7 Bending of Laminated Plates and Shells

The purpose of this chapter is to provide information and guidance for analyzing laminated plates and shells. Classical Lamination Plate Theory (CLPT), discussed in Section 4, describes the stress/strain state of a panel along a line that is always perpendicular to the mid-surface of the plate. This section introduces plates and shells bending theory which focuses on determining deflections, forces and moments due to bending at all points on a plate or a shell. Topics include plate bending theory, bending of laminated shells, membrane effects and shear deformation. The focus is on rectangular panel shapes. However, highly irregular panels have to be analyzed using Finite Element Analysis (FEA).
The stability analysis of laminated plates and shells will be discussed later in Section 8. Sandwich panels are addressed in Section 14.

7.1 References

7-1. Flansburg, L.K., *Lockheed Martin Aeronautics Procedures Manual 4057 Metallic Structural Analysis Manual*, Lockheed Martin Aeronautics, Fort Worth, TX, 2008
7-2. Staff, *LMTAS Policy and Methods for Stress Analysis*, ST-94-08, 1997.
7-3. Staff, *LTV Composite Structures Analysis Manual*, LTV Aircraft Products Group, Grand Prairie, TX, August, 1984.
7-4. Kassapoglou, C., *Design and Analysis of Composite Structures with Applications to Aerospace Structures*, John Wiley & Sons, Ltd., 2010.
7-5. Reddy, J.N., *Mechanics of Laminated Composite Plates – Theory and Analysis*, CRC Press, 1997.
7-6. Vinson, J.R. and Sierakowksi, R.L., *The Behavior of Structures Composed of Composite Materials*, Martinus Nijhoff Publishers, 1987.
7-7. Jones, R.M., *Mechanics of Composite Materials*, 2nd Edition, Brunner-Routledge, 1999.

7-8. Whitney, J.M., *Structural Analysis of Laminated Anisotropic Plates*, Technomic, 1987.
7-9. Hyer, M.W., *Stress Analysis of Fiber-Reinforced Composite Materials*, McGraw-Hill, 1997.
7-10. Vinson, J.R. and Chou, T.W., *Composite Materials and Their Use in Structures*, Wiley, 1975.
7-11. Kollár, L.P. and Springer, G.S., *Mechanics of Composite Structures*, 2003 Cambridge University Press.
7-12. Hartman, T.B., Hyer, M.W. and Case, S.W., *Stress Recovery in Composite Laminates*, AIAA 2011-1856, 52nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference.
7-13. Ashton, J.E., *Laminated Trapezoidal Plate Analysis*, FZM-5549, 1970.
7-14. Wilkins, D.J., *Anisotropic Curve Panel Analysis*, FZM-5567, 1973.
7-15. Ashton, J.E., *Anisotropic Plate Analysis – Boundary Conditions*, J. Composite Materials, Vol. 4, April 1972, pg. 162
7-16. Ashton, J.E., *An Analogy for Certain Anisotropic Plates*, J. Composite Materials, Vol. 3, April 1969, pg. 355
7-17. Chou, P.C., Carleone, J. and Hsu, C.M., *Elastic Constants of Layered Media*, J. Composite Materials, Vol. 6, pg. 80.
7-18. Pagano, N.J., *Exact Moduli of Anisotropic Laminates*, Mechanics of Composite Material, Sendeckyj, G.P., 1974.
7-19. Reed, D.L., *Advanced Composite Technology Point Stress Laminate Analysis*, FZM-594.
7-20. F-22 Structural Development Tests Coupon Tests Results and Evaluation, Vol. 5, 5PPYA081-05, Dec. 1994.
7-21. Finite Element Analysis Procedures Manual, PM 4061 (in preparation).

Table 7.1-1 Symbols and Nomenclature

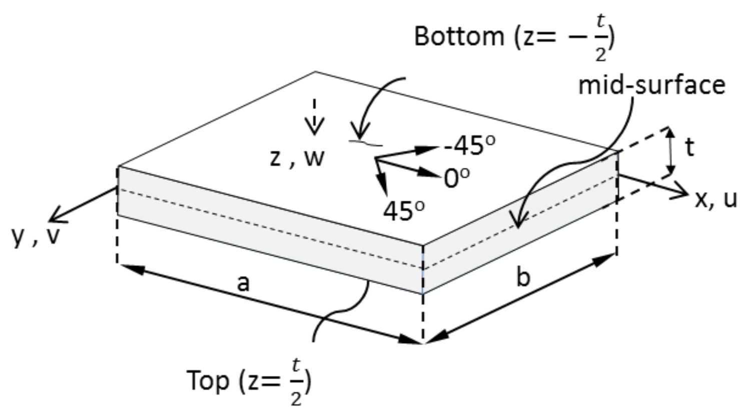
Symbol	Description	Units
a	Length of panel in x direction	in
α,β,Γ_{mn}	Coefficients of the double Fourier series	
$A_n,B_n,C_n,D_n,E_n,H_i,\widehat{Q_n}$	Displacement constants	
A_{ij}	Laminate in-plane extensional stiffness matrix	lb/in
b	Length of panel in y direction	in
B_{ij}	Laminate in-plane bending coupling stiffness matrix	lb
CSW	Crow-foot Satin Weave	
D_{ij}	Laminate bending stiffness matrix	in-lb
DLL	Design Limit Load	
Δp	Pressure differential	lb/in^2
E_b	Bending modulus of laminate	lb/in^2
γ_{xz}	Engineering shear strain in the xz plane	in/in
γ_{yz}	Engineering shear strain in the yz plane	in/in
g	Acceleration due to gravity	in/s^2
G_{13},Q_{55}	Transverse shear modulus of the ply 13 plane	lb/in^2
G_{23},Q_{44}	Transverse shear modulus of the ply 23 plane	lb/in^2
G_{xz}	Effective transverse shear modulus of the laminate in the xz plane	lb/in^2
G_{yz}	Effective transverse shear modulus of the laminate in the yz plane	lb/in^2
IDAT	Integrated Detailed Analysis Toolset	
L	Length	in
λ_1,λ_2	Roots of characteristic equation	

m	Number of double Fourier series terms in the x direction	
M_n	Running moment along an edge perpendicular to the n axis	in-lb/in
M_x	Running moment along an edge perpendicular to the x axis	in-lb/in
$M_{x\text{-PRESS}}$	Running moment on an edge perpendicular to the x direction determined using PRESS	in-lb/in
M_{xy}	Running twisting moment	in-lb/in
M_y	Running moment along an edge perpendicular to the y axis	in-lb/in
$M_{y\text{-PRESS}}$	Running moment on an edge perpendicular to the y direction determined using PRESS	in-lb/in
$\frac{\partial M_{ns}}{\partial s} + Q_n$	Kirchhoff condition	lb/in
n	Number of double Fourier series terms in the y direction	
N_x	Running axial load in x direction	lb/in
$N_{x\text{-FEA}}$	Running axial load in x direction using FEA	lb/in
$N_{x\text{-PRESS}}$	Running axial load in x direction using PRESS	lb/in
N_{xy}	Running shear load in xy plane	lb/in
N_y	Running axial load in y direction	lb/in
N_n	Running axial load in n direction	lb/in
N_{ns}	Running shear load in s direction	lb/in
p, p_o	Pressure	lb/in^2
Plate	A plate is a structure bounded by two flat surfaces on the top and bottom separated by the thickness of the plate	
Q_n	Coefficient for various transverse loading profiles	lb/in^2
Q_x	Resultant shear force along a face perpendicular to x axis	lb/in
Q_y	Resultant shear force along a face perpendicular to y axis	lb/in
$\bar{Q}_{ij}^{(k)}$	Transformed reduced stiffness matrix for k^{th} ply	lb/in^2
R	Radius of curvature of the panel	in
RTR	Ratio-to-Requirement	
s	Panel aspect ratio = a/b	
SPAM	Stiffened Panel Modeler	
Shell	A shell is a structure bounded by two curved surfaces on the top and bottom separated by the thickness of the shell	
σ_{xx}	Normal stress in the x direction in the global coordinate system	lb/in^2
σ_{yy}	Normal stress in the y direction in the global coordinate system	lb/in^2
σ_{11}	Stress in the 1 direction in the material coordinate system	lb/in^2
σ_{22}	Stress in the 2 direction in the material coordinate system	lb/in^2
t	thickness	in
τ_{xy}	Shear stress in the global coordinate system	lb/in^2
τ_{12}	Principal shear stress in the material coordinate system	lb/in^2
u	In-plane axial displacement in the x direction	in
u_n^o	In-plane displacement in n direction	in
u_s^o	In-plane displacement in s direction	in
v	In-plane axial displacement in the y direction	in

w	Displacement in z direction	in
$w_{alllowable}$	Allowable maximum displacement	in
w_{max}	Maximum displacement	in
$\frac{\partial w}{\partial n}$	Edge rotation	
x	x axis coordinate	in
y	y axis coordinate	in
z	z axis coordinate	in
z_{k-1}	z location for the k^{th} ply, k=1 is the bottom ply. (Figure 4.3-2)	in
z_{k-1}	z location for the k-1 ply	in

7.2 Definitions

Composite laminated plates are made of bonded elastic layers each of which may have different properties. The plate is bounded by two flat surfaces on the top and bottom. The surface that is located equal distance between the top and the bottom is known as the middle surface. The plate thickness is assumed to be small (the ratio of thickness to the next smallest dimension varies between 0.1 and 0.0125) compared to the two in-plane dimensions. The coordinate system, geometric parameters and displacements for a laminated plate are depicted in Figure 7.2-1 and described in Table 7.1-1.



u,v,w are displacements in x,y and z directions respectively
Figure 7.2-1 Plate Geometry and Displacements

The x, y and z axes define the global plate coordinate system. The 0° and ±45° fiber directions are also depicted in the figure. The material coordinate system, which defines the fiber direction, is defined by axis 1 and 2 and is shown in Figure 7.2-2.

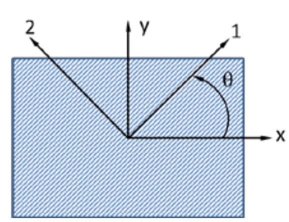


Figure 7.2-2 Material Coordinate System (1-2) and Global Coordinate System (x-y)

As discussed in Section 4 the extensional stiffness matrix A, extensional bending coupling stiffness matrix B and bending stiffness matrix D are defined in global coordinate system. It should be noted that the sign conventions used in this section, which is consistent with IDAT tools, could be different than the sign conventions used in Finite Element Analysis (FEA) tools. The force and moment sign conventions used in IDAT tools, are discussed in Section 4.4.1.4.

7.3 General Design Guidelines

The design guidelines for a composite laminate are discussed in Section 2 of this manual. The analyst should give special attention in choosing the thickness of the laminate since the laminate bending and stability are sensitive to this parameter. For composite materials that are autoclave cured and available in the IDAT composite material database the nominal thickness given in IDAT/MATUTL times the number of plies can be used for stress analysis purposes. However if the manufacturing process is different or if a new material system is used then the nominal thickness and tolerance of the part should be determined in consultation with a Material and Processes representative. In situations where the part thickness tolerances are difficult to control (High Variability Structure) care should be exercised in determining the analysis part thickness. Section 2.4.3.2 and PM4057 Section 2.6.2 and 2.6.3, Reference 7-1, provide guidelines for determining analysis thickness assumptions for structures including High Variability Structure.

The stress analyst should also be careful to ensure that the thickness and tolerance of the laminates in the curved portion and regions of ply buildup is achievable and the correct thickness is used in the analysis. This is ensured by being aware of the manufacturing processes that will be used and also by ensuring that the critical dimensions are correctly indicated in the drawing. This requires coordination with the design and manufacturing engineers. Furthermore, the analyst has to ensure that the tolerance of the total thickness of the laminate is within 8 to 10% of the nominal laminate thickness.

Often program specific external smoothness requirements govern the allowable deflection of the aircraft surface under load. The program may specify a maximum allowable plate deflection or surface waviness for a specific location. The program may zone the aircraft with a maximum allowable deflection specific to each zone.

The panel deflection or surface waviness is determined at Limit Load, *i.e.*, at 100 % Design Limit Load (DLL). The allowable maximum deflections in each zone are typically specified for a reduced flight envelope. Outside the reduced envelope other deflection criteria may be specified. Surface waviness is required to satisfy aerodynamic and LO requirements. Specific program guidelines should be followed when implementing this requirement.

In general, all composite structural laminates in bending shall be sized such that all strength related margins are greater than zero at 150% DLL. The minimum thickness for laminates on the external surface must satisfy producibility requirements and a number of functional criteria outlined in Section 2.3.1 including lightning strike, fuel containment and water intrusion. In addition to the strength requirements of this section, the thickness requirements may be affected by the durability and damage tolerance or repairability requirements of Section 4.

7.3.1 Loads

All loads such as in-plane and pressure loads should be accounted for when performing deflection /moment analysis or stability analysis. When determining in-plane deflections, reaction forces and moments due to pressure loads, the effect of membrane loads should be considered. This is discussed further in Section 7.6. Furthermore, when calculating the pressure, the net pressure load should be considered. Net pressure is the sum of the pressure on either surface of the panel. When adding pressure loads to the maneuver loads, case-consistent pressure loads should be used regardless of whether or not it adds-to or relieves maneuver loads. However, the lowest case-consistent pressure should be used when the pressure relieves the maneuver load and the highest case-consistent pressure load should be used when it adds to the maneuver load. Some programs require pressures to be ignored if they are relieving. Program guidance should be referenced for the appropriate approach.

7.4 Laminated Plate Bending Theory

The forces, moments and pressure loads acting on a plate are shown in Figure 7.4-1 and Figure 7.4-2.

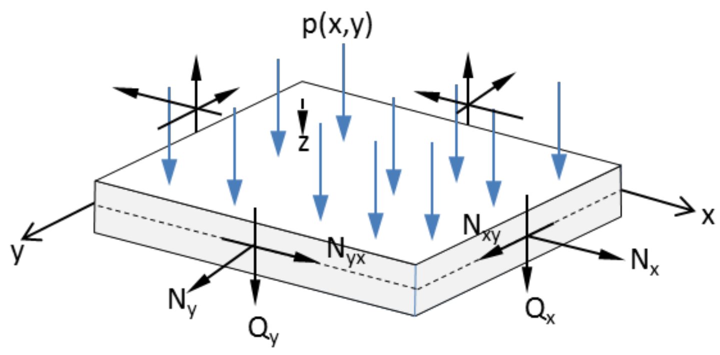


Figure 7.4-1 Forces and Pressure Loads Acting on the Plate

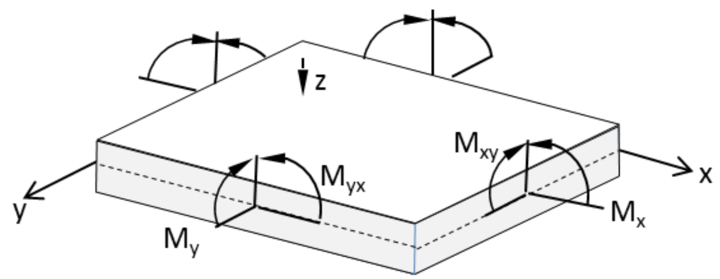


Figure 7.4-2 Moments Acting on the Plate

The symbols are described in Table 7.4-1.

Table 7.4-1 Definition of Symbols in **Figure 7.4-1** and **Figure 7.4-2**.

Symbol	Description
N_x (lb/in)	In-plane running axial load in x direction
$N_{xy} = N_{yx}$ (lb/in)	In-plane running shear load in xy plane
N_y (lb/in)	In-plane running axial load in y direction
Q_x (lb/in)	Resultant shear force on the face perpendicular to x axis
Q_y (lb/in)	Resultant shear force on the face perpendicular to y axis
p (psi)	Pressure in z direction
M_x (in-lb/in)	Running moment along edge perpendicular to x axis
M_y (in-lb/in)	Running moment along edge perpendicular to y axis
M_{xy} (in-lb/in)	Running twisting moment about x and y axis

In plate bending theory the three dimensional solid plate is reduced to a two dimensional mid-surface through Kirchhoff's assumptions (References 7-5 and 7-9) that are discussed below in Table 7.4-2.

Table 7.4-2 Kirchhoff's Assumptions

Assumptions	Consequences
Lines normal to the mid-surface remain straight before and after deformation. Also the length in the thickness direction does not change.	The transverse displacements are independent of the transverse direction (w(x,y) = 0) and the strain in the transverse direction is zero ($\varepsilon_{zz} = 0$).
The transverse normal to the mid-surface remains normal before and after deformation.	This implies that the transverse shear strains are zero ($\gamma_{xz}=\gamma_{yz}=0$).

Kirchhoff's assumptions, which are valid for thin plates, ignore the transverse shear and normal stresses ($\sigma_{xz}=\sigma_{yz}=\sigma_{zz}$) that may be important for composites since the transverse strength of the composites are lower than the in-plane strength. Therefore, as the plate thickness increases, the above assumptions are no longer valid and transverse shear deformation effects have to be considered as discussed in Section 7.7.

In addition to the above assumptions it is also assumed that each individual ply in the laminate stack is orthotropic and the principal fiber direction need not be aligned with the global plate axis.

Employing the above assumptions in addition to assuming that the laminate is symmetric ($B_{ij}=0$), balanced ($A_{16}=A_{26}=0$) and specially orthotropic ($D_{16}=D_{26}=0$) results in significant simplification. Furthermore, ignoring in-plane loads and inertia terms, the governing differential equation for the plate in the transverse direction can be written as follows (Reference 7-8).

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = p$$ **Equation 7.4-1**

where,

A_{ij} = in-plane stiffness, lb/in (Equation 4.4-14)
B_{ij} = in-plane bending coupling, lb (Equation 4.4-15)
D_{ij} = bending stiffness, in-lb (Equation 4.4-16)

The solution to the governing equation depends on the boundary conditions. At least one boundary condition out of the pairs of boundary condition defined in Table 7.4-3 should be specified along the plate edge. The local coordinate system is defined in Figure 7.4-3.

Table 7.4-3 Boundary Conditions Pairs and Definition of Symbols

$u_n^o; N_n$	$u_s^o; N_{ns}$	$\frac{\partial w}{\partial n}; M_n$	$\frac{\partial M_{ns}}{\partial s} + Q_n; w$
Symbol		Description	
u_n^o		In-plane displacement in n direction	
u_s^o		Tangential displacement in s direction	
N_n		In-plane running load in n direction	
N_{ns}		In-plane running shear load in s direction	
$\frac{\partial w}{\partial n}$		Edge rotation	
M_n		Moment about edge perpendicular to n axis	
w		Transverse displacement	
$\frac{\partial M_{ns}}{\partial s} + Q_n$		Kirchhoff condition	

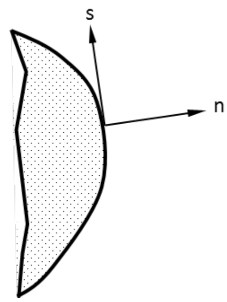


Figure 7.4-3 Normal (n) and Tangential (s) Direction Along the Plate Boundary.

The different boundary conditions that can exist along the plate edges are shown in Table 7.4-4.

Table 7.4-4 Boundary Conditions

Simply-Supported	$N_n = N_{ns} = w = M_n = 0$
Hinged-Free in the normal direction	$N_n = u_s^o = w = M_n = 0$
Hinged-Free in the tangential direction	$N_{ns} = u_n^o = w = M_n = 0$
Clamped	$u_s^o = u_n^o = w = \frac{\partial w}{\partial n} = 0$
Free	$N_{ns} = M_n = N_n = \frac{\partial M_{ns}}{\partial s} + Q_n = 0$

There are two general methods to solve the governing Equation 7.4-1, the exact direct methods and the approximate energy methods. The direct methods provide solution for a very limited number of cases such as the simply supported boundary condition on all edges or opposite edges simply supported. For all other boundary conditions the approximate energy methods are used. A good discussion on the solution methods is provided in Reference 7-6.

A discussion on the solution of the governing Equation 7.4-1 is provided for different cases below.

The thickness of the plate and shell is an important parameter to consider in the design process for composite laminates since the transverse shear effects could have a significant effect on plate/shell deformation. Transverse shear effects are further considered in Section 7.7.

7.4.1 Bending Factor

Tests have shown that the static strength of unnotched laminates in bending is higher than the unnotched tension or compression allowable. Therefore, when performing bending analysis of unnotched laminates the applied moments are divided by a bending factor to account for the increase in the unnotched strain capability. This factor is determined by comparing the unnotched laminate bending strength, determined using a four point bend test, to axial strength based on tension and compression coupon tests. The bending factor has typically been specified by individual Programs. For the material systems used on F-22 and F-35 a factor of 1.2 has been used and is recommended in lieu of specific program guidance. This bending factor of 1.2 was obtained by averaging the bending factors of all laminates at a given environment across all material types and forms and then determining the lower bound.

The bending factor should be applied only when using classical laminated plate theory and a Maximum Strain failure criterion, such as in IDAT/SQ5. Currently, this factor is applied by the analyst to the SQ5 input moments. Alternatively, if sufficient test data were available, individual bending factors could be determined separately for each material form and environment and included as an empirical factor in each IDAT material file. If material specific factors are built into IDAT material files then SQ5 could be modified to automate the process for future Programs.

For other IDAT tools such as IBOLT and CDADT bending factors are already included internal to the analysis software. Details of each empirical derivation and implementation are discussed in Section 11 and Section 12.

7.4.2 Cylindrical Bending of Laminated Plate

For cylindrical bending the plate is assumed to be long in one direction (Figure 7.4-4) such that it has a very high length-to-width ratio which allows the two-dimensional problem to be simplified to a one-dimensional problem.

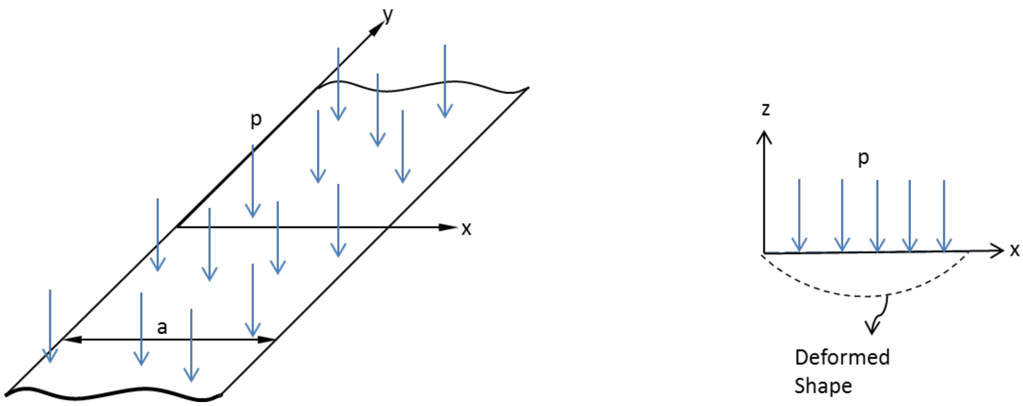


Figure 7.4-4 Plate (Infinite Strip) Subjected to Cylindrical Bending

This analysis, also known as cylindrical bending analysis, can be used to solve for displacements and moments in plates where the supports of at least one pair of opposite edges are located fairly far apart from each other so that there is no interaction between the boundaries. This condition is satisfied if the following condition is true (Reference 7-11).

$$\frac{b}{a} > 3\left(\frac{D_{11}}{D_{22}}\right)^{\frac{1}{4}}$$

where,
a is the length of the panel along x axis, in
b is the length of the panel along y axis, in
D_{ij} is the bending stiffness matrix, in-lb

The solution for the cylindrical bending problem for different boundary conditions is given in Reference 7-8. A summary of the deflection, force and bending moments at the supports for a uniformly loaded infinite strip that is either simply-supported or clamped is provided in Table 7.4-5.

Table 7.4-5 Displacements and Bending Moments for Cylindrical Bending for a Uniformly Loaded Infinite Strip without Membrane Effects

Boundary Condition	Simply-supported	Clamped
Transverse displacement w(x)	$\frac{Apx}{24D}(x^3 - 2ax^2 + a^3)$	$\frac{Apx^2}{24D}(x^2 - 2ax + a^2)$
w_{max}	$\frac{5pa^4}{384D_{11}}$	$\frac{pa^4}{384D_{11}}$
u, v	0	0
N_x, N_y, N_{xy}	0	0
M_x	$-\frac{px}{2}(x-a)$	$-\frac{p}{12}(6x^2 - 6ax + a^2)$
M_y	$\frac{px}{2D}(-AD_{12})(x-a)$	$\frac{p}{12D}(-AD_{12})(6x^2 - 6ax + a^2)$
M_{xy}	$\frac{px}{2D}(-AD_{16})(x-a)$	$\frac{p}{12D}(-AD_{16})(6x^2 - 6ax + a^2)$
$A = A_{11}A_{66} - A_{16}^2$		
$D = D_{11}A$		
Assumptions: • Laminate is symmetric and balanced. • The laminate is not assumed to be specially orthotropic *i.e.* $D_{16} \neq 0$ and $D_{26} \neq 0$ See additional restrictions in Table 7.4-6.		

The displacements, edge reaction forces and moments given in Table 7.4-5 assume that the transverse displacements are small enough to justify ignoring membrane effects. Membrane effects cannot be ignored if the conditions given in Table 7.4-6 are satisfied. IDAT tools SS8 and SO0 calculate displacements and bending moments (SO0 only) ignoring membrane effects and therefore should not be used to calculate displacements or bending moments when the membrane effects are significant.

Table 7.4-6 Guidelines for Application of Membrane Theory (Reference 7-1)

(i)	w > t/2
(ii)	$\frac{p}{E_b}\left(\frac{a}{t}\right)^4 > 100$
Where w is the deflection based on linear theory, in p is the applied pressure, psi E_b is the bending modulus of the laminate, psi $E_b = \frac{12}{t^3 D_{11}^{-1}}$ D_{11}^{-1} is the (1,1) element of the inverse of D matrix, $(\text{in-lb})^{-1}$ a is the narrow width of the plate, in t is the thickness of the panel, in	

If either of the conditions in Table 7.4-6 is satisfied then non-linear membrane effects can be determined using IDAT/PRESS or FEA as discussed in Section 7.6.

7.4.2.1 Example Problem - Cylindrical Bending of Plate

Determine the following for a laminated fuselage panel. • Maximum displacement and bending moments • Ratio-to-Requirement (RTR) for the surface waviness requirement. Assume cylindrical bending and compare the results for simply-supported and fixed boundary condition. The geometric and material properties are as follows: a = 10.9 in Δp = 0.1 psi @ 100% DLL. This is the maximum differential pressure which is the sum of the internal and external pressure across the panel. Material: IM7/977-3 CSW fabric with $[45/0/45/90/-45/0]_s$ layup; total thickness, t = 0.0996 in The guideline for maximum deflection is $w_{allowable}/a \le 0.01$ NOTE: Bending factor of Section 7.4.1 is not applied in a deflection analysis.	
Calculate in-plane [A] and bending stiffness [D] matrix using IDAT/Laminate Tool	$[A] = \begin{bmatrix} 822173 & 261341 & 0 \\ 261341 & 822173 & 0 \\ 0 & 0 & 280416 \end{bmatrix} lb/in$ $[D] = \begin{bmatrix} 635 & 261 & 0 \\ 261 & 635 & 0 \\ 0 & 0 & 276 \end{bmatrix} in-lb$
Calculate bending modulus E_b	$E_b = \frac{12}{t^3 D_{11}^{-1}}$ $[D]^{-1} = \begin{bmatrix} 1.893\times10^{-3} & -7.77\times10^{-4} & 0 \\ -7.77\times10^{-4} & 1.893\times10^{-3} & 0 \\ 0 & 0 & 3.618\times10^{-3} \end{bmatrix} in-lb^{-1}$ $E_b = \frac{12}{0.0996^3 \times 1.893\times10^{-3}} = 6.41\times10^6$ psi
The equations given in Table 7.4-5 are used to calculate displacements and bending moments	
Calculate maximum displacement	Simply-Supported $w_{max} = \frac{5pa^4}{384D_{11}} = \frac{5x0.1x10.9^4}{384x635} = 0.0289\ in$ Clamped $w_{max} = \frac{pa^4}{384D_{11}} = \frac{0.1x10.9^4}{384x635} = 0.0058\ in$
Calculate Margins for Surface Waviness	From the given guideline $\frac{w_{allowable}}{a} = 0.01$ $w_{allowable} = 0.01x10.9 = 0.109\ in$ Simply-Supported $RTR = \frac{w_{allowable}}{w_{max}} = \frac{0.109}{0.0289} = 3.77$ Clamped $RTR = \frac{0.109}{0.0058} = 18$
Check if membrane effects can be ignored	$(w_{max} = 0.0289) < \left(\frac{t}{2} = \frac{0.0996}{2} = 0.0498\ in\right)$ $\left(\frac{P}{E_b}\left(\frac{a}{t}\right)^4 = 2.24\right) < 100$ Therefore membrane effects can be ignored

Calculate Bending Moments

For simply-supported boundary condition maximum moments occur at the center, x=a/2

From Table 7.4-5

$$M_x = -\frac{px}{2}(x-a)$$

For x=a/2

$$M_x = -\frac{p}{2}\frac{a}{2}\left(\frac{a}{2}-a\right)$$

$$M_x = \frac{pa^2}{8}$$

Similarly

$$M_y = \frac{pa^2}{8}\frac{AD_{12}}{D}$$

$$M_{xy} = \frac{pa^2}{8}\frac{AD_{16}}{D}$$

Similarly the bending moments for the clamped condition are obtained by substituting x=0 and x=a/2 in the equations given for clamped condition in Table 7.4-5

.

Simply-Supported

$$A = A_{11}A_{66} - {A_{16}}^2 = 822173 \times 280416 - 261341^2$$

$$A = 1.6225 \times 10^{11} lb^2/in^2$$

$$D = D_{11}A = 635 \times 1.6225 \times 10^{11} = 1.03 \times 10^{14}$$

$$M_x = \frac{pa^2}{8} = \frac{0.1 \times 10.9^2}{8} = 1.485\ in - lb/in$$

$$M_y = \frac{pa^2}{8}\frac{AD_{12}}{D} = \frac{0.1 \times 10.9^2}{8}\frac{1.6225 \times 10^{11} \times 261}{1.03 \times 10^{14}} = 0.609\ in - lb/in$$

$$M_{xy} = \frac{pa^2}{8}\frac{AD_{16}}{D} = \frac{0.1 \times 10.9^2}{8}\frac{1.6225 \times 10^{11} \times 0}{1.03 \times 10^{14}} = 0\ in - lb/in$$

	x=a/2
M_x (in-lb/in)	1.485
M_y (in-lb/in)	0.609
M_{xy} (in-lb/in)	0

Clamped

Moments are determined at the center (x=a/2) and the edges (x=0, a). The following calculations are shown at x=a/2

$$M_x = \frac{pa^2}{24} = \frac{0.1 \times 10.9^2}{24} = 0.495\ in - lb/in$$

$$M_y = \frac{pa^2}{24}\frac{AD_{12}}{D} = \frac{0.1 \times 10.9^2}{24}\frac{1.6225 \times 10^{11} \times 261}{1.03 \times 10^{14}} = 0.203\ in - lb/in$$

$$M_{xy} = \frac{pa^2}{24}\frac{AD_{16}}{D} = \frac{0.1 \times 10.9^2}{24}\frac{1.6225 \times 10^{11} \times 0}{1.03 \times 10^{14}} = 0\ in - lb/in$$

	x=0	x=a/2
M_x (in-lb/in)	-0.99	0.495
M_y (in-lb/in)	-0.406	0.203
M_{xy} (in-lb/in)	0	0

7.4.3 Specially Orthotropic Plate

The bending of symmetric specially orthotropic plates is considered in this section. The symmetric specially orthotropic assumptions result in the uncoupling of bending and axial response, and bending and twisting response, *i.e.*, $B_{ij}=D_{16}=D_{26}=0$. This results in a simplified governing Equation 7.4-1. This governing differential equation can be solved for different boundary conditions. The complexity of the solution depends on the type of boundary condition. Closed form exact solutions can be obtained for the following boundary conditions: (a) simply supported on all sides and (b) simply supported on two opposite edges and the other two edges free, simply supported or clamped. For all other boundary conditions approximate methods such as Rayleigh-Ritz, finite difference or finite element methods can be used.

Typically, non-linear finite element methods are used for analyzing composite panels for displacements and bending. The closed form solutions, which are discussed below, can be used to verify the accuracy of the finite element analysis in the linear range.

Most laminates used in aircraft structures are not specially orthotropic. The implications of this assumption are further discussed in Section 7.4.3.4.

7.4.3.1 Simply Supported on All Sides

The governing Equation 7.4-1 is solved using Navier's method, subjected to the simply supported boundary condition. This boundary condition is satisfied by enforcing the displacements, moments and in-plane forces along the edges to be zero ($w=M_x=M_y=N_x=N_y=N_{xy}=0$; Table 7.4-4). The geometry of the plate that is simply supported is shown in Figure 7.4-5.

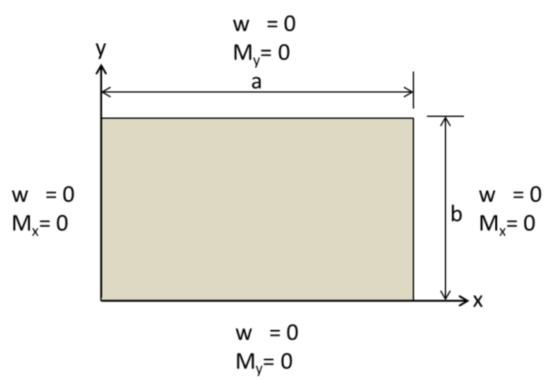


Figure 7.4-5 Geometry and Coordinate System for a Simply Supported Plate

The equations for transverse displacements for simply supported boundary conditions on all sides are given below.

$$w = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Gamma_{mn} \sin(\alpha x)\sin(\beta y)$$ **Equation 7.4-2**

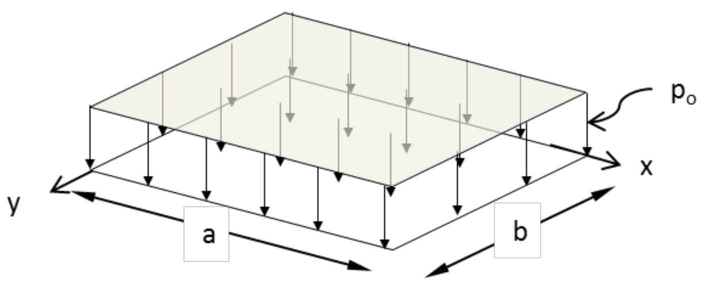
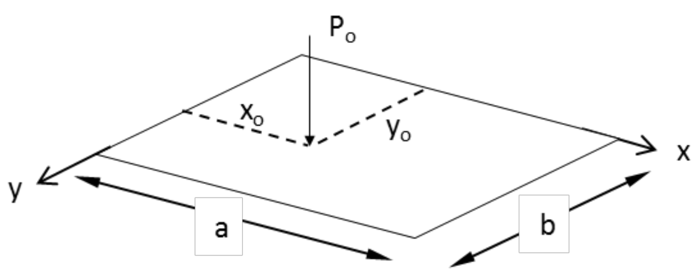
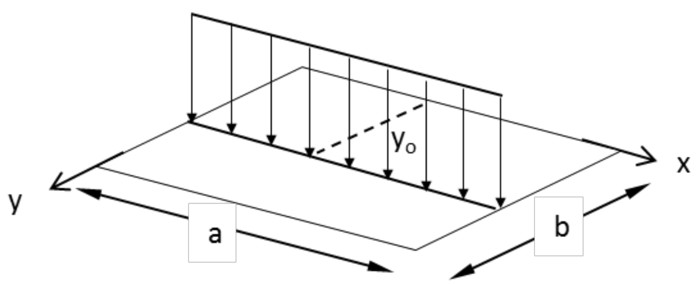
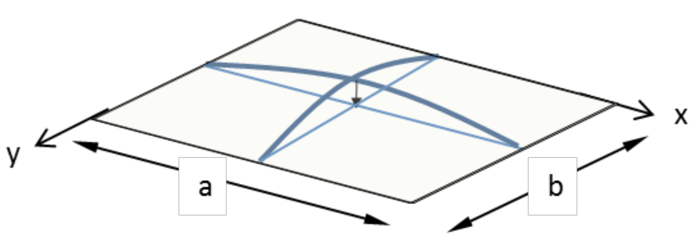
Where

$$\alpha = \frac{m\pi}{a}$$
$$\beta = \frac{n\pi}{b}$$
$$\Gamma_{mn} = \frac{Q_{mn}}{d_{mn}}$$

w = displacement in z direction, in
a = length of the plate along x axis, in
b = width of the plate along y axis, in
Q_{mn} is defined in Table 7.4-7 for various loading profiles

$$d_{mn} = \frac{\pi^4}{b^4}[D_{11}m^4s^4 + 2(D_{12}+2D_{66})m^2n^2s^2 + D_{22}n^4];\ s = \frac{b}{a}$$ **Equation 7.4-3**

Table 7.4-7 Coefficient Q_{mn} for Various Transverse Loading Profiles for a Plate Simply Supported on All Sides (Reference 7-5)

Loading Profile	Equation	Case
Uniform load p(x,y) = p_o 	$Q_{mn} = \frac{16p_o}{\pi^2 mn}; m,n = 1,3,5,\ldots$ $Q_{mn} = 0; m,n = 2,4,6,\ldots$	(a)
Point load, p(x,y) = P_o at (x_o,y_o) 	$Q_{mn} = \frac{4P_o}{ab} sin\frac{m\pi x_o}{a} sin\frac{n\pi y_o}{b};$ $m,n = 1,2,3,\ldots$	(b)
Line load, p(x,y) = p_o at y=y_o 	$Q_{mn} = \frac{4p_o}{\pi bm} sin\frac{n\pi y_o}{b};$ $m,n = 1,2,3,\ldots$	(c)
Sinusoidal load, p(x,y) = $p_o sin\frac{\pi x}{a} sin\frac{\pi y}{b}$ 	$Q_{mn} = p_o;$ $m,n = 1$	(d)

The equation for moments for simply supported boundary conditions on all sides is given as.

$$M_x = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(D_{11}\alpha^2 + D_{12}\beta^2)\Gamma_{mn}\sin(\alpha x)sin(\beta y) \quad \textbf{Equation 7.4-4}$$

$$M_y = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(D_{12}\alpha^2 + D_{22}\beta^2)\Gamma_{mn}\sin(\alpha x)sin(\beta y) \quad \textbf{Equation 7.4-5}$$

$$M_{xy} = -2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha\beta D_{66}\Gamma_{mn}\cos(\alpha x)cos(\beta y) \quad \textbf{Equation 7.4-6}$$

where,

M_x = Moment along edge perpendicular to x axis, in-lb/in
M_y = Moment along edge perpendicular to y axis, in-lb/in
M_{xy} = Twisting moment about x and y axis, in-lb/in

The equation for stresses for simply supported boundary conditions on all sides is given by

$$\sigma_{xx}{}^{(k)} = z\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Gamma_{mn}\left(\bar{Q}_{11}^{(k)}\alpha^2 + \bar{Q}_{12}^{(k)}\beta^2\right)\sin(\alpha x)sin(\beta y) \quad \textbf{Equation 7.4-7}$$

$$\sigma_{yy}{}^{(k)} = z\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Gamma_{mn}\left(\bar{Q}_{12}^{(k)}\alpha^2 + \bar{Q}_{22}^{(k)}\beta^2\right)\sin(\alpha x)sin(\beta y) \quad \textbf{Equation 7.4-8}$$

$$\tau_{xy}{}^{(k)} = -z\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\bar{Q}_{66}^{(k)}\Gamma_{mn}\alpha\beta\cos(\alpha x)cos(\beta y) \quad \textbf{Equation 7.4-9}$$

where,

$\sigma_{xx}{}^{(k)}$ = in-plane x direction normal stress in the kth ply, psi
$\sigma_{yy}{}^{(k)}$ = in-plane y direction normal stress in the kth ply, psi
$\tau_{xy}{}^{(k)}$ = in-plane kth ply shear stress, psi
$\bar{Q}_{ij}^{(k)}$= the transformed reduced stiffness matrix for the kth ply defined in Eq. 4.4-6, psi

Note that the maximum normal stress occurs at (x,y,z) = (a/2,b/2,±t/2) and maximum shear occurs at the four corners at z =±t/2.

The equation for transverse stresses for simply supported boundary conditions on all sides is given by

$$\tau_{xz}{}^{(k)} = -\left(\frac{z^2 - z_{k-1}^2}{2}\right)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}T_{12}^{(k)}\Gamma_{mn}\cos(\alpha x)\,sin(\beta y) + C_1^{(k)}(x,y) \quad \textbf{Equation 7.4-10}$$

$$\tau_{yz}{}^{(k)} = -\left(\frac{z^2 - z_{k-1}^2}{2}\right)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}T_{13}^{(k)}\Gamma_{mn}\sin(\alpha x)cos(\beta y) + C_2^{(k)}(x,y) \quad \textbf{Equation 7.4-11}$$

$$\sigma_{zz}{}^{(k)} = -\left[-\frac{z_{k-1}^3}{3} + \frac{z}{6}\left(z^2 - 3\left(\frac{h}{2}\right)^2\right)\right]\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}T_{33}^{(k)}\Gamma_{mn}\sin(\alpha x)\,sin(\beta y) + C_3^{(k)}(x,y) \quad \textbf{Equation 7.4-12}$$

where,

$T_{12}{}^{(k)} = \alpha^3\bar{Q}_{11}^{(k)} + \alpha\beta^2\left(2\bar{Q}_{66}^{(k)} + \bar{Q}_{12}^{(k)}\right)$
$T_{13}{}^{(k)} = \beta^3\bar{Q}_{22}^{(k)} + \alpha^2\beta\left(2\bar{Q}_{66}^{(k)} + \bar{Q}_{12}^{(k)}\right)$
$T_{33}{}^{(k)} = \alpha T_{12} + \beta T_{13}$
$\tau_{xz}{}^{(k)}$ = Transverse shear stress on a plane perpendicular to the x axis in the kth ply, psi
$\tau_{yz}{}^{(k)}$ = Transverse shear stress on a plane perpendicular to the y axis in the kth ply, psi
$\sigma_{zz}{}^{(k)}$ = Normal stress in the z direction in the kth ply, psi

Note that the Kirchhoff's assumptions in Section 7.4 result in transverse shear stresses being zero. Therefore, the foregoing transverse shear stresses are obtained by integrating the equilibrium equations as demonstrated in Reference 7-5. The constants C_1, C_2 and C_3 are determined by enforcing stress continuity at the interface and satisfying surface stresses at either the top or the bottom surfaces of the plate. This means that for linear analysis such as considered here, the transverse stresses satisfy the boundary condition at only one of the surfaces while it may or may not satisfy the boundary condition on the other surface (Reference 7-12). This is demonstrated in Example Problem 7.4.3.2 and illustrated is in Figure 7.4-7 of the example.

IDAT/SQ5 employs a different method to determine the transverse shear stresses using the transverse shear resultants, defined below, and the equilibrium equations after ignoring cross-derivatives of displacement. Furthermore laminates are assumed to be symmetric. Details of the method are given in References 7-8 and 7-19. Since the IDAT/SQ5 method is different the transverse shear stress profile predicted by Equation 7.4-10 and Equation 7.4-11will differ from IDAT/SQ5 predictions. This difference is demonstrated in Figure 7.4-8 and Figure 7.4-9 which are a part of the example Section 7.4.3.2.

The transverse normal and shear stresses acting on an element are depicted in Figure 7.4-6. Maximum τ_{xz} occurs at (x,y,z) = (0 or a, b/2, 0), τ_{yz} occurs at (x,y,z) = (a/2, 0 or b, 0) and maximum σ_{zz} occurs at (x,y,z)=(a/2, b/2, -t/2).

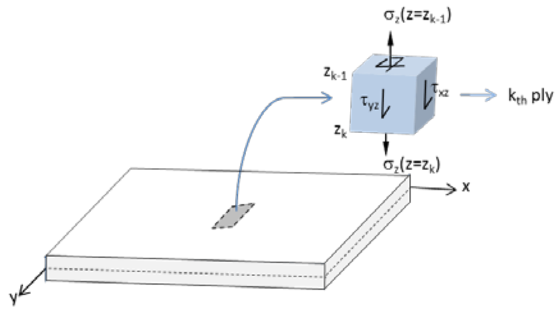


Figure 7.4-6 Transverse Normal and Shear Stresses in the kth ply

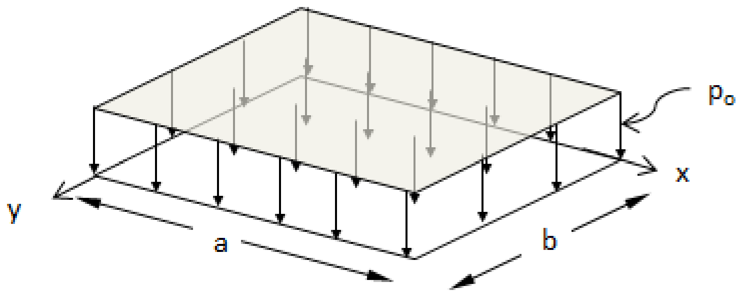
The equations for the transverse shear resultants are given as follows

$$Q_x = D_{11}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha^3\Gamma_{mn}\cos(\alpha x)\,sin(\beta y) + (D_{12}+2D_{66})\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha\beta^2\Gamma_{mn}\cos(\alpha x)\,sin(\beta y) \qquad \textbf{Equation 7.4-13}$$

$$Q_y = D_{22}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta^3\Gamma_{mn}\sin(\alpha x)\cos(\beta y) + (D_{12}+2D_{66})\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta\alpha^2\Gamma_{mn}\sin(\alpha x)\cos(\beta y) \qquad \textbf{Equation 7.4-14}$$

7.4.3.2 Example Problem – Simply Supported on All Sides

Determine the displacements, moments and stresses for the composite plate that is simply supported on all sides as shown below. Compare the solution with IDAT/SO0, IDAT/SS8, IDAT/LG020 and with FEA.



where,

a = 16.4 in, b = 10.9 in

p = 0.1 psi @ 100% DLL. This is the maximum differential pressure which is the sum of the internal and external pressure across the panel.

Material: IM7/977-3 CSW fabric with $[45/0/45/90/-45/-0]_s$ layup, t = 0.0996 in

Calculate in-plane [A] and bending stiffness [D] matrix using IDAT/Laminate Tool or the methodology described in Section 4.	$$[A] = \begin{bmatrix} 822173 & 261341 & 0 \\ 261341 & 822173 & 0 \\ 0 & 0 & 280416 \end{bmatrix} lb/in$$ $$[D] = \begin{bmatrix} 635 & 261 & 0 \\ 261 & 635 & 0 \\ 0 & 0 & 276 \end{bmatrix} in-lb$$

Calculate maximum displacement

The displacement is determined using Equation 7.4-2. The following parameters have to be calculated before the displacements can be evaluated. Since the solution is given by a series the method will be demonstrated for m=2, n=2.

Determine Q_{mn} using the Equation given in Table 7.4-7(a)

$$Q_{mn} = \frac{16p_o}{\pi^2 mn}; m,n = 1,3,5,\ldots$$

$$Q_{mn} = 0; m,n = 2,4,6,\ldots$$

$$Q_{mn} = \frac{16 \times 0.1}{\pi^2 mn} = \frac{0.16211}{mn} psi$$

m, n	Q_{mn}
1,1	0.16211/(1×1)=0.16211
1,2	0
2,1	0
2,2	0

Calculate d_{mn} given by Equation 7.4-3

$$d_{mn} = \frac{\pi^4}{b^4}[D_{11}m^4s^4 + 2(D_{12} + 2D_{66})m^2n^2s^2 + D_{22}n^4]$$

$$s = \frac{b}{a} = \frac{10.9}{16.4} = 0.6646$$

$$d_{mn} = \frac{\pi^4}{10.9^4}[635 \times m^4 \times 0.6646^4 + 2(261 + 2 \times 276)m^2n^2 \times 0.6646^2 + 635 \times n^4]$$

m, n	d_{mn}
1,1	$\frac{\pi^4}{10.9^4}[635 \times 1^4 \times 0.6646^4 + 2(261 + 2 \times 276)1^2 1^2 \times 0.6646^2 + 635 \times 1^4]$ =10.1929
1,2	89.3128
2,1	37.4729
2,2	161.2000

Calculate Γ_{mn}

From Equation 7.4-2

$$\Gamma_{mn} = \frac{Q_{mn}}{d_{mn}}$$

m, n	Γ_{mn}
1,1	$=\frac{Q_{11}}{d_{11}} = \frac{0.1621}{10.1929}$ =0.0159
1,2	0
2,1	0
2,2	0

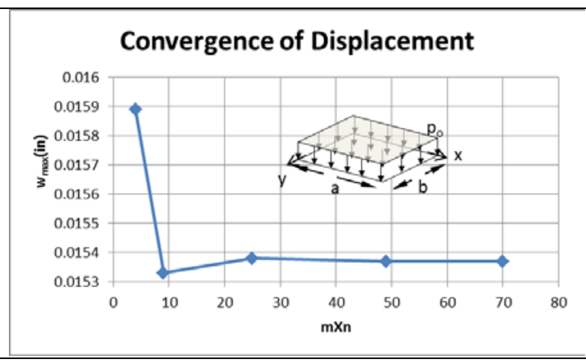
Calculate α and β

$$\alpha = \frac{m\pi}{a} = \frac{m\pi}{16.4} = 0.1916m\ in^{-1}$$
$$\beta = \frac{n\pi}{10.9} = 0.2882n\ in^{-1}$$

Calculate the maximum displacement at x=a/2 and y=b/2

$$w_{max} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Gamma_{mn} \sin\left(\alpha\frac{a}{2}\right)\sin\left(\beta\frac{b}{2}\right)$$
$$= \sum_{n=1}^{2}\sum_{m=1}^{2} \Gamma_{mn} \sin\left(0.1916 \times m \times \frac{16.4}{2}\right)\sin\left(0.2882 \times n \times \frac{10.9}{2}\right)$$
$$= \sum_{n=1}^{2}\sum_{m=1}^{2} \Gamma_{mn} \sin(1.5711 \times m)\sin(1.5711 \times n)$$
$$= \Gamma_{11}\sin(1.5711 \times 1)\sin(1.5711 \times 1) + \Gamma_{12}\sin(1.5711 \times 1)\sin(1.5711 \times 2)$$
$$+\Gamma_{21}\sin(1.5711 \times 2)\sin(1.5711 \times 1) + +\Gamma_{22}\sin(1.5711 \times 2)\sin(1.5711 \times 2)$$
$$= 0.0159 \times \sin(1.5711 \times 1)\sin(1.5711 \times 1) + 0 \times \sin(1.5711 \times 1)\sin(1.5711 \times 2)$$
$$+0 \times \sin(1.5711 \times 2)\sin(1.5711 \times 1) + 0 \times \sin(1.5711 \times 2)\sin(1.5711 \times 2)$$
$$w_{max} = 0.0159\ in$$

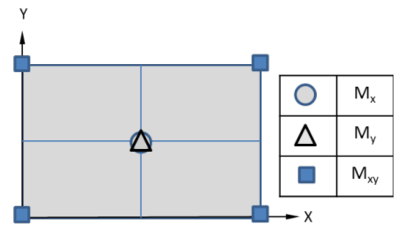
The accuracy of the solution improves when more terms (m,n) are included in the solution. For 70 terms w_{max} = 0.0154 in. The variation of the solution with the m×n terms is shown on the right. The convergence is fast since the difference between the 4 terms and 70 terms is only 3.4%.



The 2×2 series terms are used for demonstrating the method while 7×10 series terms are used to compare with the IDAT/SO0, IDAT/SS8, IDAT/LG020 and FEA solutions. m×n = 70 terms is used in the current study since it is the maximum number of terms that can be used to obtain a converged solution in SO0. Similarly for SS8 the maximum number of terms that could be used is m×n = 50. There are more number of terms in the longer direction compared to the shorter direction.

Calculate Maximum Moments

The maximum moments occur at the locations as shown on the right.



Calculate maximum M_x at x=a/2, y=b/2 using Equation 7.4-4. For demonstration purpose use the 4-term series.

$$M_x = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(D_{11}\alpha^2 + D_{12}\beta^2)\Gamma_{mn} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b}$$

$$M_x = \sum_{n=1}^{2}\sum_{m=1}^{2}(D_{11}\alpha^2 + D_{12}\beta^2)\Gamma_{mn} sin(\alpha x) sin(\beta y)$$

Since all the terms except m=n=1 are zero for the four term series

$$M_x = (635 \times 0.1916^2 + 261 \times 0.2882^2) \times 0.0159 \times \sin\left(0.1916 \times m \times \frac{16.4}{2}\right) \times \sin\left(0.2882 \times n \times \frac{10.9}{2}\right)$$

$$M_x = 0.7147\ in - lb/in$$

A more accurate solution is obtained by letting m=10 and n=7 for which

$$M_x = 0.612\ in - lb/in$$

Calculate maximum M_y at x=a/2, y=b/2 using Equation 7.4-5. The values were calculated using m=10 and n=7 terms.

$$M_y = 0.8906\ in - lb/in$$

Calculate maximum M_{xy} at x=0, y=0 using Equation 7.4-5. The values were calculated using m=10 and n=7 terms.

$$M_{xy} = -0.5659\ in - lb/in$$

Calculate maximum Stresses

The maximum normal stresses occur at (x,y,z)=(a/2,b/2,±t/2) and the maximum shear stress occurs at the four corners at z = ±t/2 . For the purpose of the discussion focus on (x,y,z)=(a/2,b/2,±t/2)

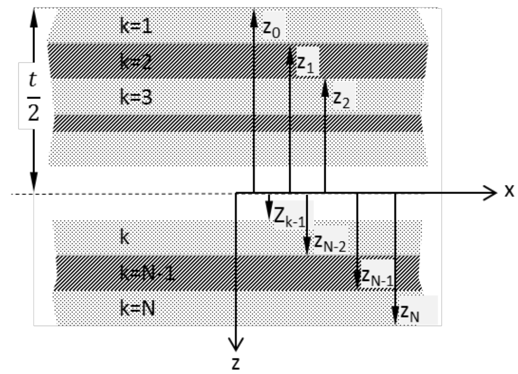
Calculate the maximum σ_{xx} using Equation 7.4-7. The normal stress σ_{xx} in the k_{th} ply is given as

$$\sigma_{xx}{}^{(k)} = z\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Gamma_{mn}\left(\bar{Q}_{11}^{(k)}\alpha^2 + \bar{Q}_{12}^{(k)}\beta^2\right) sin(\alpha x) sin(\beta y)$$

Since we are interested in only the maximum stress the above expression will be evaluated at x=a/2, y=b/2 and z=t/2. For demonstration purpose m=2 and n=2 terms will be considered. The $\bar{Q}_{ij}^{(k)}$ terms are calculated using the method described in Section 2 and is provided here.

$$\bar{Q}^{(0,90)} = \begin{bmatrix} 1.042 \times 10^7 & 4.5849 \times 10^5 & 0 \\ 4.5849 \times 10^5 & 1.042 \times 10^7 & 0 \\ 0 & 0 & 6.5 \times 10^5 \end{bmatrix} \quad \bar{Q}^{(\pm 45)} = \begin{bmatrix} 6.0893 \times 10^6 & 4.7893 \times 10^6 & 0 \\ 4.7893 \times 10^6 & 6.0893 \times 10^6 & 0 \\ 0 & 0 & 4.9808 \times 10^6 \end{bmatrix}$$

The ply arrangement and the z coordinate for the plies are defined on the right.
The outer ply (z = t/2) is a 45° ply and therefore $\bar{Q}^{(k=12)} = \bar{Q}^{(+45)}$ will be used in the analysis. Since all of the calculations that follow are for k=12th ply the index k will be dropped.



N=12

Since all the terms except m=n=1 are zero for the four term series

$$\sigma_{xx}(\frac{a}{2},\frac{b}{2},\frac{t}{2}) = \frac{0.0996}{2} \times 0.0159 \times (6.0893 \times 10^6 \times 0.1916^2 + 4.7893 \times 10^6 \times 0.2882^2)\sin(0.1916 \times 1 \times \frac{16.4}{2})\sin(0.2882 \times 1 \times \frac{10.9}{2})$$

$$\sigma_{xx}\left(\frac{a}{2},\frac{b}{2},\frac{t}{2}\right) = 498.18\, psi$$

Note: A more accurate solution is obtained by letting m=10 and n=7

$$\sigma_{xx}\left(\frac{a}{2},\frac{b}{2},\frac{t}{2}\right) = 430.32\, psi$$

Calculate maximum σ_{yy} at x=a/2, y=b/2, z=t/2 using Equation 7.4-8. The values were calculated using m=10 and n=7 terms.

$$\sigma_{yy}\left(\frac{a}{2},\frac{b}{2},\frac{t}{2}\right) = 478.5\, psi$$

Calculate maximum τ_{xy} at x=a, y=b, z=t/2 using Equation 7.4-9. The values were calculated using m=10 and n=7 terms.

$$\tau_{xy}\left(a, b, \frac{t}{2}\right) = -507.8\, psi$$

Note that the above stresses are mechanical stresses and do not include the residual stresses. The residual stresses have to be calculated separately and added to these stresses to get the total stress. Another method for getting mechanical and thermal stress is to take the moments at the locations where the moments are maximum along with the laminate and use IDAT/SQ5 to get the mechanical and thermal stresses. When doing this comparison with SQ5 care should be taken to use the secant shear modulus and secant transverse modulus to calculate the mechanical stresses as discussed in Section 4.

The following moments are obtained by using Equation 7.4-4, Equation 7.4-5 and Equation 7.4-6 along with the secant shear modulus and secant transverse modulus.

$M_x = 0.6288\ in - lb/in$

$M_y = 0.8912\ in - lb/in$

$M_{xy} = -0.5569\ in - lb/in$

These moments are compared with the moments obtained using the initial modulus in the table below.

	M_x	M_y	M_{xy}
Initial Modulus	0.6120	0.8906	-0.5659
Secant Modulus	0.6288	0.8912	-0.5569

The moments determined using secant modulus properties are used in SQ5 analysis to obtain the stresses given below which are compared to the stresses given by Equation 7.4-7, Equation 7.4-8 and Equation 7.4-9 using secant shear modulus and secant transverse modulus.

	Secant Modulus			Initial Modulus
	SQ5(Mechanical) (psi)	SQ5(Thermal) (psi)	Current Analysis (psi)	Current Analysis (psi)
$\sigma_{xx}\left(\frac{a}{2},\frac{b}{2},\frac{t}{2}\right)$	444.7	0	444.7	430.32
$\sigma_{yy}\left(\frac{a}{2},\frac{b}{2},\frac{t}{2}\right)$	474.7	0	474.7	478.50
$\tau_{xy}\left(a,b,\frac{t}{2}\right)$	-514.2	0	-514.2	-507.80

Note that in the above example the SQ5 thermal stresses are zero. If they do exist then the residual stresses have to be added to the mechanical stresses to obtain the total stress in each ply.

The stresses in the local or material coordinate system can be determined using the stress transformation equations. The maximum stresses in the local coordinate system occur at different locations and therefore the global stresses in those locations should be used to determine the local stresses as shown below. For the panel considered in this example the maximum in-plane local normal stresses occur at the corners and the maximum in-plane local shear stresses occurs at the center of the panel. The maximum stresses occur at the top or bottom 45^o plies.

For corner at x=a and y=b

$$\sigma_{xx}\left(a,b,\frac{t}{2}\right)=0$$
$$\sigma_{yy}\left(a,b,\frac{t}{2}\right)=0$$
$$\tau_{xy}\left(a,b,\frac{t}{2}\right)=507.8\,psi$$

In local 1-2 coordinate system

$$\sigma_{11}=\sigma_{xx}\cos^2\theta+\sigma_{yy}\sin^2\theta+2\tau_{xy}\sin\theta\cos\theta$$
$$\sigma_{22}=\sigma_{xx}\sin^2\theta+\sigma_{yy}\cos^2\theta-2\tau_{xy}\sin\theta\cos\theta$$
$$\sigma_{12}=-\sigma_{xx}\sin\theta\cos\theta+\sigma_{yy}\sin\theta\cos\theta+\tau_{xy}(\cos^2\theta-\sin^2\theta)$$

$$For\ \theta=45^o$$
$$\sigma_{11}(a,b,\frac{t}{2})=(0)\cos^2 45^o+(0)\sin^2 45^o+2(507.8)\sin 45^o\cos 45^o=507.8\,psi$$
$$\sigma_{22}\,(a,b,\frac{t}{2})=-507.8\,psi$$

The maximum shear stress occurs at the center in the top 45^o ply

$$\sigma_{12}\left(\frac{a}{2},\frac{b}{2},\frac{t}{2}\right)=24.1\,psi$$

Calculate Transverse Stresses

Calculate the transverse shear stress σ_{xz} using Equation 7.4-10 at x=0, y=b/2. For demonstration purpose m=2 and n=2. Since all the terms except m=n=1 are zero only one term survives. Considering the first ply k=1

$$\tau_{xz}{}^{(k=1)}=-\left(\frac{z^2-z_0^2}{2}\right)\sum_{n=1}^{2}\sum_{m=1}^{2}T_{12}^{(1)}W_{mn}\cos(\alpha x)sin(\beta y)+C_1^{(k)}(x,y)$$
$$=-\left(\frac{z^2-z_0^2}{2}\right)T_{12}^{(1)}W_{11}\cos(\alpha x)sin(\beta y)+C_1^{(1)}(x,y)$$

Calculate the constant $C_1^{(1)}$. The constant $C_1^{(1)}$ is first determined for the ply k=1 by enforcing the condition that τ_{xz}=0 on the surface.

$$\tau_{xz}{}^{(k=1)}\left(x,y,z=\frac{t}{2}=z_0\right)=-\left(\frac{z_0^2-z_0^2}{2}\right)T_{12}^{(1)}W_{11}\cos(\alpha x)\,sin(\beta y)+C_1^{(1)}(x,y)=0$$

Since the first term is zero, $C_1^{(1)}(x,y) = 0$

Calculate $T_{12}^{(1)}$

$${T_{12}}^{(k=1)} = \alpha^3\bar{Q}_{11}^{(1)} + \alpha\beta^2\left(2\bar{Q}_{66}^{(1)} + \bar{Q}_{12}^{(1)}\right)$$

$${T_{12}}^{(k=1)} = \alpha^3\bar{Q}_{11}^{(45)} + \alpha\beta^2\left(2\bar{Q}_{66}^{(45)} + \bar{Q}_{12}^{(45)}\right)$$

$${T_{12}}^{(k=1)} = (0.1916\times m)^3 \times 6.0893\times 10^6 + (0.1916\times m)(0.2882\times n)^2(2\times 4.9808\times 10^6 + 4.7893\times 10^6)$$

$${T_{12}}^{(k=1)} = 0.0428\times 10^6 \times m^3 + 0.2347\times 10^6 \times m \times n^2$$

For m=n=1

$${T_{12}}^{(k=1)} = 2.7753\times 10^5$$

$${\tau_{xz}}^{(k=1)} = -\left(\frac{z^2 - z_0^2}{2}\right)\times 2.7753\times 10^6 \times 0.0159 \times \cos\left(0.1916\times 1\times\frac{0}{2}\right) sin\left(0.2882\times 1\times\frac{10.9}{2}\right)$$

$$= -\left(\frac{z^2 - z_0^2}{2}\right)\times 2.7753\times 10^5 \times 0.0159\times 1\times 1 = -\left(\frac{z^2 - z_0^2}{2}\right)\times 0.0441\times 10^5$$

Next evaluate ${\tau_{xz}}^{(k=2)}$

$${\tau_{xz}}^{(k=2)} = -\left(\frac{z^2 - z_1^2}{2}\right)\sum_{n=1}^{2}\sum_{m=1}^{2} T_{12}^{(2)} W_{mn}\cos(\alpha x) sin(\beta y) + C_1^{(2)}(x,y)$$

Calculate ${T_{12}}^{(k=2)}$

$${T_{12}}^{(k=2)} = \alpha^3\bar{Q}_{11}^{(1)} + \alpha\beta^2\left(2\bar{Q}_{66}^{(1)} + \bar{Q}_{12}^{(1)}\right)$$

$${T_{12}}^{(k=2)} = \alpha^3\bar{Q}_{11}^{(0)} + \alpha\beta^2\left(2\bar{Q}_{66}^{(0)} + \bar{Q}_{12}^{(0)}\right)$$

$${T_{12}}^{(k=2)} = (0.1916\times 1)^3 \times 1.042\times 10^7 + (0.1916\times 1)(0.2882\times 1)^2(2\times 6.5\times 10^5 + 4.5849\times 10^5)$$

$${T_{12}}^{(k=2)} = 1.0128\times 10^5$$

$${\tau_{xz}}^{(k=2)} = -\left(\frac{z^2 - z_1^2}{2}\right)\times 1.0128\times 10^5 \times 0.0159\times 1\times 1 + C_1^{(2)}(x,y)$$

$$= \left(\frac{z^2 - z_1^2}{2}\right)\times 0.0161\times 10^5 + C_1^{(2)}(x,y)$$

Determine $C_1^{(2)}(x,y)$ by enforcing continuity at z=z_1

$${T_{12}}^{(k=1)}(z = z_1) = {T_{12}}^{(k=2)}(z = z_1)$$

$$-\left(\frac{{z_1}^2 - z_0^2}{2}\right)\times 0.0441\times 10^5 = \left(\frac{{z_1}^2 - z_1^2}{2}\right)\times 0.0161\times 10^5 + C_1^{(2)}(x,y)$$

$$C_1^{(2)}(x,y) = -\left(\frac{(-0.0415)^2 - (-0.0498)^2}{2}\right)\times 0.0441\times 10^5 = -1.6709$$

$${\tau_{xz}}^{(k=2)} = \left(\frac{z^2 - z_1^2}{2}\right)\times 0.0161\times 10^5 - 1.6709$$

This can be repeated for each interface and therefore all the 12 constants can be determined. These constants are shown below in a tabular form. Note that these constants were determined assuming m=7 and n=10 for a more accurate solution.

k	$C_1^{(k)}$
1	0
2	-1.67227
3	-2.17147
4	-3.23564
5	-3.51298
6	-3.96905
7	-4.02452
8	-3.96905
9	-3.51298
10	-3.23564
11	-2.17147
12	-1.67227

τ_{xz} can be determined for any z using the above constants. The transverse stresses are plotted below through the thickness assuming m=10 and n=17. τ_{xz} is plotted at x=0, y=b/2, τ_{yz} is plotted at x=a/2, y=0.

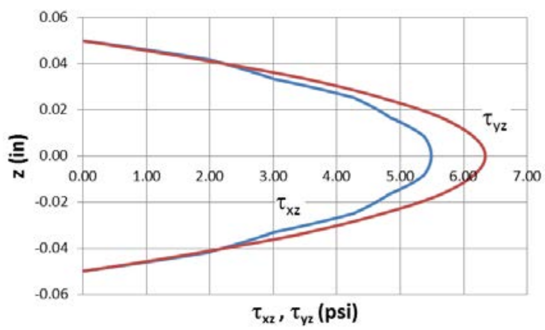


Figure 7.4-7 Variation of transverse shear stress through the thickness

The maximum transverse shear stress occurs at z=0.

$$\tau_{xz}\left(0,\frac{b}{2},0\right) = 5.52\,psi$$

$$\tau_{yz}\left(\frac{a}{2},0,0\right) = 6.37\,psi$$

The transverse shear stress profiles determined above are compared with the transverse shear stress profiles obtained using IDAT/SQ5 below. To run SQ5 the Q_x and Q_y were evaluated using Equation 7.4-13 and Equation 7.4-14. This calculation is shown below.

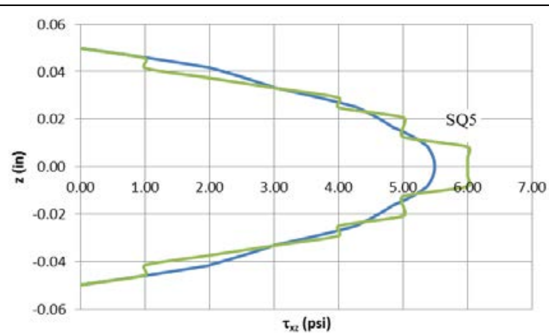


Figure 7.4-8 Comparison of τ_{xz} evaluated using SQ5 with Equation 7.4-10

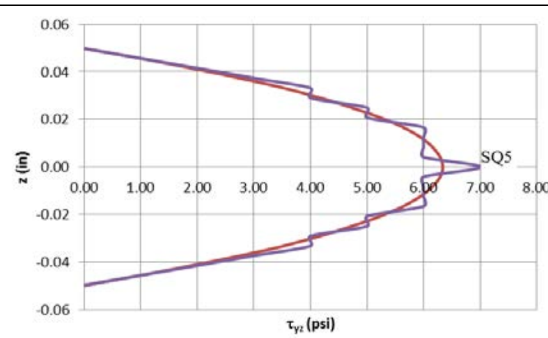


Figure 7.4-9 Comparison of τ_{yz} evaluated using SQ5 with Equation 7.4-11

Calculate Transverse Shear Resultants

Calculate the shear resultants using Equation 7.4-13 and Equation 7.4-14.

$$Q_x = D_{11}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha^3\Gamma_{mn}\cos(\alpha x)\,sin(\beta y) + (D_{12}+2D_{66})\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha\beta^2\Gamma_{mn}\cos(\alpha x)\,sin(\beta y)$$

$$Q_y = D_{22}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta^3\Gamma_{mn}\sin(\alpha x)\cos(\beta y) + (D_{12}+2D_{66})\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\beta\alpha^2\Gamma_{mn}\sin(\alpha x)\cos(\beta y)$$

Substituting values following the pattern shown in the above calculations and employing m=10 and n=7 terms
Q_x = 0.3738 lb/in
Q_y = 0.4219 lb/in

Compare with IDAT/SO0, IDAT/SS8, IDAT/LG020 and FEA

The displacements, normal and transverse stresses were compared with the IDAT/SO0, IDAT/SS8, IDAT/LG020 and FEA solutions. As discussed in Section 7.4.3.4 the SO0 and SS8 solutions are approximate solutions that are derived based on energy based Ritz method (References 7-13and 7-14). SS8 solutions are for curved plates and therefore to analyze the flat plate a radius of curvature of R=1000 inches is assumed. The FEA solutions were obtained by employing IDAT/SPAM to generate the FEM model of the panel which was then subsequently solved using ABAQUS.

Table 7.4-8 Comparison of the Series Solution with SO0, SS8, LG020 and FEA

	Series Solution (m=10,n=7)	SO0 (m=10,n=7)	SS8 (m=10,n=5)	LG020	FEA (Linear)
Max Displacement w_{max} (in)	0.0154	0.0154	0.0153	0.0154	0.0153
Max M_x(in-lb/in)	0.6120	0.6120	n.a	0.611	0.610
Max M_y(in-lb/in)	0.8906	0.8906	n.a	0.891	0.890
Max M_{xy}(in-lb/in)	-0.5659	-0.5659	n.a	n.a	-0.5784
Max σ_{11} (psi)	-507.8	n.a	n.a	n.a	-512
Max σ_{22} (psi)	507.8	n.a	n.a	n.a	512
Max σ_{12} (psi)	24	n.a	n.a	n.a	24
Max τ_{xz} (psi)	5.5167	n.a	n.a	n.a	n.a
Max τ_{yz} (psi)	6.3672	n.a	n.a	n.a	n.a
Max σ_{zz} (psi)	0.1	n.a	n.a	n.a	n.a
$Q_x = \int_{-t/2}^{t/2}\tau_{xz}dz$ lb/in	0.3738	0.3739	n.a	0.396	0.3841
$Q_y = \int_{-t/2}^{t/2}\tau_{yz}dz$ lb/in	0.4219	0.4219	n.a	0.436	0.4224

The Series solution based on specially orthotropic assumption and energy based methods (SO0, SS8) match since the laminate that is considered here is specially orthotropic and therefore there is no bending twisting coupling

7.4.3.3 Simply Supported on Two Opposite Sides

The displacements, moments and stresses for a rectangular composite panel that is simply supported on two opposite edges and free, simply supported or clamped on the other two sides can be obtained as a closed form series solution using Lévy's method (References 7-5 and 7-8). The displacements, stresses and moments for plates that are simply supported on two opposite sides and clamped on the other two sides as shown in Figure 7.4-10 are given below. The solutions for other types of boundary conditions are given in References 7-5 and 7-8.

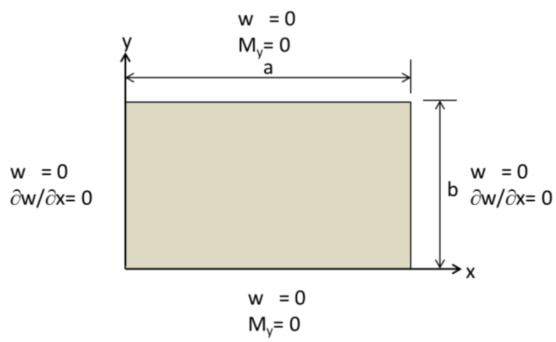


Figure 7.4-10 Geometry and Coordinate System for a Plate that is Simply Supported on Two Edges (y=0,b) and Clamped on the Other Two Sides (x=0,a)

First the roots λ for the following characteristic equation have to be determined.
$D_{11}\lambda^4 - 2(D_{12} + 2D_{66})\beta^2\lambda^2 + D_{22}\beta^4 = 0$
where

$$\alpha = \frac{m\pi}{a}; \beta = \frac{n\pi}{b}$$

[D] is the bending stiffness matrix given by Equation 4.4-16**.**

If $(D_{12} + 2D_{66})^2 > D_{11}D_{22}$, the roots are real and unequal

$$\lambda_1^2 = (-\lambda_2)^2 = \frac{\beta^2}{D_{11}}\left[D_{12} + 2D_{66} - \sqrt{(D_{12} + 2D_{66})^2 - D_{11}D_{22}}\right] \quad \textbf{Equation 7.4-15}$$

$$\lambda_3^2 = (-\lambda_4)^2 = \frac{\beta^2}{D_{11}}\left[D_{12} + 2D_{66} + \sqrt{(D_{12} + 2D_{66})^2 - D_{11}D_{22}}\right] \quad \textbf{Equation 7.4-16}$$

References 7-5 and 7-8 provides solutions for those cases where the above condition is not satisfied.

The displacements at the center of the plate in the z direction can then be calculated as

$$w(x) = \sum_{n=1}^{\infty}\Big(A_n\cosh(\lambda_1 x) + B_n\sinh(\lambda_1 x) + C_n\cosh(\lambda_3 x) + D_n\sinh(\lambda_3 x) + \frac{Q_n}{D_{22}\beta^4}\Big)\sin(\beta y) \quad \textbf{Equation 7.4-17}$$

where,

w = displacement in z direction, in

Q_n is defined in Table 7.4-9 for various loading profiles and A_n, B_n, C_n and D_n are defined below.

$$A_n = \frac{\widehat{Q_n}\lambda_3}{E_n}\left[(\lambda_1\sinh\lambda_3 a - \lambda_3\sinh\lambda_1 a)\sinh\lambda_3 a + \lambda_1(\cosh\lambda_1 a - \cosh\lambda_3 a)(\cosh\lambda_3 a - 1)\right]$$

$$B_n = \frac{\widehat{Q_n}\lambda_3}{E_n}\left[\lambda_3\sinh\lambda_3 a\,(\cosh\lambda_1 a - 1) + \lambda_1\sinh\lambda_1 a\,(1 - \cosh\lambda_3 a)\right]$$

$$C_n = -(A_n + Q_n)$$
$$D_n = -\frac{\lambda_1}{\lambda_3} B_n$$
$$E_n = -(\lambda_3 \sinh \lambda_1 a - \lambda_1 \sinh \lambda_3 a)(\lambda_1 \sinh \lambda_1 a - \lambda_3 \sinh \lambda_3 a) + \lambda_1 \lambda_3 (\cosh \lambda_3 a - \cosh \lambda_1 a)^2$$
$$\widehat{Q_n} = \frac{Q_n}{D_{22}\beta^4}$$

a = length of the plate along x axis, in
b = width of the plate along y axis, in

The moments can now be calculated. The sign convention and direction are shown in Figure 7.4-2. The M_x moment along the edge perpendicular to the x-axis (x=0,a). M_y moment along the edge perpendicular to the y-axis (y=0,b) and the M_{xy}, twisting moment about the x and y axes are given as

$$M_x = -(D_{11}\mathrm{H}_1 + D_{12}\mathrm{H}_2)$$ **Equation 7.4-18**

$$M_y = -(D_{12}\mathrm{H}_1 + D_{22}\mathrm{H}_2)$$ **Equation 7.4-19**

$$M_{xy} = -(2D_{66}\mathrm{H}_3)$$ **Equation 7.4-20**

where

[D] is the bending stiffness matrix given by Equation 4.4-16.

$$\mathrm{H}_1 = (A_n\lambda_1^2 \cosh(\lambda_1 x) + B_n\lambda_1^2 \sinh(\lambda_1 x) + C_n\lambda_3^2 \cosh(\lambda_3 x) + D_n\lambda_3^2 \sinh(\lambda_3 x)) \sin(\beta y)$$
$$\mathrm{H}_2 = -(A_n \cosh(\lambda_1 x) + B_n \sinh(\lambda_1 x) + C_n \cosh(\lambda_3 x) + D_n \sinh(\lambda_3 x))\beta^2 \sin(\beta y)$$
$$\mathrm{H}_3 = (A_n\lambda_1 \sinh \lambda_1 x + B_n \lambda_1 \cosh(\lambda_1 x) + C_n\lambda_3 \sinh(\lambda_3 x) + D_n\lambda_3 \cosh(\lambda_3 x))\beta \cos \beta y$$

And, finally, the ply level in-plane x- and y- direction normal stresses, σ_{xx} and σ_{yy}, and in-plane shear stress τ_{xy} can be calculated from

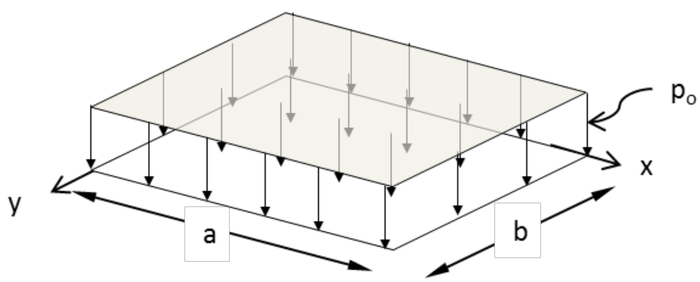
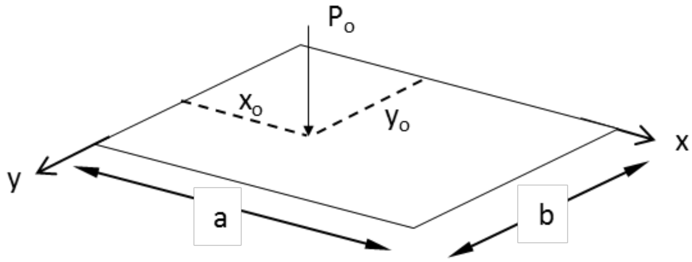
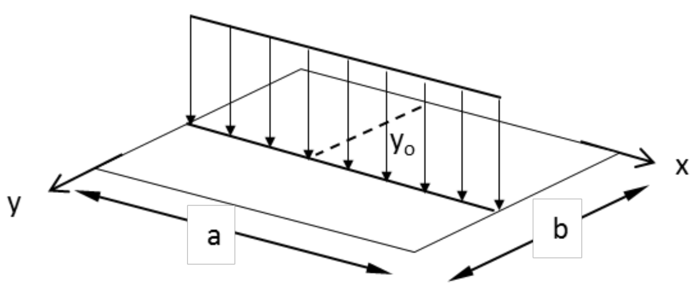
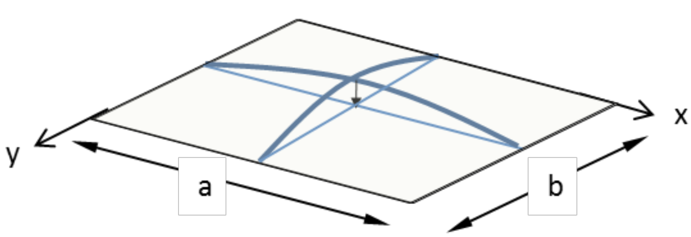
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}^k = -z \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix}^k \begin{Bmatrix} \mathrm{H}_1 \\ \mathrm{H}_2 \\ \mathrm{H}_3 \end{Bmatrix}$$ **Equation 7.4-21**

where,

$[\bar{Q}]$ is the transformed reduced stiffness matrix given by Equation 3.5-34 and Equation 3.5-35.

The transverse shear and normal stress distribution can be obtained using the methods given in References 7-5 and 7-8

Table 7.4-9 Coefficient Q_n for Various Transverse Loading Profiles for a Plate Simply Supported on Two Sides and Clamped on the Other Two Sides (Reference 7-5)

Uniform load $p(x,y) = p_o$ 	$Q_n = \frac{4p_o}{\pi m}; m = 1,3,5,\ldots$ $Q_n = 0; m,n = 2,4,6,\ldots$
Point load, $p(x,y) = P_o$ at (x_o,y_o) 	$Q_n = \frac{2P_o}{b} sin\frac{n\pi y_o}{b}; n = 1,2,3,\ldots$
Line load, $p(x,y) = p_o$ at $y=y_o$ 	$Q_n = \frac{2p_o}{b} sin\frac{n\pi y_o}{b}; n = 1,2,3,\ldots$
Sinusoidal load, $p(x,y) = p_o sin\frac{\pi x}{a} sin\frac{\pi y}{b}$ 	$Q_n = p_o; n = 1$

7.4.3.4 Limitations of Specially Orthotropic Assumption

Most laminates used in aircraft structures are symmetric, however they are not usually specially orthotropic, *i.e.* the bending twisting terms D_{16} and D_{26} do not vanish. However as the number of plies increases the D_{16} and D_{26} diminish and their effect on displacement, moments and stresses reduce. The degree of anisotropy is measured by the ratios D_{16}/D_{11} or D_{26}/D_{11}. If these ratios are small then the bending twisting coupling decreases and panel approaches the idealized approximation. The effect of anisotropy on displacement is depicted in Figure 7.4-11 (Reference 7-16).

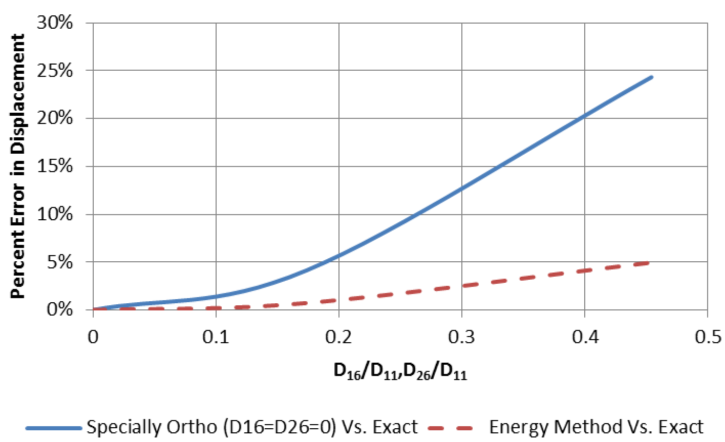


Figure 7.4-11 Effect of Anisotropy on Error in Displacement Determined Using Specially Orthotropic Assumption and Energy Method (Rayleigh Ritz used in SO0 and SS8)

Figure 7.4-11 shows that as the degree of anisotropy increases to the right the error in assuming the laminate to be specially orthotropic increases rapidly compared to the approximate energy method. For high anisotropy the error in assuming the laminate to be specially orthotropic is large because the bending-twisting coupling is neglected. In the energy based methods the bending-twisting terms are not neglected. Therefore the error in displacement is smaller. The three methods: Exact, Specially Orthotropic Solution and Energy Methods converge when the D_{16} and D_{26} terms vanish.

When using specially orthotropic assumptions for bending analysis it is prudent to check if the measure of anisotropy is small. The degree of anisotropy for some typical laminates used in aircraft structure is shown in Table 7.4-10 below.

Table 7.4-10 Degree of Anisotropy for Typical 24 ply/28 ply Symmetric Laminates for Various Ply Percentages

0°	±45°	90°	Degree of Anisotropy D_{16}/D_{11},D_{26}/D_{11}	Laminate Type
33	58	8	0.0048	“Soft”
7	43	50	0.0981	“Soft”
8	83	8	0.0131	“Soft”
42	17	42	0.0213	“Hard”
50	43	7	0.0383	“Hard”
58	33	8	0.0641	“Hard”
25	50	25	0.0912	“Quasi-Isotropic”

Based on Table 7.4-10 and Figure 7.4-11 the maximum error in displacements due to assuming specially orthotropic laminates, for typical laminates used in aircraft structure, does not exceed 3% even for high degree of anisotropy.

The energy based solution is approximate since it satisfies only the geometric boundary condition (displacements) but does not satisfy the natural boundary condition (higher order derivatives of displacements such as moments and stresses). Therefore, for the energy based solutions, convergence is assured for displacements but not for moments and stresses (Reference 7-15). Hence, in Figure 7.4-11, the error in displacements for energy based methods can be reduced by adding more terms to the series solution but this does not ensure that the stresses and moments also converge.

7.4.4 Midplane Symmetric and General Laminate

For mid-plane symmetric laminates the B_{ij} terms are zero. However, the bending-twisting coupling stiffness terms D_{16} and D_{26} which were assumed zero for the specially orthotropic laminates are included. In the case of general laminates the B_{ij} terms are also non-zero. Including these terms in the governing equations increases the complexity of the analysis. The solutions for these types of problems are obtained using approximate methods such as energy based Ritz or Galerkin methods or the finite element methods. The IDAT/SO0 and IDAT/SS8 are based on the energy based Ritz method.

7.5 Bending of Laminated Shells

Shells are structures bounded by two curved surfaces that are separated by a distance which is the thickness of the shell. The coordinate system, geometric parameters and displacements for a laminated shell are shown in Figure 7.5-1 and the parameters are described in Reference 7-8. Shells are similar to plates except for the radius of curvature R (in).

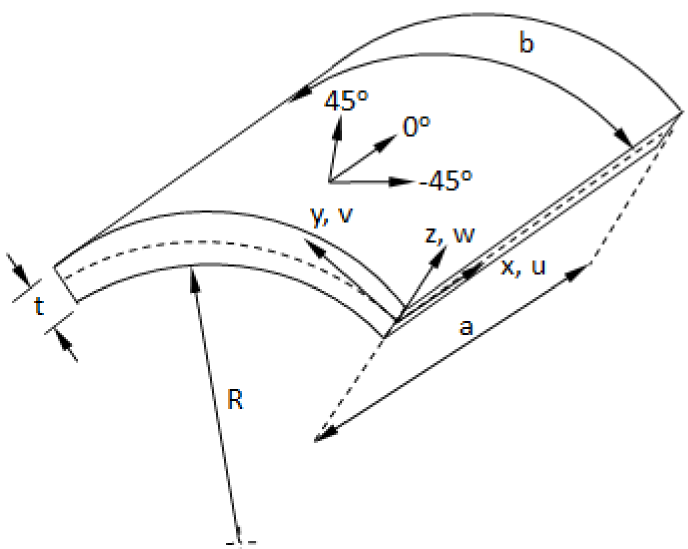


Figure 7.5-1 Shell Geometry and Displacement

Due to the presence of the 'R' term bending-extensional coupling exists for both symmetric and unsymmetric laminates. The equations for flat plates can be obtained by assuming $R \rightarrow \infty$. The solutions for different boundary conditions are discussed next.

7.5.1 Specially Orthotropic Shell

There is a very small range of problems which have closed form solutions due to the coupling of the in-plane membrane and bending of shells. The simplest case of a curved shell that is only pressurized is considered in Reference 7-8. For this situation it is assumed that only the shear coupling terms vanish and the laminate is specially orthotropic. This implies that $A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0$. A solution for this problem is obtained such that it satisfies the geometric boundary condition (displacements) and the natural boundary condition (derivatives of displacements such as moments and stresses). Details of this method are given in References 7-8 and 7-10.

7.5.2 General Laminate

For the general curved laminate all the terms in the A, B and D matrices are populated. The solution is obtained using approximate methods such as energy based series solution or using finite element analysis. IDAT/SS8 employs the energy based Rayleigh-Ritz method to develop solution for a curved panel subjected to different types of loading and constrained by various boundary conditions. Since the solution is based on approximate energy based method the displacement solutions satisfy the geometric boundary condition (displacement) but not the natural boundary condition (derivatives of displacements such as moments and stresses). Therefore convergence is assured for displacements but not for moments and stresses. Details of this method are given in Reference 7-8.

7.6 Membrane Effects in Laminated Plates and Shells

A plate under transverse loading resists external load through the development of internal bending stresses, transverse shear stresses and membrane stresses. Depending on the geometry, magnitude of the transverse deflection, bending stiffness, boundary condition and external load the bending stresses, transverse shear or membrane stresses may dominate and their effect must be included in the analysis. For example, if the membrane stresses are dominant and if they are not included in the analysis then the deflection, in-plane stresses and edge moments will not be accurate. This section focuses on the membrane effects while Section 7.7 discusses the effect of transverse shear stress.

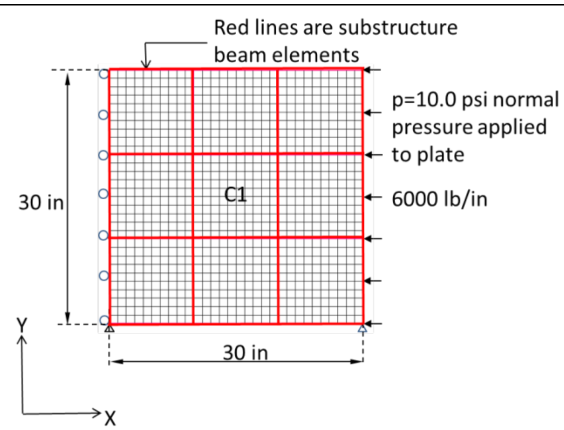
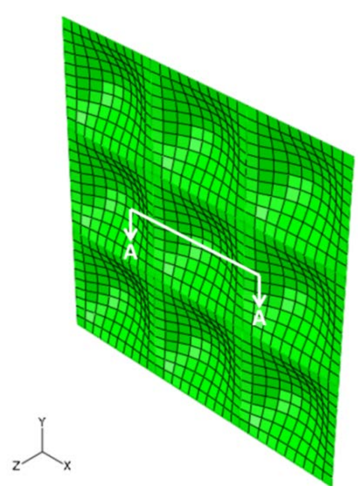
Guidelines for when to consider the membrane effects are provided in Reference 7-1, Section 10 and it is modified for composite materials and shown in Table 7.4-6. If either of the conditions given in Table 7.4-6 is met then membrane analysis must be considered. There are two methods to analyze a panel when membrane effects are significant as ascertained by the rules given in Table 7.4-6. These methods are discussed below.

The most accurate method for including membrane effects is the non-linear finite element method where the geometric non-linearity of the deformation is modeled. This method is often used when a part is carved out from the coarse grid air vehicle finite element model for further detailed analysis and the source of the panel edge moments in the coarse grid model are not entirely due to the applied net pressure across the panel surface. This method is applicable to both flat and curved panels. For curved panels IDAT/SPAM can be used to model the breakout model. Section 7.9 provides further guidelines for modeling using finite element method.

The second method is employed for flat panels when the net pressure load across the panel surface induces all the panel edge moments in the coarse grid linear finite element model. In this situation only the pressure loads and the internal running loads from the coarse grid model are retained while the edge moments are neglected and the panel is analyzed using (i) IDAT/PRESS or (ii) geometrically non-linear NASTRAN or ABAQUS fine grid analysis including follower forces. An example problem that demonstrates this method for flat panels is shown in Section 7.6.1.1 and Section 7.6.1.2 for curved panels.

Note that if the flat panel has non-uniform thickness the panel has to be analyzed using Finite Element Analysis such as NASTRAN or ABAQUS. Also, if the panel has edge build-up then a non-linear Finite Element Analysis such as IDAT/SPAM or NASTRAN is recommended. IDAT/SPAM is an ABAQUS preprocessor that is employed to generate finite element models for flat and curved quadrilateral panels subjected to in-plane and transverse loading. Panels with edge build-up can be modelled quickly in SPAM. The results of the SPAM analysis can be viewed in TMP/Vision.

7.6.1.1 Example Problem – Pressurized Flat Panel

Determine the internal running loads and moments for the 10 inx10in central panel 'C1' shown on the right. The internal loads must be determined at the location where maximum displacement occurs. <u>Material:</u> The panel is made of IM7/977-3 CSW fabric with $[45/0/45/90/-45/-0]_s$ layup and a total thickness t = 0.0996 in. <u>Boundary conditions:</u> Along the boundaries the panel is constrained not to move in the z direction. The panel is also constrained not to move in the x direction and the y direction along the left edge and the bottom edge respectively. <u>Loads:</u> A uniform transverse pressure of 10.0 psi is applied to the panel in the negative z direction. In addition, a uniform edge running load of 6000 lb/in is applied along the right edge.	 <u>Model:</u> The thick red lines are the substructure to which the skins are attached. The skin is modeled as shell elements and the substructure is modeled as very stiff beam elements which reacts a significant portion of the in-plane loads. Each panel is modeled using a 10x10 grid. A linear Finite Element Analysis is performed on the panel.
First determine if the membrane effects must be considered using the method given in Table 7.4-6. The displacement at the center of the panel C1 is determined relative to the surrounding boundaries to perform this check. The deformed model is shown on the right.	

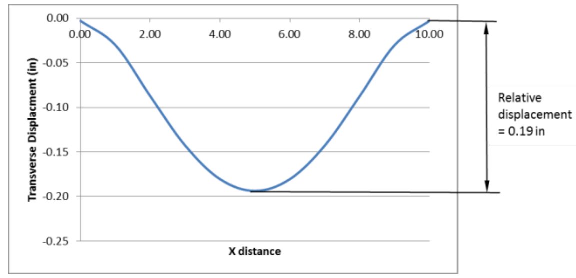
The relative displacement along the cross-section A-A is shown on the right. The center deflection (w) of the panel is measured relative to the displacement of the edges.

$$w = 0.19\ in$$

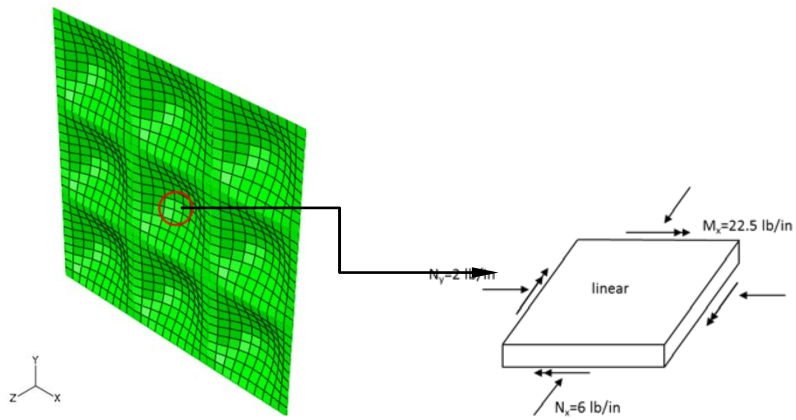
The thickness of the panel (t) = 0.0996 in
Using the first condition in Table 7.4-6

$$w > t/2$$

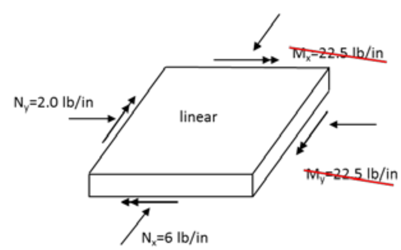
Therefore membrane effects have to be included.



Next determine the running loads and moments acting on the boundaries of the elements at the center. Note that these loads are obtained using the linear analysis. The shear loads are zero and therefore not included in the analysis.



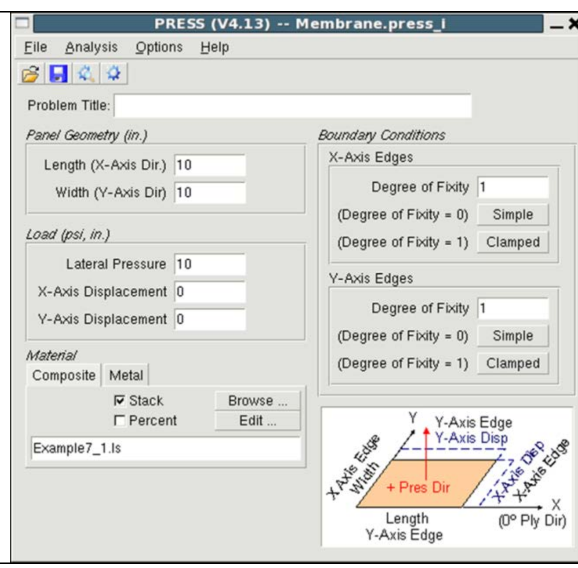
Discard the edge moments and retain only the running normal and shear loads. The edge moments are discarded because in the linear analysis the panels resist the transverse load primarily by bending while in reality the loads are resisted through membrane also, therefore the calculated moments are incorrect. The running normal and shear loads due to the applied in-plane compression load are retained since the applied in-plane loads are not included in the IDAT/PRESS analysis. The shear loads are zero and therefore are not shown on the figure.



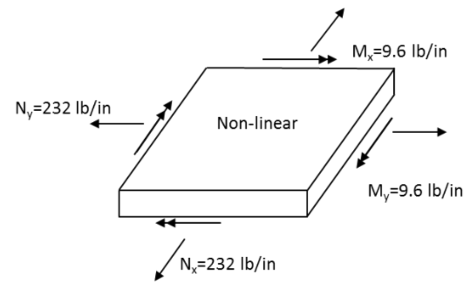
Next run the IDAT/PRESS using the center panel dimension and the applied pressure only.

Ensure that the boundary condition assumed for the PRESS analysis is consistent with the boundary condition used in the Finite Element Analysis, which in this situation is fully clamped since the substructure is assumed to be very stiff and also the adjacent panels are uniformly loaded with the same pressure.

The PRESS input is shown on the right.



The moments and running loads extracted from the PRESS analysis are shown on the right.



The total internal loads are obtained by summing the results of the linear analysis without the moments and the non-linear PRESS analysis.

$$N_x = N_{x-FEA} + N_{x-PRESS} = -6 + 232 = 226\ lb/in$$
$$N_y = -2.0 + 232 = 230\ lb/in$$
$$N_{xy} = 0 + 0 = 0\ lb/in$$
$$M_x = M_{x-PRESS} = 9.6\ in - lb/in$$
$$M_y = M_{y-PRESS} = 9.6\ in - lb/in$$

7.6.1.2 Example Problem – Pressurized Curved Panel

Determine the internal running loads and moments for the 10in x10in central panel shown on the right. The internal loads must be determined at the location where maximum displacement occurs.

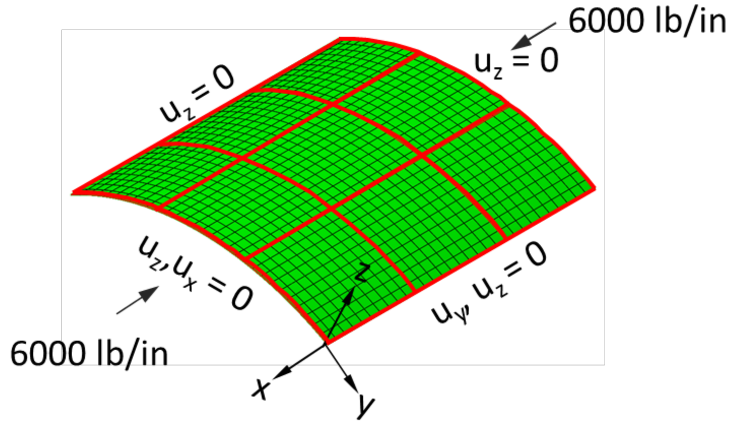
Material:
The panel is made of IM7/977-3 CSW fabric with $[45/0/45/90/-45/-0]_s$ layup and a total thickness t = 0.0996 in.

Boundary conditions:
The boundary conditions are shown in the Figure on the right.

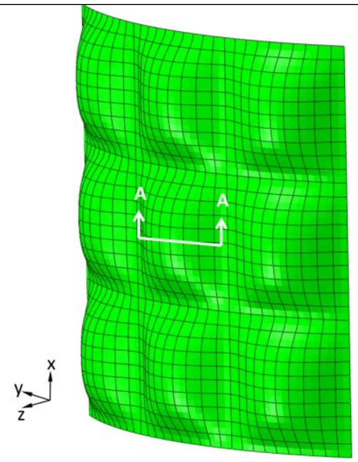
Loads:
A uniform transverse burst pressure of 30 psi is applied to the panel in the positive R direction. In addition, a uniform compressive axial edge running load of 6000 lb/in is applied as shown in the figure.

Model:
The panel has a curvature of 30in about the z axis. The thick red lines are the substructure to which the skins are attached. The skin is modeled as shell elements and the substructure is modeled as very stiff beam elements which reacts a significant portion of the in-plane loads. Each panel is modeled using a 10x10 grid.

A linear Finite Element Analysis is performed on the panel.



First determine if the membrane effects have to be considered using the method given in **Table 7.4-6**. The displacement at the center of the panel C1 is determined relative to the surrounding boundaries to perform this check. The deformed model is shown on the right.



The relative displacement along the cross-section A-A is shown on the right. The center deflection (w) of the panel is measured relative to the displacement of the edges.

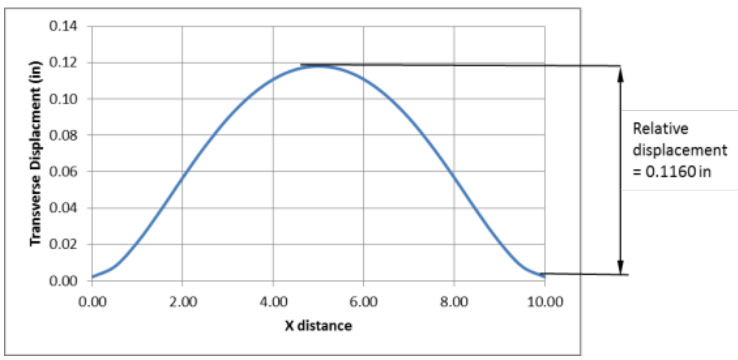
$$w = 0.116\ in$$

The thickness of the panel (t) = 0.0996 in

Using the first condition in Table 7.4-6

$$w > t/2$$

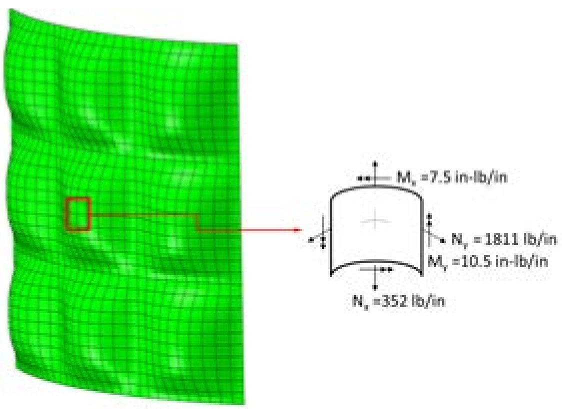
Therefore membrane effects have to be included.



Next determine the running loads and moments acting on the boundaries of the elements at the center.

Note that these loads are obtained using the linear analysis. The shear loads are zero and therefore not included in the analysis.

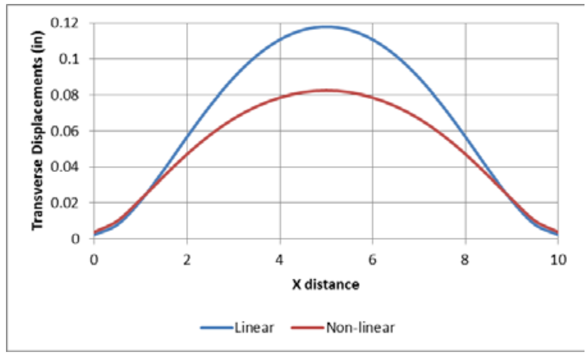
The above running loads and moments are determined only to compare with the non-linear analysis and hence need not be determined in the actual analysis.



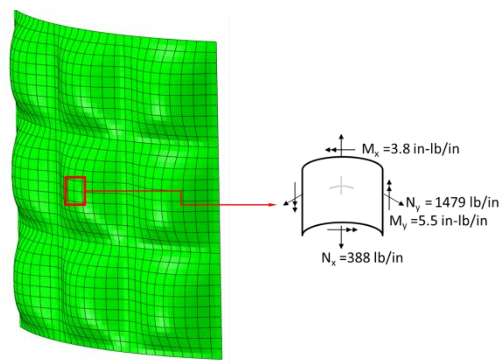
To include membrane forces the model is reanalyzed using non-linear analysis.

Note that for demonstration purpose the same model was used for the linear analysis and non-linear analysis. In reality the linear model would be a coarse grid finite element model and the 9 panels modeled here would represent the fine grid carve out model.

The non-linear displacements are compared to the linear analysis and are shown on the right. The non-linear analysis predicts lower displacement that results in reduced moments as shown below.



The running loads and moments acting on the boundaries of the elements at the center are determined next and shown on the right.



The different parameters for the linear and non-linear analysis are compared below for the curved panel.

	w	N_y (hoop direction)	N_x	M_y	M_x
	in	lb/in	lb/in	in-lb/in	in-lb/in
Linear	0.116	1811	352	10.5	7.5
Non-Linear	0.082	1479	388	5.5	3.8

7.7 Shear Deformation in Laminated Plates and Shells

Kirchhoff's assumptions discussed in Section 7.4 ignore transverse shear strains/stresses. Transverse shear stresses were instead obtained using the equilibrium equations as demonstrated in Section 7.4.3.2. These transverse shear stresses resist external loads and can increase the transverse deflection of the plate. Ignoring the transverse shear stresses will result in unconservative lower predicted displacements for the plate. Furthermore, for composites the transverse shear modulus can be considerably lower than the in-plane axial modulus when compared to isotropic materials and therefore the effects of ignoring transverse shear deformation is more significant when compared to isotropic metals. For those laminates that have high in-plane modulus to transverse shear modulus ratio and where the ratio of in-plane dimensions to laminate thickness is greater than ten the error in neglecting transverse shear can be significant.

To include the transverse shear effects one of the Kirchhoff's assumption, the transverse normal to the mid-surface remain normal before and after deformation, is relaxed. This implies that the transverse normal is not perpendicular to the mid-surface after the deformation. All the other assumptions are retained. The theory based on these set of assumptions that include the transverse shear effects is generally known as the First Order Shear Deformation Theory (FSDT). The FSDT predicts constant transverse shear strains and stresses through the thickness in each ply. Since the actual transverse shear stress variation is at least quadratic a shear correction coefficient (k), is applied to correct for this discrepancy. Details of the FSDT are given in Reference 7-5.

The transverse through-the-thickness shear and normal stress distribution are extracted from the equilibrium equations that are obtained using the corrected FSDT and therefore contain the shear correction term k. The transverse stress recovery method using equilibrium equations is further explained in Reference 7-8.

Using FSDT the resultant transverse shear loads - Q_x, Q_y (see Figure 7.4-1) can be defined as

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = k \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{yz} \\ \varepsilon_{xz} \end{Bmatrix}$$ **Equation 7.7-1**

where,

k is the shear correction coefficient

ε_{yz} is the transverse shear strain in the yz plane, in/in

ε_{xz} is the transverse shear strain in the xz plane, in/in

The shear correction is a function of ply properties and stacking sequence and cannot, in general, be determined in closed form. However, for a homogeneous orthotropic plate the shear correction used for an isotropic plate (Reference 7-8) is applicable and is given as

K = 5/6 = 0.8333 **Equation 7.7-2**

For other shear correction factors see References 7-5 and 7-8.

The in-plane stiffness terms are given as

$$A_{44} = \sum_{k=1}^{N} \bar{Q}_{44}^{(k)} (z_k - z_{k-1})$$ **Equation 7.7-3**

$$A_{45} = \sum_{k=1}^{N} \bar{Q}_{45}^{(k)} (z_k - z_{k-1})$$ **Equation 7.7-4**

$$A_{55} = \sum_{k=1}^{N} \bar{Q}_{55}^{(k)} (z_k - z_{k-1})$$ **Equation 7.7-5**

And the transformed reduced stiffness terms specific to the transverse shear calculation are given as

$\bar{Q}_{44} = Q_{44}\cos^2\theta + Q_{55}\sin^2\theta$ **Equation 7.7-6**

$\bar{Q}_{45} = (Q_{55} - Q_{44})\cos\theta\sin\theta$ **Equation 7.7-7**

$\bar{Q}_{55} = Q_{44}\sin^2\theta + Q_{55}\cos^2\theta$ **Equation 7.7-8**

where

$Q_{44} = G_{23}$, ply transverse shear modulus in 23 plane, psi
$Q_{55} = G_{13}$, ply transverse shear modulus in 13 plane, psi

The effective transverse shear moduli G_{xz} and G_{yz} for a 3D laminate are determined using the method developed in References 7-17 and 7-18.

- Reference 7-17 develops the stiffness as a summation. The summation method is used in IDAT/ANMAT routine, which is the analysis engine that is executed when the user picks 'Anisotropic 3D material report' from IDAT/Laminate. In the summation method each layer is treated as a monoclinic anisotropic layer with the plane of symmetry parallel to the layers. The properties are homogenized over the layers and effective elastic constants are determined.
- An alternate method is described in Reference 7-18 which evaluates the same quantities as an integral.

The effect of transverse shear stress on deflection for an IM7/977-3 laminate, for three different ply percentages, is shown in Figure 7.7-1.

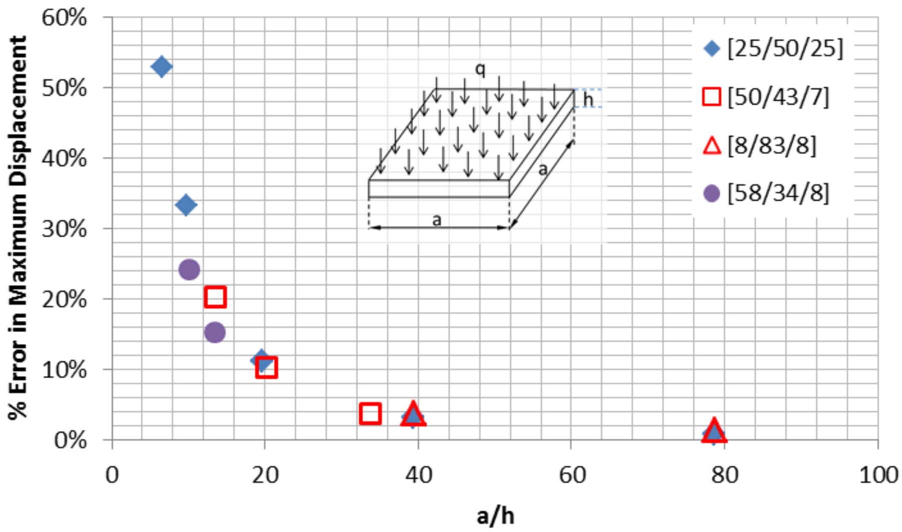


Figure 7.7-1 Effect of Transverse Shear Stress on Displacement for an IM7/977-3 Laminate Plate Simply Supported on All Sides

From the above figure it is clear that the effect of transverse shear stress is a function of the relative thickness of the laminate and is independent of the ply percentages.

7.8 Tools

A summary of the IDAT tools that can be used to calculate displacements, stresses and moments and their limitations are provided below.

Table 7.8-1 Summary of IDAT Tools for Analyzing Laminate Bending

Tool	Geometry	Material Assumption	Boundary Condition	Transverse Shear Effects Included	Stiffeners Included	Membrane Effects Included	Non-linear	Output
SO0	Flat Trapezoidal Panels	Anisotropic	Fixed, Simply Supported (SS) and Free	No	Yes	No	No	Displacements, stresses, moments and transverse shear resultant
SS8	Curved Rectangular Panels	Anisotropic	Fixed, SS, and Free	No	Yes	No	No	Displacements
SPAM	Flat/Curved Panels	Anisotropic	Fixed, SS, Free and Custom	Yes	Yes	Yes	Yes	Displacements, stresses, moment and transverse shear
PRESS	Flat Panels	Specially Orthotropic	Fixed and SS	No	No	Yes	Yes	Displacements, Stresses, Moments, Strains
COMAIN	Flat Panels Sandwich	Specially Orthotropic	SS	Yes (only for Sandwich)	No	No	No	Displacements, Stresses, Moments
LG020	Flat Panels Sandwich	Anisotropic	SS, SS & Fixed, SS & Elastic	Yes	No	No	No	Displacements, Stresses, Moments

7.9 FEA Modeling Conventions and Considerations

In general finite element analysis is employed to obtain internal loads which are used in proprietary in-house analysis software tools such as IDAT/SQ5 to obtain the final margins. The finite element model could be a coarse grid model such as the models used in complete Air Vehicle Finite Element Model suitable for linear analysis or a fine grid breakout model suitable for non-linear analysis. The breakout model should have sufficient fidelity to capture the non-linear effects, which requires that there be at least four to six elements along the narrow width of the plate and sufficient number of elements along the longer side such that an aspect ratio between 1 – 1.25 is maintained.

Furthermore the edge restraints should be modeled accurately. If the adjacent panels have the same aspect ratios and have the same pressure applied to them then the edge supports of the panel can be assumed to be clamped since the slope and displacement along the centerline of the substructure must be zero. If the adjacent panels are different such as in the case of edge panels then simply supported boundary conditions can be assumed. Clamped conditions can also be assumed for the panel when double row of fasteners are used to attach the panel to the substructure or if the substructure has sufficient torsional rigidity to prevent rotation.

When working with the breakout model it is recommended that the model should include at least an additional bay length on all sides. Also, free body boundary loads should be applied instead of displacements to the breakout model.

In a finite element model, modeling the composite material property is significantly different and complex than modeling isotropic properties. The different techniques for modeling the material properties are further discussed in Reference 7-21.