5 Properties of Laminated Composite Beam Cross-Sections

The purpose of this chapter is to provide information and guidance for calculating modulus weighted geometric properties of laminated composite beam cross-sections. The information in this chapter is intended for general guidance only. Refer to your program for specific guidance on the calculation of composite beam cross-section properties.

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[1] In 2002, administration of MIL-HDBK-17 was transferred to FAA. Future releases will be released as Composite Materials Handbook 17 (CMH-17), Materials Sciences Corporation, Secretariat. The CMH-17 organization is an all voluntary organization comprised of engineers and scientists from government, academia, and industry.

Table 5.1-1 Symbols and Nomenclature

Symbol	Description	Units
[A]	Laminate in-plane extensional stiffness matrix	lb/in
[B]	Laminate in-plane coupling stiffness matrix	lb
[D]	Laminate bending stiffness matrix	in lb
Branch/Segment	A cross-section shell segment between two junctions or between a junction and an end point.	
Junction	Intersections between branch segments	
Contour Line	The line defined by the intersection of the mid-surface of each shell unit with the plane normal to the x-axis of the beam	
$\alpha(s)$	Branch segment contour angle	radians
$\bar{y}$	y coordinate of a point on the section contour at contour coordinate s	in
$\bar{z}$	z coordinate of a point on the section contour at contour coordinate s	in
s	branch contour coordinate in the segment contour direction	in
ζ	branch contour coordinate in the segment thickness direction	in
r_R	ζ direction contour coordinate of a point on the section contour relative to an origin at the reference point R	in
q_R	s direction contour coordinate of a point on the section contour relative to an origin at the reference point R	in
y_R	Y coordinate of the reference point R	in
z_R	Z coordinate of the reference point R	in
U(x)	Axial displacement of the beam reference line element as a function of x	in
V(x)	Displacement in the y direction of the beam reference line element as a function of x	in
W(x)	Displacement in the z direction of the beam reference line element as a function of x	in
$\Phi_x(x)$	Torsional rotation of the beam reference line element as a function of x	radians
$\Phi_y(x)$	Bending rotation about the y axis of the beam reference line element as a function of x	radians
$\Phi_z(x)$	Bending rotation about the z axis of the beam reference line element as a function of x	radians
N_x	Beam resultant axial force	lbs
T_x	Beam resultant torque about the x-axis	In lbs
M_y	Beam resultant moment about the y-axis	In lbs

M_z	Beam resultant moment about the z-axis	In lbs
$M_{\omega x}$	Beam resultant Bi-moment about the x-axis	in^2 lbs
Bi-moment	Resultant force associated with warping	in^2 lbs
u	x displacement of a material point in the beam cross-section	in
Symbol	**Description**	**Units**
v	y displacement of a material point in the beam cross-section	in
w	z displacement of a material point in the beam cross-section	in
ε_x	x direction strain of a material point in the beam cross-section	in/in
ε_y	y direction strain of a material point in the beam cross-section	in/in
ε_z	z direction strain of a material point in the beam cross-section	in/in
γ_{xy}	Engineering shear strain in the x-y plane at a material point in the beam cross-section.	in/in
γ_{xz}	Engineering shear strain in the x-z plane at a material point in the beam cross-section.	in/in
γ_{yz}	Engineering shear strain in the y-z plane at a material point in the beam cross-section.	in/in
γ_{xs}	Engineering shear strain in the x-s plane at a material point in the beam cross-section.	in/in
$\gamma_{x\zeta}$	Engineering shear strain in the x-ζ plane at a material point in the beam cross-section.	in/in
$\omega(\zeta,s)$	Warping function	in^2
$\overline{\omega}$ (s)	Contour warping function component	in^2
$\widetilde{\omega}$ (s)	Thickness warping function component	in
N_s	Shell transverse resultant force in the contour coordinate system	lbs/in
M_s	Shell transverse resultant moment in the contour coordinate system	in-lbs/in
N_x	Shell axial resultant force in the contour coordinate system	lbs/in
M_x	Shell axial resultant moment in the contour coordinate system	in-lbs/in
N_{xs}	Shell shear resultant force in the contour coordinate system	lbs/in
M_{xs}	Shell twist-curvature resultant moment in the contour coordinate system	in-lbs/in
e_R	x direction strain of the beam at an axis through the reference point R	in/in
κ_{ZR}	z curvature of the beam at an axis through the reference point R	1/in
κ_{YR}	Y Curvature of the beam at an axis through the reference point R	1/in

τ'_R	Derivative of twist of the beam about an axis through the reference point R	1/in^2
τ_R	Twist of the beam about an axis through the reference point R	1/in
e_x^0	x-direction mid-surface shell strain of a material point in the cross-section in the contour coordinate system	
κ_x	mid-surface curvature about the x-axis in the contour coordinate system	
Symbol	**Description**	**Units**
γ_{xs}^0	mid-surface shear strain in the contour coordinate system	
κ_{xs}	twist curvature in the contour coordinate system	
N_{XR}	Beam axial resultant force in the y-z coordinate system at the reference point R	lbs
M_{YR}	Beam resultant moment about y in the y-z coordinate system at the reference point R	in lbs
M_{ZR}	Beam resultant moment about z in the y-z coordinate system at the reference point R	in lbs
$M_{\omega R}$	Beam resultant bi-moment in the y-z coordinate system	in^2 lbs
T_{XR}	Beam resultant torque in the y-z coordinate system	in lbs
EA	Modulus Weighted Area	lbs
ES_y	Modulus Weighted First Moment of Area about Y	in-lbs
ES_z	Modulus Weighted First Moment of Area about Z	in-lbs
EI_{yy}	Modulus Weighted Moment of Inertia about Y	in^2 lbs
EI_{zz}	Modulus Weighted Moment of Inertia about Z	in^2 lbs
EI_{zy}	Modulus Weighted Product of Inertia	in^2 lbs
GJ	Modulus Weighted Torsional Stiffness	in^2 lbs
ES_ω	Modulus Weighted First Sectorial Moment of Area	in^2 lbs
$EI_{\omega y}$	Modulus Weighted Second Sectorial Moment of Area about Y	in^3 lbs
$EI_{\omega z}$	Modulus Weighted Second Sectorial Moment of Area about Z	in^3 lbs
$EI_{\omega\omega}$	Modulus Weighted Warping Coefficient	In^4 lbs
EH	Modulus Weighted Axial-Torsional Coupling Stiffness	in-lbs
EH_c	Modulus Weighted Bending-Torsional Coupling Stiffness about Y	in^2 lbs
EH_s	Modulus Weighted Bending-Torsional Coupling Stiffness about Z	in^2 lbs

EH_q	Modulus Weighted Warping-Torsional Coupling Stiffness	in^3 lbs
I_{1i}	Geometric Branch Integral 1 for segment i	in
Symbol	**Description**	**Units**
I_{2i}	Geometric Branch Integral 2 for segment i	in^2
I_{3i}	Geometric Branch Integral 3 for segment i	in^2
I_{4i}	Geometric Branch Integral 4 for segment i	in^3
I_{5i}	Geometric Branch Integral 5 for segment i	in^2
I_{6i}	Geometric Branch Integral 6 for segment i	in^3
I_{7i}	Geometric Branch Integral 7 for segment i	in^3
I_{8i}	Geometric Branch Integral 8 for segment i	in^4
I_{9i}	Geometric Branch Integral 9 for segment i	in^3
I_{10i}	Geometric Branch Integral 10 for segment i	in^3
I_{11i}	Geometric Branch Integral 11 for segment i	in^4
I_{12i}	Geometric Branch Integral 12 for segment i	in^3
I_{13i}	Geometric Branch Integral 13 for segment i	in^4
I_{14i}	Geometric Branch Integral 14 for segment i	in^4
I_{15i}	Geometric Branch Integral 15 for segment i	in^4
y_1	"y" coordinate for end 1 of a straight branch segment	in
z_1	"z" coordinate for end 1 of a straight branch segment	in
y_2	"y" coordinate for end 2 of a straight branch segment	in
z_2	"z" coordinate for end 2 of a straight branch segment	in
l_s	Length of a straight branch segment	in
r_{fi}	Numerator of the ζ direction contour coordinate r_R for segment i	in^2
q_{fi}	Numerator of the s direction contour coordinate q_R for segment i	in^2
C_i	Branch segment warping constant of integration for segment i	in^2
y_c	"y" coordinate of cross-section centroid	in
z_c	"z" coordinate of cross-section centroid	in

y_{sc}	"y" coordinate of cross-section shear center	in
z_{sc}	"z" coordinate of cross-section shear center	in
O_N	Offset magnitude in the branch normal direction	in
Symbol	**Description**	**Units**
O_{s1}	Offset magnitude in the branch "s" direction at end 1	in
O_{s2}	Offset magnitude in the branch "s" direction at end 1	in
$y_1^{O_N}$	Normal direction offset "y" coordinate at branch end 1	in
$z_1^{O_N}$	Normal direction offset "z" coordinate at branch end 1	in
$y_2^{O_N}$	Normal direction offset "y" coordinate at branch end 2	in
$z_2^{O_N}$	Normal direction offset "z" coordinate at branch end 2	in
$y_1^{O_s}$	"s" direction offset "y" coordinate at branch end 1	in
$z_1^{O_s}$	"s" direction offset "z" coordinate at branch end 1	in
$y_2^{O_s}$	"s" direction offset "y" coordinate at branch end 2	in
$z_2^{O_s}$	"s" direction offset "z" coordinate at branch end 2	in

5.2 Introduction

Beams are an easily recognizable class of structural members in which the length dimension of the structural element is at least ten times the maximum characteristic width or height dimension of the cross-section. Beams carry axial load, bending, torsion, and in some formulations transverse shear.

Historically the idealization of structural members as beams has been successfully used to reduce the size and complexity of structural Finite Element Models and analysis. Use of beams in aircraft analysis varies with program, aircraft type, fidelity of the model, structural components, and other factors. Advancements in computing speed make the use of large higher-definition branched shell Finite Element Models a practical alternative to the extensive use of complex beam cross-sections. Current high definition branched shell analysis models typically use rectangular cross-section beam elements to represent caps and flanges on spars, ribs, and bulkheads, with more complex beam cross-sections used on a limited basis.

A beam cross-section is a two-dimensional normal view of beam material. Each cross-section is defined by a “cut” or section-view taken perpendicular to the length axis of the beam at a specified reference location. An example “cut” and the resulting beam cross-section is shown in Figure 5.2-1. A beam’s cross-section shape may be constant or may vary continuously or discontinuously along the length axis of the beam. Cross-section shape and material properties characterize the strength and stiffness properties of a beam.

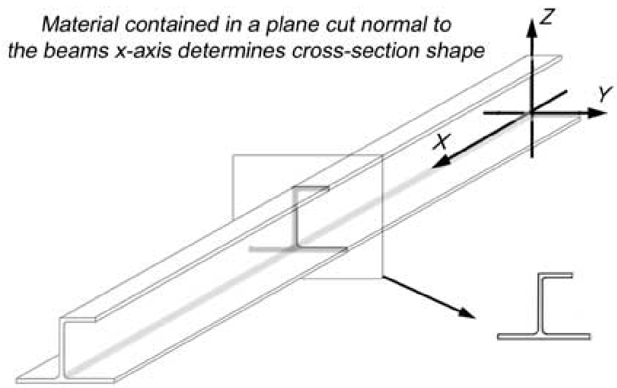

Figure 5.2-1 Cross Section

Beam cross-sections are categorized as thick or thin-walled, open or closed section, and can be formulated for straight or curved beams. These cross-section classifications are a consequence of theoretical differences in the formulation of the cross-section property equations. The addition of laminated composite materials also affects the cross-section property equations. Cross-section property theory and methodology presented in this section is applicable to laminated composite thin-walled straight beams of open or closed cross-section. The methodology also applies to thin-walled straight beam cross-sections constructed from isotropic materials.

5.2.1 Cross-Section Coordinate Systems

The beam coordinate system used in this section is a Cartesian right-hand system with coordinates (y, z, x) and is shown in Figure 5.2-2. The x-axis runs along the length axis of the beam and the beam cross-sections are defined in the y-z plane. The coordinate system used in this section is consistent with the beam coordinate systems found in NASTRAN and the IDAT utility SECTION.

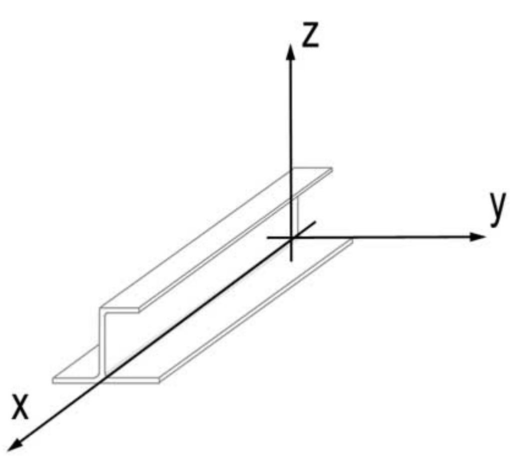

Figure 5.2-2 Beam Coordinate System (y,z,x)

The cross-section may be placed at any location in the y-z plane. The calculated results for internal loads and beam deformations are not affected by the choice of cross-section origin provided that external loads and constraints are applied consistently. The location of cross-section origin does affect the calculated values of most section properties. A J-section placed at an arbitrary location in the y-z plane is shown in Figure 5.2-3.

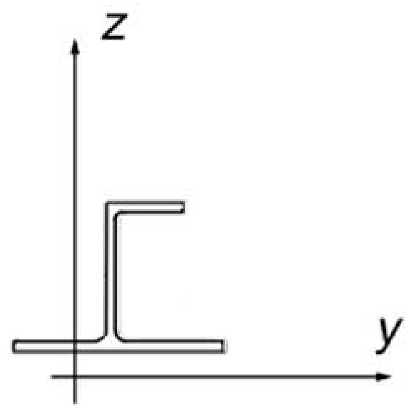

Figure 5.2-3 Cross-section Location in the y-z Plane

It is often possible to choose a location for the cross-section origin which decouples specific deformation modes of the beam. Choosing the section centroid as the section origin will decouple extension and bending, while choosing the shear center as the section origin will decouple transverse and or vertical displacement and torsion. These special locations for cross-section origin are discussed thoroughly in the following sections.

5.2.2 Cross-Section Terminology

Terminology is introduced to designate elements of the laminated thin-walled cross-section. The thin-walled cross-section is viewed as a collection of thin-shell[1] segments. Corners and intersections between shell segments are referred to as junctions. A shell segment between two junctions or between a junction and an end point is called a branch. The line defined by the intersection of the mid-surface of each shell unit and the plane normal to the x-axis of the beam is called the section contour line. The section contour line and the local wall thicknesses of each shell unit completely define the geometry of the beam cross-section. The terminology described is illustrated for a J-section in Figure 5.2-4

[1] Typically the thin-walled designation requires that the thickness of each shell segment is less than 1/10 of the maximum characteristic width and height dimension of the cross-section.

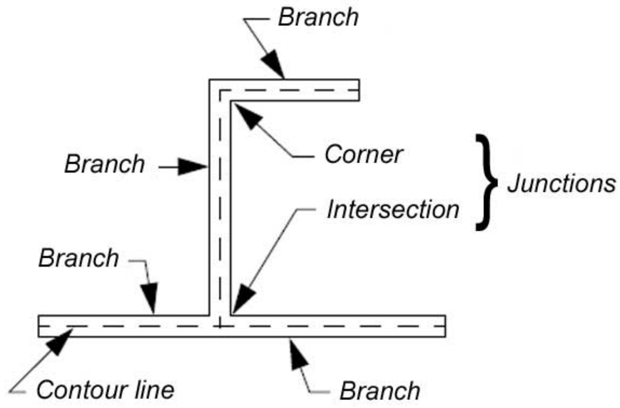

Figure 5.2-4 Cross Section Terminology

5.2.3 Cross-Section Displacements and Rotations

Structural beam theories reduce the three-dimensional structural beam problem to displacement, rotation, and torsion of a one-dimensional line element. In some theories transverse shear and/or section warping are also included. The designation and sign convention for displacements and rotations of a point on the beam x-axis are shown in Figure 5.2-5.

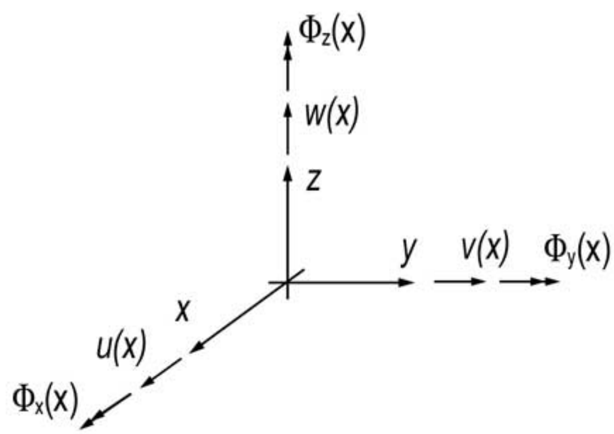

Figure 5.2-5 Cross-Section Displacements and Rotations

Where sign convention is determined by the right hand rule, and:

- $U(x)$ is axial displacement
- $V(x)$ is lateral displacement in the y-direction
- $W(x)$ is vertical displacement in the z-direction
- $\Phi_x(x)$ is torsional rotation about the x-axis
- $\Phi_y(x)$ is bending rotation about the y-axis
- $\Phi_z(x)$ is bending rotation about the z-axis

5.2.4 Cross-Section Resultant Forces and Moments

The resultant forces and moments acting on a beam cross-section are shown in Figure 5.2-6. Note that the resultant beam forces shown in Figure 5.2-6 are the static equivalent of the stress distribution acting on the beam cross-section. The resultant forces and moments represent internal loads acting on the cross-section and are distinct from external forces and reactions acting on the beam. Sign convention for moments is given by the right hand rule.

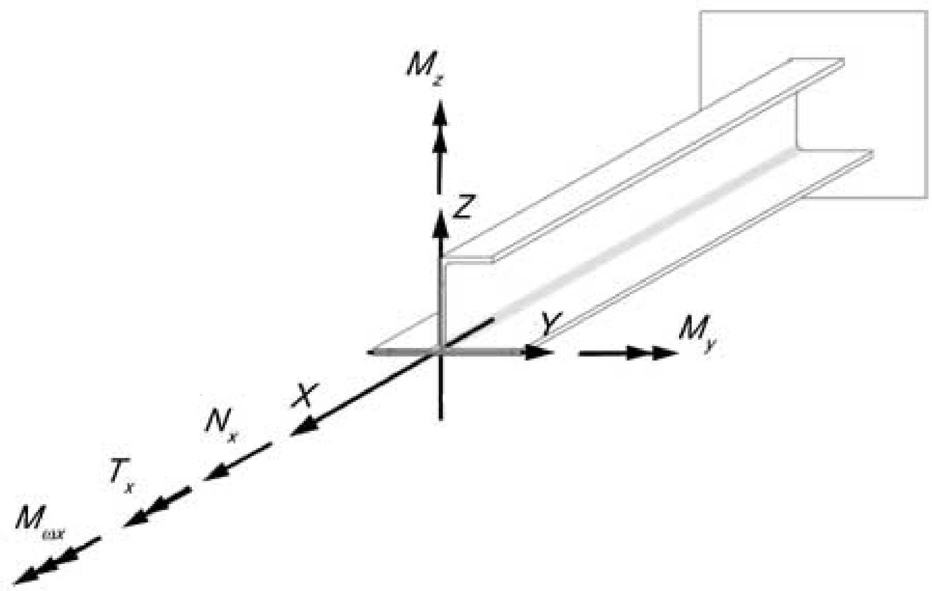

Figure 5.2-6 Cross-section Forces and Moments

Where sign convention is determined by the right hand rule, and:

N_x is axial resultant force
T_x is the resultant torque
M_y is the resultant moment about the y-axis
M_z is the resultant moment about the z-axis
$M_{\omega x}$ is the resultant bimoment about the x-axis
Internal shear forces N_y, and N_z are not shown but are considered in some beam formulations.

5.3 Calculation of Laminated Composite Beam Cross-Section Properties

Three methods for calculating laminated beam cross-section properties are described in the section. Section 5.3.1 describes the Modulus Weighted Method. The modulus weighted method is suitable for preliminary design. Specific stacking sequences are not required. Laminate axial stiffness estimates based on ply percentages are used to calculate basic modulus weighted cross-section properties. Section 5.3.2 presents the advanced method documented in FZM-9954. Laminate stacking sequences for each cross-section segment are used to capture the coupling behavior present in unsymmetric and unbalanced laminates. The FZM-9954 methodology includes formulas for calculating warping properties. Section 5.3.3 documents the modulus weighted Wojciechowski methodology implemented by the IDAT utility SECTION.

5.3.1 Modulus Weighted Area Method

The Modulus Weighted Area Method is an approximate method for calculating cross-section properties of beams constructed from laminated composite materials. This method relies on simple formulas for calculating, transforming, and combining the properties of rectangular cross-section segments. The Modulus Weighted Area Method is adapted from similar methods developed and used for concrete reinforced beams. Section 5.3.1.8 provides a summary overview of the process, and Section 5.3.1.9 presents an example problem illustrating the methodology.

The beam cross-section is represented by straight branch segments. Each straight branch of the beam cross-section is represented by a single straight rectangular segment. Curved branches are approximated using a series of straight branch segments. Accuracy of the approximation may be improved by increasing the number of segments used to represent the curved branch. Branches with thickness or material changes are also modeled using multiple segments. Figure 5.3-1 illustrates the process of representing a beam cross-section with a set of straight rectangular segments

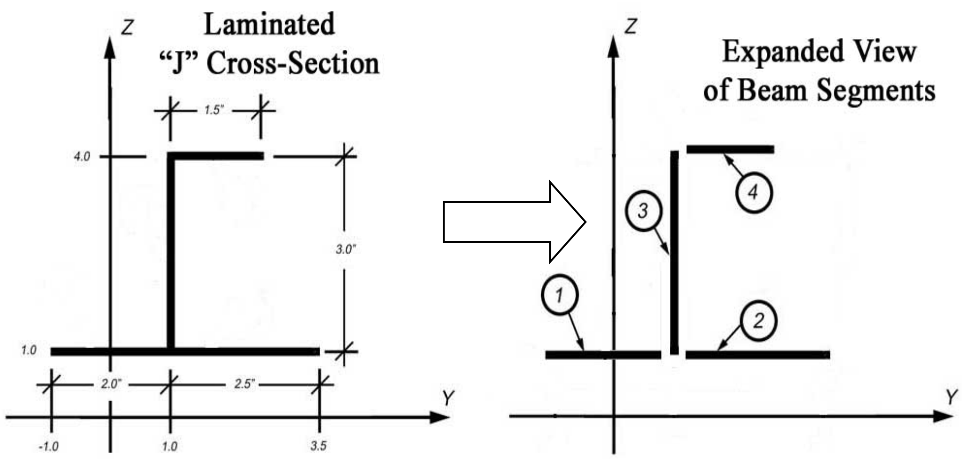

Figure 5.3-1 Expanded View of Beam Cross-Section Segments

Basic modulus weighted cross-section properties are calculated using effective axial stiffness properties. An x-direction effective stiffness modulus is required for each cross-section segment. Use of an effective axial stiffness modulus eliminates any coupling behavior that may be present due to an unsymmetric or unbalanced laminate stiffness matrix.

5.3.1.1 Laminate Axial Stiffness Modulus

The effective in-plane elastic axial stiffness (E_x) and the shear stiffness (G_{xy}) are required for the modulus weighted cross-section property method. Effective stiffness properties approximate the laminate as a homogenous orthotropic plate. Effective laminate stiffness properties neglect the laminate coupling behavior that may be present in some stacking sequences. The following paragraphs describe three alternative methods which may be used for obtaining estimates of the required laminate stiffness terms.

If the stacking sequence is known, effective elastic properties can be calculated using sub-matrix terms of the inverted Classical Laminated Plate Theory - [ABD] stiffness matrix. Sub-matrices of the inverted stiffness matrix are represented by the lower case letters "a b h d".

$$[ABD]^{-1} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ h & d \end{bmatrix}$$ **Equation 5.3-1**

The effective axial stiffness and shear stiffness of a laminate may be calculated using terms of the inverted sub-matrix "a" from Equation 5.3-1, as follows in Equation 5.3-2

$$E_x = 1/(t \cdot a_{11})$$
$$G_{xy} = 1/(t \cdot a_{66})$$ **Equation 5.3-2**

Where t is the total laminate thickness

The inverted stiffness methodology described above is implemented in the IDAT LAMINATE program.

Effective in-plane laminate stiffness may also be approximated from laminate ply percentages. Ply percentage methods for stiffness estimation are particularly useful for preliminary design since stacking sequence and laminate thickness are not required. Ply percentages may be calculated by dividing the sum of ply thickness in each allowed direction by the total thickness of all plies. The thickness of fabric plies may be allocated based on the percentage of fibers in each direction.

The IDAT utility LAM_PERCT calculates effective in-plane laminate stiffness properties using ply percentages and lamina material stiffness data. Note that the laminate thickness value required by LAM_PERCT does not affect the values calculated for effective in-plane stiffness. LAM_PERCT calculations for an example (25/50/25) laminate are shown in Figure 5.3-2.

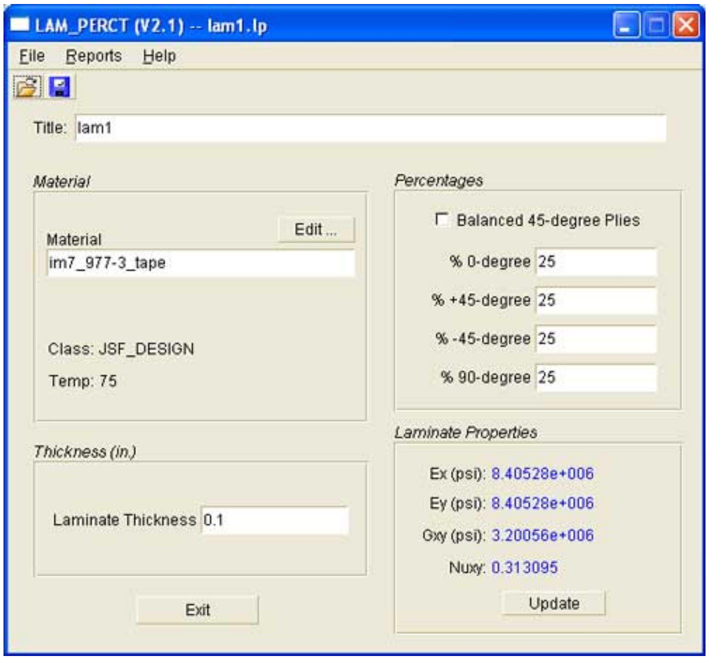

Figure 5.3-2 IDAT LAM_PERCT Effective Stiffness Properties

Effective in-plane laminate stiffness properties may also be calculated using stiffness Carpet Plots produced by the IDAT utility MATUTL. Stiffness Carpet plots are produced by selecting a material and then selecting the "Laminate Stiffness" sub-menu item from the main Carpet Plot menu as illustrated in Figure 5.3-3.

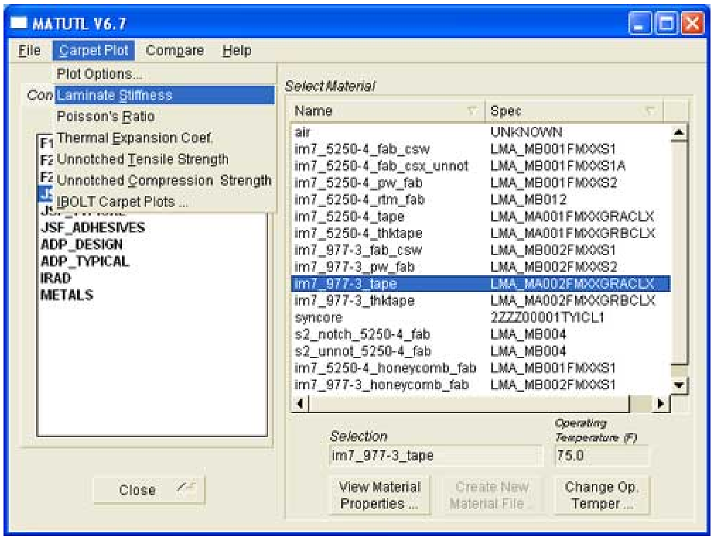

Figure 5.3-3 IDAT MATUTL access to Laminate Stiffness Carpet Plots

An example IDAT stiffness Carpet Plot is shown in Figure 5.3-4. The axial stiffness is determined by entering the chart on the horizontal axis at the percentage value for ± 45 degree plies. Proceed vertically to intercept the curve representing the number of *0* degree plies. Proceed horizontally left to read the stiffness value off the vertical axis.

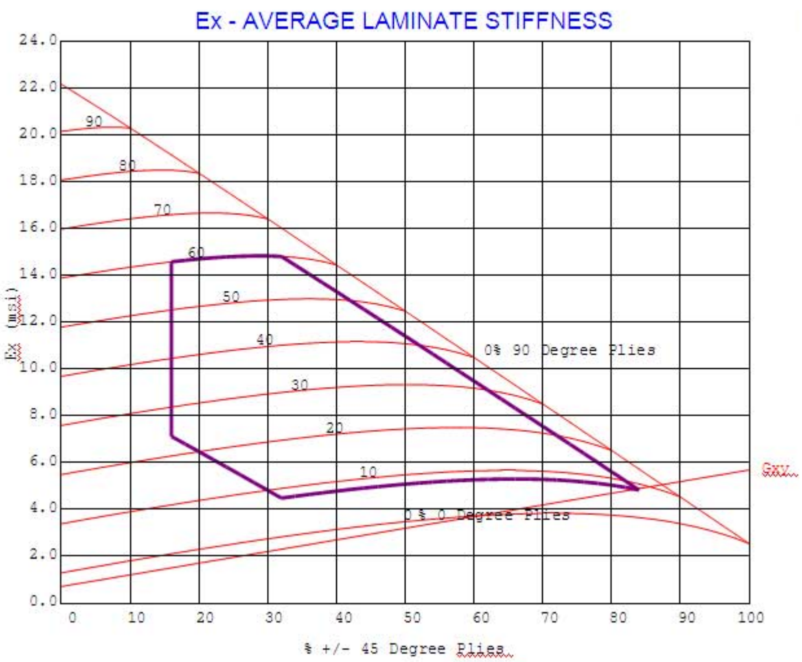

Figure 5.3-4 IDAT MATUTL Stiffness Carpet Plot

Note that the Carpet plot in Figure 5.3-4also contains a curve representing shear stiffness G_{xy}. The shear stiffness is determined by entering the chart on the horizontal axis at the percentage value for ± 45 degree plies. Proceed vertically to intercept the curve representing shear stiffness. Proceed horizontally left to read the shear stiffness value off the vertical axis.

Although less accurate, the following approximate method given by Equation 5.3-3 may be used for hand calculations of axial stiffness using ply percentages

$$E_{xx} = (E_{11}\ \%0^\circ + E_{22}\ \%90^\circ + E_{45}\ \%45^\circ) / 100 \qquad \textbf{Equation 5.3-3}$$

Where
$E_{45} = E_{xx}$ for a $[\pm45^\circ]_s$ laminate { $\approx (A_{11} - (A_{12})\ 2/A_{22})$ or approx 2.8 Msi for Carbon/Epoxy}

5.3.1.2 Modulus Weighted Area

The modulus weighted cross-sectional area is the axial stiffness modulus times the area of the segment. A rectangular cross-section segment showing the differential area element "*da*" is shown in Figure 5.3-5. In the general case, the area is obtained by integrating the differential element over the cross-section.

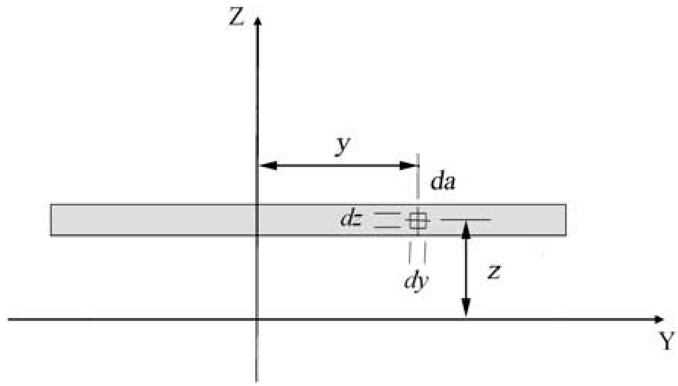

Figure 5.3-5 Differential Area Element for a Rectangular Cross-Section Segment

The modulus weighted area of a general cross-section shape is given by Equation 5.3-4.

$$EA = \int E \, da$$
$$where \quad \int da = \int\int dy \, dz$$

Equation 5.3-4

E = Axial modulus of cross-section differential area da

For a cross-section containing "n" segments, the modulus weighted area is given by the summation Equation 5.3-5

$$EA = \sum_{i=1}^{n} E_i A_i$$

Equation 5.3-5

Where, A_i is the area for segment 'i'.
E_i is the modulus for segment 'i'.

5.3.1.3 Modulus Weighted Geometric Centroid

The modulus weighted geometric centroid represents the modulus weighted center of area of the cross-section. Axial forces applied at the modulus weighted geometric centroid of a beam cross-section do not produce bending moments. Equation 5.3-6, and Equation 5.3-7 determine the modulus weighted geometric centroid $(\bar{y}, \bar{z})$ of a general cross-section shape or element.

$$\bar{y} = \frac{\int E \, y \, da}{EA}$$

Equation 5.3-6

$$\bar{z} = \frac{\int E \, z \, da}{EA}$$

Equation 5.3-7

The modulus weighted geometric centroid $(\bar{y}, \bar{z})$ of a beam cross-section containing "n" segments may be calculated using Equation 5.3-8, and Equation 5.3-9.

$$\bar{y} = \frac{\sum_{i=1}^{n} E_i\, y_i\, A_i}{EA}$$ **Equation 5.3-8**

Where, y_i is the 'y' coordinate of the centroid for segment 'i'.
A_i is area for segment 'i'.
E_i is Young's modulus for segment 'i'.

$$\bar{z} = \frac{\sum_{i=1}^{n} E_i\, z_i\, A_i}{EA}$$ **Equation 5.3-9**

Where, z_i is the 'z' coordinate of the centroid for segment 'i'.
A_i is area for segment 'i'.
E_i is Young's modulus for segment 'i'.

A coordinate system axis with its origin at the modulus weighted centroid of a cross section will be designated using a single prime.

5.3.1.4 Modulus Weighted First Moment of Area

The modulus weighted first moment of area, or "static moment" is a measure of the distribution of modulus weighted cross-sectional area in relation to a specified coordinate axis. The first moment of area is dependent on the shape of the cross section as well as the location and orientation of the cross-section relative to the coordinate system axis.The general equation for the modulus weighted first moment of area about the, "z-axis" is given by Equation 5.3-10, and the general equation for the modulus weighted first moment of area about the "y-axis" is given by Equation 5.3-11.

$$EQ_{zz} = \int E\,y\,da$$ **Equation 5.3-10**

$$EQ_{yy} = \int E\,z\,da$$ **Equation 5.3-11**

The modulus weighted first moments of area for a beam cross-section containing "n" rectangular segments may be calculated using Equation 5.3-12 and Equation 5.3-13

$$EQ_{zz} = \sum_{i=1}^{n} E_i\, y_i\, A_i$$ **Equation 5.3-12**

Where, y_i is the 'y' coordinate of the centroid for segment 'i'.
A_i is area for segment 'i'.
E_i is the Young's modulus for segment 'i'.

$$EQ_{yy} = \sum_{i=1}^{n} E_i z_i A_i$$ **Equation 5.3-13**

Where, z_i is the 'z' coordinate of the centroid for segment 'i'.
A_i is area for segment 'i'.
E_i is the Young's modulus for segment 'i'.

5.3.1.5 Modulus Weighted Area Moments of Inertia

The area moments of inertia for a beam cross-section provide a measure of the beam's bending stiffness. The area moment of inertia is also known as the second moment of area, and is dependent on the shape of the cross section as well as the location and orientation of the cross-section relative to the coordinate system axis. The general formula for the modulus weighted area moment of inertia about the "z-axis" is given by Equation 5.3-14, and the general formula for the modulus weighted area moment of inertia about the "y-axis" is given by Equation 5.3-15.

$$EI_{zz} = \int E y^2 da$$ **Equation 5.3-14**

$$EI_{yy} = \int E z^2 da$$ **Equation 5.3-15**

The modulus weighted area moments of inertia for a beam cross-section containing "n" branch segments may be calculated using Equation 5.3-16 and Equation 5.3-17

$$EI_{zz} = \sum_{i=1}^{n} E_i I_{zz\,i}$$ **Equation 5.3-16**

Where, $I_{zz\,i}$ is the area moment of inertia about the 'z' axis for segment 'i'.
E_i is the Young's modulus for segment 'i'.

$$EI_{yy} = \sum_{i=1}^{n} E_i I_{yy_i}$$ **Equation 5.3-17**

Where, I_{yy_i} is the area moment of inertia about the 'y' axis for segment 'i'.
E_i is the Young's modulus for segment 'i'.

The area product of inertia provides a measure of bend-coupling behavior present for a beam cross-section. The product of inertia is dependent on the shape of the cross section as well as the location and orientation of the cross-section relative to the coordinate system axis. The product of inertia may be positive, negative, or zero. Equation 5.3-18 gives the general formula for the modulus weighted product of inertia calculated about the “yz-axis”.

$$EI_{yz} = \int E\,yz\,da$$ **Equation 5.3-18**

The modulus weighted area product of inertia for a beam cross-section containing “n” rectangular branch segments may be calculated using Equation 5.3-19

$$EI_{yz} = \sum_{i=1}^{n} E_i\, y_i z_i A_i$$ **Equation 5.3-19**

Where, y_i is the ‘y’ coordinate of the centroid for segment ‘i’.
z_i is the ‘z’ coordinate of the centroid for segment ‘i’.
A_i is area for segment ‘i’.
E_i is the Young’s modulus for segment ‘i’.

5.3.1.6 Modulus Weighted Torsional Stiffness

The modulus weighted torsional stiffness for a cross-section is designated *GJ*. For a thin walled open cross-section, GJ may be calculated using Equation 5.3-20.

$$GJ = \frac{1}{3}\sum_{i=1}^{n} G_i\, b_i t_i^3$$ **Equation 5.3-20**

Where:
G_i is the laminated composite shear modulus for segment i.
b_i is the length of segment i
t_i is the thickness of segment i

5.3.1.7 Translation and Rotation of Segment Properties

Area moments of inertia calculated about a coordinate axis located at the section centroid of a segment may be transferred to an arbitrary set of coordinate axis using simple formulas for translation and rotation. These segment property transformation procedures are a key to using the modulus weighted area section property method. Equation 5.3-16 for EI_{zz} and Equation 5.3-17 for EI_{yy} require that segment properties be in the y-z coordinate system prior to summation. Rotation of the segment cross-section properties to a parallel coordinate system is performed first (if required), followed by translation to the y-z coordinate system of interest. The methodology for rotation and translation is explained in the following sub-sections.

5.3.1.7.1 Rotation of Segment Properties

Figure 5.3-6 shows a rectangular cross-section segment with a (y^*, z^*) principal coordinate system located at the segment centroid. The angle of rotation[1] between the (y^*, z^*) coordinate system and the (y', z') coordinate system is designated $\boldsymbol{\alpha}$. The sign of α is determined by the right hand rule.

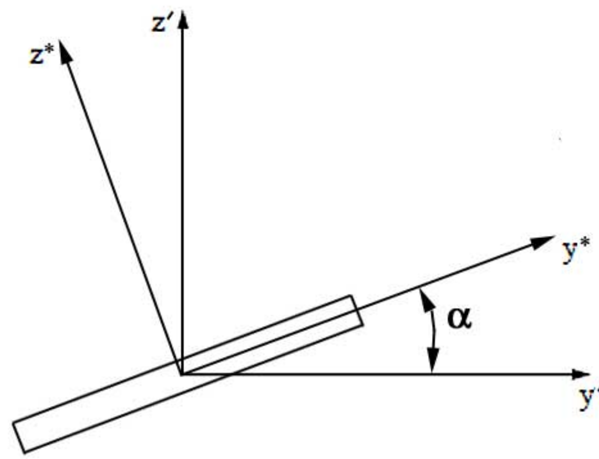

Figure 5.3-6 Rotation of Segment Axis

Area moments of inertia calculated in the (y^*, z^*) coordinate system may be transformed to the (y', z') coordinate system using the matrix Equation 5.3-21, where it is assumed that both coordinate system origins are located at the shape centroid.

$$\begin{Bmatrix} I_{y'y'} \\ I_{z'z'} \\ I_{y'z'} \end{Bmatrix} = \begin{bmatrix} \cos^2(\alpha) & \sin^2(\alpha) & 2\sin(\alpha)\cos(\alpha) \\ \sin^2(\alpha) & \cos^2(\alpha) & -2\sin(\alpha)\cos(\alpha) \\ -\dfrac{\sin(2\alpha)}{2} & \dfrac{\sin(2\alpha)}{2} & \cos(2\alpha) \end{bmatrix} \begin{Bmatrix} I_{y^*y^*} \\ I_{z^*z^*} \\ I_{y^*z^*} \end{Bmatrix} \qquad \textbf{Equation 5.3-21}$$

5.3.1.7.2 Translation of Segment Properties

Translation of axis for area moment of inertia is accomplished using the parallel axis theorem. The parallel axis theorem may be used to transform area moments of inertia from a (y', z') centroidal coordinate system to a parallel (y, z) coordinate system. Figure 5.3-7 shows a cross-section segment with a (y', z') coordinate system axis located at the segment centroid. The segment centroid and the (y', z') coordinate system origin are located at coordinates (d_y, d_z) in the (y, z) coordinate system.

[1] Positive rotation determined by the right hand rule

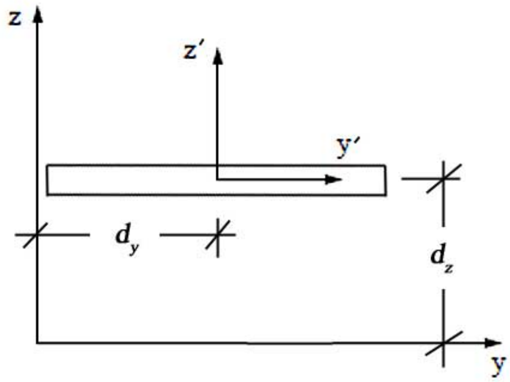

Figure 5.3-7 Translation of Segment

Area moments of inertia calculated in the (y', z') coordinate system are transformed to the parallel (y, z) coordinate system using Equation 5.3-22 and Equation 5.3-23.

$$I_{yy} = I_{y'y'} + A d_z^2$$ **Equation 5.3-22**

Where, I_{yy} is the second moment of area with respect to the *y*-axis

$I_{y'y'}$ is the second moment of area with respect to an axis parallel to the *y*-axis and passing through the centroid of the shape.

A is area of the shape

d_z is the distance between the *y*-axis and the segment centroidal axis

$$I_{zz} = I_{z'z'} + A d_y^2$$ **Equation 5.3-23**

Where, I_{zz} is the second moment of area with respect to the *z*-axis

$I_{z'z'}$ is the second moment of area with respect to an axis parallel to the *z*-axis and passing through the centroid of the shape.

A is area of the shape

d_y is the distance between the *z*-axis and the segment centroidal axis

The product of inertia calculated in the (y', z') coordinate system is transformed to the parallel (y, z) coordinate system using Equation 5.3-24.

$$I_{yz} = I_{y'z'} + A d_y d_z$$ **Equation 5.3-24**

Where, I_{yz} is the product of inertia with respect to the *z*-axis

$I_{y'z'}$ is the product of inertia with respect to an axis parallel to the *z*-axis and passing through the centroid of the shape.

A is area of the shape

d_y is the distance between the *z*-axis and the segment centroidal axis

d_z is the distance between the *y*-axis and the segment centroidal axis

5.3.1.8 Modulus Weighted Cross-Section Property Methodology

The following procedure may be used to calculate laminated beam cross-section properties using the modulus weighted methodology:

1) Establish a reference y-z coordinate system for cross-section property calculation.
2) Divide the beam cross-section into straight line rectangular segments as illustrated in Figure 5.3-1
   a. Material properties and laminate thickness must be constant for each rectangular segment
   b. Curved segments of the beam cross-section are approximated using multiple straight rectangular elements
   c. Number the segments from 1 to n
3) Determine and tabulate the axial stiffness modulus, shear stiffness modulus, dimensions, orientation, and centroidal coordinates for each rectangular segment
   a. The required laminate stiffness moduli (E_x, and G_{xy}) for each segment may be estimated using one of the techniques presented in section 5.3.1.1
   b. Dimensions are typically designated "*b*" for width, and "*h*" for height as shown in Figure 5.3-10.
   c. Refer to Figure 5.3-6 for definition of the segment orientation angle α
4) Calculate a modulus weighting factor to be applied to the properties of each segment
   a. Establish a base modulus E_b (usually set to 1.0×10^6)
   b. The modulus weighted factor $f_{xi} = E_{xi} / E_b$
   c. Note that the modulus weighting factor is dependent on segment axial stiffness and may be different for each segment of the cross-section.
5) Calculate the factor weighted principal centroidal cross-section properties of each rectangular segment using Equation 5.3-25.

$$f_{xi} A_i = f_{xi}\, b_i\, h_i$$
$$f_{xi} I_{y'y'\,i} = f_{xi} \frac{b_i\, h_i^3}{12}$$
$$f_{xi} I_{z'z'\,i} = f_{xi} \frac{h_i\, b_i^3}{12}$$
$$f_{xi} I_{y'z'\,i} = 0$$

Equation 5.3-25

6) Transform the factor weighted segment properties to the y-z coordinate axis using the rotation and translation methodology described in section 5.3.1.7
7) Combine the transformed factor weighted properties for each rectangular segment to obtain properties for the entire cross-section.

   a. Segment properties are combined using the summation equations given in sections 5.3.1.2 and 5.3.1.5
   b. Multiply each cross-section property by the base modulus E_b
   c. Combined cross-section properties are calculated with respect to the y-z coordinate system.
8) Calculate the modulus weighted cross-section centroid using Equation 5.3-8 and Equation 5.3-9 .
9) Transform the beam cross-section properties to the centroidal coordinate system using the methodology described in section 5.3.1.7.
   a. Note that the centroidal axis is denoted by the prime coordinate system in Equation 5.3-22 and Equation 5.3-23
10) Calculate the Modulus Weighted Torsional Stiffness *GJ* using the methodology presented in section 5.3.1.6.

5.3.1.9 Example Modulus Weighted Cross-Section Property Calculations

Practical application of the modulus weighted cross-section property methodology will be demonstrated by calculating properties for the example laminated beam cross-section shown in Figure 5.3-8. The y-z coordinate system for cross-section property calculation is established and segment coordinates are referenced to the mid-plane contour line of each segment laminate. The cross-section has been divided into four straight segments using the cross-section junctions[1] as natural division points and the segments have been numbered one through four. For future reference this cross-section will be designated “Cross-Section A”.

[1] See cross-section terminology in section 5.2.2

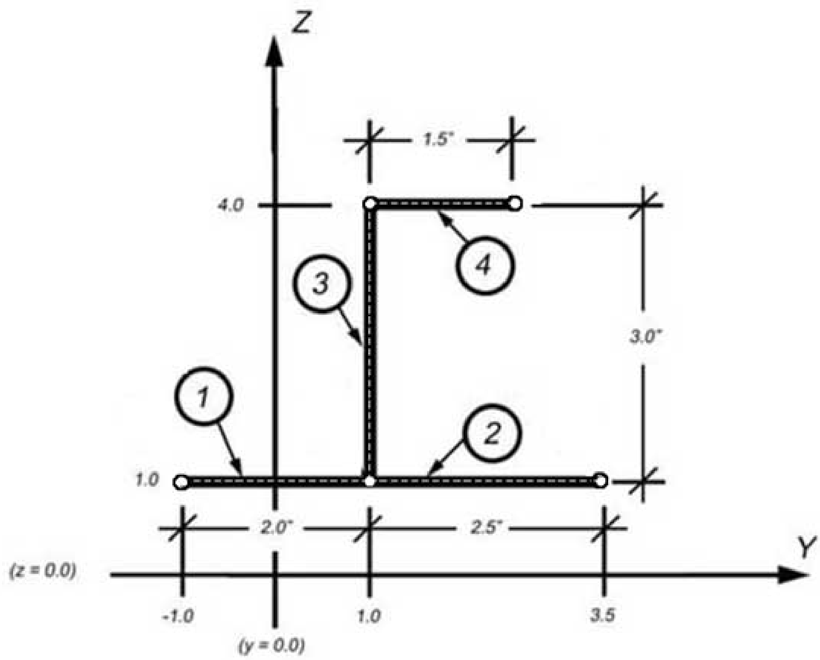

Figure 5.3-8 Example Cross-Section A

5.3.1.9.1 Calculation of Laminate Axial Stiffness Modulus

The following laminate properties are specified for segments 1, 2, and 4 of the beam cross-section:

Lamina material: B_IM7/977 Tape at 75°F, Thickness = .0053inches/ply

The laminate stacking sequence is specified by ply percentages and total laminate thickness as follows:

Laminate ply percentages = (50/40/10)
Total laminate thickness = 20 plies × .0053 inches/ply = .106”

The IDAT/MATUTL stiffness carpet plot for IM7/977-3 material at 75º F is shown in Figure 5.3-9.

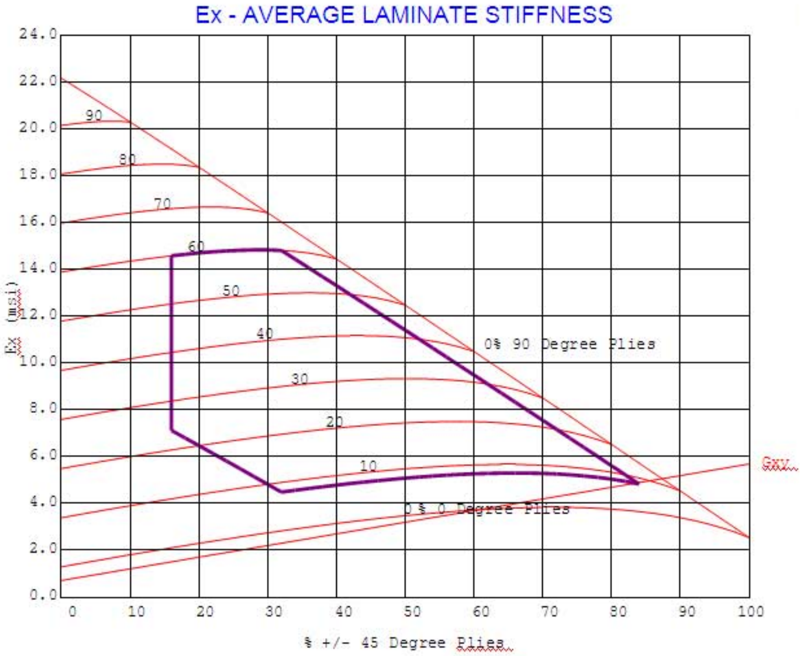

Figure 5.3-9 IDAT MATUTL Stiffness Carpet Plot for IM7/977-3

Using Figure 5.3-9, the axial modulus for segments 1, 2, and 4 is estimated to be approximately 13.0×10^6 Psi, and the shear stiffness modulus is estimated to be approximately 2.7×10^6 Psi[1]

The following laminate properties are specified for segment 3 of the beam cross-section:

Lamina material: B_IM7/977 Tape at 75°F, Thickness = .0053inches/ply

The laminate stacking sequence specified by ply percentages and total laminate thickness as follows:

Laminate ply percentages = (25/50/25)

Total laminate thickness = 16 plies × .0053 inches/ply = .0848”

Using Figure 5.3-9, the axial modulus for segment 3 is estimated to be approximately 8.4×10^6 Psi, and the shear stiffness modulus is estimated to be approximately 3.2×10^6 Psi

5.3.1.9.2 Tabulation of Cross-Section Data

The axial stiffness moduli and modulus weighted stiffness factors “f_i” for each segment “i” are listed in Table 5.3-1. A base modulus $E_b = 1.0\times10^6$ Psi has been established and is used to calculate the stiffness factors (f_{xi}). Segment dimensions and centroidal coordinates for each rectangular cross-section segment are also listed in Table 5.3-1. Because the cross-section segments are all oriented at either 0, or 90 degrees with respect to the y-z coordinate system, no rotation of segment properties is required. The segment orientation angles are not listed. Data required for segment “n” is illustrated in Figure 5.3-10.

[1] Enter the x-axis at the percent ±45º plies and proceed vertically. For E_x, intersect the %0º plies curve for the laminate and proceed left horizontally to read the axial laminate stiffness value off the y-axis. For G_{xy}, intersect the G_{xy} curve and proceed left horizontally to read the shear stiffness off the y-axis.

Table 5.3-1 Tabulation of Cross-Section Data

Segment	E_{xi}	$f_{xi} = E_{xi}/E_b$	Y_i	Z_i	$\Delta y = b_i$	$\Delta z = h_i$
1	$1.3000{\times}10^7$	13.0	0.000	1.000	2.0000	0.1060
2	$1.3000{\times}10^7$	13.0	2.250	1.000	2.5000	0.1060
3	$8.4000{\times}10^6$	8.4	1.000	2.500	0.0848	3.0000
4	$1.3000{\times}10^7$	13.0	1.750	4.000	1.5000	0.1060

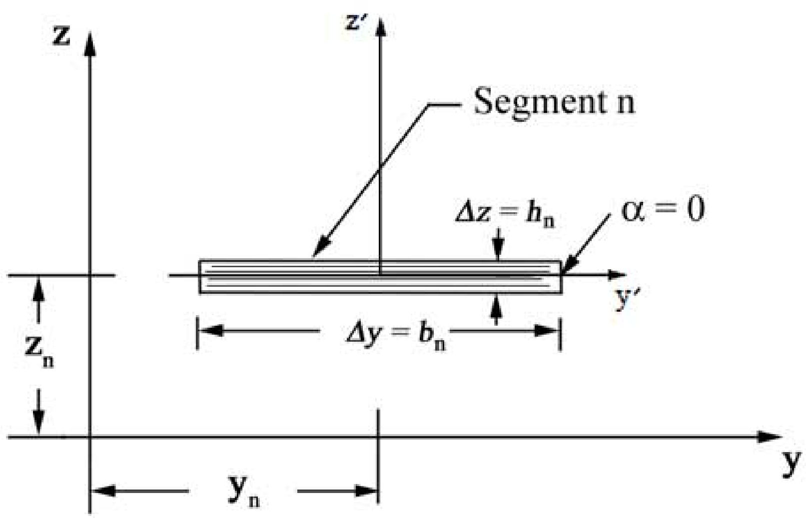

Figure 5.3-10 Data for Rectangular Segment

5.3.1.9.3 Calculate the Factor Weighted Centroidal Cross-Section Properties for Each Segment

The factor weighted principal centroidal y'-z' cross-section properties for each segment are calculated by substituting the data collected in Table 5.3-1 into Equation 5.3-25. Results of these segment cross-section property calculations are shown in Table 5.3-2.

Table 5.3-2 Factor Weighted Centroidal Properties for each Segment

Segment	f_{xi} A_i	f_{xi} $I_{y'y'i}$	f_{xi} $I_{z'z'i}$	f_{xi} $I_{y'z'i}$
1	2.756	0.0026	0.9187	0.000
2	3.445	0.0032	1.7943	0.000
3	2.137	1.6027	0.0013	0.000
4	2.067	0.0019	0.3876	0.000

Where:

$$f_{xi} A_i = f_{xi}\ b_i\ h_i$$

$$f_{xi} I_{y'y'\,i} = f_{xi} \frac{b_i\ h_i^3}{12}$$

$$f_{xi} I_{z'z'\,i} = f_{xi} \frac{h_i\ b_i^3}{12}$$

$$f_{xi} I_{y'z'\,i} = 0$$

5.3.1.9.4 Transform the Factor Weighted Centroidal Cross-Section Properties to the y-z Coordinate System

The rotation and translation methodologies presented in section 5.3.1.7 are applied to transform the y'-z' factor weighted principal centroidal segment properties to the y-z coordinate system. Only translation is required for the segments in this example.

Table 5.3-3 Transformed Factor Weighted Properties for each Segment

Segment	f_i A_i	f_i I_{yyi}	f_i I_{zzi}	f_i I_{yzi}
1	2.756	2.7586	0.9187	0.000
2	3.445	3.4482	19.2346	7.751
3	2.137	14.9587	2.1382	5.342
4	2.067	33.0739	6.7178	14.469

Where:

$$f_i I_{yyi} = f_i I_{y'y'i} + f_i A_i \cdot z_i^2$$

$$f_i I_{zzi} = f_i I_{z'z'i} + f_i A_i \cdot y_i^2$$

$$f_i I_{yzi} = f_i I_{y'z'i} + f_i A_i \cdot y_i \cdot z_i$$

5.3.1.9.5 Combine the Factor Weighted y-z Cross-Section Properties

After transformation, combine the segment properties using Equation 5.3-5, Equation 5.3-16, Equation 5.3-17, and Equation 5.3-19 to obtain factor weighted cross-section properties with respect to the y-z coordinate system. The following equations are a summation of the column data in Table 5.3-3

$$\sum_{i=1}^{4} f_{xi} A_i = 10.405$$

$$\sum_{i=1}^{4} f_{xi} I_{yy\,i} = 54.2395$$

$$\sum_{i=1}^{4} f_{xi} I_{zz\,i} = 29.0092$$

$$\sum_{i=1}^{4} f_{xi} I_{yz\,i} = 27.5627$$

Multiply by the base modulus E_b to obtain modulus weighted section properties for the laminated beam cross-section.

$$EA = E_b \sum_{i=1}^{4} f_{xi} A_i = 10.405 \times 10^6 \ lbs$$

$$EI_{yy} = E_b \sum_{i=1}^{4} f_{xi} I_{yy\,i} = 54.2395 \times 10^6 \ in^2 lbs$$

$$EI_{zz} = E_b \sum_{i=1}^{4} f_{xi} I_{zz\,i} = 29.0092 \times 10^6 \ in^2 lbs$$

$$EI_{yz} = E_b \sum_{i=1}^{4} f_{xi} I_{yz\,i} = 27.5627 \times 10^6 \ in^2 lbs$$

5.3.1.9.6 Calculate the Modulus Weighted Cross-Section Centroid

The modulus weighted cross-section centroid is calculated using Equation 5.3-8 and Equation 5.3-9. Section data needed for the centroid calculation is collected in Table 5.3-4.

Table 5.3-4 Modulus Weighted Centroid Data

Segment	$f_{xi}A_i$	Y_i	Z_i	$f_{xi}A_i$ Y	$f_{xi}A_i$ Z
1	2.756	0.000	1.000	0.000	2.756
2	3.445	2.250	1.000	7.751	3.445
3	2.137	1.000	2.500	2.137	5.342
4	2.067	1.750	4.000	3.617	8.268
$\Sigma =$	10.4050			13.5055	19.8114

$$\bar{y} = \frac{\sum_{i=1}^{4} f_{xi} A_i y_i}{\sum_{i=1}^{4} f_{xi} A_i} = \frac{13.5055}{10.4050} = 1.2980 \text{ in}$$

$$\bar{z} = \frac{\sum_{i=1}^{4} f_{xi} A_i z_i}{\sum_{i=1}^{4} f_{xi} A_i} = \frac{19.8114}{10.4050} = 1.9040 \text{ in}$$

5.3.1.9.7 Transform Modulus Weighted Cross-Section Properties to the Section Centroid

The transformation equations presented in section 5.3.1.7 may be used to transform area moment of inertia cross-section properties to the cross-section centroid. Note that within the parallel axes translation Equation 5.3-22 and Equation 5.3-23, the centroidal coordinate axes are represented by the prime symbol. Translation from the y-z coordinate axes to the centroidal coordinate axes is performed using Equation 5.3-22 and Equation 5.3-23. as follows:

$$EI_{yy} = EI_{y'y'} + EA \cdot z^2$$
$$54.2395\times10^6 = EI_{y'y'} + 10.405\times10^6 \times 1.9040^2$$
$$EI_{y'y'} = 1.6518\times10^7 \ in^2 lbs$$

$$EI_{zz} = EI_{z'z'} + EA \cdot y^2$$
$$29.0092\times10^6 = EI_{z'z'} + 10.405\times10^6 \times 1.2980^2$$
$$EI_{z'z'} = 1.1479\times10^7 \ in^2 lbs$$

$$EI_{yz} = EI_{y'z'} + EA \cdot y \cdot z$$
$$27.5627\times10^6 = EI_{y'z'} + 10.405\times10^6 \times 1.2980 \times 1.9040$$
$$EI_{y'z'} = 1.8478\times10^6 \ in^2 lbs$$

5.3.1.9.8 Calculate the Modulus Weighted Torsional Stiffness

Data needed to calculate the modulus weighted torsional stiffness is collected in Table 5.3-5.

Table 5.3-5 Modulus Weighted Torsional Stiffness Data

Segment	G_i	b_i	t_i	A_i	G_iA_i	$1/3G_ib_it_i^3$
1	2700000	2.00	0.1060	0.2120	572400.0	2143.829
2	2700000	2.50	0.1060	0.2650	715500.0	2679.786
3	3200000	3.00	0.0848	0.2544	814080.0	1951.361
4	2700000	1.50	0.1060	0.1590	429300.0	1607.872
			$\Sigma =$	0.8904	2531280.0	8382.847

The modulus weighted torsional stiffness modulus is calculated using Equation 5.3-20, which is represented by the last column in Table 5.3-5

$$GJ = \frac{1}{3}\sum_{i=1}^{n} G_i b_i t_i^3 = 8382.8\ in^2 lbs$$

$$G = \frac{\sum_{i=1}^{n} G_i A_i}{\sum_{i=1}^{n} A_i} = \frac{2531280}{.8904} = 2842857.1\ lbs/in^2$$

$$\mathrm{J} = \frac{\mathrm{GJ}}{\mathrm{G}} = \frac{8382.847}{2842857.143} = 0.00294874\ \mathrm{in}^4$$

5.3.2 FZM-9954 Branch-Integral Cross-Section Property Method

FZM-9954 documents an advanced theory and methodology for calculating the cross-section properties of thin-walled laminated composite beams. The beam cross-section is idealized as a set of straight laminated branched shell segments. Fifteen definite integrals are defined and evaluated for each branch of the cross-section. The evaluated branch integral terms are inserted into summation equations which evaluate to fifteen unique laminated composite beam cross-section properties. The methodology includes formulas for calculating warping cross-section properties. Laminate stacking sequences and Classical Laminated Plate Theory are used to increase accuracy and account for the laminate coupling behavior present in unbalanced and unsymmetric laminates.

5.3.2.1 Hooke's Law

The Branch-Integral Cross-Section Property methodology uses the laminate [ABD] stiffness matrix to calculate a reduced shell stiffness matrix for each branch laminate. The reduced form of the Hooke's law stiffness matrix contains only the terms that relate active shell resultant forces in each cross-section branch to the active mid-surface strains and curvatures in each branch segment. The reduced form of the laminate stiffness matrix for a cross-section branch is given by Equation 5.3-26.

$$\begin{bmatrix} a_{11} & b_{11} & b_{16} \\ b_{11} & d_{11} & d_{16} \\ b_{16} & d_{16} & d_{66} \end{bmatrix} =$$

$$\begin{bmatrix} A_{11} & B_{11} & B_{16} \\ B_{11} & D_{11} & D_{16} \\ B_{16} & D_{16} & D_{66} \end{bmatrix} - (\delta) \begin{bmatrix} A_{12} & A_{16} & B_{12} \\ B_{12} & B_{16} & D_{12} \\ B_{26} & B_{66} & D_{26} \end{bmatrix} \begin{bmatrix} A_{22} & A_{26} & B_{22} \\ A_{26} & A_{66} & B_{26} \\ B_{22} & B_{26} & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} A_{12} & B_{12} & B_{26} \\ A_{16} & B_{16} & B_{66} \\ B_{12} & D_{12} & D_{26} \end{bmatrix}$$

Equation 5.3-26

The dimensionless parameter δ has a value between zero and one. For $\delta = 0$, Hooke's law is based on reactive strains vanishing. Setting the reactive strains equal to zero is consistent with the deformation assumptions made in development of the FZM-9954 beam theory. For $\delta = 1$, Hooke's law is based on the reactive resultants vanishing. Setting the reactive resultants equal to zero is the approach adopted in Bernoulli-Euler beam theory. Cross-section properties are typically calculated using $\delta = 0$.

5.3.2.2 Branch Segment Contour Coordinate System and Angle

Branch contour coordinates (ζ, s, x) are defined relative to the beam (y, z, x) coordinate system and are used primarily in development of the laminated composite beam cross-section property equations for the FZM-9954 method. The contour coordinate system x-axis is oriented in the same direction as the beam coordinate system x-axis. The contour coordinate system s-axis is tangent to the branch contour line. The orientation of the s-axis within the beam y-z plane is specified by the contour angle α, with positive rotation determined by the right hand rule. Equations presented in this section are limited to straight branch segments; therefore α is constant for each segment. The local branch thickness coordinate is designated "ζ", and its orientation with respect to "s" is determined by the right hand rule. The orientation of the laminate stacking sequence is determined by the direction of the thickness coordinate ζ. The branch segment coordinate system and orientation angle α, is illustrated in Figure 5.3-11.

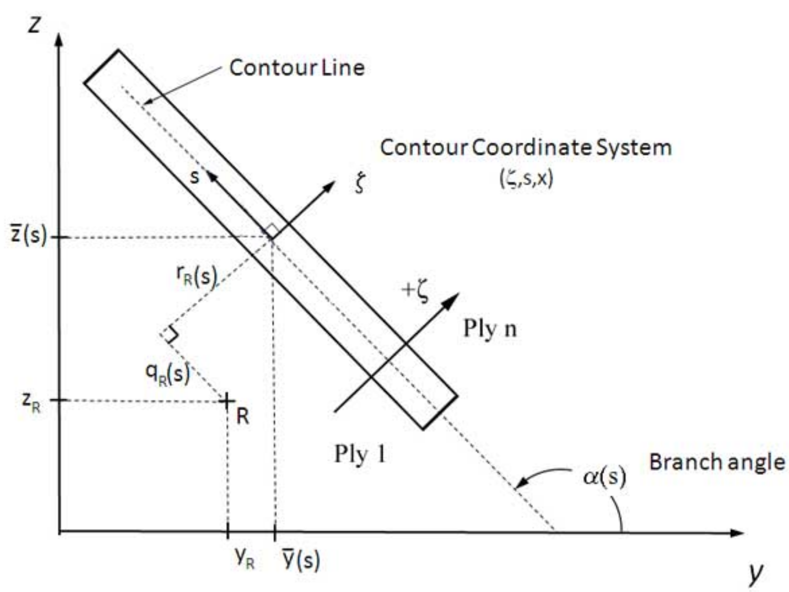

Figure 5.3-11 Branch Segment Angle α

Figure 5.3-11 introduces and illustrates the coordinate terms $\bar{y},\ and\ \bar{z}$, which appear in the FZM-9954 cross-section property equations presented in section 5.3.2. For the FZM-9954 method, $\bar{y}(s)\ and\ \bar{z}(s)$ designate the (y, z) coordinates of a point on the section contour at s. These coordinates satisfy the differential relations given by Equation 5.3-27.

$$\frac{d\bar{y}}{ds} = \cos(\alpha(s)), \quad \frac{d\bar{z}}{ds} = \sin(\alpha(s))$$

Equation 5.3-27

A material point at shell coordinates (ζ, s) in the wall of a cross-sectional element can now be described in terms of (y, z) coordinates using $\bar{y},\ \bar{z},\ \alpha,\ and\ \zeta$ as given by Equation 5.3-28

$$\begin{aligned} y(\zeta, s) &= \bar{y}(s) + \zeta\sin(\alpha(s)) \\ z(\zeta, s) &= \bar{z}(s) - \zeta\cos(\alpha(s)) \end{aligned}$$

Equation 5.3-28

Contour coordinates r_R, and q_R, and the reference point R, are also introduced in Figure 5.3-11. r_R, and q_R are coordinates in the ζ and s directions respectively of a point on the section contour relative to the reference point R, where R is an arbitrary location in the plane of the cross-section about which all displacements and rotations are referenced. The location of R in the (y, z, x) coordinate system is designated (y_R, z_R). For the FZM-9954 method, the reference point R is in effect the coordinate origin for cross-section property calculations. The direction of r_R is determined by the positive direction of ζ, and the positive direction of q_R is defined by the positive direction of s.

$$\begin{aligned} r_R(s) + \zeta &= (y - y_R)\sin(\alpha(s)) - (z - z_R)\cos(\alpha(s)) \\ q_R(s) &= (y - y_R)\cos(\alpha(s)) + (z - z_R)\sin(\alpha(s)) \end{aligned}$$

Equation 5.3-29

The contour coordinates and associated coordinate relations given by Equation 5.3-27 through Equation 5.3-29 are required for development of cross-section properties for laminated composite curved beams using the FZM-9954 methodology.

5.3.2.3 Requirements on Cross-Section Branches

The following restrictions to branch segments of a cross-section are applied to simplify evaluation of the contour integral equations for cross-section properties;

1. Cross-section branch segments are required to be straight (α = const along a branch segment)
2. Material properties, stacking sequence, and laminate thickness are required to be constant along each cross-section branch.

The specified requirements have minimal effects on accuracy and applicability of the calculated results. Curved section branches can be adequately represented by a series of connected straight branch segments. Changes in material properties, stacking sequence and shell thickness can be approximated using discrete changes from branch to branch. These approximations can be refined by adding additional junctions and dividing branches until the desired accuracy is achieved. In practice these approximate measures are generally not required since most practical cross-sections consist of straight branches and material or thickness changes along a branch are normally discrete.

5.3.2.4 Laminated Composite Beam Stiffness Equation

The laminated composite beam stiffness is given by Equation 5.3-30. The beam stiffness equation expresses beam resultant forces and moments in terms of a beam stiffness matrix multiplied by the strains and curvatures of a beam reference x-axis. Each unique element of the symmetric beam stiffness matrix represents a modulus weighted cross-section property of the laminated composite beam.

$$\begin{Bmatrix} N_{XR} \\ M_{YR} \\ M_{ZR} \\ M_{\omega R} \\ T_{XR} \end{Bmatrix} = \begin{bmatrix} EA & ES_y & -ES_z & -ES_\omega & EH \\ ES_y & EI_{yy} & -EI_{zy} & -EI_{\omega y} & EH_c \\ -ES_z & -EI_{zy} & EI_{zz} & EI_{\omega z} & -EH_s \\ -ES_\omega & -EI_{\omega y} & EI_{\omega z} & EI_{\omega\omega} & -EH_q \\ EH & EH_c & -EH_s & -EH_q & GJ \end{bmatrix} \begin{Bmatrix} e_R \\ \kappa_{yR} \\ \kappa_{zR} \\ \tau' \\ \tau_R \end{Bmatrix} \qquad \textbf{Equation 5.3-30}$$

The cross-section properties presented in Equation 5.3-30 are commonly referred to using the nomenclature given in Table 5.3-6

Table 5.3-6 Modulus Weighted Cross-Section Property Nomenclature

EA	Modulus Weighted Area
ES_y	Modulus Weighted First Moment of Area about Y
ES_z	Modulus Weighted First Moment of Area about Z
EI_{yy}	Modulus Weighted Moment of Inertia about Y
EI_{zz}	Modulus Weighted Moment of Inertia about Z
EI_{zy}	Modulus Weighted Product of Inertia
GJ	Modulus Weighted Torsional Stiffness
ES_ω	Modulus Weighted First Sectorial Moment of Area
$EI_{\omega y}$	Modulus Weighted Second Sectorial Moment of Area about Y
$EI_{\omega z}$	Modulus Weighted Second Sectorial Moment of Area about Z
$EI_{\omega\omega}$	Modulus Weighted Warping Coefficient
EH	Modulus Weighted Axial-Torsional Coupling Stiffness
EH_c	Modulus Weighted Bending-Torsional Coupling Stiffness about Y
EH_s	Modulus Weighted Bending-Torsional Coupling Stiffness about Z
EH_q	Modulus Weighted Warping-Torsional Coupling Stiffness

5.3.2.5 Cross-Section Property Equations for Laminated Composite Beams with straight branch segments

The modulus weighted cross-section properties in Equation 5.3-30 are expressed as a summation of values calculated for each branch segment. Section properties are calculated using Equation 5.3-31 through Equation 5.3-45.

$$EA = \sum_{i=1}^{n} a_{11i} I_{1i}$$ **Equation 5.3-31**

$$ES_y = \sum_{i=1}^{n} a_{11i} I_{2i} - b_{11i} \cos(\alpha_i) I_{1i}$$ **Equation 5.3-32**

$$ES_z = \sum_{i=1}^{n} a_{11i} I_{3i} - b_{11i} \sin(\alpha_i) I_{1i}$$ **Equation 5.3-33**

$$ES_\omega = \sum_{i=1}^{n} a_{11i} I_{4i} - b_{11i} I_{5i}$$ **Equation 5.3-34**

$$EH = 2\sum_{i=1}^{n} b_{16i} I_{1i}$$ **Equation 5.3-35**

$$EI_{yy} = \sum_{i=1}^{n} a_{11i} I_{6i} - 2b_{11i} \cos(\alpha_i) I_{2i} + d_{11i} \cos^2(\alpha_i) I_{1i}$$ **Equation 5.3-36**

$$EI_{zy} = \sum_{i=1}^{n} a_{11i} I_{7i} - b_{11i} \sin(\alpha_i) I_{2i} - b_{11i} \cos(\alpha_i) I_{3i} - d_{11i} \sin(\alpha_i)\cos(\alpha_i) I_{1i}$$ **Equation 5.3-37**

$$EI_{\omega y} = \sum_{i=1}^{n} a_{11i} I_{8i} + b_{11i} I_{9i} - b_{11i} \cos(\alpha_i) I_{4i} - d_{11i} \cos(\alpha_i) I_{5i}$$ **Equation 5.3-38**

$$EH_c = 2\sum_{i=1}^{n} b_{16i} I_{2i} - d_{16i} \cos(\alpha_i) I_{1i}$$ **Equation 5.3-39**

$$EI_{zz} = \sum_{i=1}^{n} a_{11i} I_{10i} + 2\, b_{11i} \sin(\alpha_i) I_{3i} + d_{11i} \sin^2(\alpha_i) I_{1i}$$ **Equation 5.3-40**

$$EI_{\omega z} = \sum_{i=1}^{n} a_{11i} I_{11i} + b_{11i} I_{12i} + b_{11i} \sin(\alpha_i) I_{4i} - d_{11i} \sin(\alpha_i) I_{5i}$$ **Equation 5.3-41**

$$EH_s = 2\sum_{i=1}^{n} b_{16i} I_{3i} + d_{16i}\sin(\alpha_i) I_{1i}$$ **Equation 5.3-42**

$$EI_{\omega\omega} = \sum_{i=1}^{n} a_{11i} I_{13i} + 2b_{11i} I_{14i} + d_{11i} I_{15i}$$ **Equation 5.3-43**

$$EH_q = 2\sum_{i=1}^{n} b_{16i} I_{4i} + d_{16i} I_{5i}$$ **Equation 5.3-44**

$$GJ = 4\sum_{i=1}^{n} d_{66i} I_{1i}$$ **Equation 5.3-45**

Where a_{ij}, b_{ij}, and d_{ij} are the Hooke's law stiffness terms given by Equation 5.3-26, and I_{1i} through I_{15i} represent fifteen geometric branch integrals which are defined by Equation 5.3-46 through Equation 5.3-60. The coordinate terms $\bar{y}, \bar{z}, y_R,$ and z_R are defined in section 5.3.2.2. $\overline{\omega}$, and $\tilde{\omega}$ are defined in section 5.3.2.7.

$$I_{1i} = \int_0^{l_{si}} ds$$ **Equation 5.3-46**

$$I_{2i} = \int_0^{l_{si}} (\bar{z} - z_R)\, ds$$ **Equation 5.3-47**

$$I_{3i} = \int_0^{l_{si}} (\bar{y} - y_R)\, ds$$ **Equation 5.3-48**

$$I_{4i} = \int_0^{l_{si}} \overline{\omega}(s)\, ds$$ **Equation 5.3-49**

$$I_{5i} = \int_0^{l_{si}} \tilde{\omega}(s)\, ds$$ **Equation 5.3-50**

$$I_{6i} = \int_0^{l_{si}} (\bar{z} - z_R)^2\, ds$$ **Equation 5.3-51**

$$I_{7i} = \int_0^{l_{si}} (\bar{z} - z_R)(\bar{y} - y_R)\, ds$$ **Equation 5.3-52**

$$I_{8i} = \int_0^{l_{si}} (\bar{z} - z_R)\,\overline{\omega}(s)\, ds$$ **Equation 5.3-53**

$$I_{9i} = \int_0^{l_{si}} (\bar{z} - z_R)\,\tilde{\omega}(s)\, ds$$ **Equation 5.3-54**

$$I_{10i} = \int_0^{l_{si}} (\bar{y} - y_R)^2\, ds$$ **Equation 5.3-55**

$$I_{11i} = \int_0^{l_{si}} (\bar{y} - y_R)\,\overline{\omega}(s)\, ds$$ **Equation 5.3-56**

$$I_{12i} = \int_0^{l_{si}} (\bar{y} - y_R)\,\tilde{\omega}(s)\, ds$$ **Equation 5.3-57**

$$I_{13i} = \int_0^{l_{si}} (\overline{\omega}(s))^2\, ds$$ **Equation 5.3-58**

$$I_{14i} = \int_0^{l_{si}} \overline{\omega}(s)\,\tilde{\omega}(s)\, ds$$ **Equation 5.3-59**

$$I_{15i} = \int_0^{l_{si}} (\tilde{\omega}(s))^2\, ds$$ **Equation 5.3-60**

The modulus weighted cross-section properties in Equation 5.3-30 can be grouped into three categories:1.) the conventional cross-section properties EA, ES_y, ES_z, EI_{yy}, EI_{zz}, EI_{zy}, and GJ, 2.) the warping properties ES_ω, $EI_{\omega y}$, $EI_{\omega z}$, and $EI_{\omega\omega}$, and 3.) the coupling stiffness properties EH, EH_c, EH_s, and EH_q. Note that an examination of the equations will show that the "conventional property" equations contain a contribution from the laminate b_{11} stiffness term and are distinct from the equivalent isotropic material form of the cross-section property equations. The warping coupling terms ($EI_{\omega y}$, $EI_{\omega z}$,) are used to compute the location of the shear center, but are not in general useful as stiffness for analysis since commercial FEA programs do not typically support a warping degree of freedom. ES_ω is needed to establish the "Principal Contour Origin[1]", which is required to calculate the correct value for the warping coefficient $EI_{\omega\omega}$. Use of the warping coefficient $EI_{\omega\omega}$ is supported by most FEA programs including NASTRAN. The "EH" terms are unique to laminated composite beams. These terms represent the stiffness contribution from the coupling terms b_{16} and d_{16} in the reduced laminate stiffness matrices (Equation 5.3-26). If the laminate construction for each shell is specially orthotropic[2], then b_{16} and d_{16} are equal to zero and the "EH" terms will all equal zero. The EH terms cannot be entered into NASTRAN using conventional beam properties (PBAR, PBEAM, PBEAML etc.). In cases where the EH terms are significant, the laminated composite beam stiffness matrix can be input directly into NASTRAN using direct matrix input via a DMIG bulk data entry.

Note that the modulus weighted cross-section properties calculated using Equation 5.3-31 through Equation 5.3-45 reduce to the classic isotropic cross-section properties if isotropic materials are used for the cross-section.

[1] See section 5.3.2.7 for a discussion of the Principal Contour Origin.

[2] Laminates which are symmetric (B_{ij} = 0), and for which the bend-twist coupling terms vanish ($D_{16} = D_{26}$=0) are classified as "specially orthotropic".

5.3.2.6 Geometric relations for Straight Branch Segments

Before evaluating the cross-section properties given by summation Equation 5.3-31 through Equation 5.3-45 and the associated branch integral Equation 5.3-46 through Equation 5.3-60 it is helpful to derive some useful relations for straight branch segments. Consider a straight branch segment between coordinates (y_1, z_1) and (y_2, z_2) as shown in Figure 5.3-12.

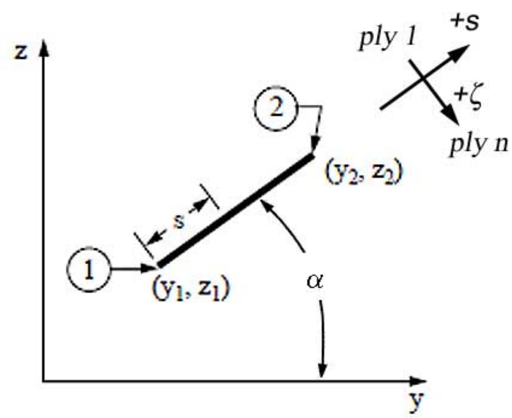

Figure 5.3-12 Straight Branch Segment

Take the branch coordinate "*s*" such that $s = 0$ at (y_1, z_1), and $s = l_s$ at (y_2, z_2). Note that if the stacking sequence for the segment is unsymmetric, the choice for s = 0 is not arbitrary. The orientation of the laminate stack is determined by the direction of positive "ζ". The direction of positive "ζ" is defined as normal to the direction of positive "*s*" using the right hand rule and the coordinate ordering (ζ, *s*, *x*), The direction for positive "*s*" is determined by the choice for end one and end two of the segment. The orientation of "ζ" relative to "*s*" is illustrated for the example segment in Figure 5.3-12. Note also that the branch segment coordinates reference the contour line or mid-plane of the laminate ($\zeta = 0$)

The length of the branch segment l_s can be expressed in terms of the end point coordinates as given by Equation 5.3-61

$$l_s = \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}$$ **Equation 5.3-61**

The transcendental functions sine, and cosine in Equation 5.3-31through Equation 5.3-45 may be expressed in terms of straight branch coordinate end points as given by Equation 5.3-62.

$$\cos(\alpha(s)) = \frac{y_2 - y_1}{l_s} \quad , \sin(\alpha(s)) = \frac{z_2 - z_1}{l_s}$$ **Equation 5.3-62**

Expressions for $\bar{y}$, *and* $\bar{z}$ found in Equation 5.3-46 through Equation 5.3-60 are derived from Equation 5.3-27, combined with Equation 5.3-62, and are given by Equation 5.3-63 and Equation 5.3-64.

$$\bar{y} = s\frac{(y_2 - y_1)}{l_s} + y_1$$ **Equation 5.3-63**

$$\bar{z} = s\frac{(z_2 - z_1)}{l_s} + z_1$$ **Equation 5.3-64**

The contour coordinates r_R, and q_R given by Equation 5.2 3 can also be expressed in a form that will simplify calculation of the warping function of Equation 5.3 5. Substituting Equation 5.3-63 and Equation 5.3-64 into the expressions for $r_R(s)$ and $q_R(s)$ given by Equation 5.3-29 yields the straight branch form for r_R given by Equation 5.3-65 and the straight branch form for $q_R(s)$ given by Equation 5.3-66. Note that r_R is no longer a function of the contour coordinate s.

$$\begin{aligned} r_R &= \frac{(y_1 - y_R)(z_2 - z_1) - (z_1 - z_R)(y_2 - y_1)}{l_s} \\ &let \quad r_f = (y_1 - y_R)(z_2 - z_1) - (z_1 - z_R)(y_2 - y_1) \\ &r_R = \frac{r_f}{l_s} \end{aligned}$$ **Equation 5.3-65**

$$\begin{aligned} q_R(s) &= s + \frac{(y_1 - y_R)(y_2 - y_1) + (z_1 - z_R)(z_2 - z_1)}{l_s} \\ &let \quad q_f = (y_1 - y_R)(y_2 - y_1) + (z_1 - z_R)(z_2 - z_1) \\ &q_R(s) = s + \frac{q_f}{l_s} \end{aligned}$$ **Equation 5.3-66**

Using Equation 5.3-65 and Equation 5.3-66, the warping functions found in Equation 5.3-46 through Equation 5.3-60 can be expressed for straight branch segments as follows in Equation 5.3-67

$$\begin{aligned} \bar{\varpi} &= s\, r_R + C_i \\ \tilde{\varpi} &= -q_R(s) \end{aligned}$$ **Equation 5.3-67**

Where $\bar{\varpi}$ is the contour warping function, $\tilde{\varpi}$ is the thickness warping function, and C_i is the constant of integration.

5.3.2.7 Evaluation of Branch Segment Warping Constants

Warping represents the out-of-plane displacement of the cross-section due to torsional loads. Since the out-of-plane displacement of the cross-section is continuous at the cross-section junctions, the value of the warping function must also be continuous at the cross-section junctions[1]. The constants C_i in the branch segment warping Equation 5.3-67 are used to establish continuity of the warping function at each branch

[1] Note that the derivative of the out-of-plane displacement is not in general continuous at the cross-section junctions.

junction. Each branch segment will in general have a unique constant C_i. The process of determining values for the branch constants C_i required to establish continuity of the warping function is illustrated in Figure 5.3-13 and is demonstrated in section 5.3.2.11.8 of the FZM-9954 example problem.

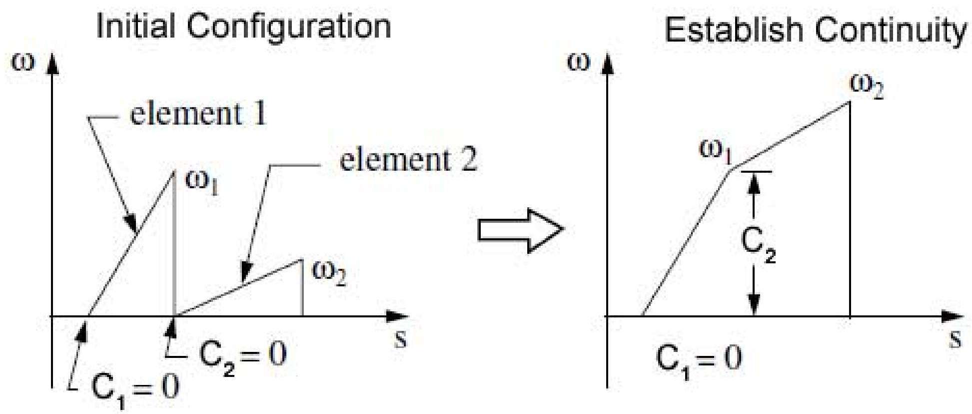

Figure 5.3-13 Warping Function Continuity

The bimoment[1] caused by warping is assumed to be equivalent to zero axial force and zero moment. This assumption is valid if warping properties are calculated at the **Principal Contour Origin**. The **contour origin** is the location (or locations) on the cross-section where the warping function evaluates to zero. The Principal Contour Origin is established by the condition $ES_\omega = 0$, and is considered to be a section property similar to the Centroid, or Shear Center. When $ES_\omega = 0$, axial force is de-coupled from warping strain in Equation 5.3-30 and thus the bimoment does not create axial force. ES_ω is set to zero by adding a common constant to each segment warping constant as illustrated in Figure 5.3-14. The value of the common constant is determined by calculating the ratio $-ES_\omega/EA$ after continuity at the junctions has been established. **Cross-section warping properties must always be calculated using the Principal Contour Origin.**

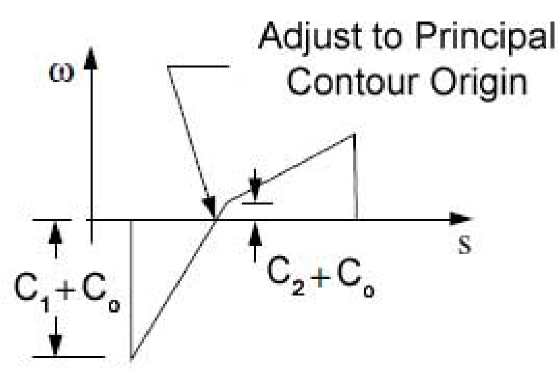

Figure 5.3-14 Principal Contour Origin

Example calculations for the warping function constants are provided as part of an example problem in section 5.3.2.11.8.

[1] See FZM-9954 section 3.1 for an explanation of the warping bimoment.

5.3.2.8 Geometric Integral Equations for Straight Branch Segments

Substituting the straight branch Equation 5.3-61 through Equation 5.3-67 into the branch geometric integral Equation 5.3-46 through Equation 5.3-60 yields a manageable set of branch integral equations for laminated composite beam cross-section properties. The straight branch geometric integral equations with evaluation of the definite integrals are given by Equation 5.3-68 through Equation 5.3-82 that follow.

$$\begin{aligned} I_{1i} &= \int_0^{l_{si}} ds \\ &= l_{si} \end{aligned}$$ **Equation 5.3-68**

$$\begin{aligned} I_{2i} &= \int_0^{l_{si}} (\bar{z} - z_R)\, ds \\ &= \int_0^{l_{si}} ((s\frac{(z_2 - z_1)}{l_{si}} + z_1) - z_R)\, ds \\ &= l_{si}\left(\frac{(z_2 + z_1)}{2} - z_R\right) \end{aligned}$$ **Equation 5.3-69**

$$\begin{aligned} I_{3i} &= \int_0^{l_{si}} (\bar{y} - y_R)\, ds \\ &= \int_0^{l_{si}} ((s\frac{(y_2 - y_1)}{l_{si}} + y_1) - y_R)\, ds \\ &= l_{si}\left(\frac{(y_2 + y_1)}{2} - y_R\right) \end{aligned}$$ **Equation 5.3-70**

$$\begin{aligned} I_{4i} &= \int_0^{l_{si}} \overline{\varpi}(s)\, ds \\ &= \int_0^{l_{si}} s\frac{r_{fi}}{l_{si}} + C_i\, ds \\ &= l_{si}\left(\frac{r_{fi}}{2} + C_i\right) \end{aligned}$$ **Equation 5.3-71**

$$\begin{aligned} I_{5i} &= \int_0^{l_{si}} \tilde{\varpi}(s)\, ds \\ &= \int_0^{l_{si}} -\left(s + \frac{q_{fi}}{l_{si}}\right) ds \\ &= -\frac{l_{si}^2}{2} - q_{fi} \end{aligned}$$ **Equation 5.3-72**

$$I_{6i} = \int_0^{l_{si}} (\bar{z} - z_R)^2 \, ds = \int_0^{l_{si}} \left(s \frac{(z_2 - z_1)}{l_{si}} + z_1 - z_R \right)^2 ds = \left(\frac{z_2^2}{3} + \frac{z_1^2}{3} + \frac{z_1 z_2}{3} - z_R z_1 - z_R z_2 + z_R^2 \right) l_{si}$$

Equation 5.3-73

$$I_{7i} = \int_0^{l_{si}} (\bar{z} - z_R)(\bar{y} - y_R) \, ds = \int_0^{l_{si}} \left(s \frac{(z_2 - z_1)}{l_{si}} + z_1 - z_R \right)\left(s \frac{(y_2 - y_1)}{l_{si}} + y_1 - y_R \right) ds = l_{si}\left(\frac{(y_2 - y_1)(z_2 - z_1)}{3} + \frac{(y_1 - y_R)(z_2 - z_1)}{2} \right) + l_{si}\left(\frac{(z_1 - z_R)(y_2 - y_1)}{2} + (y_1 - y_R)(z_1 - z_R) \right)$$

Equation 5.3-74

$$I_{8i} = \int_0^{l_{si}} (\bar{z} - z_R)\,\overline{\varpi}(s) \, ds = \int_0^{l_{si}} \left(s \frac{(z_2 - z_1)}{l_{si}} + z_1 - z_R \right)\left(s \frac{r_{fi}}{l_{si}} + C_i \right) ds = l_{si}\left(\left(\frac{(z_2 - z_1)}{3} + \frac{(z_1 - z_R)}{2} \right) r_{fi} + \frac{(z_2 - z_1)}{2} C_i + (z_1 - z_R) C_i \right)$$

Equation 5.3-75

$$I_{9i} = \int_0^{l_{si}} (\bar{z} - z_R)\,\tilde{\varpi}(s) \, ds = \int_0^{l_{si}} \left(s \frac{(z_2 - z_1)}{l_{si}} + z_1 - z_R \right)\left(-s - \frac{q_{fi}}{l_{si}} \right) ds = l_{si}^2 \left(-\frac{z_2}{3} - \frac{z_1}{6} + \frac{z_R}{2} \right) - \left(\frac{(z_2 - z_1)}{2} + (z_1 - z_R) \right) q_{fi}$$

Equation 5.3-76

$$I_{10i} = \int_0^{l_{si}} (\bar{y} - y_R)^2 \, ds = \int_0^{l_{si}} \left(s \frac{(y_2 - y_1)}{l_{si}} + y_1 - y_R \right)^2 ds = \left(\frac{y_2^2}{3} + \frac{y_1^2}{3} + \frac{y_1 y_2}{3} - y_R y_1 - y_R y_2 + y_R^2 \right) l_{si}$$

Equation 5.3-77

$$
\begin{aligned}
I_{11i} &= \int_0^{l_{si}} (\bar{y} - y_R)\,\overline{\omega}(s)\,ds \\
&= \int_0^{l_{si}} \left(s\frac{(y_2 - y_1)}{l_{si}} + y_1 - y_R \right)\left(s\frac{r_{fi}}{l_{si}} + C_i \right) ds \\
&= l_{si}\left(\left(\frac{(y_2 - y_1)}{3} + \frac{(y_1 - y_R)}{2} \right) r_{fi} + \frac{(y_2 - y_1)}{2} C_i + (y_1 - y_R)\,C_i \right)
\end{aligned}
$$

Equation 5.3-78

$$
\begin{aligned}
I_{12i} &= \int_0^{l_{si}} (\bar{y} - y_R)\,\tilde{\omega}(s)\,ds \\
&= \int_0^{l_{si}} \left(s\frac{(y_2 - y_1)}{l_{si}} + y_1 - y_R \right)\left(-s - \frac{q_{fi}}{l_{si}} \right) ds \\
&= l_{si}^2\left(-\frac{y_2}{3} - \frac{y_1}{6} + \frac{y_R}{2} \right) - \left(\frac{(y_2 - y_1)}{2} + (y_1 - y_R) \right) q_{fi}
\end{aligned}
$$

Equation 5.3-79

$$
\begin{aligned}
I_{13i} &= \int_0^{l_{si}} (\overline{\omega}(s))^2\,ds \\
&= \int_0^{l_{si}} \left(s\frac{r_{fi}}{l_{si}} + C_i \right)^2 ds \\
&= l_{si}\left(\frac{(r_{fi} + C_i)^3 - C_i^3}{3\,r_{fi}} \right)
\end{aligned}
$$

Equation 5.3-80

$$
\begin{aligned}
I_{14i} &= \int_0^{l_{si}} \overline{\omega}(s)\,\tilde{\omega}(s)\,ds \\
&= \int_0^{l_{si}} \left(s\frac{r_{fi}}{l_{si}} + C_i \right)\left(-s - \frac{q_{fi}}{l_{si}} \right) ds \\
&= \left(-\frac{r_{fi}}{3} - \frac{C_i}{2} \right) l_{si}^2 - \frac{r_{fi}\,q_{fi}}{2} - C_i\,q_{fi}
\end{aligned}
$$

Equation 5.3-81

$$
\begin{aligned}
I_{15i} &= \int_0^{l_{si}} (\tilde{\omega}(s))^2\,ds \\
&= \int_0^{l_{si}} \left(-s - \frac{q_{fi}}{l_{si}} \right)^2 ds \\
&= \frac{l_{si}^3}{3} + q_{fi}\,l_{si} + \frac{q_{fi}^2}{l_{si}}
\end{aligned}
$$

Equation 5.3-82

5.3.2.9 Modulus Weighted Cross-Section Centroid

The modulus weighted centroid of a laminated composite cross-section is strictly defined as the point through which applied axial loads do not create bending moments. Note that this is not always equivalent to the center of area as is the case for isotropic beam cross-sections. Modulus weighting and the presence of b matrix terms in unsymmetric laminates may shift the cross-section centroid away from the geometric center of area. The modulus weighted centroid of a laminated composite beam cross-section will be designated (y_c, z_c) for the FZM-9954 method, and the location can be calculated using Equation 5.3-83.

$$y_c = y_R + \frac{\int_c a_{11}(\bar{y} - y_R) - b_{11}\sin(\alpha)\,ds}{\int_c a_{11}\,ds} = y_R + \frac{ES_z}{EA}$$

$$z_c = z_R + \frac{\int_c a_{11}(\bar{z} - z_R) - b_{11}\cos(\alpha)\,ds}{\int_c a_{11}\,ds} = z_R + \frac{ES_y}{EA}$$

Equation 5.3-83

Where;

(y_R, z_R) is the location of the reference point "R"
(y_c, z_c) is the location of the cross-section centroid.

5.3.2.10 Modulus Weighted Shear Center

The modulus weighted shear center is defined as the point in the cross-section through which transverse flexural shear forces must act in order to produce zero torsional moment. The shear center is the "center of twist" or "axis of rotation" for a bar that is subjected to torsional loads only. The shear center may also be referred to as the "Principal Pole". The location of the shear center for a laminated composite beam cross-section can be calculated using Equation 5.3-84

$$y_{sc} = \Delta y + y_c$$
$$z_{sc} = \Delta z + z_c$$

Where ;

$$\Delta y = \frac{EI_{\omega y} EI_{zy} - EI_{\omega y} EI_{zz}}{EI_{zy}^2 - EI_{yy} EI_{zz}}$$

Equation 5.3-84

$$\Delta z = \frac{EI_{\omega z} EI_{yy} - EI_{\omega y} EI_{zy}}{EI_{zy}^2 - EI_{yy} EI_{zz}}$$

(y_{sc}, z_{sc}) is the location of the shear center
(y_c, z_c) is the location of the centroid.

Note that Equation 5.3-84 assumes that cross-section properties used in the calculation Δz and Δy are calculated at the section centroid and are adjusted to the principal sectorial origin.

5.3.2.11 Example FZM-9954 Cross-Section Property Calculations

Practical application of the laminated composite beam cross-section property theory and methodology will be demonstrated by the following example problem. Consider the example laminated beam cross-section shown in Figure 5.3-15. Dimensions are referenced to the mid-plane of the laminate. Balloon integers refer to segment ID's. Balloon characters refer to junction ID's. For future reference this cross-section will be designated "Cross-Section A"

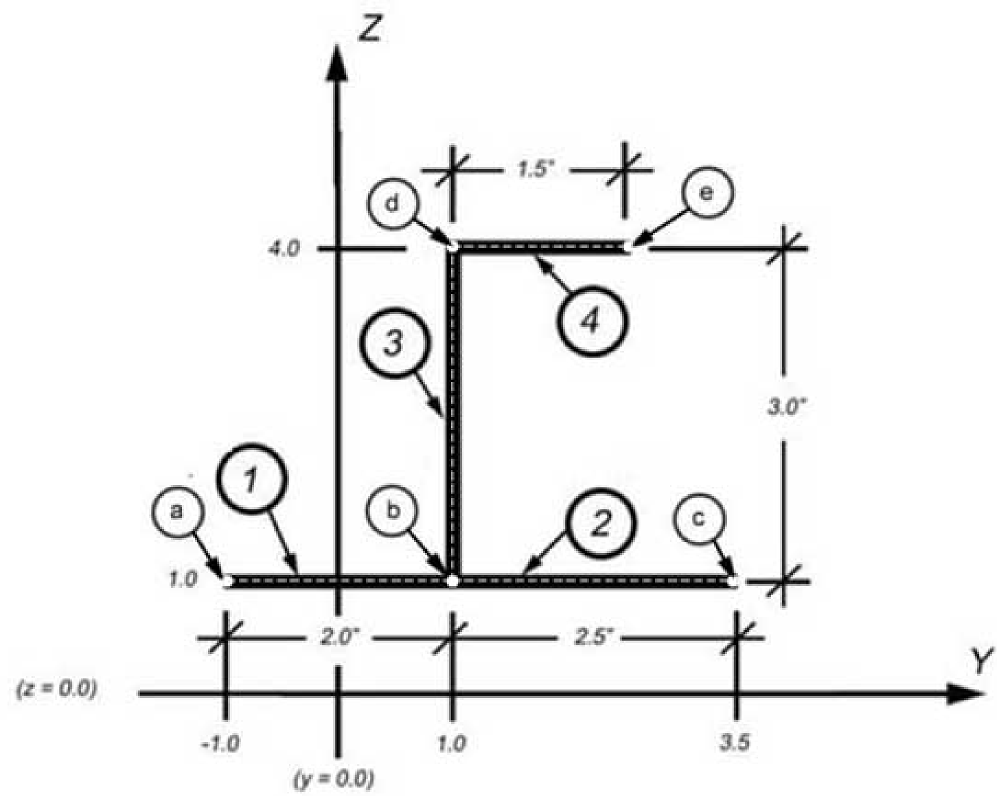

Figure 5.3-15 Example Cross-Section A

5.3.2.11.1 Laminate Properties and Preliminary Calculations

Laminate properties for segments 1, 2, and 4;

Lamina material: B_IM7/977 Tape at 75°F, Thickness = .0053inches/ply

Laminate stacking sequence;

(50/40/10) = $[+45/0/0/-45/0/0/+45/0/-45/90]_s$
Total laminate thickness = 20 plies × .0053 inches/ply = .106"

Laminate [A][B][D] matrix;

$$\begin{bmatrix} A & B \\ B & D \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1486265. & 256472. & 0. \\ 256472. & 594917. & 0. \\ 0. & 0. & 286460. \end{bmatrix} & \begin{bmatrix} 0. & 0. & 0. \\ 0. & 0. & 0. \\ 0. & 0. & 0. \end{bmatrix} \\ B & \begin{bmatrix} 1534. & 261. & 91. \\ 261. & 374. & 91. \\ 91. & 91. & 289. \end{bmatrix} \end{bmatrix}$$

For δ=0, Hooke's law (section 5.3.2) for the laminated beam segments 1, 2, and 4, is;

$$\begin{bmatrix} a_{11} & b_{11} & b_{16} \\ b_{11} & d_{11} & d_{16} \\ b_{16} & d_{16} & d_{66} \end{bmatrix} = \begin{bmatrix} 1486265. & 0. & 0. \\ 0. & 1534. & 91. \\ 0. & 91. & 289. \end{bmatrix}$$

Laminate properties for segment 3;

Lamina material: B_IM7/977 Tape at 75°F, Thickness = .0053inches/ply

Laminate stacking sequence;

(25/50/25) = $[+45/0/-45/90/+45/0/-45/90]_s$
Total laminate thickness = 16 plies × .0053 inches/ply = .0848"

Laminate [A][B][D] matrix;

$$\begin{bmatrix} A & B \\ B & D \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 790233. & 247418. & 0. \\ 247418. & 790233. & 0. \\ 0. & 0. & 271408. \end{bmatrix} & \begin{bmatrix} 0. & 0. & 0. \\ 0. & 0. & 0. \\ 0. & 0. & 0. \end{bmatrix} \\ B & \begin{bmatrix} 537. & 172. & 56. \\ 172. & 362. & 56. \\ 56. & 56. & 186. \end{bmatrix} \end{bmatrix}$$

For δ=0, Hooke's law (section 5.3.2) for the laminated beam segment 3 is;

$$\begin{bmatrix} a_{11} & b_{11} & b_{16} \\ b_{11} & d_{11} & d_{16} \\ b_{16} & d_{16} & d_{66} \end{bmatrix} = \begin{bmatrix} 790233. & 0. & 0. \\ 0. & 537. & 56. \\ 0. & 56. & 186. \end{bmatrix}$$

As the first step in calculation of the modulus weighted cross-section properties, the geometric and material properties of the cross-section are organized as follows in Table 5.3-7 and Table 5.3-8. Where l_s is calculated using Equation 5.3-61, and $\sin(\alpha_i)$ and $\cos(\alpha_i)$ are calculated using Equation 5.3-62.

Table 5.3-7 Coordinates and Preliminary Calculations for Cross-Section A

Segment	y_1	z_1	y_2	z_2	l_s	$\cos(\alpha_i)$	$\sin(\alpha_i)$
1	-1.0	1.0	1.0	1.0	2.0	1.0	0.0
2	1.0	1.0	3.5	1.0	2.5	1.0	0.0
3	1.0	1.0	1.0	4.0	3.0	0.0	1.0
4	1.0	4.0	2.5	4.0	1.5	1.0	0.0

Table 5.3-8 Hooke's Law for Cross-Section A

Segment	a11	b11	b16	d11	d16	d66
1	1486265.0	0.0	0.0	1533.6	90.8	289.0
2	1486265.0	0.0	0.0	1533.6	90.8	289.0
3	790233.0	0.0	0.0	537.5	56.3	186.4
4	1486265.0	0.0	0.0	1533.6	90.8	289.0

Traditional cross-section properties (EA, ES_y, ES_z, EI_{yy}, EI_{zz}, EI_{zy}, and GJ) will be calculated first. For initial cross-section property calculations the reference point for cross-section properties will be the origin of the cross-section coordinate system ($y_R = 0.0$, $z_R = 0.0$). The required geometric integrals I_1, I_2, I_3, I_6, I_7, and I_{10} are calculated using Equation 5.3-68 through Equation 5.3-71 and the results of the calculations are summarized in Table 5.3-9

Table 5.3-9 Geometric integrals I_1, I_2, I_3, I_6, I_7, and I_{10}

Segment	I_1	I_2	I_3	I_6	I_7	I_{10}
1	2.000	2.000	0.000	2.000	0.000	0.667
2	2.500	2.500	5.625	2.500	5.625	13.958
3	3.000	7.500	3.000	21.000	7.500	3.000
4	1.500	6.000	2.625	24.000	10.500	4.875

Where;

$$I_{1i} = l_{si}$$

$$I_{2i} = l_{si}\left(\frac{(z_2 + z_1)}{2} - z_R\right)$$

$$I_{3i} = l_{si}\left(\frac{(y_2 + y_1)}{2} - y_R\right)$$

$$I_{6i} = \left(\frac{z_2^2}{3} + \frac{z_1^2}{3} + \frac{z_1 z_2}{3} - z_R z_1 - z_R z_2 + z_R^2\right) l_{si}$$

$$I_{7i} = l_{si}\left(\frac{(y_2 - y_1)(z_2 - z_1)}{3} + \frac{(y_1 - y_R)(z_2 - z_1)}{2}\right) + l_{si}\left(\frac{(z_1 - z_R)(y_2 - y_1)}{2} + (y_1 - y_R)(z_1 - z_R)\right)$$

$$I_{10i} = \left(\frac{y_2^2}{3} + \frac{y_1^2}{3} + \frac{y_1 y_2}{3} - y_R y_1 - y_R y_2 + y_R^2\right) l_{si}$$

5.3.2.11.2 *EA* - Modulus Weighted Area

The modulus weighted area of a laminated composite beam cross-section is given by Equation 5.3-31.

$$EA = \sum_{i=1}^{n} a_{11i} I_{1i}$$

The component of *EA* for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-8, and Table 5.3-9

Segment	$a_{11i} I_{1i}$
1	2972530.0
2	3715662.5
3	2370699.0
4	2229397.5

The contributions from each segment are summed to obtain the value for *EA*.

EA = 11288289.0 lbs

5.3.2.11.3 Modulus Weighted First Moments of Area

The Modulus weighted first moments of area are calculated about the reference point (y_R, z_R) using the entire cross section. The results from first moment of area calculations are combined with the value for EA and used to establish the cross-section centroid (section 5.3.2.9). Note that the formulas for first moment of area (ES_y, ES_z) are also used to calculate Q_y, and Q_z. Q_y, and Q_z are typically calculated by making a “cut” through the cross-section and using only those portions of the branch segments lying on one side of the “cut”.

5.3.2.11.3.1 ES$_y$

The modulus weighted first moment of area about y_R is given by Equation 5.3-32.

$$ES_y = \sum_{i=1}^{n} a_{11i} I_{2i} - b_{11i} \cos(\alpha_i) I_{1i}$$

The component of ES_y for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-7, Table 5.3-8, and Table 5.3-9**.**

Segment	$a_{11i} I_{2i} - b_{11i} \cos(\alpha_i) I_{1i}$
1	2972530.0
2	3715662.5
3	5926747.5
4	8917590.0

The contributions from each segment are summed to obtain the value for ES_y.

ES_y = 21532530.0 in-lbs

5.3.2.11.3.2 ES$_z$

The modulus weighted first moment of area about z_R is given by Equation 5.3-33.

$$ES_z = \sum_{i=1}^{n} a_{11i} I_{3i} - b_{11i} \sin(\alpha_i) I_{1i}$$

The component of ES_z for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-7, Table 5.3-8, and Table 5.3-9

Segment	$a_{11i} I_{3i} - b_{11i} \sin(\alpha_i) I_{1i}$
1	0.0
2	8360240.6
3	2370699.0
4	3901445.6

The contributions from each segment are summed to obtain the value for ES_z.

ES_z = 14632385.3 in lbs

5.3.2.11.4 EI_{yy} - Modulus Weighted Moment of Inertia about Y_R

The modulus weighted moment of inertia about y_R is given by Equation 5.3-36.

$$EI_{yy} = \sum_{i=1}^{n} a_{11i} I_{6i} - 2b_{11i}\cos(\alpha_i) I_{2i} + d_{11i}\cos^2(\alpha_i) I_{1i}$$

The component of EI_{yy} for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-7, Table 5.3-8, and Table 5.3-9

Segment	$a_{11i} I_{6i} - 2b_{11i}\cos(\alpha_i) I_{2i} + d_{11i}\cos^2(\alpha_i) I_{1i}$
1	2975597.2
2	3719496.5
3	16594893.0
4	35672660.4

The contributions from each segment are summed to obtain the value for EI_{yy}.

EI_{yy} = 58962647.1 in^2-lbs

5.3.2.11.5 EI_{zy} - Modulus Weighted Product of Inertia

The modulus weighted product of inertia is given by Equation 5.3-37.

$$EI_{zy} = \sum_{i=1}^{n} a_{11i} I_{7i} - b_{11i}\sin(\alpha_i) I_{2i} - b_{11i}\cos(\alpha_i) I_{3i} \\ - d_{11i}\sin(\alpha_i)\cos(\alpha_i) I_{1i}$$

The component of EI_{zy} for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-7, Table 5.3-8, and Table 5.3-9

Segment	$a_{11i} I_{7i} - b_{11i}\sin(\alpha_i) I_{2i} - b_{11i}\cos(\alpha_i) I_{3i} - d_{11i}\sin(\alpha_i)\cos(\alpha_i) I_{1i}$
1	0.0
2	8360240.6
3	5926747.5
4	15605782.5

The contributions from each segment are summed to obtain the value for EI_{zy}.

EI_{zy} = 29892770.6 in^2-lbs

5.3.2.11.6 EI_{zz} - Modulus Weighted Moment of Inertia about Z

The modulus weighted moment of inertia about z_R is given by Equation 5.3-40.

$$EI_{zz} = \sum_{i=1}^{n} a_{11i} I_{10i} + 2\, b_{11i} \sin(\alpha_i) I_{3i} + d_{11i} \sin^2(\alpha_i) I_{1i}$$

The component of EI_{zz} for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-7, Table 5.3-8, and Table 5.3-9

Segment	$a_{11i} I_{10i} + 2\, b_{11i} \sin(\alpha_i) I_{3i} + d_{11i} \sin^2(\alpha_i) I_{1i}$
1	990843.3
2	20745782.3
3	2372311.4
4	7245541.9

The contributions from each segment are summed to obtain the value for EI_{zz}.

EI_{zz} = 31354478.9 in^2-lbs

5.3.2.11.7 GJ - Modulus Weighted Torsional Stiffness

The modulus weighted torsional stiffness is given by Equation 5.3-45.

$$GJ = 4 \sum_{i=1}^{n} d_{66i} I_{1i}$$

The component of GJ for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-8, and Table 5.3-9

Segment	$4\, d_{66i} I_{1i}$
1	2311.9
2	2889.9
3	2236.5
4	1733.9

The contributions from each segment are summed to obtain the value for GJ.

GJ = 9172.1 in^2-lbs

5.3.2.11.8 ES$_\omega$ - Modulus Weighted First Sectorial Moment of Area

The modulus weighted first Sectorial moment of area is given by Equation 5.3-34.

$$ES_\omega = \sum_{i=1}^{n} a_{11i} I_{4i} - b_{11i}\, I_{5i}$$

Where;

$$I_{4i} = l_{si}\left(\frac{r_{fi}}{2} + C_i\right)$$

$$I_{5i} = -\frac{l_{si}^2}{2} - q_{fi}$$

The equation for I_{4i} is dependent on the constant of integration in the branch segment warping function Equation 5.3-67. The constant of integration in the warping function must be calculated for each branch before the value for ES_ω can be calculated. The equation for the warping function (Equation 5.3-67) is shown here in expanded form.

$$\overline{\varpi} = s\frac{(y_1 - y_R)(z_2 - z_1) - (z_1 - z_R)(y_2 - y_1)}{l_s} + C_i$$

Equations for the branch warping function ($\overline{\varpi}$) are formulated at each end point of each branch segment. Results of these calculations are shown in Table 5.3-10.

Table 5.3-10 Warping Constant Equations

Segment	$\overline{\varpi}$ (s = 0)	$\overline{\varpi}$ (s = l_s)
1	C_1	-2.000 + C_1
2	C_2	-2.500 + C_2
3	C_3	3.000 + C_3
4	C_4	-6.000 + C_4

A connectivity matrix is populated with the warping function equations calculated at the branch segment end points. The warping connectivity matrix is shown in Table 5.3-11.

Table 5.3-11 Continuity of Warping Constants

Segments	Junctions				
	a	b	c	d	e
1	C_1	-2.0+C_1			
2		C_2	-2.5+C_2		
3		C_3		3.0+C_3	
4				C_4	-6.0+C_4

Columns of Table 5.3-11 containing values from more than one row yield sets of simultaneous equations that can be solved for continuity of the branch segment warping constants. Relative value of the constants is fixed by the equations in Table 5.3-11, but for open-cross-sections the system of equations will be underdetermined. Recall from Figure 5.3-14 that the value of the constants can be adjusted up or down by a common constant provided that the conditions for continuity at each junction remain satisfied.

Columns of the example Table 5.3-11 yield the following equations for continuity of the branch segment warping constants.

Continuity equations from column “b”;

$-2.0+C_1 = C_2 \quad => -2.0 = -C_1+C_2$

$-2.0+C_1 = C_3 \quad => -2.0 = -C_1+C_3$

$C_2 = C_3 \quad => 0.0= -C_2+C_3$

Continuity equations from column “d”;

$3.0+C_3=C_4 \quad => 3.0 = -C_3+C_4$

Assemble the independent equations in matrix form $[A]x = b$;

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} -2.0 \\ -2.0 \\ 3.0 \end{Bmatrix}$$

A solution can be obtained using the general least squares method for an underdetermined system of equations $[A]x = b$.

Assume $x = A^t t$

$$[A][A^t]t = b$$

$$t = \left([A][A^t]\right)^{-1} b$$

$$x = A^t t$$

The least squares solution produces the initial values for the warping constants given in Table 5.3-12. Substitution of the constant values in Table 5.3-12 into the continuity equations will confirm that this set of constants satisfies the requirements for warping continuity at the branch junctions.

Table 5.3-12 Warping Constants, C_i

C_1	0.750
C_2	-1.250
C_3	-1.250
C_4	1.750

The branch integrals I_4, and I_5, and the initial value for ES_ω can now be calculated as follows;

Segment	I_4	I_5	$a_{11i} I_{4i} - b_{11i} I_{5i}$
1	2.200	0.000	-743132.5
2	-2.875	-5.625	-9289156.3
3	4.800	-7.500	592674.7
4	0.150	-2.625	-2786746.9

Sum the contribution from each segment to obtain the value for ES_ω.

$$ES_\omega = -12226360.9 \text{ in}^2\text{-lbs}$$

As stated in section 5.3.2.7, the warping properties must be calculated at the "Principal Contour Origin". The "Principal Contour Origin" is located by adjusting the branch warping constants until $ES_\omega = 0$. The adjustment required is calculated by the ratio of $-ES_\omega/EA$.

Warping constant adjustment = $-ES_\omega/EA = -(-12226360.9/11288289.0) = 1.083101334$

Table 5.3-13 Warping Constants, C_i at Principal Contour Origin

C_1	0.750+1.0831	= 1.833
C_2	-1.250+1.0831	= -0.167
C_3	-1.250+1.0831	= -0.167
C_4	1.750+1.0831	= 2.833

The final value for ES_ω calculated using the adjusted constants can be verified to equal zero (see Table 5.3-14 for values of I4, and I5 calculated using the warping constants in Table 5.3-13)

Segment	$a_{11i} I_{4i} - b_{11i} I_{5i}$
1	2476418.7
2	-5264717.2
3	3160382.0
4	-372083.5

The contribution from each segment is summed to obtain the adjusted value for ES_ω.

$$ES_\omega = 0.0 \text{ in}^2\text{-lbs}$$

The remaining cross-section properties for this example cross-section and reference point (y_R=0, z_R=0) are calculated using the warping constants from Table 5.3-13 which are calculated to yield properties at the "Principal Contour Origin". At this point in the example, the remaining branch integrals can be calculated.

Table 5.3-14 Contour Integrals, I_4, I_5, I_8, and I_9

Segment	I_4	I_5	I_8	I_9
1	2.200	0.000	2.200	0.000
2	-2.875	-5.625	-2.875	-5.625
3	4.800	-7.500	14.250	-21.000
4	0.150	-2.625	0.600	-10.500

Where;

$$I_{4i} = l_{si}\left(\frac{r_{fi}}{2} + C_i\right)$$

$$I_{5i} = -\frac{l_{si}^2}{2} - q_{fi}$$

$$I_{8i} = l_{si}\left(\left(\frac{(z_2 - z_1)}{3} + \frac{(z_1 - z_R)}{2}\right) r_{fi} + \frac{(z_2 - z_1)}{2} C_i + (z_1 - z_R) C_i\right)$$

$$I_{9i} = l_{si}^2\left(-\frac{z_2}{3} - \frac{z_1}{6} + \frac{z_R}{2}\right) - \left(\frac{(z_2 - z_1)}{2} + (z_1 - z_R)\right) q_{fi}$$

Table 5.3-15 Contour Integrals, I_{11}, I_{12}, I_{13}, I_{14}, and I_{15}

Segment	I_{11}	I_{12}	I_{13}	I_{14}	I_{15}
1	-0.667	-0.667	3.087	0.667	0.667
2	-7.771	-13.958	4.608	7.771	13.958
3	4.800	-7.500	9.930	-14.250	21.000
4	-0.862	-4.875	4.515	0.862	4.875

Where;

$$I_{11i} = l_{si}\left(\left(\frac{(y_2 - y_1)}{3} + \frac{(y_1 - y_R)}{2}\right) r_{fi} + \frac{(y_2 - y_1)}{2} C_i + (y_1 - y_R) C_i\right)$$

$$I_{12i} = l_{si}^2\left(-\frac{y_2}{3} - \frac{y_1}{6} + \frac{y_R}{2}\right) - \left(\frac{(y_2 - y_1)}{2} + (y_1 - y_R)\right) q_{fi}$$

$$I_{13i} = l_{si}\left(\frac{(r_{fi} + C_i)^3 - C_i^3}{3\, r_{fi}}\right)$$

$$I_{14i} = \left(-\frac{r_{fi}}{3} - \frac{C_i}{2}\right) l_{si}^2 - \frac{r_{fi}\, q_{fi}}{2} - C_i\, q_{fi}$$

$$I_{15i} = \frac{l_{si}^3}{3} + q_{fi}\, l_{si} + \frac{q_{fi}^2}{l_{si}}$$

5.3.2.11.9 $EI_{\omega y}$ - Modulus Weighted Second Sectorial Moment about Y_R

The modulus weighted second sectorial moment about y_R is given by Equation 5.3-38.

$$EI_{\omega y} = \sum_{i=1}^{n} a_{11i} I_{8i} + b_{11i} I_{9i} - b_{11i} \cos(\alpha_i) I_{4i} - d_{11i} \cos(\alpha_i) I_{5i}$$

The component of $EI_{\omega y}$ for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-7, Table 5.3-8, Table 5.3-14, and Table 5.3-15

Segment	$a_{11i} I_{8i} + b_{11i} I_{9i} - b_{11i} \cos(\alpha_i) I_{4i} - d_{11i} \cos(\alpha_i) I_{5i}$
1	2476418.7
2	-5256090.7
3	9678979.2
4	-1484308.2

The contributions from each segment are summed to obtain the value for $EI_{\omega y}$.

$EI_{\omega y}$ = 5414999.0 in^3-lbs

5.3.2.11.10 $EI_{\omega z}$ - Modulus Weighted Second Sectorial Moment about Z_R

The modulus weighted second sectorial moment about z_R is given by Equation 5.3-41.

$$EI_{\omega z} = \sum_{i=1}^{n} a_{11i} I_{11i} + b_{11i} I_{12i} + b_{11i} \sin(\alpha_i) I_{4i} - d_{11i} \sin(\alpha_i) I_{5i}$$

The component of $EI_{\omega z}$ for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-7, Table 5.3-8, Table 5.3-14, and Table 5.3-15

Segment	$a_{11i} I_{11i} + b_{11i} I_{12i} + b_{11i} \sin(\alpha_i) I_{4i} - d_{11i} \sin(\alpha_i) I_{5i}$
1	-990843.3
2	-13780854.7
3	3164412.9
4	-2323194.2

The contributions from each segment are summed to obtain the value for $EI_{\omega z}$.

$EI_{\omega z}$ = -13930479.3 in^3-lbs

5.3.2.11.11 $EI_{\omega\omega}$ - Modulus Weighted Warping Coefficient

The modulus weighted warping coefficient is given by Equation 5.3-43

$$EI_{\omega\omega} = \sum_{i=1}^{n} a_{11i} I_{13i} + 2b_{11i} I_{14i} + d_{11i} I_{15i}$$

The component of $EI_{\omega\omega}$ for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-8, and Table 5.3-15

Segment	$a_{11i} I_{13i} + 2b_{11i} I_{14i} + d_{11i} I_{15i}$
1	3054973.5
2	9416218.2
3	6002420.2
4	6757769.0

The contributions from each segment are summed to obtain the value for $EI_{\omega\omega}$.

$EI_{\omega\omega}$ = 25231380.9 in^4-lbs

5.3.2.11.12 EH - Modulus Weighted Axial-Torsional Coupling Stiffness

The modulus weighted axial-torsional coupling stiffness is given by Equation 5.3-35

$$EH = 2\sum_{i=1}^{n} b_{16i} I_{1i}$$

The component of *EH* for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-8, and Table 5.3-9. For the example problem the laminates are symmetric and b_{16} equals zero, therefore EH will equal zero for this cross-section.

Segment	$2b_{16i} I_{1i}$
1	0.0
2	0.0
3	0.0
4	0.0

The contributions from each segment are summed to obtain the value for *EH*.

EH 0.0 in-lbs

5.3.2.11.13 EH_c - Modulus Weighted Bending-Torsional Coupling Stiffness about Y_R

The modulus weighted bending-torsional coupling stiffness about y_R is given by Equation 5.3-39

$$EH_c = 2\sum_{i=1}^{n} b_{16i} I_{2i} - d_{16i}\cos(\alpha_i) I_{1i}$$

The component of EH_c for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-7, Table 5.3-8, and Table 5.3-9.

Segment	$2\left(b_{16i} I_{2i} - d_{16i}\cos(\alpha_i) I_{1i}\right)$
1	-363.0
2	-453.8
3	0.0
4	-272.3

The contributions from each segment are summed to obtain the value for EH_c.

EH_c = -1089.1 in^2-lbs

5.3.2.11.14 EH_s - Modulus Weighted Bending-Torsional Coupling Stiffness about Z_R

The modulus weighted bending-torsional coupling stiffness about z_R is given by Equation 5.3-42.

$$EH_s = 2\sum_{i=1}^{n} b_{16i} I_{3i} + d_{16i}\sin(\alpha_i) I_{1i}$$

The component of EH_s for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-7, Table 5.3-8, and Table 5.3-9.

Segment	$2\left(b_{16i} I_{3i} + d_{16i}\sin(\alpha_i) I_{1i}\right)$
1	0.0
2	0.0
3	338.0
4	0.0

The contributions from each segment are summed to obtain the value for EH_s.

EH_s = 338.0 in^2-lbs

5.3.2.11.15 EH_q - Modulus Weighted Warping-Torsional Coupling Stiffness

The modulus weighted warping-torsional coupling stiffness is given by Equation 5.3-44.

$$EH_q = 2\sum_{i=1}^{n} b_{16i} I_{4i} + d_{16i} I_{5i}$$

The component of EH_q for each segment is calculated using the indicated equation combined with the pre-calculated values in Table 5.3-8, and Table 5.3-14.

Segment	$2\left(b_{16i} I_{4i} + d_{16i} I_{5i}\right)$
1	0.0
2	-1021.1
3	-845.0
4	-476.5

The contributions from each segment are summed to obtain the value for EH_q.

$$EH_q = -2342.6 \text{ in}^3\text{-lbs}$$

5.3.2.11.16 Summary of Section Properties at Reference (y_R, z_R)

Properties for the example cross-section "A" calculated at the reference point (y_R = 0.0, z_R=0.0) are summarized in Table 5.3-16

Table 5.3-16 Section "A" Properties Calculated at (y_R, z_R) using the Principal Sectorial Origin

EA	ES_y	ES_z	ES_ω	Eh
1.1288×10^7	2.1533×10^7	1.4632×10^7	0.0000	0.0000
EI_{yy}	EI_{zy}	$EI_{\omega y}$	EH_c	EI_{zz}
5.8963×10^7	2.9893×10^7	5.4150×10^6	-1.0891×10^3	3.1354×10^7
$EI_{\omega z}$	EH_s	$EI_{\omega\omega}$	EH_q	GJ
-1.3930×10^7	3.3801×10^2	2.5231×10^7	-2.3426×10^3	9.1721×10^3

5.3.2.11.17 Location of the Cross-Section Centroid

The modulus weighted cross-section centroid is calculated using Equation 5.3-83. Values for the reference point (y_R, z_R), the modulus weighted area EA, and the modulus weighted first moments of area ES_y, and ES_z are required for calculating the centroid location.

$$y_c = y_R + \frac{ES_z}{EA}$$

$$y_c = 0.0 + \frac{14632385.3}{11288289.0} = 1.2962 \text{ in}$$

$$z_c = z_R + \frac{ES_y}{EA}$$

$$z_c = 0.0 + \frac{21532530.0}{11288289.0} = 1.9075 \text{ in}$$

The cross-section centroid is located at (1.2962, 1.9075). The location of the modulus weighted centroid of the example cross-section is represented by the target shown in Figure 5.3-16

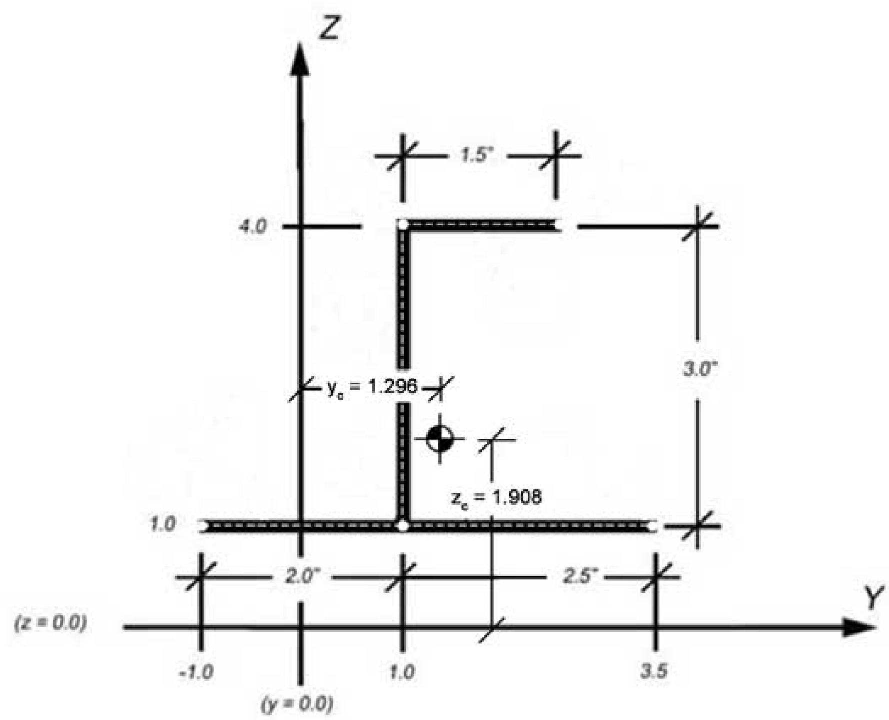

Figure 5.3-16 Cross-Section A Modulus Weighted Geometric Centroid

Properties are re-calculated at the cross-section centroid by setting the reference point (y_R, z_R) in Equation 5.3-68 through Equation 5.3-82 equal to calculated value for the cross-section centroid ($y_c = 1.2962$, $z_c = 1.9075$). The principal sectorial origin is re-calculated and new values for warping function constants are established prior to calculating values for the warping properties. The results of the cross-section property calculations at the cross-section centroid are given in Table 5.3-17

Table 5.3-17 Section "A" Properties Calculated at Section Centroid (y_c, z_c) using the Principal Sectorial Origin

EA	ES_y	ES_z	ES_ω	Eh
1.1288×10^7	0.0000	0.0000	0.0000	0.0000
EI_{yy}	EI_{zy}	$EI_{\omega y}$	EH_c	EI_{zz}
1.7889×10^7	1.9813×10^6	-1.3994×10^7	-1.0891×10^3	1.2387×10^7
$EI_{\omega z}$	EH_s	$EI_{\omega\omega}$	EH_q	GJ
7.1240×10^6	3.3801×10^2	2.3350×10^7	-2.8604×10^2	9.1721×10^3

5.3.2.11.18 Location of the Cross-Section Shear Center

Using Properties calculated at the section centroid, the modulus weighted shear center is located using Equation 5.3-84

$$\begin{aligned}\Delta y &= \frac{EI_{\omega y} EI_{zy} - EI_{\omega y} EI_{zz}}{EI_{zy}^2 - EI_{yy} EI_{zz}} \\ &= \frac{-1.3994\text{E}+07 \times 1.9813\text{E}+06 - (-1.3994\text{E}+07) \times 1.2387\text{E}+07}{(1.9813\text{E}+06)^2 - 1.7889\text{E}+07 \times 1.2387\text{E}+07} \\ &= -0.86123 \text{ in}\end{aligned}$$

$$\begin{aligned}\Delta z &= \frac{EI_{\omega z} EI_{yy} - EI_{\omega y} EI_{zy}}{EI_{zy}^2 - EI_{yy} EI_{zz}} \\ &= \frac{7.1240\text{E}+06 \times 1.7889\text{E}+07 - (-1.3994\text{E}+07) \times 1.9813\text{E}+06}{(1.9813\text{E}+06)^2 - 1.7889\text{E}+07 \times 1.2387\text{E}+07} \\ &= -0.71286 \text{ in}\end{aligned}$$

$$y_{sc} = \Delta y + y_c = -0.86123 + 1.2962 = 0.4350 \text{ in}$$

$$z_{sc} = \Delta z + z_c = -0.71286 + 1.9075 = 1.1947 \text{ in}$$

The cross-section shear center is located at (0.4346, 1.1946). The location of the modulus weighted shear center of the example cross-section is represented by the target shown in Figure 5.3-17

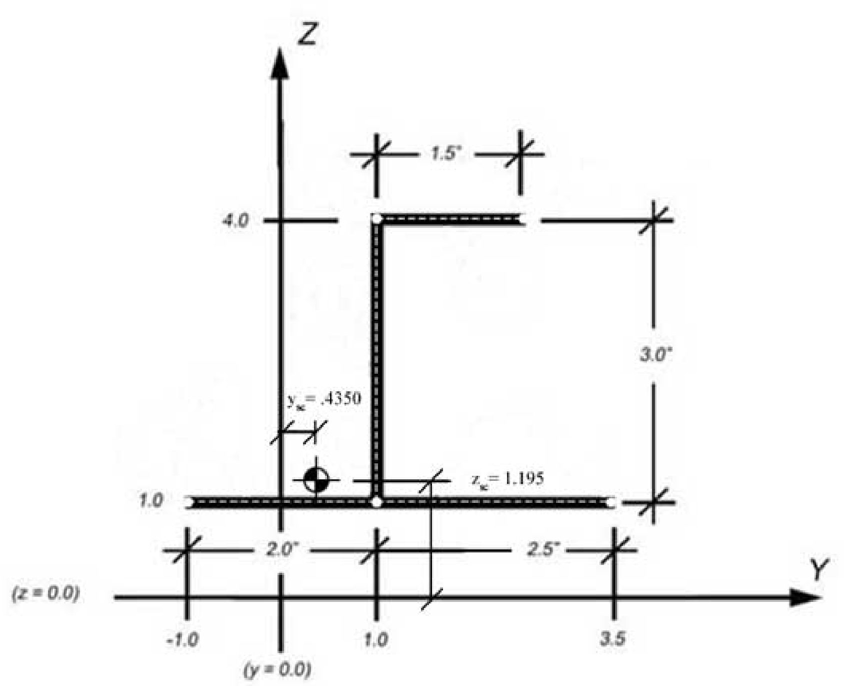

Figure 5.3-17 Cross-Section A Modulus Weighted Shear Center

Properties are re-calculated at the cross-section shear center by setting the reference point (y_R, z_R) in Equation 5.3-68 through Equation 5.3-82 equal to calculated value for the cross-section shear center (y_{sc} = 0.4346, z_{sc} = 1.1946). The principal sectorial origin is re-calculated and new values for warping function constants are established prior to calculating new values for the warping properties. The results of the cross-section property calculations at the cross-section shear center are given in Table 5.3-18

Table 5.3-18 Section "A" Properties Calculated at the Section Shear Center (y_{sc}, z_{sc}) using the Principal Sectorial Origin

EA	***ES_y***	***ES_z***	***ES_ω***	***Eh***
1.1288×10^7	8.0478×10^6	9.7268×10^6	0.0000	0.0000
EI_{yy}	***EI_{zy}***	***$EI_{\omega y}$***	***EH_c***	***EI_{zz}***
2.3625×10^7	8.9116×10^6	0.0000	-1.0891×10^3	2.0760×10^7
$EI_{\omega z}$	***EH_s***	***$EI_{\omega\omega}$***	***EH_q***	***GJ***
2.2980×10^3	3.3801×10^2	6.2218×10^6	-1.4655×10^3	9.1721×10^3

5.3.3 IDAT Section Method

The cross-section property methodology used by IDAT SECTION is based on a research paper by Wojciechowski (ref. 5-19). A closed circuit of ordered boundary points connected by straight lines defines a trapezoidal area for section property calculation. Complex cross-section shapes may be created by combining the properties of multiple trapezoidal area elements.

Each trapezoidal shape may make either positive or negative contribution to cross-section properties. Positive cross-section areas are defined by traversing the top most boundary points in order of increasing y. That is, the ordering of points must be clockwise for a positive area. The point chosen for the segment origin of a positive area should be nearest to the origin and have the lowest z coordinate. Negative areas may be defined to represent holes or cutouts in positive cross-section areas. Negative areas are defined using a counterclockwise ordering of points. If a negative area is defined it must be located entirely within or on the boundary defined by a positive area.

Complex cross-section shapes which include detail features such as fillet radii, bulbs, or chamfers may be defined with a small number of trapezoidal areas using the IDAT method. For example, properties for a complex shape such as a hat section may be calculated using one outside positive area loop and one inside negative area loop provided that the axial stiffness is the same for all elements. Each trapezoidal shape element must be located entirely within the first quadrant of the y-z coordinate system. That is, the coordinates of all vertices must be positive. Example loops for a hat section are illustrated in Figure 5.3-18.

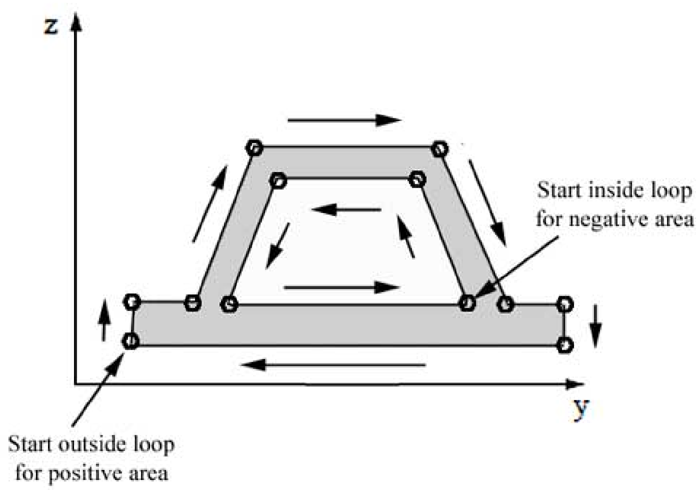

Figure 5.3-18 Cross-Section Idealization for IDAT SECTION Method

5.3.3.1 Basic Equations

Equation 5.3-85 through Equation 5.3-98 are used to calculate the contribution to cross-section properties from a single trapezoidal shape bounded by "n" points connected by straight line segments. All boundary points must be located in the first coordinate quadrant. Points on the upper boundary of the shape must be traversed in order of increasing y for a positive area, and in order of decreasing y for a negative area. Cross-section properties are calculated with respect to the y-z coordinate axis.

Area

$$\Delta A_i = (y_{i+1} - y_i)(z_{i+1} + z_i)/2$$ **Equation 5.3-85**

$$A = \sum_{i=1}^{n} \Delta A_i$$ **Equation 5.3-86**

First Moment of Area

$$\Delta M_{AY\,i} = [(y_{i+1} - y_i)/8][(z_{i+1} + z_i)^2 + (z_{i+1} - z_i)^2/3]$$ **Equation 5.3-87**

$$M_{AY} = \sum_{i=1}^{n} \Delta M_{AY\,i}$$ **Equation 5.3-88**

$$\Delta M_{AZ\,i} = [-(z_{i+1} - z_i)/8][(y_{i+1} + y_i)^2 + (y_{i+1} - y_i)^2/3]$$ **Equation 5.3-89**

$$M_{AZ} = \sum_{i=1}^{n} \Delta M_{AZ\,i}$$ **Equation 5.3-90**

Area Moment of Inertia

$$\Delta I_{Y\,i} = [(y_{i+1} - y_i)(z_{i+1} + z_i)/24][(z_{i+1} + z_i)^2 + (z_{i+1} - z_i)^2]$$ **Equation 5.3-91**

$$I_Y = \sum_{i=1}^{n} \Delta I_{Y\,i}$$ **Equation 5.3-92**

$$\Delta I_{Z\,i} = [-(z_{i+1} - z_i)(y_{i+1} + y_i)/24][(y_{i+1} + y_i)^2 + (y_{i+1} - y_i)^2]$$ **Equation 5.3-93**

$$I_Z = \sum_{i=1}^{n} \Delta I_{Z\,i}$$ **Equation 5.3-94**

Area Product of Inertia

$$\begin{aligned} &if\ (y_{i+1} - y_i) = 0 \\ &\quad \Delta I_{YZ\,i} = 0 \\ &else \\ &\quad \Delta I_{YZ\,i} = (\,((z_{i+1} - z_i)^2 (y_{i+1} + y_i)(y_{i+1}^2 + y_i^2)/8) \\ &\qquad + ((z_{i+1} - z_i)(y_{i+1} z_i - y_i z_{i+1})(y_{i+1}^2 + y_i^2 + y_{i+1} y_i)/3) \\ &\qquad + ((y_{i+1} z_i - y_i z_{i+1})^2 (y_{i+1} + y_i)/4)\,)/(y_{i+1} - y_i) \end{aligned}$$ **Equation 5.3-95**

$$I_{YZ} = \sum_{i=1}^{n} \Delta I_{YZ\,i}$$ **Equation 5.3-96**

Segment Centroid

$$\bar{y} = \frac{M_{AZ}}{A}$$ **Equation 5.3-97**

$$\bar{z} = \frac{M_{AY}}{A}$$ **Equation 5.3-98**

For the predefined section shapes, IDAT section uses Equation 5.3-20 to calculate torsional stiffness. IDAT SECTION does not calculate torsional stiffness for general shapes created from a composite of elements.

5.3.3.2 Modulus Weighted Properties

Laminated composite beam cross-sections containing multiple stacking sequences are represented in IDAT SECTION by arranging rectangular elements to create the desired cross-section shape. Each rectangular area is then associated with a laminate stack file. IDAT SECTION uses the IDAT laminate stack file combined with the inverted [ABD] laminate stiffness matrix methodology presented as the first option in section 5.3.1.1 to calculate equivalent axial stiffness properties for the cross-section segment. Properties for each segment are calculated using Equation 5.3-85 through Equation 5.3-98 and then multiplied by the equivalent axial stiffness to yield modulus weighted properties. Modulus weighted properties for each segment are combined and transformed using the same methodology presented in section 5.3.1.7. An example rectangular element at an arbitrary orientation angle is shown in Figure 5.3-19. The example segment corner points are numbered clockwise with the origin closest to the y-z coordinate origin.

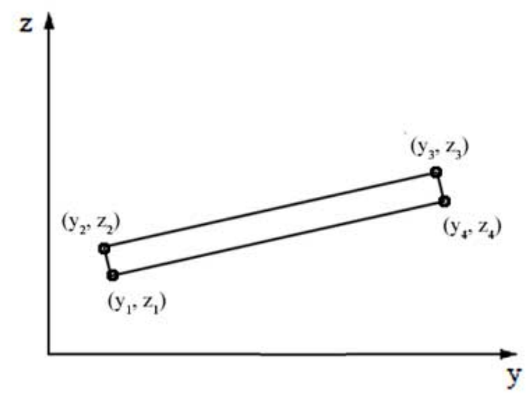

Figure 5.3-19 Cross-Section Segment for Modulus Weighted Properties

5.3.3.3 Example Calculations

Consider the example segment shown in Figure 5.3-19, with the coordinates as given in Table 5.3-19

Table 5.3-19 Example Coordinates

Point	y	z
1	0.500	1.000
2	0.465	1.197
3	2.435	1.544
4	2.470	1.347

The segment origin is chosen as the point nearest the origin, and the boundary points are ordered clockwise to create a positive segment. Area of the segment is calculated using Equation 5.3-85 and Equation 5.3-86. Results of these calculations are shown in Table 5.3-20.

Table 5.3-20 Area

i	ΔA_i
1	-0.038
2	2.700
3	0.050
4	-2.312

$$A = \sum_{i=1}^{n} \Delta A_i \qquad = 0.400 \text{ in}^2$$

The first moment of area is calculated using Equation 5.3-87 through Equation 5.3-90. The results of these calculations are shown in Table 5.3-21

Table 5.3-21 First Moment of Area

i	ΔM_{AYi}	ΔM_{AZi}
1	-0.021	-0.023
2	1.860	-0.421
3	0.036	0.592
4	-1.366	0.439

$$M_{AY} = \sum_{i=1}^{n} \Delta M_{AY\,i} \qquad = 0.509 \text{ in}^3$$

$$M_{AZ} = \sum_{i=1}^{n} \Delta M_{AZ\,i} \qquad = 0.587 \text{ in}^3$$

Location of the Segment Centroid in the y-z coordinate system may now be calculated using Equation 5.3-97 and Equation 5.3-98

$$\bar{z} = \frac{M_{AY}}{A} \qquad = 1.272 \text{ in}$$

$$\bar{y} = \frac{M_{AZ}}{A} \qquad = 1.467 \text{ in}$$

The Area Moment of Inertia with respect to the y-z coordinate system is calculated using Equation 5.3-91 through Equation 5.3-94. The results of these calculations for the example segment are shown in Table 5.3-22

Table 5.3-22 Area Moment of Inertia

i	ΔI_{Yi}	ΔI_{Zi}	ΔI_{YZi}
1	-0.015	-0.007	-0.010
2	1.718	-0.516	2.851
3	0.035	0.968	0.089
4	-1.085	0.546	-2.161

$$I_Y = \sum_{i=1}^{n} \Delta I_{Y\,i} \qquad = 0.653 \text{ in}^4$$

$$I_Z = \sum_{i=1}^{n} \Delta I_{Z\,i} \qquad = 0.991 \text{ in}^4$$

$$I_{YZ} = \sum_{i=1}^{n} \Delta I_{YZ\,i} \qquad = 0.769 \text{ in}^4$$

The translation and rotation equation presented in Section 5.3.1.7 may be used to transform these calculated y-z properties to the segment centroidal. A parallel translation to the segment y'-z' axis is performed first as follows:

$$I_y = I_{y'} + A \cdot z^2$$

$$0.653 = I_{y'} + 0.400 \cdot 1.272^2$$

$$I_{y'} = 0.005314$$

$$I_z = I_{z'} + A \cdot y^2$$
$$0.991 = I_{z'} + 0.400 \cdot 1.467^2$$
$$I_{z'} = 0.129353$$

$$I_{yz} = I_{y'z'} + A \cdot y \cdot z$$
$$0.769 = I_{y'z'} + 0.400 \cdot 1.467 \cdot 1.272$$
$$I_{y'z'} = 0.022573$$

A 10º rotation using Equation 5.3-21yields the following principal centroidal moments of inertia for the segment. These values can be verified using the common equations for a rectangular segment.

$$I_y = 0.001333$$

$$I_z = 0.133333$$

$$I_{yz} = 0.0$$

5.3.3.4 IDAT SECTION - Example J-Section Analysis

The laminated composite J cross-section previously analyzed as an example problem in sections 5.3.1.9, and 5.3.2.11 is divided into three rectangular elements for the purpose of analysis in IDAT SECTION. Four boundary points are used to define each cross-section element. Following the rules for a positive section area discussed in section 5.3.3, the boundary points for each element are ordered clockwise with the first point placed at the corner closest to the coordinate origin. The cross-section elements, boundary points, and associated identification numbers are shown in Figure 5.3-20.

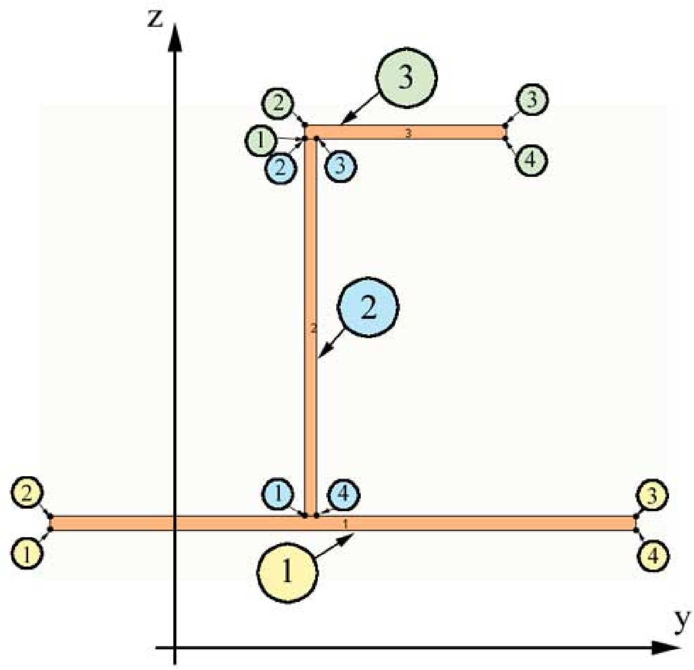

Figure 5.3-20 IDAT J-Section Analysis Elements

Laminate ply percentages, total laminate thickness, material identification and coordinates of the boundary points for the "J" cross-section shown in Figure 5.3-20, are given in Table 5.3-23.

Table 5.3-23 IDAT J-Section Example Coordinates

Element	Corner	y	z
1	1	-1.000	0.947
(50/40/10)	2	-1.000	1.053
t=.106"	3	3.500	1.053
IM7 977-3 tape	4	3.500	0.947
2	1	0.958	1.053
(25/50/25)	2	0.958	3.947
t=.0848"	3	1.042	3.947
IM7 977-3 tape	4	1.042	1.053
3	1	0.958	3.947
(50/40/10)	2	0.958	4.053
t=.106"	3	2.500	4.053
IM7 977-3 tape	4	2.500	3.947

The three cross-section elements specified in Table 5.3-23 are entered into the IDAT SECTION program using the "General Shapes" rectangle icon located on the left side of the IDAT SECTION application window (see Figure 5.3-21). Note that the "Standard Shapes" icons may only be used if material properties do not vary among elements of the cross-section. Next, a laminate stack file must be created and assigned to each rectangular cross-section element. Laminate stack files (.ls files) are typically created using the IDAT utility LAMINATE. To assign a laminate stack file: first select the desired cross-section element with a left mouse click, and then select the "Material" button located on the left side of the IDAT SECTION application window. When the pop-up window appears, select "laminate file" and browse to select the appropriate laminate stack file. After the data for each cross-section element has been entered and the material files have been assigned, choose one of the "Section Analysis" options from the top level "Analysis" menu to generate properties for the cross-section. Section property calculations for the example "J" section are shown in the IDAT SECTION window which is reproduced here as Figure 5.3-21

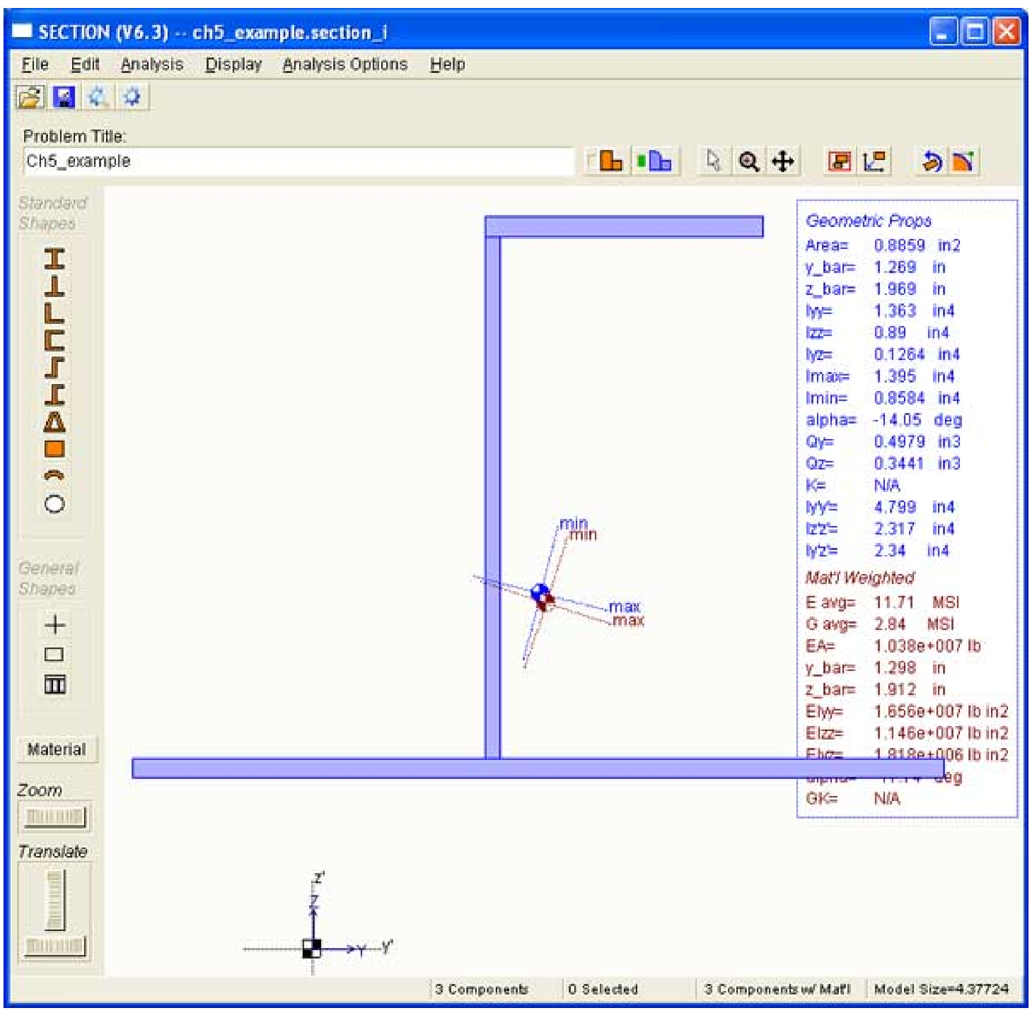

Figure 5.3-21 IDAT J-Section Analysis Results

5.3.4 Comparison of Section-Property Calculation Methods

Three methods for calculating the cross-section properties of a laminated composite beam are presented in section 5.3. The comparative advantages and disadvantages of each method along with suggestions for use are discussed in this section. A summary of the J-section example problem results is presented as Table 5.3-24, and the percentage comparison to baseline FZM-9954 δ=0 results is presented as Table 5.3-25.

Table 5.3-24 Summary of Section Property Example J-Section Results

	Section Property Method			
	FZM-9954 (Baseline δ=0.0)	**FZM-9954 (δ=1.0)**	**Modulus Weighted Area (MWA)**	**IDAT SECTION**
EA	11.288×10^6	10.393×10^6	10.405×10^6	10.380×10^6
EI_{yy}	58.963×10^6	54.184×10^6	54.230×10^6	56.196×10^6
EI_{zz}	31.354×10^6	28.966×10^6	29.009×10^6	27.133×10^6
EI_{yz}	29.893×10^6	27.529×10^6	27.563×10^6	27.401×10^6
GJ	9.172×10^3	8.538×10^3	8.383×10^3	N/A
y_r	1.2960	1.2980	1.2980	1.2690
z_r	1.9080	1.9040	1.9040	1.9690

Table 5.3-25 Section Property Example J-Section Results Comparison

	Section Property Results Comparison				
	FZM(δ=1) vs. FZM(δ=0)	**MWA vs. FZM(δ=0)**	**MWA vs. FZM(δ=1)**	**IDAT vs. FZM(δ=0)**	**MWA vs. IDAT**
EA	-7.93%	-7.82%	0.12%	-8.04%	0.24%
EI_{yy}	-8.11%	-8.03%	0.08%	-4.69%	-3.50%
EI_{zz}	-7.62%	-7.48%	0.15%	-13.46%	6.91%
EI_{yz}	-7.91%	-7.80%	0.12%	-8.34%	0.59%
GJ	-6.92%	-8.60%	-1.81%	N/A	N/A
y_r	0.154%	0.154%	0.00%	-2.08%	2.29%
z_r	-0.210%	-0.210%	0.00%	3.20%	-3.30%

An examination of the comparison in Table 5.3-25 shows that the FZM-9954 baseline method with δ=0 produces section properties that are on average about 8% stiffer than either the FZM-9954 method with δ=1 or the MWA method. The FZM-9954 "δ" parameter controls the calculation of stiffness in Hooke's law and is discussed in section 5.3.2.1. Setting δ=0 assumes that the reactive strains are zero and is consistent with the assumptions made in derivation of the FZM-9954 theory. Setting δ=1 in the FZM-9954 Hooke's law assumes that the reactive stresses are zero and is consistent with the assumptions made in Bernoulli Euler beam theory. The comparison shows that for the balanced symmetric laminates and section properties compared, the FZM-9954 method with δ=1 produces results similar to the Modulus Weighted Area (MWA) method. For the example problem considered, results from the IDAT method vary up to 7% from the FZM-9954 (δ=1) and the MWA method. The IDAT method results variation appears to be a result of the Wojciechowski calculation methodology and is not related to modeling differences.

The FZM-9954 theory and cross-section property methodology provides the most complete and accurate description of laminated composite straight beam stiffness possible within the confines of a conventional beam theory. A unique feature of the FZM-9954 methodology is that material coupling terms for shear and bending are included and accounted for in the section-property theory. The included methodology for calculating warping stiffness of laminated cross-sections is also unique. The beam stiffness matrix calculated using the FZM-9954 methodology includes coupling terms between extension, bending, warping, and torsion that are in general expected to increase accuracy of the analysis results. The increase in accuracy should be most noticeable for beams containing unsymmetric or unbalanced laminates. The fully populated stiffness matrix calculated by FZM-9954 was presented in section 5.3.2.4 as Equation 5.3-30 and is repeated below as Equation 5.3-99.

$$\begin{Bmatrix} N_{XR} \\ M_{YR} \\ M_{ZR} \\ M_{\omega R} \\ T_{XR} \end{Bmatrix} = \begin{bmatrix} EA & ES_y & -ES_z & -ES_\omega & EH \\ ES_y & EI_{yy} & -EI_{zy} & -EI_{\omega y} & EH_c \\ -ES_z & -EI_{zy} & EI_{zz} & EI_{\omega z} & -EH_s \\ -ES_\omega & -EI_{\omega y} & EI_{\omega z} & EI_{\omega\omega} & -EH_q \\ EH & EH_c & -EH_s & -EH_q & GJ \end{bmatrix} \begin{Bmatrix} e_R \\ \kappa_{yR} \\ \kappa_{zR} \\ \tau' \\ \tau_R \end{Bmatrix} \qquad \textbf{Equation 5.3-99}$$

The warping coupling terms in the FZM-9954 stiffness matrix are not in general useful unless warping is included as an independent degree of freedom in the analysis. After eliminating the warping coupling terms , the usable terms in the FZM-9954 stiffness matrix are given by the non-zero terms in the matrix Equation 5.3-100.

$$\begin{Bmatrix} N_{XR} \\ M_{YR} \\ M_{ZR} \\ M_{\omega R} \\ T_{XR} \end{Bmatrix} = \begin{bmatrix} EA & ES_y & -ES_z & 0 & EH \\ ES_y & EI_{yy} & -EI_{zy} & 0 & EH_c \\ -ES_z & -EI_{zy} & EI_{zz} & 0 & -EH_s \\ 0 & 0 & 0 & EI_{\omega\omega} & 0 \\ EH & EH_c & -EH_s & 0 & GJ \end{bmatrix} \begin{Bmatrix} e_R \\ \kappa_{yR} \\ \kappa_{zR} \\ \tau' \\ \tau_R \end{Bmatrix} \qquad \textbf{Equation 5.3-100}$$

The Modulus Weighted Area method (MWA) presented in section 5.3.1 has been used extensively by the aerospace industry for the analysis of laminated beam cross-sections. The Modulus Weighted Area method uses simple formulas with data and results that are naturally organized in a tabular format. Of the three methods presented, the Modulus Weighted Area Method is the most suitable for hand or spreadsheet calculation. Due to the complexity of the equations, the FZM-9954 method is not considered suitable for hand calculations and should be implemented using either a spreadsheet or computer programming method. The MWA method also allows for the use of ply percentage laminate specification, which is an advantage during preliminary analysis and design. The non-zero stiffness matrix terms calculated using the MWA method are identified in Equation 5.3-101. The comparison in Table 5.3-25 indicates that for the example cross-section, the section property values calculated using MWA closely match the corresponding values calculated using the FZM-9954 method if $\delta=1$. Although the compared values are in close agreement, the FZM-9954 stiffness matrix (Equation 5.3-100) contains the warping coefficient and additional coupling terms that are not calculated by the MWA method. The MWA stiffness matrix neglects coupling between extension and bending. The missing extension-bending coupling terms in the MWA stiffness matrix require that axial loads be applied at the cross-section centroid. Eccentric loading must be accounted for by the calculation and application of couples to each end of the beam. Coupling terms between torsion, bending, and extension are also neglected by the MWA method. These torsional coupling terms are typically neglected when the MWA properties are used. The FZM-9954 section property results given in sections 0, 5.3.2.11.17, and 5.3.2.11.18 show examples of the magnitude of the coupling terms when calculated at various locations in the cross-section.

$$\begin{Bmatrix} N_{XR} \\ M_{YR} \\ M_{ZR} \\ M_{\omega R} \\ T_{XR} \end{Bmatrix} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 \\ 0 & EI_{yy} & -EI_{zy} & 0 & 0 \\ 0 & -EI_{zy} & EI_{zz} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & GJ \end{bmatrix} \begin{Bmatrix} e_R \\ \kappa_{yR} \\ \kappa_{zR} \\ \tau' \\ \tau_R \end{Bmatrix}$$

Equation 5.3-101

The primary advantage of the IDAT SECTION/Wojciechowski methodology is the ability to easily represent section features such as fillet radii, chamfers, and bulbs. As with the Modulus Weighted Average method, the IDAT method also allows the use of ply percentage laminate specification. Ply percentage laminate specification is an advantage in preliminary design analysis. The stiffness matrix terms produced by the IDAT SECTION method are similar to the stiffness matrix terms produced by the MWA method except that no value is calculated for GJ. As with the MWA method, neglect of the coupling terms may affect accuracy of the analysis results. Complexity of the equations and requirements for transformations render the IDAT SECTION method marginal for hand calculations. Use of the IDAT SECTION computer program is recommended when performing section property calculations with this Wojciechowski methodology.

The FZM 9954 method is recommended for cross-sections containing unsymmetric or unbalanced laminates. The δ parameter modifies beam stiffness and may be used to correlate analysis predictions with test results. The Modulus Weighted Average method and the IDAT SECTION method are suitable for preliminary design and for analysis of cross-sections with symmetric balanced laminates. All three methods presented may be used to analyze laminates consisting of a single orthotropic layer.

5.4 General Formulas for Translation and Rotation of Coordinate Axes

Translation and rotation of axes is accomplished by moving and rotating the Cartesian coordinate system relative to the cross-section. Consider two arbitrary coordinate systems (y, z) and (y^*, z^*) as shown in Figure 5.4-1. The coordinates of the origin of the starred system in the unstarred system are (y_o, z_o), and the rotation angle of the starred system relative to the unstarred system is θ.

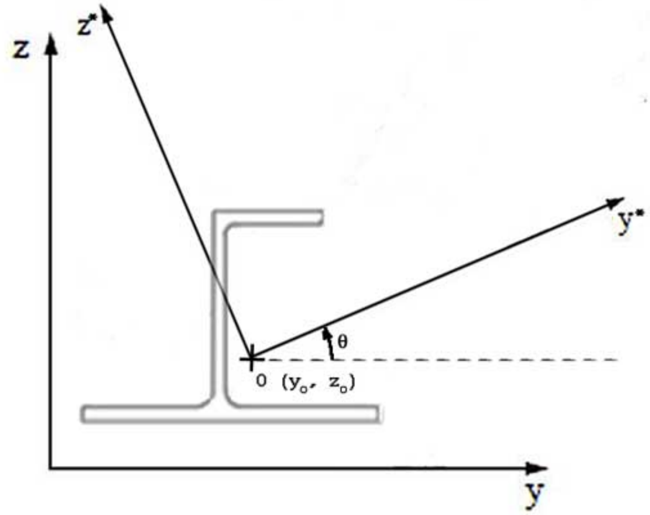

Figure 5.4-1 Coordinate Transformation and Rotation

Formulation of the modulus weighted cross-section properties presented in this section includes a reference point (y_R, z_R) that in effect serves as a method for translation of axis in the y, and z coordinate directions. The reference point (y_R, z_R) is equivalent to a translated coordinate origin ($y_0 = y_R$, $z_0 = z_R$) with a rotation angle "θ" of 0 degrees. Translation of the section properties is easily accomplished by changing the reference point to the desired value and re-calculating properties using the formulas provided. Translation and rotation of the reference coordinate axis may be accomplished using Equation 5.4-2 through Equation 5.4-7 that follow.

The coordinates of a point on the section contour in the starred system are related to coordinates in the unstarred system by Equation 5.4-1

$$\begin{aligned}
\bar{y}^*(s) &= (\bar{y}(s) - y_0)\cos(\theta) + (\bar{z}(s) - z_0)\sin(\theta) \\
\bar{z}^*(s) &= -(\bar{y}(s) - y_0)\sin(\theta) + (\bar{z}(s) - z_0)\cos(\theta) \\
\alpha^*(s) &= \alpha(s) - \theta
\end{aligned}$$

Equation 5.4-1

$EA,\ EH,\ and\ GJ$ are invariant under coordinate translation and rotation. ES_ω will always equal zero for warping properties calculated at the principal sectorial origin (as recommended) and thus will also be invariant under translation and rotation of axis. The Shear Center and the Principal sectorial origin are considered fixed relative to the cross-section contour. $EI_{\omega\omega},\ and\ EH_q$ are invariant with respect to translation and rotation provided properties are calculated at the Shear Center and the Principal Contour Origin is used.

Coordinate transformation and rotation equations for the remaining cross-section properties are obtained by substitution of the transformation relation Equation 5.4-1 into the cross-section properties given by Equation 5.3-31 through Equation 5.3-45.

Modulus Weighted First Moments of Area

$$ES_{y^*} = (ES_y - z_0 EA)\cos(\theta) - (ES_z - y_0 EA)\sin(\theta)$$
$$ES_{z^*} = (ES_y - z_0 EA)\sin(\theta) + (ES_z - y_0 EA)\cos(\theta)$$

Equation 5.4-2

Modulus Weighted Bending-Torsional Coupling Stiffness

$$EH_{c^*} = (EH_c - z_0 EH)\cos(\theta) - (EH_s - y_0 EH)\sin(\theta)$$
$$EH_{s^*} = (EH_s - z_0 EH)\sin(\theta) + (EH_z - y_0 EH)\cos(\theta)$$

Equation 5.4-3

Modulus Weighted Second Sectorial Moments

$$EI_{\omega y^*} = (EI_{\omega y} - z_0 ES_\omega)\cos(\theta) - (EI_{\omega z} - y_0 ES_\omega)\sin(\theta)$$
$$EI_{\omega z^*} = (EI_{\omega y} - z_0 ES_\omega)\sin(\theta) + (EI_{\omega z} - y_0 ES_\omega)\cos(\theta)$$

Equation 5.4-4

Modulus Weighted Moments of Inertia

$$\begin{aligned} EI_{y^*y^*} &= (EI_{yy} - 2\,z_0 ES_y + z_0^2 EA)\cos^2(\theta) \\ &\quad + (EI_{zz} - 2\,y_0 ES_z + y_0^2 EA)\sin^2(\theta) \\ &\quad - 2\,(EI_{yz} - z_0 ES_z - y_0 ES_y + y_0 z_0 EA)\sin(\theta)\cos(\theta) \end{aligned}$$

Equation 5.4-5

$$\begin{aligned} EI_{z^*z^*} &= (EI_{yy} - 2\,z_0 ES_y + z_0^2 EA)\sin^2(\theta) \\ &\quad + (EI_{zz} - 2\,y_0 ES_z + y_0^2 EA)\cos^2(\theta) \\ &\quad + 2\,(EI_{yz} - z_0 ES_z - y_0 ES_y + y_0 z_0 EA)\sin(\theta)\cos(\theta) \end{aligned}$$

Equation 5.4-6

Modulus Weighted Product of Inertia

$$\begin{aligned} EI_{y^*z^*} &= (EI_{yy} - EI_{zz} - 2\,z_0 ES_y + 2\,y_0 ES_z + z_0^2 EA - y_0^2 EA)\sin(\theta)\cos(\theta) \\ &\quad + 2\,(EI_{yz} - y_0 ES_y - z_0 ES_z + y_0 z_0 EA)\ (\cos^2(\theta) - \sin^2(\theta)) \end{aligned}$$

Equation 5.4-7

5.4.1 Rotation angle to the Principal Coordinate System

If the coordinate origin (y_0, z_0) is placed at the cross-section centroid and the angle of rotation "θ" is such that the modulus weighted product of inertia in the starred system ($EI_{y^*z^*}$) is zero, then the modulus weighted moments of inertia ($EI_{y^*y^*}$, and $EI_{z^*z^*}$) will assume maximum and minimum values. This coordinate system orientation and location is known as the principal coordinate system and determines the flexural axes of the beam cross-section. The transformation angle "θ" to the principal coordinate system is obtained by setting Equation 5.4-7 equal to zero, substituting (y_c, z_c) from Equation 5.3-83 for the coordinate origin (y_0, z_0), and solving for θ. The solution for calculating the rotation angle to the Principal Coordinate System is provided as Equation 5.4-8.

$$\tan(2\,\theta) = \frac{2\left(EI_{yz} - y_c z_c EA\right)}{\left(EI_{zz} - y_c^2 EA\right) - \left(EI_{yy} - z_c^2 EA\right)} \qquad \textbf{Equation 5.4-8}$$

5.4.2 Branch offsets and Built-Up Cross-Sections

Branch segment endpoints can be offset from the cross-section junctions. Offsets allow the calculation of properties for complex "built-up" cross-sections. Cross-section features such as fastener holes, cut-outs, and laminate thickness changes may be modeled using branch offsets. Offset branch segments maintain the connectivity required for calculation of cross-section warping properties and allow for the modeling of a continuous outer mold line. Offsets as described in this section are intended for use in the FZM-9954 section-property method and may also be used with the Modulus Weighted Method.. Branch segment offsets are implemented by repositioning the branch segment end points, without modifying segment connectivity. This concept is analogous to beam end point offsets in NASTRAN.

Normal direction offset vectors for branch segment coordinates are equal at both segment end points and are described by a signed scalar value that indicates the magnitude of offset vector in the +ζ coordinate direction(a negative value for ζ indicates an offset in the –ζ direction). A branch segment is shown with a normal offset in Figure 5.4-2.

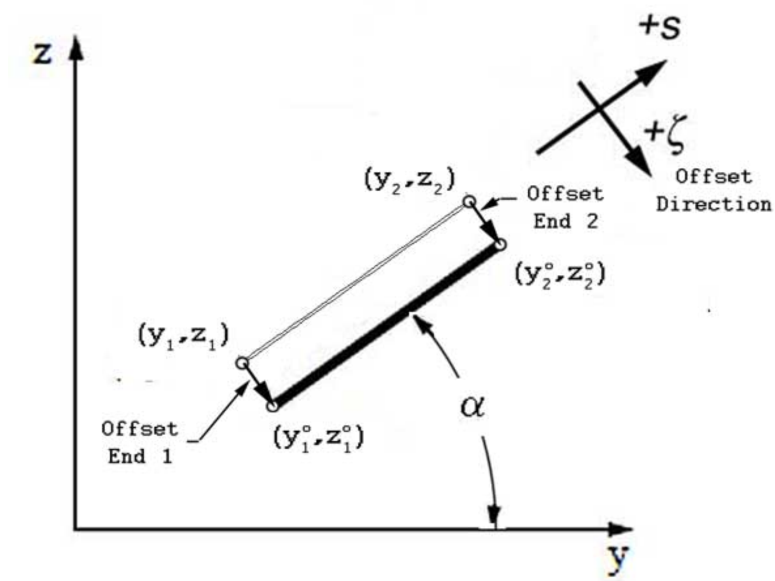

Figure 5.4-2 Branch Segment Offsets

Branch normal offset coordinates for segment endpoints are calculated using Equation 5.4-9

$$
\begin{aligned}
y_1^{on} &= y_1 + O_N \sin(\alpha) \\
z_1^{on} &= z_1 - O_N \cos(\alpha) \\
y_2^{on} &= y_2 + O_N \sin(\alpha) \\
z_2^{on} &= z_2 - O_N \cos(\alpha)
\end{aligned}
$$

Equation 5.4-9

Where:

- O_N is the offset magnitude in the branch normal (+ζ) direction.
- *Values for sin(α)* and *cos(α)* are calculated using a four quadrant value for *α.*

Segment end points may also be offset in the "s" coordinate direction to facilitate modeling of holes, or to prevent overlap of material at segment junctions. Coordinate offsets in the "s" direction are calculated using Equation 5.4-10

$$
\begin{aligned}
y_1^{os} &= y_1 + O_{s1} \cos(\alpha) \\
z_1^{os} &= z_1 + O_{s1} \sin(\alpha) \\
y_2^{os} &= y_2 + O_{s2} \cos(\alpha) \\
z_2^{os} &= z_2 + O_{s2} \sin(\alpha)
\end{aligned}
$$

Equation 5.4-10

Where:

- O_{s1} is that offset magnitude in the branch "s" direction at end 1
- O_{s2} is that offset magnitude in the branch "s" direction at end 2
- *Values for sin(α)* and *cos(α)* are calculated using a four quadrant value for *α.*

Offset branch coordinates calculated using Equation 5.4-9 and Equation 5.4-10 are used in place of (y_1, z_1), and (y_2 ,z_2) in the cross-section property geometric integral Equation 5.3-68 through Equation 5.3-82.

An example built up section modeled with a combination of segment offsets and end offsets is shown in Figure 5.4-3. There are six cross-section junctions labeled with characters "a" through "f". There are five branch segments labeled 1through 5. Branch segments 2, and 3 are a combination of skin and the lower flange of an angle section, and are offset in the normal direction to place the combined material thickness at the proper "z" coordinate. End 2 of segment 2 and end 1 of segment 3 are offset in the "s" direction at junction "c" to represent a fastener hole in the cross-section. End 1 of segment 5 is offset from junction "d" to reduce overlap with segment 3.

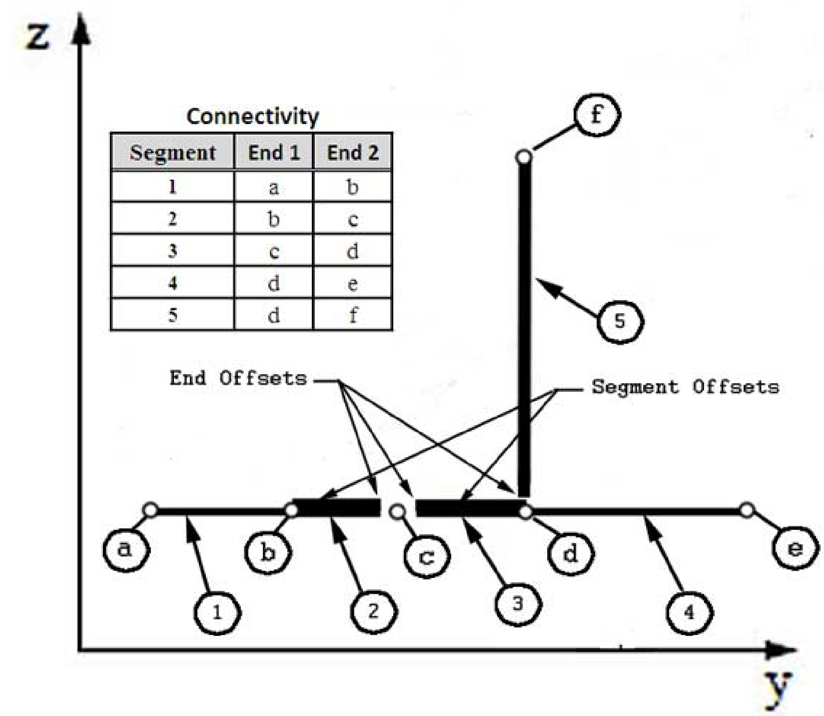

Figure 5.4-3 Built Up Cross-Section with Offsets

5.5 Unix/PC-Based Calculation

Another common approach for section property computation is through the use of stand-alone analysis tools or computational aids. The controlled tool for cross-section property calculation is IDAT SECTION. A variety of uncontrolled MS Excel- and Mathcad-based computational aids are available on individual Programs. Individual analysts are required to verify the accuracy of any such uncontrolled aids prior to initial use.

5.5.1 IDAT SECTION

The IDAT tool for cross-section property calculation is named SECTION. A complete description of SECTION’s features is found in Reference 5-18. The IDAT SECTION equations and methodology are described in section 5.3.3 of this manual. The main graphical user interface for IDAT SECTION is shown in Figure 5.5-1. The application window shows a complex multi-material cross-section modeled with fastener holes. Calculated geometric and modulus weighted section properties are listed on the right side of the application window. Distinct sub-elements of the built-up cross-section are created either from direct selection of boundary points, or from boundary points created by selecting one of the various shape buttons on the left side of the application window. Each distinct area sub-element of the cross-section shape is assigned a distinct material stiffness. Properties are calculated by selecting one of the “section analysis” menu options from the main “Analysis” top level menu.

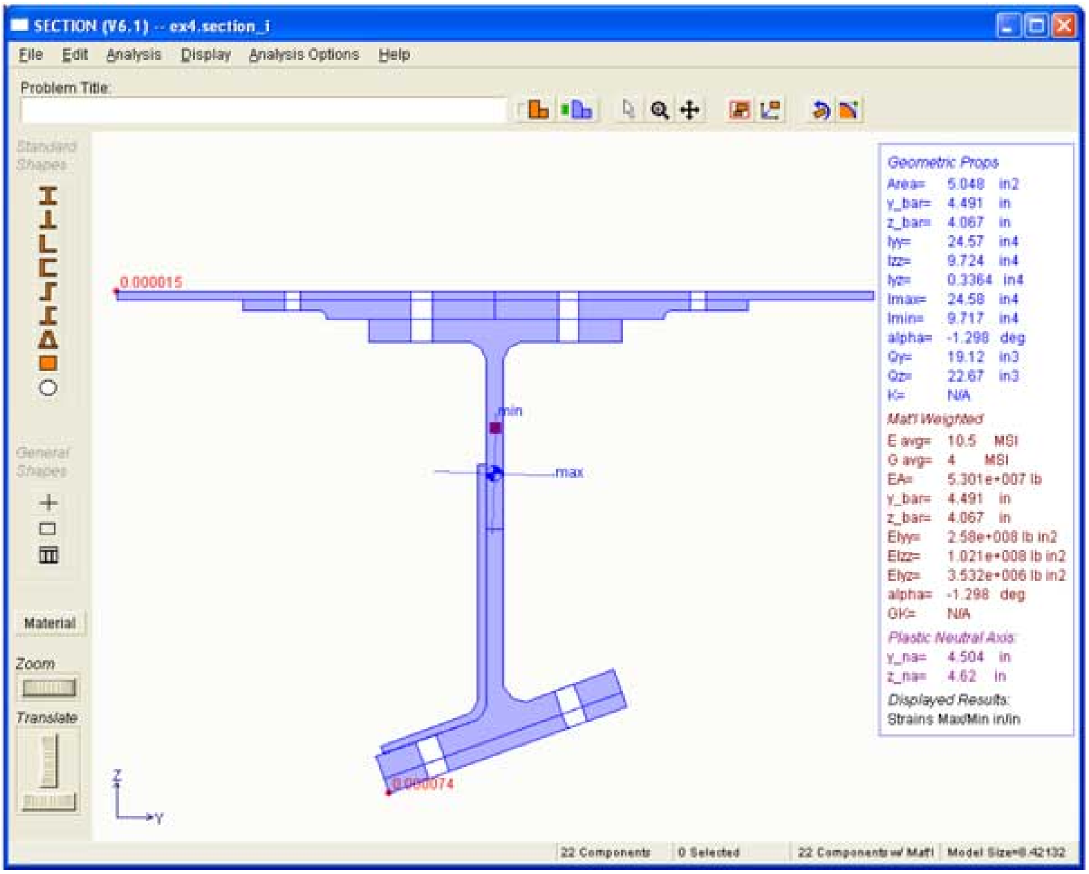

Figure 5.5-1 IDAT/SECTION Analysis Tool

5.5.2 CAD-Based Calculation

Various CAD programs, as well as CATIA, are available for the calculation of some section properties for use in analysis. At various times, these programs have not been clear about the information being provided and whether or not axes have been appropriately rotated and/or translated. The analyst is cautioned to investigate the limitations of the software and to perform checks to ensure an understanding of what information is being calculated.

5.6 Laminated Cross-Section Design Guidelines

5.6.1 Manufacturing Constraints

Cross section geometry, segment dimensions, laminate stacking sequences, and material selection will be driven by strength, stiffness, design, manufacturing, and business criteria. Modulus weighted cross-section properties such as EA, EI_{yy}, EI_{zz}, and GJ specify the beam stiffness. Cross-section properties combined with cross-section geometry and material allowables determine the strength properties of a beam. Design requirements may include penetrations and cutouts in the cross-section, as well as local attachment loads independent of the beam axial loads. Attachment loads may include spar caps that serve as a splice or termination for skin or web sections. Manufacturing criteria limit the practical methods for creating a two dimensional cross-section shape from laminated composites and thus limit the allowable combinations of laminate stacking sequence for built-up cross-section segments. Business criteria include fabrication cost and tooling requirements as factors influencing the design of a laminated composite cross-section. Common cross-section geometries for laminated composite beams are shown in Figure 5.6-1.

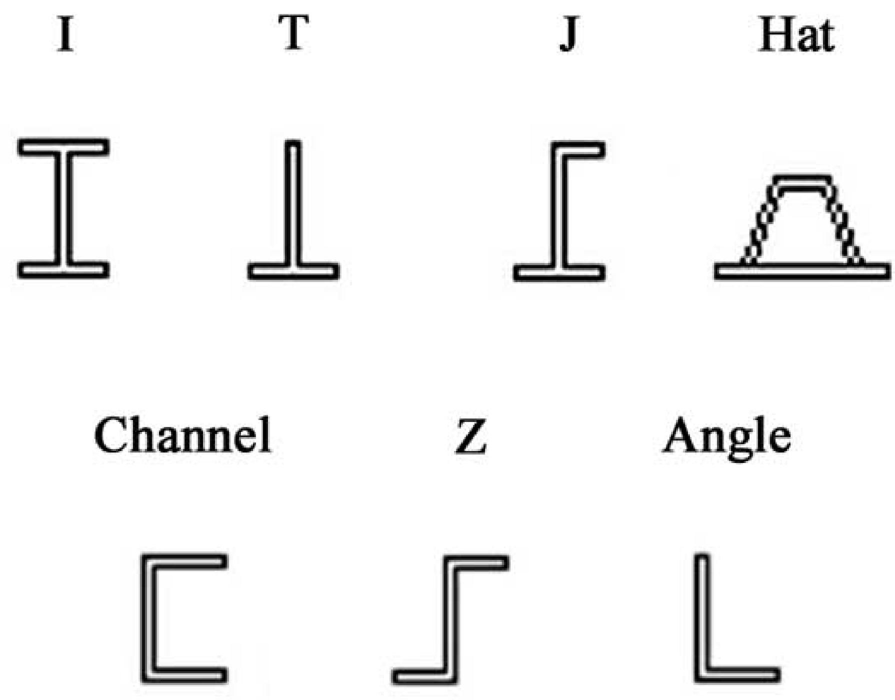

Figure 5.6-1 Common Laminated Composite Beam Cross-Section Shapes

Practical composite spar/beam cap geometries and stacking sequences are strongly influenced by manufacturability considerations. A typical “I” cross-section assembly is shown in Figure 5.6-2. Note that the flange stacking sequences are formed from a combination of half the shear web stacking sequence plus a “cap” laminate. The “noodle” is typically unidirectional pre-preg fiber and is needed to fill the void caused by bend radius limits on the shear web half laminate.

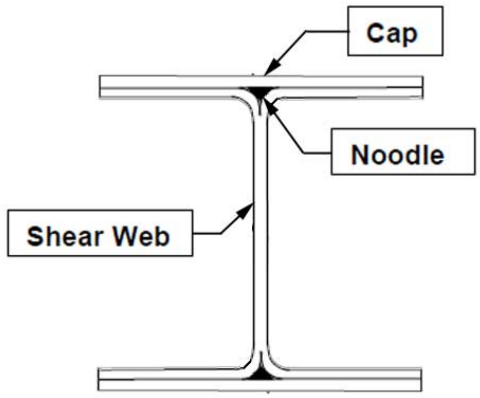

Figure 5.6-2 Manufacturability and Stacking Sequence

5.6.2 General Guidelines for Design of Laminated Composite Beam Cross-Sections:

- Laminate configuration and fiber placement guidelines outlined in section 2.3 of this manual are applicable to all segments of the beam cross-sections. These laminate guidelines include specific requirements on minimum thickness, stacking sequence, and other criteria.
- Segment b/t ratio should be less than 10.
    - Where "b" is the length of the segment and "t" is the segment thickness.
- Penetrations and cutouts larger than .25" diameter should be accounted for in all cross-section analysis.

5.6.3 Preliminary Design Procedure

Given desired *EA*, *EI*, and/or *GJ;* and estimated axial and bending loads

- Select a cross-section shape from the candidates in Figure 5.6-1
- Select a material
- Assume a cap/shear web section assembly similar to that shown for the "I" section in Figure 5.6-2
- Assume a (50/40/10) laminate for cap segments
- Assume (100/0/0) uni-tape properties for noodles
- Assume (10/80/10) for shear webs
- match skin laminate for flanges bonded to skin
- Assume b/t= 10
- Size segments to desired *EA*
- Check *EI*, *GJ*, and strains due to applied loads
- Iterate on b, t, and laminate for each segment until geometry is acceptable

5.7 Section Properties for Sandwich Construction

Reserved for future use.