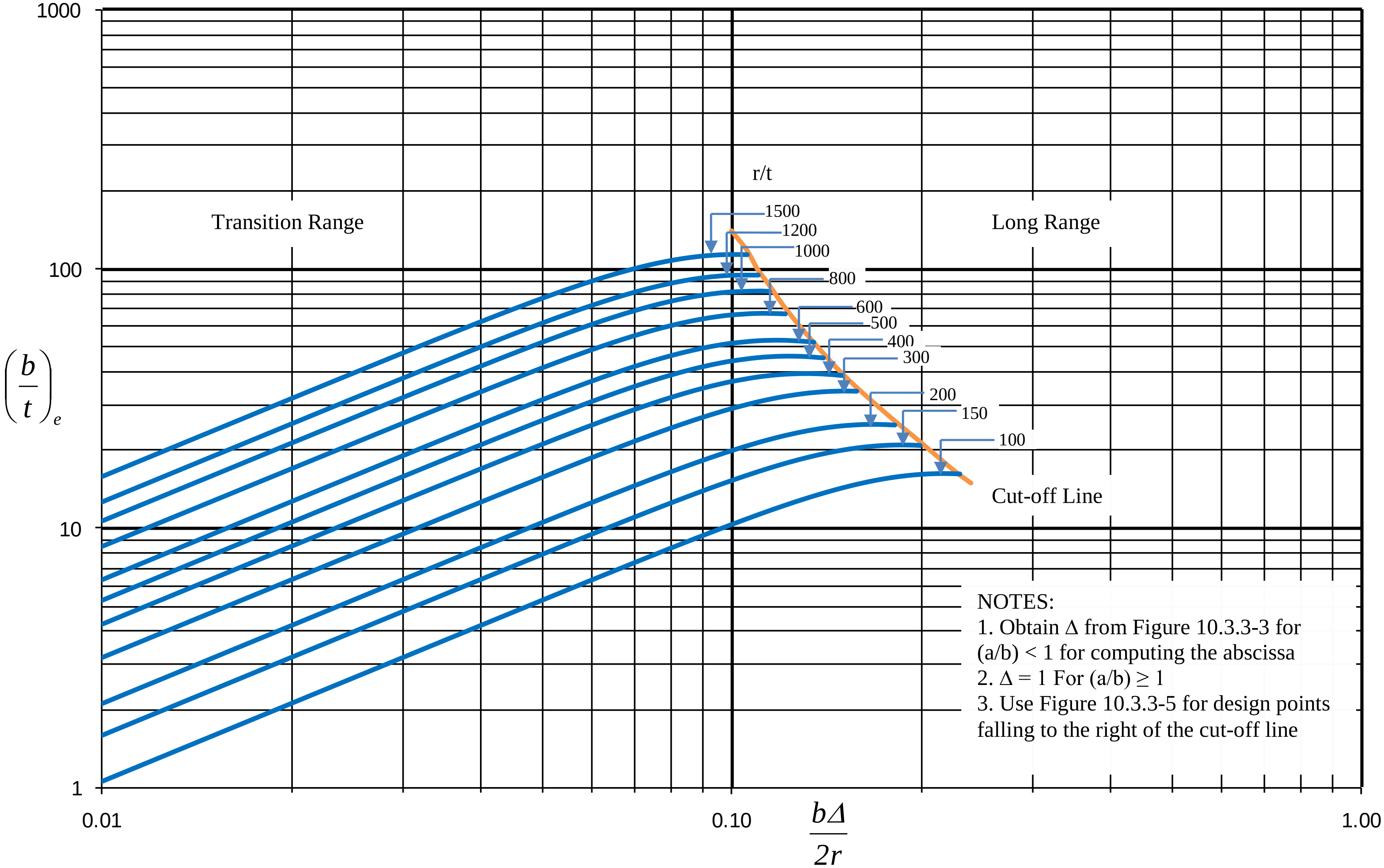


Figure 10.3.3-4: $(b/t)_e$ for a Simply Supported Curved Plate in Compression

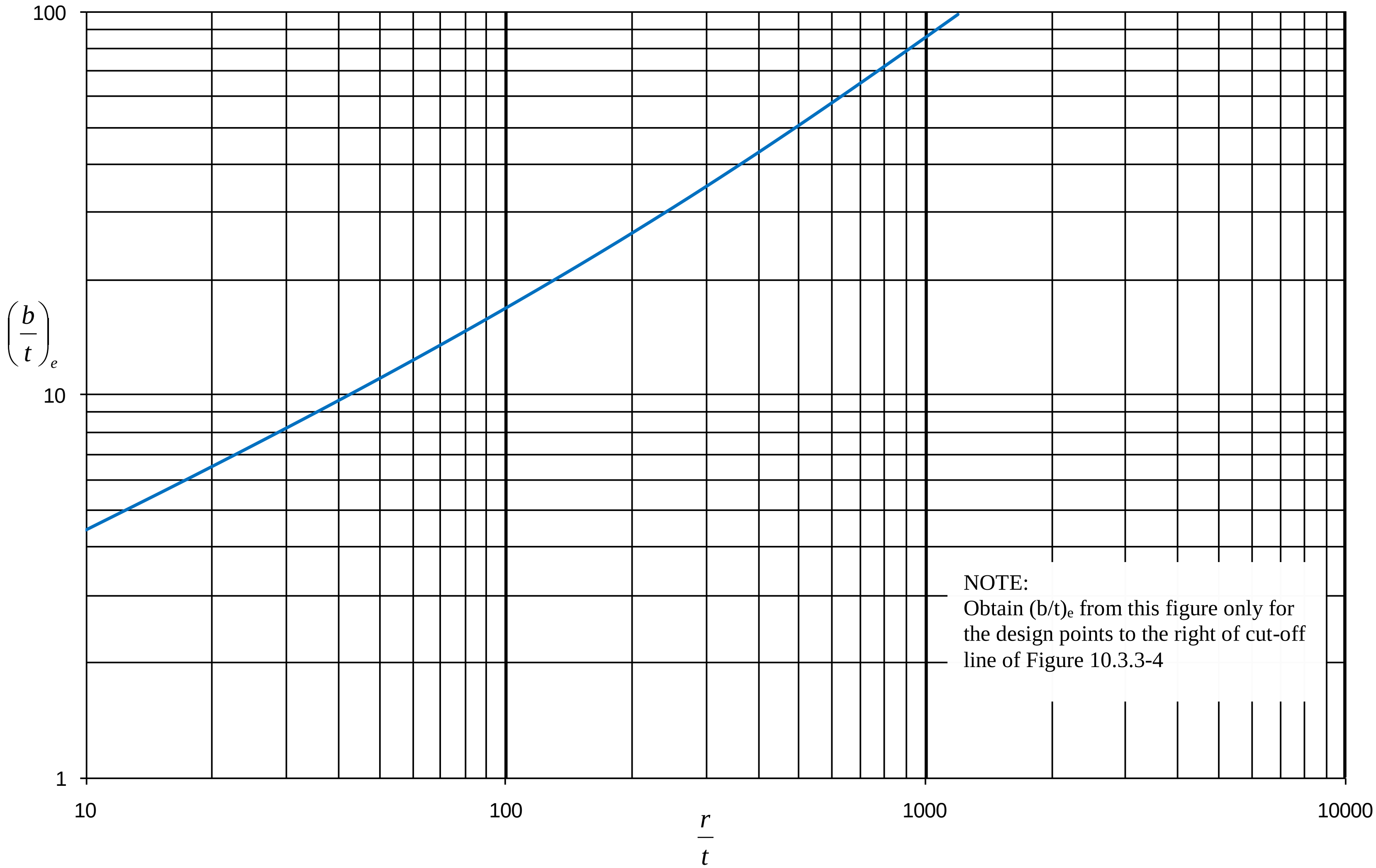


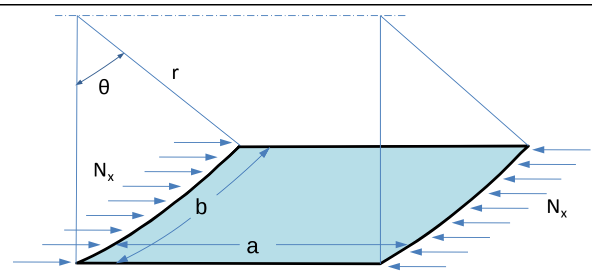
Figure 10.3.3-5: $(b/t)_e$ for Long Curved Plates in Compression

10.3.3.1.5 Example - Curved Plate under Axial Compression

Given: A 7075-T6 (bare sheet) curved plate is considered to be simply supported on all sides. The dimensions of the plate are:
t = 0.125 in, a = 10.0 in, b = 20.0 in,
r = 25 in.

Axial load, N_x = 1900 lb/in

Calculate the margin of safety in buckling.



Solution: The margin of safety is calculated either by using the allowable stress chart generated by using SM33 program of IDAT or solving the Equation 10.3.3-8 iteratively. Here the solution is presented by using the allowable initial buckling stress chart of Figure 10.3.1-10.

Material properties for 7075-T6 (bare sheet, 0.04 – 0.125) are obtained from Table 10.3.1-2 (developed from METDB) or can be obtained directly from the METDB database of IDAT.

F_{tu} = 80 ksi, F_{cy} = 71.0 ksi, F_{su} = 40.0 ksi, E_c = 10500 ksi, n_c = 12.0, ν_e = 0.33 (Poisson's ratio obtained from METDB of IDAT)

Solution: Using the Allowable Stress Chart

Calculation	**Equation/ Figure**	**Result**
Aspect ratio a/b	a/b = 10.0/20.0 = 0.5	a/b = 0.5
Plate geometric parameter, Δ	Since a/b < 1, use Figure 10.3.3-3 for Δ. For a/b = 0.5, Δ = 0.8	Δ = 0.8
Calculate (bΔ)/(2r)	(bΔ)/(2r) = (20 x 0.8)/(2 x 25)=0.32	(bΔ)/(2r) = 0.32
Obtain $(b/t)_e$	r/t = 25/.125 = 200. For (bΔ)/(2r) = 0.32 and r/t = 200, the point falls on the right side of the cut-off line in Figure 10.3.3-4. This means it is a long curved plate, thus use Figure 10.3.3-5. For r/t = 200 read $(b/t)_e$ = 26	$(b/t)_e$ = 26
Read initial buckling stress	From **Figure 10.3.3-6** for $(b/t)_e$ = 26 and curve 5 f_{cr} = 16 ksi =16,000 psi	f_{cr} = 16,000 psi
Allowable load	$P_{all} = f_{cr}$ (bt) = 16,000[(20)(0.125)] = 40,000 lb	P_{all} = 40,000 lb
Applied load	P = N_x b = 1900(20) = 38,000 lb	P = 38,000 lb
Margin of Safety	MS = (P_{all} / P) - 1 Equation 2.5.0-1 = (40,000 / 38,000) – 1 = + 0.05	MS = + 0.05

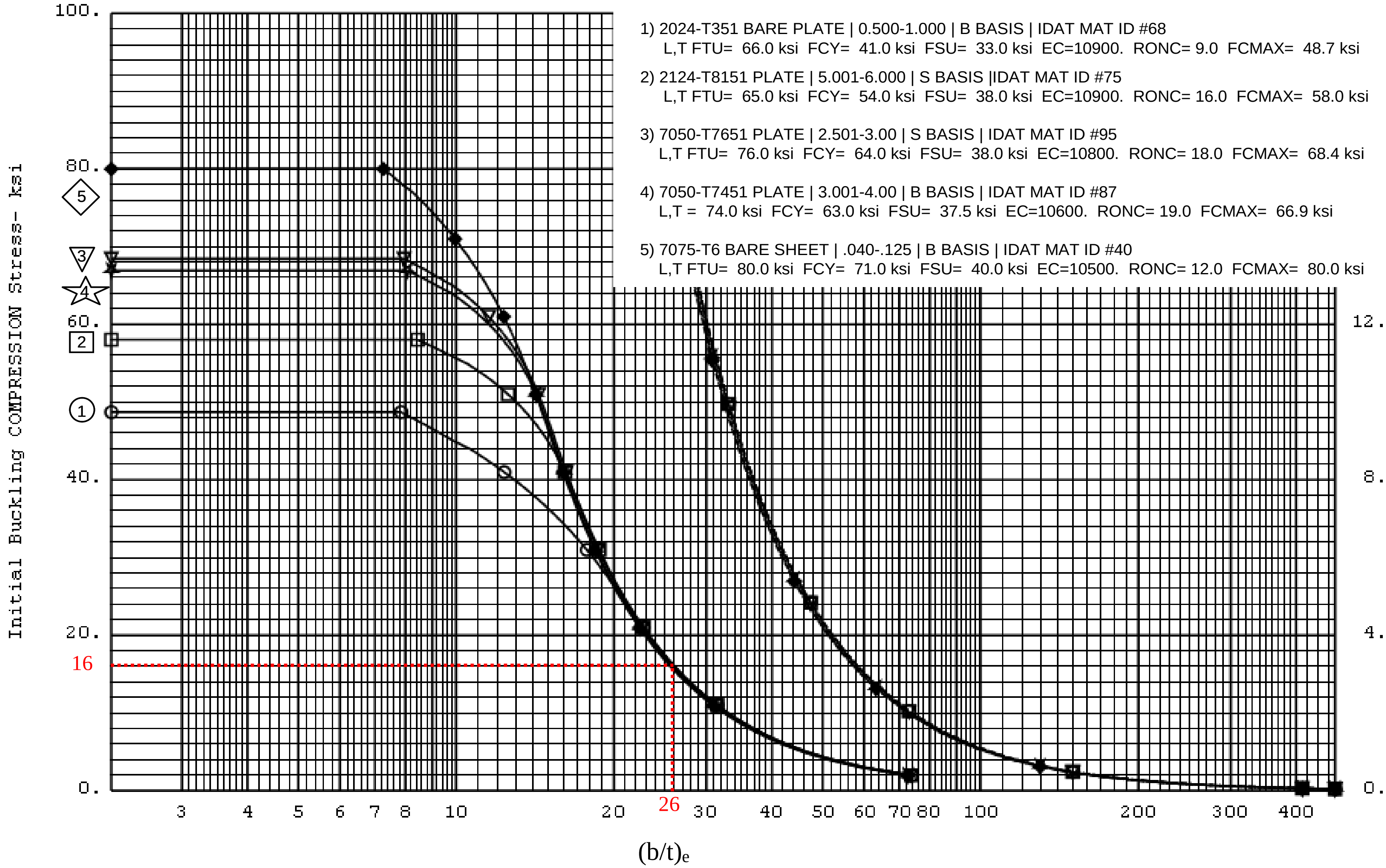
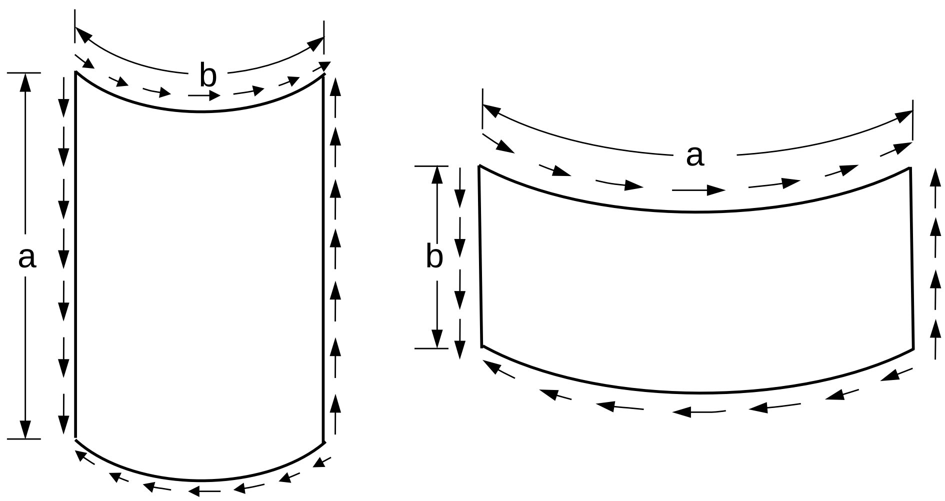


Figure 10.3.3-6: To Illustrate the use of the Figure 10.3.1-7

10.3.3.2 Curved Plates under Shear

The usual convention adopted for plates loaded in shear requires that the b dimension be the short side. When this is applied to curved plates, some ambiguity arises, since such plates may be curved either along the short edge or along the long edge. For clarity, therefore, a long curved plate is defined as one in which the long side is parallel to the axis of the cylinder of which this plate is a segment, and a wide plate is defined as one in which the long side is perpendicular to the cylinder axis. Figure 10.3.3-7 shows the definition of long and wide curved plates.



Long Curved Plate
b is circumferential
b ≤ a

Wide Curved Plate
b is axial
b ≤ a

Figure 10.3.3-7: Curved Plate Definition for Shear Loading

It was observed in Section 10.3.3.1 that the buckling behavior of a curved plate subjected to axial compression does not correlate well with the theory and empirical data in conjunction with the theory was used to improve the correlation with the test data. When the curved plate is subjected to pure shear load, Batdorf et al., Reference 10-31, found that the buckling behavior of a curved plate predicted by the small-deflection theory is in good agreement with the test data.

The buckling behavior of a long curved plate under shear is discussed in Section 10.3.3.2.1 and the buckling behavior of a wide curved plate under shear is discussed in Section 10.3.3.2.2.

10.3.3.2.1 Long Curved Plate under Shear

Figure 10.3.3-2 shows the generic cylindrical element subjected to membrane loads: axial load, N_x, circumferential load, N_y and shear load, N_{xy}. For pure shear load the loads other than shear are zero, i.e., $N_x = N_y = 0$. For a long curved plate the coordinate system is defined as shown in **Figure 10.3.3-8**, with the X- axis along the axial direction and Y-axis along the circumferential direction. The short dimension b is along the circumferential direction and the long dimension a is along the axial direction.

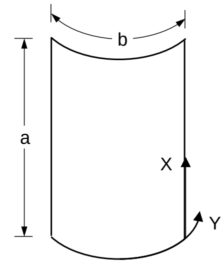


Figure 10.3.3-8: Coordinate System for a Long Curved Plate

Small deflection theory is used which assumes that the deflections are infinitesimal for predicting the behavior of a long plate subjected to a shear load. The mathematical formulation of the governing differential equation is the same as discussed in Section 10.3.3.1.1 and the same governing Equation 10.3.3-2 holds for the long curved plate subjected to shear. The governing assumptions are the same as listed in Section 10.3.3.1.1. Thus, the solution of the Equation 10.3.3-2 for shear loading is given by the following equation, Reference 10-31:

$$f_{cre} = \frac{\pi^2 E_c}{12(1-\nu_e^2)}\left(\frac{t}{b}\right)^2 k_{su} \qquad \textbf{Equation 10.3.3-20}$$

Where,
f_{cre} is the initial elastic buckling stress (psi)
k_{su} is the shear buckling coefficient for the curved plate
E_c is the compression modulus of the material (psi)
ν_e is the elastic Poisson's ratio of the material

Define $(b/t)_e$ as follows:

$$\left(\frac{b}{t}\right)_e^2 = \frac{12\left(1-\nu_e^2\right)}{\pi^2}\left(\frac{b}{t}\right)^2\left(\frac{1}{k_{su}}\right) \qquad \textbf{Equation 10.3.3-21}$$

Using Equation 10.3.3-21, buckling Equation 10.3.3-20 is written as:

$$f_{cre} = \frac{E_c}{\left(\frac{b}{t}\right)_e^2} \qquad \textbf{Equation 10.3.3-22}$$

Equation 10.3.3-22 is the plate buckling equation in the elastic region and is modified in the inelastic region by using plasticity correction factors. The plasticity correction factor used for curved plates under axial compression, Section 10.3.3.1.1.1, is also valid for curved plates subjected to shear. Using the plasticity correction factor η defined in Equation 10.3.1-8, the initial buckling stress Equation 10.3.3-22 becomes

$$f_{cr} = \frac{\eta E_c}{\left(\frac{b}{t}\right)_e^2} \qquad \textbf{Equation 10.3.3-23}$$

The shear buckling coefficient, k_{su}, for a simply supported and clamped long curved plate is shown in Figure 10.3.3-9 and Figure 10.3.3-10, respectively.

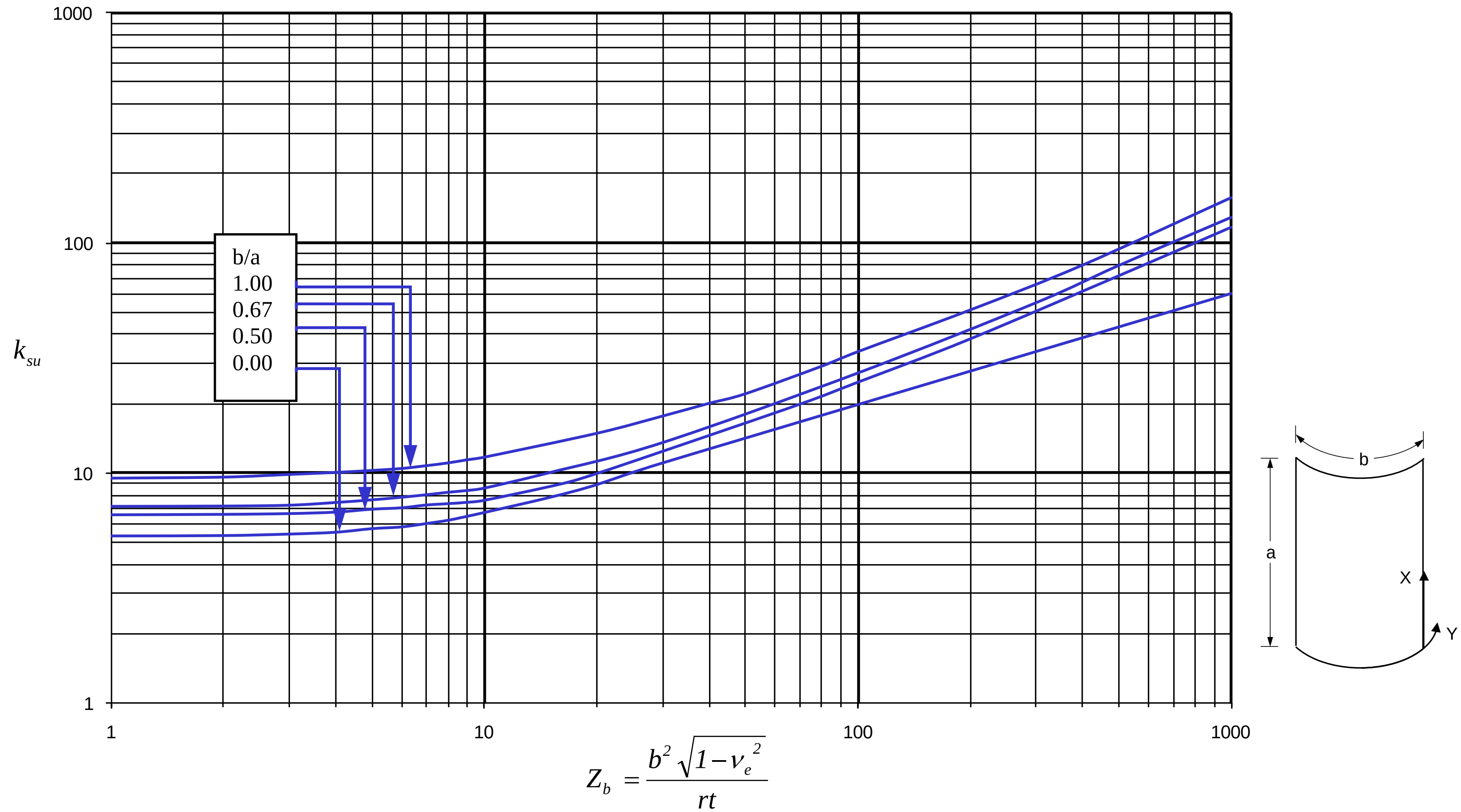
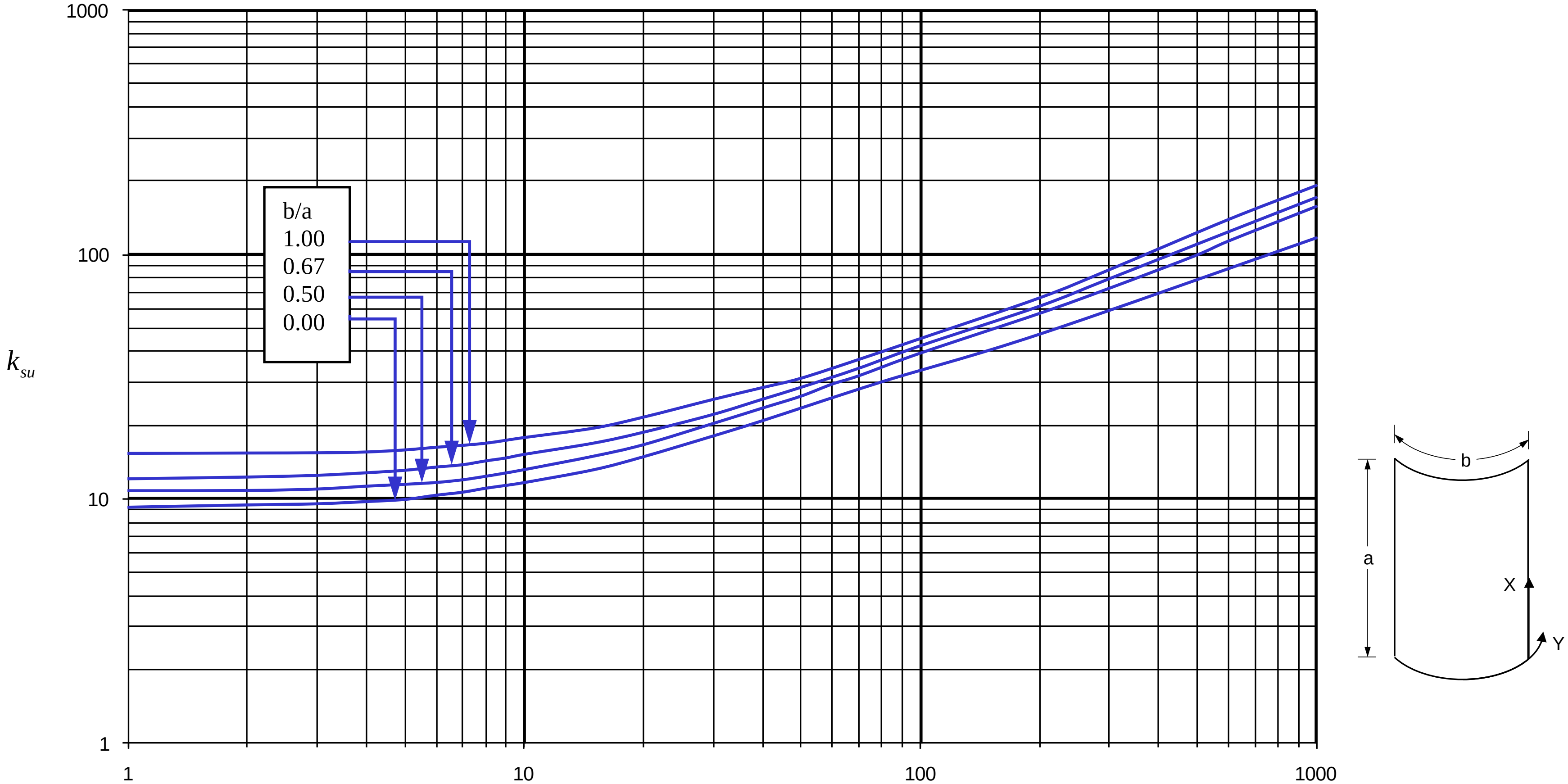


Figure 10.3.3-9: Shear Buckling Coefficient for a Simply Supported Long Curved Plate



$$Z_b = \frac{b^2\sqrt{1-\nu_e^{\,2}}}{rt}$$

Figure 10.3.3-10: Shear Stress Buckling Coefficient for a Clamped Long Curved Plate

Knowing the dimension b, radius r, thickness t and material's Poisson ratio ν_e, the plate parameter Z_b can be calculated using Equation 10.3.3-1. This calculated value of Z_b is used to obtain the corresponding value of shear stress buckling coefficient k_{su} from Figure 10.3.3-9 for a simply supported curved plate. For the curved plate having fixed boundary conditions, Figure 10.3.3-10 is used to obtain shear stress buckling coefficient k_{su} for a given value of Z_b. Knowing k_{su}, Equation 10.3.3-21 is used to calculate $(b/t)_e$ of a curved plate. For a given $(b/t)_e$, the procedure for computing an initial buckling shear stress is the same as for a flat rectangular plate covered in Section 10.3.2.2. Allowable initial shear buckling stress curves can be used for obtaining the allowable shear buckling stress for a given value of $(b/t)_e$ for a particular material. The allowable curves are generated by using program SM33 of the IDAT suite of programs. Sample curves are shown in Figures 10.3.1-10 to 10.3.1-12 for the materials shown in Table 10.3.1-2.

10.3.3.2.2 Wide Curved Plate under Shear

A wide curved plate is defined as one in which the length of the curved edge is greater than that of the straight edge as shown in Figure 10.3.3-11. The coordinate system and the dimensions used for a wide rectangular plate are shown in Figure 10.3.3-11. The mathematical formulation for a wide curved plate is same as for a long curved plate, per Section 10.3.3.2.1 if short dimension b of a wide plate is used. It is to be noted that short dimension b of a wide curved plate is parallel to the axis of the cylinder of which this plate is a segment unlike for a long curved plate where short dimension b is circumferential. Thus, all the equations of the long curved plate developed using the shortest dimension b are valid for the wide curved plate.

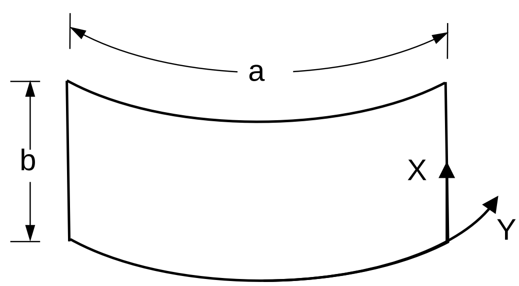
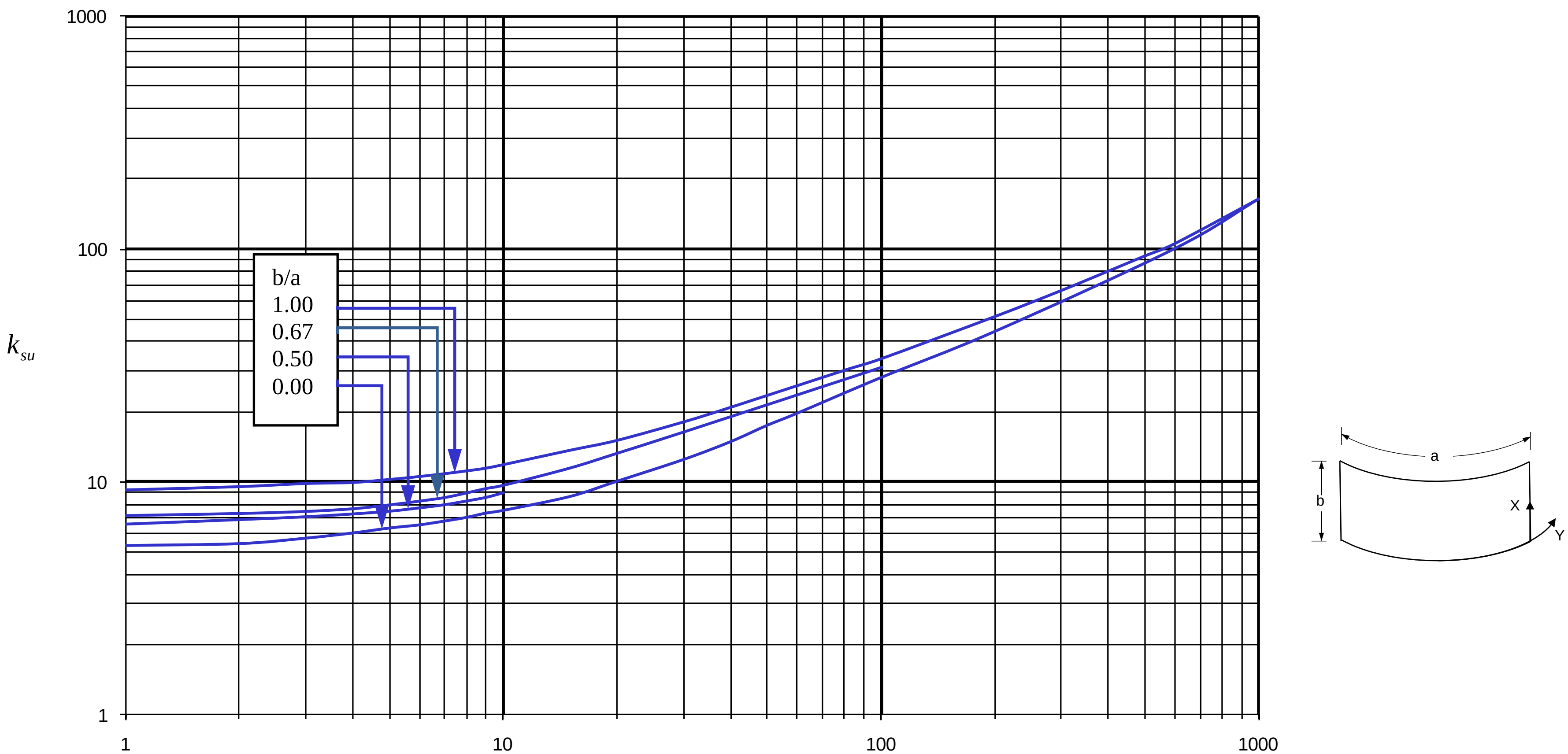


Figure 10.3.3-11: Coordinate System for a Wide Curved Plate

The shear stress buckling coefficient k_{su} is presented for two cases of plate boundary conditions. Figure 10.3.3-12 presents k_{su} for a wide curved plate simply supported on all four sides and Figure 10.3.3-13 presents k_{su} for a wide curved plate fixed on all four sides.



$$Z_b = \frac{b^2 \sqrt{1-\nu_e^{\,2}}}{rt}$$

Figure 10.3.3-12: Shear Stress Buckling Coefficient for a Simply Supported Wide Curved Plate

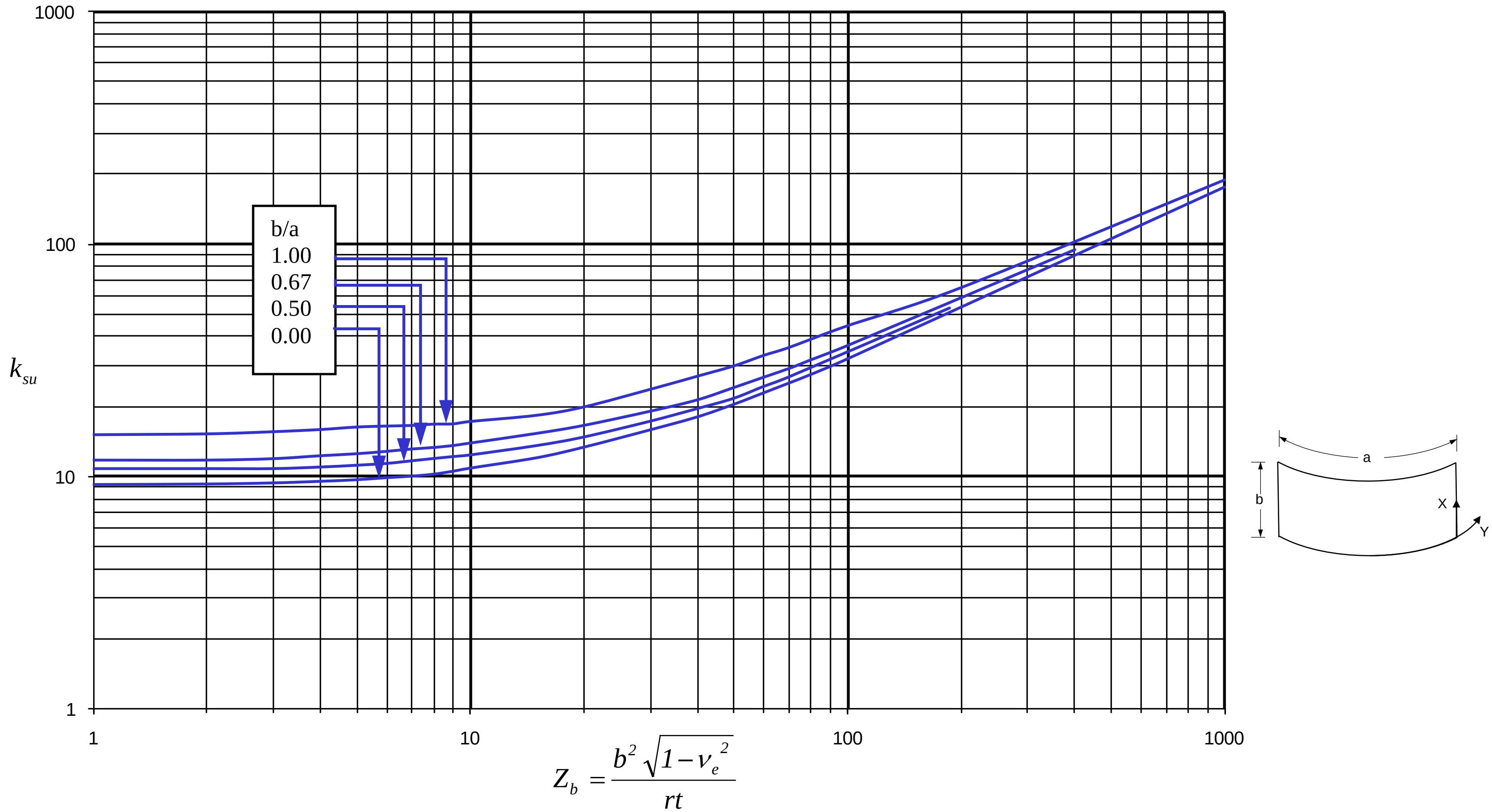
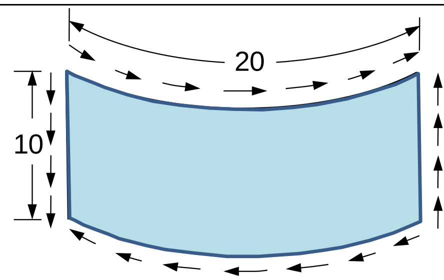


Figure 10.3.3-13: Shear Stress Buckling Coefficient for a Clamped Wide Curved Plate

10.3.3.2.3 Example - Curved Plate under Shear

Given: A 7075-T6 (bare sheet) curved plate is considered to be clamped on all sides. The thickness and radius of the plate are: t = 0.125 in, r = 25 in. Shear load, N_{xy} = 3,100 lb/in Calculate the margin of safety in buckling.	

Solution:
Material properties for 7075-T6 (bare sheet, 0.04 – 0.125) are obtained from Table 10.3.1-2 (developed from METDB) or can be obtained directly from the METDB database of IDAT.

F_{tu} = 80 ksi, F_{cy} = 71.0 ksi, F_{su} = 40.0 ksi, E_c = 10500 ksi, n_c = 12.0, ν_e = 0.33 (Poisson's ratio obtained from METDB of IDAT)

Solution: Using the Allowable Stress Chart of Figure 10.3.3-14

Calculation	Equation/ Figure	Result
Determine whether it is a wide plate or a long plate	Short dimension is 10 in and long dimension is 20 in Long side is normal to the axis of the cylinder; therefore it is a wide plate. b = 10 in, and a = 20 in b/a = 10/20 = 0.5	a=20.0 in., b = 10.0 in. A wide plate b/a = 0.5
Calculate plate parameter, Z_b	From Equation 10.3.3-1 $$Z_b = \frac{b^2\sqrt{1-\nu_e^2}}{rt}$$ $$Z_b = \frac{10^2\sqrt{1-0.33^2}}{25 \bullet 0.125} = 30.207$$	Z_b = 30
Shear buckling coefficient, k_{su}	From Figure 10.3.3-13, for Z_b = 30, and b/a = 0.5 k_{su} = 17	k_{su} = 17
Calculate $(b/t)_e$	From Equation 10.3.3-21 $$\left(\frac{b}{t}\right)_e^2 = \frac{12\left(1-\nu_e^2\right)}{\pi^2}\left(\frac{b}{t}\right)^2\left(\frac{1}{k_{su}}\right)$$ $$\left(\frac{b}{t}\right)_e^2 = \frac{12\left(1-0.33^2\right)}{\pi^2}\left(\frac{10.0}{0.125}\right)^2\left(\frac{1}{17}\right)$$ $(b/t)_e$ = 20.196	$(b/t)_e$ = 20.2

Read initial buckling stress	From Figure 10.3.3-14, curve 5, for $(b/t)_e$ = 20.2: f_{cr} = 25.5 ksi =25,500 psi	f_{cr} = 25,500 psi
Allowable load/in	$P_{all} = f_{cr}$ (t) = 25,500 (0.125) = 3,187.5 lb/in	P_{all} = 3,188 lb/in
Applied load/in	P = N_{xy} = 3,100 = 3,100 lb/in	P = 3,100 lb/in
Margin of Safety	MS = (P_{all} / P) - 1 Equation 2.5.0-1 = (3,188 / 3,100) – 1 = + 0.028	MS = + 0.03

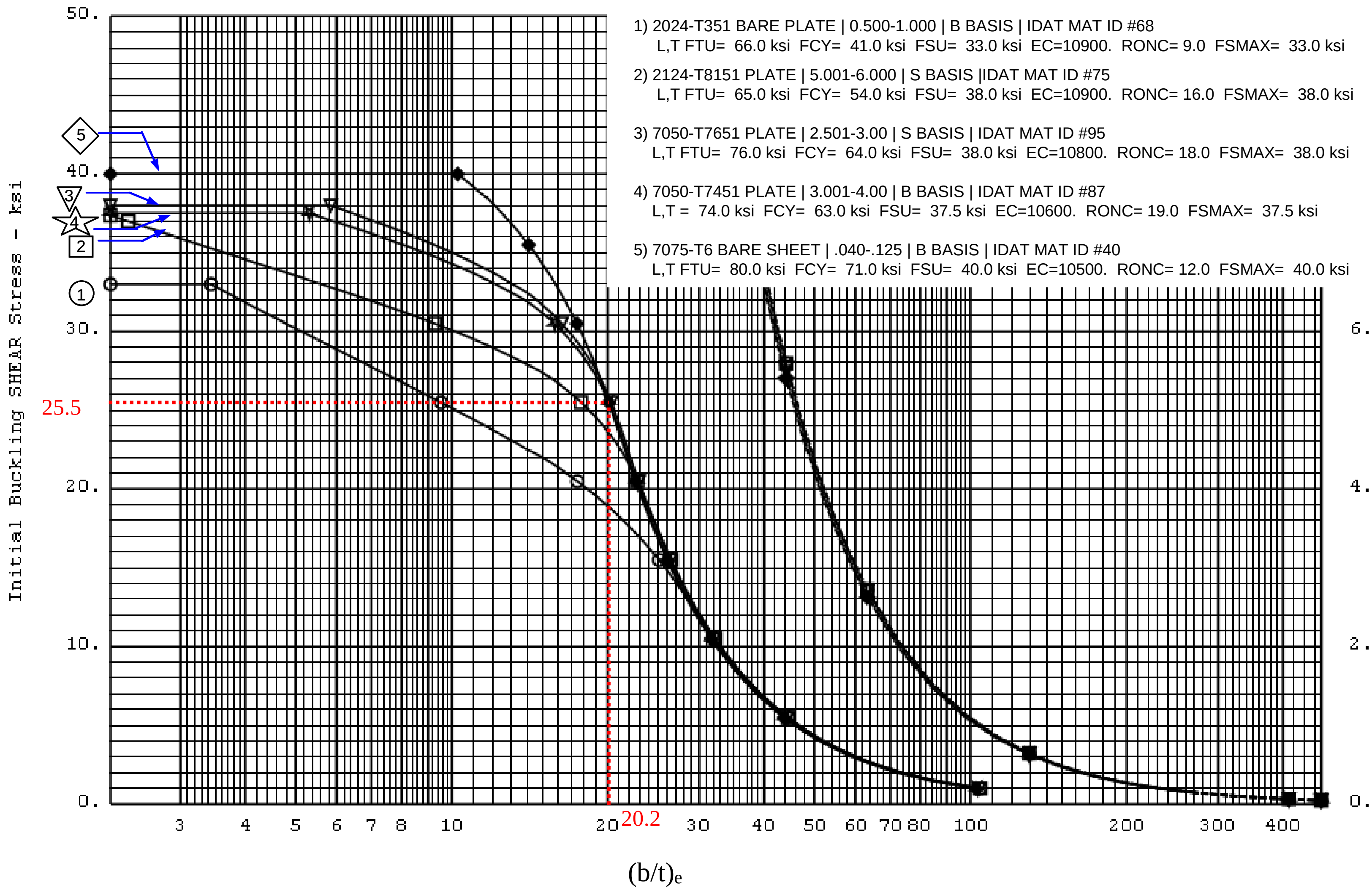


Figure 10.3.3-14: To Illustrate the use of the Figure 10.3.1-10 Initial Buckling Shear Stress for Aluminum Plates

10.3.3.3 Curved Plate under Combined Shear and Longitudinal Compression

Sections 10.3.3.1 and 10.3.3.2, respectively, provide the necessary design aids for computing the allowable compressive and shear loads on a curved plate. It was indicated earlier that the predicted load from the linear theory and test data do not agree for axially loaded curved plate. However, curved plate behavior under shear load is in close agreement with the theory. Consequently, empirical data are needed to formulate the interaction equation between shear and axial load. Batdorf *et al*, Reference 10-32, developed the theoretical solution for the buckling stresses of a curved plate subjected to the combined axial and shear load. The interaction equation was validated by Schildcrout *et al*, Reference 10-33, by tests on curved plates subjected to combined axial and shear loads. The interaction equation is presented in Section 2.5.2, case number 9, and is repeated here.

$$R_c + R_s^2 = 1$$

Where,
R_c is the compressive axial stress ratio
R_s is the shear stress ratio

Margin of safety is calculated per the equation presented in Section 2.5.2, case number 9, and is repeated here.

$$MS = \frac{2}{R_c + \sqrt{R_c^2 + 4R_s^2}} - 1$$

The axial stress ratio R_c, is calculated by dividing the applied compressive stress by the allowable compressive stress obtained per Section 10.3.3.1. Section 10.3.3.1.4 provides an example to show the procedure needed to obtain the allowable compressive stress. The shear stress ratio, R_s, is calculated by dividing the applied shear stress by the allowable shear stress obtained per Section 10.3.3.2. Section 10.3.3.2.3 provides an example to show the procedure needed to obtain the allowable shear stress. With known R_c and R_s, the margin of safety is calculated using the above equation.

10.3.3.4 Stiffened Curved Plates

Refer to Program-specific and customer-generated guidance, GD Structures Analysis Manual, Volume I, section 2.2.2 (Reference 10-2), and Timoshenko, chapter 11 (Reference 10-10).

10.3.4 FEM Based Calculation

A common approach for plate/shell stability analysis, in cases where geometry and/or boundary conditions do not lend themselves to hand analysis, is finite element modeling. FEM tools commonly used for this purpose include MSC/NASTRAN, ABAQUS, STAGS, and PANDA2. As with all FEM analyses, care must be taken in modeling details, in order to produce accurate results. The Integrated Detailed Analysis Tool (IDAT) suite includes the parametric FEM pre/postprocessor SPAM, which automates much of the FE model-building process.

10.3.5 Unix/PC-Based Calculation

Another common approach for plate/shell stability analysis is through the use of stand-alone analysis tools such as IDAT or computational aids. Additionally, a variety of uncontrolled MSExcel- and Mathcad-based computational aids are available on individual Programs. Individual analysts are required to verify the accuracy of any such uncontrolled aids prior to initial use.

10.4 Snap-Through Instability

Panel snap-through can occur for thin panels when loaded by an external normal loading toward the center of curvature creating hoop compression loads in the mid-bay region. The panel experiences a large displacement through a point of geometric instability due to the pressure-induced deflected shape combined with the compressive panel load. During snap-through, the panel experiences an abrupt change in deflection after passing through the instability point until the panel reaches a stable, deformed shape in which the panel curvature is reversed. Figure 10.4.0-1 illustrates this effect for the analogous case of a simply supported arch with a lateral pressure. As the pressure is increased from point *a* to *b*, the rise of the arch is reduced to *b*, *i.e.*, the pressure-induced deflection reduces the initial rise of the arch. When point *b* is reached, any additional increase in load causes the arch to jump to some deflection *b′*. The deflection and load continue to increase along the curve to the right of *b′*. Since only a slight increase in load is required to change the deflection from *b* to *b′*, this region is unstable.

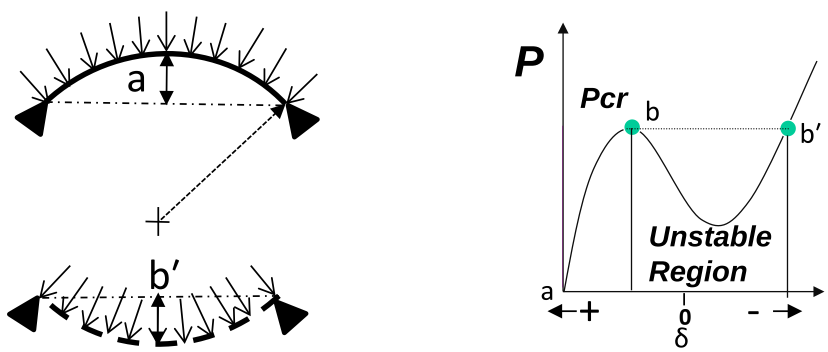


Figure 10.4.0-1 Snap-Through Instability for a Simply Supported Arch

10.4.1 Introduction and Criteria

Snap-through action, described above, results in a dynamic reduction in strain energy as the deflected shape changes rapidly with relatively constant applied load. The analysis of the dynamic behavior and the possible consequences of permanent set in metals, delamination in composites or overloading of adjacent structure is in the time domain and is beyond the scope of normal nonlinear static strength analysis. The preferred analysis approach is the use of ABAQUS RIKS-solution models.

However, it is recognized that because much of airframe structure is already modeled as NASTRAN FEM, analysts may prefer to continue with the use of a NASTRAN model. Thus, to aid in the NASTRAN analysis, a slope-change abruptness test was developed for use with NASTRAN Solution 106/600 model output in the prediction/avoidance of the dynamic snap-through situations. Compliance with the Slope Change criteria of Section 10.4.2 will prevent the violent dynamic response, although geometric nonlinearity may still occur and shall be considered in the analysis including effects on adjacent structure due to load redistribution in strength and life calculations.

In the absence of specific program criteria on snap-through, it is recommended that structure be sized to prevent snap-through at load levels below ultimate load. This requirement may be reduced to no panel snap-through at Design Limit Load or some percent of design limit load depending on the program detrimental deformation criteria, on a by-exception basis if approved by cognizant engineering management and customer. If snap-through is allowed below ultimate load, then nonlinear effects shall be accounted for in the analysis, including any redistribution of load to adjacent structure.

All curved panels subjected to radially inward acting point loads or crushing pressure loads, *i.e.*, pressures that induce hoop compression into the panel, shall be analyzed for snap-through in addition to the typical panel stability in-plane analysis. At a minimum, the panel snap-through response must be checked for the maximum crushing pressure load condition, the maximum crushing point load condition, *e.g.*, step loading, and the mechanical load condition producing the minimum eigenvalue, combined with case-consistent pressure, mechanical, and thermal loading, as applicable. While the traditional buckling eigenvalue of the panel is not an indicator of load levels associated with snap-through, studies have shown that the load case that produces the lowest eigenvalues are often, but not always, critical for snap-through.

It has been observed that distribution of the pressure load can make a significant difference in snap-through susceptibility. For example, for the same total load, a sinusoidally distributed pressure on a panel will result in a higher critical snap-through load than if the pressure is uniformly distributed. In addition, panels analyzed for snap-through using a distributed pressure gradient have been observed to snap-through in static testing with a lower, uniformly distributed load.

Note: Panels that meet the snap-through requirements when loaded with a pressure gradient as might be seen on-aircraft, can undergo snap-through when loaded with a uniform pressure as might be found in static or fatigue testing. This should be evaluated during test planning and prior to testing.

Thermal loading adds another complexity to the analysis of snap-through buckling. Often, for hybrid structures of graphite composite skins and metallic substructure, elevated temperature usually produces tension in the composite skins due to the coefficient of thermal expansion mismatch between the skin and the metallic substructure. Because of the differences in the coefficients of thermal expansion, aluminum-graphite composite is usually the most severe. This tension is beneficial for the composite skin snap-through thus the critical cases are often room temperature. While thermal loading, especially low temperatures, may cause thermally-induced buckling of composite panels, it does not produce the pressure-driven snap-through response discussed in this section. For purposes of snap-through analysis, thermal loads are held constant and not varied with the mechanical loads in the equations of Section 10.4.2.

The critical lateral loading depends on the dimensions and rigidity of the panel, the flexibility of the support structure, the type of load distribution and the initial curvature of the arch. The buckling mode of the arch is dependent on the rise of the arch as is shown in Figure 10.4.1-1. For a large rise, the effect of any axial load resulting in extension of the centerline of the curved panel is negligible and the panel is considered inextensible. This results in an anti-symmetrical buckling mode initially and then if there is further loading after buckling, snap-through can occur. For a shallow rise, the effect of axial load resulting in the centerline extending is not negligible and the panel is considered extensible. This results in a symmetric buckling mode and snap-through occurs.

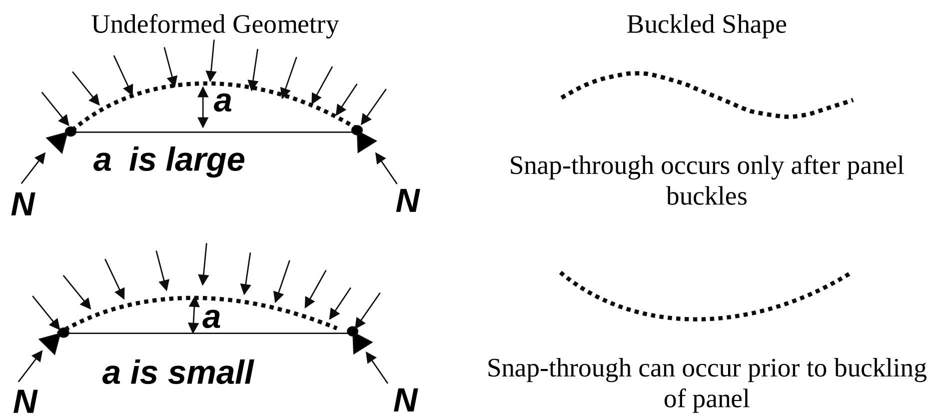


Figure 10.4.1-1 Arch Geometry and Buckling Mode Shape

Although classical methods exist which predict snap-through, they examine limited geometries and loading. Reference 10-10 and 10-34 provide a discussion of these methods. They also typically examine only the uniformly applied pressures; however, pressure gradients, in-plane stresses and biaxial stresses change the snap-through response.

Because of these considerations, the recommended approach to final analysis is a nonlinear fine-grid finite element model of the panel and adjacent bays. This allows for application of the correctly distributed pressure loads, along with the in-plane biaxial loads, the modeling of any complex curvature which may be present and accounts for any membrane loads which might develop. It also aids in capturing the correct boundary stiffness at the edges of the panel. Some guidelines for the nonlinear finite element approach are provided in Section 10.4.4.

A screening procedure, developed using nonlinear ABAQUS RIKS solution parametric analyses, is provided in Section 10.4.3 for panels which are or can be approximated as circular loaded with either a uniformly distributed or a concentrated load typically representative of aircraft panels. This method provides a conservative estimate of the critical snap-through load, is available in IDAT/SNAPTHRU, and works best on panels with single near-circular curvature. It would not be a good predictor of spherical panel behavior or panels with complex curvature and these should be analyzed using the methods of Section 10.4.4. If a panel, screened using the method in Section 10.4.3 does not exhibit snap-through behavior, no further analysis is required. If it does, further analysis using the methods of Section 10.4.4 is recommended prior to panel resizing.

10.4.2 Slope Change Abruptness Method for Determining Panel Snap-Through

Determining exactly when snap-through occurs can be an analysis challenge. Panels, which have significant biaxial curvature approaching spherical, have delayed, but violent, snap-through. Conversely, panels that are nearly, but not quite flat have gradual nonlinearity at very low load levels. This requires definition of a threshold to determine where detrimental snap-through has occurred when using NASTRAN Solution 106/600 solution sequences. ABAQUS RIKS solution models have an algorithm and similar criterion built-in and do not require the analyst to perform the post processing outlined in this section.

While the ABAQUS RIKS-solution model is the preferred approach, if a NASTRAN model is used, the post-processing and criteria described in this section is necessary to determine whether snap-through occurs. The criteria described in this section were developed at Lockheed Martin based on NASTRAN Solution 600/106 FEA studies and have been successfully implemented on a major aircraft program. Because snap-through is not just a nonlinear deflection problem, but is determined from the abruptness of that deflection change, a Slope Change Abruptness Criteria, in conjunction with the results from a nonlinear, large displacement NASTRAN FE model is required.

The Slope Change Abruptness Method describes an approach to quantitatively determine when the response of the structure is abrupt enough to be called snap-through. Snap-through can result in permanent deformation or fatigue of metallic structure or delamination of composite structure. Initially the defined approach used the slopes of the load-deflection curve. If the slope changed an order of magnitude from one load step to the next, snap-through was said to have occurred. However, a comparison of animations from fine-grid finite element models to the prediction of snap-through for the same structure indicated that approach was insufficiently sensitive. The abruptness parameter defined in this section incorporates both the change in the slope as well as the magnitude of the deflection from one incremental load step to the next into the calculation. To determine the recommended slope abruptness parameter, Δm_{abrupt}, a number of models/loads/deflection/ animations were examined. Based on this, a recommended value of 0.80 was set for the Δm_{abrupt} cutoff. If the calculated value for the maximum Δm_{abrupt} for a panel exceeded this value, snap-through has occurred. This eliminated the cases where the deflection gradually increased to a maximum value and focused on the cases where there was little to no deflection and then due to one incremental load change, the panel inverted. Full scale static testing later verified that the selected cutoff was reasonable.

Figure 10.4.2-1 shows a cross-plot of the ratio of slopes of the load-deflection curve and the slope abruptness parameter with the recommended abruptness parameter cutoff indicated. A more conservative approach would be to lower the cutoff to the linear part of the curve. A less conservative approach is not recommended, *i.e.*, increasing the cutoff, as there were panels which were known to have snap-through buckling that were not correctly predicted if the cutoff was greater than 0.80.

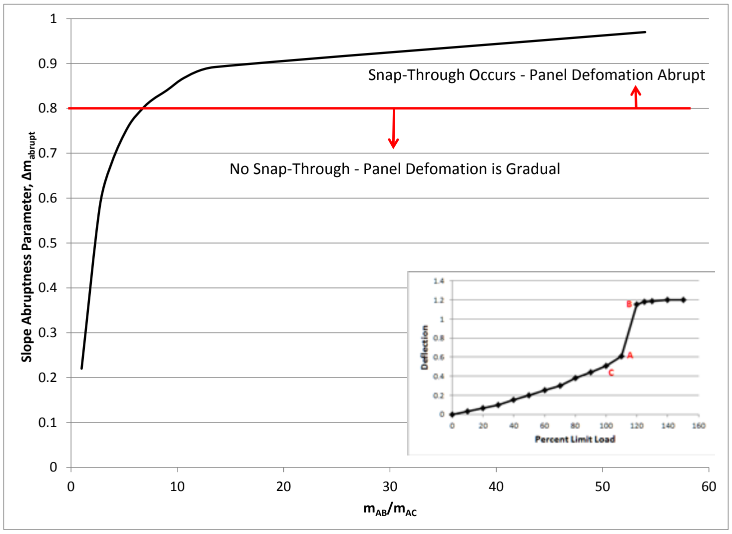


Figure 10.4.2-1 Cross-plot of Slope Abruptness Parameter and Ratio of Subsequent Slopes of Deflection Curve

The approach is illustrated in Figure 10.4.2-2, which is used in the analysis procedure described in Table 10.4.2-1.

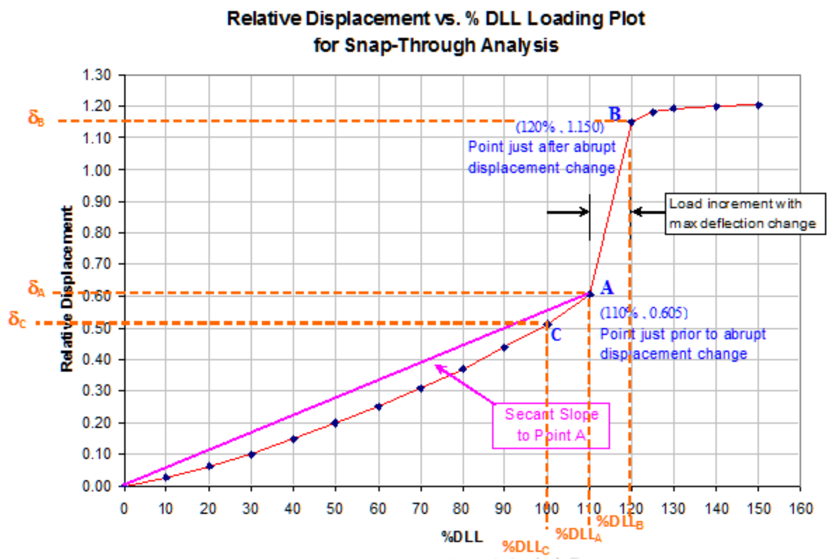


Figure 10.4.2-2 Snap-Through Buckling Analysis Parameters

Table 10.4.2-1 Slope Change Abruptness Criteria Method for Determining Snap-Through Buckling

Step	Discussion
1. Review a deflected plot of the panel under pressure loading to determine visually if the panel deflection at ultimate load results in a reversed curvature. This plot should result from a NASTRAN Solution 106 (if convergence is achieved) or Solution 600 non-linear model run.	If there is no curvature reversal in the deflected shape, then the panel does not experience a snap-through condition. No further analysis for snap-through is required.
2. Using the load increment which experiences the most abrupt change in displacement, determine the FE node which experiences the largest deflection relative to its deflection at	If the load increment is no more than 20% below, at or above ultimate load, ultimate load may be used. If it is

Step	Discussion
the previous load step in the direction normal to the panel surface relative to the nearest panel edge on both sides of the deflected grid	significantly below ultimate load, then the grid selection should be consistent with load increment of the most abrupt change.
3. Create a deflection-load level plot for a single node using the relative, normal deflection of the node point chosen in Step 2 and the incrementally applied load levels. <u>Ensure the load increment in the area of the abrupt change is ≤ 5%</u>	This step provides a graphical representation of the snap-through occurrence. See Figure 10.4.2-1
4. Determine the location where the most abrupt change in the deflection occurs between load increments	See Figure 10.4.2-1
5. Define the relative displacement plot points: • A is the point just prior to the most abrupt displacement change. Formulate as: $(\%DLL_A,\delta_A)$ • B is the load increment point just after most abrupt displacement change. Formulate as: $(\%DLL_B,\delta_B)$ • C is the point a load increment below Point A. Formulate as: $(\%DLL_C,\delta_C)$	See Figure 10.4.2-1. Points are labeled A, B and C to aid in discussion.
6. Calculate the slope of a straight line between the origin and Point A per Equation 10.4.2-1.	This is the secant slope of Point A
7. Calculate the incremental slope between Points A and B per Equation 10.4.2-2.	
8. Calculate δ_C', the equivalent deflection at Point C using Equation 10.4.2-3.	This accounts for uneven load increments if the load increments are evenly spaced, use the deflection at c, δ_C, directly.
9. Calculate the slope abruptness parameter, Δm_{abrupt}	Equation 10.4.2-4
10. Using Equation 10.4.2-4, determine if panel snap-through has occurred.	Equation 10.4.2-4 accounts for the abruptness of the deflection change.
11. Calculate a ratio-to-requirement per Equation 10.4.2-5 if Step 9 indicates snap-through has occurred.	Note composite panels will need to be analyzed for possible interlaminar tension or interlaminar shear failure. See PM4056 for discussion.

Calculate the secant slope of a straight line between the origin and Point A as shown

$$m_{secant} = \frac{\delta_A}{\%DLL_A}$$ **Equation 10.4.2-1**

The incremental slope between points A and B is calculated as

$$m_{AB} = \frac{\delta_B - \delta_A}{\%DLL_B - \%DLL_A}$$ **Equation 10.4.2-2**

To account for uneven load increments, the equivalent deflection at Point C, δ_C', is calculated as

$$\delta_C' = \delta_A + \left(\frac{\%DLL_B - \%DLL_A}{\%DLL_A - \%DLL_C}\right)(\delta_C - \delta_A)$$ **Equation 10.4.2-3**

Calculate the slope abruptness parameter

$$\Delta m_{abrupt} = \frac{m_{AB} - m_{secant}}{m_{AB}} \cdot \frac{(\delta_B - \delta_A)}{(\delta_B - \delta_C')}$$ **Equation 10.4.2-4**

The portion of the equation involving the differences in the slopes is the slope abruptness factor and normally varies from 0 to 1. The portion of the equation involving the differences in deflection is an indicator of the magnitude of the deflection change. Typically, if snap-through is present the curve will have a similar trend to what is shown in Figure 10.4.2-2, where there are one or more load steps where there is a large increase in the deflection.

To determine if snap-through has occurred, compare the slope abruptness parameter to the recommended maximum value. If the ratio is greater than or equal to 0.80, then snap-through occurs.

$$1.0 \geq \Delta m_{abrupt} \geq 0.80 \qquad \textbf{Equation 10.4.2-5}$$

The recommendation of a maximum value of 0.80 was discussed above. Note the slope abruptness parameter should always be less than or equal to 1.0. If values calculated are greater than 1.0, that indicates the slope is negative and this approach does not work. Smaller load increments may be required in the model to remedy this issue.

If the panel is exhibiting snap-through per Equation 10.4.2-5, then the ratio-to-requirement is calculated from Equation 2.5.5-1 to show how much below the Criteria Load Level the instability occurred.

$$R_{SnapThrough} = \frac{\%DLL_{SnapThrough}}{\%DLL_{criteria}} \geq 1.0 \qquad \textbf{Equation 10.4.2-6}$$

where
$\%DLL_{SnapThrough}$ is the load level at which snap-through instability is predicted
$\%DLL_{criteria}$ is the criteria load level below which snap through shall not occur (150 or per program requirements. See discussion in Section 10.4.1)

To meet the criteria load level of no snap-through, $R_{snap\text{-}through}$ must be greater than or equal to 1.0. If not, the panel thickness is increased to meet the criteria. Whether or not snap-through occurs and irrespective of the criteria load level, an analysis of the structure at ultimate is required.

10.4.3 Snap-Through Screening Method using a Parametric Solution

This section provides a closed-form screening approach, derived from a parametric FEA ABAQUS nonlinear study, to determine if further, more detailed analysis is required. This is the method used in IDAT/SNAPTHRU. This approach can be used in initial design to determine if snap-through is suspected. If it is, the final configuration of the structure is analyzed using an ABAQUS RIKS solution model (preferred) or a NASTRAN Sol 106/600 with the criteria presented in Section 10.4.2.

For the screening analyses, a 1.0 in strip of the panel is used. It is constrained to allow motion only in the plane of curvature. The following assumptions are made:

- Panel symmetry
- Panel forms a circular arch, see below for complex curvature panels
- End supports are simply-supported
- Panel thickness, modulus, and Poisson's ratio are taken at the center of the panel
- Direction of the material properties taken in the curvature hoop direction

For complex curvature panels, the plane of curvature is chosen as the section through the panel center and parallel to a panel edge that gives the maximum radius of curvature. If there is significant curvature along both the panel axes or the panel is spherical, the screening method shall not be used.

Snap-through instability is expressed in terms of a geometric eigenvalue parameter, λ, and a loading parameter, R_{cr}, as described in Equations 10.4.3-1 and 10.4.3-2. Panel geometry is shown in Figure 10.4.3-1.

$$\lambda = \frac{2L^2}{\pi^3 r}\sqrt{\frac{A}{I}}$$ **Equation 10.4.3-1**

where
L is the chord length of the curved panel (in)
r is the radius (in)
A is the cross-sectional area (in^2) of the panel
I is the moment of inertia of the panel (in^4) about the panel centerline

$$R_{cr} = \frac{wL^4}{2\pi^4 EI}\sqrt{\frac{A}{I}}$$ **Equation 10.4.3-2**

where
w is the uniform applied running load (lb/in)
E is the panel Young's modulus (psi)

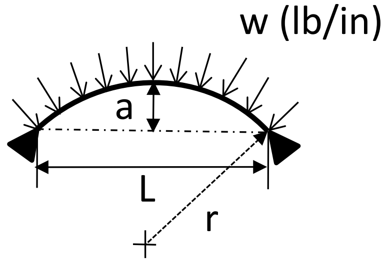


Figure 10.4.3-1 Panel Geometry for Distributed Uniform Loading

Using ABAQUS, a parametric study of panels of constant radius was conducted for various panel geometries and rise dimensions to determine the critical snap-through pressure load and the results were compared with classical methods described in References 10-10 and 10-34, where applicable. These studies were run on a unit width model with uniform pressure and a single concentrated load at panel center with simply supported boundary conditions. The results were plotted as λ versus R_{cr} and a curve fit was performed. The resulting equations and ranges of applicability are given as Equation 10.4.3-3 for uniform pressure load and the curves are plotted in Figure 10.4.3-2

$$R_{cr} = 1.7477\lambda^2 - 2.993\lambda + 2.1125 \qquad \lambda \le 3.74$$
$$R_{cr} = 7.8984\lambda - 14.167 \qquad 3.74 < \lambda \le 8.8$$
$$R_{cr} = 6.3738\lambda - 0.7399 \qquad 8.8 \le \lambda$$
Equation 10.4.3-3

For eigenvalues equal-to or less than 3.74, snap-through buckling is the critical buckling mode. For eigenvalues above 3.74, asymmetrical buckling precedes snap-through buckling; however, these equations still provide a prediction of the snap-through buckling pressure which may be below the criteria load level. The advantage of using the Riks method in ABAQUS is that the solution can proceed after panel buckling to determine the snap-through pressure. <u>Recall that in addition to any snap-through analysis that is done, traditional panel buckling must always be checked.</u>

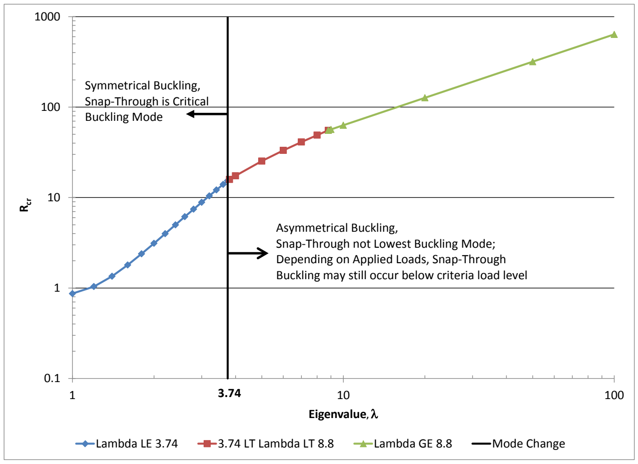


Figure 10.4.3-2 Snap-Thru Buckling Prediction Curve for Uniform Running Load on Constant Radius Panel

Similarly, for a concentrated load, as shown in Figure 10.4.3-3, a set of equations has been derived and is given as Equation 10.4.3-4. For these ABAQUS solutions, the concentrated load, applied over a 3 in. square area, simulated a typical step-load on a panel.

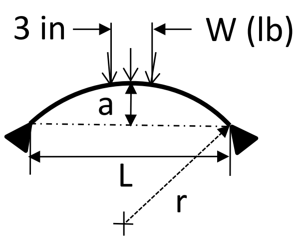


Figure 10.4.3-3 Panel Geometry for Concentrated Loading

$$R_{cr} = \frac{1.7477\lambda^2 - 2.993\lambda + 2.1125}{L_1\lambda_1} C_1 \qquad \lambda \le 3.74$$

$$R_{cr} = \frac{7.8984\lambda - 14.167}{L_1\lambda_1} C_1 \qquad 3.74 < \lambda \le 8.8 \qquad \textbf{Equation 10.4.3-4}$$

$$R_{cr} = \frac{6.3738\lambda - 0.7399}{L_1\lambda_1} C_1 \qquad 8.8 \le \lambda$$

where
$C_1 = 0.1097L + 0.1439$
$\lambda_1 = 0.0017L^3 - 0.0397L^2 + 0.3261L + 0.0358$
$L_1 = 1$ $\qquad L \ge 10$
$L_1 = -0.0119L^2 + 0.2133L + 0.0017$ $\qquad L < 10$

To utilize the approximate approach the value for λ is calculated from Equation 10.4.3-1 and then R_{cr} is determined from Equation 10.4.3-3 for uniform running load or from Equation 10.4.3-4 for a concentrated load. The critical snap-through buckling running load, w_{cr} (lb/in), is then determined from Equation 10.4.3-2, reformulated as

$$w_{cr} = \frac{2R_{cr}\pi^4 EI}{L^4}\sqrt{\frac{I}{A}} \qquad \textbf{Equation 10.4.3-5}$$

where
R_{cr} is the critical bucking value determined from Equation 10.4.3-3 or 10.4.3-4

If the analysis is done using a unit width strip, the pressure, p_{cr}, is equal to the calculated running load, w_{cr}. If not, the critical buckling pressure can be calculated by dividing w_{cr} by the panel width. The critical snap-through buckling pressure is then compared to the ultimate applied pressure and a ratio-to-requirement written.

$$R_{tr} = \frac{p_{cr}}{p} \qquad \textbf{Equation 10.4.3-6}$$

where
p_{cr} is the critical snap-through buckling pressure from Equation 10.4.1-5 (psi)
p is the applied pressure at the appropriate criteria load level, *i.e.*, ultimate, limit, or some percent of DLL. (Refer to Section 10.4.1 for discussion)(psi)

This approach has been implemented in IDAT/SNAPTHRU Option E.

IDAT/SNAPTHRU Option A uses a nonlinear Riks solution finite element analysis method with follower forces. It is a similar model as described above to derive the closed-form equation and the same cautions and restrictions apply. It does allow for the use of fixed boundary conditions.

10.4.4 Snap-Through Analysis Using Nonlinear Finite Element Analysis

The most accurate snap-through analysis is performed using a nonlinear, large displacement fine grid finite element (FE) model of the panel under analysis as well as at least one adjacent bay in all directions. The model should include nonlinear material properties for metallic panels and for panel stiffeners that are loaded to stress levels above the proportional limit of the panel material.

The panel loading should include all case-consistent mechanical, pressure, and thermally induced loads applicable to the critical load condition under analysis. Typically, the thermal load is applied to the finite element model first, and then the in-plane mechanical and out-of-plane pressure loads are applied incrementally to evaluate the deflection response. The load increment near expected large deflection changes should be around 5% DLL or less to get accurate results. Larger increments up to a maximum of 10% DLL increments are permissible elsewhere in the model.

Snap-through analysis has been successfully accomplished using NASTRAN Solution 106/600 and ABAQUS RIKS-solution models. The ABAQUS RIKS-solution model is the preferred technical approach; however, NASTRAN is often the approach preferred by analysts because they already may have a fine-grid NASTRAN model. Table 10.4.4.-1 provides a look at the advantages of each approach.

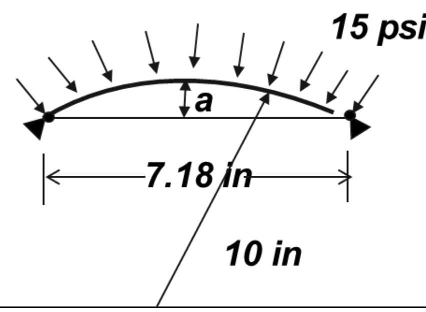
Table 10.4.4-1 Advantages and Disadvantages for Use of Different Finite Element Codes

ABAQUS RIKS Solution	NASTRAN SOLUTION 106/600
ABAQUS RIKS solution is a more robust solution sequence and it well suited to this type of problem. (Advantage)	Many aircraft parts exist as fine grid NASTRAN models for other required analyses, so not having to convert the geometry to an ABAQUS solution can save the analyst time (Advantage)
An ABAQUS run will provide a snap-through solution even when the panel has buckled, (it still might be at a load level of concern). (Advantage)	Many analysts are familiar with making NASTRAN runs and post-processing the information. (Advantage)
ABAQUS has faster run times. (Advantage)	NASTRAN Solution 106 tends to have longer run times and thus, selection of and minimizing the number of load cases is important. (Disadvantage)
ABAQUS provides that solution directly from an examination of the load proportionality factor. Snap-through occurs when the load proportionality factor starts to decrease. (Advantage)	To obtain the snap-through load level from the NASTRAN model, post-processing is required per Section 10.4.2. (Disadvantage)

Other finite element model codes can be used for solution of this class of problem; however, at LM these are two codes in common usage.

10.4.5 Example Problems

10.4.5.1 Example Snap-Through Buckling Calculation Using the Screening Method

Given: A panel of radius 10 in.; 0.040 in thick; with a chord length of 7.18. Material: Aluminum 7475-T61 Sheet. E=10.0E6 ν = 0.33 The applied pressure is 15 psi ultimate Panel is Simply Supported	
Determine the Ratio to Requirement using the method of Section 10.4.3 and assume a unit width strip.	
Screening Solution Section 10.4.3	
Calculate I	$\mathrm{I} = \frac{bt^3}{12} = \frac{1 \cdot 0.040^3}{12} = 5.333x10^{-6} in^4$
For a rectangular cross-section skin A=wt and I=wt^3/12	$\sqrt{\frac{A}{I}} = \sqrt{\frac{wt}{\left(\frac{wt^3}{12}\right)}} = \sqrt{\frac{12}{t^2}}$
Calculate λ	$\lambda = \frac{2L^2}{\pi^3 r}\sqrt{\frac{A}{I}} = \frac{2 \cdot 7.18^2}{\pi^3 \cdot 10}\sqrt{\frac{12}{0.040^2}} = 28.80$
Determine R_{cr} from Equation 10.4.3-3 for λ≥8.8	$R_{cr} = 6.3738\lambda - 0.7399 = 6.3738 \cdot 28.80 - 0.7399 = 182.83$
Determine w_{cr} from Equation 10.4.3-5	$w_{cr} = \frac{2R_{cr}\pi^4 EI}{L^4}\sqrt{\frac{I}{A}} = \frac{2 \cdot 182.83 \cdot \pi^4 \cdot 10x10^6 \cdot 5.333x10^{-6}}{7.18^4} \cdot \sqrt{\frac{0.040^2}{12}} = 8.25\ lb/in$
$p_{cr}=w_{cr}/b=w_{cr}/1.0$	p_{cr}=8.25 psi
Determine the RTR	$R_{tr} = \frac{p_{cr}}{p} = \frac{8.25}{15} = 0.55$
Per the screening method, the panel exhibits snap-through buckling well below ultimate load. A finite element solution described in Section 10.4.4 should be considered prior to redesigning. If the biaxial membrane response and boundary conditions more representative of having adjacent panels present are used, the panel may not require as substantial a change as this RTR would indicate.	
From IDAT/SNAPTHRU, using the ABAQUS option, the following result is obtained	

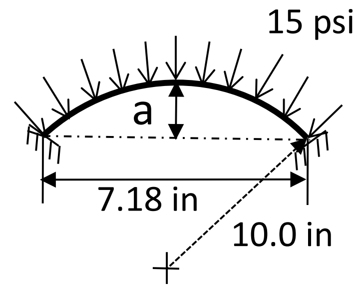
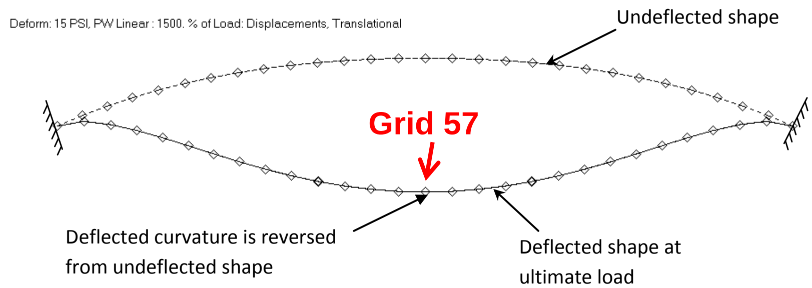
```
                 +-----------------------------------------------+
                 |                                               |
                 |                 SNAPTHRU                      |
                 |                   V2.0                    |
                 |       Curved Panel Snap Through Analysis      |
                 |               LM AERONAUTICS CO.              |
                 |                                               |
                 +-----------------------------------------------+

Tue Aug  6 13:49:04 2013

METHOD: ABAQUS Riks Analysis

INPUT:        'test_simple.sta'
THICKNESS(in) =     0.04
CHORD(L,in)   =     7.18
RADIUS(R,in)  =     10
MODULUS(E,psi)=     1e7
POISSONS      =     0.33
PRESSURE(psi) =     15
BOUNDARY COND =     'SIMPLE'
INCREMENTS    =     100
# OF ELEMENTS =     20

150% DLL Load, psi                               =  15.000
CRITICAL PRESSURE [first max(LPF)*Pressure, psi] =  8.460
% DUL                                            =  56.40%
LPF*                DISP@Center
=====               ===========
0.0499              -0.000300
0.0995              -0.000603
0.173               -0.00106
0.281               -0.00179
0.428               -0.00303
0.518               -0.00488
0.541               -0.00650
0.551               -0.00810
0.556               -0.00969
0.559               -0.0113
0.561               -0.0129
0.562               -0.0145
0.563               -0.0161
0.564               -0.0177
0.564               -0.0193
0.564               -0.0209
0.564               -0.0226
0.564               -0.0242
0.564               -0.0258
0.564               -0.0275
0.564               -0.0291
0.564               -0.0308
0.563               -0.0324
0.563               -0.0341
...
----------------------------
MAX LPF                :             0.564
MAX DISP@CENTER        :             -0.0308

*LPF: Load Proportionality Factor
```

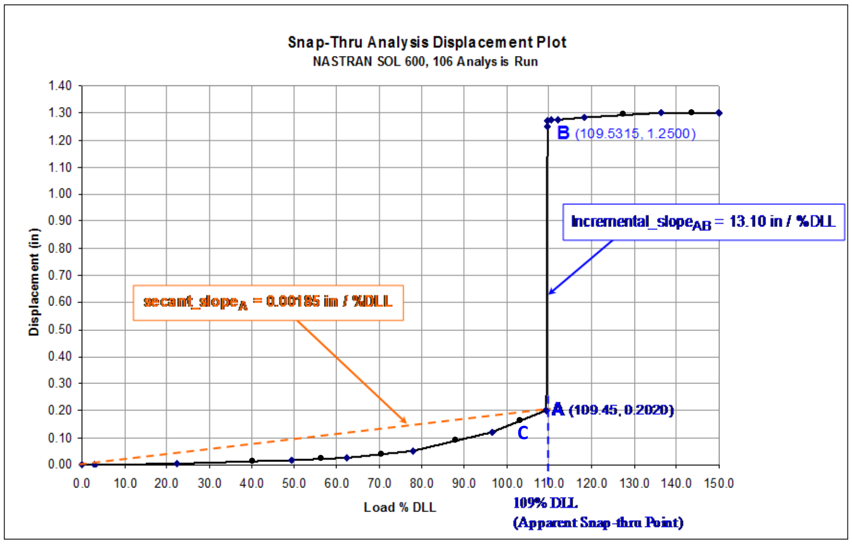
The results show the load proportionality factor (LPF) and deflection associated with each step. Snap-through occurs when the LPF reaches its first maximum and begins to decrease. This is shown in red in the output provided above and summarized at the end of the SNAPTHRU output listing.
The results show snap-through occurs at 56.4% of the applied ultimate load: 0.564x15=8.46 psi using the IDAT/SNAPTHRU ABAQUS Solution and at 8.25 psi or 55.1% of the applied ultimate load using the IDAT/SNAPTHRU closed form equations.
Note: the IDAT/SNAPTHRU ABAQUS Solution is based on a strip model and is not the method described in Section10.4.4. It is intended as a screening tool per Section 10.4.3.

10.4.5.2 Example of Slope Change Abruptness Criteria Method for Determination of Snap-Through Buckling

<table>
<tr><td>Given:
A panel of radius 10 in.; 0.040 in thick;
Chord length of 7.18.
Material: Aluminum 7475-T61 Sheet.
E=10.0E6
ν = 0.33

The applied pressure is 15 psi ultimate.

Edges are fixed.</td><td>
</td></tr>
<tr><td colspan="2">A NASTRAN FEM was built, <u>for illustration of the Slope Change Abruptness Method only</u>, using a 0.25 in wide strip model. For real structure this is not the preferred approach and a full panel should be modeled with adjacent panels and substructure stiffness represented.</td></tr>
<tr><td>1. Review Deflected Shape
2. Examining the output it is determined that Grid 57 experiences the largest relative deflection after snap-through.</td><td>
</td></tr>
</table>

3. Plot Displacement vs. Load Level for Grid 57 Also check from the output that the load increment at the abrupt deflection change meets the recommended load increment size (5% or less). In this case, it is less than 5%.	
4. Determine the location where the most abrupt increase in the absolute value of deflection occurs	The change occurs between load increments 109.45%DLL and 109.53%DLL
5. Define Coordinates for Points A, B, and C using Equation 10.4.2-1	Point A: (109.45, 0.2020) Point B: (109.53, 1.250) Point C: (104.00, 0.165)
6. Calculate the secant slope using Equation 10.4.2-1	$$m_{secant} = \frac{\delta_A}{\%DLL_A} = \frac{0.2020}{109.45} = 0.00185$$
7. Calculate the incremental slope using Equation 10.4.2-2	$$m_{AB} = \frac{\delta_B - \delta_A}{\%DLL_B - \%DLL_A} = \frac{1.250 - 0.2020}{109.53 - 109.45} = 13.10$$
8. Calculate the equivalent deflection at C using Equation 10.4.2-3	$$\delta_C' = \delta_A + \left(\frac{\%DLL_B - \%DLL_A}{\%DLL_A - \%DLL_C}\right)(\delta_c - \delta_A)$$ $$= 0.2020 + \left(\frac{109.53 - 109.45}{109.45 - 104.00}\right)(0.165 - 0.2020) = 0.2015$$
9. Calculate the slope abruptness parameter Equation 10.4.2-4	$$\Delta m_{abruptness} = \frac{m_{AB} - m_{secant}}{m_{AB}} \cdot \frac{(\delta_B - \delta_A)}{(\delta_B - \delta_C')}$$ $$= \frac{13.10 - 0.00185}{13.10} \cdot \frac{(1.250 - 0.2020)}{(1.250 - 0.2015)} = 1.0$$
10. Determine if panel snap-through has occurred using Equation 10.4.2-5	Is $0.80 \leq \Delta m_{abruptness} \leq 1.0$ => Yes; therefore snap-through has occurred
10. Calculate the Ratio-to-Requirement using Equation 10.4.2-5	$$R_{snap-through} = \frac{\%DLL_{SnapThrough}}{\%DLL_{criteria}} = \frac{109.45}{150} = 0.72$$
Snap-through is predicted at 72% of Ultimate Load using NASTRAN Solution with the Slope Change Method	
This problem solution varies from the problem defined in Section 10.4.5.1, in that this panel is analyzed with fixed boundary constraints. The closed form solution of Section 10.4.3 is only for simply supported panels; however, the ABAQUS option of IDAT/SNAPTHRU does allow for fixed end constraints. IDAT/SNAPTHRU was run for this example and the results are shown below. It predicts snap-through at a load of 80%, slightly higher than the NASTRAN prediction of 72% of load.	

The simply supported panel would snap-through at approximately 55-56% of DUL and the fixed edge panel would snap-through at 72-80% of DUL.

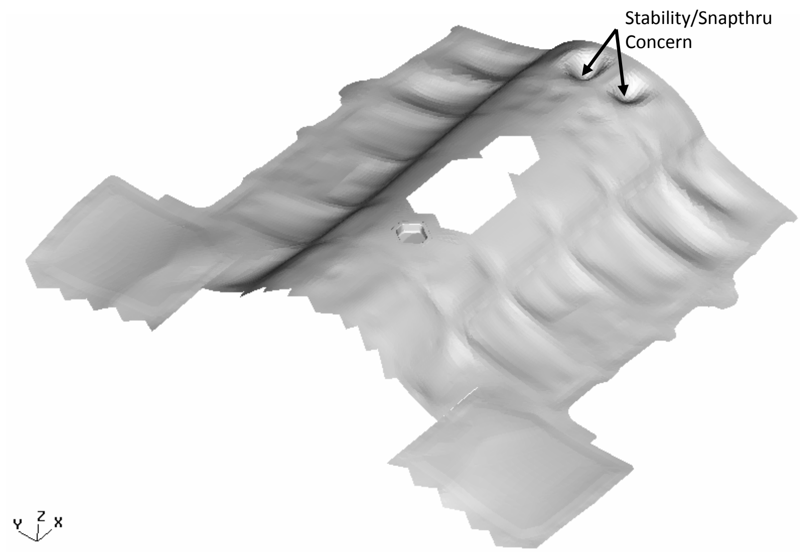
Neither of these approaches indicates a panel that would meet the recommended criterion of no snap-through buckling below ultimate load

```
+----------------------------------------------+
|                                              |
|                   SNAPTHRU                   |
|                     V2.0                 |
|       Curved Panel Snap Through Analysis     |
|               LM AERONAUTICS CO.             |
|                                              |
+----------------------------------------------+

Tue Aug  6 13:46:50 2013

METHOD: ABAQUS Riks Analysis

INPUT:       'test_clamp.sta'
THICKNESS(in) =     0.04
CHORD(L,in)   =     7.18
RADIUS(R,in)  =     10
MODULUS(E,psi)=     1e7
POISSONS      =     0.33
PRESSURE(psi) =     15
BOUNDARY COND =     'CLAMPED'
INCREMENTS    =     100
# OF ELEMENTS =     20

150% DLL Load, psi                             =  15.000
CRITICAL PRESSURE [first max(LPF)*Pressure, psi] =  12.000
% DUL                                          =  80.00%
```

…Note – output is truncated.

10.4.5.3 Example Panel – Use of Screening Tool

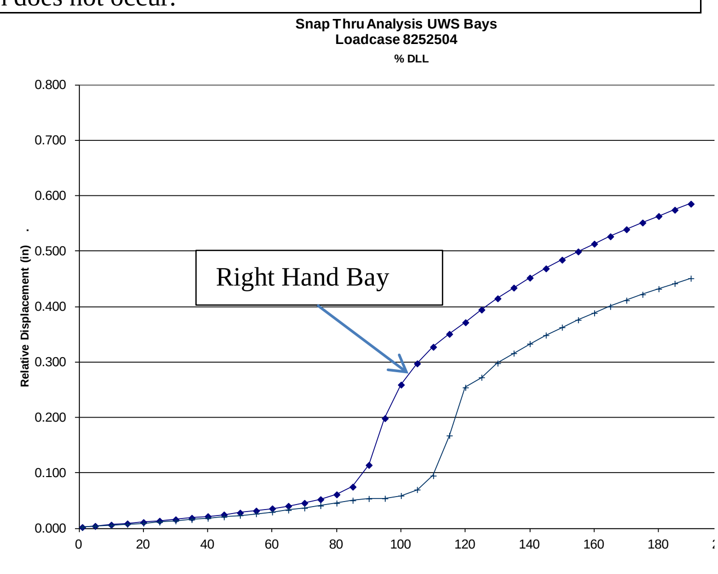
The upper skin panel, shown below, was suspected of exhibiting snap-through and was analyzed using a fine grid NASTRAN finite element approach. The applied pressure load is approximately 11.64 psi (limit) crushing the skin. Use the screening method to predict whether this panel will snap-though.



FEM Solution

The fine grid model solution show an onset of non-linearity at 85% of DLL (56.7% DUL) or approximately 9.9 psi, which was examined using the slope change abruptness criteria method of Section 10.4.2. This resulted in a slope change ratio of 0.68. Since this is below the recommended slope abruptness ratio of 0.80, the nonlinearity is not snap-through. So per the FEM solution, snap through does not occur.

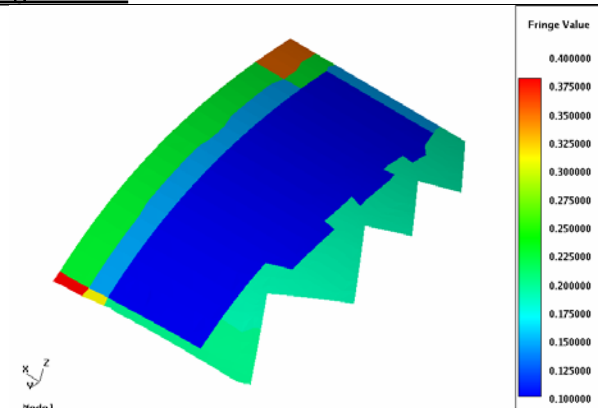
Percent of DLL	Slope Change Ratio
50	0.142
55	0.170
60	0.204
65	0.242
70	0.293
75	0.360
80	0.464
85	0.680
90	0.576
95	0.338
100	0.246
105	0.224
110	0.132
115	0.159
120	0.143
125	0.097
130	0.075
135	0.040
140	0.007
145	-0.024
150	-0.052
155	-0.081



Screening Solution

The panel geometry is defined by the contour points:

x	y	z
548.232	-0.003	126.6
548.19	-4.446	126.442
548.15	-8.138	126.029
548.112	-11.661	125.334
548.07	-15.561	124.12
548.033	-18.942	122.551
547.994	-22.34	120.263
547.7	-25.755	116.803



All of these points are at approximatey the same x location, so only the y and z coordinates will be used to develop an approximate circular panel radius.

Material: IM7/5250-7 Tape with outer plies of Fabric; 18 plies total. t=0.1014 in
E_x= 9.57x10^6 psi (Material-x is circumferential)
ν_{avg}=0.379

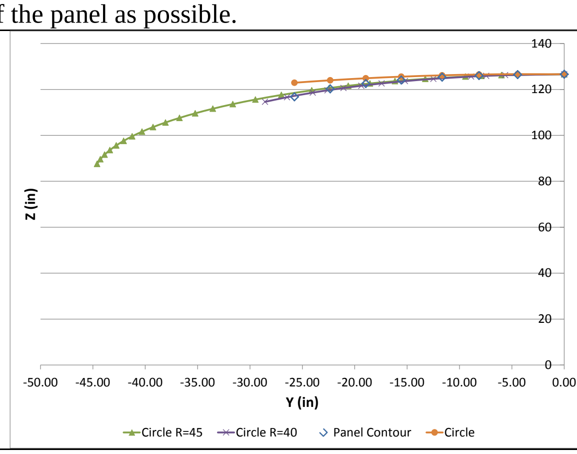
The first step is to develop the approximate radius. This can be done in CATIA or by hand. It, in general will not match the entire panel but the goal is to match as much of the panel as possible.

The Diamond shaped points are the original panel contour

The green triangle-symbol circle curve, with an average radius of 70.09 in was derived by determining the equation of a circle using 3 points from the skin contour.

The r=45 in and r=40 in curves were determined by trial and error, plotting different radii to see which best modeled the skin panel.

r=40 in (purple crosses) was selected for analysis



The IDAT/SNAPTHRU screening tool provides for the option of using Simple or Fixed Boundary conditions, while the hand method described in Section 10.4.3 is for Simple Boundary Conditions only. For pressurized skins, where the pressure is also on adjacent panels, fixed boundary conditions are usually assumed. For illustration purposes, this example will also show the method of Section 10.4.3 with Simple boundary conditions.

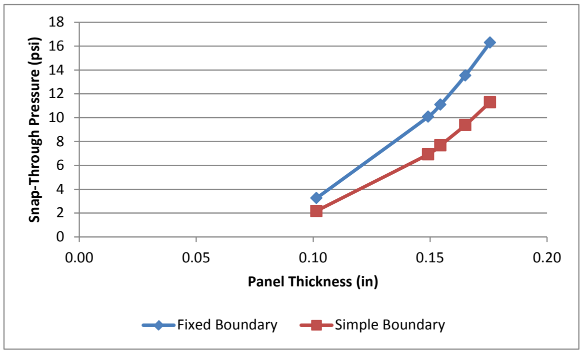
Screening Solution Section 10.4.3

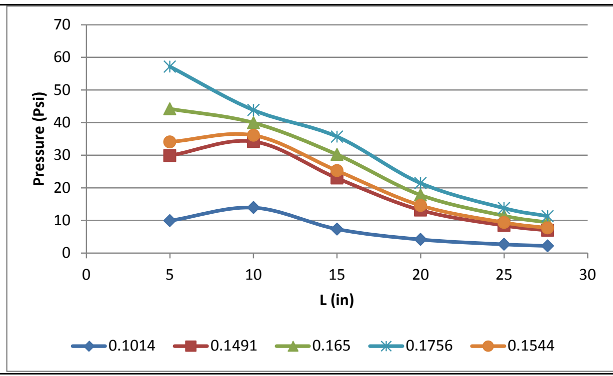
Calculate the Chord Length	Endpoints: (-0.003, 126.6), (-25.755,116.803)
	$L = \sqrt{(-25.755 + 0.003)^2 + (116.803 - 126.6)^2} = 27.553$
Calculate I	$\mathrm{I} = \frac{bt^3}{12} = \frac{1 \cdot 0.1014^3}{12} = 8.688x10^{-5}in^4$
For a rectangular cross-section skin A=wt and I=wt^3/12	$\sqrt{\frac{A}{I}} = \sqrt{\frac{wt}{\left(\frac{wt^3}{12}\right)}} = \sqrt{\frac{12}{t^2}} = 34.163$

Calculate λ	$\lambda = \frac{2L^2}{\pi^3 r}\sqrt{\frac{A}{I}} = \frac{2 \cdot 27.553^2}{\pi^3 \cdot 40} \cdot 34.164 = 41.824$
Determine R_{cr} from Equation 10.4.3-3 for λ≥8.8	$R_{cr} = 6.3738\lambda - 0.7399 = 6.3738 \cdot 41.824 - 0.7399 = 265.84$
Determine w_{cr} from Equation 10.4.3-5	$w_{cr} = \frac{2R_{cr}\pi^4 EI}{L^4}\sqrt{\frac{I}{A}} = \frac{2 \cdot 265.84 \cdot \pi^4 \cdot 9.57x10^6 \cdot 8.688x10^{-5}}{27.553^4} \cdot 0.02927 = 2.187\ lb/in$
$p_{cr}=w_{cr}/b=w_{cr}/1.0$	p_{cr}=2.187 psi
This calculation for Simple Boundary conditions would indicate the panel snaps-through at 2.187 psi. This a conservative approach would trigger additional analysis requirements. Next look at fixed boundary conditions using IDAT/SNAPTHRU to run an ABAQUS strip model: P_{cr}=3.248 psi	

```
From IDAT/SNAPTHRU Output V2.0:
           Thu Sep 26 09:46:13 2013

           METHOD: ABAQUS Riks Analysis

           INPUT:     'Example_Final.sta'
           THICKNESS(in) = 0.1014
           CHORD(L,in)   = 27.553
           RADIUS(R,in)  = 40
           MODULUS(E,psi)= 9.57e6
           POISSONS      = 0.379
           PRESSURE(psi) = 11.64 (Limit)
           BOUNDARY COND = 'CLAMPED'
           INCREMENTS    = 100
           # OF ELEMENTS = 20

           100% DLL Load, psi                                 =  11.640
           CRITICAL PRESSURE [first max(LPF)*Pressure, psi] =  3.248
           % DLL                                              =  27.90%

           LPF*             DISP@Center
           =====            ===========
           0.0494                -0.00184
           0.0970                -0.00379
           0.162            -0.00700
           0.228            -0.0128
           0.261            -0.0212
           0.273            -0.0309
           0.277            -0.0405
           0.279            -0.0502
           0.279            -0.0600
```

The degree of conservatism of the screening method varies depending on pressure, thickness and rise of the arch. The figures below explore panel thickness versus snap-through pressure for this particular panel geometry. The left figure is thickness versus pressure for a panel with a chord length of 27.77 in. The figure on the right is a plot of chord length versus pressure for different thickness panels. In both of these comparisons, the E is held constant for comparison purposes. With composite materials, depending on the plies added there is some variation in E for different thicknesses.





10.5 Large Deflection of Plates

Large deflections of plates, sometimes called membrane action, results from the nonlinear behavior of thin sheets under applied pressure loads. As a thin sheet is pressure-loaded, initially the deflection increases linearly with the pressure. At some point, the sheet deforms sufficiently and starts to behave as a thin shell-membrane reacting the load as in-plane hoop tension. At this point, the plate-bending-only solution is no longer valid. When the sheet is undergoing membrane response, the deflections increase nonlinearly at a much slower rate than linear analysis would predict. If the pressure continues to increase, the onset of permanent set occurs and the deflection and the degree of permanent set increase linearly with increasing pressure. If the pressures are sufficiently high, when the pressure is removed, the sheet exhibits permanent deformation. The methods described in this section are valid only prior to any permanent set and within the elastic range of the material stress-strain response.

Fully membrane behavior is somewhat rare in aircraft structure. Most often, there is a combination of plate bending and membrane loads present. Experimental investigations have shown that edge constraints and stiffness of substructure, along with the magnitude of pressure loads that may or may not be present on adjacent panels have a large effect on the panel behavior. As such, the most accurate solution for determining membrane effects is geometrically nonlinear finite element analysis with sufficient detail to capture the nonlinear behavior. For the panel, that means a minimum of four to six elements across the narrow width between supports, sufficient elements along the height to result in aspect ratios of 1-1.25. In addition, the supports should be modeled to capture the correct stiffness, including the torsional stiffness, since this is the primary stiffness involved in determining the edge constraint fixity. If the model is a breakout model, it is recommended that the model extend at least one bay beyond the bay under analysis. Loads from the parent model may be applied as enforced displacements or free-body grid point forces at the boundaries or by using a condensed stiffness and loads super-element approach and the method selected needs to result in a breakout model response consistent with the parent model.

An alternative for performing the detail finite element analysis is the use of IDAT/PRESS and/or IDAT/LG025 for flat panels and IDAT/SPAM for curved panels. All of these programs provide membrane and bending stresses at several critical locations: the center of the panel, the center of the long side and the center of the short side. This allows for analysis of the panel and the attachments. These programs are most useful for rectangular or near-rectangular panels where it is reasonable to assume fixed or simple boundary constraints.

Note that a means of approximating the correct internal loads for pressure loaded flat or near-flat panels for which only coarse grid FE internal loads are available, is to combine the in-plane loads from the coarse grid model with membrane and bending loads from one of the IDAT solutions or a hand analysis. This is illustrated in Example 10.5.1.1.

Because the transition from plate bending behavior to membrane behavior is gradual, there is no definitive theoretical criterion to determine when a membrane analysis is used. There are two practical guidelines, based on testing of thin flat aluminum panels, which are in common use. These are summarized in Table 10.5.0-1 and if either guideline is met, then the degree of membrane stresses present is sufficient to warrant a membrane analysis.

Table 10.5.0-1 Guidelines for Application of Membrane Theory

If Either Check is True, Use Membrane Analysis Methods:
$\delta_{linear} > t/2$
$\frac{p}{E}\left(\frac{b}{t}\right)^4 > 100$
where δ_{linear} is the deflection (in) based on linear theory or linear FEM output with sufficient element resolution p is the applied pressure (psi) E is the Young's modulus of the material (psi) b is the narrow width of the panel (in) t is the thickness of the panel (in)

Testing has verified that if adjacent panels are of similar aspect ratios and under approximately the same pressure, the panel acts as if it is clamped or fully fixed because the in-plane displacement and slope of the sheet relative to the elastic network of substructure must be zero where it passes over the centerline of the substructure. If, however, adjacent bays are not loaded with similar pressures, as in an end bay, then the panel acts as simply supported unless the substructure has sufficient torsional rigidity to be able to develop large moments at the edges.

Section 10.5.1 presents a classical analysis approach for simply supported rectangular membranes based on information in Reference 10-41 supplemented by results from Reference 10-5. The predicted deflection and stresses compare favorably to testing of metal sheet and provides membrane loads similar to those obtained from IDAT/PRESS. Section 10.5.1.1 provides an example problem comparison for a typical panel of aspect ratio 1.5 between the results obtained by the method outlined here and the results from IDAT/PRESS. Section 10.5.2 presents similar classical approach for rectangular membranes, which are fully fixed at the edges based on information in References 10-18 and 10-42.

When analyzing panels under pressure load it is important to obtain the net applied pressure. The net pressure is the summation of pressures from all sources such as fuel tank, cockpit, cabin, pressurized cavity and aerodynamic pressures. It is important to ensure that the sign convention is understood as it may not be consistent for the various pressure sources. Irrespective of the source of the pressure-induced membrane loads and moments, *i.e.*, FEM, IDAT tool or hand analysis, they should be combined with the in-plane applied loads when calculating panel margins of safety.

10.5.1 Membrane Analysis of Simply Supported Rectangular Panels with Uniform Pressure Load

Geometry for a typical rectangular membrane is shown in Figure 10.5.1-1, along with the coordinate system. This coordinate system is consistent with IDAT/PRESS.

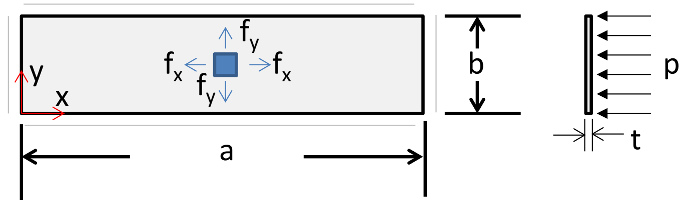


Figure 10.5.1-1 Rectangular Panel Geometry for Membrane Analysis

A square panel, aspect ratio of *a/b=1*, will have equal membrane stresses in both the x and y directions. As the panel elongates in one axis, *x*, the stresses in the long-axis direction, f_x, remain unchanged and the stresses in the short axis direction, f_y, increase in magnitude. Reference 10-41 indicates that, from a practical standpoint, at approximately *a/b=2*, the panel behaves like a plate of infinite length.

Figure 10.5.1-2 provides a plot of deflection/thickness against the nondimensional pressure coefficient $pb^4/(Et^4)$ for different aspect ratio panels for simply supported boundary conditions (held but not fixed). Interpolation between curves based on aspect ratio is acceptable.

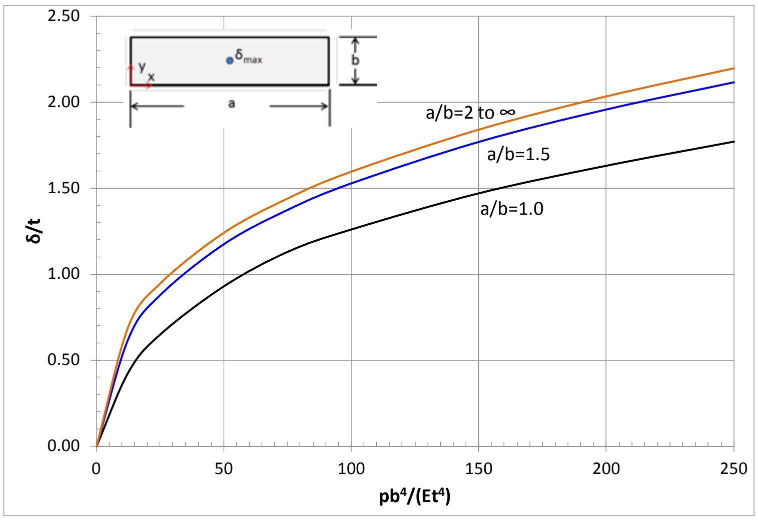


Figure 10.5.1-2 Deflection Coefficients for Flat Plates with Simply Supported Boundary Constraints

Figures 10.5.1-3 and 10.5.1-4 provide plots of a nondimensional stress parameters *X* and *Y* for different aspect ratio panels for simple supported boundary conditions. This boundary condition restrains the edges from in-plane motion but the edges are allowed to rotate and provide no moment resistance. To calculate the stress, the parameters *X* and *Y* are defined, generically, as

$$X = \frac{f_x b^2}{E t^2}$$

$$Y = \frac{f_y b^2}{E t^2}$$

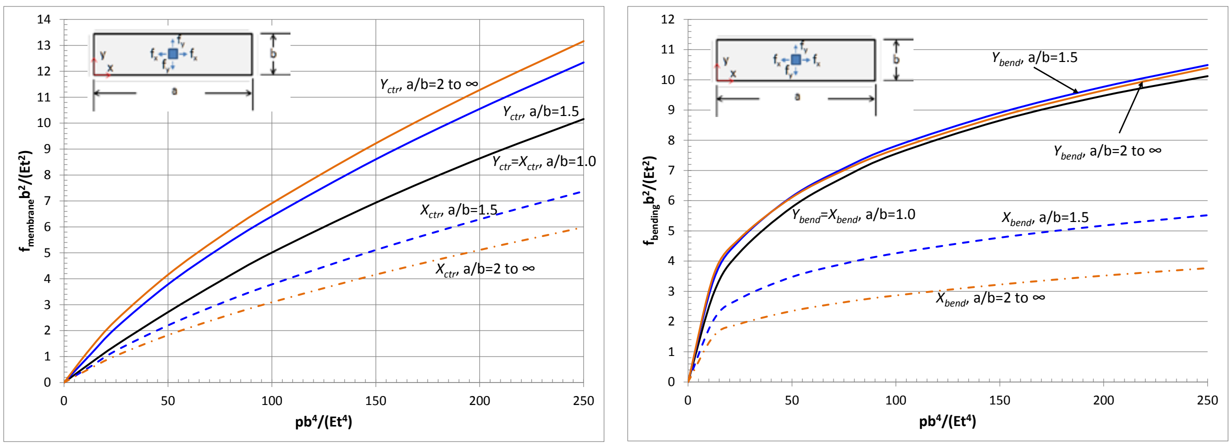
Equation 10.5.1-1

where
X, Y are the stress parameters in the x and y directions, respectively, from Table 10.5.1-1
f_x, f_y is the stress in the x and y directions, respectively(psi)
E is the Young's Modulus (psi)
b is the panel short side dimension, parallel to the y-axis (in)
t is the panel thickness (in)

Note: The subscripts in Figures 10.5.1-1 and 10.5.1-2 indicate whether the stresses are membrane or bending stresses and the location: center of panel (ctr), center of long edge (ctr-Ledge) or center of short edge (ctr-Sedge). All parameters are normalized to the panel short side dimension.

Equations 10.5.1-2 through 10.5.1-5 provide further explanation of how to use these stress parameters. Figure 10.5.1-3 can be used to calculate the f_x and f_y membrane and bending stresses in the center of the panel. The total stress in the panel is the sum of the two contributions: membrane plus bending.

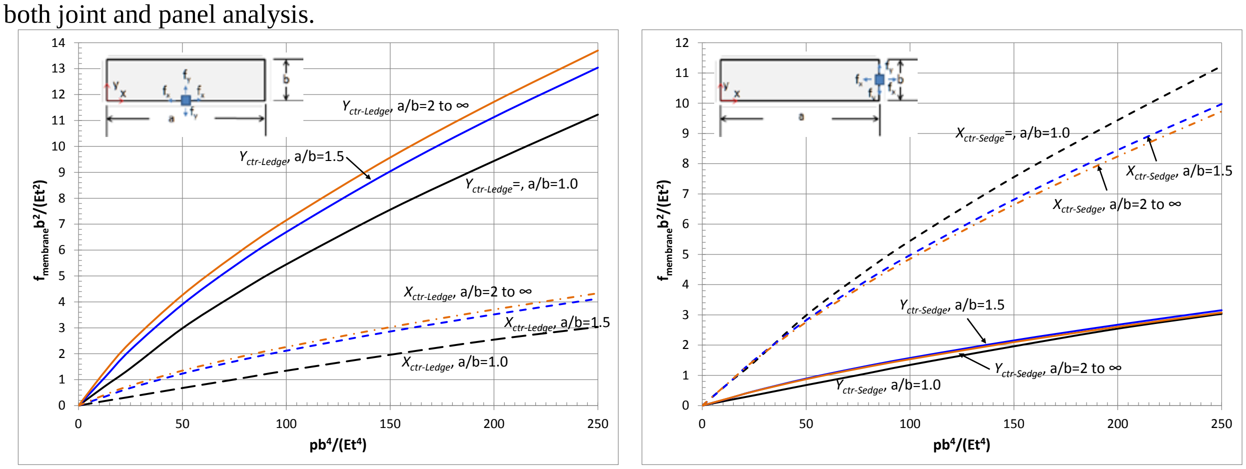
It is interesting to note that for the membrane stresses, the stress parameter, and thus, the stress in the *y*-direction is larger for the higher aspect ratio panels but that the *x*-direction stress parameter gets smaller as the panel aspect ratio increases. Note also that the difference in the *y*-direction bending stress parameter for various aspect ratio panels is small, particularly at high $pb^4/(Et^4)$ ratios, where the panel behaves more fully as a membrane.



a) Membrane Stress b) Bending Stress

Figure 10.5.1-3 Membrane and Bending Stresses at the Center of the Panel for Flat Plates with Simply Supported Boundary Constraints

Figure 10.5.1-4 can be used to calculate the f_x and f_y membrane stresses at the center of the panel edge for use in both joint and panel analysis.



a) Center of Long Edge b) Center of Short Edge

Figure 10.5.1-4 Membrane Stresses at the Edges of Flat Plates with Simply Supported Boundary Constraints

Note that while the stress parameters at the center of the long edge follow the same trends as the membrane stress parameters in the center of the panel, the stress parameters at the center of the short side do not, with the square panel having the largest stress parameter, resulting in the largest stress.

Tables 10.5.1-1 and 10.5.1-2 present the stress parameters from Figures 10.5.1-2 through 10.5.1-4 in numerical form.

Table 10.5.1-1 Deflection and Stress Parameters for the Center of Simply Supported Rectangular Plates under Uniform Pressure Load[1]

Deflection Parameter δ/t At Center of Plate			
	a/b=1	a/b =1.5	a/b =2.0
$pb^4/(Et^4)$	δ/t	δ/t	δ/t
0	0.000	0.000	0.000
12.5	0.430	0.625	0.696
25	0.650	0.879	0.946
50	0.930	1.175	1.241
75	1.130	1.374	1.440
100	1.260	1.528	1.596
150	1.470	1.769	1.840
200	1.630	1.957	2.033
250	1.770	2.115	2.196

	Membrane Stress Parameter At Center of Plate						Bending Extreme Fiber Stress Parameter At Center of Plate				
	a/b =1.0	a/b =1.5		a/b =2 to ∞			a/b =1.0	a/b =1.5		a/b =2 to ∞	
$pb^4/(Et^4)$	$X_{ctr}=Y_{ctr}$	X_{ctr}	Y_{ctr}	X_{ctr}	Y_{ctr}		$X_{bend}=Y_{bend}$	X_{bend}	Y_{bend}	X_{bend}	Y_{bend}
0	0.000	0.000	0.000	0.000	0.000		0.000	0.000	0.000	0.000	0.000
12.5	0.736	0.605	1.064	0.542	1.293		2.964	2.075	3.413	1.532	3.577
25	1.440	1.231	2.109	1.035	2.403		4.300	2.761	4.700	1.925	4.761
50	2.713	2.202	3.783	1.827	4.155		5.786	3.476	6.137	2.349	6.108
75	3.903	3.039	5.176	2.496	5.612		6.775	3.914	7.072	2.637	7.005
100	5.010	3.784	6.410	3.094	6.911		7.552	4.260	7.813	2.864	7.703
150	6.934	5.110	8.605	4.160	9.220		8.652	4.777	8.909	3.224	8.792
200	8.639	6.292	10.550	5.111	11.280		9.475	5.177	9.770	3.519	9.651
250	10.158	7.375	12.340	5.989	13.160		10.120	5.519	10.490	3.766	10.390

[1] Reference 10-5: a/b=1 Stress, Reference 10-6: a/b=1 Deflection, Reference 10-41: a/b=1.5 ad 2 to infinity

Table 10.5.1-2 Stress Parameters for Edges of Simply Supported Rectangular Plates under Uniform Pressure Load

	Membrane Stress At Center of Short Edge						Membrane Stress At Center of Long Edge					
	a/b =1.0		a/b =1.5		a/b =2 to ∞		a/b =1.0		a/b =1.5		a/b =2 to ∞	
$pb^4/(Et^4)$	$X_{ctr-Sedge}$	$Y_{ctr-Sedge}$	$X_{ctr-Sedge}$	$Y_{ctr-Sedge}$	$X_{ctr-Sedge}$	$Y_{ctr-Sedge}$	$X_{ctr-Ledge}$	$Y_{ctr-Ledge}$	$X_{ctr-Ledge}$	$Y_{ctr-Ledge}$	$X_{ctr-Ledge}$	$Y_{ctr-Ledge}$
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12.5	0.736	0.175	0.733	0.232	0.750	0.237	0.175	0.736	0.339	1.072	0.411	1.298
25	1.440	0.339	1.510	0.478	1.492	0.472	0.339	1.440	0.680	2.150	0.772	2.441
50	2.983	0.678	2.824	0.893	2.761	0.873	0.678	2.983	1.234	3.903	1.349	4.264
75	4.267	1.006	3.955	1.251	3.859	1.220	1.006	4.267	1.702	5.381	1.830	5.788
100	5.446	1.345	4.977	1.574	4.852	1.534	1.345	5.446	2.117	6.694	2.261	7.149
150	7.557	1.960	6.811	2.154	6.640	2.100	1.960	7.557	2.858	9.037	3.026	9.570
200	9.439	2.545	8.452	2.673	8.240	2.606	2.545	9.439	3.518	11.130	3.707	11.720
250	11.227	3.035	9.972	3.153	9.728	3.076	3.035	11.227	4.125	13.040	4.334	13.700

The membrane stress in the center of the panel can be calculated by solving Equation 10.5.1-1 in conjunction with parameters from Table 10.5.1-1 or Table 10.5.2-1 depending on panel edge constraints, as

$$f_{membrane-x} = \frac{X_{ctr}Et^2}{b^2} \qquad f_{membrane-y} = \frac{Y_{ctr}Et^2}{b^2}$$ **Equation 10.5.1-2**

where
X_{ctr}, Y_{ctr} are from Table 10.5.1-1 or 10.5.2-1 or Figure 10.5.1-3 or Figure 10.5.2-3 for the appropriate edge constraint, aspect ratio and $pb^4/(Et^4)$ ratio. Interpolation is acceptable.

The bending stress at the center of the panel can be determined from

$$f_{bending-x} = \frac{X_{bend}Et^2}{b^2} \qquad f_{bending-y} = \frac{Y_{bend}Et^2}{b^2}$$ **Equation 10.5.1-3**

where
X_{bend}, Y_{bend} are from Table 10.5.1-1 or 10.5.2-1 or Figure 10.5.1-3 or 10.5.2-3 for the appropriate edge constraint, aspect ratio and $pb^4/(Et^4)$ ratio. Interpolation is acceptable.

The membrane stress at the center of the long edge can be determined from

$$f_{membrane-x} = \frac{X_{ctr-Ledge}Et^2}{b^2} \qquad f_{membrane-y} = \frac{Y_{ctr-Ledge}Et^2}{b^2}$$ **Equation 10.5.1-4**

where
$X_{ctr\text{-}Ledge}$, $Y_{ctr\text{-}Ledge}$ are from Table 10.5.1-2 or 10.5.2-2 or Figure 10.5.1-4 or 10.5.2-3 for the appropriate edge constraint, aspect ratio and $pb^4/(Et^4)$ ratio. Interpolation is acceptable.

The membrane stress at the center of the short edge for simply supported edge constraints can be determined from

$$f_{membrane-x} = \frac{X_{ctr-Sedge}Et^2}{b^2} \qquad f_{membrane-y} = \frac{Y_{ctr-Sedge}Et^2}{b^2}$$ **Equation 10.5.1-5**

where
$X_{ctr\text{-}Sedge}$, $Y_{ctr\text{-}Sedge}$ are from Table 10.5.1-2 or Figure 10.5.1-4 for the appropriate aspect ratio and $pb^4/(Et^4)$ ratio. Interpolation is acceptable.

To convert from stress to running load (lb/in) for membrane stresses, calculate the membrane stress and then multiply by the thickness of the panel. To convert from stress to running moment (in-lb/in) for the bending stress, Equation 10.5.1-3, calculate the bending stress and then use the following

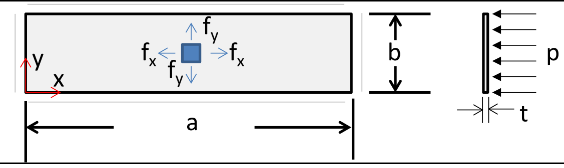
$$m_{bending} = \frac{f_{bending}t^2}{6}$$ **Equation 10.5.1-6**

where
$f_{bending}$ is calculated from Figure 10.5.1-3b or Equation 10.5.1-3 (psi)
t is the panel thickness (in)

The stresses resulting from applied pressures can be added to stresses in the panel due to in-plane loading and then a margin of safety based on a combination of the components can be calculated. Reference Section 2.5 to determine the appropriate interaction equation.

10.5.1.1 Example Problem – Panel with Uniform Pressure Load

Given: A panel, 8x12 inches, made from 2024-T62 Clad Aluminum Sheet
a=12 in
b=8 in
t=0.063 in



Determine the deflection and stress for a 10 psi applied pressure using the method of Section 10.5.1 and compare to results from IDAT/PRESS

- Assume the panel is held but not fixed, *i.e.* simply supported

Material Properties from IDAT/METDB
F_{tu} = 58000 psi
F_{ty} = 45000 psi
F_{cy} = 45000 psi
E_t= 10.5x10^6
ν=0.33

Solution

Calculate $pb^4/(Et^4)$

$$\frac{pb^4}{Et^4} = \frac{10(8)^4}{10.5x10^6(0.063)^4} = 247.63$$

Calculate b/a

$$\frac{a}{b} = \frac{12}{8} = 1.5$$

From Table 10.5.1-1 Determine δ/t and stress parameters at the center of the panel for membrane and bending
Interpolate to 247.63

$pb^4/(Et^4)$	δ/t	X_{ctr}	Y_{ctr}	X_{bend}	Y_{bend}
200	1.957	6.292	10.550	5.177	9.770
247.63	2.108	7.324	12.255	5.503	10.456
250	2.115	7.375	12.340	5.519	10.490

From Table 10.5.1-2 Determine stress parameters at the center of the panel's long and short edges
Interpolate to 247.63

$pb^4/(Et^4)$	$X_{ctr-Ledge}$	$Y_{ctr-Ledge}$	$X_{ctr-Sedge}$	$Y_{ctr-Sedge}$
200	3.518	11.130	8.452	2.673
247.63	4.096	12.950	9.900	3.130
250	4.125	13.040	9.972	3.153

Calculate the deflection

$$\frac{\delta}{t} = 2.108$$

$$\delta = 2.108t = 2.108(0.063) = 0.133 \text{ in}$$

Calculate $f_{membrane-x}$ at the center of the panel

$$f_{membrane-x} = \frac{X_{ctr}Et^2}{b^2}$$

$$f_{membrane-x} = \frac{7.324(10.5x10^6)(0.063)^2}{8^2} = 4769psi$$

Calculate the in-plane running membrane load

$$N_{x-membrane} = f_{membrane-x}t = 4769(0.063) = 300.4lb/in$$

Calculate $f_{membrane-y}$ at the center of the panel

$$f_{membrane-y} = \frac{Y_{ctr}Et^2}{b^2}$$

$$f_{membrane-y} = \frac{12.255(10.5x10^6)(0.063)^2}{8^2} = 7980psi$$

Calculate the in-plane running membrane load

$$N_{y-membrane} = f_{membrane-y}t = 7980(0.063) = 502.7lb/in$$

Calculate $f_{bending-x}$ at the center of the panel

$$f_{bending-x} = \frac{X_{bend}Et^2}{b^2}$$

$$f_{bending-x} = \frac{5.503(10.5x10^6)(0.063)^2}{8^2} = 3583psi$$

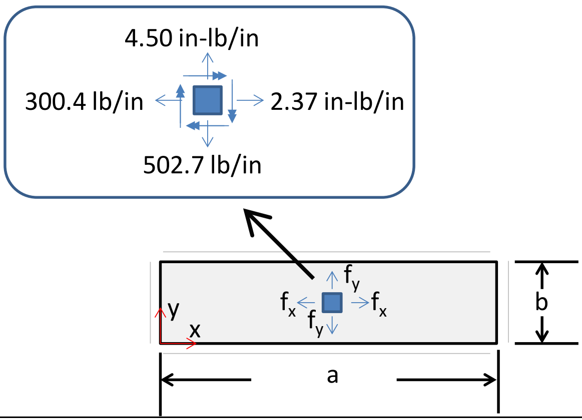
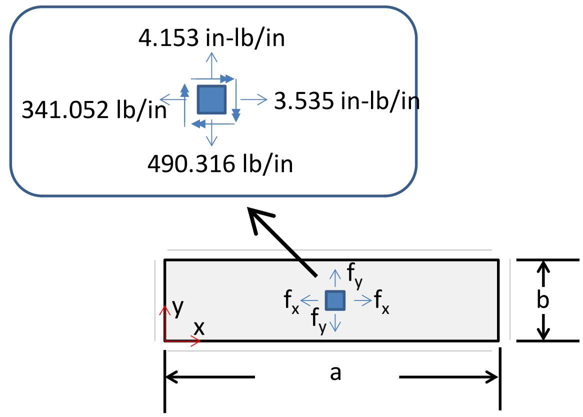
Calculate the running x bending moment at the center of the panel

$$m_{bending-x} = \frac{f_{bending-x}t^2}{6} = \frac{3583(0.063)^2}{6} = 2.37inlb/in$$

Calculate $f_{bending-y}$ at the center of the panel

$$f_{bending-y} = \frac{Y_{bend}Et^2}{b^2}$$

$$f_{bending-y} = \frac{10.456(10.5x10^6)(0.063)^2}{8^2} = 6809psi$$

Calculate the running y bending moment at the center of the panel	$m_{bending-y} = \frac{f_{bending-y}t^2}{6} = \frac{6809(0.063)^2}{6} = 4.50 inlb/in$
IDAT/PRESS output shown at the end of the example.	
Summarizing Results: Center of the Panel δ=0.132 in	**IDAT/PRESS Results: Center of the Panel** δ=0.1278 in
	
The results compare favorably for the two solution sequences, with the hand analysis giving slightly higher membrane stresses in the y-direction and slightly higher bending moments about the x-axis.	
Calculate the membrane loads for the long edge of the panel. (No bending – Simply Supported)	
Calculate $f_{membrane-x}$ at the center of the panel long side $f_{membrane-x} = \frac{X_{ctr-Ledge}Et^2}{b^2}$	$f_{membrane-x} = \frac{4.096(10.5x10^6)(0.063)^2}{8^2} = 2667psi$
Calculate the in-plane running membrane load	$N_{x-membrane} = f_{membrane-x}t = 2667(0.063) = 168.02lb/in$
Calculate $f_{membrane-y}$ at the center of the panel long side $f_{membrane-y} = \frac{Y_{ctr-Ledge}Et^2}{b^2}$	$f_{membrane-y} = \frac{12.950(10.5x10^6)(0.063)^2}{8^2} = 8433psi$
Calculate the in-plane running membrane load	$N_{y-membrane} = f_{membrane-y}t = 8433(0.063) = 531lb/in$
Summarizing Results: Center of Long Edge of Panel	**IDAT/PRESS Results: Center of Long Edge of Panel**
$N_{x-membrane}$=168 lb/in $N_{y-membrane}$ = 531 lb/in	$N_{x-membrane}$=162 lb/in $N_{y-membrane}$ = 490 lb/in
Calculate the membrane loads for the short edge of the panel. (No bending – Simply Supported)	
Calculate $f_{membrane-x}$ at the center of the panel short edge $f_{membrane-x} = \frac{X_{ctr-Sedge}Et^2}{b^2}$	$f_{membrane-x} = \frac{9.900(10.5x10^6)(0.063)^2}{8^2} = 6446psi$
Calculate the in-plane running membrane load	$N_{x-membrane} = f_{membrane-x}t = 6446(0.063) = 406lb/in$
Calculate $f_{membrane-y}$ at the center of the panel long edge $f_{membrane-y} = \frac{Y_{ctr-Sedge}Et^2}{b^2}$	$f_{membrane-y} = \frac{3.130(10.5x10^6)(0.063)^2}{8^2} = 2038psi$
Calculate the in-plane running membrane load	$N_{y-membrane} = f_{membrane-y}t = 2038(0.063) = 128lb/in$
Summarizing Results: Center of Short Edge of Panel	**IDAT/PRESS Results: Center of Short Edge of Panel**

$N_{x\text{-membrane}}$=406 lb/in $N_{y\text{-membrane}}$ = 128 lb/in	$N_{x\text{-membrane}}$=341 lb/in $N_{y\text{-membrane}}$ = 113 lb/in
The hand analysis for the edge of panel loads compare favorably but are, again, slightly higher than the results from IDAT/PRESS	

```
          +----------------------------------------------+
          |                                              |
          |                   PRESS                      |
          |                   V4.13                      |
          |         Flat Panel Pressure Analysis         |
          |            LM AERONAUTICS CO.                |
          |                                              |
          +----------------------------------------------+

DATE : 02-Jul-2013    TIME : 12:32:49

TITLE : Section 10.5.1.1 Example

INPUT DATA :
           Length (x-dir)          A =    12.00 in (x-dir.)
           Width  (y-dir)          B =     8.00 in (y-dir.)
           Laminate Thickness      T =   0.0630 in
           Applied Pressure        Q =    10.00 psi
           Edge Disp (x-dir)       U =   0.0000 in
           Edge Disp (y-dir)       V =   0.0000 in
           X-Axis Edge Fixity        =    0.000    (simple)
           Y-Axis Edge Fixity        =    0.000    (simple)

ISOTROPIC MATERIAL PROPERTIES :
             E =    10500000. (psi)
            Nu =       0.330
           Thk =       0.063 (in)


           Maximum Deflection      W =   0.1278 in
```

```
           y ^
             |
             +-------------+
      Mx ^   |             |
         ^   |             | |
  Nx <---|   |             | |---> Nx
         |   |             | V
             |             | V Mx
             +-------------+ ------> x

              X - Direction
            Left & Right Edges
```

Item		Edge	Midspan	
Membrane Load	Nx	341.052	341.052	lb/in
Reaction Load	Ny	112.547		lb/in
Bending Moment	Mx	0.000	3.535	in-lb/in
Reaction Moment	My	0.000		in-lb/in
Reaction Moment	Mxy	0.000		in-lb/in
Membrane Strain		0.000459	0.000459	in/in
Bending Strain		0.000000	0.000454	in/in
Total Strain (top)		0.000459	0.000006	in/in
Total Strain (bot)		0.000459	0.000913	in/in

```
                 Ny ^
           y ^      |
             |  ----->> My
             +-------------+
             |             |
             |             |
             |             |
             |             |
             +-------------+ --> x
               My <<-----
                    |
                    V Ny

              Y - Direction
            Top & Bottom Edges
```

Item		Edge	Midspan	
Membrane Load	Ny	490.316	490.316	lb/in
Reaction Load	Nx	161.804		lb/in
Bending Moment	My	0.000	4.153	in-lb/in
Reaction Moment	Mx	0.000		in-lb/in
Reaction Moment	Mxy	0.000		in-lb/in
Membrane Strain		0.000660	0.000660	in/in
Bending Strain		0.000000	0.000533	in/in
Total Strain (top)		0.000660	0.000128	in/in
Total Strain (bot)		0.000660	0.001193	in/in

10.5.2 Membrane Analysis of Fixed Rectangular Panels with Uniform Pressure Load

The geometry for a typical rectangular membrane with fixed edge restraints is shown in Figure 10.5.2-1, along with the coordinate system. This solution assumes the edge constraint clamps the plate rigidly against rotations and translations normal to the edge but permit displacements parallel to the edge.

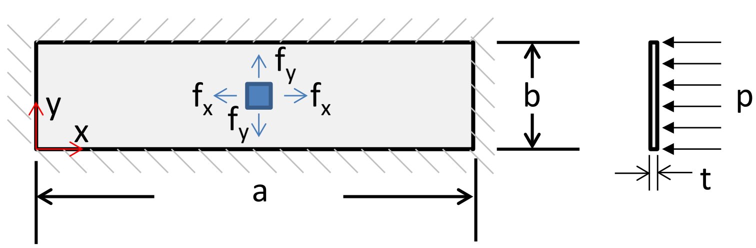


Figure 10.5.2-1 Fixed Edge Restraint Rectangular Panel Geometry for Membrane Analysis

A square panel, aspect ratio of *a/b=1*, will have equal membrane stresses in both the x and y directions. As the panel elongates in one axis, *x*, the stresses in the long-axis direction, f_x, remain unchanged and the stresses in the short axis direction, f_y, increase in magnitude. Reference 10-42 indicates that for fixed edge panels with an aspect ratio of 1.5

or greater, the stresses and center-of-panel deflection under high pressures differs from infinitely long panels by less than 3 percent.

Figure 10.5.2-2 provides a plot of deflection/thickness against the nondimensional pressure coefficient $pb^4/(Et^4)$ for different aspect ratio panels for fixed boundary conditions (fixed and held). Interpolation between curves based on aspect ratio is acceptable.

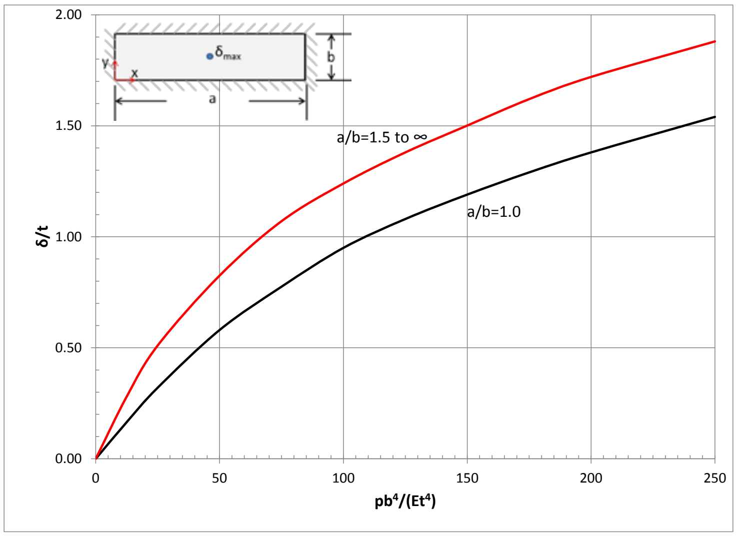


Figure 10.5.2-2 Deflection Coefficients for Flat Plates with Fixed Boundary Constraints

Figure 10.5.2-3 provides plots of the nondimensional stress parameter, Y, for different aspect ratio panels for fixed boundary conditions. This boundary condition restrains the edges from in-plane motion as well as rotation providing moment resistance. The parameter Y is defined by Equation 10.5.1-1.

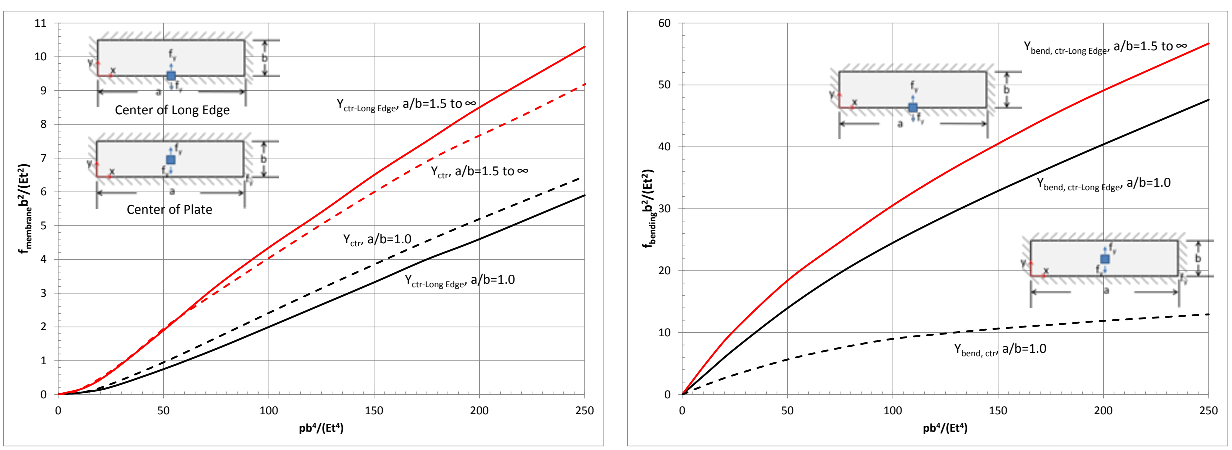


Figure 10.5.2-3 Membrane and Bending Stresses, f_y,
at the Center of the Panel and Center of Long Edge for Flat Plates with Fixed Boundary Constraints

Tables 10.5.2-1and 10.5.2-2 present the deflection coefficients and stress parameters from Figures 10.5.2-2 and 10.5.2-3 in numerical form. Equations 10.5.1-2 through 10.5.1-4 and 10.5.1-6 can then be used to determine the f_y membrane and bending stresses at the center of the panel and the center of the long edge and the running moment due to the bending stress.

Bending and membrane stresses in the x direction, f_x, for all aspect ratios is equal to f_y for a/b=1.0 for the appropriate $pb^4/(Et^4)$ ratio and location. That is to say, at a/b=1, f_x=f_y and as the panel elongates, f_x remains unchanged and f_y increases per Figure 10.5.2-3.

As with simply supported panels, the stresses resulting from applied pressures can be added to stresses in the panel due to in-plane loading and then a margin of safety based on a combination of the components can be calculated. Reference Section 2.5 to determine the appropriate interaction equation.

No example problem is provided as it will be similar to Example 10.5.1.1

Table 10.5.2-1 Deflection and Stress Parameters for the Center of Fixed Rectangular Plates under Uniform Pressure Load

At Center of Panel					
	Deflection		**Membrane Stress**		**Bending Extreme Fiber Stress**
	a/b=1.0	**a/b=1.5 to ∞**	**a/b=1.0**	**a/b=1.5 to ∞**	**a/b=1.0**
$pb^4/(Et^4)$	δ/t	δ/t	Y_{ctr}	Y_{ctr}	Y_{ctr}
0	0	0.000	0.000	0.000	0.000
12.5	0.165	0.280	0.075	0.228	1.725
25	0.320	0.510	0.309	0.705	3.200
50	0.580	0.825	0.951	1.935	5.650
75	0.775	1.070	1.675	3.020	7.550
100	0.950	1.240	2.415	4.037	9.000
125	1.080	1.380	3.142	5.027	9.900
150	1.190	1.501	3.848	5.976	10.700
175	1.290	1.620	4.541	6.890	11.300
200	1.380	1.720	5.197	7.655	11.900
250	1.540	1.880	6.459	9.189	12.900

Table 10.5.2-2 Stress Parameters for Center of Long Edge of Fixed Plates under Uniform Pressure Load

	Membrane Stress At Center of Long Side		Bending Extreme Fiber Stress At Center of Long Side	
	a/b=1.0	a/b=1.5 to ∞	a/b=1.0	a/b=1.5 to ∞
$pb^4/(Et^4)$	$Y_{ctr\text{-}Lside}$	$Y_{ctr\text{-}Lside}$	$Y_{ctr\text{-}Lside}$	$Y_{ctr\text{-}Lside}$
0	0.000	0.000	0.000	0.000
12.5	0.070	0.200	3.730	5.550
25	0.220	0.660	7.380	10.460
50	0.750	1.900	13.950	18.400
75	1.350	3.200	19.650	24.640
100	2.000	4.350	24.500	30.550
125	2.660	5.400	28.840	35.800
150	3.320	6.500	32.880	40.500
175	4.000	7.500	36.700	45.000
200	4.600	8.500	40.400	49.100
250	5.900	10.300	47.600	56.700

10.6 Post Buckling Behavior

10.6.1 Flat Plates

Post buckling analysis of flat plates and the subsequent load redistribution, sometimes referred to as Diagonal Tension Analysis, is covered extensively in PM4057 Section 6.6.

10.6.2 Curved Plates

Refer to Program-specific and customer-generated guidance, Lockheed Martin Engineering Stress Memo Manual SM106f (Reference 10-1), NACA-TN 2661 (Reference 10-35) and 2662 (Reference 10-36), and Bruhn, Section C11.2 (Reference 10-37)

10.7 Shells

10.7.1 Shell Stability – Unreinforced

Basic theory is discussed in Timoshenko, chapter 10 (Reference 10-10).

10.7.1.1 Cylindrical

Refer to Program-specific and customer-generated guidance, GD Structures Analysis Manual, Volume I, Section 10 (Reference 10-2), and Timoshenko, chapter 10 (Reference 10-10).

10.7.1.2 Truncated Cones

Refer to Program-specific and customer-generated guidance, GD Structures Analysis Manual, Volume I, Section 10 (Reference 10-2), and NASA SP-8019 (Reference 10-38).

10.7.1.3 Doubly Curved

Refer to Program-specific and customer-generated guidance, GD Structures Analysis Manual, Volume I, Section 10 (Reference 10-2), and NASA SP-8032 (Reference 10-39).

10.7.2 Local Loads Effects – Unreinforced

10.7.2.1 Spherical

Refer to Program-specific and customer-generated guidance and GD Structures Analysis Manual, Volume I, Section 10 (Reference 10-2).

10.7.2.2 Ellipsoidal

Refer to Program-specific and customer-generated guidance and GD Structures Analysis Manual, Volume I, Section 10 (Reference 10-2).

10.7.2.3 Doubly Cylindrical

Refer to Program-specific and customer-generated guidance and GD Structures Analysis Manual, Volume I, Section 10 (Reference 10-2).

10.7.3 Pressure Vessel – Unreinforced

10.7.3.1 Cylindrical

Refer to Program-specific and customer-generated guidance, Lockheed Martin Engineering Stress Memo Manual SM127 (Reference 10-1), and Timoshenko, chapter 15 (Reference 10-10).

10.7.3.2 Spherical

Refer to Program-specific and customer-generated guidance, GD Structures Analysis Manual, Volume I, Section 9 (Reference 10-2), and Timoshenko, chapter 16 (Reference 10-10).

10.7.3.3 Conical

Refer to Program-specific and customer-generated guidance, GD Structures Analysis Manual, Volume I, Section 9 (Reference 10-2), and Timoshenko, chapter 16 (Reference 10-10).

10.7.4 Pressure Vessel – Reinforced with Stringers

Refer to Program-specific and customer-generated guidance, Lockheed Martin Engineering Stress Memo Manual SM127 (Reference 10-1), and Air Force Stress Analysis Manual (Reference 10-8).

10.7.5 Pressure Vessel – Reinforced with or without Stringers but with Rings

Refer to Program-specific and customer-generated guidance, Lockheed Martin Engineering Stress Memo Manual SM127 (Reference 10-1), and Air Force Stress Analysis Manual (Reference 10-8).

10.7.6 Post-Buckling Strength of a Pressurized Cylinder

Refer to Program-specific and customer-generated guidance, GD Structures Analysis Manual, Volume I, Section 10 (Reference 10-2), and D. J. Peery's paper (Reference 10-40).