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DYNAMICS OF STRUCTURES

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DYNAMICS OF STRUCTURES
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To Erma and Zafrira

into the subject of dynamics of multi-degree-of-freedom structural systems. To gain maximum benefit from the book, the reader should understand the elementary theory of structures and the basic elements of the theory of vibrations. An elementary knowledge of calculus, linear differential equations, and Laplace transforms, in addition to the rudiments of matrix algebra, is sufficient mathematical background. Additional mathematical tools such as Fourier transforms, calculus of variations, and probability are used in their fundamental form and are explained when first presented.

As a text, the book should be of greatest usefulness at the lower graduate level. However, some of the material may also be used to advantage in upper level undergraduate courses. For example, Chapter 1 has been used at the University of California as the basis for a structures course at the senior level. The book can be used for a one-year course or a one-semester course. A one-semester course may consist of Chapters 1, 2, 3, 4, 5, 7, and 8, or it may cover, in addition, Chapters 9 and 10 for classes in which the material of Chapter 1 needs only to be reviewed.

A brief survey of the book is as follows. Chapter 1 contains a part of the theory of structures with emphasis on energy theorems and methods which form a basis for much of the material to follow. A discussion of matrix stiffness and flexibility methods is included in order to bring together important material from a large number of published papers. Chapter 2 develops principles of analytical mechanics which have proven to be very powerful in a wide variety of problems in mechanics. Chapter 3 presents a treatment of natural vibrations of structural systems having an arbitrary number of degrees of freedom. Chapters 4, 5, and 6 deal with various analytical methods that are useful in treating natural vibrations, with attempts to explain and clarify some of the phenomena involved. Chapter 7 presents commonly accepted concepts of damping as related to structures. Chapters 8, 9, and 10 give methods for finding the response of structures to periodic and nonperiodic excitations, including proportional and nonproportional damping and computations for dynamic stresses. In these chapters the excitations are deterministic; i.e., they are specific functions of time. Chapter 11 extends the treatment to random excitations. This chapter presents the background in statistics and the theory of probability necessary to the development of the subject matter. In Chapter 12 some of the considerations which are important in using digital computers to solve problems in structural dynamics are discussed. An example is presented in depth to illustrate the points discussed and to show the development of a computer program.

As is often true in efforts of this kind extending over a consider-

able period of time, the authors are indebted to many persons. In particular they acknowledge the contribution of Professor W.T. Thomson of the Department of Engineering at UCLA, whose lectures in Dynamics added significantly to the subject matter in the book. For part of the background in matrix methods, particularly as related to transfer matrices, a series of lectures presented by Dr.-Ing. Eduard Pestel, Professor of Mechanics at the Technische Hochschule in Hannover, Germany were very important. The authors are grateful for the encouragement of Dean L.M.K. Boelter of the UCLA College of Engineering. Acknowledged also are the efforts of students who contributed significantly by solving problems, pointing out errors, and making suggestions for improved clarity in presentation. Finally, the book and the course notes which formed the basis for it would not have been completed without the cooperation and efforts of the Reports Group of the Department under the supervision of Estelle E. Dorsey. The typing and assistance of Misses Jacquie Davis and Sharlene Belanger are gratefully acknowledged as are the secretarial services of Mrs. Naomi Krueger.

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DYNAMICS OF STRUCTURES

CHAPTER 1

Force-Deflection Properties of Structures

1.1 Introduction

A study of the dynamic behavior of a structure may begin logically with an investigation of its properties which relate deflections to applied forces. These are called *force-deflection properties* and are related to and depend upon three properties: the stress-strain properties of the materials from which the structure is made, the geometry and sectional properties of the structure, and constraints which exist at the boundaries and internally within the structure.

These properties are usually found by investigating the structure under static conditions, either by determining deflections resulting from the application of a known set of forces in equilibrium or by determining a set of forces required to equilibrate a prescribed deflection configuration. These investigations may be made by testing the actual structure, or by using a structurally similar model. Usually, however, they are made by using known methods of analysis and synthesis based on the theory of structures and related theories in elasticity, plasticity, and strength of materials.

Force-deflection properties based on the static equilibrium state are not necessarily applicable to the same structure in the dynamic state. Implied in the term force-deflection property, as used here, is a unique relationship between the two not influenced by the path of motion leading to the deflected shape. For such a relationship to exist the material must be ideally elastic over the range of stress and strain involved in the motion, and other forces dependent upon velocity must not exist. However, in actual structures materials are not ideally elastic over a large range of stress and strain, and there exist velocity-dependent forces such as friction forces in structural connections, drag forces induced by motion of the structure through the surrounding air or other media, etc.

In Fig. 1.1 a stress-strain curve characteristic of many structural materials[1] is shown with the position of the proportional limit indi-

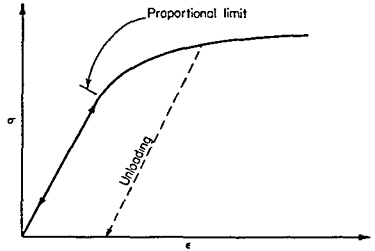


Figure 1.1

cated. Below this limit the material is very nearly elastic, that is, a one-to-one correspondence exists between stress and strain independent of direction of loading. Within this nearly elastic region it is reasonable to postulate an ideal material making up an ideal structure in which force and deflection are uniquely related. In this ideal structure friction forces and other extraneous velocity-dependent forces are zero. Such an ideal structure is conservative and its force-deflection properties are characterized by a potential function usually referred to as the *strain energy function*. To approximate the behavior of the actual structure, the energy dissipative forces are determined separately and superposed on the elastic and inertial forces associated with the ideal structure. The determination of these dissipative or "damping" forces is discussed in Chapter 7. The remainder of this chapter is devoted to a discussion of the force-deflection properties of the ideal structure.

In view of the linear relationship between stress and strain it may be supposed that the forces and deflections are necessarily related linearly as well. Such a linear relationship is appealing to the analyst because it permits the use of the principle of superposition, that is, a total deflection caused by a set of forces may be found by adding deflections caused by individual forces of the set acting separately. To investigate further the question of linearity between force and deflection, we must consider, in addition to the stress-strain properties of the material, the other two factors on which force-deflection properties depend, namely, geometry and constraints.

It can be shown by rigorous analysis, taking into account the changing geometrical configuration of the structure as it deflects under load, that forces and deflections are not related linearly even though ideal elasticity is postulated. This result is due to the fact that the distribution of internal forces or stresses and, hence, the distribution of strains changes with changes in geometry associated with increasing load and deflection. For example, if we consider a single deflection u resulting from a single applied force F, the ratio of deflection to force may be written in an infinite series having the form

$$\frac{u}{F} = a_0 + a_1F + a_2F^2 + a_3F^3 + \dots \tag{1.1}$$

where $a_0, a_1, a_2, a_3, \dots$ are constants which depend upon the dimensions of the structure and upon the moduli of elasticity of the materials. The constant a_0 is the limiting ratio of deflection to force as the force and the deflection become vanishingly small. This is the same constant that results when the force-deflection analysis is based on the unchanging geometrical configuration of the unloaded structure. For most structures the terms following the first constant in Eq. (1.1) are small and it is customary to neglect them. Thus, the customary deflection analysis is accurate for small deflections but decreases in accuracy with large deflections.

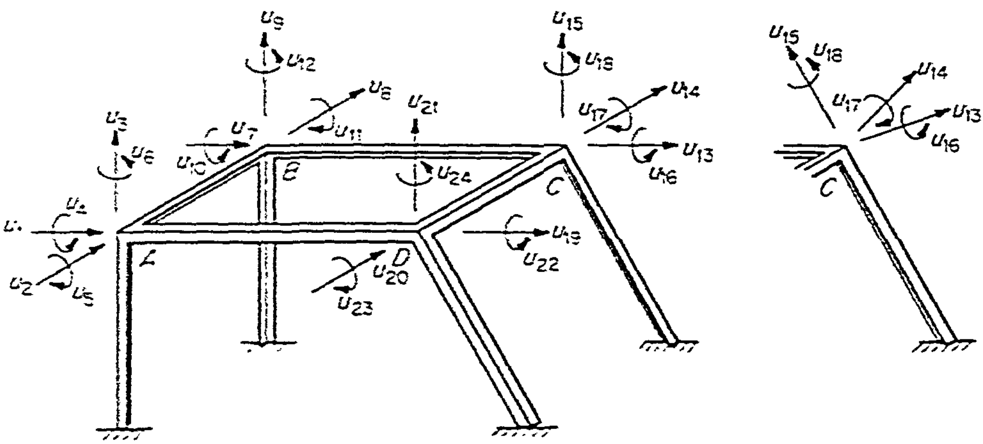
It is possible, also, for a system of constraints to change as the structure is loaded, thus providing another source of nonlinearity in force-deflection properties. For example, analyses of building structures are usually based on the supposition that foundation constraints are fixed, whereas in fact a certain amount of settling and shifting is inevitable. Depending upon the nature of the constraints, nonlinearities may or may not be introduced as a result. The methods of analysis and the theory underlying them discussed in this chapter apply to ideal elastic structures under sufficiently small deflections that the force-deflection properties may be considered linear.

Considering the vast amount of material that may be included in the mechanics of structures which relates to force-deflection properties, a complete discussion is beyond the scope of a single chapter. It is intended here to treat only the subject matter developed in recent years and related primarily to the properties of structural systems. It is not intended to discuss structural elements such as rods, beams, plates, etc., except to show how the properties of these elements are used to synthesize the properties of the structural system they compose. It is intended that the treatment be considered as general, not relating to particular structural types such as beams, frames, or shells although illustrative examples are taken chiefly from the first two categories for simplicity and clarity.

1.2 Systems of Coordinates

One of the first matters to be dealt with in the solution of a structural problem is the establishment of a system of coordinates which serves to identify and to order the deflections and forces. The choice of coordinate system depends upon the geometry of the structure, the location and nature of constraints, distribution of forces, and the required information concerning deflections. In what follows, two types of coordinates will be discussed. One type, referred to as *discrete coordinates*, defines displacements and forces at a set of discrete points in terms of components having specified directions. In general, the component directions at each point are mutually perpendicular so that an orthogonal triad is associated with each point in the coordinate system. The triad serves to define components of rotation as well as translation (considering rotations to be vanishingly small so that they may be considered as vectors), and components of couples as well as forces. We use the generic terms *displacement* to include rotations, and *force* to include couples. Thus, in three-dimensional space there are six coordinates identified with each point in the system, whereas in two-space there are only three. For structural systems composed of elements which intersect at points, such as in frames, it is useful to identify the coordinate system with these points of intersection. Such a coordinate system is shown in Fig. 1.2 in which a system of twenty-four coordinates is associated with a simple, three-dimension frame structure. The coordinate displacements are $u_1, u_2, \ldots, u_{24}$ and the forces are $F_1, F_2, \ldots, F_{24}$.

It should be noted that the directional orientation of the triads is immaterial but will usually be indicated by preferred directions in the structure. In Fig. 1.2(a) they are all shown having the same direction. However, any one of them may be rotated, as shown for point C in Fig. 1.2(b), if desired. A second example of this type of coordinate



(a) (b)

Figure 1.2

system is shown in Fig. 1.3 as applied to a reinforced shell structure typical of airplane or space vehicle structures. Here the coordinates are associated with the points of intersection of longitudinal stiffeners with transverse frames. In this example, however, the elements are not all one dimensional but include two-dimensional sheet panels or plates. Therefore, there are line connections as well as point connections between elements, and it is logical to locate coordinate points anywhere along these lines. In fact, for any kind of structure it is possible to select any number of points located arbitrarily. In addition to the structural configuration the number and location of these

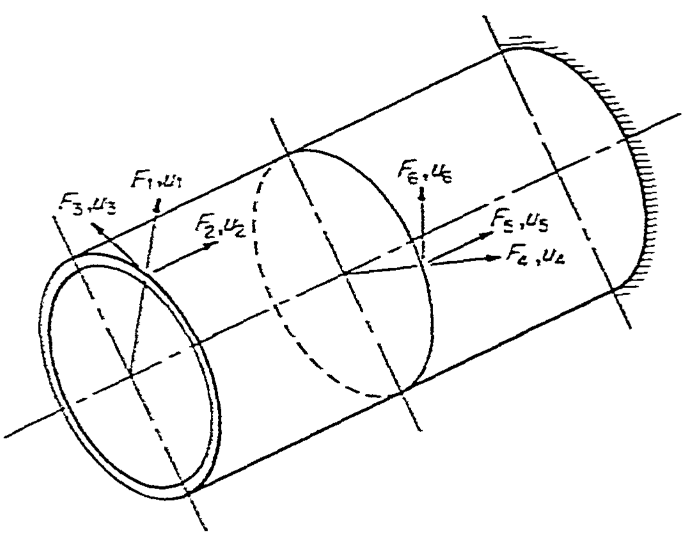


Figure 1.3

points are determined by considering accuracy as well as ease of analysis.

The second type of coordinate system to be distinguished is one in which the coordinates [p_i in Eq. (1.2)] specify the magnitude of prescribed space distributions [such as $\phi_i(x)$ in Fig. 1.4]. These, hereafter referred to as *distributed coordinates*, are often applied to structures whose properties are continuously distributed in space and to problems in which the forces are distributed. However, it is possible and practical to use it also for structural systems and forces which are discretized. To illustrate this coordinate system consider a structural element of one dimension such as the beam shown in Fig. 1.4. The beam is a cantilever of length l and is built in at the end $x = 0$. A number of functions $\phi_1(x), \ldots, \phi_4(x)$ are shown, representing a set of shapes in which the beam may bend. The actual deflection of the beam $w(x)$ is represented by the expression

$$\begin{aligned} w(x) &= \phi_1(x)\,p_1 + \phi_2(x)\,p_2 + \phi_3(x)\,p_3 + \phi_4(x)\,p_4 \\ &= \sum_{i=1}^{4} \phi_i(x)\,p_i \end{aligned} \tag{1.2}$$

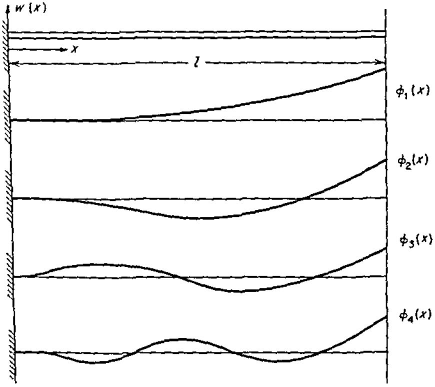


Figure 1.4

The coordinates $p_1, \ldots, p_i$ determine the amplitudes of the respective functions which contribute to the total deflection.

Although only four functions were used in Fig. 1.4, any arbitrary number may be used. In general, a larger number leads to greater accuracy in the solution of the problem at the expense of greater labor.

Deflections $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$, in three-space are given by the equation

$$u, v, w = \sum_{i=1}^{m} \phi_i(x, y, z)\, p_i \tag{1.3}$$

where m is any arbitrary integer.

1.3 Systems with Constraints: Generalized Coordinates

Most structural systems are subject to constraints either at their boundaries or at points internal to the system. Constraints may result from relationships among forces, or they may be kinematical in character, that is, result from relationships among displacements. In either case such constraining relationships must be taken into consideration in making use of coordinate systems.

Relationships among the coordinates that exist because of constraints on the structural system are called *constraint equations.* If constraints exist, these equations will relate not only the displacements involved but also the force components on the same coordinates. Thus, it is useful to think of constraints as relating not just displacements or forces but as relating the coordinates themselves. Therefore, we shall use the term *constrained* coordinates in discussing coordinate systems in which there exist one or more equations of constraint.

Implied in the above discussion is the thought that systems of *unconstrained* or *independent* coordinates exist. In general, this is true in structures and one may derive such a system from a system of constrained coordinates. For example, if the displacements of a system are defined in terms of m coordinates and if there exist r equations of constraint among these displacements, then r of the coordinates may be expressed in terms of the remaining $m - r$ coordinates which are independent. Thus, if

$$n = m - r \tag{1.4}$$

n independent coordinates exist and the displacements and forces may be completely defined in terms of these n coordinates. Coordinates which are completely independent in this way are called *generalized coordinates*, although the term *independent coordinates* may be more logical in context with this discussion.

To digress briefly it is appropriate to point out that constraints may exist for which it is not possible to determine, by elimination of coordinates, a set of independent ones. For example, constraint equations may arise from force equilibrium considerations in which the forces involved depend upon velocities rather than upon displacements. Friction forces in lubricated structural joints would exemplify this situation. If relative motion were possible in the joints these forces would be velocity-dependent and the equations of constraint would involve velocities. If these equations are nonintegrable it is not possible to determine a set of constraint equations among the displacements and, hence, the elimination of dependent coordinates is not possible. Constraints of this type are called *nonholonomic* in contrast to the former type of *holonomic* constraints for which it is possible to derive a set of independent coordinates. Nonholonomic constraints are rarely encountered or dealt with in structures and it is not our purpose to consider them further.

To return to the subject of generalized coordinates consider a set of coordinates related to a set of discrete points, as illustrated in Fig. 1.2. For a completely deformable structure displacements on each coordinate may exist independent of all the others. For example, we may visualize in this illustration twenty-four independent deflection configurations, each one obtained by imposing an arbitrary displacement on one coordinate while keeping all other displacements zero. Thus, the coordinate system is *generalized*. It is customary to speak of such independent deflection configurations as *degrees of freedom*. Hence, a structure has as many degrees of freedom as it has generalized coordinates. It is important to remember, however, that this is an arbitrary number since the coordinate system is arbitrary in that we may select as many reference points on the structure as we wish. Since, in principle, this number may approach infinity, we think of a structure as having, in fact, an infinity of degrees of freedom. In restricting this to a finite number, such as 24 in Fig. 1.2, we must bear in mind that we are concerning ourselves only with those particular deflection configurations which are defined by the coordinate system adopted.

Structural elements are generally not equally flexible with respect to all axes, hence in systems it is sometimes appropriate to neglect certain deflections which are small when compared with others. This amounts to taking the flexibility of some elements with respect to

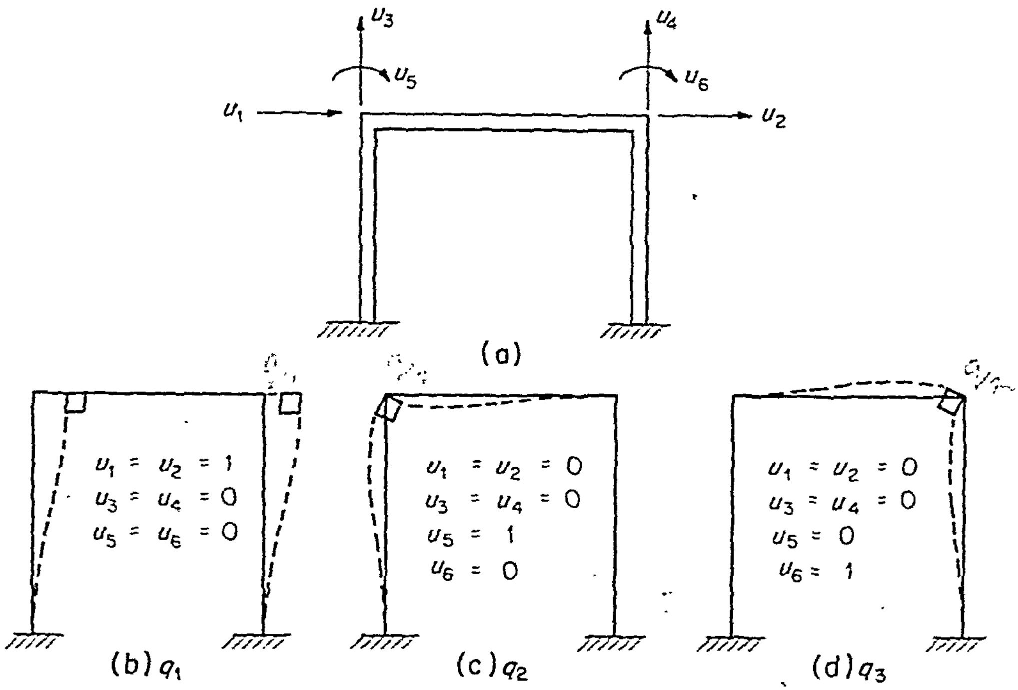


Figure 1.5

certain axes as zero, or to imposing kinematical constraints on a set of coordinates. Thus, constraints may be introduced because of certain idealizations which may be made for convenience of analysis. Consider, for example, a rigid plane frame as shown in Fig. 1.5.

Six coordinates suffice to define the displacements of this structure at the corners. If the members are considered to be flexible, both in the axial and lateral directions, the structure will have six degrees of freedom. However, since axial deflections of beams or columns are generally much smaller than lateral deflections, let us idealize the structure by considering all axial flexibilities to be zero, that is, the lengths of all the members will remain unchanged under load. The following three equations of constraint are deduced.

$$\left.\begin{aligned} u_1 - u_2 &= 0 \\ u_3 &= 0 \\ u_4 &= 0 \end{aligned}\right\} \tag{1.5}$$

Then, there are, from Eq. (1.4), three generalized coordinates; hence, three degrees of freedom. Three independent deflection configurations are shown in Fig. 1.5 (b), (c), (d). If we call the three generalized displacements q_1, q_2, q_3, they are then related to the u's as

$$\left.\begin{aligned} q_1 &= u_1 = u_2 \\ q_2 &= u_5 \\ q_3 &= u_6 \end{aligned}\right\} \tag{1.6}$$

Alternately, it may be useful to adopt a different set of generalized coordinates $\bar{q}_1, \bar{q}_2, \bar{q}_3$ related to the q's as

$$\left.\begin{aligned} \bar{q}_1 &= q_1 \\ \bar{q}_2 &= q_2 + q_3 \\ \bar{q}_3 &= -q_2 + q_3 \end{aligned}\right\} \tag{1.7}$$

This transformation gives generalized displacements which are either symmetrical or anti-symmetrical with respect to a vertical axis through the center of the frame, as shown in Fig. 1.6. Other combinations are also possible, giving in any case three generalized coordinates.

A second example will illustrate a more general procedure for determining a coordinate transformation from a set of constrained coordinates to a set of generalized coordinates. Consider again, a

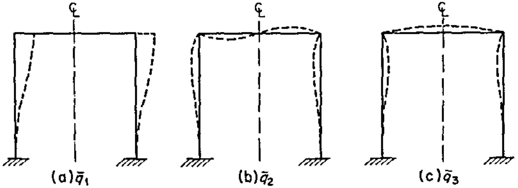


Figure 1.6

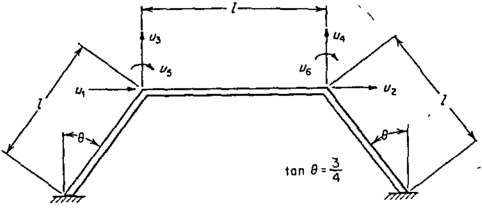


Figure 1.7

plane frame similar to that of the last example except that the columns are not vertical. This frame is shown in Fig. 1.7.
Again, the axial deflections are considered to be zero. In the resulting three equations of constraint only the displacements u_1 through u_4 are involved. These equations are

$$\begin{aligned} u_1 - u_2 &= 0 \\ u_1 \sin\theta + u_3 \cos\theta &= 0 \\ -u_2 \sin\theta + u_4 \cos\theta &= 0 \end{aligned} \tag{1.8}$$

In matrix form these equations are

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1.9}$$

If u_1 is chosen as the independent coordinate, we may write this equation as

$$\begin{bmatrix} 1 \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1.10}$$

If we transfer the first term to the right side and premultiply both sides by the inverse of the 3×3 matrix we obtain

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{1}{\cos^2\theta} \begin{bmatrix} \cos^2\theta & 0 & 0 \\ 0 & -\cos\theta & 0 \\ \sin\theta\cos\theta & 0 & -\cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ \sin\theta \\ 0 \end{bmatrix} u_1$$

Carrying out the multiplication, the resulting equation becomes

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 \\ -\tan\theta \\ \tan\theta \end{bmatrix} u_1 \tag{1.11}$$

The complete set of six displacements may be written in terms of the independent coordinates u_1, u_5, u_6. Here $\tan\theta$ is set equal to $\frac{3}{4}$.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -\frac{3}{4} & 0 & 0 \\ \frac{3}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_5 \\ u_6 \end{Bmatrix} \tag{1.12}$$

Again, it may be convenient to use generalized coordinates which represent symmetrical and anti-symmetrical displacements. Thus, we may choose

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_5 \\ u_6 \end{Bmatrix} \tag{1.13}$$

Inverting this last 3×3 matrix and substituting $\{u_1\, u_5\, u_6\}$ into Eq. (1.12) results in the following tranformation:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 \\ \frac{3}{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \tag{1.14}$$

As the final example in this section we shall consider a problem in which deflections are defined in terms of a set of functions as in Eq. (1.2). Again a square, plane frame is used so that we may later compare results with the previous example. Figure 1.8 shows the frame with an exploded view to aid in defining terms.

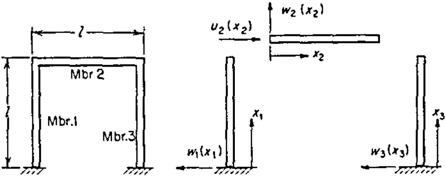


Figure 1.8

A simple analysis shows that the lateral displacement $w(x)$ of a uniform beam loaded at its ends contains terms which depend upon x^2 and x^3.* In addition, if the slope at $x = 0$ is not zero, another term exists which depends upon x and, if the displacement $w(x)$ at $x = 0$ is not zero, a constant term is added. Consequently, the functions $\phi(x)$ are chosen to satisfy these requirements. For each of the three members the functions are as follows.

*See Fig. 1.11 and corresponding Eqs. (1.41).

$$\left.\begin{aligned}
&\text{Member 1.} \quad w_1(x_1) = \phi_1(x_1)\,p_1 + \phi_2(x_1)\,p_2 \\
&\text{Member 2.} \quad w_2(x_2) = \phi_3(x_2)\,p_3 + \phi_4(x_2)\,p_4 + \phi_5(x_2)\,p_5 \\
&\qquad\qquad\quad\; u_2(x_2) = \phi_6(x_2)\,p_6 \\
&\text{Member 3.} \quad w_3(x_3) = \phi_7(x_3)\,p_7 + \phi_8(x_3)\,p_8
\end{aligned}\right\} \tag{1.15}$$

where

$$\begin{aligned}
\phi_1 &= \left(\frac{x}{l}\right)^2 \\
\phi_2 &= \left(\frac{x}{l}\right)^3 \\
\phi_3 &= \left(\frac{x}{l}\right) \\
\phi_4 &= \left(\frac{x}{l}\right)^2 \\
\phi_5 &= \left(\frac{x}{l}\right)^3 \\
\phi_6 &= 1 \\
\phi_7 &= \left(\frac{x}{l}\right)^2 \\
\phi_8 &= \left(\frac{x}{l}\right)^3
\end{aligned} \tag{1.16}$$

Note that ϕ_3 represents a rigid-body rotation of member 2 and that ϕ_6 represents a rigid-body horizontal translation of that member. The remaining functions represent deformations. Note also that the functions chosen satisfy the boundary conditions at the base of the structure, that is

$$w(0) = 0 \quad \text{and} \quad w'(0) = 0$$

The prime in $w'(0)$ designates a derivative with respect to a space coordinate.

The kinematical equations of constraint arise from a consideration of the compatibility of deflections at the two corners with respect to both translation and rotation. Six equations exist as follows:

$$\left.\begin{aligned}
w_1(l) + u_2(0) &= 0 \\
w_3(l) + u_2(l) &= 0 \\
w_2(0) &= 0 \\
w_2(l) &= 0 \\
w_1'(l) - w_2'(0) &= 0 \\
w_3'(l) - w_2'(l) &= 0
\end{aligned}\right\} \tag{1.17}$$

Substituting Eqs. (1.15) and (1.16) into (1.17) yields the following five equations of constraint among the coordinates p. [The third of Eqs. (1.17) results in $0 = 0$ and, consequently, is eliminated in Eqs. (1.18).]

$$\left.\begin{aligned}
&p_1 + p_2 + p_6 &&= 0\\
&p_7 + p_8 + p_6 &&= 0\\
&p_3 + p_4 + p_5 &&= 0\\
&\frac{2}{l}p_1 + \frac{3}{l}p_2 - \frac{1}{l}p_3 &&= 0\\
&\frac{2}{l}p_7 + \frac{3}{l}p_8 - \frac{1}{l}p_3 - \frac{2}{l}p_4 - \frac{3}{l}p_5 &&= 0
\end{aligned}\right\} \tag{1.18}$$

In matrix form these equations, after multiplying the last two by l, are

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\ 2 & 3 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & -2 & -3 & 0 & 2 & 3 \end{bmatrix} \begin{Bmatrix} p_1\\ p_2\\ p_3\\ p_4\\ p_5\\ p_6\\ p_7\\ p_8 \end{Bmatrix} = \begin{Bmatrix} 0\\ 0\\ 0\\ 0\\ 0 \end{Bmatrix} \tag{1.19}$$

Coordinates p_2, p_6, and p_8 are selected as independent coordinates and are transferred to the right side of the equation, leaving the remaining five terms involving the dependent coordinates on the left side. The independent coordinates must be chosen so that the 5×5 matrix on the left side has an inverse. Inverting this matrix and proceeding as in the last example, leads to the following transformation.

$$\begin{Bmatrix} p_1\\ p_2\\ p_3\\ p_4\\ p_5\\ p_6\\ p_7\\ p_8 \end{Bmatrix} = \begin{bmatrix} -1 & -1 & 0\\ 1 & 0 & 0\\ 1 & -2 & 0\\ -2 & 6 & -1\\ 1 & -4 & 1\\ 0 & 1 & 0\\ 0 & -1 & -1\\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} p_2\\ p_6\\ p_8 \end{Bmatrix} \tag{1.20}$$

In general, if in any structural system with holonomic constraints there is established a set of m coordinates ($u_1, u_2, \ldots, u_m$ or $p_1, p_2, \ldots, p_m$)

and if there exist among these coordinates r equations of constraint, it is possible to find a coordinate transformation which will relate the constrained coordinates to a set of $n = m - r$ generalized coordinates $(q_1, q_2, \ldots, q_n)$. In matrix form the transformation is

$$\{u\} = [C]\ \{q\} \tag{1.21}$$

where $[C]$ is the transformation matrix and is of order $m \times n$.

1.4 Virtual Work and Generalized Forces

In coordinate transformations connected with structural problems we are concerned not only with the transformation as related to displacements but also as related to forces. If a transformation such as expressed by Eq. (1.21) exists among displacements there will then also exist a transformation which expresses a relationship among forces $\{F\}$ in the u coordinate system and forces $\{Q\}$ in the q coordinate system. The latter forces in the generalized coordinate system are called, appropriately, *generalized forces.*

To derive the associated force transformation it is useful to appeal to the concept of virtual work. To clarify this concept it is necessary, first, to define a virtual displacement. Stated simply, a virtual displacement is an arbitrarily small change in an independent or generalized displacement configuration of a structure. With a structure having n degrees of freedom one may associate n different and independent virtual displacements. Virtual work is the work done by the applied forces acting through a virtual displacement.

Consider the coordinate system associated with a set of discrete points on the structure as in Fig. 1.2. Associated with the coordinate displacements $u_1, u_2, \ldots, u_m$ are a set of forces $F_1, F_2, \ldots, F_m$. Consider now a vector $\{u\}$ which, for the purpose of this section, is to be thought of as a set of m displacements associated with a general virtual displacement of the structure. The virtual work done by these forces on their respective displacements is given by the equation

$$\text{Virtual work} = \{F\}^T\{u\} \tag{1.22}$$

Now, the same general virtual displacement may be expressed by a vector $\{q\}$ having n components. The virtual work done by the same set of forces, expressed in the generalized coordinate system, acting through the virtual displacement $\{q\}$ is given by

$$\text{Virtual work} = \{Q\}^T\{q\} \tag{1.23}$$

Note that in these equations virtual work is a scalar quantity. In each equation we are considering the same set of forces acting through the same virtual displacement. The only difference between the two equations lies in the fact that we are expressing those forces and displacements in different coordinate systems. Based on physical consideration the virtual work is the same regardless of the coordinate system used to express it. Virtual work is said to be an invariant under the coordinate transformation. It follows, therefore, that

$$\{Q\}^T\{q\} = \{F\}^T\{u\} \tag{1.24}$$

Now, the displacements are related by Eq. (1.21). This relationship exists for virtual displacements as well as for real displacements. Substituting this equation into Eq. (1.24) results in

$$\{Q\}^T\{q\} = \{F\}^T[C]\{q\} \tag{1.25}$$

Since the virtual displacements represented by the components of $\{q\}$ are all independent it follows that the relationship between the force vectors is given by

$$\{Q\}^T = \{F\}^T[C]$$

The transposed equation

$$\{Q\} = [C]^T\{F\} \tag{1.26}$$

may alternately be used. This equation expresses the force transformation associated with the displacement transformation given by Eq. (1.21).

To illustrate, consider the problem of the last section involving the plane frame shown in Fig. 1.7. The displacement transformation with which we are concerned is given by Eq. (1.14). From Eq. (1.26) the force transformation is written immediately as follows.

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 2 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} \tag{1.27}$$

Of particular interest is the generalized force Q_1 which is separately expressed as

$$Q_1 = F_1 + F_2 - \tfrac{3}{4}F_3 + \tfrac{3}{4}F_4 \tag{1.28}$$

In Fig. 1.9 the first generalized displacement q_1 is shown for this structure, with the four contributing forces applied. It is seen that each of these forces does work on the displacement and, hence, contributes to the generalized force Q_1. Note that all forces do positive

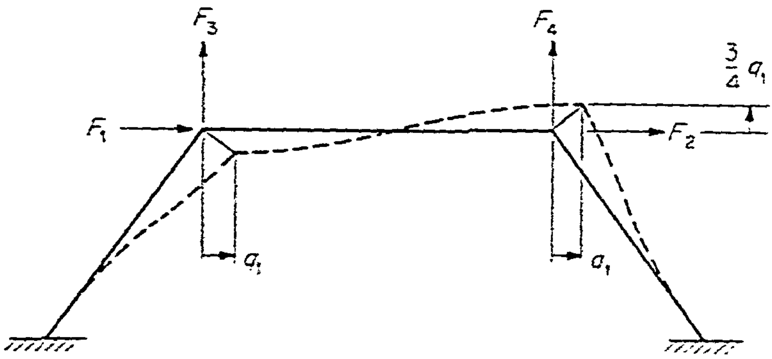


Figure 1.9 Generalized displacement q_1.

work except F_3 for which the work done is negative; hence the negative sign in Eq. (1.28).

The same procedure, based on the concept of virtual work, may be used to determine generalized forces associated with continuously distributed force systems. To illustrate, we consider a structure with elements of one dimension such as the simple frame shown in Fig. 1.8. Deflections may be represented as in Eq. (1.3) which for one dimension is written as

$$w(x) = \sum_{i=1}^{m} \phi_i(x)\, p_i \tag{1.29}$$

This equation applies also if we consider $w(x)$ as a virtual displacement function and the p's as "virtual changes" in the coordinates associated with that virtual displacement.

Now, consider a distributed force function $f(x)$ such that a positive force at every value of x has the same direction as a positive displacement $w(x)$. The virtual work expression in terms of these forces is

$$\text{Virtual work} = \int f(x)\, w(x)\, dx \tag{1.30}$$

where the integral is taken over the entire length of the structure, including all its branches. Substituting Eq. (1.29) and interchanging the order of integration and summation yields the following expression.

$$\text{Virtual work} = \sum_{i=1}^{m} p_i \int f(x)\, \phi_i(x)\, dx \tag{1.31}$$

If, for convenience, we use the letter Γ to represent the force integral

$$\Gamma_i = \int f(x)\, \phi_i(x)\, dx \tag{1.32}$$

Eq. (1.31) then takes the following matrix form.

$$\text{Virtual work} = \{\Gamma\}^T \{p\} \tag{1.33}$$

Also, the virtual work may be written in terms of the generalized forces as in Eq. (1.23). Equating the two expressions gives

$$\{Q\}^T\{q\} = \{\Gamma\}^T\{p\} \tag{1.34}$$

Now, we write the transformation of displacements

$$\{p\} = [C]\{q\} \tag{1.35}$$

Note that this transformation is exemplified by Eq. (1.20) where the independent coordinates $\{p_2, p_6, p_8\}$ are generalized coordinates. If we substitute Eq. (1.35) into Eq. (1.34) and note, as before, that the components of the vector $\{q\}$ are independent, we are led to the following equation of force transformation

$$\{Q\} = [C]^T\{\Gamma\} \tag{1.36}$$

From this relationship it follows that any one of the generalized forces, say the kth one, may be written as follows.

$$Q_k = \sum_{i=1}^{m} C_{ik} \int f(x)\,\phi_i(x)\,dx \tag{1.37}$$

Distributed forces applied to structural systems may also be transformed when the coordinate system relates to a set of discrete points. In this case the transformation yields a set of discrete generalized forces which produce the same gross deflection of the structure as do the distributed forces. By gross deflection is meant that part of the deflection characterized by deflections of the coordinate points. In addition to this gross deflection, those members to which the distributed forces are applied directly will undergo deflections relative to the zero deflection configuration of the generalized coordinate system. *The total deflection of the structure is the sum of the two.*

To illustrate this type of problem consider the frame shown in Fig. 1.5 and the coordinate transformation given by Eq. (1.6). We *shall consider a lateral wind force applied to the left-hand column* of the frame as shown in Fig. 1.10.

The distributed force $f(x)$ will produce the same deflections q_1, q_2, q_3

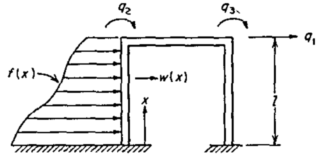


Figure 1.10

as would be produced by a set of generalized forces Q_1, Q_2, Q_3. In addition, however, the left-hand column will undergo a further deflection even with $q_1 = q_2 = q_3 = 0$. The total lateral deflection $w(x)$ of the loaded column may be considered as the sum of the deflection $w_0(x)$, which is independent of the generalized displacements, and deflections dependent upon those displacements which, in this problem, include only q_1 and q_2. Thus, we write

$$w(x) = w_0(x) + \phi_1(x)\, q_1 + \phi_2(x)\, q_2 \tag{1.38}$$

where

$w_0(x)$ = deflection relative to zero generalized displacement

$\phi_1(x)$ = deflection associated with a unit displacement on coordinate q_1

$\phi_2(x)$ = deflection associated with a unit displacement on coordinate q_2

The virtual work equation is derived by equating the virtual work expressed in terms of the distributed force $f(x)$ to that in terms of the generalized forces, where q_1, q_2, q_3 and $w_0(x)$ are regarded as virtual displacements. In considering this equation it must be borne in mind that a virtual displacement in q_1 and q_2 is associated with a virtual displacement in $w(x)$, which involves only the last two terms on the right side of Eq. (1.38).

$$\begin{aligned} &Q_1 q_1 + Q_2 q_2 + Q_3 q_3 \\ &\quad = q_1 \int_0^l f(x)\, \phi_1(x)\, dx + q_2 \int_0^l f(x)\, \phi_2(x)\, dx \end{aligned} \tag{1.39}$$

From this equation it is seen that the generalized forces are as follows:

$$\begin{aligned} Q_1 &= \int_0^l f(x)\, \phi_1(x)\, dx \\ Q_2 &= \int_0^l f(x)\, \phi_2(x)\, dx \\ Q_3 &= 0 \end{aligned} \tag{1.40}$$

The deflection functions $\phi_1(x)$ and $\phi_2(x)$ are shown in Fig. 1.11. They may be calculated by applying forces and moments (corresponding to the generalized forces) necessary to produce the required unit displacements in each case. For a uniform beam of length l, they are

$$\left.\begin{aligned} \phi_1(x) &= 3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3 \\ \phi_2(x) &= -l\left(\frac{x}{l}\right)^2\left[1 - \frac{x}{l}\right] \end{aligned}\right\} \tag{1.41}$$

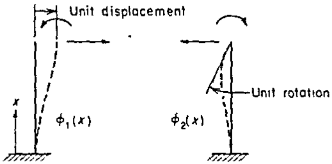


Figure 1.11

It can be shown, using the principle of virtual displacements or using the first theorem of Castigliano[2] (to be discussed later in this chapter), that the forces Q_1 and Q_2 (Eq. (1.40)) are equal in magnitude and opposite in direction to the reactions at the upper end of the loaded column necessary to equilibrate force $f(x)$ when the displacements q_1 and q_2 are zero.

1.5 Force and Deflection Influence Coefficients

Where forces and deflections are defined in a system of discrete coordinates they are related to each other through a set of force-deflection influence coefficients. These coefficients are uniquely determined by the force-deflection properties of the structure and the coordinate system. They are of two kinds, *flexibility coefficients* and *stiffness coefficients* and they relate forces and deflections as shown in the following two equations.

$$\{q\} = [a]\{Q\} \tag{1.42}$$

$$\{Q\} = [k]\{q\} \tag{1.43}$$

where

$\{q\}$ = a generalized displacement vector of n components
$\{Q\}$ = a generalized force vector of n components
$[a]$ = a square matrix of flexibility coefficients of order n
$[k]$ = a square matrix of stiffness coefficients of order n

Substituting Eq. (1.43) into (1.42) the following results are obtained.

$$\{q\} = [a][k]\{q\} \tag{1.44}$$

Because vector $\{q\}$ is entirely arbitrary it is necessary that

$$[a][k] = [I] \tag{1.45}$$

in which $[I]$ is the identity matrix. Therefore, the two matrices bear the reciprocal or inverse relationship

$$\begin{aligned}[a] &= [k]^{-1} \\ [k] &= [a]^{-1}\end{aligned} \tag{1.46}$$

It is important to acquire a clear understanding of the physical significance of these coefficients. To aid in interpretation, Eqs. (1.42) and (1.43) are restated in expanded form. First, we shall discuss the flexibility coefficients and restate Eq. (1.42) as follows:

$$\left.\begin{aligned}
q_1 &= a_{11}Q_1 + a_{12}Q_2 + \ldots + a_{1j}Q_j + \ldots + a_{1n}Q_n \\
q_2 &= a_{21}Q_1 + a_{22}Q_2 + \ldots + a_{2j}Q_j + \ldots + a_{2n}Q_n \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
q_i &= a_{i1}Q_1 + a_{i2}Q_2 + \ldots + a_{ij}Q_j + \ldots + a_{in}Q_n \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
q_n &= a_{n1}Q_1 + a_{n2}Q_2 + \ldots + a_{nj}Q_j + \ldots + a_{nn}Q_n
\end{aligned}\right\} \tag{1.47}$$

To determine the typical coefficient a_{ij} we compute the deflection q_i which results from the application of a force Q_j equal to unity, with all other forces set equal to zero. From the ith equation of the above set, the resulting deflection is the desired coefficient a_{ij}. In fact, all the elements of the jth column $(a_{1j}, a_{2j}, \ldots, a_{ij}, \ldots, a_{nj})$ are found by determining all of the deflections resulting from the unit force Q_j.

In a similar manner Eq. (1.43) is expanded.

$$\left.\begin{aligned}
Q_1 &= k_{11}q_1 + k_{12}q_2 + \ldots + k_{1j}q_j + \ldots + k_{1n}q_n \\
Q_2 &= k_{21}q_1 + k_{22}q_2 + \ldots + k_{2j}q_j + \ldots + k_{2n}q_n \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
Q_i &= k_{i1}q_1 + k_{i2}q_2 + \ldots + k_{ij}q_j + \ldots + k_{in}q_n \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
Q_n &= k_{n1}q_1 + k_{n2}q_2 + \ldots + k_{nj}q_j + \ldots + k_{nn}q_n
\end{aligned}\right\} \tag{1.48}$$

To determine a typical column, say the jth one, of the stiffness matrix we produce a deflection configuration in which all deflections at the chosen coordinates are zero except q_j which is given a unit value. A set of forces, $Q_1, Q_2, \ldots, Q_i, \ldots, Q_n$, is required to establish and equilibrate this deflected configuration. From Eqs. (1.48) these forces are identically $k_{1j}, k_{2j}, \ldots, k_{ij}, \ldots, k_{nj}$, i.e., the jth column of the stiffness matrix.

For simple structures it is relatively easy to determine deflection influence coefficients by the procedures suggested above. If the structure is statically determinate, flexibility coefficients are readily evaluated. Stiffness coefficients may be determined without regard to static indeterminacy, but the ease with which they may be evaluated

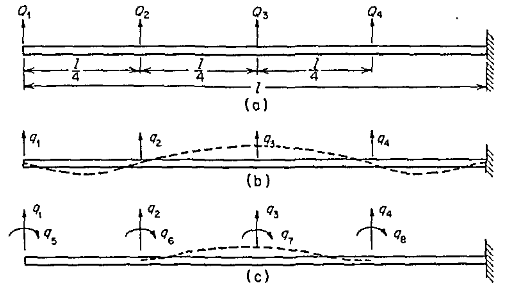


Figure 1.12

depends significantly on the nature and extent of the coordinate system adopted. This point is illustrated by the following example.

Consider a simple cantilever beam, as shown in Fig. 1.12(a), with four equally spaced transverse forces $Q_1 \ldots Q_4$ applied to it. If our problem is to determine the four lateral deflections at the forces, a convenient and adequate coordinate system would include $q_1 \ldots q_4$, as shown in Fig. 1.12(b). A flexibility matrix using these four coordinates may, indeed, be determined readily. If the stiffness matrix associated with the same coordinate system were desired it could be found by inverting the flexibility matrix according to Eq. (1.46). However, let us suppose that we wish to determine this stiffness matrix without first finding the flexibility matrix. We might then use the direct approach already suggested, in which we determine forces required *to produce a set of prescribed deflection configurations. If, for example,* we were to determine the third column of the stiffness matrix, $\{k_{13}, k_{23}, k_{33}, k_{43}\}$, a configuration in which q_3 is unity and $q_1 = q_2 = q_4 = 0$ is required. This configuration is shown dotted in Fig. 1.12(b). To compute the necessary forces we must solve the problem of a beam loaded at point 3, simply supported at points 1, 2, and 4, and clamped at the right end. Coefficient k_{33} is the force at point 3 required to produce a unit deflection at that point. Coefficients k_{13}, k_{23}, and k_{43} are the required reactions at the simple supports. While this problem is not difficult to solve, the purpose of this example is to point out that the desired stiffness matrix can be generated more easily by expanding the coordinate system to include rotations q_5, q_6, q_7, q_8. Although this expands the stiffness matrix to order eight, which would seem to indicate greater difficulty, the task of finding the individual stiffness coefficients is greatly reduced. For example, the third column of the matrix is now determined by considering, as before, a configuration in which q_3 is unity and all other displacements are zero. Since these zero displacements now include rotations, the configuration is as shown dashed in Fig. 1.12(c). Now, the problem reduces to that of finding the required force on the end of a cantilever beam (in fact, two beams) to produce a unit deflection, and an end moment required to produce a zero slope ($q_6 = q_7 = q_8 = 0$). Had our force system included couples applied at the four load points it would have been clearly necessary to include the four rotation coordinates at the outset. However, since no couples actually exist it is possible now to reduce our eighth-order stiffness matrix to a fourth-order one by the following procedure.

Let us write Eq. (1.43) showing the matrix $[k]$ partitioned to permit separation of the translations and rotations.

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ \hline Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix} = \left[\begin{array}{c:c} [k]_{11} & [k]_{12} \\ \hdashline [k]_{21} & [k]_{22} \end{array}\right] \begin{Bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \\ \hline \begin{Bmatrix} q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} \end{Bmatrix} \tag{1.49}$$

In this example, the submatrices $[k]_{11}$, $[k]_{12}$, $[k]_{21}$, and $[k]_{22}$ are all of order four. This equation may be written, alternately, as two matrix equations involving these four submatrices as follows:

$$\begin{aligned} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} &= [k]_{11} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} + [k]_{12} \begin{Bmatrix} q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} \\ \begin{Bmatrix} Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix} &= [k]_{21} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} + [k]_{22} \begin{Bmatrix} q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} \end{aligned} \tag{1.50}$$

Since the forces (couples) $Q_5 \ldots Q_8$ are zero, the second equation above may be used to express the rotations $q_5 \ldots q_8$ in terms of the translations $q_1 \ldots q_4$.

$$\begin{Bmatrix} q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = -[k]_{22}^{-1}[k]_{21} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \tag{1.51}$$

If this result is substituted into the first equation of Eq. (1.50) there is obtained

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = ([k]_{11} - [k]_{12}[k]_{22}^{-1}[k]_{21}) \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \tag{1.52}$$

The reduced matrix $[k]^*$ in parenthesis, is square and of order four.

$$[k]^* = [k]_{11} - [k]_{12}[k]_{22}^{-1}[k]_{21} \tag{1.53}$$

This is the stiffness matrix relating transverse forces to the lateral deflections which was desired. It is the inverse of the fourth-order flexibility matrix discussed at the outset of the problem.

For many structures the procedures illustrated by the foregoing example are adequate for the determination of deflection influence coefficients. However, in structural systems which are both highly redundant and geometrically complicated, additional procedures may be found useful in which the flexibility and/or stiffness matrices for the complete system are synthesized from the properties of the elements of the system. These additional procedures will be discussed later.

1.6 Reciprocity Relationship Among Forces and Deflections

In structures having linear force-deflection properties the flexibility and stiffness coefficients have the following property, often referred to as *Maxwell's reciprocity relationship.*

$$\left.\begin{aligned} a_{ij} &= a_{ji} \\ k_{ij} &= k_{ji} \end{aligned}\right\} \tag{1.54}$$

Therefore, the flexibility and stiffness matrices are symmetric. This important property has a profound bearing on the response of structures in the linear force-deflection range in terms of both static and dynamic behavior.

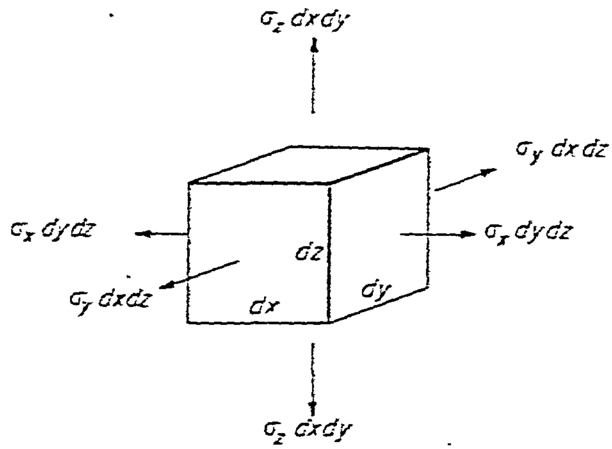


Figure 1.13

To prove the reciprocity relationship expressed by Eq. (1.54) consider a structure as a set of connected elemental volumes $dV(dx, dy, dz)$, one of which is shown in Fig. 1.13. Without loss of generality and to simplify the analysis, consider these elements to be oriented along the principal axes of stress and strain. Principle stresses $\sigma_x, \sigma_y, \sigma_z$ are shown as applied to the element in Fig. 1.13.
Hooke's law relates stress and strain in elastic bodies and is written as follows for principal stresses.

$$\{\epsilon\} = \frac{1}{E}[N]\{\sigma\} \tag{1.55}$$

where

$\{\epsilon\}$ = a vector of the three strain components
$\{\sigma\}$ = a vector of the three stress components
E = the modulus of elasticity

$$[N] = \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \tag{1.56}$$

ν = Poisson's ratio.

Now, consider coordinates r and s of a set of generalized coordinates, with associated deflections q_r and q_s and forces Q_r and Q_s. Each of the two forces will cause deflections to exist on all coordinates of the system, in particular on coordinates r and s. Denote by $q_r^{(s)}$ the deflection on coordinate r caused by force Q_s, and by $q_s^{(r)}$ the deflection on coordinate s caused by force Q_r. Also, denote by $\{\sigma^{(r)}\}$ and $\{\epsilon^{(r)}\}$ the stress and strain vectors resulting from force Q_r, and by $\{\sigma^{(s)}\}$ and $\{\epsilon^{(s)}\}$ those resulting from force Q_s. Thus, deflection $q_r^{(s)}$

is compatible with strains $\{\epsilon^{(s)}\}$ and deflection $q_s^{(r)}$ is compatible with strains $\{\epsilon^{(r)}\}$.

Using the concept of virtual work, the virtual work done by stresses $\{\sigma^{(r)}\}$ on virtual strains $\{\epsilon^{(s)}\}$ is given for a single element by the following expression:

$$\text{Virtual work on element } dV = \{\sigma^{(r)}\}^T\{\epsilon^{(s)}\}\,dx\,dy\,dz$$

When integrated over the entire structure this total virtual work is equal to that done by force Q_r on a virtual displacement $q_r^{(s)}$.

$$Q_r q_r^{(s)} = \int_V \{\sigma^{(r)}\}^T\{\epsilon^{(s)}\}\,dx\,dy\,dz \tag{1.57}$$

Substituting Hooke's law into this equation gives the result

$$Q_r q_r^{(s)} = \int_V \frac{1}{E}\{\sigma^{(r)}\}^T[N]\{\sigma^{(s)}\}\,dx\,dy\,dz \tag{1.58}$$

In a similar way the virtual work done by force Q_s on a virtual displacement $q_s^{(r)}$ is determined to be

$$Q_s q_s^{(r)} = \int_V \frac{1}{E}\{\sigma^{(s)}\}^T[N]\{\sigma^{(r)}\}\,dx\,dy\,dz \tag{1.59}$$

The triple matrix products in the integrands of Eqs. (1.58) and (1.59) are scalars and, therefore, these products may be transposed without altering the result. Thus

$$\{\sigma^{(r)}\}^T[N]\{\sigma^{(s)}\} = \{\sigma^{(s)}\}^T[N]\{\sigma^{(r)}\} \tag{1.60}$$

Therefore, the two integrals are identical. It must be noted, however, that this is true only because the matrix $[N]$ is symmetric in consequence of Hooke's law. Thus, the following resulting relationship is dependent upon Hooke's law.

$$Q_r q_r^{(s)} = Q_s q_s^{(r)} \tag{1.61}$$

This interesting and important relationship is known as *Betti's law*. If we now use flexibility coefficients to relate force and deflection, we find

$$\left.\begin{aligned} q_r^{(s)} &= a_{rs} Q_s \\ q_s^{(r)} &= a_{sr} Q_r \end{aligned}\right\} \tag{1.62}$$

Substitution of these equations into Eq. (1.61) leads to the reciprocity relationship

$$a_{rs} = a_{sr}$$

Similarly, the force-deflection relationships may be expressed in terms of stiffness coefficients, leading to the reciprocity relationship among those coefficients.

$$k_{rs} = k_{sr}$$

The importance of the reciprocity relationship in the theory of structures will be in evidence repeatedly in following sections.

1.7 Force-Deflection Influence Functions

For distributed systems, forces and deflections are related by a deflection influence function which in three space has the form

$$a = a(x, y, z; \xi, \eta, \zeta) \tag{1.63}$$

in which (x, y, z) and (ξ, η, ζ) are a dual set of position coordinates. Since this function plays the same role in distributed systems as the flexibility coefficient a_{ij} plays in discretized systems, it may appropriately be called a *flexibility influence function*. In one dimension it relates force and deflection as

$$u(x) = \int_L a(x, \xi) f(\xi)\, d\xi \tag{1.64}$$

where

$u(x)$ = a deflection at x, which may be an axial or transverse displacement or a rotation

$f(\xi)$ = force per unit of length at ξ

ξ = a variable of integration (the integration extends over the length of the one-dimensional structure)

If we write this equation in differential form we have

$$du(x) = a(x, \xi) f(\xi)\, d\xi \tag{1.65}$$

Now, $f(\xi)\, d\xi$ is the force applied to the differential element at ξ, and $du(x)$ is the incremental deflection at x caused by this force. Therefore, the function $a(x, \xi)$ may be regarded as the increment of deflection at x produced by a unit force at ξ. Hence, this function may be constructed by applying a unit point force on the structure at distance ξ from the origin and computing the resulting deflection at distance x from the origin. Note that if the structure is divided into a number of increments of finite length $\Delta\xi$, the integral goes over to a sum of a finite number of terms and Eq. (1.64) becomes

$$u(x_i) = \sum_j a(x_i, \xi_j) f(\xi_j)\, \Delta\xi_j$$

where

$u(x_i)$ = the deflection at the ith element

$f(\xi_j)\, \Delta\xi_j = F_j$ is the force on the jth element

If we write $u(x_i)$ as u_i for simplicity and $a(x_i, \xi_j)$ as a_{ij}, the equation above is

$$u_i = \sum_j a_{ij} F_j \tag{1.66}$$

Except for the change in notation, this equation is identical to that for the ith deflection component derived from Eq. (1.42).

Because of the direct relationship that exists between the function $a(x, \xi)$ *and the coefficient* a_{ij}, *it is to be expected that they both* have the same property of reciprocity, that is

$$a(x, \xi) = a(\xi, x) \tag{1.67}$$

From the foregoing discussion it is evident that the function $a(x, \xi)$ related to a given structure must satisfy the differential equation governing the deflections of that structure together with its boundary conditions. For example, the function for a beam must satisfy the Bernoulli-Euler differential beam equation assuming, of course, that we adopt that equation which includes only the deflection due to flexure of the beam. Consider a uniform cantilever beam as shown in Fig. 1.14 with a unit force at ξ and the deflection $w(x)$ to be determined.

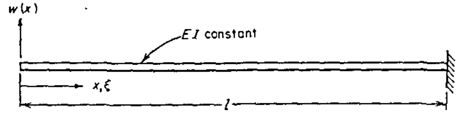


Figure 1.14

The Bernoulli-Euler equation for a uniform beam is[2]

$$\frac{d^4 w(x)}{dx^4} - \frac{f(x)}{EI} = 0 \tag{1.68}$$

Integrating twice yields the equation

$$\frac{d^2 w}{dx^2} = \frac{M(x)}{EI}$$

where $M(x)$ is the bending moment. In terms of the influence function $a(x, \xi)$, $M(x)$ is interpreted as the bending moment at x caused by a unit force at ξ, and the last equation becomes

$$\frac{d^2 a(x, \xi)}{dx^2} = \frac{M(x)}{EI}$$

Since the variables x and ξ range independently from 0 to l, two cases must be considered

Case 1. $a_1(x, \xi)$ for $0 < \xi < x < l$

Case 2. $a_2(x, \xi)$ for $0 < x < \xi < l$

Considering the first case we solve the equation

$$\frac{d^2 a_1(x, \xi)}{dx^2} = \frac{1}{EI}(x - \xi)$$

together with the boundary conditions

$$a_1(l, \xi) = \frac{da_1}{dx}(l, \xi) = 0.$$

The solution is

$$a_1(x, \xi) = \frac{1}{6EI}(l - x)^2 (2l + x - 3\xi) \tag{1.69}$$

To find a solution for the second case we note the following relationship which follows from the fact that the deflection curve is a straight line to the left of the load point.

$$a_2(x, \xi) = a_1(\xi, \xi) + (x - \xi) \frac{da_1}{dx}(\xi, \xi) \tag{1.70}$$

From Eq. (1.69) the following values are found:

$$a_1(\xi, \xi) = \frac{1}{3EI}(l - \xi)^3$$

$$\frac{da_1}{dx}(\xi, \xi) = -\frac{1}{2EI}(l - \xi)^2$$

Substitution of these values into Eq. (1.70) gives the following solution for Case 2:

$$a_2(x, \xi) = \frac{1}{6EI}(l - \xi)^2 (2l + \xi - 3x) \tag{1.71}$$

The complete function is given by Eqs. (1.69) and (1.71). This complete function is plotted in Fig. 1.15, with the range indicated over which the two parts $a_1(x, \xi)$ and $a_2(x, \xi)$ apply.

Comparing the two expressions 1.69 and 1.71 it is seen that the property of reciprocity is stated as

$$\left.\begin{aligned} a_1(x, \xi) &= a_2(\xi, x) \\ a_2(x, \xi) &= a_1(\xi, x) \end{aligned}\right\} \tag{1.72}$$

In Fig. 1.15 the reciprocal function $a(\xi, x)$ is also shown, with the first equality above indicated for the particular values of x, ξ shown.

If a distributed force is applied to the beam, Eq. (1.64) may be used to compute the deflection function $w(x)$ using the influence function which has been constructed. The equation is written as

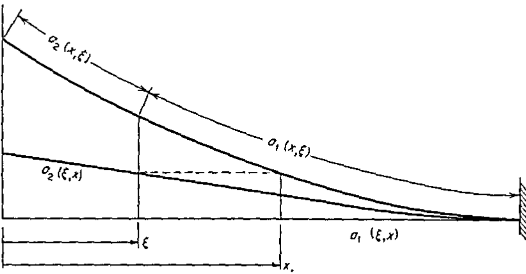


Figure 1.15

follows to show the intervals of integration in ξ for the two separate functions.

$$w(x) = \int_0^x a_1(x,\xi) f(\xi)\, d\xi + \int_x^l a_2(x,\xi) f(\xi)\, d\xi \qquad (1.73)$$

The influence function as discussed in this section is not restricted to applications in structural problems. Known also as *Green's function* it is generally useful in the solution of linear differential equations. Books in applied mathematics[3] show that the function related to a linear differential equation of nth order has the following properties:

1. $a(x,\xi)$ satisfies the differential equation.
2. $a(x,\xi)$ satisfies the prescribed boundary conditions.
3. $a(x,\xi)$ and its first $(n-2)$ derivatives taken with respect to x are continuous at $x = \xi$.
4. The $(n-1)$th derivative in x is discontinuous.

With respect to the last property the discontinuity has a definite magnitude which depends on the differential equation, but this point will not be discussed further.

1.8 Strain Energy

It has been pointed out that an ideal elastic structure is conservative, that is, work done by applied forces acting through the deflections produced is stored in the deflected structure as potential or strain energy. Therefore, the strain energy U may be determined by computing the work done by those forces. Consider an elastic structure with a set of forces $F_1, F_2, \ldots, F_m$ acting through deflections u_1,

$u_2, \ldots, u_m$, where these forces and deflections represent terminal values, that is, those values attained after the forces are all applied and the structure is in static equilibrium in the deflected state. We represent by $F_i(t)$ and $u_i(t)$ intermediate time-dependent values of the terminal force F_i and terminal deflection u_i. Then, the work-energy equation is expressed as

$$U = \sum_{i=1}^{m} \int_0^{u_i} F_i(t)\, du_i(t) \tag{1.74}$$

When the forces $F_i(t)$ are applied slowly so that the inertial forces are negligible, then the deflections $u_i(t)$ are given by

$$u_i(t) = u_i(F_1(t), F_2(t), \ldots, F_m(t))$$

We may express $du_i(t)$ as

$$du_i(t) = \sum_{j=1}^{m} \frac{\partial u_i}{\partial F_j}(t)\, dF_j(t) \tag{1.75}$$

For linear force-deflection properties

$$\frac{\partial u_i}{\partial F_j}(t) = a_{ij} \tag{1.76}$$

where a_{ij} is the flexibility influence coefficient which is constant throughout the loading process. Substituting Eqs. (1.75) and (1.76) into Eq. (1.74) we obtain the equation

$$U = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} \int_0^{F_i} F_i(t)\, dF_j(t)$$

At this point it is convenient to relate the intermediate values of force to the terminal values through a dimensionless time function $f(t)$ as

$$F_i(t) = F_i f_i(t)$$

This relationship is substituted into the work-energy equation

$$U = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} F_i F_j \int_0^1 f_i(t)\, df_j(t) \tag{1.77}$$

Note that the initial and terminal values of $f(t)$ are 0 and 1, respectively. The relative values of any two of these functions $f_i(t)$ and $f_j(t)$ depends upon the loading sequence, which may be entirely arbitrary. If we plot one of these functions against the other, as in Fig. 1.16, the path between the initial point (0,0) and final point (1,1) is arbitrary.

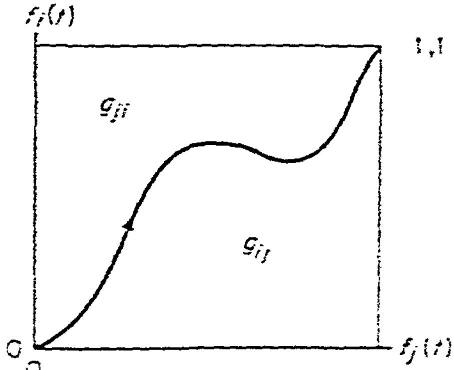


Figure 1.16

If, for convenience, we define the integral in Eq. (1.77) as g_{ij}

it follows that

$$\left.\begin{aligned} g_{ij} &= \int_0^1 f_i(t)\, df_j(t) \\ g_{ji} &= \int_0^1 f_j(t)\, df_i(t) \end{aligned}\right\} \tag{1.78}$$

The two areas represented by these integrals are indicated in Fig. 1.16. The following relationship is clear from that figure and may also be deduced by carrying out the integrations in Eq. (1.78) by parts.

$$g_{ij} + g_{ji} = 1 \tag{1.79}$$

We may now write Eq. (1.77) as

$$U = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} g_{ij} F_i F_j \tag{1.80}$$

which has the matrix form

$$U = \{F\}^T [ag] \{F\} \tag{1.81}$$

where the square matrix $[ag]$ is

$$[ag] = \begin{bmatrix} a_{11}g_{11} & a_{12}g_{12} & \cdots & a_{1m}g_{1m} \\ a_{21}g_{21} & a_{22}g_{22} & \cdots & a_{2m}g_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1}g_{m1} & a_{m2}g_{m2} & \cdots & a_{mm}g_{mm} \end{bmatrix}$$

In consideration of Eq. (1.79) and of the reciprocity relationship of Eq. (1.54) it is a simple exercise to show that the strain energy is independent of the loading sequence and, hence, depends only upon the terminal values of the applied forces. This is done by showing that upon expansion of a few terms of Eq. (1.80) these terms combine so that the sum is independent of the coefficients g. In view of this fact and from the fact deduced from Eq. (1.79) that the diagonal elements g_{ii} are equal to $\frac{1}{2}$, it is permissible and convenient to assign the value $\frac{1}{2}$ to all the g's.

$$g_{ij} = g_{ji} = \tfrac{1}{2}$$

This makes the matrix $[ag]$ proportional to matrix $[a]$ and, hence, symmetric. Therefore, the strain energy U may be expressed in either of the following forms which follow from Eqs. (1.80) and (1.81).

$$\begin{aligned} U &= \tfrac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} F_i F_j \\ &= \tfrac{1}{2} \{F\}^T [a] \{F\} \end{aligned} \tag{1.82}$$

If the coordinates u_i are generalized coordinates, then the strain energy may also be expressed in terms of generalized forces Q.

$$U = \tfrac{1}{2}\{Q\}^T [a] \{Q\} \tag{1.83}$$

where the matrix $[a]$ is now associated with the generalized q-Q coordinate system. Under these conditions that matrix will have an inverse $[k]$ [Eq. (1.46)]. [It will be shown later that constrained systems may exist in which the matrix $[a]$ associated with Eq. (1.82) is singular.] If we substitute Eq. (1.43) into Eq. (1.83) and take into account the inverse relationship between $[k]$ and $[a]$, we obtain the following expression for strain energy.

$$\begin{aligned} U &= \tfrac{1}{2}\{q\}^T [k] \{q\} \\ &= \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} q_i q_j \end{aligned} \tag{1.84}$$

Since strain energy is invariant under coordinate transformations, Eqs. (1.82), (1.83), and (1.84) are useful in showing how the flexibility and stiffness matrices transform. Consider, for example, the matrix $[a]$ in Eq. (1.83), associated with the generalized coordinate system q-Q. Under a transformation such as given by Eq. (1.26) the following result is obtained.

$$\begin{aligned} U &= \tfrac{1}{2}\{Q\}^T [a]_q \{Q\} \\ U &= \tfrac{1}{2}\{F\}^T [C] [a]_q [C]^T \{F\} \\ &= \tfrac{1}{2}\{F\}^T [a]_u \{F\} \end{aligned}$$

where

$$[a]_u = [C][a]_q[C]^T$$

The subscripts u and q are used to stipulate the coordinate system with which the flexibility matrices are associated. If the u's are constrained, as implied in arriving at the transformation matrix $[C]$ in Eq. (1.21), the order of the two flexibility matrices will be different. In this example the order of $[a]_q$ is n and the order of $[a]_u$ is m, where $m > n$.

A second example is given to illustrate a problem which arises frequently in dynamics, namely a transformation which relates a set of generalized coordinates to a set of *normal* coordinates. (See Chapter 3, Section 7.) The transformation is written as

$$\{q\} = [\gamma] \{\eta\}$$

where

$[\gamma]$ = a square transformation matrix

$\{\eta\}$ = a displacement vector expressed in normal coordinates

A stiffness matrix associated with the normal coordinates is obtained by substituting this transformation into Eq. (1.84).

$$U = \tfrac{1}{2}\{\eta\}^T[\gamma]^T[k]_q[\gamma]\{\eta\}$$
$$= \tfrac{1}{2}\{\eta\}^T[k]_\eta\{\eta\}$$

in which

$$[k]_\eta = [\gamma]^T[k]_q[\gamma]$$

As will be shown later this transformation has the property of making $[k]_\eta$ a diagonal matrix. Therefore, it is a useful transformation in structural and dynamical analysis.

Two theorems of great importance in the theory of structures are based on strain energy considerations. These theorems are known generally as the first and second theorems of Castigliano.[2] The first is based on the fact that the strain energy function is uniquely defined by the strains and, therefore, the deflections of the structure. In terms of generalized deflections,

Strain Energy

$$U = U(q_1, q_2, \ldots, q_n) \tag{1.85}$$

A variation, δU, associated with an arbitrary set of virtual displacements, δq, is

$$\delta U = \sum_{i=1}^{n} \frac{\partial U}{\partial q_i}\,\delta q_i \tag{1.86}$$

Based on the concept of virtual work, however, this same change in strain energy may be expressed in terms of the virtual work done by the generalized forces acting through these same virtual displacements.

Q_i

$$\delta U = \sum_{i=1}^{n} Q_i\,\delta q_i \tag{1.87}$$

Because the deflections $q_1 \ldots q_n$ are all independent, the corresponding virtual displacements may be arbitrarily defined. Therefore, the two expressions in Eqs. (1.86) and (1.87) may be compared term by term. The resulting relationship expresses the first theorem of Castigliano.

$$\frac{\partial U}{\partial q_k} = Q_k \tag{1.88}$$

It is important to note that this relationship holds only in a generalized coordinate system because virtual displacements expressed in terms of constrained coordinates are not independent. Hence, the conclusions leading to Eq. (1.88) cannot be drawn. Note that Eq. (1.88) is obtainable from the Lagrange equations (to be developed in Chapter 2) for conservative systems with zero kinetic energy. Note also that in the development of this theorem it was not necessary to stipulate that the force-deflection properties be linear.

To develop the second theorem it is necessary to base the proof on linear properties. We begin with Eq. (1.82) and form the partial derivative of U with respect to, say, F_k.

$$\frac{\partial U}{\partial F_k} = \frac{1}{2} \sum_{i=1}^{m} a_{ik} F_i + \frac{1}{2} \sum_{j=1}^{m} a_{kj} F_j = \frac{1}{2} \sum_{i=1}^{m} (a_{ik} + a_{ki}) F_i$$

The second step above follows from replacing the summation index j with i and combining the terms, since the letter used to designate a summation index is immaterial. Now, using the property of reciprocity related to the flexibility coefficients, we replace a_{ik} with a_{ki} to obtain

$$\frac{\partial U}{\partial F_k} = \sum_{i=1}^{m} a_{ki} F_i$$

From the definition of the flexibility coefficient, the right side of this equation is identically equal to the deflection u_k, hence

$$\frac{\partial U}{\partial F_k} = u_k \tag{1.89}$$

This is the second theorem of Castigliano. It is not restricted to generalized coordinates but applies only to structures whose force-deflection properties are linear.

The theorems expressed by Eqs. (1.88) and (1.89) are useful in the determination of flexibility and stiffness coefficients, particularly in statically indeterminate structures. They may also be used to determine forces in redundant constraints or in redundant members in such structures. In particular, Eq. (1.89) takes the form

$$\frac{\partial U}{\partial F_k} = 0 \tag{1.90}$$

when used to determine a force F_k in a redundant constraint or member where the deflection u_k may be measured in such a way that it must be zero. In this form the theorem is known as the *minimum strain energy theorem*. Its application will be found in many books on the theory of structures.

1.9 Singular Flexibility and Stiffness Matrices

We now consider cases in which the flexibility or stiffness matrix of a system is singular. A matrix is singular if two or more of its rows or columns are linearly dependent.[4]

A. *Flexibility Matrix with Constrained Coordinates*

We will now show that when constraints exist among the coordinates, as discussed in Section 1.3, the resulting flexibility matrix is singular. The constrained displacements are designated as $u_1, \ldots, u_m$ or $p_1, \ldots, p_m$, depending on whether they are discrete or distributed. The relationship between the constrained and generalized coordinates

$q_1, \ldots, q_n$ is given by Eq. (1.21) and is characterized by a transformation or "constraining" matrix $[C]$. Examples of such a matrix are given in Eqs. (1.14) and (1.20). Consider now a flexibility matrix as defined in Eq. (1.42), except that we now wish to consider this matrix in the constrained $u - F$ coordinates rather than in the generalized $q - Q$ coordinates. Thus, for our present purpose we define $[a]$ by the equation

$$\{u\} = [a]\{F\} \tag{1.91}$$

If the force vector $\{F\}$ is arbitrary, it can be seen from the discussion of Section 1.5 that the flexibility matrix exists even though there are kinematic constraints among the displacements. Suppose that an equation of constraint exists in the form

$$c_1 u_1 + c_2 u_2 + \ldots + c_m u_m = 0 \tag{1.92}$$

Any one of the equations in the sets (1.8) or (1.18) will serve as an example. Noting that the ith displacement component may be expressed in the form

$$u_i = \sum_{j=1}^{m} a_{ij} F_j \tag{1.93}$$

we may substitute in Eq. (1.92) and form the equation

$$c_1 \sum_j a_{1j} F_j + c_2 \sum_j a_{2j} F_j + \ldots + c_m \sum_j a_{mj} F_j = 0$$

If we collect all terms under a single sum over j this equation appears as

$$\sum_{j=1}^{m} (c_1 a_{1j} + c_2 a_{2j} + \ldots + c_m a_{mj}) F_j = 0$$

Since the F's are independent, it follows that for each value of j we may write

$$c_1 a_{1j} + c_2 a_{2j} + c_m a_{mj} = 0 \tag{1.94}$$

This equation tells us that there exists, under these circumstances, a linear combination of rows in matrix $[a]$ which will yield a null row, or a row of zeros. From the theory of determinants it follows that the determinant of this matrix is zero, and $[a]$ is singular.[4] If a single equation of constraint exists, the rank of $[a]$ will be $m - 1$. If more than one equation of constraint exists the rank will be reduced accordingly. Clearly, the inverse of $[a]$ does not exist. Hence, the following statement may be made in summary. If constraints exist among the coordinates of a system then the flexibility matrix is singular. Hence a unique stiffness matrix does not exist in the same coordinate system.

When a flexibility matrix $[a]_q$ associated with a generalized coordinate system q-Q has been derived, it is possible to transform it into a flexibility matrix $[a]$ associated with constrained coordinates u-F. The reverse is also possible. These tranformations are obtained as follows. From Eq. (1.42)

$$\{q\} = [a]_q \{Q\} \tag{1.42}$$

Premultiply each side of Eq. (1.42) by the constraining matrix $[C]$, defined by Eq. (1.21)

$$[C]\{q\} = [C][a]_q \{Q\} \tag{1.95}$$

Introduce Eqs. (1.21) and (1.26) into Eq. (1.95)

$$\{u\} = [C][a]_q [C]^T \{F\} \tag{1.96}$$

From Eq. (1.91) $\{u\}$ can also be written as

$$\{u\} = [a]_u \{F\} \tag{1.91}$$

Comparing Eqs. (1.96) and (1.91)

$$[a]_u = [C][a]_q [C]^T \tag{1.97}$$

The reverse procedure, namely obtaining $[a]_q$ from $[a]_u$, is also possible. Premultiply and postmultiply each side of Eq. (1.97) by $[C]^T$ and $[C]$, respectively.

$$[C]^T [a]_u [C] = [C]^T [C][a]_q [C]^T [C] \tag{1.98}$$

The product $(\underset{n\times m}{[C]^T}\underset{m\times n}{[C]})$ appearing in Eq. (1.98) is a square matrix of order n which is equal to the number of generalized coordinates. Premultiply and postmultiply each side of Eq. (1.98) by $([C]^T[C])^{-1}$ to obtain an expression for $[a]_q$ in terms of $[a]_u$

$$[a]_q = ([C]^T [C])^{-1} [C]^T [a]_u [C] ([C]^T [C])^{-1} \tag{1.99}$$

B. *Stiffness Matrix for an Unconstrained Structure*

A structure is unconstrained when a force, however small, can be applied so as to cause an indefinitely large displacement. Structures may be unconstrained internally or externally. In either case a set of forces can be applied which will cause an indefinitely large displacement, as stated above. A common example of an unconstrained structure is one capable of virtual displacements in one or more rigid-body modes, that is, modes which do not induce strains in the structure. Since the structure is in either static or dynamic equilibrium, the applied forces (including inertial forces in the case of dynamic equilibrium) will do no work on a virtual displacement in a rigid-body mode according to the principle of virtual work.[5]

Consider a system of generalized coordinates together with a stiffness matrix $[k]$ which relates generalized forces and displacements according to Eq. (1.43). Let us suppose that virtual displacement

$\{\delta q\}$ exists on which the forces $\{Q\}$ do no work. Then, we may write

$$\{Q\}^T\{\delta q\} = 0 \tag{1.100}$$

Substituting Eq. (1.43) we have

$$\{q\}^T[k]\{\delta q\} = 0 \tag{1.101}$$

This relationship is considered to hold for any arbitrary displacement vector $\{q\}$, that is, the virtual displacements are measured from an arbitrary deflected configuration of the structure. Hence, the following equation holds.

$$[k]\{\delta q\} = \{0\} \tag{1.102}$$

We now recall the following theorem of matrix algebra.[6] Let $[A]$ be a square matrix. Then, a necessary and sufficient condition for $[A]^{-1}$ to exist is that there be no nonzero vector $\{x\}$ such that $[A]\{x\} = \{0\}$. The converse also holds. If there exists a nonzero vector $\{x\}$ such that $[A]\{x\} = \{0\}$, then $[A]$ does not have an inverse.* In Eq. (1.102) we have postulated a nonzero virtual displacement $\{\delta q\}$. Hence, it follows that $[k]$ is singular. This result is summarized in the following statement. If, in a given system of generalized coordinates a virtual displacement is possible on which the virtual work vanishes, the associated stiffness matrix is singular. It follows that in this same coordinate system a flexibility matrix does not exist.

1.10 Synthesis of Force-Deflection Properties of a Structural System

In this and the remaining sections of this chapter, we shall be concerned with methods of synthesizing force-deflection properties of a complete structural system from those of the elements that comprise the system. Although we are concerned primarily with stiffness and flexibility matrices because they are essential in dynamic analysis, the methods extend to the determination of internal forces and deflections or stresses and strains. These results are useful in dynamic stress-strain analysis.

Two methods are distinguished in the following discussion, the *stiffness* method and the *flexibility* method. They are also called the *displacement* method and the *force* method, respectively. Insofar as the matrix equations are concerned, a complete duality exists between the two methods as first pointed out by Argyris.[7] In application,

*The existence of a nonzero vector $\{x\}$ such that $[A]\{x\} = \{0\}$ is equivalent to saying that two or more rows or columns of $[A]$ are linearly dependent.

significant and essential differences exist which may indicate a preferred method for a particular structure.

Three pieces of information are required in the synthesis to be accomplished. These are

1. element coordinate system
2. a transformation of coordinates relating element coordinates to structural system coordinates
3. element stiffnesses and flexibilities

The element coordinate system defines and orders the displacements and forces related to all of the individual elements of the structure. A sub-group of coordinates will relate to each element and the total number of coordinates in the system will be the sum of the coordinates in these subgroups. Displacements are designated by the vector $\{\delta\}$ and forces by $\{P\}$. In terms of the sub-groups these vectors are shown in the following partitioned form:

$$\{\delta\} = \left\{\begin{array}{c} \{\delta\}_1 \\ \hline \{\delta\}_2 \\ \hline \vdots \\ \hline \{\delta\}_i \\ \hline \vdots \\ \hline \{\delta\}_m \end{array}\right\} \quad \text{and} \quad \{P\} = \left\{\begin{array}{c} \{P\}_1 \\ \hline \{P\}_2 \\ \hline \vdots \\ \hline \{P\}_i \\ \hline \vdots \\ \hline \{P\}_m \end{array}\right\} \tag{1.103}$$

In these vectors it is implied that the structure consists of m elements.

The number of coordinates assigned to an element depends upon the type of element and will be different, in general, for the stiffness and flexibility methods. Consider, for example, a beam such as shown in Fig. 1.17.

This beam may be an element in a frame such as shown in Fig. 1.2. Therefore, its coordinates will logically define displacements and

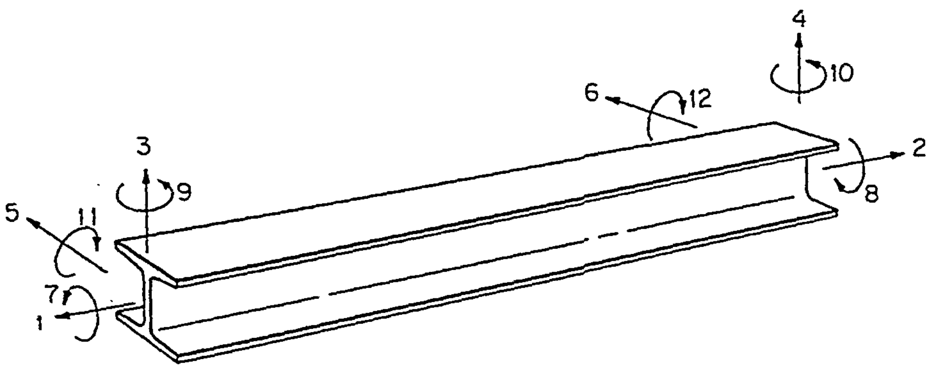


Figure 1.17

forces at the ends where it is joined to other elements. In three space, the system would include three translations and three rotations at each end for a total of twelve coordinates. If the stiffness method is used the twelve coordinates shown would be included in $\{\delta\}$ and $\{P\}$ for this beam. On the other hand, the number of coordinates would be reduced to six if the flexibility method were used. As will be made clear in subsequent discussion the number of coordinates related to a single element in the stiffness method will exceed the number involved in the flexibility method by the number of rigid-body degrees of freedom of the unconstrained element.

The second piece of information required in the synthesis process is a transformation of coordinates relating element coordinates and structural system coordinates. Related to displacements this transformation is

$$\{\delta\} = [\beta]\{q\} \tag{1.104}$$

It relates specifically to generalized displacements $\{q\}$ since only in a generalized coordinate system are the displacement components independent. This independence is essential to the existence of the transformation matrix $[\beta]$ which may be constructed, column by column, by assigning values of unity to each component q, in turn, thus defining sets of generalized deflection configurations with corresponding sets of δ's.

The existence of the transformation is also dependent upon compatibility of the coordinate systems related. This means that it must be possible to define every element displacement δ in each generalized displacement configuration of the system. It also means that the element coordinate system must be complete enough to describe fully the deflection of each element in each generalized displacement configuration. Equation (1.104) may be considered as an equation of *displacement compatibility which insures the connectivity of the system.*

Related to forces, the coordinate transformation with which we shall be concerned is written

$$\{P\} = [b]\{F\} \tag{1.105}$$

In this transformation we relate internal forces to the applied force vector $\{F\}$. Because the applied force components are independent, each may be applied separately. Therefore, each column of $[b]$, say the jth one, is obtained by computing the internal forces which result from the application of a unit value of the jth force component (with all other components set equal to zero). Note that even if the displacements in the u-F coordinate system are kinematically constrained, as discussed in Section 1.3, it is possible to construct the transformation matrix $[b]$ so long as the forces are independent. Equation (1.105) may be considered as an equation of force equilibrium in which the internal forces are determined so as to be in equilibrium with the applied forces.

The last piece of information with which we are concerned in this section relates to the stiffness and flexibility matrices of the elements. The element stiffness matrix $[\kappa]$ connects the element displacements and forces as follows

$$\{P\} = [\kappa]\{\delta\} \tag{1.106}$$

If the element coordinates are ordered so that the numbers are in sequence for each element, as implied in Eq. (1.103), the matrix $[\kappa]$ may then be partitioned as indicated in the following form of Eq. (1.106).

$$\begin{Bmatrix} \{P\}_1 \\ \hline \{P\}_2 \\ \hline \vdots \\ \hline \{P\}_i \\ \hline \vdots \\ \hline \{P\}_m \end{Bmatrix} = \begin{bmatrix} [\kappa]_{11} & & & \text{Elements not shown are zero.} & & \\ & [\kappa]_{22} & & & & \\ & & \ddots & & & \\ & & & [\kappa]_{ii} & & \\ & & & & \ddots & \\ & & & & & [\kappa]_{mm} \end{bmatrix} \begin{Bmatrix} \{\delta\}_1 \\ \hline \{\delta\}_2 \\ \hline \vdots \\ \hline \{\delta\}_i \\ \hline \vdots \\ \hline \{\delta\}_m \end{Bmatrix} \tag{1.107}$$

It is seen that $[\kappa]$ is of nearly diagonal form with non-zero elements in only a relatively few co-diagonals adjacent to the principal diagonal. Formation of this matrix is carried out by arranging the stiffness matrices for the individual elements as indicated. For the ith element the force-deflection equation is

$$\{P\}_i = [\kappa]_{ii}\{\delta\}_i \tag{1.108}$$

If, in this equation, $\{\delta\}_i$ includes all the displacements at the connecting points of element i, and consequently, $\{P\}_i$ includes all the forces, then the following is true. The vector $\{\delta\}_i$ represents a generalized displacement vector in which rigid-body displacements of the element are included, and the forces in vector $\{P\}_i$ are in equilibrium. For example, if the element were a beam in three-space, such as shown in Fig. 1.17, and if $\{\delta\}$ and $\{P\}$ contained all twelve components indicated, then there would be six rigid-body degrees of freedom and six equations of equilibrium among the forces. Hence, the element is unconstrained in the sense of Section 1.9 and the matrix $[\kappa]_{ii}$ is singular. In fact, the number of singularities will be equal to the number of rigid-body degrees of freedom, which is equal to the number of force equilibrium equations. Hence, in such a complete coordinate system the inverse of $[\kappa]_{ii}$ does not exist.

The element flexibility matrix $[\alpha]$ expresses the following force-deflection relationship.

$$\{\delta\} = [\alpha]\{P\} \tag{1.109}$$

Again, following a sequential numbering of the coordinates from one element to another, the matrix may be partitioned in the same way as the stiffness matrix.

$$\begin{Bmatrix} \{\delta\}_1 \\ \hline \{\delta\}_2 \\ \hline \vdots \\ \hline \{\delta\}_i \\ \hline \vdots \\ \hline \{\delta\}_m \end{Bmatrix} = \begin{bmatrix} [\alpha]_{11} & & & & \\ & [\alpha]_{22} & & & \\ & & \ddots & & \\ & & & [\alpha]_{ii} & \\ & & & & \ddots \\ & & & & & [\alpha]_{mm} \end{bmatrix} \begin{Bmatrix} \{P\}_1 \\ \hline \{P\}_2 \\ \hline \vdots \\ \hline \{P\}_i \\ \hline \vdots \\ \hline \{P\}_m \end{Bmatrix} \tag{1.110}$$

For the ith element the following equation holds.

$$\{\delta\}_i = [\alpha]_{ii}\{P\}_i \tag{1.111}$$

From the foregoing discussion it is recalled that $[\alpha]_{ii}$ is not, in general, the inverse of $[\kappa]_{ii}$ for the same element because the latter matrix does not have an inverse. From Eq. (1.111) it is seen that the highest order of $[\alpha]_{ii}$ that can exist is equal to the largest number of independent forces in vector $\{P\}_i$. This is equal to the total number of forces minus the number of equations of equilibrium which serve to define some of the forces in terms of others. Therefore, the order of $[\alpha]_{ii}$ will be equal to the rank of $[\kappa]_{ii}$ and may be obtained by inverting a submatrix of $[\kappa]_{ii}$ corresponding to the same coordinates.

We shall consider a simple example to clarify the foregoing discussion. Consider a uniform inextensible beam of length l and section bending modulus EI as shown in Fig. 1.18.

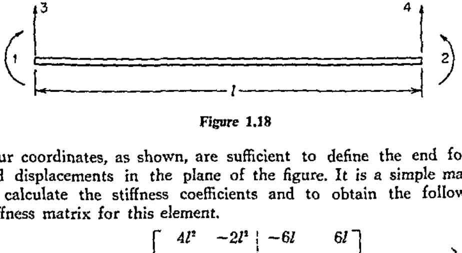


Figure 1.18

Four coordinates, as shown, are sufficient to define the end forces and displacements in the plane of the figure. It is a simple matter to calculate the stiffness coefficients and to obtain the following stiffness matrix for this element.

$$[\kappa] = \frac{EI}{l^3}\left[\begin{array}{cc|cc} 4l^2 & -2l^2 & -6l & 6l \\ -2l^2 & 4l^2 & 6l & -6l \\ \hline -6l & 6l & 12 & -12 \\ 6l & -6l & -12 & 12 \end{array}\right] \tag{1.112}$$

The matrix is clearly singular and, because there are two rigid-body degrees of freedom, the rank is $(4 - 2)$. If we now wish to find a flexibility matrix for the beam whose order will be two, we have a choice in the selection of two coordinates in which to express it. The following five combinations are possible: 1, 2; 1, 3; 1, 4; 2, 3; 2, 4. The combination 3, 4 is not possible since the two forces P_3 and P_4 are not independent; they must be equal in magnitude and opposite in sign. If we choose the combination 1, 2, for example, it is easy to construct the flexibility matrix by computing the rotations at the two ends caused by unit couples applied separately at the ends. The result is

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \frac{l}{EI}\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \tag{1.113}$$

It is easily verified that the same result is obtained by inverting the submatrix of $[\kappa]$ [in the upper-left corner of Eq. (1.112)] pertaining to coordinates 1 and 2.

1.11 Stiffness Method

In this method the stiffness matrix $[k]$ for the complete structure is synthesized, using the stiffnesses $[\kappa]$ of the elements. We use the coordinate transformation equation together with the related force transformation

$$\{Q\} = [\beta]^T\{P\} \tag{1.114}$$

Substituting Eqs. (1.43), (1.106), and (1.104) into (1.114) we obtain

$$[k]\{q\} = [\beta]^T[\kappa][\beta]\{q\}$$

Since the generalized displacement vector $\{q\}$ is arbitrary the following equation must hold.

$$[k] = [\beta]^T[\kappa][\beta] \tag{1.115}$$

Thus, the stiffness matrix for the complete structure is obtained. If the structure is adequately constrained, this matrix may be inverted to give deflections in terms of the generalized forces.

$$\{q\} = [k]^{-1}\{Q\} \tag{1.116}$$

Substituting this into Eq. (1.104) gives the element deflections or strains $\{\delta\}$; thus

$$\{\delta\} = [\beta][k]^{-1}\{Q\} \tag{1.117}$$

Substitution of this result into Eq. (1.106) gives the internal forces or stresses $\{P\}$ in terms of external forces.

$$\{P\} = [\kappa][\beta][k]^{-1}\{Q\} \tag{1.118}$$

Further, if the generalized forces were obtained from a set of forces $\{F\}$, which may also be generalized forces or which may be related to a constrained coordinate system by means of a transformation, as in Eq. (1.26), then that transformation may be substituted into Eq. (1.118) to yield the internal forces in terms of $\{F\}$.

$$\{P\} = [\kappa][\beta][k]^{-1}[C]^T\{F\} \tag{1.119}$$

Thus, the stiffness method provides a means for determining the stiffness of a structural system and, by inversion, its flexibility as well. In addition, one may find stresses and strains in the structure caused by applied forces.

In most problems there are many coordinates required to define displacements on which there are no forces. This situation was illustrated in the beam problem discussed in Section 1.5. In this case the stiffness matrix may be partitioned, as shown in Eq. (1.49), and reduced as in Eq. (1.53). Thus, the order of the matrix may often be reduced, bringing about a significant reduction in the amount of subsequent computation. With this in mind the stiffness method may be presented in a somewhat different form.

First, we distinguish between those displacements in $\{q\}$ associated with applied forces and those associated with zero forces. Thus,

$$\{q\} = \begin{Bmatrix} \{q\}^* \\ \hline \{q\}^0 \end{Bmatrix} \tag{1.120}$$

where

$\{q\}^*$ = displacements on coordinates on which applied forces $\{Q\}^*$ exist.

$\{q\}^0$ = displacements on selected coordinates on which applied forces are zero.

Equation (1.49) is now rewritten using the above notation.

$$\begin{Bmatrix} Q^* \\ \hline 0 \end{Bmatrix} = \left[\begin{array}{c|c} k_{11} & k_{12} \\ \hline k_{21} & k_{22} \end{array}\right] \begin{Bmatrix} q^* \\ \hline q^0 \end{Bmatrix} \tag{1.121}$$

This equation is equivalent to the two following equations, written for convenience without the brackets which have been used heretofore in denoting matrices.

$$\left.\begin{aligned} Q^* &= k_{11}q^* + k_{12}q^0 \\ 0 &= k_{21}q^* + k_{22}q^0 \end{aligned}\right\} \tag{1.122}$$

The second of these equations is solved for q^0 and substituted into the first to give the reduced form

$$Q^* = (k_{11} - k_{12}k_{22}^{-1}k_{21})q^* = k^*q^* \tag{1.123}$$

where

$$k^* = k_{11} - k_{12} k_{22}^{-1} k_{21} \tag{1.124}$$

The submatrices of $[k]$ may be identified in terms of the matrix $[\kappa]$ by expressing Eq. (1.115) with the matrix $[\beta]$ partitioned as follows.

$$[\beta] = [\beta^* \,\vdots\, \beta^0]$$

$$[k] = \begin{bmatrix} \beta^{*T} \\ \hline \beta^{0T} \end{bmatrix} [\kappa] \quad [\beta^* \,\vdots\, \beta^0] \tag{1.125}$$

The submatrices of $[k]$ are determined by carrying out the indicated matrix multiplication.

$$\left.\begin{aligned} k_{11} &= \beta^{*T} \kappa \beta^* \\ k_{12} &= \beta^{*T} \kappa \beta^0 = k_{21}^T \\ k_{21} &= \beta^{0T} \kappa \beta^* = k_{12}^T \\ k_{22} &= \beta^{0T} \kappa \beta^0 \end{aligned}\right\} \tag{1.126}$$

Substituting these results into Eq. (1.124) we obtain the reduced stiffness matrix in the form

$$k^* = \beta^{*T} \kappa \beta^* - \beta^{*T} \kappa \beta^0 (\beta^{0T} \kappa \beta^0)^{-1} \beta^{0T} \kappa \beta^* \tag{1.127}$$

Using the inverse of this matrix, Eq. (1.123) may be used to give displacements in terms of applied forces.

$$q^* = k^{*-1} Q^* \tag{1.128}$$

We may now determine the element deflections from Eq. (1.104) using partitioned matrices.

$$\{\delta\} = [\beta^* \,\vdots\, \beta^0] \begin{Bmatrix} q^* \\ \hline q^0 \end{Bmatrix} = \beta^* q^* + \beta^0 q^0 \tag{1.129}$$

Substituting for q^0 its value obtained from the second of Eqs. (1.122) we obtain the following equation for element deflections.

$$\begin{aligned} \delta &= (\beta^* - \beta^0 k_{22}^{-1} k_{21}) q^* \\ &= [\beta^* - \beta^0 (\beta^{0T} \kappa \beta^0)^{-1} \beta^{0T} \kappa \beta^*] q^* \end{aligned} \tag{1.130}$$

Using Eq. (1.128) these element deflections can be found from the applied forces. Premultiplication by the elemental stiffness matrix $[\kappa]$ yields the following equation which expresses the internal forces, or stresses, in terms of the applied forces.

$$P = \kappa [\beta^* - \beta^0 (\beta^{0T} \kappa \beta^0)^{-1} \beta^{0T} \kappa \beta^*] k^{*-1} Q^* \tag{1.131}$$

This equation is the counterpart of Eq. (1.118) written in reduced form. The latter equation has the advantage over the former one in that the matrices to be inverted are of lower order; the reduction in

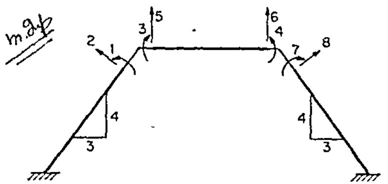


Figure 1.19

order being dependent upon the number of displacement coordinates with zero forces.

We shall apply the stiffness method to the frame shown in Fig. 1.7 to determine its stiffness and also to find the internal shears and bending moments in the members caused by the forces F_1 through F_4, applied as shown in Fig. 1.9. We shall have to establish a system of element coordinates from which we may construct the $[\beta]$ matrix. A system of element coordinates is shown in Fig. 1.19.

For our generalized coordinates we shall use the system established by the transformation of Eq. (1.12), where the u displacements are shown in Fig. 1.7 and

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_5 \\ u_6 \end{Bmatrix} \tag{1.132}$$

The generalized displacement q_1 is shown in Fig. 1.9. Using this information the $[\beta]$ matrix is constructed.

$$[\beta] = \begin{bmatrix} 0 & -1 & 0 \\ -\frac{5}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ -\frac{3}{4} & 0 & 0 \\ \frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{5}{4} & 0 & 0 \end{bmatrix} \tag{1.133}$$

The $[\kappa]$ matrix may be written from Eq. (1.112) which gives the 4×4 matrix for a single beam. Note, however, that the two columns are fixed at the ground so that translation and rotation at those points are zero. Hence, the matrices for those two elements are of order 2 and may be obtained by selection of the appropriate rows and columns of the 4×4 matrix in Eq. (1.112). All members are considered to be of equal length l and equal flexural stiffness EI.

$$[\kappa] = \frac{EI}{l^3}\left[\begin{array}{cc|cccc|cc} 4l^2 & -6l & & & & & & \\ -6l & 12 & & & & & & \\ \hline & & 4l^2 & -2l^2 & -6l & 6l & & \\ & & -2l^2 & 4l^2 & 6l & -6l & & \\ & & -6l & 6l & 12 & -12 & & \\ & & 6l & -6l & -12 & 12 & & \\ \hline & & & & & & 4l^2 & -6l \\ & & & & & & -6l & 12 \end{array}\right] \tag{1.134}$$

The stiffness matrix $[k]$ obtained from Eq. (1.115) is the 3×3 matrix show below.

$$[k] = \frac{EI}{2l^3}\left[\begin{array}{c|cc} 129 & 3l & 3l \\ \hline 3l & 16l^2 & 4l^2 \\ 3l & 4l^2 & 16l^2 \end{array}\right] \tag{1.135}$$

Since the torques Q_2 and Q_3 are zero in this problem, this matrix may be reduced according to Eq. (1.124). The reduced matrix k^* is a scalar which relates the single generalized force Q_1 and the generalized displacement q_1.

$$\begin{aligned} q_1 &= \frac{20l^3}{1281EI}Q_1 \\ &= \frac{20l^3}{1281EI}\left[F_1 + F_2 - \frac{3}{4}(F_3 - F_4)\right] \end{aligned} \tag{1.136}$$

If the rotations q_2 and q_3 are desired, they may be obtained from q_1 using the second equation of Eq. (1.122). In this problem we obtain

$$\begin{aligned} q_2 = q_3 &= -\frac{3}{20l}q_1 \\ &= -\frac{3l^2}{1281EI}\left[F_1 + F_2 - \frac{3}{4}(F_3 - F_4)\right] \end{aligned} \tag{1.137}$$

The internal forces (shears and bending moments) are obtained using Eq. (1.131). The results are given in Eq. (1.138) without showing the steps in matrix multiplication.

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \end{Bmatrix} = \frac{2}{1281} \begin{bmatrix} 81l \\ -159 \\ 81l \\ -81l \\ -162 \\ 162 \\ -81l \\ 159 \end{bmatrix} \left[F_1 + F_2 - \frac{3}{4}(F_3 - F_4) \right] \tag{1.138}$$

Here the P's are internal forces corresponding to the ordering of the element coordinates shown in Fig. 1.19. They are shears and bending moments whose positive directions as applied to the elements are given by the arrows. Shears and bending moments at other points may be computed from equilibrium requirements as can the axial forces in the elements.

1.12 Stiffness Method with Distributed Coordinates

In Section 1.2 we referred to distributed coordinates which may be used to specify displacements in a distributed structure, or one on which the forces are distributed. By means of these coordinates a distributed displacement function is specified in terms of a set of prescribed functions $\phi_i(x)$ $i = 1, 2, 3, \ldots$, as in Eq. (1.2). The coordinates p_i specify the magnitudes of the displacements whose distributions are given by the corresponding fuctions $\phi_i(x)$. Thus, these coordinates p_i play the same role in the distributed coordinate system as the coordinates u_i play in the discretized systems.

The concept of distributed coordinates may be applied to the separate elements of a structural system, as was illustrated by the example of Section 1.3, using the portal frame structure shown in Fig. 1.8. As related to the displacements of the unconnected elements, the coordinates p are generalized coordinates and play the same role as the discretized displacements δ defined in Section 1.10. When the elements are connected together to form the structural system, these coordinates are constrained by the conditions of deflection compatibility. These conditions are referred to as kinematic constraints. The equations of constraint [exemplified by Eq. (1.18) in the example of Section 1.3] may be used to relate the complete set of coordinates, which apply to all the elements of the system, to a reduced set of generalized coordinates which apply to the connected structure. The construction of such a coordinate transformation is illustrated in Section 1.3, and the transformation is given in Eq. (1.20). In this light Eq. (1.20) would appear in the form

$$\{p\} = [\beta]\{q\} \tag{1.139}$$

Comparison of Eq. (1.104) with Eq. (1.139) clarifies the use of $[\beta]$ as the transformation matrix in the latter equation.

The stiffness method discussed in Section 1.11, using discretized element coordinates, may now be extended to include distributed element coordinates. The element stiffness matrix $[\kappa]$ may be constructed from the stiffness properties of the elements and the prescribed displacement functions ϕ. For example, the stiffness coefficient κ_{ij} of a beam in pure bending is determined from the bending stiffness modulus EI and the pair of functions $\phi_i(x)$ and $\phi_j(x)$ as follows. (See Problem 12, Chapter 1.)

$$\kappa_{ij} = \int_0^l EI(x)\,\phi_i''(x)\,\phi_j''(x)\,dx \tag{1.140}$$

Eq. (1.115) may be used to find the stiffness matrix for the connected structure in terms of the generalized coordinates q. The remaining equations of Section 1.11 apply, without change, to the case of distributed coordinates.

To illustrate the method we return to the portal frame of Fig. 1.8. The functions chosen for the three members of the frame are given in Eqs. (1.15) and (1.16); the equations of constraint are Eqs. (1.19). The transformation, Eq. (1.139), is given by Eq. (1.20) for the problem where

$$[\beta] = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & -2 & 0 \\ -2 & 6 & -1 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.141}$$

and the generalized coordinates q_1, q_2, q_3 for the frame are identified as

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} p_2 \\ p_6 \\ p_8 \end{Bmatrix} \tag{1.142}$$

The stiffness coefficients are found for the beam elements from Eq. (1.140) and the complete element stiffness matrix is obtained. All members have the same stiffness modulus EI in bending.

$$[\kappa] = 2\frac{EI}{l^3}\left[\begin{array}{cc|cccc|cc} 2 & 3 & & & & & & \\ 3 & 6 & & & & & & \\ \hline & & 0 & 0 & 0 & 0 & & \\ & & 0 & 2 & 3 & 0 & & \\ & & 0 & 3 & 6 & 0 & & \\ & & 0 & 0 & 0 & 0 & & \\ \hline & & & & & & 2 & 3 \\ & & & & & & 3 & 6 \end{array}\right] \qquad (1.143)$$

Applications of Eq. (1.115) yields the stiffness matrix $[k]$ of order three.

$$[k] = 2\frac{EI}{l^3}\begin{bmatrix} 4 & -7 & 1 \\ -7 & 28 & -7 \\ 1 & -7 & 4 \end{bmatrix} \qquad (1.144)$$

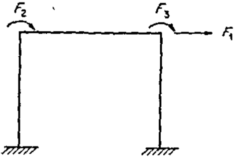


Figure 1.20

At this point we shall introduce a set of applied forces to the frame and continue the example to show how deflections are obtained. Consider the forces F_1, F_2, and F_3 shown in Fig. 1.20. The generalized forces Q_1, Q_2, Q_3 may be found by application of the method of virtual work in which the virtual work δW is found in terms of the forces F and the forces Q,

$$\delta W = F_1 \delta u_2(l) - F_2 \delta w_1'(0) - F_3 \delta w_3'(l).$$

Referring to Eqs. (1.15) and (1.16) this equation may be written in terms of the coordinates p as

$$\delta W = F_1 \delta p_8 - F_2 \cdot \frac{1}{l}\delta p_3 - F_3 \cdot \frac{1}{l}(\delta p_3 + 2\delta p_4 + 3\delta p_5)$$

From Eq. (1.20) the virtual displacements $\delta p_3, \delta p_4, \delta p_5$ may be found in terms of the generalized displacements $\delta p_2, \delta p_6, \delta p_8$ which correspond, respectively, to $\delta q_1, \delta q_2, \delta q_3$. Thus, the virtual work takes the form

$$\delta W = -\frac{F_2}{l}\delta q_1 + \left(F_1 + 2\frac{F_2 + F_3}{l}\right)\delta q_2 - \frac{F_3}{l}\delta q_3$$

Writing the virtual work also in the form

$$\delta W = Q_1 \delta q_1 + Q_2 \delta q_2 + Q_3 \delta q_3$$

permits the following identification of the generalized forces:

$$Q_1 = -\frac{F_2}{l}$$

$$Q_2 = F_1 + 2\frac{F_2 + F_3}{l}$$

$$Q_3 = -\frac{F_3}{l}$$

The displacements $\{q\}$ may be found in terms of the forces $\{Q\}$ by inverting the matrix $[k]$. The desired equation is

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \frac{l^3}{84EI} \begin{bmatrix} 21 & 7 & 7 \\ 7 & 5 & 7 \\ 7 & 7 & 21 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} \tag{1.145}$$

If we consider the particular case in which the moments F_2 and F_3 are zero, we note that Q_1 and Q_3 are zero and $Q_2 = F_1$. The following deflections are then obtained:

$$q_1 = p_2 = \frac{7l^3}{84EI} F_1$$

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$$q_2 = p_6 = \frac{5l^3}{84EI} F_1$$

K7

$$q_3 = p_8 = \frac{7l^3}{84EI} F_1$$

From the coordinate transformation all other coordinates, p_1 through p_8 may be found. Then, using Eq. (1.15) the deflections $w(x)$ may be determined at every point on the frame.

1.13 Flexibility Method

In this method, the elemental flexibility matrix $[\alpha]$ as defined from Eq. (1.109)

$$\{\delta\} = [\alpha]\{P\} \tag{1.109}$$

is used to synthesize the flexibility matrix $[a]$ for the complete structure.

$$\{u\} = [a]\{F\} \tag{1.91}$$

Here, the matrix $[a]$ is related to the u-F coordinate system in which the u's may be either constrained or generalized coordinates, but the

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F's must be independent for matrix $[a]$ to exist. (See Section 1.5). Using the principle of virtual work

$$\{F\}^T\{u\} = \{P\}^T\{\delta\} \tag{1.146}$$

Eqs. (1.109) and (1.91) are substituted to give

$$\{F\}^T[a]\{F\} = \{P\}^T[\alpha]\{P\} \tag{1.147}$$

At this point the internal forces must be related to the applied forces by the transformation.

$$\{P\} = [b]\{F\} \tag{1.148}$$

where $[b]$ is a transformation matrix of order $p' \times m$ if there are p' components in vector $\{P\}$ and m components in $\{F\}$. It is noted, again, that the order p' of $\{P\}$ in the flexibility method will not be equal, in general, to the order p of the same vector in the stiffness method. Substituting Eq. (1.148) into (1.147) we obtain

$$\{F\}^T[a]\{F\} = \{F\}^T[b]^T[\alpha][b]\{F\}.$$

Comparing the two sides of the last equation, the flexibility matrix $[a]$ of the complete structure is given by

$$[a] = [b]^T[\alpha][b] \tag{1.149}$$

For statically determinate structures, the matrix $[b]$ may be constructed by finding the internal forces caused by unit values of the applied forces using equations of equilibrium. In other words, Eq. (1.148) satisfies the equations of equilibrium for the structure. For statically indeterminate structures the matrix can be constructed by determining the internal forces, using any one of the methods available for analysis of such structures. However, the matrix procedure to be outlined provides in itself a useful and compact method for the analysis of indeterminate structures.

To develop this procedure, Eq. (1.148) is viewed in a slightly different light. The vector $\{F\}$ is extended to include, in addition to the applied forces, a set of self-equilibrating forces corresponding to and equal in number to the redundancies in the structure. These forces may be viewed as internal forces at cuts made in the structure in sufficient number to make it statically determinate. When considered in this light the concept involved in this method is a familiar one to structural analysts. Alternately, one may view the extension of the $\{F\}$ vector as a transformation of coordinates in which the deflection vector $\{u\}$ is correspondingly extended, although the added deflection components are all zero. This corresponds, obviously, to the condition that the relative deflections across the cuts must be zero. With the extension of the force vector, Eq. (1.148) appears as

$$\{P\} = [b]\begin{Bmatrix} \{F\}^* \\ \hline \{F\}^0 \end{Bmatrix} \tag{1.150}$$

where $\{F\}$ appears in partitioned form with $\{F\}^*$, the applied force vector, and $\{F\}^0$, the vector whose components are the unknown redundant forces. The matrix $[b]$ may also be partitioned.

$$[b] = [[b]^* \mid [b]^0] \tag{1.151}$$

Equation (1.150) is then rewritten in the equivalent form

$$\{P\} = [b]^* \{F\}^* + [b]^0 \{F\}^0 \tag{1.152}$$

Before proceeding with the further development of the flexibility method, an example will be introduced to illustrate the construction of the matrix $[b]$. This example will later be continued to include a complete analysis by this method.

Consider a two-story plane building frame, as shown in Fig. 1.21. Figure 1.21(a) shows the applied forces and the dimensions of the frame. Also, the members are designated by number and the relative moments of inertia of the members are given. Figure 1.21(b) shows the choice of redundant forces, F_3 through F_8, considered as the shears, bending moments, and axial forces at the centers of the two horizontal members. Figure 1.21(c) indicates the desired internal forces $\{P\}$.

To construct the matrix $[[b]^* \mid [b]^0]$ for the frame of Fig. 1.21, we express the elements of vector $\{P\}$ in terms of the applied loads $\{F\}^*$ and the redundant forces $\{F\}^0$ acting on the reduced, statically-determinate structure of Fig. 1.21(b).

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \\ P_{11} \\ P_{12} \end{Bmatrix} = \left[\begin{array}{cc:cccccc} -l & 0 & 1 & -l & -l & 0 & 0 & 0 \\ 0 & 0 & 1 & -l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & l & -l & 0 & 0 & 0 \\ -l & 0 & 1 & -l & -l & 1 & -l & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & l & -l & 1 & l & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & l & 0 \end{array}\right] \begin{Bmatrix} F_1 \\ F_2 \\ \hline F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} \tag{1.153}$$

It is seen from this example that many other choices of redundant force systems could be made; the criterion for the choice being that

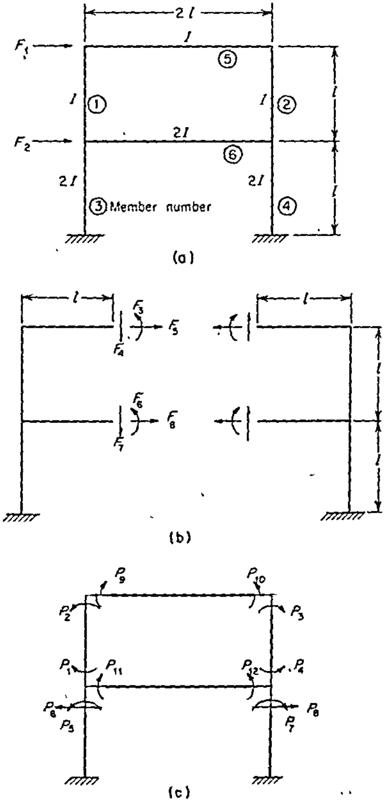


Figure 1.21 (a) Applied forces $\{F\}^*$. (b) Choice of redundants $\{F\}^\circ$ (reduced, statically-determinate structure). (c) Desired internal forces $\{P\}$.

the structure should be statically determinate when these forces are zero. This reduction to a statically-determinate structure is a matter of convenience. It must be noted, however, that this is not essential. Any method of dealing with indeterminate structures could be used to find the matrix $[b]$ associated with a reduced indeterminate structure.

At this point it is necessary to consider the order of the matrix $[a]$. Because the force vector $\{F\}$ was extended to include the redundant forces, the order of $[b]$ was altered. If there are m applied forces and r redundant forces, the number of components in $\{F\}$ is $m + r$. Now, taking p' as the number of components in $\{P\}$, the order of matrix $[b]$ is $p' \times (m + r)$. Also, the order of the elemental flexibility matrix $[\alpha]$ is $p' \times p'$. Hence, from Eq. (1.149) the order of $[a]$ is $(m + r) \times (m + r)$. The deflection-force equation

$$\{u\} = [a]\{F\}$$

may be written in the form

$$\left\{\begin{array}{c}\{u\}^* \\ \hline \{0\}\end{array}\right\} = \left[\begin{array}{c|c}[a]_{11} & [a]_{12} \\ \hline [a]_{21} & [a]_{22}\end{array}\right] \left\{\begin{array}{c}\{F\}^* \\ \hline \{F\}^0\end{array}\right\} \tag{1.154}$$

where the orders of the submatrices in $[a]$ are

$$\begin{aligned} &[a]_{11} \text{ of order } m \times m \\ &[a]_{12} \text{ of order } m \times r \\ &[a]_{21} \text{ of order } r \times m \\ &[a]_{22} \text{ of order } r \times r \end{aligned}$$

The lower part of the deflection vector $\{u\}$ is zero inasmuch as these are deflections related to the self-equilibrating redundant forces, or relative deflections across the cuts. Equation (1.154) may be separated into the two following equations.

$$\begin{aligned} \{u\}^* &= [a]_{11}\{F\}^* + [a]_{12}\{F\}^0 \\ \{0\} &= [a]_{21}\{F\}^* + [a]_{22}\{F\}^0 \end{aligned} \tag{1.155}$$

The second equation may be solved for the redundant forces

$$\{F\}^0 = -[a]_{22}^{-1}[a]_{21}\{F\}^* \tag{1.156}$$

When this is substituted into the first of Eqs. (1.155) there follows

$$\begin{aligned} \{u\}^* &= ([a]_{11} - [a]_{12}[a]_{22}^{-1}[a]_{21})\{F\}^* \\ &= [a]^*\{F\}^* \end{aligned} \tag{1.157}$$

where

$$[a]^* = [a]_{11} - [a]_{12}[a]_{22}^{-1}[a]_{21} \tag{1.158}$$

is the reduced flexibility matrix whose order is $m \times m$.

In Eq. (1.158) submatrix $[a]_{22}$ corresponds to the redundant force vector $\{F\}^0$. Since $[a]_{22}$ must be inverted, it becomes apparent that the choice of redundants is a subject of great importance. A poor choice of redundants may cause $[a]_{ss}$ to be nearly singular. For instance, for a multi-story frame, cuts at the columns would be preferred to cuts at the girders in order to avoid a nearly singular matrix.*

To carry out the foregoing analysis in connection with the building frame of Fig. 1.21, it is necessary first to construct the elemental flexibility matrix $[\alpha]$. Methods for computing the flexibility coefficients for beam elements have already been discussed. Using these methods this matrix is constructed and appears in Eq. (1.159). Remember that in this problem the various elements have different lengths and moments of inertia.

All terms not shown are zero.

$$[\alpha] = \frac{l}{12EI}\begin{bmatrix} 4 & 2 & & & & & & & & & & \\ 2 & 4 & & & & & & & & & & \\ & & 4 & 2 & & & & & & & & \\ & & 2 & 4 & & & & & & & & \\ & & & & 6 & 3l & & & & & & \\ & & & & 3l & 2l^2 & & & & & & \\ & & & & & & 6 & 3l & & & & \\ & & & & & & 3l & 2l^2 & & & & \\ & & & & & & & & 8 & 4 & & \\ & & & & & & & & 4 & 8 & & \\ & & & & & & & & & & 4 & 2 \\ & & & & & & & & & & 2 & 4 \end{bmatrix} \tag{1.159}$$

By carrying out the multiplications in Eq. (1.149) the following matrix $[a]$ is obtained.

$$[a] = \frac{l}{12EI}\left[\begin{array}{cc|cccccc} 18l^2 & 5l^2 & -15l & 15l^2 & 18l^2 & -9l & 9l^2 & 5l^2 \\ 5l^2 & 2l^2 & -3l & 3l^2 & 5l^2 & -3l & 3l^2 & 2l^2 \\ \hline -15l & -3l & 60 & 0 & -30l & 12 & 0 & -6l \\ 15l^2 & 3l^2 & 0 & 44l^2 & 0 & 0 & 12l^2 & 0 \\ 18l^2 & 5l^2 & -30l & 0 & 36l^2 & -18l & 0 & 10l^2 \\ -9l & -3l & 12 & 0 & -18l & 24 & 0 & -6l \\ 9l^2 & 3l^2 & 0 & 12l^2 & 0 & 0 & 16l^2 & 0 \\ 5l^2 & 2l^2 & -6l & 0 & 10l^2 & -6l & 0 & 4l^2 \end{array}\right] \tag{1.160}$$

*See Problem 17, Chapter 1.

Here the submatrices indicated in Eq. (1.154) are shown where, in this example, $m = 2$ and $r = 6$. To compute the reduced flexibility matrix $[a]^*$ by Eq. (1.158) and to find the redundant forces by Eq. (1.156), it is necessary to invert submatrix $[a]_{22}$. Since, in this example, its elements have different dimensions, it is convenient to first rearrange this matrix by interchanging the second and fourth rows and the second and fourth columns. This leaves the matrix symmetric and permits partitioning into four submatrices, each having consistent dimensions. This rearranged and partitioned $[a]_{22}$ matrix is shown below.

$$\left[\begin{array}{cc|cccc} 60 & 12 & -30l & 0 & 0 & -6l \\ 12 & 24 & -18l & 0 & 0 & -6l \\ \hline -30l & -18l & 36l^2 & 0 & 0 & 10l^2 \\ 0 & 0 & 0 & 44l^2 & 12l^2 & 0 \\ 0 & 0 & 0 & 12l^2 & 16l^2 & 0 \\ -6l & -6l & 10l^2 & 0 & 0 & 4l^2 \end{array}\right]$$

This matrix may now be inverted by the method of partitioning,[8] and the resulting matrix rearranged to restore the original ordering of rows and columns. The following inverted submatrix $[a]_{22}$ results.

$$[a]_{22}^{-1} = \frac{12EI}{l}\begin{bmatrix} \frac{13}{396} & 0 & \frac{1}{22l} & \frac{1}{396} & 0 & \frac{-2}{33l} \\ 0 & \frac{4}{140l^2} & 0 & 0 & -\frac{3}{140l^2} & 0 \\ \frac{1}{22l} & 0 & \frac{7}{44l^2} & \frac{1}{44l} & 0 & -\frac{13}{44l^2} \\ \frac{1}{396} & 0 & \frac{1}{44l} & \frac{28}{396} & 0 & \frac{7}{132l} \\ 0 & -\frac{3}{140l^2} & 0 & 0 & \frac{11}{140l^2} & 0 \\ -\frac{2}{33l} & 0 & -\frac{13}{44l^2} & \frac{7}{132l} & 0 & \frac{43}{44l^2} \end{bmatrix} \quad (1.161)$$

Insertion of this matrix together with submatrices of $[a]$ in Eq. (1.158) gives the following reduced flexibility matrix.

$$[a]^* = \frac{l^3}{1680EI}\begin{bmatrix} 279 & 89 \\ 89 & 59 \end{bmatrix} \quad (1.162)$$

Internal forces may be found in the flexibility method by inserting Eq. (1.156) into Eq. (1.152) to give

$$\{P\} = ([b]^* - [b]^0[a]_{22}^{-1}[a]_{21})\{F\}^* \quad (1.163)$$

In our example, this equation results in

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \\ P_{11} \\ P_{12} \end{Bmatrix} = \frac{1}{140} \begin{bmatrix} -37l & 3l \\ 33l & 3l \\ -33l & -3l \\ 37l & -3l \\ 17l & 27l \\ -70 & -70 \\ -17 & -27 \\ 70 & 70 \\ 33 & -3 \\ -33 & -3 \\ 54 & 24 \\ -54 & -24 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{1.164}$$

It is clear that from these results, stresses at all points in the structure can be found. Also, by insertion of this equation into Eq. (1.109), an analysis of strains may be made.

This completes the procedures involved in the analysis of structures by the flexibility method. However, it is interesting to note that in parallel with the foregoing development the results may be expressed directly in terms of the elemental flexibility matrix $[\alpha]$ and the submatrices $[b]^*$ and $[b]^0$ of $[b]$. If Eq. (1.149) is written with the matrices partitioned it becomes

$$\left[\begin{array}{c|c} [a]_{11} & [a]_{12} \\ \hline [a]_{21} & [a]_{22} \end{array}\right] = \left[\begin{array}{c} [b]^{*T} \\ \hline [b]^{0T} \end{array}\right] [\alpha] \left[\begin{array}{c|c} [b]^* & [b]^0 \end{array}\right] \tag{1.165}$$

Expanding the triple product on the right side of this equation and identifying the resulting submatrices with the corresponding ones in $[a]$ gives

$$\left.\begin{aligned} [a]_{11} &= [b]^{*T}[\alpha][b]^* \\ [a]_{12} &= [b]^{*T}[\alpha][b]^0 = [a]_{21}^T \\ [a]_{21} &= [b]^{0T}[\alpha][b]^* = [a]_{12}^T \\ [a]_{22} &= [b]^{0T}[\alpha][b]^0 \end{aligned}\right\} \tag{1.166}$$

Substitution of these equations into Eq. (1.158) gives the reduced flexibility matrix in the form

$$\begin{aligned} [a]^* &= [b]^{*T}[\alpha][b]^* - [b]^{*T}[\alpha][b]^0([b]^{0T}[\alpha][b]^0)^{-1}[b]^{0T}[\alpha][b]^* \\ &= [b]^{*T}[\alpha][B] \end{aligned} \tag{1.167}$$

where

$$[B] = [b]^* - [b]^0 ([b]^{0T} [\alpha] [b]^0)^{-1} [b]^{0T} [\alpha] [b]^* \tag{1.168}$$

It is clear that the transposed form is equally valid.

$$[a]^* = [B]^T [\alpha] [b]^* \tag{1.169}$$

Substitution of Eq. (1.166) into (1.163) gives the following form for the internal force equation.

$$\begin{aligned} \{P\} &= [[b]^* - [b]^0 ([b]^{0T} [\alpha] [b]^0)^{-1} [b]^{0T} [\alpha] [b]^*] \{F^*\} \\ &= [B] \{F\}^* \end{aligned} \tag{1.170}$$

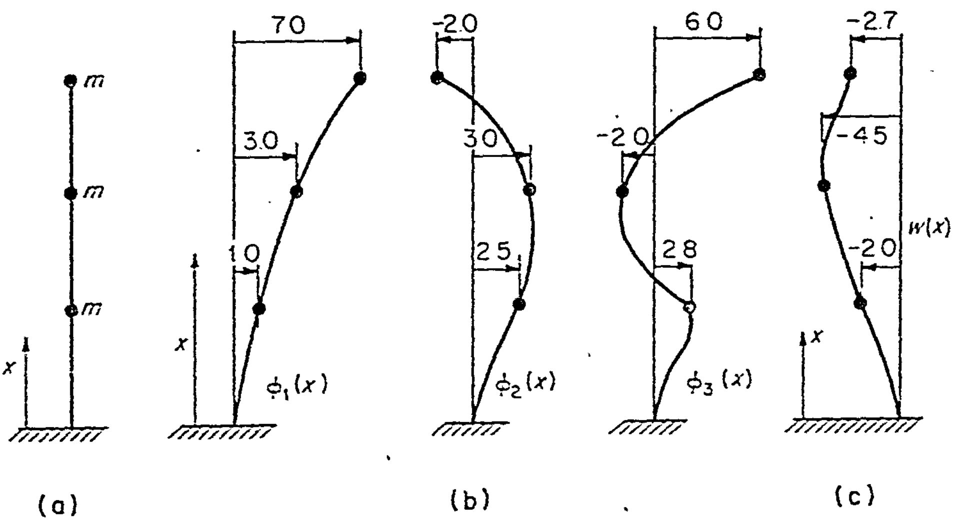
Regardless of which form one wishes to use in dealing with a given problem, the computions are essentially the same in either case.

It is interesting to point out that there is a parallel between the stiffness and flexibility methods. This parallel can be demonstrated by comparing Sections 1.11 and 1.13. Note, for instance, the parallel between Eqs. (1.115) and (1.149), between Eqs. (1.124) and (1.158), or between displacements $\{q\}^0$ and redundant forces, $\{F\}^0$ and so on.

PROBLEMS

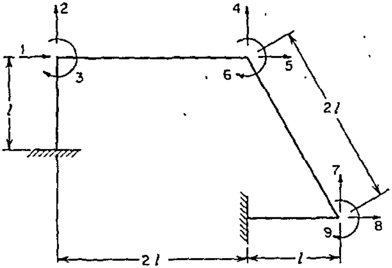
1. The structure shown has three characteristic displacement functions given by $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ in Problem 1(b). These functions are defined at the centers of the three masses. Use functions $\phi_i(x)$ $i = 1, 2, 3$ to express the arbitrary displacement configuration of the three masses as shown in Problem 1(c).

$$w(x) = \sum_{i=1}^{3} \phi_i(x) p_i$$



Problem 1

2. The members in the plane frame shown are considered inextensible. Select a set of generalized displacement coordinates $\{q\}$ and derive the transformation matrix $[C]$ relating coordinates $\{u\}$ (shown in Probl— ²) to your selected $\{q\}$ coordinates.



Problem 2

3. *Given the force transformation*

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Derive the transformation matrix $[C]$ in the relation

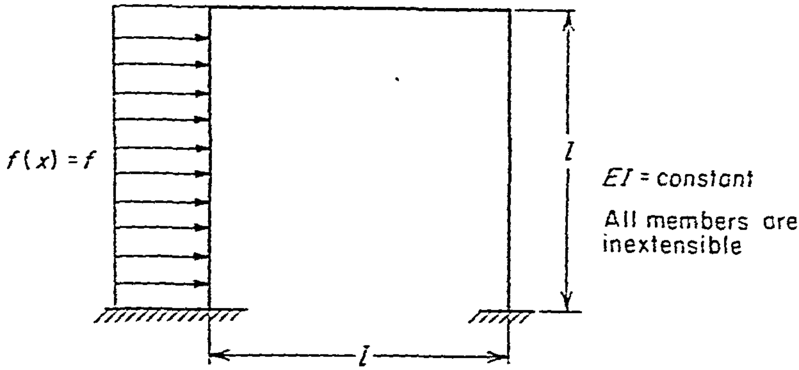
$$\{u\} = [C]\{q\}$$

4. Equation (1.88)

$$Q_k = \frac{\partial U}{\partial q_k}$$

was derived without stipulating linearity of the force-deflection properties. Show that the relation

However you must reconsider the functions expressing the displacement of the left-hand column in view of the uniformly distributed load.

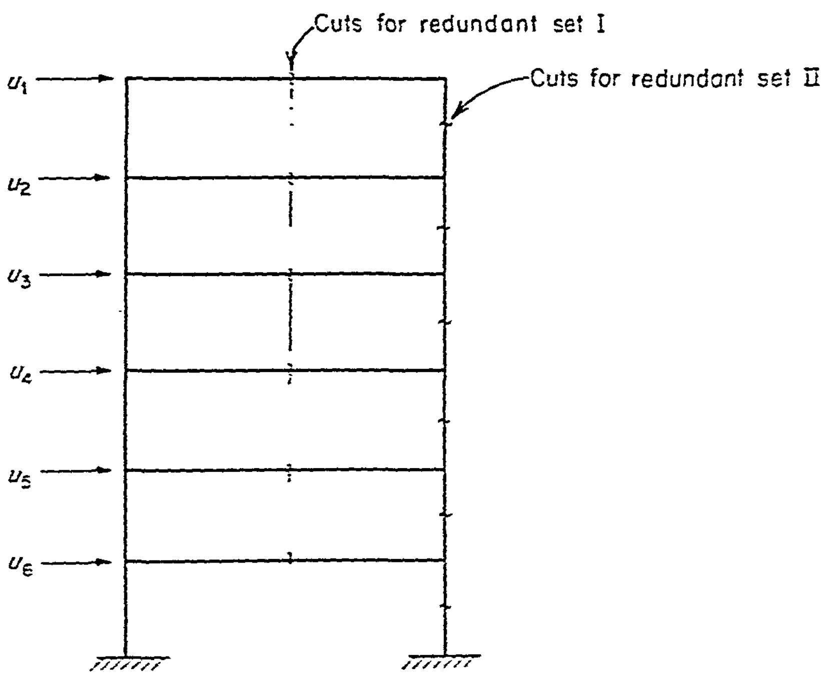


Problem 14

15. Derive the expressions of the displacement functions $w(x)$ for the three members in the portal frame of Problem 14.

16. Derive the flexibility matrix of order 2×2 for the structure of Problem 11, using the method of Section 1.13. Check your result by comparing with the inverse of the stiffness matrix in Problem 11.

17. Which set of redundants would you choose to construct the reduced 6×6 flexibility matrix for the multi-story building frame shown. Redundant Set I results from cutting all girders at midspan. Redundant Set II results from cutting all right-hand side columns at mid-height. Discuss.



Problem 17

bodies. To describe the motion of each mass in three-dimensional space, we need a minimum of 6 coordinates or a total of 36 independent coordinates for all six masses. Following the discussion in Chapter 1, these 36 independent or generalized coordinates represent the number of degrees of freedom of motion of the mass system of Fig. 2.1(b). Note the distinction between mass system and structural system. In Fig. 2.1(b) the mass of the beam has been lumped in six separate rigid bodies; the force displacement properties of the beam are still distributed. The system of Fig. 2.1(b) may be considered as one possible model representing approximately the actual physical system of Fig. 2.1(a). In choosing this model we will obtain the information regarding the motion of the idealized model at the points where coordinates are defined. More accurate information can be obtained by lumping the distributed mass at closer intervals and defining more coordinates.

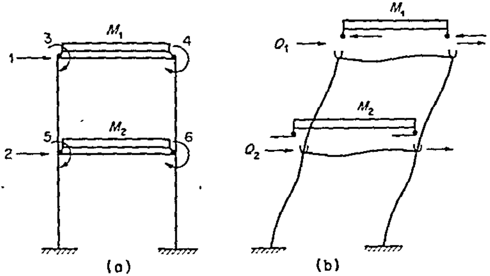


Figure 2.2

We will now demonstrate the use of coordinates to formulate the equations of motion of a system. The two-story building frame of Fig. 2.2 is considered to move in the plane of the paper. We idealize the system by assuming the mass to be concentrated in two rigid lumps at the floor levels. We further assume that the mass is mounted on end rollers so that it can only translate laterally. Rotation of the mass cannot take place since we also assume the members of the building frame to be inextensible. The force displacement properties of the massless frame members are distributed. Six coordinates are defined in Fig. 2.2(a). Coordinates 1 and 2 describe the motion of masses m_1 m_2, respectively. Consider the building frame to be displaced initially to the right, and then suddenly released. The frame will continue to vibrate with no external forces acting on it. In the absence of damping forces the motion will persist indefinitely. Let us assume that no damping exists. If we snap a picture of the vibrating system at some instant after the motion has started, we may get a deformed shape as shown in Fig. 2.2(b). The masses are drawn removed from the frame for convenience of discussion. The forces acting between the frame and the masses are at coordinates 1 and 2 only. Writing Newton's second law for masses m_1 and m_2, we obtain

$$-Q_1 = m_1 \ddot{q}_1$$
$$-Q_2 = m_2 \ddot{q}_2$$

q - displacement

or in matrix form

$$-\{Q\}_A = [m]\{\ddot{q}\}_A \tag{2.1}$$

in which q_1, q_2, Q_1, and Q_2 are positive to the right as indicated by our choice of coordinates in Fig. 2.2(a). [ ] designates a diagonal matrix. Since action and reaction must be equal and opposite at the points where the masses and frame make contact, Q_1, Q_2 are shown acting on the frame in the positive direction [Fig. 2.2(b)] while forces of the same magnitude act in the negative direction on the corresponding masses m_1 and m_2.

Using the stiffness matrix $[k]$ defined for the coordinates of Fig. 2.2(a), we write

$$\{Q\} = [k]\{q\} \tag{2.2}$$

Since the equations of motion (2.1) are concerned only with coordinates 1 and 2 where inertial forces exist, we partition Eq. (2.2) as follows.

$$\begin{Bmatrix} \{Q\}_A \\ \hline \{Q\}_B \end{Bmatrix} = \left[\begin{array}{c|c} [k]_{11} & [k]_{12} \\ \hline [k]_{21} & [k]_{22} \end{array}\right] \begin{Bmatrix} \{q\}_A \\ \hline \{q\}_B \end{Bmatrix} \tag{2.3}$$

where

$$\{Q\}_A = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} \qquad \{Q\}_B = \begin{Bmatrix} Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix}$$

Since no external or inertial forces act along coordinates 3, 4, 5, 6, $\{Q\}_B = \{0\}$. We proceed now to express $\{Q\}_A$ in terms of $\{q\}_A$, following the steps in reducing Eq. (1.49), Section 1.5, to obtain

$$\{Q\}_A = [k]^* \{q\}_A \tag{2.4}$$

in which

$$[k]^* = [k]_{11} - [k]_{12}[k]_{22}^{-1}[k]_{21} \quad \text{[See Eq. (1.53), Section 1.5.]}$$

Substituting Eq. (2.4) in Eq. (2.1)

$$[m]\{\ddot{q}\}_A + [k]^* \{q\}_A = \{0\} \tag{2.5}$$

The equations of motion (2.5) can also be derived from the flexibility *matrix* $[a]$ *of the system. Write*

$$\begin{Bmatrix} \{q\}_A \\ \hline \{q\}_B \end{Bmatrix} = \left[\begin{array}{c|c} [a]_{11} & [a]_{12} \\ \hline [a]_{21} & [a]_{22} \end{array}\right] \begin{Bmatrix} \{Q\}_A \\ \hline \{Q\}_B \end{Bmatrix}$$

But since $\{Q\}_B = \{0\}$ *it follows that*

$$\{q\}_A = [a]_{11} \{Q\}_A \tag{2.6}$$

The only forces acting at frame coordinates 1 and 2 are inertial forces $-m_1 \ddot{q}_1$ and $-m_2 \ddot{q}_2$, respectively, or

$$\{Q\}_A = -[m] \{\ddot{q}\}_A \tag{2.7}$$

Substituting Eq. (2.7) into (2.6)

$$\{q\}_A = -[a]_{11}[m]\{\ddot{q}\}_A \tag{2.8}$$

The equations of motion as expressed in Eq. (2.8) are identical to Eq. (2.5). This can be verified by multiplying each side of Eq. (2.8) by $[k]^*$ of Eq. (2.4) and recognizing that

$$[k]^* [a]_{11} = [I]$$

In the formulation of the equations of motion (Eqs. 2.5 or 2.8) for the system of Fig. 2.2, the displacement coordinates defined at massless points on the frame were not present because no inertia forces were associated with these coordinates. This is true even though external forces may be applied at the massless points such as, for instance, time varying moments $\{Q(t)\}_B$ at the coordinates 3, 4, 5, 6 in Fig. 2.2. (See Problem 2.2.)

2.3 Constraints

To illustrate dynamic systems with constraints, let us consider a single elastically-supported rigid body connected to a fixed point by a massless spring. In three-space the position of this body would be given by six coordinates—three for translation and three for rotation. Let us suppose that the system is constrained so that it must

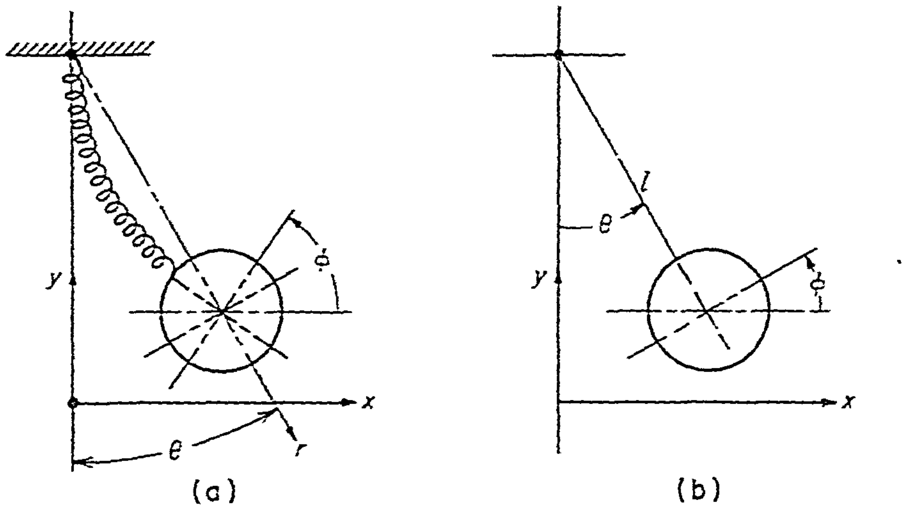


Figure 2.3

undergo plane motion, i.e., every point in the system must move along paths parallel to a given plane. This constrained system is shown in Fig. 2.3(a).

A rigid body in plane motion requires three coordinates. Referring to Fig. 2.3(a) these coordinates may be r, θ, ϕ or x, y, ϕ. In either case the first two coordinates (r, θ in polar coordinates or x, y in rectangular coordinates) give the displacement of the center of mass, and the last coordinate ϕ gives the angular displacement. Other choices are possible, but in any case the number of coordinates will be three. Now, if the flexible spring is replaced by a rigid rod of fixed length l (allowed to pivot freely at the fixed point) two equations of constraint are introduced [Fig. 2.3(b)]. In polar coordinates these are

$$r = l, \qquad \phi = \theta$$

In rectangular coordinates they are

$$x^2 + y^2 = l^2, \qquad \phi = \theta$$

The number of essential coordinates is thus reduced to one [$n = m - r = 1$, see Eq. (1.4)], and we have in the last case a single-degree-of-freedom pendulum.

2.4 Kinetic Energy and Generalized Mass in Discrete Coordinates

In formulating equations of motion it is natural to use coordinates in which the motion of the structure can be interpreted most readily.

However, the most logical choice of coordinates for physical interpretation is not always the best choice for solution of the equations. Therefore, it is useful to consider coordinate transformation in dynamic systems.

Let us consider a system of connected rigid bodies and let a discrete set of coordinates $u_1, u_2, u_3, \ldots, u_m$ (which may be constrained) represent the displacements (rotation as well as translation) of these bodies. The kinetic energy of the system is written as

$$T = \tfrac{1}{2} \sum_j m_j \dot{u}_j^2 \tag{2.9}$$

in which m_j represents mass or mass moments of inertia, depending on whether the corresponding coordinate u_j represents translation or rotation. (See Problem 2.3.)

We now apply a coordinate transformation

$$u_j = u_j(q_1, q_2, \ldots, q_n) \tag{2.10}$$

$$j = 1, 2, 3, \ldots, m \qquad m > n$$

to take us from the constrained u coordinates to generalized coordinates q. In dealing with kinetic energy this transformation requires some discussion. From Eqs. (2.10) we can write

$$du_j = \sum_{k=1}^{n} \frac{\partial u_j}{\partial q_k} dq_k \tag{2.11}$$

or

$$d\{u\} = [C]\, d\{q\} \tag{2.12}$$

in which

$$C_{jk} = \frac{\partial u_j}{\partial q_k} \tag{2.13}$$

When the transformation expressed by Eq. (2.10) is linear, namely the u's are linear functions of the q's, then in relation (2.13)

$$C_{jk} = \frac{\partial u_j}{\partial q_k} = \text{constant}$$

$$j = 1, 2, 3, \ldots, m$$

$$k = 1, 2, 3, \ldots, n$$

Such a linear transformation was considered in Section 1.3 and expressed by Eq. (1.21).

$$\{u\} = [C]\{q\}$$

In general, the coordinate transformation as expressed by Eq. (2.11) requires only that the $C_{jk} = (\partial u_j / \partial q_k)$ exist. For a nonlinear transformation the C_{jk} may be functions of the q's. This will be demonstrated by an example.

The spring pendulum of Fig. 2.4 has a mass particle m suspended by an elastic spring of stiffness k and free length l. We consider this system constrained to move without friction in the plane of the paper. Hence, we are dealing with a two-degree-of-freedom system. Displacements of the mass particle may be expressed in terms of the coordinates u_1, u_2 or q_1, q_2.

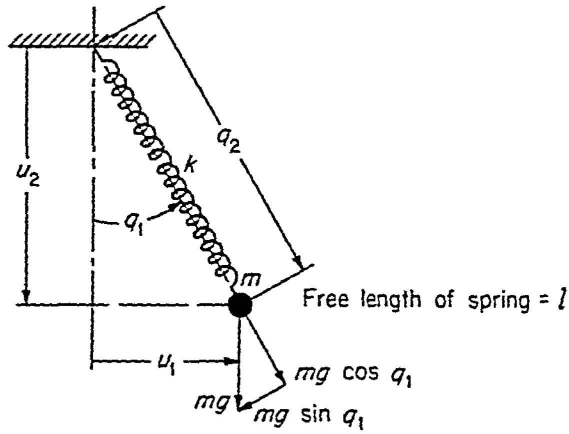


Figure 2.4

The coordinate transformation

$$u_j = u_j(q_1, q_2)$$

is given by

$$u_1 = q_2 \sin q_1$$
$$u_2 = q_2 \cos q_1$$

from which we obtain

$$\begin{aligned} du_1 &= q_2 (\cos q_1)\, dq_1 + (\sin q_1)\, dq_2 \\ du_2 &= -q_2 (\sin q_1)\, dq_1 + (\cos q_1)\, dq_2 \end{aligned} \tag{2.14}$$

or in matrix form

$$d\{u\} = [C]\, d\{q\} \tag{2.15}$$

where

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_1}{\partial q_1} & \dfrac{\partial u_1}{\partial q_2} \\ \dfrac{\partial u_2}{\partial q_1} & \dfrac{\partial u_2}{\partial q_2} \end{bmatrix} = \begin{bmatrix} q_2 \cos q_1 & \sin q_1 \\ -q_2 \sin q_1 & \cos q_1 \end{bmatrix} \tag{2.16}$$

From Eq. (2.16) it is seen that the terms C_{ij} are functions of the q's.

Returning now to the coordinate transformation expressed by Eq. (2.10) and taking first derivatives with respect to time, we write

$$\dot{u}_j = \sum_{k=1}^{n} \frac{\partial u_j}{\partial q_k} \dot{q}_k \tag{2.17}$$

The square of the velocity $\dot{u}_j$ is

$$\dot{u}_j^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial u_j}{\partial q_k} \frac{\partial u_j}{\partial q_l} \dot{q}_k \dot{q}_l \tag{2.18}$$

Substituting Eq. (2.18) into Eq. (2.9) we write the kinetic energy in the q coordinate system as

$$T = \frac{1}{2} \sum_j m_j \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial u_j}{\partial q_k} \frac{\partial u_j}{\partial q_l} \dot{q}_k \dot{q}_l$$

Interchanging the order of summation

$$T = \frac{1}{2} \sum_k \sum_l \dot{q}_k \dot{q}_l \sum_j m_j \frac{\partial u_j}{\partial q_k} \frac{\partial u_j}{\partial q_l}$$

Defining generalized mass m_{kl} by

$$m_{kl} = \sum_j m_j \frac{\partial u_j}{\partial q_k} \frac{\partial u_j}{\partial q_l} \tag{2.19}$$

the kinetic energy becomes

$$T = \frac{1}{2} \sum_k \sum_l m_{kl} \dot{q}_k \dot{q}_l \tag{2.20}$$

We now compare the expressions for kinetic energy in the u coordinate system, Eq. (2.9), with the same energy expressed in the generalized coordinate system q, Eq. (2.20). In Eq. (2.9) the m_j's are constants representing the masses and mass moments of inertia of the rigid bodies in the system. Hence, T is only a function of the velocities $\dot{u}_j$ or

$$T = T(\dot{u}_1, \dot{u}_2, \dot{u}_3, \ldots, \dot{u}_m) \tag{2.21}$$

In Eq. (2.20) the generalized mass terms m_{kl} are a function of the partial derivatives $\partial u_j/\partial q_k$. These partial derivatives may, in general, be functions of the q's as was demonstrated in the example of Fig. 2.4. Consequently, the kinetic energy expressed in the generalized coordinate system and given by Eq. (2.20) may, in general, be a function of the generalized displacements q as well as the generalized velocities $\dot{q}$, or

$$T = T(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; q_1, q_2, \ldots, q_n) \tag{2.22}$$

In the special case when the transformation

$$u_j = u_j(q_1, q_2, \ldots, q_n)$$

is linear, the kinetic energy T in Eq. (2.22) will reduce to a function of the $\dot{q}_j$ $(j = 1, 2, \ldots, n)$ only.

In matrix form Eq. (2.9) is written as

$$T = \tfrac{1}{2}\{\dot{u}\}^T[m]\{\dot{u}\} \tag{2.23}$$

and transformation Eq. (2.17) is given by

$$\{\dot{u}\} = [C]\{\dot{q}\} \tag{2.24}$$

Substituting Eq. (2.24) into (2.23) we obtain the kinetic energy in terms of coordinates q

$$T = \tfrac{1}{2}\{\dot{q}\}^T[C]^T[m][C]\{\dot{q}\} \tag{2.25}$$

This expression is identical to Eq. (2.20) which has the matrix form

$$T = \tfrac{1}{2}\{\dot{q}\}^T[m_{kl}]\{\dot{q}\} \tag{2.26}$$

Comparing Eqs. (2.25) and (2.26) we write the generalized mass matrix $[m_{kl}]$ in the form

$$[m_{kl}] = [C]^T[m][C] \tag{2.27}$$

$[m_{kl}]$ is no longer a diagonal matrix. It is symmetric and in general of lower order than $[m]$ since $[C]$ is of order $m \times n$ where $m > n$. Equation (2.27) is a restatement of Eq. (2.19) in matrix form.

It is interesting to note the similarity between the expression for generalized mass matrix [Eq. (2.27)] and that of generalized stiffness and flexibility matrix discussed in Chapter 1. For a linear transformation

$$\{u\} = [C]\{q\}$$

the generalized stiffness matrix in the q coordinate system can be obtained from the corresponding stiffness matrix $[k]_u$ in the u coordinate system.*

$$[k]_q = [C]^T[k]_u[C]$$

Similarly, for a linear force transformation

$$\{F\} = [C]\{Q\}$$

the generalized flexibility matrix in the q coordinate system takes the form

$$[a]_q = [C]^T[a]_u[C]$$

We now extend the example of Fig. 2.4 and write the kinetic energy in the u and q coordinates. From Fig. 2.4 we write

$$\begin{aligned} T = T(\dot{u}_1, \dot{u}_2) &= \tfrac{1}{2}\{\dot{u}\}^T[m]\{\dot{u}\} \\ &= \tfrac{1}{2}m(\dot{u}_1^2 + \dot{u}_2^2) \end{aligned}$$

To express the kinetic energy in the q coordinate system, we first derive the generalized mass matrix $[m_{kl}]$ using Eq. (2.27). Transformation matrix $[C]$ is given by Eq. (2.16). Hence

*Provided $[k]_u$ exists. (See Section 1.9.)

coordinates $q_j (j = 1, 2, \ldots, n)$, we can express the displacement at any point along the column as

$$w(x) = \sum_j \phi_j(x)\, q_j \tag{2.36}$$

Taking first derivatives with respect to time, Eq. (2.36) becomes

$$\dot{w}(x) = \sum_j \phi_j(x)\, \dot{q}_j \tag{2.37}$$

The square of the velocity $\dot{w}(x)$ is given by

$$\dot{w}^2(x) = \sum_k \sum_l \phi_k(x)\, \phi_l(x)\, \dot{q}_k \dot{q}_l \tag{2.38}$$

The kinetic energy for the distributed mass system can be written as

$$T = \tfrac{1}{2} \int m(x)\, \dot{w}^2(x)\, dx \tag{2.39}$$

where the integral extends over the entire length of the column in Fig. 2.5(b), or over all members of a more complex system. Substituting Eq. (2.38) into (2.39)

$$T = \tfrac{1}{2} \int m(x) \left(\sum_k \sum_l \phi_k(x)\, \phi_l(x)\, \dot{q}_k \dot{q}_l \right) dx$$

Interchanging the order of summation and integration and rearranging, we write

$$T = \tfrac{1}{2} \sum_k \sum_l \dot{q}_k \dot{q}_l \int m(x)\, \phi_k(x)\, \phi_l(x)\, dx \tag{2.40}$$

or

$$T = \tfrac{1}{2} \sum_k \sum_l m_{kl}\, \dot{q}_k \dot{q}_l$$

In matrix form

$$T = \tfrac{1}{2} \{\dot{q}\}^T [m_{kl}] \{\dot{q}\} \tag{2.41}$$

in which

$$m_{kl} = \int m(x)\, \phi_k(x)\, \phi_l(x)\, dx \tag{2.42}$$

is the generalized mass m_{kl} expressed in terms of the distributed mass function $m(x)$, and the modal shapes $\phi_k(x)$ and $\phi_l(x)$ corresponding to generalized coordinates q_k and q_l, respectively. We summarize then, that when distributed coordinates are used with a lumped mass system, the generalized mass m_{kl} is expressed by Eq. (2.35).

$$m_{kl} = \sum_j m_j\, \phi_{jk}\, \phi_{jl} \tag{2.35}$$

But when the mass of the system is distributed, the summation in Eq. (2.35) is transformed into an integral given by Eq. (2.42).

The discussion in this section to this point dealt with generalized distributed coordinates. It is instructive to consider constrained distributed coordinates.

Lumped Mass: We first consider lumped mass systems. Let the displacement of mass m_i in a specified direction be given by

$$w(x_i) = \sum_j \phi_j(x_i)\, p_j \tag{2.43}$$

where the p_j ($j = 1, 2, \ldots, m$) are constrained distributed coordinates. The displacements and corresponding velocities of all masses in the system can be written as

$$\{w\} = [\phi]\{p\} \tag{2.44}$$

and

$$\{\dot{w}\} = [\phi]\{\dot{p}\} \tag{2.45}$$

respectively.
For the kinetic energy of the system, we write

$$T = \tfrac{1}{2}\{\dot{w}\}^T[m]\{\dot{w}\}$$

Substituting for the velocities $\{\dot{w}\}^T$ and $\{\dot{w}\}$ from Eq. (2.45)

$$T = \tfrac{1}{2}\{\dot{p}\}^T[\phi]^T[m][\phi]\{\dot{p}\} \tag{2.46}$$

We now apply a coordinate transformation

$$p_j = p_j(q_1, q_2, \ldots, q_n) \tag{2.47}$$

$$j = 1, 2, \ldots, m \quad m > n$$

to take us from constrained coordinates p_j to generalized coordinates q_j. Differentiating Eq. (2.47) with respect to time

$$\dot{p}_j = \sum_k \frac{\partial p_j}{\partial q_k}\dot{q}_k \tag{2.48}$$

In matrix form

$$\{\dot{p}\} = [C]\{\dot{q}\} \tag{2.49}$$

in which

$$C_{jk} = \frac{\partial p_j}{\partial q_k} \qquad \begin{array}{l} j = 1, 2, \ldots, m \\ k = 1, 2, \ldots, n \end{array}$$

may be constants or functions of q, depending on whether the transformation of Eq. (2.47) is linear or nonlinear. Substituting from Eq. (2.49) into Eq. (2.46), we write

$$T = \tfrac{1}{2}\{\dot{q}\}^T[C]^T[\phi]^T[m][\phi][C]\{\dot{q}\} \tag{2.50}$$

or

$$T = \tfrac{1}{2}\{\dot{q}\}^T[m_{kl}]\{\dot{q}\} \tag{2.51}$$

in which the generalized mass matrix is given by

$$[m_{kl}] = [C]^T[\phi]^T[m][\phi][C] \tag{2.52}$$

Distributed Mass: When the mass of the system is distributed we write the displacement and velocity at any point on the system as

$$w(x) = \sum_{j=1}^{m} \phi_j(x)\, p_j \tag{2.53}$$

and

$$\dot{w}(x) = \sum_{j} \phi_j(x)\, \dot{p}_j \tag{2.54}$$

respectively. From Eq. (2.54) we write

$$\dot{w}^2(x) = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} \phi_\alpha(x)\, \phi_\beta(x)\, \dot{p}_\alpha \dot{p}_\beta \tag{2.55}$$

Using Eq. (2.55), the kinetic energy

$$T = \tfrac{1}{2} \int m(x)\, \dot{w}^2(x)\, dx$$

is written in the form

$$T = \tfrac{1}{2} \int m(x) \left(\sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} \phi_\alpha(x) \phi_\beta(x) \dot{p}_\alpha \dot{p}_\beta \right) dx$$

Interchanging order of summation and integration and rearranging

$$T = \tfrac{1}{2} \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} \dot{p}_\alpha \dot{p}_\beta \int m(x) \phi_\alpha(x) \phi_\beta(x)\, dx$$

or

$$T = \tfrac{1}{2} \sum_{\alpha} \sum_{\beta} m_{\alpha\beta} \dot{p}_\alpha \dot{p}_\beta$$

In matrix form

$$T = \tfrac{1}{2} \{\dot{p}\}^T [m_{\alpha\beta}] \{\dot{p}\} \tag{2.56}$$

in which

$$m_{\alpha\beta} = \int m(x) \phi_\alpha(x) \phi_\beta(x)\, dx \tag{2.57}$$

with the integral extending over all members of the system.

We now apply the coordinate transformation of Eq. (2.49)

$$\{\dot{p}\} = [C]\{\dot{q}\} \tag{2.49}$$

Substituting Eq. (2.49) into Eq. (2.56)

$$T = \tfrac{1}{2} \{\dot{q}\}^T [C]^T [m_{\alpha\beta}] [C] \{\dot{q}\} \tag{2.58}$$

or

$$T = \tfrac{1}{2} \{\dot{q}\}^T [m_{kl}] \{\dot{q}\} \tag{2.59}$$

in which the generalized mass matrix is given by

$$\underset{n\times n}{[m_{kl}]} = \underset{n\times m}{[C]^T} \underset{m\times m}{[m_{\alpha\beta}]} \underset{m\times n}{[C]} \tag{2.60}$$

2.6 D'Alembert's Principle

Consider the ith mass m_i of a system of connected rigid bodies, and the force components F_j ($j = 1, 2, \ldots, 6$) acting upon it in three-dimensional space. The F_j's include externally applied forces and internal forces applied to m_i by the interconnections with other masses of the system. If the mass m_i is in equilibrium at rest, then

$$\sum_{j=1}^{6} F_j = 0 \tag{2.61}$$

We emphasize again that the F's may include moments as well as forces, and represent all forces acting on mass m_i. Let the displacements of the system be represented by a set of constrained coordinates $u_1, u_2, \ldots, u_m$. The displacement of each rigid mass m_i is described by displacement components u_j ($j = 1, 2, \ldots, 6$), representing three translations and three rotations in three-dimensional space. These displacement components correspond to forces F_j ($j = 1, 2, \ldots, 6$) in Eq. (2.61).

Consider now a displacement of m_i with components δu_j ($j = 1, 2, \ldots, 6$). The virtual work δW_i done by force components F_j ($j = 1, 2, \ldots, 6$) along these displacements is given by

$$\delta W_i = \sum_j F_j \, \delta u_j \tag{2.62}$$

Since the rigid mass m_i is in equilibrium, the virtual work δW_i vanishes. Hence, a criterion for static equilibrium of m_i is given by

$$\delta W_i = 0 \tag{2.63}$$

Extended over the complete system of masses m_i, it follows that for static equilibrium of the system

$$\delta W = \sum_i \delta W_i = 0 \tag{2.64}$$

or using Eq. (2.62)

$$\delta W = \sum_j^{m} F_j \, \delta u_j = 0 \tag{2.65}$$

where the summation extends over all coordinates of the system. Equation (2.65) is a statement of the principle of virtual work.

If mass m_i is not in equilibrium it will accelerate in accordance with Newton's second law. The equation of motion associated with the jth coördinate of mass m_i is given by

$$m_j \ddot{u}_j = F_j \tag{2.66}$$

or

$$m_j \ddot{u}_j - F_j = 0 \tag{2.67}$$

in which F_j includes all the forces (or moments) acting along coordinate j. The mass or moment of inertia of mass m_i is represented by m_j, depending on whether $\ddot{u}_j$ represents linear or angular acceleration, respectively. For linear acceleration each of the m_j pertaining to the same rigid body has the same value. For angular acceleration, m_j representing the mass moment of inertia will, in general, have a different value for each of the principal axes of inertia. Equation (2.66) applies then to angular as well as linear acceleration. However, the coordinate system u_j associated with m_i must have its origin at the center of mass and lie in the directions of the principal axes of inertia.

While the system is in motion, we look at it at some instant t (i. e., we take a photograph of the system). We now consider at this instant of time t a virtual displacement with components $\delta u_j (j = 1, 2, \ldots, m)$. Then using Eq. (2.67), we may write for each coordinate j

$$(m_j \ddot{u}_j - F_j)\, \delta u_j = 0 \tag{2.68}$$

Summing over the entire system of m coordinates, Eq. (2.68) becomes

$$\sum_{j=1}^{m} (m_j \ddot{u}_j - F_j)\, \delta u_j = 0 \tag{2.69}$$

Equation (2.69) is a statement of the principle of D'Alembert for the system. Rewriting Eq. (2.69) in the form

$$\sum_j m_j \ddot{u}_j \delta u_j = \sum_j F_j \delta u_j$$

and recalling that the right-hand side represents the virtual work δW done by forces F_j acting along virtual displacements δu_j, we write

$$\delta W = \sum_j m_j \ddot{u}_j \,\delta u_j = \sum_j F_j \,\delta u_j \tag{2.70}$$

Equation (2.70) is a statement of virtual work for a system in motion.

2.7 Hamilton's Principle

The Irish mathematician and physicist, Sir William Hamilton (1805–1865), formulated his celebrated principle in dynamics in which the governing equation depends explicitly on the energy of the system. Hamilton's principle is stated as an integral equation in which the energy is integrated over an interval in time.

We will now introduce the concept of the varied path in order to facilitate the derivation of Hamilton's principle. Let us consider, again, the ideal structural system of interconnected rigid masses whose displacements are measured by coordinates $u_j (j = 1, 2, \ldots, m)$. We confine our attention to an interval of time during which the system moves from configuration 1 at $t = t_1$ to configuration 2 at $t = t_2$. During this time interval each point will move along a space path such that Newton's second law holds; this path is called the Newtonian path. To fix ideas consider the simply supported beam with four lumped mass particles, as shown in Fig. 2.6. The system is moving in the plane of the paper with its position described by coordinates u_1, u_2, u_3, u_4. Figures 2.6(b), (c), (d) (solid curve) show

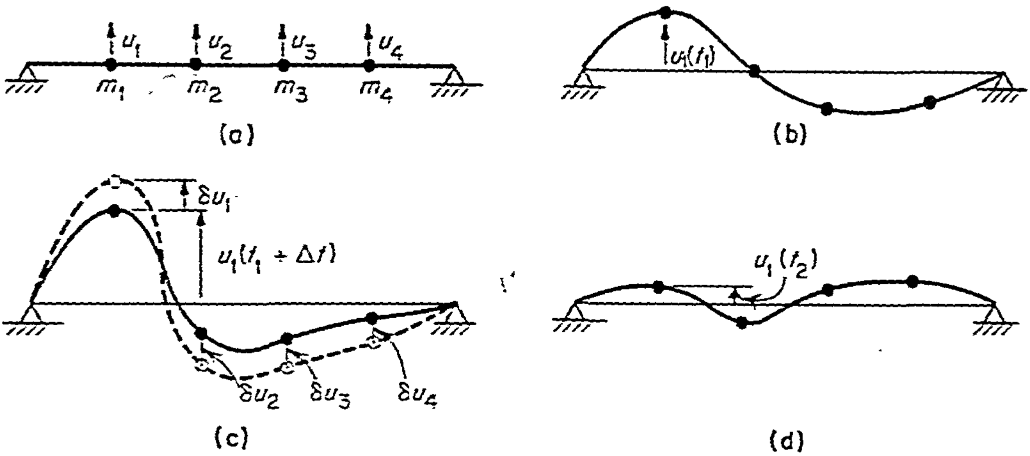


Figure 2.6 (b) Displacement configuration at time $t = t_1$. (c) Displacement configuration at time $t = t_1 + \Delta t$. (d) Displacement configuration at time $t = t_2$.

the displaced configuration of the system at times t_1, $t_1 + \Delta t$ and t_2, respectively. These configurations may be visualized as photographs of the moving system snapped at the times indicated. If we now plot displacement u_1 as a function of time, the curve may look as shown

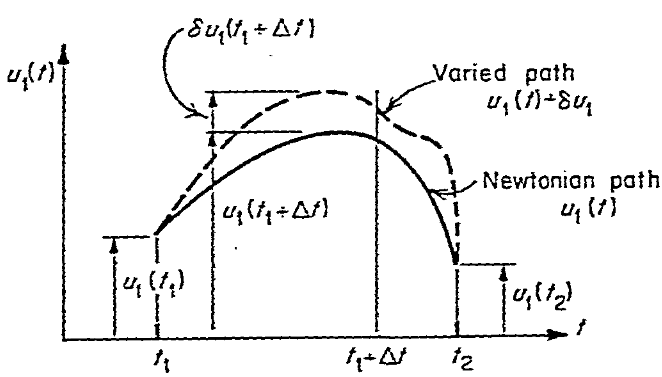


Figure 2.7 Newtonian path and varied path for coordinate u_1 of Fig. 2.6.

in Fig. 2.7 (solid line). This curve represents the Newtonian path for mass m_1 in the time interval between t_1 and t_2. A similar path can be drawn for each mass particle of Fig. 2.6(a).

Consider now a virtual displacement with components δu_j applied to the system at the instant $t_1 + \Delta t$. The displacement configuration may then appear as shown by the dashed curve in Fig. 2.6(c). A corresponding change $\delta u_1(t + \Delta t)$ is shown in the Newtonian path for mass m_1 in Fig. 2.7. We repeat the process of taking photographs of the moving system in the time interval between t_1 and t_2 [as in Fig. 2.6(c)] and apply a virtual displacement δu_j to each configuration. We can then plot along with the Newtonian path for each point, a "varied path" representing a plot of the displacement $u + \delta u$ as a function of time. Such a varied path for mass m_1 of Fig. 2.6 is shown by the dashed line in Fig. 2.7.

In Fig. 2.7 the varied path is shown to coincide with the Newtonian path at t_1 and t_2. This follows from our deliberate choice of the virtual displacements $\delta u_1(t_1) = \delta u_1(t_2) = 0$. If we choose $\delta u_j(t_1) = \delta u_j(t_2) = 0$ for all j, the varied path will coincide with the Newtonian path at all points in the system at times $t = t_1$ and $t = t_2$.

Before proceeding, we emphasize that the virtual displacements δu_j are always selected so as to satisfy the boundary conditions ($\delta u = 0$ at the beam supports of Fig. 2.6). This is discussed further in the next section.

Consider now the velocity of a point on the varied path for coordinate j. This velocity can be written as the velocity on the Newtonian path $\dot{u}_j$ plus the variation $\delta\dot{u}_j$ or

$$\dot{u}_j + \delta\dot{u}_j \tag{2.71}$$

The velocity is also obtained by taking the time derivative of the displacement $(u_j + \delta u_j)$ or

$$\frac{d}{dt}(u_j + \delta u_j)$$

which can be written as

$$\dot{u}_j + \frac{d}{dt}(\delta u_j) \tag{2.72}$$

Equating (2.71) and (2.72) we obtain

$$\dot{u}_j + \delta\dot{u}_j = \dot{u}_j + \frac{d}{dt}(\delta u_j)$$

or

$$\delta\dot{u}_j = \delta\left(\frac{d}{dt}u_j\right) = \frac{d}{dt}(\delta u_j) \tag{2.73}$$

That is, the operators δ and d/dt are commutative.

We return now to our ideal system of rigid masses whose motion is described by coordinates $u_j (j = 1, 2, 3, \ldots, m)$. The kinetic energy of the system is given by

$$T = \tfrac{1}{2} \sum_{j=1}^{m} m_j \dot{u}_j^2$$

Associated with virtual displacements δu_j are virtual velocities $\delta \dot{u}_j$. Hence, the virtual kinetic energy or variation in kinetic energy corresponding to the $\delta \dot{u}_j$'s is written as

or $\delta \dot{u}_j^2 = 2\dot{u}$

$$\delta T = \sum_j \frac{\partial T}{\partial \dot{u}_j} \delta \dot{u}_j$$

$$= \sum_j m_j \dot{u}_j \, \delta \dot{u}_j$$

or using relation (2.73)

$$\delta T = \sum_j m_j \dot{u}_j \frac{d}{dt} \delta u_j \tag{2.74}$$

Multiplying each side of the identity

$$\frac{d}{dt} (\dot{u}_j \, \delta u_j) = \ddot{u}_j \, \delta u_j + \dot{u}_j \frac{d}{dt} \delta u_j$$

by m_j, which is considered to be constant, and summing over all m coordinates of the system, we write

$$\frac{d}{dt} \sum_{j=1}^{m} m_j \dot{u}_j \, \delta u_j = \sum_{j=1}^{m} m_j \ddot{u}_j \, \delta u_j + \sum_{j=1}^{m} m_j \dot{u}_j \frac{d}{dt} \delta u_j \tag{2.75}$$

Substituting from Eqs. (2.70) and (2.74) in Eq. (2.75), we obtain

$$\frac{d}{dt} \sum_{j=1}^{m} m_j \dot{u}_j \, \delta u_j = \delta T + \delta W \tag{2.76}$$

Integrating both sides of Eq. (2.76) over the time interval between t_1 and t_2, we write

$$\sum_{j=1}^{m} m_j \dot{u}_j \, \delta u_j \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} (\delta T + \delta W) \, dt \tag{2.77}$$

When we consider the instantaneous virtual displacements δu_j which define the varied path over a time continuum from t_1 to t_2, we select

$$\delta u_j(t_1) = \delta u_j(t_2) = 0 \qquad \text{for} \quad j = 1, 2, 3, \ldots, m \tag{2.78}$$

so that the Newtonian and varied path coincide at $t = t_1$ and $t = t_2$ for all coordinates $u_j (j = 1, 2, 3, \ldots, m)$. This was demonstrated in Fig. 2.7 for coordinate u_1. Using relation (2.78) it follows that

$$\sum_j m_j \dot{u}_j \, \delta u_j \Big|_{t_1}^{t_2} = 0 \tag{2.79}$$

and Eq. (2.77) becomes

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0 \tag{2.80}$$

Equation (2.80) is a statement of Hamilton's principle.

For a conservative system in which all forces F_j $(j = 1, 2, \ldots, m)$ are derivable from a potential energy V

$$\delta W = -\delta V \tag{2.81}$$

As a simple example of Eq. (2.81) consider Fig. 2.8. The work done by the force of gravity F_j in Fig. 2.8(a) on virtual displacement δu_j is

$$\delta W = F_j\, \delta u_j$$

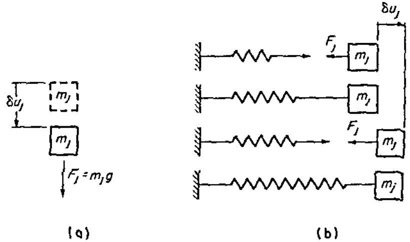


Figure 2.8 (a) Variation in gravitational potential. (b) Variation in strain energy potential.

The corresponding variation in potential energy is

$$\delta V = -F\, \delta u_j$$

hence

$$\delta W = -\delta V$$

Similarly, in Fig. 2.8(b), the virtual work done by spring force F_j acting on the mass m_j through a virtual displacement δu_j is

$$\delta W = -F_j\, \delta u_j$$

The minus sign is a result of F_j and δu_j being oppositely directed. *The corresponding change in potential energy (represented entirely by the change in strain energy in the spring) is given by*

$$\delta V = F_j\, \delta u_j$$

so that again

$$\delta W = -\delta V$$

Potential energy V may also arise from other sources, such as electrical and magnetic fields.

Using Eq. (2.81) Eq. (2.80) becomes

$$\int_{t_1}^{t_2} (\delta T - \delta V)\, dt = \delta \int_{t_1}^{t_2} (T - V)\, dt = \delta \int_{t_1}^{t_2} L\, dt = 0 \tag{2.82}$$

where

$$L = T - V \tag{2.83}$$

is termed the *Lagrangian* or the *kinetic potential.* When strain energy U is the only potential source

$$V = U$$

and the Lagrangian takes the form

$$L = T - U$$

We emphasize that the statement of Hamilton's principle in the form of Eq. (2.80) or (2.82) depends upon the energies of the system and is invariant under a coordinate transformation.

We observe also that Eq. (2.82) states that of all the possible paths of motion of a system during an interval of time from t_1 to t_2, the actual path will be that one for which the integral

$$\int_{t_1}^{t_2} L\, dt$$

has a stationary value. It can be shown that this stationary value will, in fact, be the *minimum* value of the integral.

2.8 Formulation of Beam Problems by Using Hamilton's Principle

As an example of the use of Hamilton's principle in formulating equations of motion let us consider a uniform beam. We shall consider the free vibration of the beam, and shall first consider only the strain energy of flexure. We write expressions for kinetic and strain energies

$$T = \tfrac{1}{2} \int_{x=0}^{l} m \dot{w}^2\, dx \tag{2.84}$$

$$U = \tfrac{1}{2} \int_{x=0}^{l} EI w''^2\, dx \tag{2.85}$$

Here, m is the mass per unit length, EI is the flexural rigidity, w is the deflection, and x is the length coordinate. The integrations are taken over the beam length l. In the kinetic energy integral

Hence, we isolate it and interchange the order of the two integrations to give

$$\int_{x=0}^{l} m \left\{ \int_{t_1}^{t_2} \dot{w} \delta \dot{w} \, dt \right\} dx$$

We integrate the time integral by parts using $\delta \dot{w} = d/dt \, \delta w$ and the fact that δw vanishes at both ends of the time interval.

$$\int_{t_1}^{t_2} \dot{w} \delta \dot{w} \, dt = \dot{w} \, \delta w \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{w} \, \delta w \, dt = -\int_{t_1}^{t_2} \ddot{w} \delta w \, dt$$

Putting this back into Eq. (2.93) that equation becomes

$$\int_{t_1}^{t_2} \int_0^l (-m\ddot{w} - [EIw'']'') \, \delta w \, dx \, dt = 0$$

Because the time interval $t_1 \to t_2$ is arbitrary, the space integral must vanish, i. e.,

$$\int_0^l (-m\ddot{w} - [EIw'']'') \, \delta w \, dx = 0$$

Now, δw is arbitrary everywhere except at the boundaries; hence, the above equation requires that the integrand vanish. We thus obtain the following partial differential equation of motion for the beam in bending.

$$[EIw'']'' + m\ddot{w} = 0 \tag{2.94}$$

For our second example, let us reconsider the foregoing beam problem with the stipulation that we must account for shear as well as flexural deformations, and we must also account for the angular velocities of the beam sections which are not considered to have negligible depth. In this example the power of Hamilton's principle is more evident than in the previous one. The energy integrals in this example are

$$T = \tfrac{1}{2} \int_{x=0}^{l} m\dot{w}^2 \, dx + \tfrac{1}{2} \int_{x=0}^{l} m\rho^2 \dot{\psi}^2 \, dx \tag{2.95}$$

$$U = \tfrac{1}{2} \int_{x=0}^{l} EI\psi'^2 \, dx + \tfrac{1}{2} \int_{x=0}^{l} \alpha GA\gamma^2 \, dx \tag{2.96}$$

where

ρ = radius of gyration of mass element $m\,dx$ about the neutral axis

G = the modulus of elasticity in shear

A = the area of the cross-section

α = a dimensionless number which depends upon the shape of the section

αGA = the effective shear modulus.

ψ = angle of rotation of the beam section

γ = angle of shear deformation

The variables w, ψ, γ are not independent, but are related by

$$w' = \psi + \gamma \tag{2.97}$$

Proceeding as before we form

$$\delta T = \int_0^l m\dot{w}\,\delta\dot{w}\,dx + \int_0^l m\rho^2\dot{\psi}\delta\dot{\psi}\,dx\,. \tag{2.98}$$

Further

$$\begin{aligned}\int_{t_1}^{t_2} \delta T\,dt &= \int_{t_1}^{t_2}\int_0^l m\dot{w}\,\delta\dot{w}\,dx\,dt + \int_{t_1}^{t_2}\int_0^l m\rho^2\dot{\psi}\delta\dot{\psi}\,dx\,dt \\ &= \int_0^l\int_{t_1}^{t_2} m\dot{w}\,\delta\dot{w}\,dt\,dx + \int_0^l\int_{t_1}^{t_2} m\rho^2\dot{\psi}\delta\dot{\psi}\,dt\,dx\end{aligned} \tag{2.99}$$

As before, we integrate by parts using

$$\delta\dot{w} = \frac{d}{dt}\delta w, \qquad \delta\dot{\psi} = \frac{d}{dt}\delta\psi \qquad \text{[See Eq. (2.73).]}$$

Then

$$\int_{t_1}^{t_2} m\dot{w}\,\delta\dot{w}\,dt = -\int_{t_1}^{t_2} m\ddot{w}\,\delta w\,dt$$

Similarly

$$\int_{t_1}^{t_2} m\rho^2\dot{\psi}\delta\dot{\psi}\,dt = -\int_{t_1}^{t_2} m\rho^2\ddot{\psi}\delta\psi\,dt$$

Hence

$$\int_{t_1}^{t_2} \delta T\,dt = -\int_{t_1}^{t_2}\int_0^l m\,(\ddot{w}\,\delta w + \rho^2\ddot{\psi}\,\delta\psi)\,dx\,dt \tag{2.100}$$

Now, we form

$$\delta U = \int_0^l EI\,\psi'\,\delta\psi'\,dx + \int_0^l \alpha GA\gamma\,\delta\gamma\,dx \tag{2.101}$$

Evaluating the first integral we obtain

$$\int_0^l EI\psi'\,\delta\psi'\,dx = -\int_0^l [EI\psi']'\,\delta\psi\,dx$$

subject to the boundary conditions

$$EI\psi'\delta\psi\Big|_0^l = 0 \tag{2.102}$$

namely

(bending moment) (virtual rotation) = 0, at $x = 0$ and $x = l$

Using the relationship (2.97) the second integral of Eq. (2.101) is evaluated as

$$\int_0^l \alpha GA\gamma\,\delta\gamma\,dx = \int_0^l \alpha GA(w' - \psi)\,\delta\,(w' - \psi)\,dx$$
$$= \int_0^l \alpha GA(w' - \psi)\,\delta w'\,dx - \int_0^l \alpha GA(w' - \psi)\,\delta\psi\,dx$$
$$= -\int_0^l [\alpha GA(w' - \psi)]'\,\delta w\,dx$$
$$-\int_0^l \alpha GA(w' - \psi)\,\delta\psi\,dx$$

Here, we used the further boundary condition

$$\alpha GA(w' - \psi)\,\delta w\Big|_0^l = 0 \tag{2.103}$$

or

(shear) (virtual displacement) = 0, at $x = 0$ and $x = l$

We now have δU in the form

$$\delta U = -\int_0^l (\{[EI\psi']' + \alpha GA(w' - \psi)\}\,\delta\psi + [\alpha GA(w' - \psi)]'\,\delta w)\,dx \tag{2.104}$$

Hamilton's principle yields

$$\delta\int_{t_1}^{t_2} (T - U)\,dt = \int_{t_1}^{t_2}\int_0^l (\{-m\ddot{w} + [\alpha GA(w' - \psi)]'\}\,\delta w$$
$$+ \{-m\rho^2\ddot{\psi} + [EI\psi']' + \alpha GA(w' - \psi)\}\,\delta\psi)\,dx\,dt = 0 \tag{2.105}$$

Now, w and ψ are independent variables, hence, δw and $\delta\psi$ are independent. Therefore, the coefficients of δw and $\delta\psi$ in the integrand above must vanish separately, and we are led to the two equations

$$\left.\begin{aligned} -m\ddot{w} + [\alpha GA(w' - \psi)]' &= 0 \\ -m\rho^2\ddot{\psi} + [EI\psi']' + \alpha GA(w' - \psi) &= 0 \end{aligned}\right\} \tag{2.106}$$

Equations (2.106) are the partial differential equations of motion* for a beam fixed at its ends $x = 0$ and $x = l$ (boundary conditions Eqs. 2.102, 2.103), and in which shear, flexural deformation and angular velocities of the beam sections were accounted for.

2.9 Lagrange's Equations

The French mathematician Lagrange (1736–1813) discovered a relationship which provides a method of great power and versatility

*Beams which are governed [illegible]ifferential Eqs. (2.106) are referred to in the literature as Timosh[illegible] Timoshenko who first studied them.

for the formulation of the equations of motion for any dynamical system. The relationship involves the energies of the system, viz, kinetic energy, strain energy, and work performed by forces on displacements of the system. The versatility of the method lies in the ease with which equations of motion can be formulated in terms of generalized coordinates for which the direct application of Newton's second law is difficult. Unlike Hamilton's principle, which is stated as an integral equation in which the total energy is integrated over an interval in time, the Lagrange equations are differential equations in which one considers the energies of the system instantaneously in time.

Hamilton's principle may be used to develop Lagrange's equations in a set of generalized coordinates. As previously noted, Hamilton's principle is invariant with respect to coordinate systems. Hence, if we begin with Hamilton's principle, as we shall do in the following development, and derive the Lagrange equations, it is useful to transform from a set of coordinates which may be constrained, to a set of generalized coordinates.

Consider the system of connected rigid bodies described in the foregoing sections (for instance, Section 2.4). The kinetic energy T expressed in a set of generalized coordinates $q_j (j = 1, 2, \ldots, n)$ is, in general, a function of the generalized displacement q_j as well as generalized velocities $\dot{q}_j$. This was discussed in Section 2.4, expressed by Eq. (2.22), and repeated here for convenience.

$$T(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n;\ q_1, q_2, \ldots, q_n)$$

It follows that

$$\delta T = \frac{\partial T}{\partial \dot{q}_1}\delta\dot{q}_1 + \frac{\partial T}{\partial \dot{q}_2}\delta\dot{q}_2 + \cdots + \frac{\partial T}{\partial \dot{q}_n}\delta\dot{q}_n + \frac{\partial T}{\partial q_1}\delta q_1 + \frac{\partial T}{\partial q_2}\delta q_2 + \cdots + \frac{\partial T}{\partial q_n}\delta q_n$$

or, in more concise form

$$\delta T = \sum_{j=1}^{n}\left(\frac{\partial T}{\partial \dot{q}_j}\delta\dot{q}_j + \frac{\partial T}{\partial q_j}\delta q_j\right) \tag{2.107}$$

It has already been noted (Chapter 1) that generalized forces may be obtained from the forces F_j by the principle of virtual work.

$$\{Q\}^T\{\delta q\} = \{F\}^T\{\delta u\} \text{ [See Eq. (1.24)]}$$

or

$$\sum_{j=1}^{n} Q_j\delta q_j = \sum_{j=1}^{n} F_j\,\delta u_j \tag{2.108}$$

We recall that the F's represent all forces acting on the rigid masses and, hence, may include internal, as well as external, forces. The Q's also will include both in the generalized coordinate systems. Each side of Eq. (2.108) represents the virtual work δW done by forces Q_j or F_j on virtual displacements δq_j or δu_j, respectively, or

$$\delta W = \sum_j Q_j \delta q_j \tag{2.109}$$

Now, we proceed with the development of Lagrange's equations in generalized coordinates, starting with Hamilton's principle. Write

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = \int_{t_1}^{t_2} \sum_j \left(\frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j + \frac{\partial T}{\partial q_j} \delta q_j + Q_j \delta q_j\right) dt = 0 \tag{2.110}$$

in which δT and δW were substituted from Eqs. (2.107) and (2.109), respectively. Consider the first term in Eq. (2.110)

$$\begin{aligned}\int_{t_1}^{t_2} \sum_j \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j\, dt &= \sum_j \int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j\, dt \\ &= \sum_j \left\{\frac{\partial T}{\partial \dot{q}_j} \delta q_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) \delta q_j\, dt\right\} \\ &= -\int_{t_1}^{t_2} \sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) \delta q_j\, dt \end{aligned} \tag{2.111}$$

remembering that

$$\frac{\partial T}{\partial \dot{q}_j} \delta q_j \Big|_{t_1}^{t_2} = 0$$

for $j = 1, 2, \ldots, n$ (See Eq. (2.78))

Using identity (2.111) in Eq. (2.110) we obtain

$$\int_{t_1}^{t_2} \sum_j \left\{-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j} + Q_j\right\} \delta q_j\, dt = 0 \tag{2.112}$$

Since the q's are generalized coordinates, the δq_j's are arbitrary except at $t = t_1$ and $t = t_2$, at which instants they are set equal to zero. Consequently, the expression in the brackets in Eq. (2.112) must vanish.

$$-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j} + Q_j = 0$$

or

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j} \tag{2.113}$$

There are n of these equations ($j = 1, 2, \ldots, n$). They are the Lagrange equations expressed in generalized coordinates.

The generalized force Q_j is generally considered to be composed of three parts

$$Q_j = Q_{A_j} + Q_{E_j} + Q_{D_j} \tag{2.114}$$

where

Q_{A_j} = applied force, externally reacted

Q_{E_j} = internal elastic force

Q_{D_j} = damping force; may be internally or externally reacted.

Consider the elastic force components Q_{E_j} which have potential U —the strain energy. According to the Castigliano theorem

$$Q_{E_j} = -\frac{\partial U}{\partial q_j} \tag{2.115}$$

This is the same expression as that derived in Chapter 1, Eq. (1.88), except for the sign. The sign change results from the fact that in the theorem of Castigliano the force is considered as applying to the elastic element as in the spring in the simple spring-mass system shown in Fig. 2.10. As considered in the foregoing discussion, however, Q_{E_j} is applied to the mass, hence it has the opposite sign.

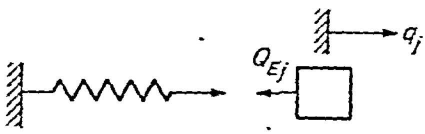


Figure 2.10

Using relations (2.114) and (2.115), the Lagrange Equation (2.113) may be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial U}{\partial q_j} - Q_{D_j} = Q_{A_j} \tag{2.116}$$

The damping forces Q_{D_j} are considered in Chapter 7.

Note that in Eq. (2.116) the applied forces Q_{A_j} may contain forces derivable from potential energy sources other than strain energy U. It is sometimes convenient to include with the strain energy term U the potential energy associated with those external forces arising from potential fields of all kinds, e.g., gravity, electric, magnetic, etc. If, as before, we denote by V the potential energy arising from all potential sources, including U, then the rth Lagrange equation for a *conservative* system (in which all external forces and damping forces adding or removing energy from the system are zero) has the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = 0 \tag{2.117}$$

This equation may also be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0$$

where $L = T - V$ is the Lagrangian defined earlier in Section 2.7, Eq. (2.83) and

$$V = V(q_1, q_2, \ldots, q_n)$$

As an example we will use the Lagrange equations to formulate the equations of motion for the mass particle of Fig. 2.4. Since no damping forces are acting on the system we set $Q_{D_1} = Q_{D_2} = 0$ in Eq. (2.116), and write for coordinates q_1 and q_2.

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial U}{\partial q_1} &= Q_{A_1}\\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial U}{\partial q_2} &= Q_{A_2}\end{aligned}\right\} \qquad (2.118)$$

in which Q_{A_1} and Q_{A_2} arise from the gravitational field. The generalized velocities of the mass particle are given by

$$q_2 \dot{q}_1 \quad \text{and} \quad \dot{q}_2$$

Hence, the kinetic energy T becomes

$$T = \tfrac{1}{2} m (q_2 \dot{q}_1)^2 + \tfrac{1}{2} m \dot{q}_2^2$$

A variation δT in T, corresponding to virtual displacements $\delta q_1, \delta q_2$ and virtual velocities $\delta \dot{q}_1$, $\delta \dot{q}_2$, is given by

$$\delta T = m q_2^2 \dot{q}_1 \delta \dot{q}_1 + m q_2 \dot{q}_1^2 \delta q_2 + m \dot{q}_2 \delta \dot{q}_2$$

Comparing this expression for δT with

$$\delta T = \sum_{j=1}^{2} \left(\frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j + \frac{\partial T}{\partial q_j} \delta q_j\right) \qquad \text{[See Eq. (2.107)]}$$

we write

$$\left.\begin{aligned}&\frac{\partial T}{\partial \dot{q}_1} = m q_2^2 \dot{q}_1, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = 2 m q_2 \dot{q}_1 \dot{q}_2 + m q_2^2 \ddot{q}_1\\ &\frac{\partial T}{\partial \dot{q}_2} = m \dot{q}_2, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m \ddot{q}_2\\ &\frac{\partial T}{\partial q_1} = 0\\ &\frac{\partial T}{\partial q_2} = m q_2 \dot{q}_1^2\end{aligned}\right\} \qquad (2.119)$$

The variation δU in strain energy U, corresponding to virtual displacements δq_1, δq_2, is written as

$$\delta U = k(q_2 - l)\, \delta q_2$$

Comparing this expression with

$$\delta U = \sum_{j=1}^{2} \frac{\partial U}{\partial q_j} \delta q_j$$

we obtain

$$\left.\begin{aligned} \frac{\partial U}{\partial q_1} &= 0 \\ \frac{\partial U}{\partial q_2} &= k(q_2 - l) \end{aligned}\right\} \tag{2.120}$$

Virtual work δW, done by the force of gravity on virtual displacements $q_2\delta q_1$ and δq_2, is obtained by adding the products of these displacements by the corresponding components of mg. (see Fig. 2.4.)

$$\delta W = mg(\cos q_1)\,\delta q_2 - mg(\sin q_1)\,q_2\delta q_1$$

Comparing this expression with

$$\delta W = \sum_{j=1}^{2} Q_{A_j}\,\delta q_j$$

we write

$$\left.\begin{aligned} Q_{A_1} &= -mg(\sin q_1)\,q_2 \\ Q_{A_2} &= mg\cos q_1 \end{aligned}\right\} \tag{2.121}$$

Using Eqs. (2.119), (2.120), and (2.121) in Eq. (2.118), the equations of motion for the mass particle of Fig. 2.4 become

$$\left.\begin{aligned} mq_2^2\ddot{q}_1 + 2mq_2\dot{q}_1\dot{q}_2 &= -mg(\sin q_1)\,q_2 \\ m\ddot{q}_2 - mq_2\dot{q}_1^2 + k(q_2 - l) &= mg\cos q_1 \end{aligned}\right\}$$

Note that if we use Eqs. (2.117) to formulate the equations of motion we write

$$\delta V = \delta U + (-\delta W) = mg(\sin q_1)q_2\delta q_1 + \{k(q_2 - l) - mg\cos q_1\}\,\delta q_2$$

Comparing with

$$\delta V = \sum_{j=1}^{2} \frac{\partial V}{\partial q_j}\,\delta q_j$$

we obtain

$$\frac{\partial V}{\partial q_1} = mg(\sin q_1)q_2$$

$$\frac{\partial V}{\partial q_2} = k(q_2 - l) - mg\cos q_1$$

Using these results together with (2.119) in Eqs. (2.117) we obtain the equations of motion for the mass particle of Fig. 2.4.

2.10 Lagrange's Equations in Constrained Coordinates

We pointed out in Section 2.7 that Hamilton's principle

$$\int_{t_1}^{t_2} (\delta T + \delta W)\,dt = 0 \quad \text{[See Eq. (2.80).]}$$

is invariant with respect to coordinate transformation. Therefore, it holds true for constrained as well as generalized coordinates. Let us now consider a system with its motion described by coordinates $u_j(j = 1, 2, \ldots, m)$ that may be constrained, and develop Lagrange's equations in these coordinates.

If we follow the steps of Section 2.9, leading to Eq.(2.112), we write

$$\int_{t_1}^{t_2} \sum_j \left\{-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_j}\right) + \frac{\partial T}{\partial u_j} + F_j\right\} \delta u_j \, dt = 0 \qquad (2.122)$$

Equation (2.122) is identical with Eq. (2.112) except that q_j and Q_j are replaced by u_j and F_j, respectively. Since the u's may be constrained, the $\delta u_j(j = 1, 2, \ldots, m)$ may not all be assigned arbitrarily. Therefore, the expression in the brackets in Eq. (2.122) may not vanish separately for each j, and a set of m independent Lagrange equations may not exist.

We now apply a coordinate transformation

$$du_j = \sum_k \frac{\partial u_j}{\partial q_k} dq_k \quad \text{[See Eq. (2.11).]}$$

or in terms of virtual displacements

$$\delta u_j = \sum_{k=1}^{n} \frac{\partial u_j}{\partial q_k} \delta q_k \qquad (2.123)$$

in which the u's are expressed in terms of generalized coordinates $q_k(k = 1, 2, \ldots, n) \, n \leqq m$. Substituting Eq. (2.123) into (2.122) and multiplying by (-1)

$$\int_{t_1}^{t_2} \sum_j \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_j}\right) - \frac{\partial T}{\partial u_j} - F_j\right\} \sum_k \frac{\partial u_j}{\partial q_k} \delta q_k \, dt = 0$$

Interchanging the order of summation

$$\int_{t_1}^{t_2} \sum_k \delta q_k \left\{\sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_j}\right) - \frac{\partial T}{\partial u_j} - F_j\right] \frac{\partial u_j}{\partial q_k}\right\} dt = 0$$

Since the δq_k are arbitrary, it follows that

$$\sum_j \frac{\partial u_j}{\partial q_k} \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_j}\right) - \frac{\partial T}{\partial u_j} - F_j\right\} = 0 \quad \text{for } k = 1, 2, \ldots, n \quad (2.124)$$

There are n of these equations. These are Lagrange's equations written in terms of the u coordinate system.

For the special case in which the kinetic energy T in the u coordinate system is a function of the velocities $\dot{u}_j(j = 1, 2, \ldots, m)$ only, Eq. (2.124) can be simplified. It is convenient to begin with Eq. (2.122) and generate the first term

$$\int_{t_1}^{t_2} \sum_j \left\{ -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_j}\right)\right\} \delta u_j \, dt$$

for the special case under consideration.
Since

$$T = T(\dot{u}_1, \dot{u}_2, \ldots, \dot{u}_m)$$

or

$$T = \tfrac{1}{2} \sum_j m_j \dot{u}_j^2$$

it follows that

$$\delta T = \sum_j \frac{\partial T}{\partial \dot{u}_j} \delta \dot{u}_j$$

or

$$\delta T = \sum_j m_j \dot{u}_j \, \delta \dot{u}_j$$

Equating the two expressions for δT and integrating between t_1 and t_2, we write

$$\int_{t_1}^{t_2} \sum_j \frac{\partial T}{\partial \dot{u}_j} \delta \dot{u}_j \, dt = \int_{t_1}^{t_2} \sum_j m_j \dot{u}_j \delta \dot{u}_j \, dt \tag{2.125}$$

Remembering that

$$\frac{d}{dt}(\delta u_j) = \delta \dot{u}_j \qquad \text{[See Eq. (2.73)]}$$

we integrate each side of Eq. (2.125) by parts

$$\begin{aligned} &\sum_j \frac{\partial T}{\partial \dot{u}_j} \delta u_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_j}\right) \delta u_j \, dt \\ &= \sum_j m_j \dot{u}_j \, \delta u_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_j m_j \ddot{u}_j \, \delta u_j \, dt \end{aligned} \tag{2.126}$$

From Section 2.7 it is recalled that on the varied path

$$\delta u_j(t_1) = \delta u_j(t_2) = 0 \qquad j = 1, 2, \ldots, m \quad \text{[See Eq. (2.78).]}$$

hence

$$\sum_j \frac{\partial T}{\partial \dot{u}_j} \delta u_j \Big|_{t_1}^{t_2} = \sum_j m_j \dot{u}_j \, \delta u_j \Big|_{t_1}^{t_2} = 0$$

and Eq. (2.126) reduces to

$$\int_{t_1}^{t_2} \sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_j}\right) \delta u_j \, dt = \int_{t_1}^{t_2} \sum_j m_j \ddot{u}_j \delta u_j \, dt \tag{2.127}$$

Since T is a function of the velocities $\dot{u}_j$ only, we also have

$$\frac{\partial T}{\partial u_j} = 0, \qquad j = 1, 2, \ldots, m$$

Substituting this and relation (2.127) into Eq. (2.122) we obtain

$$\int_{t_1}^{t_2} \sum_j (-m_j \ddot{u}_j + F_j)\, \delta u_j\, dt = 0 \tag{2.128}$$

We now apply a coordinate transformation

$$\delta u_j = \sum_{k=1}^{n} \frac{\partial u_j}{\partial q_k} \delta q_k \quad \text{[See Eq. (2.123)]}$$

and rewrite Eq. (2.128) in the form

$$\int_{t_1}^{t_2} \sum_j (-m_j \ddot{u}_j + F_j) \sum_k \frac{\partial u_j}{\partial q_k} \delta q_k\, dt = 0.$$

Interchanging the order of summation and multiplying by (-1)

$$\int_{t_1}^{t_2} \sum_k \delta q_k \left[\sum_j (m_j \ddot{u}_j - F_j) \frac{\partial u_j}{\partial q_k} \right] dt = 0$$

Since the δq_k are arbitrary it follows that

$$\sum_j \frac{\partial u_j}{\partial q_k} (m_j \ddot{u}_j - F_j) = 0 \quad \text{for } k = 1, 2, \ldots, n \tag{2.129}$$

For a linear transformation

$$\{u\} = [C]\{q\}$$

in which

$$C_{jk} = \frac{\partial u_j}{\partial q_k} \qquad \text{[See Eq. (2.13)]}$$

Equations (2.129) become

$$\sum_j C_{jk} (m_j \ddot{u}_j - F_j) = 0 \qquad \text{for } k = 1, 2, \ldots, n$$

or in matrix form

$$[C]^T[m]\{\ddot{u}\} - [C]^T\{F\} = \{0\} \tag{2.130}$$

If, as was done in the last section, we separate all forces $\{F\}$ acting on the system into

applied forces	$\{F\}_A$
internal elastic forces	$\{F\}_E$
damping forces	$\{F\}_D$

we write

$$\{F\} = \{F\}_A + \{F\}_E + \{F\}_D$$

and Eq. (2.130) becomes

$$[C]^T[m]\{\ddot{u}\} - [C]^T\{F\}_E - [C]^T\{F\}_D = [C]^T\{F\}_A \tag{2.131}$$

From Chapter 1

$$\{F\}_E = -[k]\{u\}$$

where the minus sign follows from the fact that the F_{E_j} as used here are the reactive elastic forces applied to the mass. (See Fig. 2.10.)

Using this expression for $\{F\}_E$ and the relation

$$\{\ddot{u}\} = [C]\{\ddot{q}\}$$

Equation (2.131) becomes

$$[C]^T[m][C]\{\ddot{q}\} + [C]^T[k][C]\{q\} - [C]^T\{F\}_D = [C]^T\{F\}_A$$

or

$$[m_{kl}]\{\ddot{q}\} + [k_{kl}]\{q\} - \{Q\}_D = \{Q\}_A \qquad (2.132)$$

in which

$[m_{kl}]$ = the generalized mass matrix (See Section 2.4.)

$[k_{kl}]$ = the generalized stiffness matrix

$\{Q\}_D = [C]^T\{F\}_D$ are the generalized damping forces

$\{Q\}_A = [C]^T\{F\}_A$ are the generalized applied forces

Equations (2.132) are the equations of motion in generalized coordinates q. We emphasize that to formulate these equations we must use the generalized mass as discussed in Sections 2.4 and 2.5, and the generalized stiffnesses and generalized forces discussed in Chapter 1. all of which are referred now to the q coordinate system. Equations (2.132) can also be generated by applying the coordinate transformation to energies T and U and to forces $\{F\}_D$, $\{F\}_A$, and writing Lagrange's equations directly in terms of generalized coordinates. We also call attention to the fact that matrix $[C]$ in Eqs. (2.132) may be a square matrix for the special case when the u's as well as the q's are generalized coordinates.

Illustrative Examples:

As a first example of the application of Eqs. (2.132), we consider the two-degree-of-freedom conservative system illustrated in Fig. 2.11.

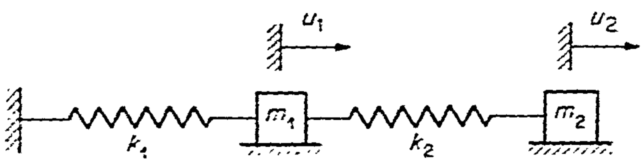


Figure 2.11

The two rigid bodies of masses m_1 and m_2 are constrained by frictionless guides to move in translation in the horizontal direction. Motion is in the plane of the paper. The connecting springs are elastic and have stiffnesses k_1 and k_2 as indicated. One coordinate is required to measure the displacement of each body and the reference data are such that the coordinates are zero when the springs are unstrained. Since no external or damping forces act on the system, the equations of motion in terms of coordinates u are given by

$$[m]\{\ddot{u}\} + [k]\{u\} = \{0\}$$

where

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

and

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Let us suppose that our problem requires that the equations of motion be written in a new set of coordinates q_1, q_2 given the linear transformation

$$\{u\} = [C]\{q\}, \qquad [C] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

To formulate these equations we must evaluate the generalized mass and stiffness matrices. Using the relations

$$[m_{kl}] = [C]^T[m][C]$$

and

$$[k_{kl}] = [C]^T[k][C]$$

we obtain

$$[m_{kl}] = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}$$

$$[k_{kl}] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

and the equations of motion in coordinates q become

$$\begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Figure 2.12

These equations may be obtained by applying the transformation to the energies T and U and writing the Lagrange equations directly in terms of the generalized coordinates q. We leave this as an exercise to the reader.

As a second example we consider the system of Fig. 2.12. The frame is identical with that shown in Fig. 1.7 of Chapter 1. The axial deflections are considered to be zero, hence, the coordinates u are constrained. The equations of constraint are the same as those given by Eqs. (1.8) in Section 1.3 of Chapter 1. Selecting again as generalized coordinates

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_5 \\ u_6 \end{Bmatrix}$$

we write from Eqs. (1.14) in Chapter 1

$$\{u\} = [C]\{q\}$$

where

$$[C] = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 \\ \frac{3}{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

The mass matrix for coordinates u is given by

$$[m] = \begin{bmatrix} 2m & & & & & \\ & m & & & & \\ & & 2m & & & \\ & & & m & & \\ & & & & 8J & \\ & & & & & 3J \end{bmatrix}$$

The system is subjected to external forces $F_j (j = 1, 2, \ldots, 6)$ acting along coordinates $u_j (j = 1, 2, \ldots, 6)$.

Our problem requires that we formulate the equations of motion of the system in terms of coordinates q. Since no damping forces are acting on the system we set $\{F\}_{D} = \{0\}$ in Eq. (2.132). The generalized mass matrix and the generalized applied force vector can be computed from the relations

$$[m_{kl}] = [C]^T [m][C]$$

and

$$\{Q\}_u = [C]^T\{F\}_u$$

However, the generalized stiffness matrix $[k_{kl}]$ cannot be computed from

$$[k_{kl}] = [C]^T[k][C]$$

since $[k]$ does not exist in the u coordinate system. Namely, since the u's are constrained it is not possible to generate, for instance, the first column of $[k]$ by setting $u_1 = 1$ and $u_j = 0$ for $j = 2, 3, \ldots, 6$ because $u_1 = u_2$. To get around this difficulty we recall that in the q coordinate system

$$[a]_q^{-1} = [k]_q$$

where $[k]_q = [k_{kl}]$ is the desired generalized stiffness matrix. Hence, we follow the following steps. First, we derive $[a]_u$. Then, we apply Eq. (1.99) to get $[a]_q$ from which $[k_{kl}]$ is obtained by inversion.*

2.11 Coupling

In Section 1.8, Chapter 1, we mentioned a linear transformation from a set of generalized coordinates q to *normal* coordinates η. The transformation was written in the form

$$\{q\} = [\gamma]\{\eta\} \tag{2.133}$$

where

$[\gamma]$ = a square transformation matrix

$\{\eta\}$ = a displacement vector expressed in normal coordinates

The equations of motion in the q coordinate system

$$[m]_q\{\ddot{q}\} + [k]_q\{q\} - \{Q\}_D = \{Q\}_I \quad \text{[See Eq. (2.132).]}$$

can be transformed to corresponding equations of motion in the η coordinate system by substituting from Eq. (2.133) into Eq. (2.132) and premultiplying by $[\gamma]^T$. The resulting equations have the form

$$[m]_\eta\{\ddot{\eta}\} + [k]_\eta\{\eta\} - [\gamma]^T\{Q\}_D = [\gamma]^T\{Q\}_I \tag{2.134}$$

in which

$$[m]_\eta = [\gamma]^T[m]_q[\gamma]$$
$$[k]_\eta = [\gamma]^T[k]_q[\gamma]$$

are diagonal matrices. Hence, transformation (2.133) caused the equations of motion to become decoupled; namely only one unknown, $\ddot{\eta}_j$ and η_j, appears in any one of Eqs. (2.134). This property of the

*$[a]_u$ and $[a]_q$ for this problem are given in Problem 1.8 Chapter 1.

transformation is stated here without proof but will be discussed in Chapter 3.

When $[m]_\eta$ in Eq. (2.132) is not a diagonal matrix, the system is said to be dynamically coupled. Similarly, when $[k]_\eta$ in Eq. (2.132) is not a diagonal matrix, the system is said to be statically coupled. It is possible to obtain a set of coordinates in which the system is statically coupled, dynamically coupled, or both, while in normal coordinates η the system is neither statically nor dynamically coupled. In any event the nature of coupling for any system depends upon the coordinate system. Hence, coupling *is not* an inherent property of the system.

PROBLEMS

1. Given a structure with its stiffness matrix in the generalized $q_j (j=1, 2, \ldots, n)$ coordinate system partitioned as

$$\begin{array}{cc} & \begin{array}{cc} r & s \end{array} \\ \begin{array}{c} r \\ s \end{array} & \left[\begin{array}{c|c} [k]_{11} & [k]_{12} \\ \hline [k]_{21} & [k]_{22} \end{array}\right] \end{array} \qquad r + s = n$$

and a corresponding flexibility matrix

$$\begin{array}{cc} & \begin{array}{cc} r & s \end{array} \\ \begin{array}{c} r \\ s \end{array} & \left[\begin{array}{c|c} [a]_{11} & [a]_{12} \\ \hline [a]_{21} & [a]_{22} \end{array}\right] \end{array} \qquad r + s = n$$

Show that

$$[k]^* [a]_{11} = [I]$$

where

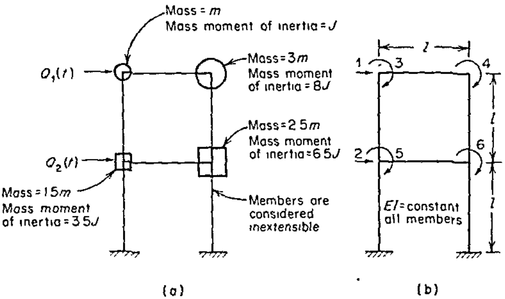
$$[k]^* = [k]_{11} - [k]_{12} [k]_{22}^{-1} [k]_{21}$$

2. Consider the frame of Fig. 2.2(a) to be acted upon by time varying forces

$$\begin{Bmatrix} Q_1(t) \\ Q_2(t) \\ Q_3(t) \\ Q_4(t) \\ Q_5(t) \\ Q_6(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assuming masses m_1 and m_2 to be restricted to lateral motion described by u_1, u_2, respectively, formulate the equations of motion of the system. Compare with Eq. (2.5) and discuss.

3. Write the mass and stiffness matrices for the system shown. Formulate the equations of motion. $Q_1(t)$ and $Q_2(t)$ are the only forces acting on the system.



Problem 3 (a) Masses, mass moments of inertia, and acting forces. (b) Coordinates.

4. Show that the generalized mass matrix is always a symmetric matrix

$$[m_{kl}]^T = [m_{kl}]$$

5. The coordinates $u_j (j = 1, 2, \ldots, 6)$ in Problem 3 are transformed by

$$\{u\} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \{q\}$$

Derive the generalized stiffness and mass matrices in the q coordinate system.

6. Derive the generalized mass matrix in the distributed q coordinate system for the system of Problem 1, Chapter 1. Functions ϕ_j $(j = 1, 2, 3)$ in the figure are associated with coordinates $q_j (j = 1, 2, 3)$.

7. The beam in the figure is inextensible. It is restrained at its ends by two elastic springs. We express the displacement of the beam by

$$u(x) = \phi_1(x) q_1$$

$$w(x) = \phi_2(x)q_2 + \phi_3(x)q_3 + \phi_4(x)q_4$$
$$= \sum_{i=2}^{4} \phi_i(x)q_i$$

where

$$\phi_1(x) = 1$$
$$\phi_2(x) = \left(\frac{x}{l}\right)$$
$$\phi_3(x) = \left(\frac{x}{l}\right)^2$$
$$\phi_4(x) = \left(\frac{x}{l}\right)^3$$

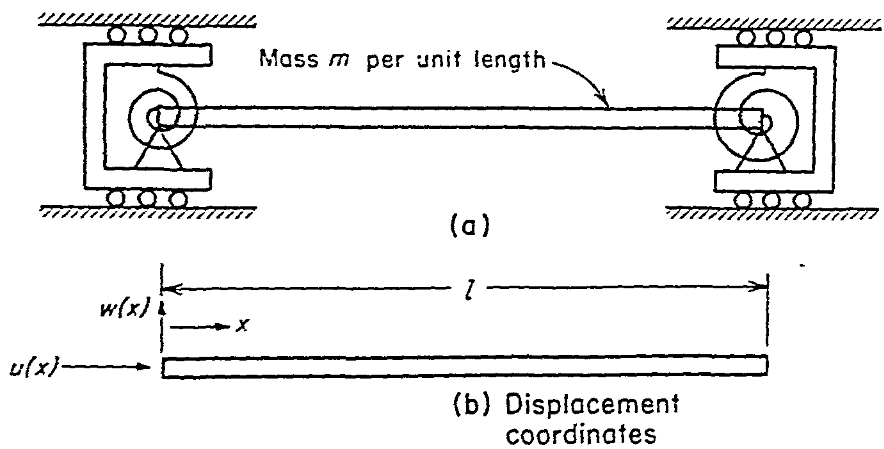
The kinetic energy of the beam in motion is given by

$$T = \tfrac{1}{2}\int_0^l m(\dot{w}^2 + \dot{u}^2)\,dx$$

Starting with this expression, derive the generalized mass matrix of the beam in the q coordinate system. In the resulting matrix

$$m_{j1} = m_{1j} = 0 \qquad \text{for } j = 2, 3, 4$$

Discuss why this is to be expected in the present problem.



Problem 7

8. Derive the generalized mass matrix in the q coordinate system for the frame of Fig. 1.8, Chapter 1.
The constrained distributed coordinates $p_j(j = 1, 2, \ldots, 8)$ and corresponding functions $\phi_j(j = 1, 2, \ldots, 8)$ are defined by Eqs. (1.15) and (1.16), respectively. The coordinate transformation to generalized distributed coordinates $q_j(j = 1, 2, 3)$ is given by Eq. (1.20) where

$$\begin{Bmatrix} p_2 \\ p_6 \\ p_8 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Note that in the light of Problem 7

$$m_{\alpha 6} = m_{6\alpha} = 0 \quad \text{for all } \alpha = 1, 2, \ldots, 5, 7, 8 \text{ (except } \alpha = 6)$$

in $[m_{\alpha\beta}]$.

9. In Eq. (2.73) it was shown that the operators δ and d/dt are commutative, namely

$$\delta \dot{u}_j = \delta\left(\frac{d}{dt} u_j\right) = \frac{d}{dt}(\delta u_j)$$

Similarly for

$$w = w(x, t)$$

$$w' = \frac{\partial}{\partial x} w(x, t)$$

and

$$\delta w' = \delta\left(\frac{\partial}{\partial x} w\right) = \frac{\partial}{\partial x} \delta w$$

or the operators δ and $\partial/\partial x$ are commutative. Using these relations, generate the identity expressed by Eq. (2.88).

10. For normal mode vibration (discussed in Chapter 3) the linear and angular accelerations $\ddot{w}$ and $\ddot{\psi}$ for the beam considered in Section 2.8 can be written as

$$\ddot{w} = -\omega^2 w$$

and

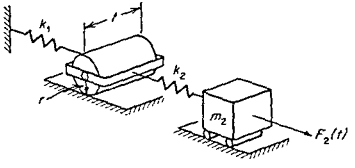
$$\ddot{\psi} = -\omega^2 \psi$$

where ω represents the frequency of vibration.
Using these relations reduce Eqs. (2.106) to a fourth-order differential equation in w only. Show that this equation reduces to Eq. (2.94) (with $\ddot{w}$ replaced by $\omega^2 w$) when the shear stiffness increases without limit and the radius of gyration goes to zero, namely

$$GA \to \infty$$

$$\rho \to 0$$

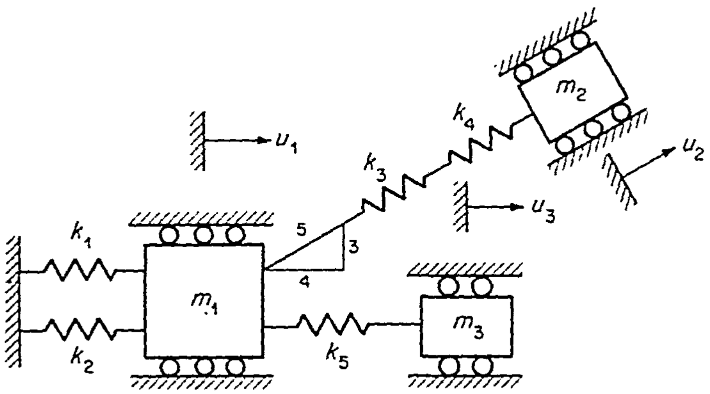
11. Using Lagrange's equations, formulate the equations of motion for the mass spring system shown. The uniform disk has a mass m per unit



Problem 11

volume and rolls without slip in the plane of the paper. Mass m_2 translates without friction in the horizontal direction and is acted upon by force $F_2(t)$.

12. Write the stiffness matrix for the system shown using the indicated coordinates. Using this stiffness matrix formulate the equations of motion. Check your results by regenerating them through the use of Lagrange's equations.

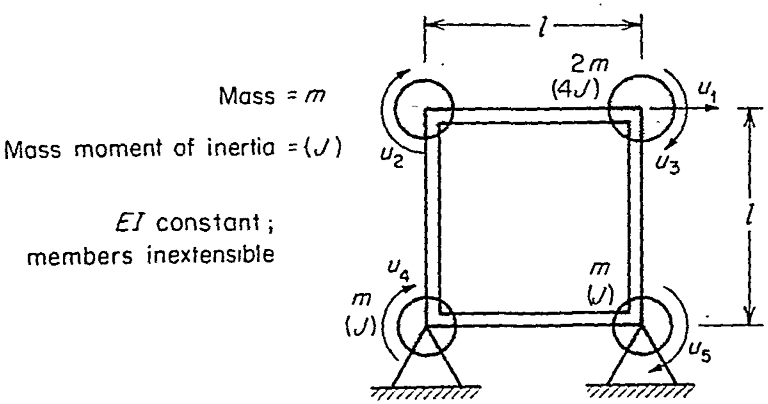


Problem 12

13. In Problem 12 write the equations of motion in the q coordinate system given

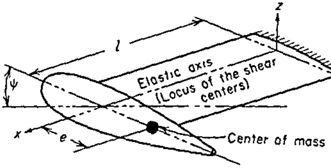
$$\{u\} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \{q\}$$

14. Derive the stiffness matrix and formulate the equations of motion for the system with masses lumped at the four corners of the frame.



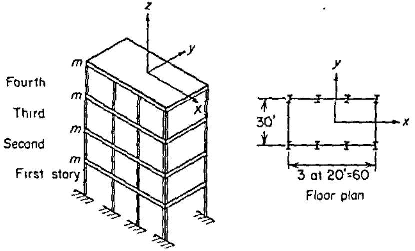
Problem 14

15. Consider a *uniform* airplane wing of semispan l to be cantilevered, as shown. The wing carries a tip-tank of mass m and mass radius of gyration ρ taken relative to a lateral axis through its center of mass. The center of mass of the tank lies behind the elastic axis of the wing at a distance e. The stiffness moduli of the wing are constant and are EI and GJ in flexure and torsion, respectively. Considering only vertical bending and twisting of the wing and neglecting its mass, formulate the equations of motion using Lagrange's equations.



Problem 15

16. The four story building shown has two frames with four columns in each in the $x - z$ plane and four frames of two columns each in the $y - z$ plane. The columns are inextensible and restrained against rotation



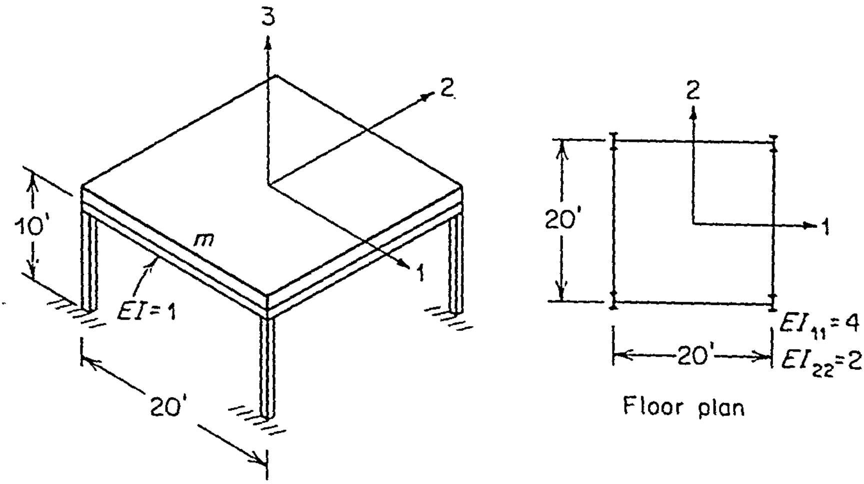
Problem 16

at their ends. All columns are identical within any given story with flexural rigidities given by

Typical Column at Story Number	EI_{yy}	EI_{xx}
1	1	2
2	1.5	3.5
3	2	5
4	2.5	6.5

The mass of the building is lumped at the floor levels and is evenly distributed over the floor area. Each floor mass m has three degrees of freedom of motion, two translations along x and y, and a rotation about z. Derive the stiffness matrix of the building and formulate the equations of motion (neglect the torsional resistance of the individual columns).

17. The one-story space frame shown has four identical columns with flexural rigidities $EI_{11} = 4$, $EI_{22} = 2$. The flexural rigidity of each of the four girders is $EI = 1$. All members are inextensible, however the frame joints are free to rotate maintaining the 90° angle between columns and girders. The rigid mass m is evenly distributed over the roof area and has three degrees of freedom of motion, two translations u_1, u_2 and a rotation about axis 3. Derive the stiffness matrix for the space frame (neglect the torsional rigidity of the individual columns) and formulate the equations of motion.



Problem 17

18. Show that the equations of motion are dynamically uncoupled when

$$T = \frac{1}{2} \sum_j m_j \dot{q}_j^2$$

and statically uncoupled when

$$U = \frac{1}{2} \sum_j k_{jj} q_j^2$$

CHAPTER 3

Natural Modes of Vibration

3.1 Definitions

An ideal elastic structure is one in which no internal damping forces exist. Such a structure may vibrate for an indefinitely long period of time without the application of external forces other than those required to initiate the motion. This is possible only in theory because no such ideal structure actually exists. However, many structures and structural materials in practical use have such a small amount of internal damping that free vibrations may persist for relatively long periods of time with only gradual diminution in amplitude. The character of the free vibrations of an ideal structure depends upon its mass distribution, its load-deflection properties, and the manner in which motion is initiated.

If the initial conditions are properly imposed it is possible to cause vibration in any one of several *natural modes** which are characteristic of the structure. In a natural mode each point in the structure executes harmonic motion about a position of static equilibrium, every point passing through its equilibrium position at the same instant and reaching its extremum at the same instant. Obviously,

*Also called *normal modes* or *principal modes*.

then, the frequency of the oscillation is the same at every point and this is the natural frequency of the structure in the particular mode involved. If one considers the structure at an instant when all the points in it reach an extremum and, hence, are momentarily stationary, he visualizes the structure in a particular deformation configuration that is a peculiar property of a natural mode. It is convenient, then, to think of a natural mode as a deflected configuration in which the motion of each point is harmonic and in which the vibration has a specific natural frequency associated with that mode.

An elastic structure may have many natural modes. In fact, a structure having distributed properties has an infinite number of them in theory. In general, each mode is distinct from all others and its frequency is also distinct. It is possible for several modes to have the same frequency, however, this condition is unusual in structures. Not so unusual is the case in which several mode frequencies are very nearly the same; so close together, in fact, that very great computational or experimental accuracy is required to distinguish them. It will be shown in this chapter that the number of natural modes which may be determined for an ideal structure is equal to its number of degrees of freedom.

A knowledge of natural modes and frequencies is basic to an understanding of the dynamic response of a structure under any kind of excitation. In many problems the formulation and solution of equations of motion is greatly facilitated by using natural mode information. In this chapter we shall consider the equations of motion for free vibration and their solution which yields the modes and frequencies. In subsequent chapters, we shall consider various principles and methods by which the modes may be determined.

3.2 Equations of Motion for Free Vibration

Let us consider a structural system whose displacements are given by generalized coordinates q_i $(i = 1, 2, \ldots, n)$. The force-deflection relationship in terms of flexibility and stiffness matrices are given by

$$\{q\} = [a]\{Q\} \tag{3.1}$$

$$\{Q\} = [k]\{q\} \tag{3.2}$$

(See Chapter 1, Eqs. 1.42 and 1.43.)
In free vibration the only forces acting on the mass of the system are those imposed by the internal elastic forces $\{Q\}$ as expressed by Eq. (3.2): According to the principle of D'Alembert, these forces are equal and opposite to the inertial forces

$$[m]\{\ddot{q}\}$$

Hence

$$[k]\{q\} = -[m]\{\ddot{q}\} \tag{3.3}$$

This also follows directly from Eq. (2.132), Chapter 2, when the generalized damping forces $\{Q\}_D$ and the generalized applied forces $\{Q\}_A$ are set equal to zero. We emphasize again that $[a]$, $[k]$, and $[m]$ are, respectively, the matrices of generalized flexibility, stiffness, and mass in the q coordinate system. Equations (3.3) form a set of linear second-order differential equations whose solution is given by

$$\ddot{q}_i = -\omega^2 q_i \qquad i = 1, 2, \ldots, n$$

or

$$\{\ddot{q}\} = -\omega^2 \{q\} \tag{3.4}$$

where ω is the angular natural frequency of vibration of the structural system in any one of its natural modes. Substituting from Eq. (3.4) in (3.3)

$$\boxed{[k]\{q\} = \omega^2[m]\{q\}} \tag{3.5}$$

Premultiplying each side by $[a]$ and recalling that

$$[a][k] = [I] \qquad \text{(identity matrix)}$$

we obtain

$$\{q\} = \omega^2[a][m]\{q\} \tag{3.6}$$

The matrix $[a][m]$ is called a *dynamical matrix*, designated as $[D]$. Using this matrix and dividing by ω^2 Eq. (3.6) becomes

$$[D]\{q\} = \frac{1}{\omega^2}\{q\} \tag{3.7}$$

An alternate form of this equation is

$$\left([D] - \frac{1}{\omega^2}[I]\right)\{q\} = \{0\} \tag{3.8}$$

Equations (3.8) form a set of n homogeneous algebraic equations in q with $1/\omega^2$ unknown. The problem presented here is called an *eigenvalue* or *characteristic value* problem in which the quantities $1/\omega^2$ are called eigenvalues or characteristic values. The relationships among the amplitudes q are called *eigenfunctions* or characteristic functions. The vector $\{q\}$ corresponding to a particular natural mode of vibration is called an *eigenvector*, characteristic vector, or *modal column*.

If we premultiply each side of Eq. (3.5) by $[m]^{-1}$ we obtain

$$[m]^{-1}[k]\{q\} = \omega^2\{q\} \tag{3.9}$$

Since

$$[D] = [a][m] \qquad \text{and} \qquad [a]^{-1} = [k]$$

it follows that

$$\begin{aligned}[D]^{-1} &= [m]^{-1}[a]^{-1} \\ &= [m]^{-1}[k]\end{aligned}$$

and Eq. (3.9) takes the form

$$[D]^{-1}\{q\} = \omega^2\{q\} \tag{3.10}$$

This form of the eigenvalue problem can be obtained directly from Eq. (3.7) by premultiplying by $\omega^2[D]^{-1}$. The two forms (3.7) and (3.10) thus stand in inverse relationship.

We note that $[D]$ will exist only when $[a]$ exists, or what amounts to the same thing when $[k]$ is a nonsingular matrix. $[k]$ will be singular when the system has rigid body degrees of freedom as discussed in Chapter 1, Section 9.* This subject will be discussed in detail in Chapter 4, Section 10.

3.3 Solution of the Eigenvalue Problem

Let us consider the set of n equations represented in matrix form by Eq. (3.8). If, for convenience, we designate $1/\omega^2$ as λ, these equations appear as follows (ijth element of the dynamical matrix is designated as D_{ij})

$$\left.\begin{array}{l}(D_{11} - \lambda)q_1 + D_{12}q_2 + D_{13}q_3 + \ldots + D_{1n}q_n = 0 \\ D_{21}q_1 + (D_{22} - \lambda)q_2 + D_{23}q_3 + \ldots + D_{2n}q_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ D_{n1}q_1 + D_{n2}q_2 + D_{n3}q_3 + \ldots + (D_{nn} - \lambda)q_n = 0\end{array}\right\} \tag{3.11}$$

A solution of this homogeneous set of equations for the q's by any method, such as Cramer's rule, leads to zero values if the determinant Δ of the coefficients of the q's in Eq. (3.11) is different from zero. A nontrivial solution can exist only if

$$\Delta = 0 \tag{3.12}$$

The determinant Δ is called the *characteristic determinant* and is expressed in terms of the unknown λ's. If we expand Δ we will obtain a polynomial in λ of degree n. Thus, Eq. (3.12) may be expressed in the form[9]

*See also Problem 3, Chapter 3.

$$\lambda^n + C_1\lambda^{n-1} + C_2\lambda^{n-2} + \ldots + C_{n-1}\lambda + C_n = 0 \tag{3.13}$$

This is called the *characteristic equation* or *frequency equation*, and may also be written in the form

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)\ldots(\lambda - \lambda_n) = 0 \tag{3.14}$$

The n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real[4] and, except in unusual cases, distinct. These roots are related to the natural frequencies by

$$\omega_i^2 = \frac{1}{\lambda_i} \tag{3.15}$$

Notice that the largest λ root yields the smallest frequency, called the *fundamental frequency* or *first-mode frequency*. Hence, we designate it as ω_1, and the remaining frequencies ω_i in ascending mode numbers. Hence, larger subscripts designate greater frequencies of higher modes.

The solution proceeds by finding the n roots in λ by the use of any one of a number of methods not discussed here. For the rth mode the set of Eqs. (3.11) may be written as follows by inserting the known value λ_r

$$\left\{\begin{array}{l}
(D_{11} - \lambda_r)q_1 + D_{12}q_2 + \ldots + D_{1k}q_k + \ldots + D_{1n}q_n = 0 \\
D_{21}q_1 + (D_{22} - \lambda_r)q_2 + \ldots + D_{2k}q_k + \ldots + D_{2n}q_n = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
D_{s1}q_1 + D_{s2}q_2 + \ldots + D_{sk}q_k + \ldots + (D_{ss} - \lambda_r)q_s + \ldots + D_{sn}q_n = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
D_{n1}q_1 + D_{n2}q_2 + \ldots + D_{nk}q_k + \ldots + (D_{nn} - \lambda_r)q_n = 0
\end{array}\right\} \tag{3.16}$$

It can be seen that n sets of equations of this form may be obtained by insertion of the n values of λ. We find the n mode configurations by determining the q's which satisfy these sets. There are, then, n sets of q's, each set containing n values. Because the sets of Eqs. (3.16) are homogeneous, unique solutions in the q's do not exist. We can obtain only ratios among the q's. In other words, there exist for each mode unique solutions for $n - 1$ of the q's if we assign an arbitrary value to one of them.

The physical interpretation of this mathematical result is that the deflected configuration of a structure which describes a natural mode is defined by known ratios among the amplitudes of motion at the various points. The actual amplitude of motion of the system as a whole is arbitrary and not a property of a natural mode. If we consider one of the q's in Eq. (3.16), say q_k, to be known, we may find the remaining unknowns by solving any set of $n-1$ equations. Suppose we eliminate the sth equation and write the set as

$$\left\{\begin{array}{lllll}
(D_{11}-\lambda_r)q_1 + & D_{12}q_2 + \ldots + & D_{1n}q_n = & & -D_{1k}q_k \\
D_{21}q_1 + & (D_{22}-\lambda_r)q_2 + \ldots + & D_{2n}q_n = & & -D_{2k}q_k \\
\cdots & \cdots & \cdots & & \cdots \\
D_{k1}q_1 + & D_{k2}q_2 + \ldots + & D_{kn}q_n = & -(D_{kk}-\lambda_r)\,q_k \\
\cdots & \cdots & \cdots & & \cdots \\
D_{n1}q_1 + & D_{n2}q_2 + \ldots + & (D_{nn}-\lambda_r)\,q_n = & & -D_{nk}q_k
\end{array}\right\} \tag{3.17}$$

In the preceding equations the terms D_{sk} are not included since we have rejected the sth equation. We may now obtain a solution for any one of the q's, say q_p using Cramer's rule.

$$q_p = \frac{\begin{array}{c} \hspace{6em} p\text{th} \downarrow \text{column} \\ \begin{vmatrix}
(D_{11}-\lambda_r) & D_{12} \ldots\ldots & -D_{1k} \ldots\ldots & D_{1n} \\
D_{21} & (D_{22}-\lambda_r) \ldots & -D_{2k} \ldots\ldots & D_{2n} \\
 & s\text{th row missing} \ldots & \ldots\ldots & \\
D_{n1} & D_{n2} \ldots\ldots & -D_{nk} \ldots\ldots & (D_{nn}-\lambda_r)
\end{vmatrix}\end{array}}{\begin{vmatrix}
(D_{11}-\lambda_r) & D_{12} & \vdots & \ldots D_{1p} \ldots\ldots & D_{1n} \\
D_{21} & (D_{22}-\lambda_r) & k\text{th column missing} & \ldots D_{2p} \ldots\ldots & D_{2n} \\
 & \ldots\ldots & \vdots & s\text{th row missing} \ldots & \\
D_{n1} & D_{n2} \ldots & \vdots & \ldots D_{np} \ldots\ldots & (D_{nn}-\lambda_r)
\end{vmatrix}}\, q_k \tag{3.18}$$

The determinant in the denominator is the same as the characteristic determinant for $\lambda = \lambda_r$ and with the sth row and kth column missing. Thus, it is precisely the minor of the (s, k) element of Δ_r, where we designate by Δ_r the determinant of the coefficients of the q's in Eq. (3.16) with $\lambda = \lambda_r$. The determinant in the numerator is obtained by crossing out row s and column p, and thus is derived from the minor of the (s, p) element of Δ_r. However, the determinant thus obtained has the kth column out of position since it occupies the position of the missing pth column. The required change in column position can be accomplished by a succession of interchanges of adjacent columns, the process being continued until the kth column stands adjacent to the pth one. The order of the other columns will thus be undisturbed. Since each column change alters the sign of the determinant, the sign change accompanying the required interchanges will be given by

$$(-1)^{[p-(k+1)]} = (-1)^{p+k+1}$$

This equality can be verified by trying the four possible combinations of (p, k): (odd, odd), (odd, even), (even, odd), (even, even). The sign must be changed once again because the elements of the kth column have their signs changed in the determinant. Hence the total sign change is given by $(-1)^{p+k}$. If we designate as $\tilde{M}_{ij}$ the minors of the determinant Δ, the desired amplitude q_p is given by

$$q_p^{(r)} = (-1)^{p+k} \left(\frac{\tilde{M}_{sp}}{\tilde{M}_{sk}}\right)^{(r)} q_k^{(r)} \tag{3.19}$$

The superscript (r) denotes the rth normal mode, obtained explicitly by substituting $\lambda = \lambda_r$ in Δ, and, hence, in the minors $\tilde{M}_{sp}$ and $\tilde{M}_{sk}$.

A cofactor, C_{ij}, of a determinant is defined as a signed minor given by

$$C_{ij} = (-1)^{i+j} \tilde{M}_{ij} \tag{3.20}$$

Using cofactors in place of minors the amplitudes may be found by

$$q_p^{(r)} = \left(\frac{C_{sp}}{C_{sk}}\right)^{(r)} q_k^{(r)} \tag{3.21}$$

The modal column $\{q^{(r)}\}$ may be found in the following way. From the foregoing equation we see that

$$\{q^{(r)}\} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \\ \vdots \\ q_n \end{Bmatrix}^{(r)} = \left(\frac{q_k}{C_{sk}}\right)^{(r)} \begin{Bmatrix} C_{s1} \\ C_{s2} \\ \vdots \\ C_{sk} \\ \vdots \\ C_{sn} \end{Bmatrix}^{(r)} \tag{3.22}$$

The column matrix on the right side of the above equation is precisely the sth column of the adjoint matrix of the matrix

$$([D] - \lambda_r [I])$$

Since s is arbitrary, it follows that the columns of the adjoint matrix must be proportional for each value of λ, i.e., for each mode.

The foregoing relationship can be deduced somewhat more concisely by the following argument. First, for simplicity, let us define a matrix $[B]$ by

$$[B^{(r)}] = [D] - \lambda_r [I] \tag{3.23}$$

Then, from matrix algebra*

$$[B^{(r)}][B^{(r)}]^{-1} = [I]$$

where

$$[B^{(r)}]^{-1} = \frac{[C_{ji}^{(r)}]}{\Delta_r}$$

$[C_{ji}^{(r)}]$ is the adjoint of matrix $[B^{(r)}]$ which is the matrix of the cofactors of $[B^{(r)}]$ transposed.[8] Δ_r is the determinant of $[B^{(r)}]$, and, hence, we may write

$$[B^{(r)}][C_{ji}^{(r)}] = \Delta_r[I] \tag{3.24}$$

But from Eq. (3.12) $\Delta_r = 0$ if λ_r is a characteristic root. In this case the right side of Eq. (3.24) is zero. Considering then the sth column, $\{C_{si}^{(r)}\}$, of $[C_{ji}^{(r)}]$ and noting that the indices i, j, s refer to positions in the original matrix $[B]$, we may write

$$[B^{(r)}]\{C_{si}^{(r)}\} = \{0\} \tag{3.25}$$

This is a set of homogeneous equations in the C_{si} which determines each one to within an arbitrary constant. In other words, if we arbitrarily assign a value to one of the C_{si} the remaining $n-1$ elements of the column $\{C_{si}^{(r)}\}$ are determined uniquely. Now, from Eqs. (3.16), which in matrix form appear as

$$([D] - \lambda_r[I])\{q^{(r)}\} = \{0\} \tag{3.26}$$

we obtain the following form when we substitute Eq. (3.23).

$$[B^{(r)}]\{q^{(r)}\} = \{0\} \tag{3.27}$$

We therefore have a second set of homogeneous equations in the q's which determines each one to within an arbitrary multiplicative constant. Thus, we deduce from Eqs. (3.25) and (3.27) that the columns

$$\{q^{(r)}\} \quad \text{and} \quad \{C_{si}^{(r)}\}$$

are proportional, i.e.,

$$\{q^{(r)}\} = \alpha_r\{C_{si}^{(r)}\} \tag{3.28}$$

where α_r is a constant of proportionality. This is a restatement of Eq. (3.22).

Example: Torsional Vibration of a Shaft with Three Disks.

Consider a uniform weightless shaft of length $3l$ with three disks located at equal intervals along its length, as shown in Fig. 3.1. The shaft is fixed at the right end.

Let I_1, I_2, I_3 be the mass moments of inertia of the three disks and

*At this point we select λ_r different from any of the characteristic roots so that $\Delta_r \neq 0$ and the inverse $[B^{(r)}]^{-1}$ exists.

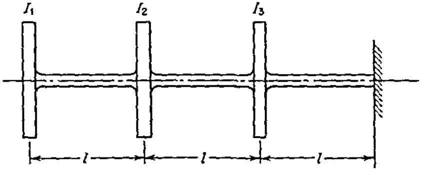


Figure 3.1

let $\theta_1, \theta_2, \theta_3$ be their angular displacements from the position of static equilibrium. The torsional stiffness of the shaft is given by the modulus GJ. The torque-displacement equation is

$$\frac{d\theta}{dx} = \frac{M}{GJ}$$

where M designates torque.

The flexibility matrix $[a]$ is formed by considering unit torques ($M = 1$) to be applied at the points 1, 2, 3 in succession. The elements of the matrix are

$$a_{11} = \frac{3l}{GJ} \quad a_{12} = \frac{2l}{GJ} \quad a_{13} = \frac{l}{GJ}$$

$$a_{21} = \frac{2l}{GJ} \quad a_{22} = \frac{2l}{GJ} \quad a_{23} = \frac{l}{GJ}$$

$$a_{31} = \frac{l}{GJ} \quad a_{32} = \frac{l}{GJ} \quad a_{33} = \frac{l}{GJ}$$

The mass matrix is given by

$$[m] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

The following equations result from the application of Eq. (3.8).

$$\begin{Bmatrix} \left(a_{11}I_1 - \frac{1}{\omega^2}\right)\theta_1 + a_{12}I_2\theta_2 + a_{13}I_3\theta_3 = 0 \\ a_{21}I_1\theta_1 + \left(a_{22}I_2 - \frac{1}{\omega^2}\right)\theta_2 + a_{23}I_3\theta_3 = 0 \\ a_{31}I_1\theta_1 + a_{32}I_2\theta_2 + \left(a_{33}I_3 - \frac{1}{\omega^2}\right)\theta_3 = 0 \end{Bmatrix}$$

To solve the above set of equations we must assign relative values to I_1, I_2, and I_3. For our present purposes we shall consider them to be equal. Also, we shall set $(Il/GJ) = \mu$ (a constant). The characteristic determinant Δ may be written

$$\Delta = \begin{vmatrix} \left(3\mu - \frac{1}{\omega^2}\right) & 2\mu & \mu \\ 2\mu & \left(2\mu - \frac{1}{\omega^2}\right) & \mu \\ \mu & \mu & \left(\mu - \frac{1}{\omega^2}\right) \end{vmatrix}$$

It will be convenient, in this example, to set

$$\frac{1}{\omega^2} = \mu\lambda$$

Then, we may write Δ in a more convenient form

$$\Delta = \begin{vmatrix} (3-\lambda)\mu & 2\mu & \mu \\ 2\mu & (2-\lambda)\mu & \mu \\ \mu & \mu & (1-\lambda)\mu \end{vmatrix}$$

The frequency equation becomes

$$\lambda^3 - 6\lambda^2 + 5\lambda - 1 = 0$$

Roots of this equation are

$$\lambda_1 = 5.0489, \quad \lambda_2 = 0.6431, \quad \lambda_3 = 0.3080$$

The natural frequencies follow from

$$\omega_i^2 = \frac{1}{\mu\lambda_i}$$

and are

$$\omega_1 = 0.445\sqrt{\frac{GJ}{Il}}$$

$$\omega_2 = 1.247\sqrt{\frac{GJ}{Il}}$$

$$\omega_3 = 1.802\sqrt{\frac{GJ}{Il}}$$

The matrix $[B] = [D] - \lambda[I]$ may be written as

$$\mu\begin{bmatrix} (3-\lambda) & 2 & 1 \\ 2 & (2-\lambda) & 1 \\ 1 & 1 & (1-\lambda) \end{bmatrix}$$

It will be instructive to write the complete adjoint matrices for $\lambda = \lambda_1, \lambda_2, \lambda_3$. Note that only one column of each adjoint matrix need be computed to determine the modal columns. For $\lambda = \lambda_1 = 5.0489$,

$$\frac{1}{\mu}[B^{(1)}] = \begin{bmatrix} -2.0489 & 2 & 1 \\ 2 & -3.0489 & 1 \\ 1 & 1 & -4.0489 \end{bmatrix}$$

The adjoint matrix $[C_{ij}^{(1)}]$ is

$$\begin{bmatrix} 11.3447 & 9.0978 & 5.0489 \\ 9.0978 & 7.2958 & 4.0489 \\ 5.0489 & 4.0489 & 2.2469 \end{bmatrix}$$

If the modal column $\{\theta^{(1)}\}$ is normalized to make $\theta_1^{(1)}$ equal to unity, the following column may be obtained from any one of the three columns in $[C_{ij}^{(1)}]$.

$$\{\theta^{(1)}\} = \begin{Bmatrix} 1.0000 \\ 0.8019 \\ 0.4450 \end{Bmatrix}$$

For $\lambda = \lambda_2 = 0.6431$

$$\frac{1}{\mu}[B^{(2)}] = \begin{bmatrix} 2.3569 & 2 & 1 \\ 2 & 1.3569 & 1 \\ 1 & 1 & 0.3569 \end{bmatrix}$$

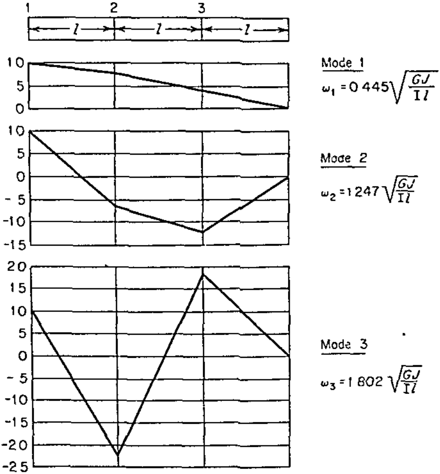


Figure 3.2

$$\frac{1}{\mu^2}[C_{\mu}^{(2)}] = \begin{bmatrix} -0.5147 & 0.2862 & 0.6431 \\ 0.2862 & -0.1578 & -0.3569 \\ 0.6431 & -0.3569 & -0.8009 \end{bmatrix}$$

The modal column for the second mode is

$$\{\theta^{(2)}\} = \begin{Bmatrix} 1.0000 \\ -0.5561 \\ -1.2490 \end{Bmatrix}$$

Finally, for $\lambda = \lambda_3 = 0.3080$

$$\frac{1}{\mu}[B^{(3)}] = \begin{bmatrix} 2.6920 & 2 & 1 \\ 2 & 1.6920 & 1 \\ 1 & 1 & 0.6920 \end{bmatrix}$$

$$\frac{1}{\mu^2}[C_{\mu}^{(3)}] = \begin{bmatrix} 0.1709 & -0.3840 & 0.3080 \\ -0.3840 & 0.8629 & -0.6920 \\ 0.3080 & -0.6920 & 0.5549 \end{bmatrix}$$

$$\{\theta^{(3)}\} = \begin{Bmatrix} 1.000 \\ -2.247 \\ 1.803 \end{Bmatrix}$$

In summary, the three modes are shown in Fig. 3.2.

3.4 Orthogonality of the Natural Modes

In the previous section it was shown that the natural modes and frequencies of an ideal structure are found by solution of the equations of motion for free harmonic vibrations. The normal modes are expressed as a relationship among the amplitudes of the structure in which all the ratios are determined but the absolute values are not. For example, Eq. (3.28) determines the normal mode amplitudes to within an arbitrary multiplicative constant α which is often selected such that one of the amplitudes is unity. The column matrix $\{q^{(r)}\}$ is called the modal column in the rth mode and is also called the rth eigenvector. It is useful to think of a modal column for an n degree-of-freedom system as a vector in an n dimension space where each element of the column is a component of the vector in one of the n coordinate directions. Then, for an n degree-of-freedom system there are n such eigenvectors. It is the purpose of this section to show that this set of eigenvectors is orthogonal in a particular sense.*

To show the orthogonality relations we recall that each eigenvector and associated eigenvalue satisfies the equations of motion for free vibration. Let us write these equations in matrix form for the rth normal mode, using Eq. (3.5) for this purpose.

$$\omega_r^2[m]\{q^{(r)}\} = [k]\{q^{(r)}\} \tag{3.29}$$

We now premultiply both sides of this equation by the transposed modal column for one of the other modes, say $\{q^{(s)}\}$.

$$\omega_r^2\{q^{(s)}\}^T[m]\{q^{(r)}\} = \{q^{(s)}\}^T[k]\{q^{(r)}\} \tag{3.30}$$

Note that the matrices to be multiplied are conformable and that the triple products are scalars. We now apply the rule of matrix algebra which states that the transpose of a product of several matrices is obtained by taking the product of the tranposed matrices in the reverse order. Then, equating the transposes of the products on each side of Eq. (3.30) leads to the equation

$$\omega_r^2\{q^{(r)}\}^T[m]\{q^{(s)}\} = \{q^{(r)}\}^T[k]\{q^{(s)}\} \tag{3.31}$$

In performing the above operation we used the symmetry property of the matrices $[m]$ and $[k]$ which makes each one equal to its transpose. Next, we shall write for the sth mode an equation corresponding to Eq. (3.29).

$$\omega_s^2[m]\{q^{(s)}\} = [k]\{q^{(s)}\} \tag{3.32}$$

Now, we premultiply both sides of this equation by the transpose of the rth modal column to give

$$\omega_s^2\{q^{(r)}\}^T[m]\{q^{(s)}\} = \{q^{(r)}\}^T[k]\{q^{(s)}\} \tag{3.33}$$

Focusing our attention on Eqs. (3.31) and (3.33) we see that they are the same except that the left members contain frequencies corresponding to different modes. If these frequencies are different we may subtract Eq. (3.33) from (3.31) to give

$$(\omega_r^2 - \omega_s^2)\{q^{(r)}\}^T[m]\{q^{(s)}\} = 0$$

It follows that for $\omega_r \neq \omega_s$

$$\{q^{(r)}\}^T[m]\{q^{(s)}\} = 0 \tag{3.34}$$

And, therefore, from Eqs. (3.31) or (3.33)

$$\{q^{(r)}\}^T[k]\{q^{(s)}\} = 0 \tag{3.35}$$

Equations (3.34) and (3.35) express the orthogonality relationship among the natural modes. Note that two modes having the same frequency are not necessarily orthogonal; however, as stated previously, this represents an unusual case in structures. For a given mode, say the rth, the triple product in Eq. (3.34) is equal to a constant different from zero, say M_r.

$$\{q^{(r)}\}^T[m]\{q^{(r)}\} = M_r \tag{3.36}$$

Setting $s = r$, Eq. (3.31) gives

$$\{q^{(r)}\}^T [k] \{q^{(r)}\} = \omega_r^2 M_r \tag{3.37}$$

A further comment concerning the orthogonality relation may be instructive. Recalling our representation of the natural modes as eigenvectors in n space, we usually think of orthogonality as a condition among the vectors that they be mutually perpendicular. This requirement would be expressed by the vanishing of the scalar products of all pairs of vectors. Thus

$$q^{(r)} \cdot q^{(s)} = 0 \tag{3.38}$$

In matrix form this requirement would appear as

$$\{q^{(r)}\}^T \{q^{(s)}\} = 0 \tag{3.39}$$

Therefore, we see that the orthogonality relation among the natural modes of vibration is not the direct orthogonality relation visualized above. Equations (3.34) and (3.35) state that the natural modes are orthogonal with respect to the matrices $[m]$ and $[k]$, respectively. These are called weighting matrices. Following this idea a step further one could say that the two vectors of Eq. (3.39) are orthogonal with respect to the identity matrix, since

$$\{q^{(r)}\}^T \{q^{(s)}\} = \{q^{(r)}\}^T [I] \{q^{(s)}\}$$

Here, the identity matrix is the weighting matrix.

The modal columns are, as stated before, determined to within a multiplicative constant. Hence, the magnitudes of the eigenvectors are not determined. It is convenient sometimes to *normalize* them, that is, to adjust their magnitudes so as to make the constant M_r in Eq. (3.36) equal to unity. The orthogonal eigenvectors so normalized are sometimes called *orthonormal*. In general, the process of normalizing the modal columns in practical engineering problems is not of very great importance.

3.5 Matrix Iteration

A convenient and frequently used method for the solution of the eigenvalue problem makes use of an iteration technique[33] applicable to the equations of motion expressed in matrix form. Either Eq. (3.7) or (3.10) may be used. The iteration is started by selecting a trial modal column which is premultiplied by $[D]$ or $[D]^{-1}$, according to the equation to be used. The resulting column matrix is then normalized, usually by reducing one of its components to unity. The normalized column is premultiplied as before to yield a third column matrix. This matrix is normalized in the same way as before and becomes still another trial column. The process is repeated until the successive normalized column matrices converge to a common matrix. Successive iterations then yield the same column, and the normalizing factor attains a constant value. Should Eq. (3.7) be used the factor converges to the largest value of $\lambda = 1/\omega^2$, which corresponds to the lowest frequency or to the first mode. The corresponding column matrix is, then, the modal column or eigenvector for the first mode.

In case Eq. (3.10) is used, the iteration converges to the largest value of ω^2 or to the highest mode. This process, then, yields either the first mode or the nth mode (for an n degree-of-freedom system) leaving intermediate modes undetermined. An extension of the technique makes it possible to constrain the equation of motion so as to make possible convergence to successively higher or successively lower modes. In the following discussion the iteration method is covered in detail and a proof for convergence is given. The extension to the method permitting convergence to intermediate modes is discussed in the next section.

To demonstrate the method we shall use Eq. (3.7). Hence, the following discussion will result in a proof for convergence to the first mode. As stated before, the trial column to be used in the first iteration may be arbitrary except that it will not, in general correspond to one of the natural modes. It is expedient to select a trial column reasonably similar to the anticipated first mode to hasten convergence. In the following proof we shall represent the arbitrary trial column by linear superposition of the n eigenvectors of the system. This is possible because the n orthogonal eigenvectors are linearly independent in n space.[49]

$$\{q_1\} = C_1\{q^{(1)}\} + C_2\{q^{(2)}\} + \cdots + C_n\{q^{(n)}\} \tag{3.40}$$

where

$$\{q_1\} = \text{the first trial column}$$

$$\{q^{(1)}\}, \{q^{(2)}\}, \ldots, \{q^{(n)}\} = \text{the } n \text{ eigenvectors}$$

$$C_1, C_2, \ldots, C_n = \text{constants}$$

The trial column is selected arbitrarily, hence it is a known vector. The modal columns are known in the sense that they are dependent only upon the properties of the system. The constants $C_1 \ldots C_n$ are unknown numbers to be determined by the above equation. According to the iteration procedure we premultiply the trial column by the dynamical matrix which, in view of Eq. (3.40), results in the equation

$$[D]\{q_1\} = C_1[D]\{q^{(1)}\} + C_2[D]\{q^{(2)}\} + \cdots + C_n[D]\{q^{(n)}\} \tag{3.41}$$

Now, according to Eq. (3.7) the following holds

$$
\begin{aligned}
[D]\{q^{(1)}\} &= \frac{1}{\omega_1^2}\{q^{(1)}\} \\
[D]\{q^{(2)}\} &= \frac{1}{\omega_2^2}\{q^{(2)}\} \\
&\vdots \\
[D]\{q^{(r)}\} &= \frac{1}{\omega_r^2}\{q^{(r)}\} \\
&\vdots \\
[D]\{q^{(n)}\} &= \frac{1}{\omega_n^2}\{q^{(n)}\}
\end{aligned}
\tag{3.42}
$$

Substituting these equations into Eq. (3.41) yields the equation

$$
\begin{aligned}
[D]\{q_1\} &= \{q_2\} \\
&= \frac{C_1}{\omega_1^2}\{q^{(1)}\} + \frac{C_2}{\omega_2^2}\{q^{(2)}\} + \cdots + \frac{C_n}{\omega_n^2}\{q^{(n)}\}
\end{aligned}
\tag{3.43}
$$

where $\{q_2\}$ is the second trial column. We now repeat the process by premultiplying Eq. (3.43) by the dynamical matrix and substituting Eqs. (3.42) to give the results of the second iteration.

$$
\begin{aligned}
[D]\{q_2\} &= \{q_3\} \\
&= \frac{C_1}{\omega_1^4}\{q^{(1)}\} + \frac{C_2}{\omega_2^4}\{q^{(2)}\} + \cdots + \frac{C_n}{\omega_n^4}\{q^{(n)}\}
\end{aligned}
\tag{3.44}
$$

It is easy to see that if the process is repeated, the pth iteration will yield

$$
\begin{aligned}
[D]\{q_p\} &= \{q_{p+1}\} \\
&= \frac{C_1}{\omega_1^{2p}}\{q^{(1)}\} + \frac{C_2}{\omega_2^{2p}}\{q^{(2)}\} + \cdots + \frac{C_n}{\omega_n^{2p}}\{q^{(n)}\}
\end{aligned}
\tag{3.45}
$$

Notice that nothing has been said about normalizing the successive trial columns. This is not necessary in order to establish proof of convergence but, in any event, it amounts to nothing more than a readjustment of the constants $C_1 \ldots C_n$ at each step. Referring now to Eq. (3.45) we recall that the modes are numbered beginning with the one having the lowest frequency. Therefore

$$
\omega_1 < \omega_2 < \omega_3 < \cdots < \omega_n \tag{3.46}
$$

If p is large enough, i.e., if the iteration process has been carried through a sufficient number of steps it follows that

$$
\frac{1}{\omega_1^{2p}} \gg \frac{1}{\omega_2^{2p}} \gg \cdots \gg \frac{1}{\omega_n^{2p}} \tag{3.47}
$$

The first term, then, in the right-hand member of Eq. (3.45) becomes the only significant one. The number of required iterations depends upon the accuracy desired, i.e., the number of significant figures desired in the mode numbers and frequencies. Assuming that p iterations are required to achieve the desired accuracy the resulting equation is

$$\{q_{p+1}\} = \frac{C_1}{\omega_1^{2p}}\{q^{(1)}\} \tag{3.48}$$

This means that the $(p + 1)$th trial column becomes identical to the first natural mode column to within a multiplicative constant, i.e., the iteration converges to the first modal column. An additional iteration yields

$$\{q_{p+2}\} = \frac{C_1}{\omega_1^{2p+2}}\{q^{(1)}\}$$

If the columns $\{q_{p+1}\}$ and $\{q_{p+2}\}$ are normalized in the same way, the ratio of the normalizing constant for the latter column to that for the former is $1/\omega_1^2$. Thus, the iteration procedure leading to the first modal column also yields the first mode frequency.

Using the same method as the foregoing, it can be shown that the iteration process starting with Eq. (3.10) converges to the modal column corresponding to the highest mode number $\{q^{(n)}\}$, and yields the corresponding frequency. It is clear from the proof given above that iteration on either Eq. (3.7) or (3.10) will not converge to an intermediate mode.

3.6 Convergence to Intermediate Modes by Matrix Iteration

By use of the orthogonality relationships it is possible to modify the equations of motion so that iteration will converge to intermediate modes in either ascending or descending succession. If, for example, Eq. (3.7) is used to find the first mode it is possible next to find the second, then the third, and so on to the nth mode. It is not possible by this technique to go immediately to any desired intermediate mode.

Returning to Eq. (3.40) we have considered the trial column as a linear superposition of all the modal columns. We may say that the n modal columns are components of the trial column and that the contribution of each component is measured by the magnitude of its coefficient C. Let us consider now that having started with Eq. (3.7) we have determined the first normal mode $\{q^{(1)}\}$. We wish next to proceed, by iteration, to the second normal mode. As before, we select a trial column $\{q_1\}$ to begin the iteration. However, we impose a condition of constraint on this trial column by requiring that it be orthogonal to the first normal mode which we have just found. The equation of constraint is

$$\{q^{(1)}\}^T[m]\{q_1\} = 0 \tag{3.49}$$

Substituting Eq. (3.40) into this equation we write

$$C_1\{q^{(1)}\}^T[m]\{q^{(1)}\} + C_2\{q^{(1)}\}^T[m]\{q^{(2)}\} + \cdots + C_n\{q^{(1)}\}^T[m]\{q^{(n)}\} = 0 \tag{3.50}$$

From the orthogonality condition we see that all the terms on the left side of Eq. (3.50) are zero except the first. Hence, the equation of constraint becomes

$$C_1\{q^{(1)}\}^T[m]\{q^{(1)}\} = 0 \tag{3.51}$$

Since the triple matrix product in Eq. (3.51) is, in general, not zero, this equation demands that the constant C_1 be zero. Therefore, subject to this constraint, Eq. (3.40) becomes

$$\{q_1\} = C_2\{q^{(2)}\} + C_3\{q^{(3)}\} + \cdots + C_n\{q^{(n)}\} \tag{3.52}$$

We see that the equation of constraint (3.49) has the effect of reducing to zero the first mode component of the trial column. It is said that the first mode is swept out of the trial column and, hence, out of the iterated solution. This process is referred to as *sweeping* for this reason.

It is now necessary to apply this condition of constraint to the solution of Eq. (3.7). We notice that Eq. (3.49) determines a relationship among the elements of the trial column $\{q_1\}$. This relationship is determined by expanding the triple matrix product. First, let us define the elements of the trial column by writing

$$\{q_1\} = \begin{Bmatrix} q_{11} \\ q_{21} \\ q_{31} \\ \vdots \\ q_{n1} \end{Bmatrix} \tag{3.53}$$

Here, we let the first subscript denote the position in the column, and the second subscript the trial number. Actually, all that is said in the following development relating to the first trial column will apply equally well to all other trial columns. Therefore, there is no need to single out any one trial column in this part of the discussion. In terms of the matrix elements the expanded form of Eq. (3.49) is

$$q_{11}\sum_i m_{i1}q_i^{(1)} + q_{21}\sum_i m_{i2}q_i^{(1)} + q_{31}\sum_i m_{i3}q_i^{(1)} + \cdots + q_{n1}\sum_i m_{in}q_i^{(1)} = 0 \tag{3.54}$$

This equation may be used to express any one of the trial q's in terms of all the others. It is customary, although a matter of arbitrary choice, to express q_{11} in terms of the others. Thus

$$q_{11} = -\frac{\sum_i m_{i2} q_i^{(1)}}{\sum_i m_{i1} q_i^{(1)}} q_{21} - \frac{\sum_i m_{i3} q_i^{(1)}}{\sum_i m_{i1} q_i^{(1)}} q_{31} \cdots \cdots - \frac{\sum_i m_{in} q_i^{(1)}}{\sum_i m_{i1} q_i^{(1)}} q_{n1} \tag{3.55}$$

In writing Eq. (3.55), q_{11} is regarded as the constrained element of the trial column. All other elements remain arbitrary. Thus

$$\begin{aligned} q_{21} &= q_{21} \\ q_{31} &= q_{31} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ q_{n1} &= q_{n1} \end{aligned} \tag{3.56}$$

Equations (3.55) and (3.56) may be combined in a matrix form relating the constrained trial column to the arbitrary one.

$$\begin{bmatrix} 0 & -\frac{\sum_i m_{i2} q_i^{(1)}}{\sum_i m_{i1} q_i^{(1)}} & \cdots & -\frac{\sum_i m_{in} q_i^{(1)}}{\sum_i m_{i1} q_i^{(1)}} \\ 0 & 1 & 0 \cdots\cdots & 0 \\ 0 & 0 & 1 \cdots\cdots & 0 \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ 0 & 0 & \cdots\cdots & 1 \end{bmatrix} \begin{Bmatrix} q_{11} \\ q_{21} \\ q_{31} \\ \vdots \\ q_{n1} \end{Bmatrix} = \begin{Bmatrix} q_{11} \\ q_{21} \\ q_{31} \\ \vdots \\ q_{n1} \end{Bmatrix}_c \tag{3.57}$$

The column matrix on the left side of Eq. (3.57) is an arbitrary trial column; the one on the right side, distinguished by subscript c, is a constrained trial column. The square matrix, called the sweeping matrix $[S]$, is the identity matrix except for the first row which is determined from Eq. (3.55). Equation (3.57) is rewritten in the more compact form

$$[S]\{q_1\} = \{q_1\}_c \tag{3.58}$$

The same equation applies to successive trial columns

$$[S]\{q_p\} = \{q_p\}_c \tag{3.59}$$

By following the iteration procedure of the last section it becomes clear that if we apply the constrained trial column having the components given by Eq. (3.52) to the equation of motion (3.7), the iteration will converge to the second mode. This procedure is started by premultiplying the constrained trial column by the dynamical matrix, thus determining the second trial column

$$[D]\{q_1\}_c = [D][S]\{q_1\} = \{q_2\}$$

The process is repeated as many times as necessary.

$$\begin{aligned} [D]\{q_2\}_c &= [D][S]\{q_2\} = \{q_3\} \\ &\cdots\cdots\cdots\cdots \\ [D]\{q_p\}_c &= [D][S]\{q_p\} = \{q_{p+1}\} \end{aligned} \tag{3.60}$$

After convergence to the required accuracy further iterations lead to the following relationships stemming from Eq. (3.7).

$$[D]\{q^{(2)}\}_c = [D][S]\{q^{(2)}\} = \frac{1}{\omega_2^2}\{q^{(2)}\} \tag{3.61}$$

The first equality expressed by this equation states that

$$\{q^{(2)}\}_c = [S]\{q^{(2)}\}$$

Since there is no distinction between $\{q^{(2)}\}_c$ and $\{q^{(2)}\}$ this equation is simply a restatement of the orthogonality relationship

$$\{q^{(1)}\}^T[m]\{q^{(2)}\} = 0$$

The second equality states that iteration on Eq. (3.7) will lead to convergence on the second mode if the dynamical matrix is post multiplied by the sweeping matrix. Thus, the equation of constraint (3.49) may be regarded as a constraint on the dynamical matrix such that the first mode is swept out of the solution. Hence, Eq. (3.7) may be used for the second mode solution with the original dynamical matrix $[D]$ replaced by a constrained one, $[D]_1$, given by

$$[D]_1 = [D][S]_1 \tag{3.62}$$

where $[S]_1$ denotes the sweeping matrix generated above, which serves to sweep out the first normal mode. Then, the second mode emerges from iteration on the equation

$$[D]_1\{q\} = \frac{1}{\omega^2}\{q\} \tag{3.63}$$

where the trial column matrix used to begin the iteration may be an arbitrary one.

To cause convergence to the third mode, two orthogonality relationships are used to constrain the trial columns. These are

$$\left.\begin{aligned} \{q^{(1)}\}^T[m]\{q_1\} &= 0 \\ \{q^{(2)}\}^T[m]\{q_1\} &= 0 \end{aligned}\right\} \tag{3.64}$$

Substitution of the Eq. (3.40) into these equations leads to the requirement

$$C_1 = C_2 = 0 \tag{3.65}$$

Therefore, iteration on a trial column constrained by use of Eqs. (3.64) will converge to the third mode. If we expand these two equations we obtain the following pair, the first of which is obviously the same as Eq. (3.54).

$$\left.\begin{cases} q_{11} \sum_i m_{i1} q_i^{(1)} + q_{21} \sum_i m_{i2} q_i^{(1)} + \cdots + q_{n1} \sum_i m_{in} q_i^{(1)} = 0 \\ q_{11} \sum_i m_{i1} q_i^{(2)} + q_{21} \sum_i m_{i2} q_i^{(2)} + \cdots + q_{n1} \sum_i m_{in} q_i^{(2)} = 0 \end{cases}\right\} \tag{3.66}$$

If these two equations are used to define q_{11} and q_{21} (again this is an arbitrary choice as any two of the q's may be chosen) these two elements of the trial column are constrained. The remaining elements may be chosen arbitrarily; thus

$$\left.\begin{array}{l} q_{31} = q_{31} \\ q_{41} = q_{41} \\ \cdots\cdots\cdots \\ q_{n1} = q_{n1} \end{array}\right\} \tag{3.67}$$

Equations (3.66) and (3.67) may be combined to form a second sweeping matrix $[S]_2$ which sweeps out the first and second modes. The first two rows of this matrix are determined by Eqs. (3.66) and the remaining rows correspond to the last $n - 2$ rows of the identity matrix. This second sweeping matrix is used, as was the first, to operate on the dynamical matrix according to the equation

$$[D][S]_2 = [D]_2 \tag{3.68}$$

The modified dynamical matrix is such as to cause convergence to the third mode when used in the equation

$$[D]_2\{q\} = \frac{1}{\omega^2}\{q\} \tag{3.69}$$

The foregoing discussion is sufficient to indicate the method of solution for higher modes beyond the third. Also, application of this method to the inverse problem stated by Eq. (3.10) requires no further explanation.

Example—Matrix Iteration

To illustrate the method of matrix iteration for determining normal modes we shall use the example of Section 3.3. The system is shown in Fig. 3.1. The following matrices are required and may be found from the previous work.

$$[a] = \frac{l}{GJ}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$[m] = I\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The mass moments of inertia of the three disks are equal, i.e.,

$$I_1 = I_2 = I_3 = I$$

The dynamical matrix is

$$[D] = [a][m]$$

$$= \mu \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

where $\mu = Il/GJ$.

We shall determine the first normal mode by using Eq. (3.7) which is, for this example

$$[D]\{\theta\} = \frac{1}{\omega^2}\{\theta\}$$

The equation is written in the form

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \lambda \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

where, as previously, $\lambda = 1/\mu\omega^2$.

From experience we realize that in the first mode the three angular displacements have the same sign and that $\theta_1 > \theta_2 > \theta_3$. We shall select for our trial column the values $\theta_1 = \theta_2 = \theta_3 = 1$. The steps in the iteration are shown below. In each step the amplitude θ_1 is reduced to unity.

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 5 \\ 3 \end{Bmatrix} = 6.000 \begin{Bmatrix} 1.0000 \\ 0.8333 \\ 0.5000 \end{Bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} 1.0000 \\ 0.8333 \\ 0.5000 \end{Bmatrix} = \begin{Bmatrix} 5.1667 \\ 4.1667 \\ 2.3333 \end{Bmatrix} = 5.1677 \begin{Bmatrix} 1.0000 \\ 0.8065 \\ 0.4516 \end{Bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} 1.0000 \\ 0.8065 \\ 0.4516 \end{Bmatrix} = \begin{Bmatrix} 5.0646 \\ 4.0646 \\ 2.2581 \end{Bmatrix} = 5.0646 \begin{Bmatrix} 1.0000 \\ 0.8026 \\ 0.4459 \end{Bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} 1.0000 \\ 0.8026 \\ 0.4459 \end{Bmatrix} = \begin{Bmatrix} 5.0511 \\ 4.0511 \\ 2.2485 \end{Bmatrix} = 5.0511 \begin{Bmatrix} 1.0000 \\ 0.8020 \\ 0.4451 \end{Bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} 1.0000 \\ 0.8020 \\ 0.4451 \end{Bmatrix} = \begin{Bmatrix} 5.0491 \\ 4.0491 \\ 2.2471 \end{Bmatrix} = 5.0491 \begin{Bmatrix} 1.0000 \\ 0.8019 \\ 0.4450 \end{Bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} 1.0000 \\ 0.8019 \\ 0.4450 \end{Bmatrix} = \begin{Bmatrix} 5.0488 \\ 4.0488 \\ 2.2469 \end{Bmatrix} = 5.0488 \begin{Bmatrix} 1.0000 \\ 0.8019 \\ 0.4450 \end{Bmatrix}$$

The iteration has converged within the five-significant-figure accuracy used in this example. The results are almost identical to those found from the previous solution of this problem.

$$\lambda_1 = 5.0488; \qquad \{\theta^{(1)}\} = \begin{Bmatrix} 1.0000 \\ 0.8019 \\ 0.4450 \end{Bmatrix}$$

Next, we wish to proceed to the second mode. Hence, we apply the following orthogonality constraint on the trial column $\{\theta\}$ to be used.

$$\{\theta^{(1)}\}^T [m] \{\theta\} = I \begin{Bmatrix} 1.0000 \\ 0.8019 \\ 0.4450 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = 0$$

In expanded form this equation appears as

$$\theta_1 + 0.8019 \quad \theta_2 + 0.4450 \quad \theta_3 = 0$$

If we constrain the trial value θ_1 and consider θ_2 and θ_3 to be arbitrary, we may write

$$\theta_1 = -0.8019 \quad \theta_2 - 0.4450 \quad \theta_3$$
$$\theta_2 = \theta_2$$
$$\theta_3 = \theta_3$$

In matrix form these equations are

$$\begin{bmatrix} 0 & -0.8019 & -0.4450 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

The square matrix is the sweeping matrix $[S]_1$ which, when premultiplied by the dynamical matrix $[D]$, yields the matrix $[D]_1$ to be used for convergence on the second normal mode. Thus

$$[D]_1 = \mu \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -0.8019 & -0.4450 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \mu \begin{bmatrix} 0 & -0.4057 & -0.3350 \\ 0 & 0.3962 & 0.1100 \\ 0 & 0.1981 & 0.5550 \end{bmatrix}$$

In the second mode we expect a node to exist somewhere on the axis of the disks. If we assume, for trial, that this occurs at the second disk, we might adopt trial values $\theta_1 = 1$, $\theta_2 = 0$, $\theta_3 = -1$. The iteration then will proceed as for the first mode except that we now use the constrained dynamical matrix.

$$\begin{bmatrix} 0 & -0.4057 & -0.3350 \\ 0 & 0.3962 & 0.1100 \\ 0 & 0.1981 & 0.5550 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0.3350 \\ -0.1100 \\ -0.5550 \end{Bmatrix} = 0.3350 \begin{Bmatrix} 1.0000 \\ -0.3284 \\ -1.6567 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & -0.4057 & -0.3350 \\ 0 & 0.3962 & 0.1100 \\ 0 & 0.1981 & 0.5550 \end{bmatrix} \begin{Bmatrix} 1.0000 \\ -0.3284 \\ -1.6567 \end{Bmatrix} = \begin{Bmatrix} 0.6882 \\ -0.3123 \\ -0.9845 \end{Bmatrix} = 0.6882 \begin{Bmatrix} 1.0000 \\ -0.4538 \\ -1.4305 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & -0.4057 & -0.3350 \\ 0 & 0.3962 & 0.1100 \\ 0 & 0.1981 & 0.5550 \end{bmatrix} \begin{Bmatrix} 1.0000 \\ -0.4538 \\ -1.4305 \end{Bmatrix} = \begin{Bmatrix} 0.6633 \\ -0.3372 \\ -0.8838 \end{Bmatrix} = 0.6633 \begin{Bmatrix} 1.0000 \\ -0.5084 \\ -1.3324 \end{Bmatrix}$$

. . . .

$$\begin{bmatrix} 0 & -0.4057 & -0.3350 \\ 0 & 0.3962 & 0.1100 \\ 0 & 0.1981 & 0.5550 \end{bmatrix} \begin{Bmatrix} 1.0000 \\ -0.5553 \\ -1.2475 \end{Bmatrix} = \begin{Bmatrix} 0.6433 \\ -0.3572 \\ -0.8024 \end{Bmatrix} = 0.6433 \begin{Bmatrix} 1.0000 \\ -0.5553 \\ -1.2473 \end{Bmatrix}$$

Twelve iterations are required to cause convergence to the indicated number of significant figures. We obtain for the second mode

$$\lambda_2 = 0.6433; \qquad \{\theta^{(2)}\} = \begin{Bmatrix} 1.0000 \\ -0.5553 \\ -1.2473 \end{Bmatrix}$$

Comparison with the previous results shows that agreement between the modal columns is obtained in the first three significant figures. Agreement between the two eigenvalues is very good. In general, the iteration procedure leads to some loss in accuracy because of rounding off errors, so that one should use a greater number of significant figures in the analysis than are required in the results.

The third mode can be obtained readily by inverting the equation as explained before, but to further illustrate the sweeping procedure we shall follow it through the next step. The two equations of constraint required result from the two orthogonality relations

$$\{\theta^{(1)}\}^T[m]\{\theta\} = 0$$
$$\{\theta^{(2)}\}^T[m]\{\theta\} = 0$$

The expanded equations are

$$\theta_1 + 0.8019 \quad \theta_2 + 0.4450 \quad \theta_3 = 0$$
$$\theta_1 - 0.5553 \quad \theta_2 - 1.2473 \quad \theta_3 = 0$$

Eliminating θ_2 we obtain

$$\theta_1 = 0.5549 \quad \theta_3$$

Eliminating θ_1 we obtain

$$\theta_2 = -1.2469 \quad \theta_3$$

Hence, the sweeping matrix $[S]_2$ is

$$[S]_2 = \begin{bmatrix} 0 & 0 & 0.5549 \\ 0 & 0 & -1.2469 \\ 0 & 0 & 1 \end{bmatrix}$$

The new dynamical matrix is

$$[D]_2 = [D][S]_2 = \mu \begin{bmatrix} 0 & 0 & 0.1709 \\ 0 & 0 & -0.3840 \\ 0 & 0 & 0.3080 \end{bmatrix}$$

Iteration in this case becomes a trivial problem when one realizes that the first two elements of the trial column are of no consequence in the matrix multiplication. Therefore, we shall start with a trial column in which the third element only is defined, and its value will be taken as unity. The computation proceeds to yield

$$\begin{bmatrix} 0 & 0 & 0.1709 \\ 0 & 0 & -0.3840 \\ 0 & 0 & 0.3080 \end{bmatrix} \begin{Bmatrix} -- \\ - \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.1709 \\ -0.3840 \\ 0.3080 \end{Bmatrix} = 0.3080 \begin{Bmatrix} 0.5549 \\ -1.2469 \\ 1.0000 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0.1709 \\ 0 & 0 & -0.3840 \\ 0 & 0 & 0.3080 \end{bmatrix} \begin{Bmatrix} 0.5549 \\ -1.2469 \\ 1.0000 \end{Bmatrix} = \begin{Bmatrix} 0.1709 \\ -0.3840 \\ 0.3080 \end{Bmatrix} = 0.3080 \begin{Bmatrix} 0.5549 \\ -1.2469 \\ 1.0000 \end{Bmatrix}$$

By normalizing the column with respect to the third element, since only that one is of consequence in the calculations, convergence is immediate. The results are

$$\lambda_3 = 0.3080; \qquad \{\theta^{(3)}\} = \begin{Bmatrix} 0.5549 \\ -1.2469 \\ 1.0000 \end{Bmatrix}$$

We can make the first element of the column unity by dividing through by 0.5549, thus giving a proportional modal column $\{\theta^{(3)}\}$.

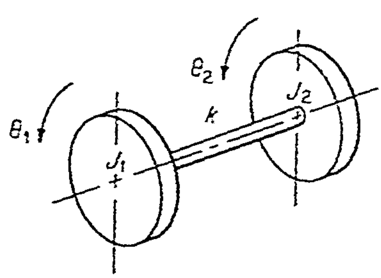
$$\{\theta^{(3)}\} = \begin{Bmatrix} 1.0000 \\ -2.2470 \\ 1.8021 \end{Bmatrix}$$

This column agrees closely with that obtained previously, and the eigenvalues are identical in the two solutions.

and a rotation about a vertical axis. The weight of the first floor is 100 lb/ft^2, and the second floor 75 lb/ft^2. The columns are fixed at their ends against rotation about a horizontal axis. The force required to displace the top of a typical column one inch with respect to its bottom in the x or y direction is 10 kips for the first story columns and 8 kips for the second story columns. Derive the characteristic equation and compute the natural frequencies and corresponding modes for free vibration of the structure in the xz yz planes. Draw the mode shapes.

2. Derive the characteristic equation for torsional vibration of the structure in Problem 1, and compute the natural frequencies and corresponding modes. Neglect the torsional stiffness of the individual columns.

3. Two disks having mass moments of inertia J_1 and J_2 are joined by a light shaft of stiffness k. Formulate the equations of motion using Lagrange's equations and compute the natural modes and frequencies by solving the characteristic equation. Note that the [k] matrix is singular and, hence, the frequency corresponding to the rigid body rotation of the two disks and shaft is zero. This is true despite the fact that the system has two degrees of freedom of motion.



Problem 3

4. Derive the characteristic equation and compute the natural frequencies and modes for the system of Problem 11, Chapter 2.

$$k_1 = 2 \text{ kips/inch}$$
$$k_2 = 1 \text{ kip/inch}$$
$$m_2 = 3000 \text{ lb}$$
$$r = 3 \text{ ft}$$
$$t = 2 \text{ ft}$$

The weight per unit volume of the disk $300/\pi$ (lb/ft^3).

5. Derive the characteristic equation and compute the natural modes and frequencies for the system of Problem 12, Chapter 2. Use the equations of motion in the q coordinate system as obtained in Problem 13, Chapter 2.

$$k_1 = 40 \text{ lb/in.} \qquad m_1 = 20 \frac{\text{lb sec}^2}{\text{ft}}$$
$$k_2 = 60 \text{ lb/in.} \qquad m_2 = 8 \frac{\text{lb sec}^2}{\text{ft}}$$
$$k_3 = 30 \text{ lb/in.} \qquad m_3 = 12 \frac{\text{lb sec}^2}{\text{ft}}$$
$$k_4 = 50 \text{ lb/in.}$$
$$k_5 = 80 \text{ lb/in.}$$

6. The displacements $w(x_i)$, $i = 1, 2, \ldots, n$ of a lumped mass system $[m]$ are described by

$$w(x_i) = \sum_{j=1}^{n} \phi_j(x_i) q_j \qquad \text{(See Eq. 2.28, Chapter 2.)}$$

or

$$\{w\} = [\phi]\{q\} \qquad \text{(See Eq. 2.29, Chapter 2.)}$$

where the q's are distributed coordinates and the $\phi_j(x)$'s are displacement functions with $\phi_j(x_i)$ representing the displacement of the ith mass located at x_i in the displacement function corresponding to the jth coordinate.

The eigenvectors of the system, computed in the q coordinates, are denoted by

$$\{q^{(r)}\} \qquad r = 1, 2, \ldots, n$$

The corresponding mode shapes $\{\Phi^{(r)}\}$ $r = 1, 2, \ldots, n$ describing the displacements of the lumped masses vibrating in the natural frequencies of the system are computed from

$$\{\Phi^{(r)}\} = [\phi]\{q^{(r)}\}$$

Using the orthogonality relations, Eq. (3.34), show that

$$\begin{aligned} \{\Phi^{(r)}\}^T[m]\{\Phi^{(s)}\} &= 0, \qquad \text{for } r \neq s \text{ and } \omega_r \neq \omega_s \\ &= M_r, \quad \text{for } r = s. \end{aligned}$$

Using Eqs. (3.36) and (3.37) show also that

$$\begin{aligned} \{\Phi^{(r)}\}^T[k]\{\Phi^{(s)}\} &= 0, \qquad \text{for } r \neq s \text{ and } \omega_r \neq \omega_s \\ &= \omega_r^2 M_r, \quad \text{for } r = s. \end{aligned}$$

7. The displacement $w(x)$ of a distributed mass system with mass $m(x)$ per unit length is described by

$$w(x) = \sum_{j=1}^{n} \phi_j(x) q_j \qquad \text{(See Eq. 2.36, Chapter 2.)}$$

where the q_j's are distributed coordinates and the $\phi_j(x)$'s are continuous displacement functions in x. The eigenvectors computed in the q coordinate system are denoted by

$$\{q^{(r)}\} \qquad r = 1, 2, \ldots, n$$

The corresponding mode shapes $\Phi_r(x)$ $r = 1, 2, \ldots, n$ describing the displacement configuration of the distributed mass system vibrating in its natural frequencies are computed from

$$\Phi_r(x) = \sum_{j=1}^{n} \phi_j(x) q_j^{(r)}$$

Using the orthogonality relations (Eq. 3.34) show that

$$\int_0^l m(x)\Phi_i(x)\Phi_j(x)\,dx = 0 \qquad i \neq j \quad \omega_i \neq \omega_j$$

where the integral extends over the entire mass of the system. Using Eq. (3.36) and (3.37), show also that

$$\int_0^l EI(x)\,\Phi_i''(x)\,\Phi_j''(x)\,dx = 0 \qquad \text{for } i \neq j$$
$$= \omega_i^2 \int_0^l m(x)\,\Phi_i^2(x)\,dx, \qquad \text{for } i = j$$

where $U = \frac{1}{2}\int_0^l EI(x)[w''(x)]^2\,dx$ is the strain energy of the system and, hence

$$k_{ij} = \int_0^l EI(x)\,\phi_i''(x)\,\phi_j''(x)\,dx$$

is the generalized stiffness. (See Chapter 1, Problem 12.)

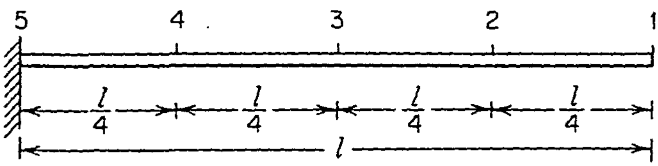
8. The cantilever beam shown is uniform, having flexural rigidity EI and mass m per unit length. Consider the mass lumped at points 1, 2, 3, 4, 5 with

$$M_1 = M_5 = \tfrac{1}{2}M_j \qquad j = 2, 3, 4$$

and

$$\sum_{j=1}^{5} M_j = ml$$

Compute the first natural mode and frequency of vibration using matrix iteration. Check your results by computing the adjoint matrix of $[D - \lambda_1 I]$.



Problem 8

9. Construct the appropriate sweeping matrices and compute the second and third natural modes and frequencies of the beam in Problem 8.
10. Repeat Problem 5 using matrix iteration. Check with results of Problem 5.
11. Compute the fifth and fourth modes and frequencies for the structure of Problem 14, Chapter 2. Set $J = \frac{1}{6}ml^2$. Use matrix iteration.
12. Compute the highest natural mode of torsional vibration for the structure of Problem 16, Chapter 2.
13. Compute the first and second natural modes of vibration in the xz plane for the structure of Problem 16, Chapter 2.
14. Compute the four natural modes and frequencies of vibration in the yz plane for the structure in Problem 16, Chapter 2. Start with the dynamical matrix $[D]$ and iterate to yield the results. Then, proceed with $[D]^{-1}$ and iterate. Compare the results. Plot the mode shapes.

15. Compute the natural modes and frequencies for the structure of Problem 17, Chapter 2. Use matrix iteration. Check the results by generating the modal columns from the adjoints of $[D - \lambda_r I]$.

16. Write the characteristic equation and compute the roots by plotting an approximate curve $f(\lambda)$ for the structure of Problem 3, Chapter 2. The left side of the characteristic Eq. (3.13) is $f(\lambda)$. The correct roots are obtained when $f(\lambda) = 0$. Set

$$m = J \qquad EI = 1 \qquad l = 1$$

17. Derive generalized mass and stiffness matrices in diagonal form for the structure of Problem 14.

18. Write the equations of motion for the structure of Problem 11, Chapter 2, so that *only one unknown appears in each equation (decoupled form)*. (You may use your results from Problem 4 of this chapter.)

CHAPTER 4

Energy Methods

4.1 Classification of Methods for Determining Natural Modes

In this chapter and in Chapters 5 and 6, we shall consider methods of formulating and solving equations of motion for natural vibrations. In view of the fact that such vibrations never actually occur in physical structures, this may seem to place undue emphasis on the subject. However, knowledge of the natural modes and frequencies of structures is important in considering their response to various kinds of excitation. This is particularly true when structures and force systems are complex and when the excitation is not periodic.

The methods to be considered are classified under three general categories according to the mathematical form of the governing equations. These categories are *energy methods*, *differential equation methods*, and *integral equation methods*. This classification, used by Bisplinghoff, Ashley, and Halfman[1c] is a logical one for the analyst inasmuch as the mathematical methods and, hence, to some extent the techniques of computation are thus distinguished. It must be pointed out, however, that the form of governing equations can be changed quite readily by methods described in many books on applied mathematics. Indeed, we have seen in Chapter 2 that differential equations may be derived from energy forms. The feature that distinguishes the three categories most clearly is the manner in which boundary conditions are handled. These differences will be discussed briefly.

Energy methods are based on the use of one or more of the energy principles of mechanics, e.g., conservation of energy, virtual work, Hamilton's principle, Lagrange's equations, etc. They make use of one or more displacement functions selected somewhat arbitrarily and which approximate the natural mode functions. If the natural modes determined by the analysis are to satisfy the prescribed boundary conditions, it is necessary that the selected function or functions satisfy them. However, the analysis may proceed in a formal way whether or not the boundary conditions are thus satisfied. Therefore, we may say that energy methods of analysis are not concerned with boundary conditions. This is true *only in a formal sense because the significance and accuracy of the solution depends very much upon whether or* not boundary conditions are satisfied.

Differential equation methods require solution of governing differential equations. To obtain unique solutions, the prescribed boundary conditions must be satisfied explicitly. In natural mode vibrations, the frequency equation stems directly from mathematical conditions imposed by considering the boundary conditions. Therefore, it is impossible by these methods to obtain modes which do not satisfy boundary conditions.

Integral equation methods involve the solution of governing integral equations. Kernels[3] of these equations include influence functions (Green's functions) which will, by the usual methods of derivation, satisfy boundary conditions. Solution of the *integral equation does not, however, require consideration of boundary conditions.* Therefore, we may say that they are satisfied implicitly through the use of appropriate influence functions.

Some of the methods of analysis discussed in this and subsequent chapters are exact methods; others are approximate. The distinction to be made here is based primarily on mathematical considerations. Since physical structures are always more or less continuous, exact governing equations must be in the form of differential or integral equations in which the independent space variable or variables may vary continuously. However, in all but certain exceptional cases, solutions *of these equations are not possible and the space variables* are discretized so that *approximate solutions may* be obtained. Thus, differential equations are approximated by difference equations and integrals are approximated by finite sums. In the energy methods an equivalent approximation is made through the use of a finite number of functions in representing a natural mode.

It is important that the analyst gain an understanding of the magnitudes of the approximations which result from the use of various methods on various types of structures. This knowledge will often dictate his choice of method. It should be kept in mind also that all methods of analysis are approximate in an engineering sense. Many factors, such as variations in material properties, dimensional tolerances, etc., cause differences to exist between the physical structure and the ideal structure (or mathematical model) used in analysis. These differences may at times overshadow smaller differences which attend the choice of method of analysis. In the present state of the art, judgment in these matters can be gained only through experience.

4.2 Rayleigh's Method

This method is based on a principle stated by Lord Rayleigh in his famous work *The Theory of Sound*[11] first published in 1877. In the terminology used here Rayleigh's principle may be stated as follows. *In a natural mode of vibration of a conservative system the frequency of the vibration is stationary.* We shall consider this principle and its application to vibration problems in some detail. To do so it is convenient to determine the frequency in terms of the kinetic and potential energies of the system by use of the principle of conservation of energy.

$$T(t) + V(t) = E \qquad \text{(a constant)} \tag{4.1}$$

In natural mode vibrations the displacements vary harmonically with frequency ω. Hence, T and V are also harmonic with frequency 2ω. We may write

$$\begin{aligned} T(t) &= T_{\max}\cos^2\omega t = \frac{T_{\max}}{2}(1 + \cos 2\omega t) \\ V(t) &= V_{\max}\sin^2\omega t = \frac{V_{\max}}{2}(1 - \cos 2\omega t) \end{aligned} \tag{4.2}$$

where we have elected to measure time from an instant when the system passes through its equilibrium position. Kinetic energy is proportional to the square of the velocities. Therefore, in simple harmonic motion it is proportional to the square of the frequency of vibration. Thus, we may write

$$T_{\max} = A\omega^2 \tag{4.3}$$

where A is a constant which depends upon the distribution of mass and upon the mode shape. If we substitute Eq. (4.2) into (4.1) at two different instants in time, namely, when the system passes through its equilibrium position and when it passes through its extreme position, we obtain the equation

$$T_{\max} = V_{\max} \tag{4.4}$$

Using Eq. (4.3) we determine the frequency

$$\omega^2 = \frac{V_{max}}{A} \tag{4.5}$$

Since both V_{max} and A depend upon the mode configuration, we have here an equation that relates the natural frequency with the corresponding natural mode. This equation is basic to the Rayleigh method in which approximations to natural frequencies are computed using mode functions which approximate the natural modes. The method is applied in practice to the first mode for which suitable approximations may easily be made, especially for simple structural elements. As will be seen later it may be difficult to make satisfactory approximations even to first mode configurations for complex structures. The success of the method lies in the stationary behavior of ω^2 in the vicinity of a natural mode because, as shown by Rayleigh,[11] variations in ω^2 with changes in the mode function are of second order in this vicinity. Therefore, accurate estimates of frequency may be made even though the estimated mode functions are in error.

To help fix ideas, let us apply the foregoing equations to the bending vibrations of a slender beam. We shall consider the motion to be horizontal so that the gravity potential may be neglected. Hence, the potential energy V is identical to strain energy U. Kinetic and strain energy expressions for the beam are given in Eqs. (2.84) and (2.85), and the differential equation of motion to which we shall refer later is given in Eq. (2.94). Since the deflection $w(x, t)$ is harmonic in time with angular frequency ω, we may write

$$T_{max} = \tfrac{1}{2}\omega^2 \int_0^l m(x) w^2(x)\, dx \tag{4.6}$$

$$U_{max} = \tfrac{1}{2} \int_0^l EI(x) [w''(x)]^2\, dx \tag{4.7}$$

where $w(x)$ is the maximum amplitude of vibration. In this example Eq. (4.5) becomes

$$\omega^2 = \frac{\int_0^l EI(x)[w''(x)]^2\, dx}{\int_0^l m(x) w^2(x)\, dx} \tag{4.8}$$

In a natural mode, say the rth one, the amplitude function $w(x)$ becomes the rth mode shape function $\Phi_r(x)$. Equation (4.8) then yields the exact frequency in that mode.

$$\omega_r^2 = \frac{\int_0^l EI(x)[\Phi_r''(x)]^2\, dx}{\int_0^l m(x) \Phi_r^2(x)\, dx} \tag{4.9}$$

Inasmuch as a conservative system can undergo free vibrations in only one (or a combination) of its natural modes, one may question the interpretation of Eq. (4.8) in which it has been implied that $w(x)$ may vary from the natural mode functions $\Phi(x)$, and that the frequency changes in consequence of this variation. This question may be clarified by thinking of the varied function as being caused by a constraint on the system. Consider, for example, a beam fixed at both ends undergoing free vibration in its second mode, as shown by the solid line in Fig. 4.1. At the node the amplitude is zero and

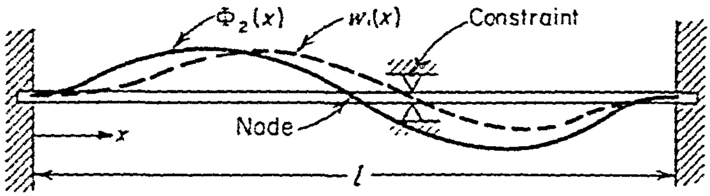


Figure 4.1

the vibration would be unaffected by the insertion of a constraint to prevent motion at that point. However, if a constraint were applied near that point then the mode of vibration would be changed, as indicated by the dashed line. As a result the frequency would also be changed and would be given by Eq. (4.8) if the function $w(x)$ were known.

As has been pointed out it is often possible to estimate functions, especially for the first mode, such that very accurate frequencies are obtained by Rayleigh's method. As an example, we shall consider a simply-supported slender beam of length l, uniform section modulus EI, and mass m per unit length. A rigid body of mass M is attached at the center. This beam is shown in Fig. 4.2. In considering func-

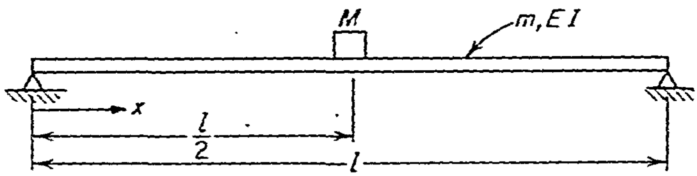


Figure 4.2

tions $w(x)$ suitable as approximations to the first mode $\Phi_1(x)$, two appear as good choices, based on the following logic. It is known that the first mode function for the beam without the center mass is

$$w(x) = \sin \frac{\pi x}{l} \tag{4.10}$$

On the other hand, if the beam itself were massless, and we had only the center mass, the first mode would be obtained exactly by computing the deflections resulting from a statical point force applied at the center. This function is

$$
\begin{aligned}
w(x) &= \frac{x}{l}\left[3 - 4\left(\frac{x}{l}\right)^2\right], && \text{for } 0 \le \frac{x}{l} \le \frac{1}{2} \\
&= \left(1 - \frac{x}{l}\right)\left[3 - 4\left(1 - \frac{x}{l}\right)^2\right], && \text{for } \frac{1}{2} \le \frac{x}{l} \le 1
\end{aligned}
\tag{4.11}
$$

We use these functions to compute approximate values of ω_1 by means of Eq. (4.8). We must, however, consider the kinetic energy of the center mass by adding to the denominator the term $Mw^2(l/2)$. The first function, Eq. (4.10), leads to the frequency

$$\omega^2 = \frac{\pi^4 EI}{2l^3} \frac{1}{\frac{ml}{2} + M} \tag{4.12}$$

As would be expected, the exact frequency is obtained from this equation for the case $M = 0$. Thus

$$\omega^2_{M=0} = \frac{\pi^4 EI}{ml^4} \tag{4.13}$$

The maximum error in frequency given by Eq. (4.12) would be expected when $m = 0$ for which case

$$\omega^2_{m=0} = \frac{\pi^4 EI}{2Ml^3} \tag{4.14}$$

Now, the second function, Eq. (4.11), gives the frequency

$$\omega^2 = \frac{48EI}{l^3} \frac{1}{\frac{17}{35}ml + M} \tag{4.15}$$

Here, the exact frequency is yielded for the case $m = 0$. Thus

$$\omega^2_{m=0} = \frac{48EI}{Ml^3} \tag{4.16}$$

The maximum error is expected when $M = 0$ for which case the frequency is

$$\omega^2_{M=0} = \frac{98.824EI}{ml^4} \tag{4.17}$$

A test of the Rayleigh method is afforded in this example by comparing Eq. (4.17) with (4.13) for $M = 0$, and Eq. (4.14) with (4.16) for $m = 0$. In each case the approximate frequency exceeds the exact value by only 1.5%.

4.3 Stationary Property of the Frequency

From the statement of Rayleigh's principle, as given in the last section, it must follow that a function which, when inserted in Eq. (4.5), renders ω^2 stationary is a natural mode. In this connection it is noted, parenthetically, that if ω^2 is stationary then ω is also stationary, except possibly at zero frequency, because

$$\delta(\omega^2) = 2\omega\,\delta\omega$$

In this section we shall investigate this matter by seeking such a function. For clarity we shall continue to use our example of the slender beam for which the frequency is given by Eq. (4.8). Thus, our problem is defined by the equation

$$\delta(\omega^2) = \delta\frac{\int_0^l EI(x)[w''(x)]^2\,dx}{\int_0^l m(x)w^2(x)\,dx} = 0 \tag{4.18}$$

This is a problem in the calculus of variations similar to that dealt with in Section 2.8. It is known in the literature as a particular type of the Sturm-Liouville problem[3] which is considered in detail in books on applied mathematics. When we carry out the indicated variation of the quotient in Eq. (4.18) the following equation results.*

$$\frac{\left(\int_0^l mw^2\,dx\right)\delta\int_0^l EIw''^2\,dx - \left(\int_0^l EIw''^2\,dx\right)\delta\int_0^l mw^2\,dx}{\left(\int_0^l mw^2\,dx\right)^2} = 0$$

We substitute Eq. (4.8), equate the numerator to zero, and divide out the nonzero integral

$$\int_0^l mw^2\,dx$$

Thus, we come to the equation

$$\delta\int_0^l EIw''^2\,dx - \omega^2\delta\int_0^l mw^2\,dx = 0 \tag{4.19}$$

The variations of the two integrals are obtained by operations similar to those of Section 2.8. When they are carried out, subject to any set of natural boundary conditions on the beam, the following equation is obtained.

$$\int_0^l \{[EIw'']'' - \omega^2 mw\}\,\delta w\,dx = 0 \tag{4.20}$$

Recalling the discussion in Chapter 2 concerning the arbitrariness of δw (except at the boundaries), it is clear that if Eq. (4.20) is to hold for all admissible functions δw, then the quantity in the paren-

* In this and following expressions, $E(x)$, $I(x)$, and $w(x)$ are written as E, I, and w, respectively, for compactness.

theses must vanish. Thus, we are led to the differential equation for the beam already written as Eq. (2.94).

$$[EIw'']'' - \omega^2 mw = 0 \tag{2.94}$$

(The relation $\ddot{w} = -\omega^2 w$ was used in the equation above.) Functions $w(x)$ which satisfy this equation are the natural modes of the beam $\Phi(x)$, as will be shown in the next chapter. Therefore, it follows that the natural mode functions are those which render the frequency stationary, as stated by Rayleigh's principle.

4.4 Variation of ω^2 Between Modes

Although it has been shown that ω^2 is stationary at the natural modes, we have not said anything as yet about the manner in which it varies in the vicinity of the modes other than that the variations are small. It was asserted by Rayleigh,[11] and proved by Temple and Bickley,[12] that at the first mode ω^2 is a minimum. This means that any constrained mode in the vicinity of the first natural mode will yield a frequency higher than the exact first mode frequency. This proved to be true in our example of Section 4.2. It was also shown in References 11 and 12 that for systems having a finite number of degrees of freedom the highest frequency is a maximum. This follows from the proof that any frequency corresponding to a constrained mode must lie between the minimum and maximum natural frequencies of the system. Beyond these statements nothing more can be said about the behavior of the ω^2 function without defining precisely the manner in which the function $w(x)$ is to be varied. In the example to follow, a particular device is used by which the mode shape may be varied smoothly from one natural mode to the next by varying a single parameter p. Thus, it is possible to plot the function $\omega^2(p)$ over any desired range, i.e., over as many natural modes as we wish. This example is included to assist in clarifying this interesting and important subject.

We refer again to Eq. (4.8) which allows us to compute ω^2 corresponding to any function $w(x)$. The Lagrange interpolation formula[9] is used to represent this function in terms of the first n natural modes of the beam; $\Phi_1(x)$, $\Phi_2(x), \ldots, \Phi_n(x)$. Thus

$$w(x) = \sum_{i=1}^{n} \Phi_i(x) \prod_{\substack{k=1 \\ k\neq i}}^{n} (p - p_k) \tag{4.21}$$

The parameter p may be varied arbitrarily over any desired range so that when Eq. (4.21) is inserted into Eq. (4.8), ω^2 may be regarded as a function of p. The Lagrange interpolation formula has the property which makes $w(x)$ proportional to any one of the natural modes when p takes on a value corresponding to that mode. For example,

$$w(x) = \Phi_r(x) \prod_{\substack{k=1 \\ k \neq r}}^{n} (p_r - p_k) \qquad \text{when } p = p_r$$

The values of the parameter at the natural modes are ordered such that $p_1 < p_2 < p_3 < \ldots < p_n$.

Omitting details of the calculation involved in substituting Eq. (4.21) into (4.8), the following equation is obtained which gives ω^2 as a function of p.

$$\omega^2(p) = \frac{\sum_{i=1}^{n} K_i \prod_{\substack{k=1 \\ k \neq i}}^{n} (p - p_k)^2}{\sum_{i=1}^{n} M_i \prod_{\substack{k=1 \\ k \neq i}}^{n} (p - p_k)^2} \tag{4.22}$$

where, for a slender beam

$$K_i = \int_0^l EI(x)[\Phi_i''(x)]^2\, dx$$

$$M_i = \int_0^l m(x)\Phi_i^2(x)\, dx$$

We note, first, that when p takes on a natural mode value p_r, $\omega^2(p)$ takes on the value ω_r^2,

$$\omega^2(p_r) = \frac{K_r \prod_{\substack{k=1 \\ k \neq r}}^{n} (p_r - p_k)^2}{M_r \prod_{\substack{k=1 \\ k \neq r}}^{n} (p_r - p_k)^2}$$

$$= \frac{K_r}{M_r} = \omega_r^2$$

The last result is obtained in consequence of Eq. (3.79). Secondly, it is found upon differentiating Eq. (4.22) with respect to p that the derivative vanishes at $p = p_r$. Thus, the function satisfies the requirement of Rayleigh's principle that its value be stationary at the natural modes.

To carry the example further we shall consider the beam to be uniform and simply-supported for which case

$$K_i = i^4 \frac{\pi^4 EI}{2l^3}$$

$$M_i = \frac{1}{2} ml \qquad \text{for } i = 1, 2, 3, \ldots, n$$

For this case Eq. (4.22) becomes

$$\omega^2(p) = \omega_1^2 \frac{\sum_{i=1}^{n} i^4 \prod_{\substack{k=1 \\ k \neq i}}^{n} (p - p_k)^2}{\sum_{i=1}^{n} \prod_{\substack{k=1 \\ k \neq i}}^{n} (p - p_k)^2} \tag{4.23}$$

where ω_1^2 is the square of the first mode frequency

$$\omega_1^2 = \frac{\pi^4 EI}{ml^4}$$

For numerical computation it is convenient to let p_k take on integral values. It is sufficient for our purposes to restrict these values to $p_k = 1, 2, 3, 4, 5$, thus considering only the first five modes of the beam. The computed values of the ratio $\omega^2(p)/\omega_1^2$ are plotted on the solid

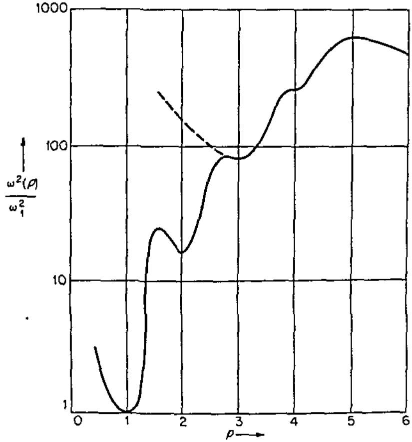


Figure 4.3

curve in Fig. 4.3. It is readily seen that the function satisfies the conditions discussed earlier. In addition, it is seen that all of the values of the natural frequencies are minima in this example except at $p = 5$ for which ω_5^2 is a maximum, as expected.

In Reference 12 it is shown that if a function can be chosen so that it is orthogonal to the first r natural modes, then the $(r+1)$th frequency is the minimum one characteristic of the system. This is equivalent to removing the first r modes by the process of "sweeping," as discussed in Section 3.6. To show the effect of sweeping in the present example, computations were carried out using Eq. (4.23), but removing the first two values of p_k; thus, $p_k = 3, 4, 5$. The results below $p = 3$ are plotted on the dashed-line curve in Fig. 4.3 which shows clearly that ω_3^2 becomes the minimum characteristic value or natural frequency.

4.5 The Rayleigh-Ritz Method

This method, which may be regarded as an extension of Rayleigh's method, is due to Ritz.[13] It is based on the premise that a number of functions may be superposed to provide a closer approximation to the exact natural modes than can be had using a single function as in Rayleigh's method. If the functions are suitably chosen, it provides not only a closer approximation to the exact first mode frequency but it allows the calculation of higher mode shapes and frequencies. An arbitrary number of functions may be used and, in general, it may be said that a larger number leads to more accurate results at the expense of greater labor in computation. If n functions $\phi_1(x)$, $\phi_2(x), \ldots, \phi_n(x)$ are chosen, the deflection $w(x)$ is represented as

$$w(x) = C_1\phi_1(x) + C_2\phi_2(x) + \cdots + C_n\phi_n(x) \tag{4.24}$$

The coefficients $C_1, C_2, \ldots, C_n$ are to be determined so that the superposed functions provide the best possible approximations to the natural modes. The Ritz method is based on the argument that these "best" approximations are obtained by adjusting the coefficients so that the frequency is made stationary at the natural modes, in conformity with Rayleigh's principle. To do this Eq. (4.24) is inserted in the Rayleigh frequency equation [Eq. (4.8) for the slender beam] and the resulting expression is partially differentiated with respect to each of the coefficients. These partial derivatives are set to zero to form a set of n equations as follows:

$$\left.\begin{aligned} \frac{\partial \omega^2}{\partial C_1} &= 0 \\ \frac{\partial \omega^2}{\partial C_2} &= 0 \\ &\vdots \\ \frac{\partial \omega^2}{\partial C_n} &= 0 \end{aligned}\right\} \tag{4.25}$$

This is a set of linear, homogeneous algebraic equations in the C's containing the undetermined number ω^2. The set defines an eigenvalue problem identical to that discussed in Chapter 3 and may be solved by the same techniques. In this case each of the n eigenvectors contains n components in the C's. For example, the rth eigenvector would contain the components $C_1^{(r)}, C_2^{(r)}, \ldots, C_n^{(r)}$. These components, when inserted into Eq. (4.24), determine the "best" approximation to the rth natural mode. Thus

$$\Phi_r(x) = C_1^{(r)}\phi_1(x) + C_2^{(r)}\phi_2(x) + \cdots + C_n^{(r)}\phi_n(x) \tag{4.26}$$

The success of the Rayleigh-Ritz method depends very much on the choice of functions. Therefore, it is important to discuss this subject. Before doing so, however, the method will be applied to the slender beam problem because it is convenient to discuss the choice of functions in connection with this problem. The eigenvalue problem for the beam can be formulated as discussed above and this has been done in detail in Reference 15. However, as is so often the case in analysis, there are other methods by which the same result can be obtained, and one of these is by use of Lagrange equations. We shall use this method in the following development and, although the reasoning does not follow that of Ritz, we shall classify it as a Rayleigh-Ritz method because the results are identical.

In using Lagrange's equations we are concerned with time-dependent energy functions. Hence, it will be necessary to consider the time-dependent deflection function $w(x, t)$. In a manner analogous to Eq. (4.24) this function is approximated by space-dependent functions $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ and time-dependent generalized displacement coordinates $q_1(t), q_2(t), \ldots, q_n(t)$. Thus,

$$\begin{aligned} w(x, t) &= \phi_1(x)q_1(t) + \phi_2(x)q_2(t) + \cdots + \phi_n(x)q_n(t) \\ &= \sum_{i=1}^{n} \phi_i(x)q_i(t) \end{aligned} \tag{4.27}$$

This function is inserted into the expressions for kinetic and strain energies of the beam. The kinetic energy includes that of translation only in the case of a slender beam. It is given in Eq. (2.84) and repeated here for convenience.

$$T(t) = \tfrac{1}{2}\int_{x=0}^{l} m(x)\dot{w}^2(x, t)\, dx \tag{4.28}$$

As in earlier chapters, the dot is used to denote partial differentiation with respect to time. The strain energy includes only that of flexure and is given by Eq. (2.85), repeated here for convenience.

$$U(t) = \tfrac{1}{2}\int_{x=0}^{l} EI(x)[w''(x, t)]^2\, dx \tag{4.29}$$

As before, the prime denotes partial differentation with respect to x. Since there are no external forces applied to the beam except at immoveable constraints where they do no work, and since the beam is considered to be perfectly elastic, the rth Lagrange equation has the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial U}{\partial q_r} = 0$$

The first term is developed as follows, by differentiating under the integral of Eq. (4.28).

$$\frac{\partial T}{\partial \dot{q}_r} = \int_0^l m(x)\,\dot{w}(x,t)\,\frac{\partial \dot{w}(x,t)}{\partial \dot{q}_r}\,dx$$

Substituting Eq. (4.27) we obtain

$$\frac{\partial T}{\partial \dot{q}_r} = \int_0^l m(x)\left[\sum_{i=1}^{n} \phi_i(x)\,\dot{q}_i\right]\phi_r(x)\,dx$$

Upon interchanging the order of summation and integration this equation becomes

$$\frac{\partial T}{\partial \dot{q}_r} = \sum_{i=1}^{n} \dot{q}_i \int_0^l m(x)\,\phi_r(x)\,\phi_i(x)\,dx$$

$$= \sum_{i=1}^{n} m_{ri}\dot{q}_i$$

in which

$$m_{ri} = \int_0^l m(x)\,\phi_r(x)\,\phi_i(x)\,dx \tag{4.30}$$

is the generalized mass. (See Section 2.5, Eq. 2.42.) Differentiating with respect to time, we obtain the first term in the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) = \sum_{i=1}^{n} m_{ri}\ddot{q}_i \tag{4.31}$$

The second term is equal to zero because the kinetic energy is a function of $\dot{q}$ only. The third term is obtained by differentiating under the integral of Eq. (4.29).

$$\frac{\partial U}{\partial q_r} = \int_0^l EI(x)\,w''(x,t)\,\frac{\partial w''(x,t)}{\partial q_r}\,dx$$

Substitution of Eq. (4.27), followed by an interchange in order of summing and integrating, leads to the equation

$$\frac{\partial U}{\partial q_r} = \sum_{i=1}^{n} q_i \int_0^l EI(x)\,\phi_r''(x)\,\phi_i''(x)\,dx$$

Here, the integral is defined as a generalized stiffness k_{ri}. (See Chapter 1, Eq. 1.140, and Problem 1.12.)

$$k_{ri} = \int_0^l EI(x)\,\phi_r''(x)\,\phi_i''(x)\,dx \tag{4.32}$$

The third term is written, accordingly, as

$$\frac{\partial U}{\partial q_r} = \sum_{i=1}^{n} k_{ri} q_i \tag{4.33}$$

The rth Lagrange equation follows from Eqs. (4.31) and (4.33).

$$\sum_{i=1}^{n} m_{ri} \ddot{q}_i + \sum_{i=1}^{n} k_{ri} q_i = 0 \tag{4.34}$$

In the present problem we are concerned with natural vibrations, in which case the q's all vary harmonically in time with the same angular frequency ω. Thus

$$\ddot{q}_i = -\omega^2 q_i$$

In this case Eq. (4.34) takes the form

$$-\omega^2 \sum_{i=1}^{n} m_{ri} q_i + \sum_{i=1}^{n} k_{ri} q_i = 0 \tag{4.35}$$

Since r may have any value from 1 to n, inclusive, it is clear that n equations of the form (4.35) exist. The set of equations may be given in matrix form by

$$\omega^2[m]\{q\} = [k]\{q\} \tag{4.36}$$

This equation is identical in form to Eq. (3.5) so that all of the discussions of Chapter 3 concerning methods of solution, orthogonality of the eigenvectors, etc., are applicable to it. It is necessary to note, however, that the matrices $[m]$ and $[k]$ are symmetric as may be deduced from Eqs. (4.30) and (4.32).

Solution of Eq. (4.36) yields a set of approximations to the natural frequencies $\omega_1, \omega_2, \ldots, \omega_n$. It also yields a set of eigenvectors $\{q^{(1)}\}$, $\{q^{(2)}\}, \ldots, \{q^{(n)}\}$, which, when substituted into Eq. (4.27), give a set of approximations to the natural modes. Thus, the rth natural mode is approximated by

$$\Phi_r(x) = \sum_{i=1}^{n} \phi_i(x) q_i^{(r)} \tag{4.37}$$

In this equation we have regarded the q's as constants, which indeed they appear to be from the solution of the eigenvalue problem. Since they are by definition harmonic functions of time, these constants can be regarded as amplitudes of the harmonic functions. In natural modes we obtain only the relative amplitudes of vibration. Thus, the q's belonging to an eigenvector pertaining to a given mode give the relative contributions of their corresponding functions $\phi(x)$ to that mode. These relative contributions are the same for every instant in time.

4.6 Application of the Rayleigh-Ritz Method to a Beam With Nonuniform Mass Distribution

The Rayleigh-Ritz method is particularly useful in treating continuous structures with nonuniform mass and/or stiffness properties. Very good approximations to the natural modes may often be obtained using modes applicable to similar structures with uniform properties, many of which may be found in the literature.[11]

As an example, the method will be applied to a simply-supported beam with nonuniform mass. For simplicity, the stiffness is considered to be uniform. The beam is symmetrical with respect to its center and only the symmetrical modes of vibration will be sought. As will be shown in Section 4.8, this may be done by introducing only symmetrical functions into the analysis. The beam is shown in Fig. 4.4. The mass $m(x)$ per unit length is taken as

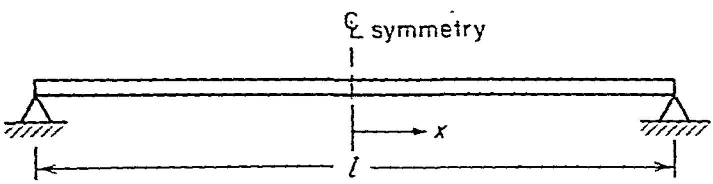


Figure 4.4

$$m(x) = m_0 \cos \frac{\pi x}{l}$$

The stiffness EI is considered to be a constant. Three functions are chosen, corresponding to the first three symmetrical modes of a simply-supported uniform beam of the same length. Thus, in the expression

$$w(x) = \phi_1(x) q_1 + \phi_2(x) q_2 + \phi_3(x) q_3$$

the functions are

$$\phi_1(x) = \cos \frac{\pi x}{l}$$

$$\phi_2(x) = \cos \frac{3\pi x}{l}$$

$$\phi_3(x) = \cos \frac{5\pi x}{l}$$

The generalized mass parameters m_{ij} are computed from Eq. (4.30) and are listed below.

$$m_{11} = \frac{4m_0 l}{3\pi}$$

$$m_{12} = m_{21} = \frac{4m_0 l}{15\pi}$$

$$m_{13} = m_{31} = -\frac{4m_0 l}{105\pi}$$

$$m_{22} = \frac{36m_0 l}{35\pi}$$

a given structure under one set of boundary constraints will be entirely different from those of the same structure under another set of constraints. Therefore, it is evident that complete confidence in the results of a Rayleigh-Ritz analysis requires functions which meet the requirements at the boundaries. In practice it is often inconvenient to satisfy all of these requirements. For example, shear discontinuities in beams with concentrated masses present difficulties, and these discontinuities are often neglected with good results. In this case the boundary (or intermediate) conditions relate to the third derivatives of the functions. It may be said, in general, that it becomes progressively more hazardous to ignore constraints on the second and first derivatives and on the displacement itself.

A second consideration in the choice of functions is brought to light by considering the Ritz process as one by which natural modes are synthesized from the chosen functions. One may think of this process as a generalized Fourier synthesis, although the analogy is not complete because the functions $\phi(x)$ are not necessarily orthogonal. It is well known that in the representation of a periodic function by a Fourier series[16] the series must contain functions having periods or wavelengths ranging from the longest to the shortest of those pertaining to all the harmonics. Rigorously interpreted, this means that the series must be infinite. Use of a finite number of terms results in an approximate representation and the accuracy improves with an increasing number of terms. In practice, a finite number of natural modes is sought. To synthesize the highest of these modes there must be included in the series functions having the general characteristics of all them. Indeed, functions corresponding to still higher modes should be included to improve accuracy. For example, in the beam problem of the last section we sought the first two modes and used functions having characteristics of the first three. In this problem the first mode is characterized by zero nodes, the second by two, and the third by four because only symmetrical modes are used.

A criticism of the Rayleigh-Ritz method, as well as other energy methods, is that a good choice of functions often requires judgment and experience. Therefore, it is not entirely amenable to treatment by routine computing procedures.

4.8 Symmetrical and Antisymmetrical Modes

If a structure is symmetrical with respect to at least one plane of symmetry the assumed modes ϕ_i can be chosen in such a way as to separate the natural modes into two groups: symmetrical modes and antisymmetrical modes. If symmetrical functions ϕ_i are chosen, the solution leads to symmetrical modes and, similarly, antisymmetrical functions lead to antisymmetrical modes. To illustrate this, let us return to the beam problem and consider a beam which is symmetrical with respect to a plane perpendicular to it through its center, as in the example of Section 4.6. If we measure x from the plane of symmetry, this property can be specified by writing

$$m(x) = m(-x) \tag{4.38}$$

and

$$EI(x) = EI(-x) \tag{4.39}$$

Let us determine the natural modes in terms of n assumed functions, s of which are symmetrical and $n - s$ antisymmetrical. Order the assumed modes thus

$$\text{Symmetrical} \quad \phi_i(x) = \phi_i(-x) \qquad \text{for } 1 \leq i \leq s \tag{4.40}$$

$$\text{Antisymmetrical} \quad \phi_i(x) = -\phi_i(-x) \quad \text{for } s < i \leq n \tag{4.41}$$

It is easy to verify the following equations relating the derivatives of symmetrical and antisymmetrical functions.

For $\phi(x)$ symmetrical or even — For $\phi(x)$ antisymmetrcal or odd

$$\left.\begin{aligned} \phi(-x) &= \phi(x) \\ \phi'(-x) &= -\phi'(x) \\ \phi''(-x) &= \phi''(x) \end{aligned}\right\} \tag{4.42}$$

$$\left.\begin{aligned} \phi(-x) &= -\phi(x) \\ \phi'(-x) &= \phi'(x) \\ \phi''(-x) &= -\phi''(x) \end{aligned}\right\} \tag{4.43}$$

The n natural modes of the beam are computed from Eq. (4.36) where we determine the m_{ij} and k_{ij} from Eqs. (4.30) and (4.32). In these equations the integrals are considered in two parts. Thus

$$m_{ij} = \int_{-l/2}^{0} m(x)\phi_i(x)\phi_j(x)\,dx + \int_{0}^{+l/2} m(x)\phi_i(x)\phi_j(x)\,dx \tag{4.44}$$

$$k_{ij} = \int_{-l/2}^{0} EI(x)\phi_i''(x)\phi_j''(x)\,dx + \int_{0}^{+l/2} EI(x)\phi_i''(x)\phi_j''(x)\,dx \tag{4.45}$$

Consider the first integral on the right side of Eq. (4.44). We make a transformation of the variable $x = -\xi$. Then

$$\begin{aligned} \int_{-l/2}^{0} m(x)\phi_i(x)\phi_j(x)\,dx &= -\int_{l/2}^{0} m(-\xi)\phi_i(-\xi)\phi_j(-\xi)\,d\xi \\ &= \int_{0}^{l/2} m(\xi)\phi_i(-\xi)\phi_j(-\xi)\,d\xi \end{aligned}$$

Inasmuch as we may designate the variable of integration by any letter we choose, m_{ij} can be written as

$$m_{ij} = \int_{0}^{+l/2} m(x)\phi_i(-x)\phi_j(-x)\,dx + \int_{0}^{+l/2} m(x)\phi_i(x)\phi_j(x)\,dx$$

Similarly

$$k_{ij} = \int_{0}^{+l/2} EI(x)\phi_i''(-x)\phi_j''(-x)\,dx + \int_{0}^{+l/2} EI(x)\phi_i''(x)\phi_j''(x)\,dx$$

Now we see that if $\phi_i(x)$ and $\phi_j(x)$ are both either symmetrical or antisymmetrical, the two integrals will add to give

$$m_{ij} = 2\int_0^{+l/2} m(x)\phi_i(x)\phi_j(x)\,dx \tag{4.46}$$

$$k_{ij} = 2\int_0^{+l/2} EI(x)\phi_i''(x)\phi_j''(x)\,dx \tag{4.47}$$

On the other hand, if one of the two functions is symmetrical and the other antisymmetrical, the two integrals sum to zero.

$$m_{ij} = 0 \tag{4.48}$$

$$k_{ij} = 0 \tag{4.49}$$

Considering the matrices $[m]$ and $[k]$, it is seen that all of the elements m_{ij} and k_{ij} for which

$$\left.\begin{matrix} 1 \le i \le s \\ 1 \le j \le s \end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix} s < i \le n \\ s < j \le n \end{matrix}\right.$$

are given by Eqs. (4.46) and (4.47). All the elements for which

$$\left.\begin{matrix} 1 \le i \le s \\ s < j \le n \end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix} s < i \le n \\ 1 \le j \le s \end{matrix}\right.$$

are given by Eqs. (4.48) and (4.49), and hence are zero.

The two matrices appear as

$$[m] = \left[\begin{array}{ccc|ccc} m_{11} & \cdots & m_{1s} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ m_{s1} & \cdots & m_{ss} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & m_{(s+1)(s+1)} & \cdots & m_{(s+1)n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & m_{n(s+1)} & \cdots & m_{nn} \end{array}\right]$$

$$[k] = \left[\begin{array}{ccc|ccc} k_{11} & \cdots & k_{1s} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ k_{s1} & \cdots & k_{ss} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & k_{(s+1)(s+1)} & \cdots & k_{(s+1)n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & k_{n(s+1)} & \cdots & k_{nn} \end{array}\right]$$

Consequently, the set of equations given in matrix form by Eq. (4.36) separates into two sets.

$$\omega^2[m]_{\text{symm}}\{q\} = [k]_{\text{symm}}\{q\} \tag{4.50}$$

$$\omega^2[m]_{\text{antisymm}}\{q\} = [k]_{\text{antisymm}}\{q\} \tag{4.51}$$

Here, $[m]_{\text{symm}}$ and $[k]_{\text{symm}}$ are square matrices of order s whose elements are derived from the symmetrical function $\phi(x) = \phi(-x)$ by use of Eqs. (4.46) and (4.47). The matrices $[m]_{\text{antisymm}}$ and $[k]_{\text{antisymm}}$ are square matrices of order $n - s$ whose elements are obtained from the same equations using the antisymmetrical functions $\phi(x) = -\phi(-x)$. Solution of Eq. (4.50) leads to symmetrical natural modes and the corresponding frequencies. Solution of Eq. (4.51) leads to antisymmetrical modes.

It may be readily verified that the same results apply to structures other than beams. For example, one might consider a problem in torsional vibrations in which the strain energy depends upon the first derivatives of the deflection. Here, the generalized stiffness coefficients are given by

$$k_{ij} = \int_0^l GJ(x)\,\phi_i'(x)\,\phi_j'(x)\,dx \tag{4.52}$$

which, for a symmetrical structure leads to

$$k_{ij} = \int_0^{+l/2} GJ(x)\,\phi_i'(-x)\,\phi_j'(-x)\,dx + \int_0^{+l/2} GJ(x)\,\phi_i'(x)\,\phi_j'(x)\,dx \tag{4.53}$$

Substitution of the first derivatives given in Eqs. (4.42) and (4.43) leads to the same results as obtained for the beam problem.

4.9 Coupled Natural Modes: Bending-Torsion Modes of a Beam

Because of the complexities of geometrical configuration and mass distribution of actual structures their natural modes seldom involve simple bending, torsion, etc. Usually natural vibrations involve the coupling of these simple motions. The purpose of this section is to show how such coupled natural modes are determined by the Rayleigh-Ritz energy method. This is most clearly done by an example. We shall choose a beam in which the mass centers do not coincide with the elastic axis, thus leading to coupling of bending and torsion.

Consider a beam defined by two axes; a straight *elastic axis* which lies on the x axis, and an *inertial axis* at distance $e(x)$ from it. The *elastic axis* is the locus of the shear centers,[1] and the *inertial axis* is the locus of the centers of mass elements cut normal to x. We wish to find the natural modes and frequencies of vibration of this beam

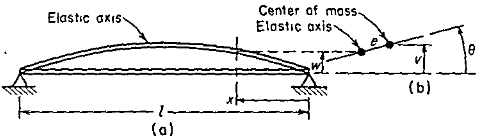


Figure 4.6

in those modes where the motion is normal to the plane of the two axes. The deflected beam is shown in Fig. 4.6 where (a) shows the deflected elastic axis and (b) shows a translated and rotated chordwise section at x. The chordwise sections are considered to be rigid, i.e., straight lines perpendicular to x and passing through the elastic and inertial axes remain straight as the beam deflects.

The displacements w and v at a given section are related through the rotation θ of that section by the equation

$$v = w + e\theta \tag{4.54}$$

We shall again use the Lagrange equations to formulate the eigenvalue problem, and must begin with the expressions for kinetic and strain energy. Kinetic energy will include that due to translation of the mass centers and that due to torsional rotation. Rotatory energy, i.e., kinetic energy of rotation in the plane of bending, is neglected in this example.

$$T = \tfrac{1}{2}\int_0^l m\dot{v}^2\,dx + \tfrac{1}{2}\int_0^l m\rho^2\dot{\theta}^2\,dx \tag{4.55}$$

where

m = mass per unit length of beam

ρ = radius of gyration of mass element mdx about the inertial axis

Strain energy includes the energy of flexure and the energy of torsion. Energy associated with lateral shear deflection of the beam is neglected.

$$U = \tfrac{1}{2}\int_0^l EIw''^2\,dx + \tfrac{1}{2}\int_0^l GJ\theta'^2\,dx \tag{4.56}$$

where

EI = section bending stiffness modulus

GJ = section torsional stiffness modulus

The energy expressions involve all three displacements (v, w, θ) which are related through the equation of constraint (4.54). We choose to express v in terms of the other two and rewrite the equation for kinetic energy as

$$T = \tfrac{1}{2}\int_0^l m(\dot{w} + e\dot{\theta})^2\,dx + \tfrac{1}{2}\int_0^l m\rho^2\dot{\theta}^2\,dx$$

This reduces to the following form which clearly shows the nature of the bending-torsion coupling

$$T = \tfrac{1}{2}\int_0^l m\dot{w}^2\,dx + \tfrac{1}{2}\int_0^l m(\rho^2 + e^2)\dot{\theta}^2\,dx + \int_0^l me\dot{w}\dot{\theta}\,dx \tag{4.57}$$

Presence of the cross product $\dot{w}\dot{\theta}$ in this expression signifies dynamic coupling. Zero static coupling is indicated by Eq. (4.56). It is easy to see that use of the coordinates v, θ would lead to static coupling without dynamic coupling. As will be seen later, the choice of w and θ used in this example leads to simpler computation if the matrix $[k]$ is to be inverted.

We come now to the definition of generalized coordinates and the choice of mode functions. Two classes of functions must be chosen—those representing bending deflections and those representing torsional deflections. An arbitrary number of each may be selected. We shall define bending in terms of the first m functions $\phi_1(x), \phi_2(x), \ldots, \phi_m(x)$ and torsion in terms of the last $n - m$ functions $\phi_{m+1}(x), \ldots, \phi_n(x)$.

In terms of the generalized coordinates $q_1, q_2, \ldots, q_n$ we write

$$w(x, t) = \sum_{i=1}^{m} \phi_i(x)q_i(t) \tag{4.58}$$

$$e_0\theta(x, t) = \sum_{i=m+1}^{n} \phi_i(x)q_i(t) \tag{4.59}$$

In the last equation we have included an arbitrary length e_0 so that all of the q's have the same dimension. This is a matter of convenience since it permits the elements of the $[m]$ and $[k]$ matrices to have consistent dimensions throughout. A specific choice of mode functions would require consideration of boundary conditions at this point. However, it will be more instructive to postpone such a choice and to formulate the equations of motion in terms of an arbitrary set of functions.

Returning to the Lagrange equations we form the rth one. The following terms are developed.

$$\frac{\partial T}{\partial \dot{q}_r} = \int_0^l m\dot{w}\frac{\partial \dot{w}}{\partial \dot{q}_r}\,dx + \int_0^l m(\rho^2 + e^2)\dot{\theta}\frac{\partial \dot{\theta}}{\partial \dot{q}_r}\,dx + \int_0^l me\dot{w}\frac{\partial \dot{\theta}}{\partial \dot{q}_r}\,dx + \int_0^l me\dot{\theta}\frac{\partial \dot{w}}{\partial \dot{q}_r}\,dx$$

Evaluation of the above integrals depends upon whether r is smaller or greater than m. For $1 \leq r \leq m$, the following holds.

$$\frac{\partial \dot{w}}{\partial \dot{q}_r} = \phi_r(x)$$

and

$$\frac{\partial \dot{\theta}}{\partial \dot{q}_r} = 0$$

For these values of r we obtain the following expression upon substitution of the above and Eqs. (4.58) and (4.59) followed by an interchange in order of summation and integration.

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_r} &= \sum_{i=1}^{m} \dot{q}_i \int_0^l m\phi_r\phi_i \, dx + \sum_{i=m+1}^{n} \dot{q}_i \int_0^l \frac{me}{e_0} \phi_r\phi_i \, dx \\ &= \sum_{i=1}^{m} m_{ri}\dot{q}_i + \sum_{i=m+1}^{n} (me)_{ri}\dot{q}_i \end{aligned} \tag{4.60}$$

where

$$m_{ri} = \int_0^l m\phi_r\phi_i \, dx \tag{4.61}$$

$$(me)_{ri} = \int_0^l \frac{me}{e_0} \phi_r\phi_i \, dx \tag{4.62}$$

Following the same procedure for $m < r \leq n$, it is seen that

$$\frac{\partial \dot{w}}{\partial \dot{q}_r} = 0 \qquad \text{and} \qquad \frac{\partial \dot{\theta}}{\partial \dot{q}_r} = \frac{1}{e_0}\phi_r$$

Then

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_r} &= \sum_{i=m+1}^{n} \dot{q}_i \int_0^l m \frac{\rho^2 + e^2}{e_0^2} \phi_r\phi_i \, dx \\ &\quad + \sum_{i=1}^{m} \dot{q}_i \int_0^l \frac{me}{e_0} \phi_r\phi_i \, dx \\ &= \sum_{i=1}^{m} (me)_{ri}\dot{q}_i + \sum_{i=m+1}^{n} I_{ri}\dot{q}_i \end{aligned} \tag{4.63}$$

where

$$I_{ri} = \int_0^l m \frac{\rho^2 + e^2}{e_0^2} \phi_r\phi_i \, dx \tag{4.64}$$

Next, we form the derivative $\partial U/\partial q_r$ using Eq. (4.56)

$$\frac{\partial U}{\partial q_r} = \int_0^l EIw'' \frac{\partial w''}{\partial q_r} \, dx + \int_0^l GJ\theta' \frac{\partial \theta'}{\partial q_r} \, dx$$

Again, two sets of values are obtained depending upon r. Following the same procedure the following equations are obtained.

For $1 \leq r \leq m$

$$\frac{\partial U}{\partial q_r} = \sum_{i=1}^{m} k_{b_{ri}} q_i \tag{4.65}$$

For $m < r \leq n$

$$\frac{\partial U}{\partial q_r} = \sum_{i=m+1}^{n} k_{t_{ri}} q_i \tag{4.66}$$

where

$$k_{b_{ri}} = \int_0^l EI \phi_r'' \phi_i'' \, dx \tag{4.67}$$

$$k_{t_{ri}} = \int_0^l \frac{GJ}{e_0^2} \phi_r' \phi_i' \, dx \tag{4.68}$$

The rth Lagrange equation is written below for each case.

For $1 \leq r \leq m$

$$\sum_{i=1}^{m} m_{ri} \ddot{q}_i + \sum_{i=m+1}^{n} (me)_{ri} \ddot{q}_i + \sum_{i=1}^{m} k_{b_{ri}} q_i = 0$$

For $m < r \leq n$

$$\sum_{i=1}^{m} (me)_{ri} \ddot{q}_i + \sum_{i=m+1}^{n} I_{ri} \ddot{q}_i + \sum_{i=m+1}^{n} k_{t_{ri}} q_i = 0$$

The entire set of equations can be compressed into a single matrix equation which, for natural vibrations, is identical in form with Eq. (4.36). The mass matrix is of order $n \times n$ and its composition may be deduced from the above equations.

$$[m] = \left[\begin{array}{ccc|ccc} m_{11} & \cdots & m_{1m} & (me)_{1(m+1)} & \cdots & (me)_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ m_{m1} & \cdots & m_{mm} & (me)_{m(m+1)} & \cdots & (me)_{mn} \\ \hline (me)_{(m+1)1} & \cdots & (me)_{(m+1)m} & I_{(m+1)(m+1)} & \cdots & I_{(m+1)n} \\ \vdots & & \vdots & \vdots & & \vdots \\ (me)_{n1} & \cdots & (me)_{nm} & I_{n(m+1)} & \cdots & I_{nn} \end{array}\right] \tag{4.69}$$

The matrix is partitioned as shown to make the fact clear that the submatrix in the 11 position (upper-left corner) is of order $m \times m$ and is identical with the mass matrix for uncoupled bending. Also, the submatrix in the 22 position (lower-right corner), which is of order $(n - m) \times (n - m)$, is identical with the mass matrix for uncoupled torsion. From Eqs. (4.61) and (4.64) it is clear that these two submatrices are symmetric. The submatrices in positions 12 and 21 contain the coupling elements $(me)_{ij}$ given by Eq. (4.62). From property of symmetry it is seen that one of these two submatrices is the transpose of the other. Therefore, the complete matrix $[m]$ is symmetric.

The $[k]$ matrix is also of order $n \times n$ and has the form

$$[k] = \left[\begin{array}{ccc|ccc} k_{b_{11}} & \cdots & k_{b_{1m}} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ k_{b_{m1}} & \cdots & k_{b_{mm}} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & k_{t_{(m+1)(m+1)}} & \cdots & k_{t_{(m+1)n}} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & k_{t_{n(m+1)}} & \cdots & k_{t_{nn}} \end{array}\right] \tag{4.70}$$

The submatrix in the upper-left corner contains elements obtained from Eq. (4.67) and pertains to uncoupled bending. Similarly, the submatrix in the lower-right corner pertains to uncoupled torsion as seen from the nature of its elements in Eq. (4.68). The remaining two submatrices are null matrices, as is expected, for zero static coupling between bending and torsion.

Determination of the coupled natural modes and frequencies of the beam now requires solution of Eq. (4.36) where matrices $[m]$ and $[k]$ are given by Eqs. (4.69) and (4.70), respectively. If the solution is to proceed by matrix iteration and the lower modes are sought, then the construction of the dynamical matrix involves $[k]^{-1}$. In this case the choice of coordinates leading to the uncoupled form of $[k]$ is a good one because the matrix is more easily inverted. If, on the other hand, the higher modes are sought, it is easier to choose coordinates leading to the uncoupled form for $[m]$ since in that case $[m]^{-1}$ is required.

Note that if the distance $e(x)$ is zero for all values of x, i.e., the inertial and elastic axes coincide, then the matrix $[m]$ also has the uncoupled form. In this case the eigenvalue problem separates into two independent ones in uncoupled bending and torsion.

In choosing the functions $\phi(x)$ for both bending and torsion, the conditions discussed in Section 4.7 must be considered. Among other things, boundary conditions in torsion as well as in bending should be satisfied. It is often useful to choose for these functions the uncoupled natural modes in bending and torsion where these are known or may be readily found.

4.10 Formulation of the Eigenvalue Problem for Unconstrained or Partially-Constrained Structures by use of Stiffness Influence Coefficients

In this section we shall consider the free vibrations of structures which are unconstrained or partially constrained so that they may move as rigid structures without undergoing deformation. A well known example is an airplane in flight which is completely unconstrained so that as a rigid body it has six degrees of freedom. Other examples may be found in which the structure is partially constrained so that it may have one or more, but less than six, degrees of freedom as a rigid body. Free vibration of structures necessarily involves deformation since strain energy is present. Therefore, selected mode functions to be used in determining the natural modes by energy methods *may* include functions which describe the possible rigid-body motions, and *must* include functions which describe the deformation. Thus, the total motion of vibration may be considered as a superposition of rigid-body motion and motion of deformation. Mode functions which describe the rigid-body motion are called *rigid-body modes*. These modes do not contribute to the strain energy of the motion. Hence, stiffness coefficients associated with them are zero.

In the case of a completely unconstrained structure there are no constraints which may impose forces on the structure. Hence, in free vibration the structure is completely force free. From impulse-momentum considerations it follows that linear and angular momenta must be conserved. In natural vibrations the complete structure comes to rest twice during each cycle, therefore these momenta which are clearly zero at those instants must be zero throughout the complete cycle. There are, therefore, six equations of linear and angular momentum which must be satisfied. If the structure is partially constrained there are one or more, but less than six, of these equations.

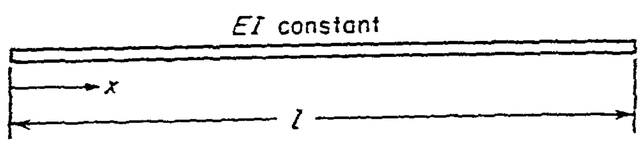


Figure 4.7

The manner in which rigid-body modes and the momentum equations may be handled in formulating the eigenvalue problem is illustrated in the following example. We shall consider the natural modes of a free-free beam having a uniform stiffness distribution but a nonuniform mass distribution. The beam is shown in Fig. 4.7. The mass per unit length is given by

$$m(x) = \frac{2}{3}\frac{M}{l}\left(1 + \frac{x}{l}\right) \tag{4.71}$$

where M = total mass of beam.

We shall be concerned with transverse motion of the beam in the plane of the figure. Therefore, rigid-body motion is confined to two degrees of freedom—vertical translation and rotation in the plane of the figure. The following two rigid-body mode functions are chosen to describe these two motions

$\phi_1(x) = 1$ vertical translation of unit amplitude

$\phi_2(x) = \frac{x}{l}$ rotation about left end of beam giving unit amplitude at right end

The above choice is an arbitrary one as any two independent linear functions could be used. For deformation modes it is convenient to consider deflections relative to a straight line through the two ends of the beam. The following four modes are chosen.

$$\phi_3(x) = \sin\frac{\pi x}{l}$$

$$\phi_4(x) = \sin\frac{2\pi x}{l}$$

$$\phi_5(x) = \sin\frac{3\pi x}{l}$$

$$\phi_6(x) = \sin\frac{4\pi x}{l}$$

The boundary conditions on the free-free beam are obtained from the fact that bending moment and shear vanish at the two ends, thus

$$w''(0) = w''(l) = w'''(0) = w'''(l) = 0$$

The last four functions fail to satisfy the boundary conditions on the third derivative. However, as will be seen later, this does not lead to noticeable error in the solution. We proceed with the computation of the m_{ij} and k_{ij} by means of Eqs. (4.30) and (4.32) and construct the mass and stiffness matrices. Since $\phi_1''(x)$ and $\phi_2''(x)$ are zero, the corresponding k's are zero; hence, the elements of the first two rows and columns of the stiffness matrix are zero. At this point the problem appears as follows.

$$M\begin{bmatrix} 1.00000 & 0.55556 & 0.63662 & -0.10610 & 0.21220 & -0.05305 \\ 0.55556 & 0.38889 & 0.33841 & -0.21220 & 0.13828 & -0.10610 \\ 0.63662 & 0.33841 & 0.50000 & -0.06004 & 0 & -0.00480 \\ -0.10610 & -0.21220 & -0.06004 & 0.50000 & -0.06485 & 0 \\ 0.21220 & 0.13828 & 0 & -0.06485 & 0.50000 & -0.06617 \\ -0.05305 & -0.10610 & -0.00480 & 0 & -0.06617 & 0.50000 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

$$= \frac{\pi^4 EI}{2l^3\omega^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 81 & 0 \\ 0 & 0 & 0 & 0 & 0 & 256 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \tag{4.72}$$

Now, let us determine the equations which result from the vanishing of the linear and angular momenta. The equation for linear momentum is

$$\int_0^l m(x)\dot{w}\,dx = 0$$

Using the transformation Eq. (4.27) and interchanging the order of summation and integration leads to

$$\sum_{j=1}^{6} \dot{q}_j \int_0^l m(x)\phi_j(x)\,dx = 0$$

Since the q_j's are all harmonic with the same frequency ω, we may write

$$\dot{q}_j = i\omega q_j$$

where i is the unit imaginary number. Therefore, the desired equation is

$$\sum_{j=1}^{6} q_j \int_0^l m(x)\phi_j(x)\,dx = 0 \tag{4.73}$$

The equation for angular momentum is obtained in a similar way starting with

$$\int_0^l m(x)\dot{w}x\,dx = 0$$

where the moment of momentum is taken with respect to the left end of the beam for convenience. Any point may be used so long as the linear momentum is zero. The equation above leads by the same steps to the following.

$$\sum_{j=1}^{6} q_j \int_0^l m(x)\phi_j(x)x\,dx = 0 \tag{4.74}$$

Because of the choice of the functions $\phi_1(x)$ and $\phi_2(x)$, where $\phi_1(x)$ represents translation only and $\phi_2(x)$ represents rotation about the left end of the beam, Eqs. (4.73) and (4.74) are identical, respectively, to the first two of Eq. (4.72). Thus, the matrix Eq. (4.72) satisfies not only the dynamic equilibrium conditions expressed by the Lagrange equations but the momentum relationships as well.

In proceeding with the solution of Eq. (4.72) we would ordinarily determine the dynamical matrix by first inverting matrix $[k]$. However, since the latter matrix is singular, its inverse does not exist. Alternately, we may solve the problem by other means such as inverting the mass matrix to form the inverse of the dynamical matrix, then iterating to converge first on the higher modes. Or, we may form the matrix $([m] - \lambda[k])$ and, from it, expand the frequency equation and, hence, obtain the frequency roots. It is possible, however, to continue the solution of this problem by introducing a coordinate transformation which removes the singularities from $[k]$ so that the dynamical matrix $[D]$ can be constructed, and the iteration will converge to the lowest modes first as is often desired.† This procedure is considered in the following discussion.

We are concerned then with the eigenvalue problem expressed in the form

$$[m]\{q\} = \frac{1}{\omega^2}[k]\{q\} \tag{4.75}$$

where the stiffness matrix is of rank equal to its order, less the number of rigid-body degrees of freedom. This property of the stiffness matrix for unconstrained structures is discussed in Chapter 1, Section 1.9. We note here, that although the stiffness matrix is singular, this will not always be evidenced clearly by the existence of a set of null rows and columns as in Eq. (4.72).‡ This latter property resulted from a fortunate choice of rigid-body mode functions. From earlier discussion it follows that we can write a number of momentum relationships equal to the number of rigid-body degrees of freedom. For the beam of Fig. 4.7 these relationships are expressed by Eqs. (4.73) and (4.74). In matrix form these relationships can be written as

$$[T]\{q\} = \{0\} \tag{4.76}$$

The elements of matrix T are the integrals in Eqs. (4.73) and (4.74), and in view of our choice of rigid-body mode functions these elements happen also to be identical to the elements in the first two rows of

†There is a method of directly constructing a flexibility matrix and, hence, a dynamical matrix for an unconstrained or partially constrained system by placing fictitious constraints on it. This method is discussed in Chapter 6, Section 8.

‡See, for instance, Eq. (1.112), Chapter 1, Section 10.

the generalized mass matrix of Eq. (4.72). Equations (4.76) may be looked upon as equations of constraint on the q's which stem from the principle of conservation of momentum. Matrix $[T]$ is of order $k \times m$ $(m > k)$ where m is the number of assumed mode functions and k is the number of rigid-body modes included in m. We can therefore partition Eq. (4.76) and express k of the q's in terms of the remaining $m - k = n$

$$\left[\underset{k\times k}{[T]_A} \,\vdots\, \underset{k\times n}{[T]_B}\right] \begin{Bmatrix} \{q\}_A \\ \hline \{q\}^* \end{Bmatrix}^{k\times 1}_{n\times 1} = \{0\}$$

or

$$[T]_A \{q\}_A + [T]_B \{q\}^* = \{0\}$$

and, hence

$$\{q\}_A = -[T]_A^{-1}[T]_B \{q\}^* \tag{4.77}$$

The order of the submatrices is as indicated. $\{q\}_A$ represents a number of displacement coordinates equal to the number of rigid-body modes, and $\{q\}^*$ is a reduced column of generalized displacements corresponding in number to the number of modes of vibration. From Eq. (4.77) we can construct a matrix which relates the column containing all the coordinates to the reduced column $\{q\}^*$.

$$\underset{m\times 1}{\{q\}} = \underset{m\times n}{[T]^*} \underset{n\times 1}{\{q\}^*} \tag{4.78}$$

where

$$[T]^* = \begin{bmatrix} -[T]_A^{-1}[T]_B \\ \hline [I] \end{bmatrix} \tag{4.79}$$

and

$$\{q\} = \begin{Bmatrix} \{q\}_A \\ \hline \{q\}^* \end{Bmatrix} \tag{4.80}$$

$[I]$ is the identity matrix. Equation (4.78) represents a transformation from coordinates q to a reduced coordinate set q^*; consequently, the generalized mass and stiffness matrices in the new coordinates are given, respectively, by

$$[m]^* = [T]^{*T}[m][T]^*$$
$$[k]^* = [T]^{*T}[k][T]^*$$

where $[m]$ and $[k]$ are, respectively, the generalized mass and stiffness matrix in the q coordinates. The eigenvalue problem in the reduced coordinates q^* takes the form

$$[m]^*\{q\}^* = \frac{1}{\omega^2}[k]^*\{q\}^* \tag{4.81}$$

The matrix $[k]^*$ is not singular, so that solution may proceed by forming the dynamical matrix in the usual fashion.

In the free-free beam of Figure 4.7

$$\{q\}_A = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}, \quad \text{and} \quad \{q\}^* = \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

$$[T]_A = [m]_{11}, \qquad [T]_B = [m]_{12}$$

and hence

$$[T]^* = \begin{bmatrix} -[m]_{11}^{-1}[m]_{12} \\ \hline I \end{bmatrix}$$

where $[m]_{11}$ and $[m]_{12}$ are submatrices of the generalized mass matrix written on the left-hand side of Eq. (4.72) and partitioned as follows.†

$$\left[\begin{array}{c|c} \underset{2\times2}{[m]_{11}} & \underset{2\times4}{[m]_{12}} \\ \hline \underset{4\times2}{[m]_{21}} & \underset{4\times4}{[m]_{22}} \end{array}\right]$$

The matrices $[m]^*$ and $[k]^*$ for the free-free beam become

$$[m]^* = M\begin{bmatrix} 0.09179 & -0.02165 & -0.13122 & 0.01439 \\ -0.02165 & 0.19604 & -0.00339 & -0.15198 \\ -0.13122 & -0.00339 & 0.44979 & -0.03544 \\ 0.01439 & -0.15198 & -0.03544 & 0.42401 \end{bmatrix}$$

$$[k]^* = \frac{\pi^4 EI}{2l^3}[k]_{22}$$

where $[k]_{22}$ is the nonsingular 4×4 submatrix at the lower-right corner of matrix $[k]$ on the right-hand side of Eq. (4.72). Using the last results in Eq. (4.81) the solution for the first four natural modes may proceed by methods discussed earlier. In each mode the coordinates q_1 and q_2 are related to the vector $\{q_3, q_4, q_5, q_6\}$ by Eq. (4.77). Thus, the complete eigenvector may be obtained for each of the four modes.

In the problem at hand the results for the first mode are

$$\omega_1 = 22.697\sqrt{\frac{EI}{Ml^3}}$$

$$\{q^{(1)}\} = \begin{Bmatrix} -0.72568 \\ 0.16413 \\ 1.00000 \\ -0.01648 \\ -0.01820 \\ 0.00074 \end{Bmatrix}$$

†We emphasize again that $[T]^*$ contains elements of the generalized mass matrix as a consequence of our choice of rigid body modes.

The first mode is given by the equation

$$w(x) = -0.72568 + 0.16413\frac{x}{l} + \sin\frac{\pi x}{l}$$

$$-0.01648\sin\frac{2\pi x}{l} - 0.01820\sin\frac{3\pi x}{l}$$

$$+0.00074\sin\frac{4\pi x}{l}$$

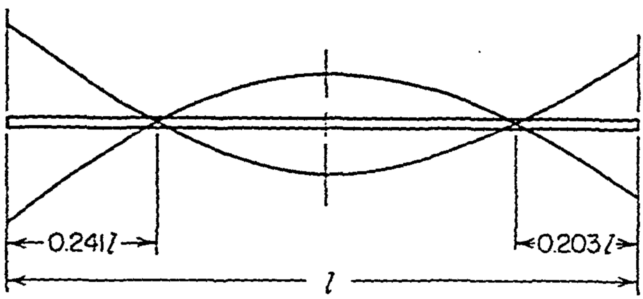


Figure 4.8

It is seen that the first three terms are the major contributors to the mode shape shown in Fig. 4.8. The frequency obtained in this solution is within 0.2% of that obtained by a separate analysis using influence coefficients and lumping the beam into 50 mass elements. This tends to justify the use of functions which do not satisfy all of the boundary conditions as discussed earlier.

4.11 Natural Vibration of Plates

The natural modes and frequencies of two dimensional structures, such as thin plates and shells, may be found by use of energy methods in the same manner as those for one-dimensional structures already considered. In principle the procedures are the same. However, in practice greater difficulties are encountered in finding suitable functions that satisfy boundary conditions. We are now concerned with functions of two independent space variables and double integration is, therefore, required in the determination of the generalized masses and stiffnesses.

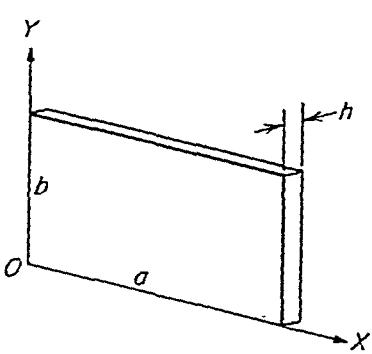


Figure 4.9

To illustrate the procedure and to

bring out these points let us consider the flexural vibrations of a flat rectangular plate shown in Fig. 4.9. We shall express the deflection w, a function of space variables (x, y) and of time t, in terms of functions $\phi(x, y)$ (which we shall select as satisfactory functions insofar as boundary conditions and other considerations of Section 4.7 are concerned) and generalized coordinates $q(t)$.

$$w(x, y, t) = \sum_{i=1}^{n} \phi_i(x, y) q_i(t) \tag{4.82}$$

The kinetic energy associated with translation of the plate is given by

$$T = \frac{1}{2} \int_0^a \int_0^b m(x, y) \dot{w}^2(x, y, t)\, dy\, dx \tag{4.83}$$

where $m(x, y)$ is the mass per unit area. The strain energy expression is given as[17]

$$\begin{aligned} U = \frac{1}{2} \int_0^a \int_0^b D(x, y) \Bigg\{ \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right]^2 \\ -2(1 - \nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \Bigg\}\, dy\, dx \end{aligned} \tag{4.84}$$

where

ν = Poisson's ratio

D = flexural stiffness modulus of the plate

$= \dfrac{Eh^3}{12(1 - \nu^2)}$

h = thickness of plate

Again, using Lagrange's equations we take the indicated partial derivatives. Terms for the rth equation of the set are obtained as follows.

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_r} &= \int_0^a \int_0^b m(x, y) \dot{w} \frac{\partial \dot{w}}{\partial \dot{q}_r}\, dy\, dx \\ &= \sum_{i=1}^{n} m_{ri} \dot{q}_i \end{aligned} \tag{4.85}$$

where

$$m_{ri} = \int_0^a \int_0^b m(x, y) \phi_r(x, y) \phi_i(x, y)\, dy\, dx \tag{4.86}$$

Development of the above term parallels that for the one-dimensional problem. From the strain energy expression we obtain

$$\begin{aligned} \frac{\partial U}{\partial q_r} = \int_0^a \int_0^b D(x, y) \Bigg\{ \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] \frac{\partial}{\partial q_r} \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] \\ - (1 - \nu) \frac{\partial}{\partial q_r} \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \Bigg\}\, dy\, dx \end{aligned} \tag{4.87}$$

From Eq. (4.82) we note that

$$\frac{\partial^2 w}{\partial x^2} = \sum_{i=1}^{n} \frac{\partial^2 \phi_i}{\partial x^2} q_i$$

and

$$\frac{\partial^2 w}{\partial y^2} = \sum_{i=1}^{n} \frac{\partial^2 \phi_i}{\partial y^2} q_i$$

Therefore

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \sum_{i=1}^{n} \left(\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial y^2}\right) q_i \tag{4.88}$$

and

$$\frac{\partial}{\partial q_r}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] = \frac{\partial^2 \phi_r}{\partial x^2} + \frac{\partial^2 \phi_r}{\partial y^2} \tag{4.89}$$

Furthermore, we can see that the product $(\partial^2 w/\partial x^2)(\partial^2 w/\partial y^2)$ is given by

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \phi_i}{\partial x^2}\frac{\partial^2 \phi_j}{\partial y^2} q_i q_j$$

Differentiating this expression term by term with respect to q_r the following derivative is obtained.

$$\frac{\partial}{\partial q_r}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) = \sum_{i=1}^{n}\left(\frac{\partial^2 \phi_i}{\partial x^2}\frac{\partial^2 \phi_r}{\partial y^2} + \frac{\partial^2 \phi_r}{\partial x^2}\frac{\partial^2 \phi_i}{\partial y^2}\right) q_i \tag{4.90}$$

Again, from Eq. (4.82) we obtain

$$\frac{\partial^2 w}{\partial x \partial y} = \sum_{i=1}^{n} \frac{\partial^2 \phi_i}{\partial x \partial y} q_i$$

The square of this derivative is required and may be expressed by

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \phi_i}{\partial x \partial y}\frac{\partial^2 \phi_j}{\partial x \partial y} q_i q_j$$

Upon differentiating this expression term by term with respect to q_r we obtain

$$\frac{\partial}{\partial q_r}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = 2\sum_{i=1}^{n} \frac{\partial^2 \phi_r}{\partial x \partial y}\frac{\partial^2 \phi_i}{\partial x \partial y} q_i \tag{4.91}$$

Now, we insert Eqs. (4.88), (4.89), (4.90), and (4.91) into Eq. (4.87) which, after some manipulation involving changing orders of integration and summation and collecting terms, yields

$$\frac{\partial U}{\partial q_r} = \sum_{i=1}^{n} k_{ri} q_i \tag{4.92}$$

where the stiffness coefficient k_{ri} is given by

$$\begin{aligned} k_{ri} = \int_0^a \int_0^b D\Bigg\{&\left(\frac{\partial^2 \phi_r}{\partial x^2} + \frac{\partial^2 \phi_r}{\partial y^2}\right)\left(\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial y^2}\right) \\ &- (1-\nu)\left(\frac{\partial^2 \phi_i}{\partial x^2}\frac{\partial^2 \phi_r}{\partial y^2} + \frac{\partial^2 \phi_r}{\partial x^2}\frac{\partial^2 \phi_i}{\partial y^2} - 2\frac{\partial^2 \phi_r}{\partial x \partial y}\frac{\partial^2 \phi_i}{\partial x \partial y}\right)\Bigg\}\, dy\, dx \end{aligned} \tag{4.93}$$

As in the case of one-dimensional structures, the mass and stiffness coefficients have the property of symmetry. In addition, we have again formulated the eigenvalue problem for the plate in the form of Eq. (4.36) so that solution of the problem will proceed as before.

To carry our example further, let us prescribe a particular set of boundary conditions and select functions which satisfy them. The only boundary conditions that permit the choice of simple functions are those corresponding to simply-supported edges. Here the boundary conditions which should be satisfied by each function $\phi(x,y)$ are

$$\phi(o,y) = \phi(a,y) = \phi(x,o) = \phi(x,b) = 0$$

$$\frac{\partial^2\phi}{\partial x^2}(o,y) = \frac{\partial^2\phi}{\partial x^2}(a,y) = \frac{\partial^2\phi}{\partial y^2}(x,o) = \frac{\partial^2\phi}{\partial y^2}(x,b) = 0$$

Sine functions satisfy these conditions and we may then take products of sine functions in x and y, two of which are given below, say

$$\phi_1(x,y) = \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$\phi_2(x,y) = \sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}$$

In general, we may take the ith function as

$$\phi_i(x,y) = \sin\frac{j\pi x}{a}\sin\frac{k\pi y}{b} \tag{4.94}$$

where j and k are any integers. A particular pair of these integers, p and n, is chosen to represent the rth function, thus

$$\phi_r(x,y) = \sin\frac{p\pi x}{a}\sin\frac{n\pi y}{b}$$

Now, the masses m_{ri} may be found from Eq. (4.86). Considering the plate to be uniform with mass m per unit area, we obtain

$$m_{ri} = m\int_0^a\int_0^b \sin\frac{p\pi x}{a}\sin\frac{j\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}\,dy\,dx$$

For $i \neq r$, which implies also that $j \neq p$ and $k \neq n$, the above integral is zero. We determine m_{rr} to be

$$m_{rr} = m\int_0^a\int_0^b \sin^2\frac{p\pi x}{a}\sin^2\frac{n\pi y}{b}\,dy\,dx$$
$$= \tfrac{1}{4}mab$$

The stiffnesses k_{ri} and k_{rr} are found from Eq. (4.93). Although the computations are more involved than those above, it may be readily determined that the $k_{ri}(i \neq r)$ are zero inasmuch as integrals of the same type as for $m_{ri}(i \neq r)$ are encountered. The stiffnesses k_{rr} are found to be

$$k_{rr} = \frac{\pi^4 D}{4}\left(\frac{p^2}{a^2} + \frac{n^2}{b^2}\right)^2 ab$$

Since the m_{ri} and k_{ri} for $i \neq r$ are all zero, the mass and stiffness matrices are of diagonal form. This tell us that the modes defined by the chosen functions $\phi(x, y)$ [Eq. (4.94)] are uncoupled; hence, those functions are the natural mode functions for the plate. The rth equation of the set is

$$\omega^2 m_{rr} q_r = k_{rr} q_r$$

Thus, the natural frequency in the rth natural mode is given by

$$\omega_r = \sqrt{\frac{k_{rr}}{m_{rr}}} = \pi^2 \sqrt{\frac{D}{m}}\left(\frac{p^2}{a^2} + \frac{n^2}{b^2}\right) \tag{4.95}$$

For other boundary conditions it will not be so simple to choose satisfactory functions and it is extremely unlikely that natural mode functions will be chosen as in the foregoing example. However, the same procedure may be used and although the matrices $[m]$ and $[k]$ will not, in general, be diagonal matrices, they may be determined and the eigenvalue problem solved to find approximate modes and frequencies. Various boundary conditions have been handled by this method,[15] choosing functions of the form

$$\phi_i(x, y) = \alpha_j(x)\beta_k(y) \tag{4.96}$$

This representation is a generalization of Eq. (4.94) where the functions $\alpha(x)$ and $\beta(y)$ are both sine functions. In general, these functions may be chosen as the natural mode functions of a uniform beam having the same boundary conditions at its ends as those at the corresponding edges of the plate.

4.12 Effects of Rotatory Inertia and Shear on Beam Vibrations

In our previous examples concerned with beam vibrations we have repeatedly neglected the effects of rotatory inertia and shear. In most problems encountered in practice these effects can safely be neglected with little error; however, for short deep beams it may be desirable to include them. In using energy methods these effects may be accounted for approximately by appropriate additions to the kinetic and strain energy expressions. We shall consider these additional energy terms in this section.

Effect of Rotatory Inertia. By *rotatory inertia* is meant the inertia associated with angular acceleration of the beam sections. In the absence of shear, these sections, according to elementary beam theory, always remain plane and perpendicular to the longitudinal axis of the beam.

Therefore, the angle of rotation ψ is equal to the slope w' of the beam axis. If shear is present this will not be strictly true but the change in angle due to shear is small and a correction for this small change is a second-order correction. In the kinetic energy expression we add to the energy associated with the linear velocity $\dot{w}$ that associated with angular velocity $\dot{\psi}$. Thus, the kinetic energy is written as

$$T = \frac{1}{2}\int_0^l m(x)\,\dot{w}^2(x,t)\,dx + \frac{1}{2}\int_0^l m(x)\,\rho^2(x)\,\dot{\psi}^2(x,t)\,dx \qquad \text{(See Eq. 2.95.)} \tag{4.97}$$

where ρ is the section radius of gyration taken about the neutral axis of bending. If we represent $w(x,t)$, as in Eq. (4.27), and make the approximation $\psi = w'$, we then have for $\dot{\psi}$ the following:

$$\dot{\psi}(x,t) = \sum_{i=1}^{n} \phi_i'(x)\dot{q}_i(t) \tag{4.98}$$

The additional energy term will result in a correction to the generalized masses m_{ri}. These may be found by taking the following derivative

$$\begin{aligned}
\frac{\partial T}{\partial \dot{q}_r} &= \int_0^l m(x)\,\dot{w}\,\frac{\partial \dot{w}}{\partial \dot{q}_r}\,dx + \int_0^l m(x)\,\rho^2(x)\,\dot{\psi}\,\frac{\partial \dot{\psi}}{\partial \dot{q}_r}\,dx \\
&= \sum_{i=1}^{n} \dot{q}_i\left\{\int_0^l m(x)\,\phi_r(x)\,\phi_i(x)\,dx + \int_0^l m(x)\,\rho^2(x)\,\phi_r'(x)\,\phi_i'(x)\,dx\right\} \\
&= \sum_{i=1}^{n} m_{ri}\dot{q}_i
\end{aligned}$$

Thus, we see that the corrected generalized mass is given by

$$m_{ri} = \int_0^l m(x)\,\phi_r(x)\,\phi_i(x)\,dx + \int_0^l m(x)\,\rho^2(x)\,\phi_r'(x)\,\phi_i'(x)\,dx \tag{4.99}$$

To get an idea of the relative importance of the second term in this expression we consider a simple example. The first mode of a simply-supported uniform beam is

$$\phi_1(x) = \sin\frac{\pi x}{l}$$

For this example we find

$$\begin{aligned}
m_{11} &= m\int_0^l \sin^2\frac{\pi x}{l}\,dx + m\rho^2\frac{\pi^2}{l^2}\int_0^l \cos^2\frac{\pi x}{l}\,dx \\
&= \frac{1}{2}ml + \frac{1}{2}ml\pi^2\frac{\rho^2}{l^2} = \frac{1}{2}ml\left(1 + \pi^2\frac{\rho^2}{l^2}\right)
\end{aligned}$$

For the fundamental frequency of the beam this will result in a correction factor of

$$\frac{1}{\sqrt{1 + (\pi\rho/l)^2}}$$

For beams in which $(\pi\rho/l)^2$ is small the above correction can be approximated by*

$$1 - \frac{1}{2}\left(\frac{\pi\rho}{l}\right)^2$$

This correction agrees with the result derived by Timoshenko in Reference 15, who used a differential equation approach.

For an I beam with a thin web, the radius of gyration ρ is nearly equal to half the beam depth h. In this case the correction factor is

$$1 - \frac{1}{8}\left(\frac{\pi h}{l}\right)^2$$

For $h/l = \frac{1}{20}$ this becomes 0.997. The correction is relatively more important in higher modes.[15]

Effect of Shear. When shear effect is included in beam vibration, the flexibility of the beam is increased and, consequently, the generalized stiffness is decreased. This will be demonstrated by the following.

The total strain energy due to flexure and shear is

$$U = \tfrac{1}{2} EI \int_0^l w_b''^2 \, dx + \tfrac{1}{2} \alpha GA \int_0^l w_s'^2 \, dx$$

in which

α = a dimensionless number dependent upon the shape of the cross section

w_b = the deflection due to bending

w_s = the deflection due to shear (hence, w_s' is the angle of shear deformation)

then

$w = w_b + w_s$ = the total deflection of the beam

Our next step is to express w_b and w_s using Eq. (4.27). Since the bending deflection curve and the shear deflection curve will not differ greatly[15] we use, as an approximation, the same $\phi_i(x)$ functions for w, w_b, and w_s and write

$$w = \sum_i \phi_i q_i, \qquad w_b = \sum_i \phi_i q_{bi}, \qquad w_s = \sum_i \phi_i q_{si}$$

from which we have

$$w_b'' = \sum_i \phi_i'' q_{bi}, \qquad w_s' = \sum_i \phi_i' q_{si}$$

*This approximation follows from $\sqrt{1 - a} \approx 1 - \frac{a}{2}$ for $a \ll 1$.

We can write the total deflection w in the form

$$w = (1 + \beta)w_b$$

where

$$\beta = \frac{w_s}{w_b}$$

or

$$\sum_i \phi_i q_i = \sum_i \phi_i(q_{bi} + q_{si}) = \sum_i \phi_i(1 + \beta_i)q_{bi}$$

with $\beta_i = q_{si}/q_{bi}$. Hence,

$$q_i = (1 + \beta_i)q_{bi}$$

Using the last two results, we obtain the relations

$$\left.\begin{aligned} q_{bi} &= \frac{1}{1 + \beta_i} q_i \\ q_{si} &= \frac{\beta_i}{1 + \beta_i} q_i \end{aligned}\right\} \tag{4.100}$$

To obtain the generalized stiffness k_{ri} we take the partial derivative, $\partial U/\partial q_r$ of the total strain energy U due to flexure and shear, and write

$$\frac{\partial U}{\partial q_r} = EI \int_0^l w_b'' \frac{\partial w_b''}{\partial q_r}\, dx + \alpha GA \int_0^l w_s' \frac{\partial w_s'}{\partial q_r}\, dx$$

Noting that

$$\frac{\partial w_b''}{\partial q_r} = \frac{\partial w_b''}{\partial q_{br}} \frac{dq_{br}}{dq_r}$$

$$\frac{\partial w_s'}{\partial q_r} = \frac{\partial w_s'}{\partial q_{sr}} \frac{dq_{sr}}{dq_r}$$

and using the expressions for w_b'', w_s', and relations (4.100), $\partial U/\partial q_r$ becomes

$$\begin{aligned} \frac{\partial U}{\partial q_r} &= \sum_i q_i \frac{EI}{(1 + \beta_i)(1 + \beta_r)} \left\{\int_0^l \phi_r''\phi_i''\, dx + \frac{\alpha GA}{EI} \beta_r\beta_i \int_0^l \phi_r'\phi_i'\, dx\right\} \\ &= \sum_i k_{ri} q_i \end{aligned}$$

Hence, the generalized stiffness with the shear term included is

$$k_{ri} = \frac{EI}{(1 + \beta_i)(1 + \beta_r)} \left\{\int_0^l \phi_r''\phi_i''\, dx + \frac{\alpha GA}{EI} \beta_r\beta_i \int_0^l \phi_r'\phi_i'\, dx\right\} \tag{4.101}$$

To show what correction factor is applied to the generalized stiffness for bending only [k_{ri} (bending only)] when shear is included, we factor out

$$EI \int_0^l \phi_r''\phi_i''\, dx = k_{ri}\,(\text{bending only})$$

in Eq. (4.101) and write

$$k_{ri}\text{ (bending and shear)} = \frac{1}{(1+\beta_r)(1+\beta_i)} \times \left(1 + \frac{\alpha GA}{EI}\beta_r\beta_i \frac{\int_0^l \phi_r'\phi_i'\,dx}{\int_0^l \phi_r''\phi_i''\,dx}\right) k_{ri}\text{ (bending only)}$$

For a simply-supported uniform beam the first mode is

$$\phi_1(x) = \sin\frac{\pi x}{l},$$

and β_1 is given by

$$\beta_1 = \frac{E}{\alpha G}\left(\frac{\pi\rho}{l}\right)^2$$

This value of β_1 can be obtained by using the method of virtual work to compute the ratio of the shear deflection w_s to the bending deflection w_b at the center of the beam. Using the expressions above for ϕ_1 and β_1, the stiffness correction factor to k_{11} of the simple beam becomes

$$\frac{1}{1+\beta_1} \approx 1 - \beta_1 \quad \text{(for small } \beta_1\text{)}$$

The correction factor is less than 1, as expected, because the stiffness decreases when the shear effect is accounted for. The stiffness correction for the simple beam results in a corresponding correction factor for the fundamental frequency, given by

$$\sqrt{1-\beta_1} \approx 1 - \frac{\beta_1}{2}$$

Again, this correction agrees with the results obtained by Timoshenko[15] who used a differential equation solution.

For an I beam with a thin web for which

$$\rho \approx \frac{h}{2} \quad \text{(where } h \text{ is the depth of the beam section)}$$

$$\frac{G}{E} = 0.375$$

$$\alpha = 1$$

and

$$\frac{h}{l} = \frac{1}{20}$$

the correction factor to the fundamental frequency is $1-(\beta_1/2)=0.992$. Here also, as in the case of rotatory inertia effects, the contribution of shear is relatively more important in higher modes.[15]

4.13 Vibration of a Rotating Beam—Centrifugal Force Field

Energy methods may be extended to include structures acted upon by potential fields other than the elastic force fields with which we have been primarily concerned. One such field is the *centrifugal force field* which acts upon a beam that vibrates in a rotating plane. Such a beam is shown in *Fig. 4.10* where the plane in which the beam vibrates is OAB. This plane rotates with constant angular velocity Ω about the fixed axis OA.

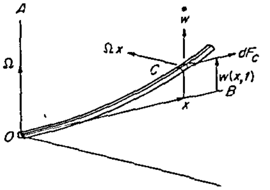


Figure 4.10

The axis of equilibrium from which the deflection w is measured, is OB. The element at C has two components of velocity—$\dot{w}$ in the plane of vibration, and Ωx normal to that plane. *Considering only* the kinetic energy in translation, as is appropriate for a slender beam, the total kinetic energy of the beam is given by

$$T = \frac{1}{2}\int_0^l m(x)\,[\dot{w}^2 + \Omega^2 x^2]\,dx \tag{4.102}$$

The strain energy may be computed in the usual way but, in addition, we must include the potential energy* due to the *centrifugal force* field.[15] This energy is equal to the negative of the total work done by the centrifugal forces dF_c acting through their displacements which occur as the beam deflects from its equilibrium position *along* OB. The component of the displacement at C parallel to the centrifugal forces is equal to the difference between the length measured along the axis of the beam from O to C and the distance x. This displacement is directed toward the axis of rotation. Hence, force dF_c does negative work on it. Consequently, the potential energy is positive. We presume that the displacement referred to is caused by the transverse deflection w, and shall neglect stretching of the beam due to

*Note that the potential energy due to gravity must also be included; however, its effect in this problem is merely to change the position of equilibrium about which the beam vibrates and does not affect the frequencies or modes. Hence, we neglect it in this problem.

the centrifugal forces. Then, the length from O to C along the deflected beam axis, is given by the integral

$$\int_0^x \sqrt{1 + \left(\frac{\partial w}{\partial \xi}\right)^2}\, d\xi$$

where ξ is a dummy variable measured along x. The differential centrifugal force dF_c is equal to

$$dF_c = \Omega^2 m(x)\, x\, dx$$

Using these expressions we write the total potential energy V.

$$V = U + \int_0^l \left\{\int_0^x \sqrt{1 + \left(\frac{\partial w}{\partial \xi}\right)^2}\, d\xi - x\right\} \Omega^2 m(x)\, x\, dx$$

where U is the usual strain energy of bending. The square root is now expanded in Taylor's series of which we use only the first two terms for small amplitude vibration. Thus

$$\sqrt{1 + \left(\frac{\partial w}{\partial \xi}\right)^2} \approx 1 + \frac{1}{2}\left(\frac{\partial w}{\partial \xi}\right)^2$$

With this simplification the potential energy becomes

$$V = U + \frac{1}{2}\Omega^2 \int_0^l m(x)\, x \int_0^x \left(\frac{\partial w}{\partial \xi}\right)^2 d\xi\, dx \qquad (4.103)$$

Again, we expand $w(x, t)$ in a series of functions, as in Eq. (4.27), using generalized coordinates. To write the rth Lagrange equation in terms of these functions and coordinates, and include the centrifugal field, we must use the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = 0 \qquad \text{(See Eq. 2.117.)} \qquad (4.104)$$

Referring to Eq. (4.102) we can see that the kinetic energy terms in the above Lagrange equation are not affected by rotation of the beam. Consequently, the generalized masses are the same as if the beam were not rotating. The generalized stiffnesses will be changed, however, as will be discovered when we operate on Eq. (4.103) to obtain $\partial V/\partial q_r$.

$$\frac{\partial V}{\partial q_r} = \frac{\partial U}{\partial q_r} + \Omega^2 \int_0^l m(x)\, x \int_0^x \left(\frac{\partial w}{\partial \xi}\right) \frac{\partial}{\partial q_r}\left(\frac{\partial w}{\partial \xi}\right) d\xi\, dx$$

To evaluate the partial derivative in this equation we must recognize that since ξ is a dummy variable replacing x, we may write

$$w(\xi, t) = \sum_{i=1}^{n} \phi_i(\xi)\, q_i(t)$$

Hence

$$\frac{\partial}{\partial q_r}\left(\frac{\partial w}{\partial \xi}\right) = \phi_r'(\xi)$$

After performing the usual operations the following equation is obtained.

$$\frac{\partial V}{\partial q_r} = \frac{\partial U}{\partial q_r} + \Omega^2 \sum_{i=1}^{n} q_i \int_0^l m(x) x \int_0^x \phi_r'(\xi) \phi_i'(\xi)\, d\xi\, dx$$

The first term on the right side of this equation is associated with the stiffness of the nonrotating beam. To this is added an "apparent" stiffness from the second term, which results from rotation of the beam. The total stiffness coefficient k_{ri} is the sum of the two terms. Using Eq. (4.32) for the first term, we obtain

$$\begin{aligned} k_{ri} = &\int_0^l EI(x) \phi_r''(x) \phi_i''(x)\, dx \\ &+ \Omega^2 \int_0^l m(x) x \int_0^x \phi_r'(\xi) \phi_i'(\xi)\, d\xi\, dx \end{aligned} \tag{4.105}$$

It is interesting to note that a perfectly rigid beam hinged at its root so that it can rotate in the plane OAB of Fig. 4.10 has an apparent stiffness due to its rotation. Hence, it may execute free vibrations at a natural frequency which depends upon the angular velocity Ω. Let us determine this frequency by using the mode function

$$\phi_1(x) = \frac{x}{l}$$

associated with such a rigid, hinged uniform beam.
The generalized mass m_{11} is obtained from

$$m_{11} = m \int_0^l \left(\frac{x}{l}\right)^2 dx = \frac{1}{3} ml$$

The generalized stiffness k_{11} is found from Eq. (4.105) where the first term of that equation contributes nothing because the second derivative $\phi_1''(x)$ vanishes.

$$k_{11} = m\Omega^2 \int_0^l x \int_0^x \frac{1}{l^2}\, d\xi\, dx = \tfrac{1}{3}\Omega^2 ml$$

The natural frequency is

$$\omega_1^2 = \frac{k_{11}}{m_{11}} = \Omega^2$$

or

$$\omega_1 = \Omega$$

Thus, we obtain the result that the beam executes one cycle of motion in plane AOB (Fig. 4.10) for each revolution around the axis of rotation. This is equivalent to saying that the beam rotates in a plane inclined to the horizontal reference plane.

4.14 The Method of Selected Modes in Terms of Influence Functions and Influence Coefficients

A limitation on the use of the Rayleigh-Ritz method lies in the fact that the determination of the generalized stiffness coefficients k_{ij} requires taking derivatives of the selected mode functions, as shown by Eqs. (4.32), (4.52) and (4.101). If these functions are expressed analytically the derivatives may be taken accurately. However, in practice this may not always be done. For example, the selected mode shapes may be obtained from experimental tests in which the amplitudes are measured at certain points on the structure. Then the curves may be plotted and the derivatives determined graphically. This, however, is a difficult and inaccurate process.

Influence Functions. An alternate procedure by which the above difficulty is circumvented is obtained by considering strain energy in terms of the influence functions of the structure. Where the applied forces are functions of time, the strain energy so considered is given by

$$\begin{aligned} U &= \tfrac{1}{2}\int_{x=0}^{l} u(x,t)\,f(x,t)\,dx \\ &= \tfrac{1}{2}\int_{x=0}^{l}\int_{\xi=0}^{l} a(x,\xi)\,f(x,t)\,f(\xi,t)\,d\xi\,dx \end{aligned} \tag{4.106}$$

where $u(x,t)$ is substituted from Eq. (1.64), Chapter 1. The distributed force $f(x,t)$ is an inertial force in the case of free vibrations, and since these vibrations are harmonic, we may express this force in terms of the mass distribution and the frequency.

$$f(x,t) = \omega^2 m(x)\,u(x,t) \tag{4.107}$$

We continue to express deflection $u(x,t)$ in terms of selected mode shapes $\phi(x)$ according to Eq. (4.27), as in the Rayleigh-Ritz method. Hence, we write Eq. (4.107) in the form

$$f(x,t) = \omega^2 m(x) \sum_{i=1}^{n} \phi_i(x)\,q_i(t)$$

Similarly, replacing x with ξ

$$f(\xi,t) = \omega^2 m(\xi) \sum_{j=1}^{n} \phi_j(\xi)\,q_j(t)$$

Substituting these forces into Eq. (4.106) leads to the equation

$$U = \tfrac{1}{2}\omega^4 \sum_{i=1}^{n}\sum_{j=1}^{n} G_{ij}\,q_i\,q_j \tag{4.108}$$

where the constants G_{ij} are defined by

$$G_{ij} = \int_{x=0}^{l} m(x)\,\phi_i(x) \int_{\xi=0}^{l} a(x,\xi)\,m(\xi)\,\phi_j(\xi)\,d\xi\,dx \tag{4.109}$$

From Eq. (4.108) the derivative $\partial U/\partial q_r$ is written as

$$\frac{\partial U}{\partial q_r} = \omega^4 \sum_{i=1}^{n} G_{ri} q_i \qquad (4.110)$$

Since the kinetic energy terms in the Lagrange equation for this case are the same as those developed in the Rayleigh-Ritz method, we are led to the following matrix form for the eigenvalue problem.

$$[m]\{q\} = \omega^2 [G]\{q\} \qquad (4.111)$$

where the elements of the matrix $[G]$ are given by Eq. (4.109).

The results obtained by this method of analysis are identical to those resulting from a method attributed to Galerkin, and discussed in Chapter 6 under integral equation methods.

Influence Coefficients. Equation (4.111) applies also to a lumped mass system in which a discrete number of influence coefficients a_{ij} are defined. We will demonstrate this by the following development.

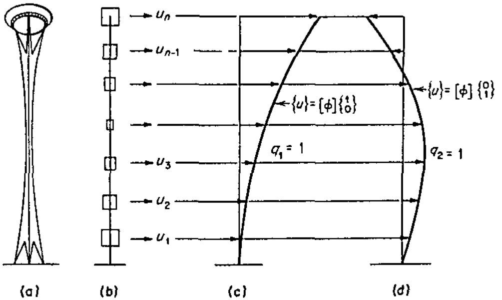


Figure 4.11. (a) Actual structure. (b) Idealized structure (lumped mass). (c) Assumed first mode shape. (d) Assumed second mode shape.

Consider a lumped mass system (Fig. 4.11) whose displacements are given by u_j $(j = 1, 2, \ldots, n)$. The strain energy of the system is

$$U = \tfrac{1}{2}\{F\}^T[a]\{F\} \qquad (4.112)$$

where $[a]$ is the flexibility matrix of the system in the u coordinates. The forces $\{F\}$ are inertial forces in the case of free vibration and can be expressed in terms of the mass matrix $[m]$ of the system and the natural frequency ω^2.

$$\{F\} = \omega^2[m]\{u\} \tag{4.113}$$

Let us suppose now that we are interested in determining the first two or three mode shapes and corresponding frequencies. Using assumed shapes for the desired modes (such as shown in Fig. 4.11) and possibly assuming additional mode shapes to improve accuracy, we can write

$$\{u\} = [\phi]\{q\} \tag{4.114}$$

where ϕ is of order $n \times k$. $n \geq k$. For the system of Fig. 4.11 $[\phi]$ is of order $n \times 2$. The first column of $[\phi]$ represents the assumed first mode shape shown in Fig. 4.11(c). while the second column of $[\phi]$ represents the assumed second mode shape shown in Fig. 4.11(d). Substituting Eq. (4.114) into Eq. (4.113) we write

$$\{F\} = \omega^2[m][\phi]\{q\}$$

Substituting this expression for the inertial forces in Eq. (4.112) the strain energy expression becomes

$$U = \tfrac{1}{2}\omega^4\{q\}^T[G]\{q\} \tag{4.115}$$

where

$$[G] = [\phi]^T[m][a][m][\phi] \tag{4.116}$$

The generalized mass matrix in the q coordinate system is given by

$$[m] = [\phi]^T[m][\phi] \qquad \text{(See Eq. 2.34, Chapter 2.)}$$

Comparing the energy expression

$$U = \tfrac{1}{2}\{q\}^T[k]\{q\} \qquad \text{(See Eq. 1.84, Chapter 1.)}$$

with Eq. (4.115), it follows that the generalized stiffness matrix $[k]$ in the q coordinates is given by

$$[k] = \omega^4[G] \tag{4.117}$$

The equations of motion for free vibration in coordinates q are

$$[m]\{\ddot{q}\} + [k]\{q\} = \{0\}$$

Using Eq. (4.117) and the relation

$$\{\ddot{q}\} = -\omega^2\{q\}$$

the equations of motion become identical to Eq. (4.111).

$$[m]\{q\} = \omega^2[G]\{q\}.$$

PROBLEMS

1. A displacement function $w(x)$ is chosen as an approximation to the first natural mode shape of a slender beam of length l undergoing bending

vibrations. In terms of the true mode shapes $\Phi_i(x)$ $(i = 1, 2, \ldots, n)$ of the beam $w(x)$ may be written as

$$w(x) = \sum_{i=1}^{n} C_i \Phi_i(x)$$

where C_i are constants.

Using the orthogonality relations

$$\left.\begin{aligned} \int_0^l m(x)\Phi_i(x)\Phi_j(x)\,dx &= 0 \\ \int_0^l EI(x)\Phi_i''(x)\Phi_j''(x)\,dx &= 0 \end{aligned}\right\} \quad \text{for } i \neq j$$

and

$$\int_0^l EI(x)\Phi_r''^{2}(x)\,dx = \omega_r^2 \int_0^l m(x)\Phi_r^2(x)\,dx \quad \text{(see Problem 7, Chapter 3),}$$

show that the insertion of $w(x)$ above in Eq. (4.8), will yield ω_1^2 only if $w(x)$ is precisely $\Phi_1(x)$, and that for any other choice of $w(x)$ the resulting frequency will always be larger than ω_1.

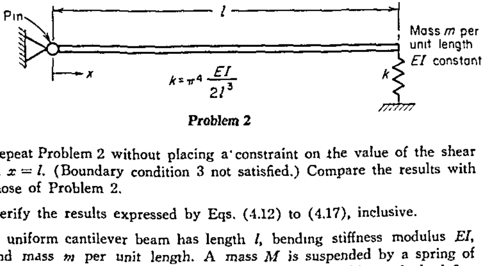
2. Compute the first two natural frequencies and mode shapes for the uniform beam shown. Use mode functions

$$\phi_1(x) = \frac{x}{l}$$

$$\phi_2(x) = \sin\frac{\pi x}{l}$$

and satisfy the boundary conditions

1. $w(0) = 0$
2. $w''(0) = w''(l) = 0$ (moments vanish at $x = 0$ and $x = l$)
3. $EIw'''(l) = kw(l)$ (shear at $x = l$ is equal to the spring force)



Problem 2

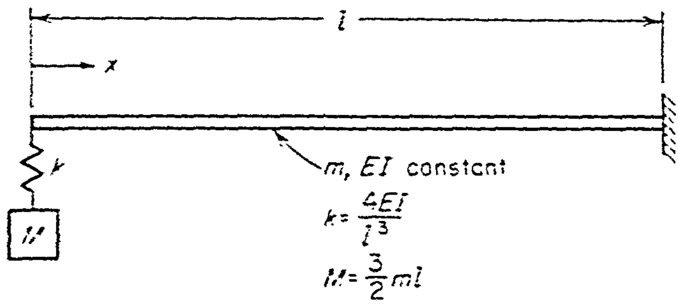
3. Repeat Problem 2 without placing a constraint on the value of the shear at $x = l$. (Boundary condition 3 not satisfied.) Compare the results with those of Problem 2.

4. Verify the results expressed by Eqs. (4.12) to (4.17), inclusive.

5. A uniform cantilever beam has length l, bending stiffness modulus EI, and mass m per unit length. A mass M is suspended by a spring of stiffness k at the free end as shown. Using Rayleigh's method, define

an approximate mode shape and compute the first natural frequency of the system.

Hint: Choose as a mode shape the deformed configuration of the cantilever and spring resulting from the application of a unit load at mass M.



Problem 5

6. Analyze the system of Problem 5 as a two-degree-of-freedom system using the Rayleigh-Ritz method. Express the displacement of the system by

$$w(x, t) = \sum_{i=1}^{2} \phi_i(x) q_i(t)$$

using

$$\phi_1(x) = \frac{1}{2}\left\{\left(\frac{x}{l}\right)^3 - 3\left(\frac{x}{l}\right) + 2\right\} \qquad \text{(for the beam)}$$

and

$$\phi_2(0) = 1 \qquad \text{(for mass } M\text{)}$$

Compute the two natural frequencies and corresponding mode shapes. Plot the mode shapes.

7. Differentiate Eq. (4.22) to obtain $(d/dp)\omega^2(p)$. Show that this derivative is zero when $p = p_r$, where p_r corresponds to the rth natural mode.

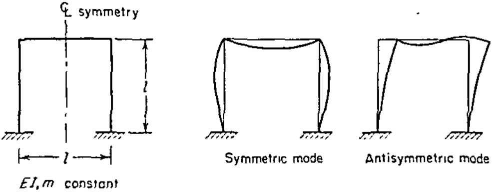
8. Carry out the operations of Eq. (4.25) for the slender beam whose natural frequencies are given by Eq. (4.8). Formulate the eigenvalue problem. The displacement $w(x)$ of the beam is expressed by Eq. (4.24).

9. Using the Rayleigh-Ritz method, compute the first two natural modes and frequencies for a cantilever beam of length l, with constant flexural stiffness EI, and a variable mass $m(x)$ per unit length given by

$$m(x) = m_0\left(1 - \frac{x}{2l}\right)$$

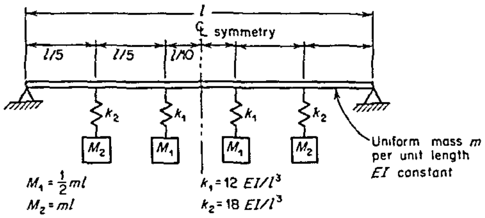
[Define at least three displacement functions $\phi_i(x)$.]

10. Assuming mode shapes, compute the first symmetric and antisymmetric natural modes and frequencies of the symmetric portal frame shown in the figure for Problem 10 on the next page. Show that the symmetrical and antisymmetrical modes are not coupled.



Problem 10

11. Using the Rayleigh-Ritz method, compute the first two symmetrical modes and frequencies for the system shown. Plot the mode shapes.



Problem 11

12. Compute the first two antisymmetric modes for the system of Problem 11.

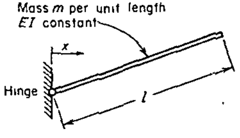
13. The uniform beam of mass m per unit length and constant EI is hinged at one end and free at the other end. The beam is free to rotate as a rigid body about the hinge in the plane of the paper. Compute the first three mode shapes and corresponding frequencies. Plot the mode shapes. Use

$$\phi_1(x) = \frac{x}{l}$$

$$\phi_2(x) = \sin\frac{\pi x}{l}$$

$$\phi_3(x) = \sin\frac{2\pi x}{l}$$

$$\phi_4(x) = \sin\frac{3\pi x}{l}$$



Problem 13

The boundary conditions are $w(0) = w''(0) = w''(l) = w'''(l) = 0$. The last boundary condition is not satisfied by the chosen functions.

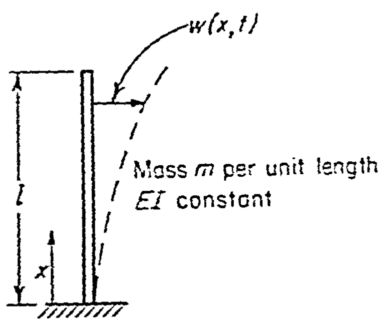
14. Verify Eqs. (4.89), (4.90), (4.91), and (4.93).

15. Fill in all the steps leading to the expression for the generalized stiffness k_{ri} as given by Eq. (4.101).

16. Compute the first three modes of vibration of a hinged beam rotating with angular velocity Ω about a vertical axis. Use $\phi_1 = x/l$, and select for ϕ_2 and ϕ_3 the first and second mode functions of a cantilever beam (or good approximations to these functions). The beam is uniform with constant EI and mass m per unit length. Let

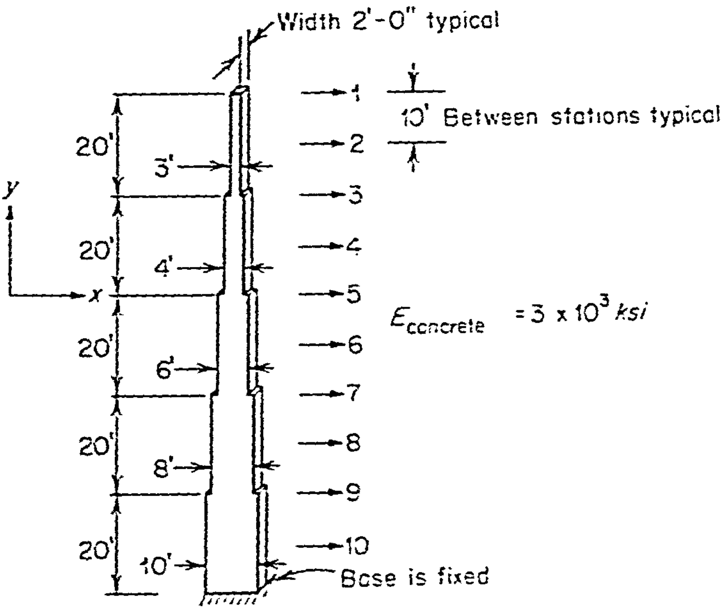
$$\Omega^2 = (\text{constant})\frac{EI}{ml^4}$$

17. A uniform column is built in at the bottom and free at the top. Approximate the lateral deflection $w(x, t)$ in a suitable set of mode functions $\phi_i(x)$ and generalized coordinates $q_i(t)$, $i = 1, 2, \ldots, n$. Using Lagrange's equations, formulate the equations of motion in normal mode vibrations, including the effect of gravity on the distributed mass. Write the equations for the generalized mass and stiffness coefficients using small-amplitude approximation.



Problem 17

18. Using Eq. (4.111) as applied to lumped mass systems, compute the first three translational natural mode shapes and frequencies for a simplified model of the structure vibrating in the x direction in the plane of the



Problem 18

paper. The structure is constructed of reinforced concrete. For simplicity, assume that the entire cross-section is effective in bending [the moment of inertia at each section $= 1/12(\text{width})(\text{depth})^3$]. Lump the mass at 10′ intervals at the ten stations shown in the figure. The weight of concrete is 150 lb/ft^3 and the modulus of elasticity $E = 3 \times 10^3$ ksi. Select the first four mode shapes. Recall that the first mode shape has one node, the second has two nodes, and so on.

CHAPTER 5

Differential Equation Methods

5.1 Introduction

In this chapter we shall consider methods for the determination of natural modes and frequencies through the solution of governing differential equations. These methods may be classified into two categories, *exact methods* and *approximate methods*. By exact methods we mean those in which an explicit solution of the differential equation is obtained in closed form. In general such solutions are possible only for linear differential equations with constant coefficients. Equations of this type are obtained in connection with small-amplitude vibrations of structures having uniformly distributed mass and stiffness properties. Where these properties are not uniformly distributed, the coefficients are variable and exact solutions are not always possible. For such cases approximate methods may be used.

It is characteristic of differential equation methods that the frequency equation results explicitly from the satisfying of boundary conditions. This is true whether the solution is an exact or an approximate one. To illustrate this point, two examples are chosen in which structural elements having uniformly distributed properties are treated by exact methods.

5.2 Torsional Vibrations of a Uniform Slender Beam

The torsional moment-deflection relationship for a slender beam is given by $M_T = GJ(\partial\theta/\partial x)$, where GJ is the torsional stiffness. The governing differential equation of motion in torsional vibration is formulated from this equation together with a consideration of torque equilibrium of an element of the beam, such as shown in Fig. 5.1.

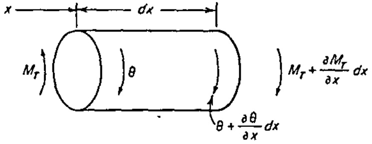


Figure 5.1

The equation of torque equilibrium is found by considering the torques at the two ends of the element and the inertial torque associated with the angular acceleration $\ddot{\theta}$.

$$I\ddot{\theta}\,dx = -M_T + \left(M_T + \frac{\partial M_T}{\partial x}dx\right)$$

where I is the mass moment of inertia per unit of length. We may reduce this equation and substitute

$$GJ\frac{\partial^2\theta}{\partial x^2} \quad \text{for} \quad \frac{\partial M_T}{\partial x}$$

to give

$$I\ddot{\theta} = GJ\frac{\partial^2\theta}{\partial x^2} \tag{5.1}$$

In a natural mode of vibration at frequency ω, we may obtain from Eq. (5.1) the following ordinary differential equation in x, where θ represents the maximum amplitude of vibration.

$$\frac{d^2\theta}{dx^2} + \frac{\omega^2 I}{GJ}\theta = 0 \tag{5.2}$$

The solution of Eq. (5.2) will be expressed in terms of two undetermined constants, C_1 and C_2, and may take the convenient form

$$\theta = C_1 \cos\sqrt{\frac{\omega^2 I}{GJ}}\,x + C_2 \sin\sqrt{\frac{\omega^2 I}{GJ}}\,x \tag{5.3}$$

It may be expected that the constants will be determined by making the above general solution satisfy two prescribed boundary conditions which must exist.

Let us consider first a beam fixed at end $x = 0$ and free at end $x = l$. The two corresponding boundary conditions are

$$\theta(0) = 0 \qquad \text{and} \qquad \frac{d\theta}{dx}(l) = 0$$

The second condition stems from the fact that the torque M_T, which is proportional to $d\theta/dx$, is zero at the free end. From the first condition the constant C_1 is determined to be zero. For the second condition the first derivative must be found. Thus

$$\frac{d\theta}{dx} = C_2\sqrt{\frac{\omega^2 I}{GJ}}\cos\sqrt{\frac{\omega^2 I}{GJ}}\,x$$

The second boundary condition leads to the equation

$$C_2\sqrt{\frac{\omega^2 I}{GJ}}\cos\sqrt{\frac{\omega^2 I}{GJ}}\,l = 0 \tag{5.4}$$

This equation can be satisfied by one of the following conditions:

1. $C_2 = 0$
2. $\omega = 0$
3. $\sqrt{\dfrac{\omega^2 I}{GJ}}\,l = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots, \dfrac{2n-1}{2}\pi \qquad$ where $n = 1, 2, 3, \ldots$

The first two conditions correspond to a state of rest and, hence, lead to a valid, but uninteresting, solution. The third condition leads to the frequency equation since it stipulates that ω must take on only certain discrete values. Thus

$$\omega_n = \frac{2n-1}{2}\frac{\pi}{l}\sqrt{\frac{GJ}{I}} \qquad n = 1, 2, 3, \ldots \tag{5.5}$$

At these natural frequencies the beam undergoes free vibrations at an amplitude equal to the constant C_2, which must remain unspecified. This is in accord with our previous finding that we may determine the amplitude of a natural mode to within a single arbitrary constant. At frequencies other than those given by Eq. (5.5), the beam cannot undergo force-free vibration since Eq. (5.4) can then be satisfied only by a state of rest. The number n in Eq. (5.5) is the mode number beginning with the first mode, or lowest frequency, at $n = 1$. Since n can take on an infinity of integral values, we see that there is an infinity of natural modes, at least in theory. In practice, only the lower modes are significant. We are led to the mode shapes by substituting Eq. (5.5) into (5.3) together with the equation $C_1 = 0$.

$$\theta = C_t \sin\left(\frac{2n-1}{2}\pi\frac{x}{l}\right) \tag{5.6}$$

We may normalize the modes so that the amplitude is unity at the free end, by setting $C_t = 1$. The first three modes, thus normalized, are shown in Fig. 5.2.

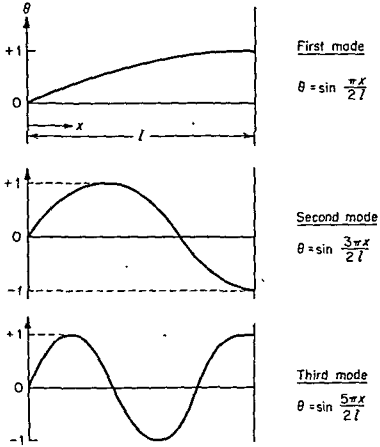


Figure 5.2

Next, we shall consider the effect of attaching a rigid body to the free end of the beam, as shown in Fig. 5.3. The body has a mass

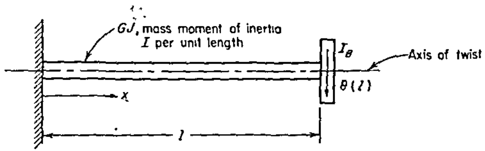


Figure 5.3

moment of inertia I_B about the axis of twist and is attached so that its center of mass lies on that axis. The boundary condition at $x = 0$ is the same as before but the torque in the shaft at the end $x = l$ is now equal to the inertial torque imposed by the attached body. This torque is equal to

$$M_T(l) = \omega^2 I_B \theta(l)$$

The internal torque in the shaft at this end is also given by

$$M_T(l) = GJ\frac{d\theta}{dx}(l)$$

Since the two torques are equal, the boundary condition is found by equating the two expressions. Thus

$$\frac{d\theta}{dx}(l) = \frac{\omega^2 I_B}{GJ}\theta(l)$$

As before, the general solution (5.3) holds and the boundary condition at $x = 0$ requires that the constant C_1 be zero. The second boundary condition, expressed above, leads to the equation

$$C_2\sqrt{\frac{\omega^2 I}{GJ}}\cos\sqrt{\frac{\omega^2 I}{GJ}}\,l = C_2\frac{\omega^2 I_B}{GJ}\sin\sqrt{\frac{\omega^2 I}{GJ}}\,l$$

or

$$\tan\sqrt{\frac{\omega^2 I}{GJ}}\,l = \frac{I}{I_B}\sqrt{\frac{GJ}{\omega^2 I}}$$

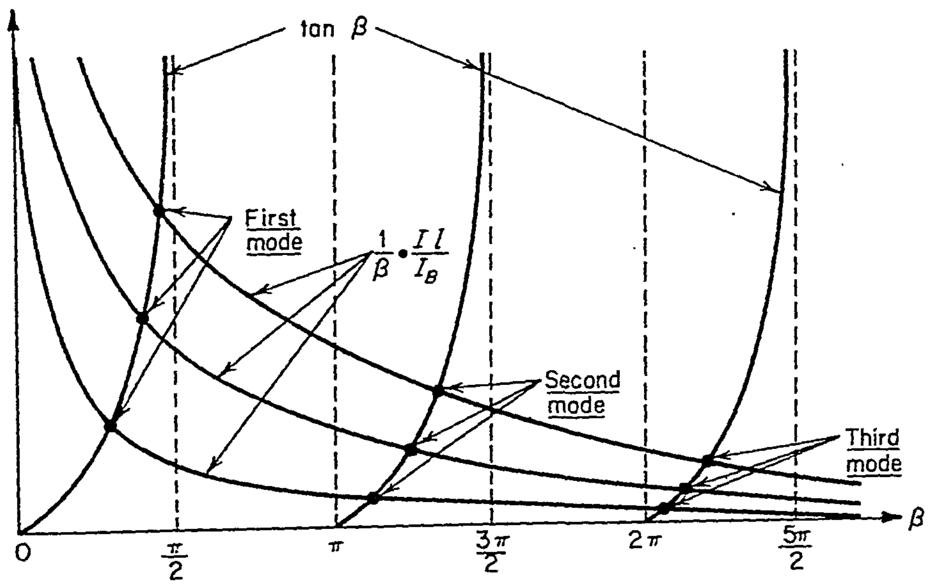


Figure 5.4

This is the frequency equation to be satisfied by the natural frequencies ω. For convenience we shall define an eigenvalue β by

$$\beta = \sqrt{\frac{\omega^2 I}{GJ}}\, l \tag{5.7}$$

The frequency equation then takes the form

$$\beta \tan \beta = \frac{Il}{I_B} \tag{5.8}$$

The roots of this equation are shown graphically in Fig. 5.4 as the intersections of $\tan \beta$ and the family of rectangular hyperbolas $(1/\beta)(Il/I_B)$ (three of which are shown) both plotted against β. The eigenvalues lie between 0 and $\pi/2$ for the first mode, between π and $3\pi/2$ for the second, and between 2π and $5\pi/2$ for the third. Higher mode intersections are not shown but their existence is obvious. For a specific value of Il/I_B the eigenvalues may be determined graphically or by numerical methods such as the Newton-Raphson[6] iterative process. From Eq. (5.7) it follows that the corresponding natural frequencies are given by

$$\omega = \frac{\beta}{l}\sqrt{\frac{GJ}{I}} \tag{5.9}$$

It is interesting to observe in Fig. 5.4 that the limiting values of β obtained as I_B goes to zero (corresponding to the vanishing of the end mass) are $\pi/2, 3\pi/2, 5\pi/2, \ldots, (2n-1)\pi/2$ for the successive modes, thus corroborating the results given by Eq. (5.5).

As before, the mode shape is found by substituting Eq. (5.9) into the general solution (5.3). Thus

$$\theta = C_2 \sin \beta \frac{x}{l} \tag{5.10}$$

5.3 Bending Vibrations of a Uniform Slender Cantilever Beam

For our second example, let us determine the natural modes and frequencies of a uniform cantilever beam. If we neglect the effects of shear deformations and rotatory inertia, Eq. (2.94) of Chapter 2 applies. For free vibrations

$$\ddot{w} = -\omega^2 w$$

Hence, Eq. (2.94) is written as

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] - \omega^2 m w = 0 \tag{5.11}$$

This equation holds for a nonuniform beam. But for the present example, in which we consider the beam to be uniform, the quantities m and EI are constants. Hence, the equation may be written in the following form for this example.

$$\frac{d^4w}{dx^4} - \frac{\omega^2 m}{EI} w = 0 \tag{5.12}$$

The solution to this fourth-order differential equation contains four constants and is written in the form

$$w(x) = C_1 \cosh \beta x + C_2 \sinh \beta x + C_3 \cos \beta x + C_4 \sin \beta x \tag{5.13}$$

where

$$\beta^4 = \frac{\omega^2 m}{EI} \tag{5.14}$$

As in the previous example the frequency equation is obtained by satisfying a set of prescribed boundary conditions. For a cantilever beam with x measured from the fixed end, these conditions are

$$w(0) = \frac{dw}{dx}(0) = 0$$

and

$$\frac{d^2w}{dx^2}(l) = \frac{d^3w}{dx^3}(l) = 0$$

The first two conditions are clear from the nature of the fixity. The last two conditions stem from the fact that the bending moment and shear vanish at the free end. To express these boundary conditions in terms of the constants $C_1 \ldots C_4$, we shall need the first three derivatives of Eq. (5.13). These are

$$\frac{dw}{dx} = \beta(C_1 \sinh \beta x + C_2 \cosh \beta x - C_3 \sin \beta x + C_4 \cos \beta x) \tag{5.15}$$

$$\frac{d^2w}{dx^2} = \beta^2(C_1 \cosh \beta x + C_2 \sinh \beta x - C_3 \cos \beta x - C_4 \sin \beta x) \tag{5.16}$$

$$\frac{d^3w}{dx^3} = \beta^3(C_1 \sinh \beta x + C_2 \cosh \beta x + C_3 \sin \beta x - C_4 \cos \beta x) \tag{5.17}$$

By inserting the four boundary conditions into Eq. (5.13) and the first three derivatives above, the following equations among the constants $C_1 \ldots C_4$ are obtained.

$$\left.\begin{aligned} C_1 \qquad\qquad\qquad + C_3 \qquad\qquad\qquad &= 0 \\ C_2 \qquad\qquad\qquad + C_4 \qquad &= 0 \\ C_1 \cosh \beta l + C_2 \sinh \beta l - C_3 \cos \beta l - C_4 \sin \beta l &= 0 \\ C_1 \sinh \beta l + C_2 \cosh \beta l + C_3 \sin \beta l - C_4 \cos \beta l &= 0 \end{aligned}\right\} \tag{5.18}$$

For arbitrary values of βl, the above set of equations can be satisfied only by letting all four constants vanish. This corresponds to a state of rest equilibrium. If all four constants are not to vanish, Eq. (5.18) can be satisfied only by the vanishing of the determinant of the set. When we set this determinant to zero we obtain the frequency equation

$$\cosh \beta l \cos \beta l + 1 = 0 \tag{5.19}$$

Values of βl which satisfy this equation are the eigenvalues corresponding to the natural frequencies of vibration. Again, we show these eigenvalues graphically by writing Eq. (5.19) in the form

$$\cosh \beta l = -\sec \beta l$$

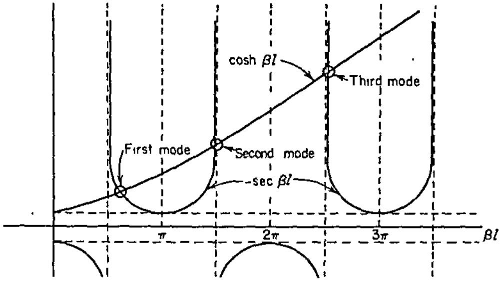


Figure 5.5

These two functions are plotted against βl in Fig. 5.5, and their intersections locate the eigenvalues. Using the eigenvalues βl the natural frequencies are determined from Eq. (5.14). Thus

$$\omega = (\beta l)^2 \sqrt{\frac{EI}{ml^4}} \tag{5.20}$$

Using Eqs. (5.18), any three of the constants $C_1 \ldots C_4$ can be expressed in terms of the fourth one. Thus, the deflection $w(x)$ in Eq. (5.13) may be expressed in terms of any one of the constants. If that constant, which is then arbitrary, is adjusted so that the tip deflection is unity, we obtain the following equation defining the mode shapes.

$$\begin{aligned} w\left(\frac{x}{l}\right) = \frac{1}{2} & \frac{\left[\cosh \beta l + \cos \beta l\right]\left[\sinh \beta l\left(\frac{x}{l}\right) - \sin \beta l\left(\frac{x}{l}\right)\right]}{\sinh \beta l \cos \beta l - \cosh \beta l \sin \beta l} \\ & - \frac{1}{2} \frac{\left[\sinh \beta l + \sin \beta l\right]\left[\cosh \beta l\left(\frac{x}{l}\right) - \cos \beta l\left(\frac{x}{l}\right)\right]}{\sinh \beta l \cos \beta l - \cosh \beta l \sin \beta l} \end{aligned} \tag{5.21}$$

By means of this equation any mode shape may be found by substituting the eigenvalue βl for that mode.

In higher modes, it is seen from Fig. 5.5 that the eigenvalues βl will be approximated by

$$\beta l \approx \frac{2n-1}{2}\pi \tag{5.22}$$

where n is the mode number.

5.4 Natural Frequencies of Single-Span Uniform Beams with Various End Constraints

The examples of the last two sections have shown how the frequency equation follows from the application of boundary conditions to the general solution of the differential equation governing the motion of the structure. From this it follows that two structures, identical in all respects except for boundary constraints, will have different natural modes and frequencies of vibration. As will be seen, there are cases in which the frequencies are identical but, in such cases, the mode shapes differ.

In this section we shall summarize the natural frequencies for simple, uniform beams with various kinds of constraints at the ends. We shall be concerned with only the "natural" boundary conditions in which the constraint forces at the ends do no work. For cases in which constraint forces do work on the beam, we must have information concerning the force-deflection properties of the attached structure which reacts these forces. This, in effect, requires that we extend the boundaries of our structure beyond those of the simple beam itself.

Constraints which do no work on an arbitrary displacement of the beam must be such that either the displacements at the constraints or the constraint forces are zero. Here we use the terms *displacement* and *force* in the general sense—displacement including rotation, force including moment or torque. In beams, work at constraints might be computed from either or both of the products:

(Transverse shear) · (transverse displacement)

(Bending moment) · (rotation)

For both products to be zero at the end of a beam, one term in each product must vanish. Thus, the boundary conditions must occur in pairs at each end. The following four combinations are possible.

$$\left.\begin{array}{lll} \text{Transverse displacement} = 0, & w = 0 \\ \text{Bending moment} = 0, & w'' = 0 \end{array}\right\} \quad \text{hinged end}$$

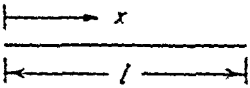
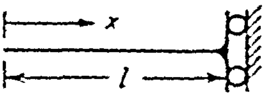
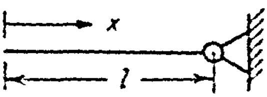
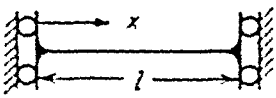
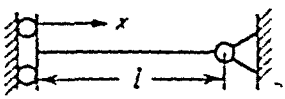
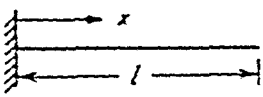
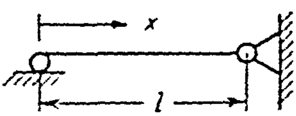
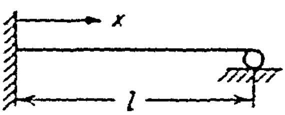
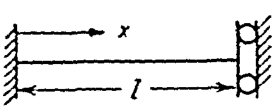
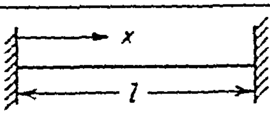
Transverse displacement $= 0$, Rotation $= 0$,	$w = 0$, $w' = 0$	clamped end
Bending moment $= 0$, Transverse shear $= 0$,	$w'' = 0$, $w''' = 0$	free end
Rotation $= 0$, Transverse shear $= 0$,	$w' = 0$, $w''' = 0$	guided end

Considering both ends, we have four boundary conditions as we would expect to have in a problem governed by a differential equation of fourth order. Since any pair of conditions at one end may be combined with any one of the four pairs at the other end, then we have ten possible combinations or ten beam types where the beam is uniform. If the beam is nonuniform in such a way that it is not symmetrical with respect to its midpoint, then sixteen different combinations are possible. Each one is typified by a unique eigenvalue problem and, hence, by its own unique set of natural modes. In all cases the natural frequencies can be expressed in terms of eigenvalues βl according to Eq. (5.20). In Table 5.1 the eigenvalues corresponding to the first three modes are listed for all ten beam types.

The following points are of interest in connection with the computation of the values in the following table. The first point to be mentioned has to do with the periodicity of the eigenvalues. Those for the hinged-hinged, guided-guided, and guided-hinged boundary conditions are completely periodic beginning with the first mode, so that the expressions giving $(\beta l)_n$ in terms of the mode number are exact in all modes. In all other cases the eigenvalues do not occur in periodic intervals as has been seen, for example, in the cantilever or clamped-free case treated in Section 5.3. However, in these cases the eigenvalues are very nearly equally spaced for modes higher than the third so that the expression giving $(\beta l)_n$ in terms of mode number is, although approximate, quite accurate beyond $n = 3$.

The second point to be brought out is that the boundary conditions can be grouped in pairs, each pair having the same frequency equation. Thus, the first and last cases have the same frequency equation, and the second and ninth, the third and eighth, and the fourth and seventh may be similarly paired. In all these cases, zero eigenvalues exist. In the first four cases it is apparent that rigid-body motion can occur. Hence, in these cases a zero eigenvalue or frequency is considered to correspond to this motion. The first mode number, then, represents rigid-body motion. In the last four cases the zero frequency corresponds to a state of rest equilibrium. In the fifth and sixth cases, namely the guided-hinged and clamped-free, zero eigenvalues do not occur. For higher modes the frequencies in these two cases approach equality.

TABLE 5.1 EIGENVALUES FOR SINGLE-SPAN UNIFORM BEAMS

Case	Boundary Conditions		$(\beta l)_n$			
			$n=1$	$n=2$	$n=3$	$n>3$
1		Free-Free $w''(0)=0 \quad w''(l)=0$ $w'''(0)=0 \quad w'''(l)=0$	0	4.730	7.853	$\frac{2n-1}{2}\pi$ (approx)
2		Free-Guided $w''(0)=0 \quad w'(l)=0$ $w'''(0)=0 \quad w'''(l)=0$	0	2.365	5.498	$\frac{4n-5}{4}\pi$ (approx)
3		Free-Hinged $w''(0)=0 \quad w(l)=0$ $w'''(0)=0 \quad w''(l)=0$	0	3.927	7.069	$\frac{4n-3}{4}\pi$ (approx)
4		Guided-Guided $w'(0)=0 \quad w'(l)=0$ $w'''(0)=0 \quad w'''(l)=0$	0	3.142	6.283	$(n-1)\pi$ (exact)
5		Guided-Hinged $w'(0)=0 \quad w(l)=0$ $w'''(0)=0 \quad w''(l)=0$	1.571	4.712	7.854	$\frac{2n-1}{2}\pi$ (exact)
6		Clamped-Free $w(0)=0 \quad w''(l)=0$ $w'(0)=0 \quad w'''(l)=0$	1.875	4.694	7.855	$\frac{2n-1}{2}\pi$ (approx)
7		Hinged-Hinged $w(0)=0 \quad w(l)=0$ $w''(0)=0 \quad w''(l)=0$	3.142	6.283	9.425	$n\pi$ (exact)
8		Clamped-Hinged $w(0)=0 \quad w(l)=0$ $w'(0)=0 \quad w''(l)=0$	3.927	7.069	10.210	$\frac{4n+1}{4}\pi$ (approx)
9		Clamped-Guided $w(0)=0 \quad w'(l)=0$ $w'(0)=0 \quad w'''(l)=0$	2.365	5.498	8.639	$\frac{4n-1}{4}\pi$ (approx)
10		Clamped-Clamped $w(0)=0 \quad w(l)=0$ $w'(0)=0 \quad w'(l)=0$	4.730	7.853	10.996	$\frac{2n+1}{2}\pi$ (approx)

$$\omega_n = (\beta l)_n^2 \sqrt{\frac{EI}{ml^4}}$$

5.5 The Holzer Method

From the historical point of view it is appropriate to begin a discussion of approximate methods for solution of differential equations with the method developed by Holzer[20] for treating torsional vibrations. Mathematically, this method approximates differential equations of motion by difference equations in which the independent space variable takes on only selected discrete values. Physically interpreted, the method replaces the distributed structure with one in which the masses and connecting elastic elements are discretized or "lumped." We shall consider the torsional vibrations of a non-uniform shaft for which a part of the lumped system is shown in Fig. 5.6.

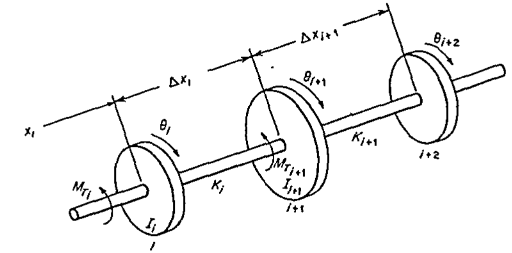


Figure 5.6

The equation of moment equilibrium for the ith mass is

$$M_{T_{i+1}} - M_{T_i} + \omega^2 I_i \theta_i = 0$$

where

$M_{T_{i+1}}$ = torque applied to the mass $i + 1$ by shaft section i

M_{T_i} = torque applied to mass i by shaft section $i - 1$

I_i = moment of inertia of mass i

θ_i = angular displacement of mass i

ω = angular frequency of vibration

A second equation may be written expressing the twist of shaft section i in terms of the torque on that section.

$$\theta_{i+1} - \theta_i = \frac{M_{T_{i+1}}}{k_i}$$

where

$$k_i = \text{stiffness of shaft section } i$$

These equations are written in the following form for the analysis

$$M_{T_{i+1}} = M_{T_i} - \omega^2 I_i \theta_i \tag{5.23}$$

$$\theta_{i+1} = \theta_i + \frac{M_{T_{i+1}}}{k_i} \tag{5.24}$$

Although Eqs. (5.23) and (5.24) were derived by direct reference to the system of Fig. 5.6, they may be derived, alternately, from the differential Eq. (5.2) for a distributed shaft. The parametric form of this equation is written as

$$\left.\begin{aligned} \frac{dM_T}{dx} &= -\omega^2 I(x)\,\theta \\ \frac{d\theta}{dx} &= \frac{M_T}{GJ(x)} \end{aligned}\right\}$$

The first equation above comes from Eq. (5.1) for natural mode vibration. The second one expresses the torsional moment-deflection relationship. The first derivatives are approximated by difference forms as

$$\frac{dM_T}{dx} \approx \frac{M_{T_{i+1}} - M_{T_i}}{\Delta x_i}$$

$$\frac{d\theta}{dx} \approx \frac{\theta_{i+1} - \theta_i}{\Delta x_i}$$

Using these forms and the definitions

$$I_i = I(x)\,\Delta x_i$$

and

$$k_i = \frac{GJ(x)}{\Delta x_i}$$

the Eqs. (5.23) and (5.24) are obtained as a set of first-order difference equations replacing differential Eq. (5.2).

To lump the distributed structure we define an arbitrary number of points, say n, and associate with these points a set of discrete masses having moments of inertia $I_1, I_2, \ldots, I_n$. The lumped system will approximate the distributed system more and more closely as n becomes larger and larger. Since only n coordinates $\theta_1, \theta_2, \ldots, \theta_n$ are required to define an arbitrary displacement of the system, the lumped system has only n degrees of freedom. Therefore, the analysis will yield only n natural modes instead of an infinite number which, in theory, exist for a distributed system, as was seen in the example of Section 5.2.

To proceed to a solution of Eqs. (5.23) and (5.24), using the Holzer method, we must prescribe and satisfy a set of boundary conditions. An example to be followed will clarify the procedure. Let us consider a shaft of length l which is free at the end $x = 0$ and fixed at the end $x = l$. If we number the n points starting with 1 at the free end, these boundary conditions are then defined by the equations

$$\left.\begin{aligned} M_{T_1} &= 0 \\ \theta_n &= 0 \end{aligned}\right\} \tag{5.25}$$

The Holzer procedure begins by selecting a trial value of ω and, starting at one boundary (say at $x = 0$), progressing point by point along the shaft using Eqs. (5.23) and (5.24) until the other boundary is reached. If the selected value of ω corresponds to a natural frequency, the boundary condition at the far end will be satisfied. In general, this will not be attained on the first trial, and successive trials will be necessary. At the first point ($i = 1$) the equations are

$$M_{T_2} = 0 - \omega^2 I_1 \theta_1$$

$$\theta_2 = \theta_1 + \frac{M_{T_2}}{k_1}$$

An arbitrary value of θ_1 (say 1 radian) may be selected since we are free to choose the amplitude at one point on the shaft. This gives us a value for M_{T_2} from the first equation which is then substituted into the second to yield a value for θ_2. The second pair of equations are written for $i = 2$, into which these values for M_{T_2} and θ_2 are entered. Thus

$$M_{T_3} = M_{T_2} - \omega^2 I_2 \theta_2$$

$$\theta_3 = \theta_2 + \frac{M_{T_3}}{k_2}$$

Proceeding point by point we arrive finally at point $i = n - 1$ for which we write

$$M_{T_n} = M_{T_{n-1}} - \omega^2 I_{n-1} \theta_{n-1}$$

$$\theta_n = \theta_{n-1} + \frac{M_{T_n}}{k_{n-1}}$$

Into these equations, previously obtained values for $M_{T_{n-1}}$ and θ_{n-1} are substituted, giving values for M_{T_n} and θ_n. If θ_n satisfies the second prescribed boundary condition (5.25), the trial value for ω is one of the natural frequencies. If not, a new trial value is selected and the computations are repeated. Identification of the natural frequencies and selection of successive trial values are aided by plotting, in each case, the residual value for θ_n against the corresponding trial fre-

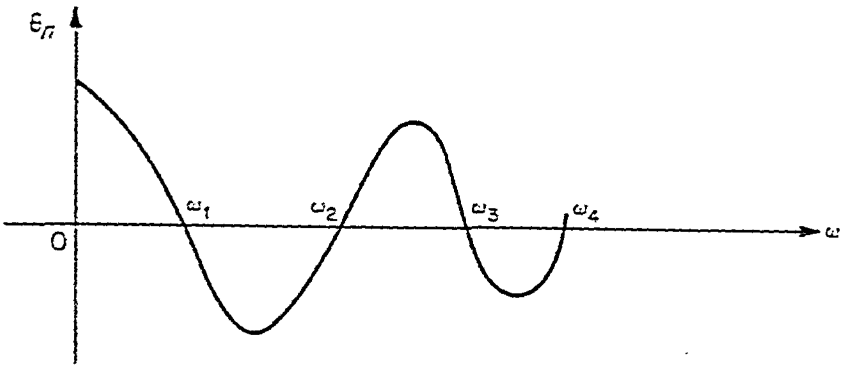


Figure 5.7

quency as in Fig. 5.7. The natural frequencies are the intercepts at which $\theta_n = 0$. There will be n of these intercepts; however, in practice it is generally necessary to determine only a small number of the lowest frequencies.

The modes are determined by computing the values $\theta_1, \ldots, \theta_n$ using the same procedure, but substituting for ω, the natural frequencies which have been determined.

In torsional vibrations, shafts supported at the ends have one boundary condition prescribed at each end. Natural boundary conditions require that the ends be either free or fixed. Hence, four possible sets of boundary conditions exist: free-free, fixed-free, free-fixed, and fixed-fixed. In the free-free case, a zero natural frequency will exist corresponding to rigid-body rotation. (See Problem 3, Chapter 3.) For this case there will be a frequency intercept at the origin in Fig. 5.7.

5.6 The Myklestad-Thomson Method

A method quite analogous to the Holzer method was originated by Myklestad[21] for the treatment of beams. The applicable equations were rearranged and simplified by Thomson[22] to permit a systematic tabular computation and to extend the applicability of the method to more general problems. In this section we concern ourselves only with flexural vibrations of beams supported at the ends and whose motions are governed by the differential Eq. (5.11). More general problems are considered in later sections.

In this procedure, as in the case of torsional vibrations, the beam is lumped by selecting n points along its length at which masses

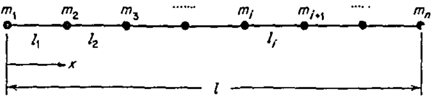


Figure 5.8

are concentrated, as shown in Fig. 5.8. Various methods for calculating the discrete masses and locating their positions are possible. Some of these methods are considered in Reference 10 with a discussion of their relative accuracies. Sections of the beam between the masses are considered to be uniform, and the stiffness properties for any section are taken as the mean values for the nonuniform beam over that section.

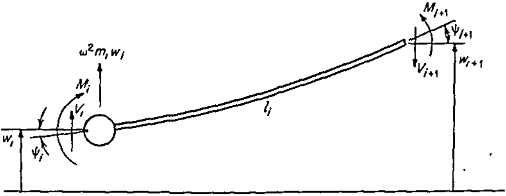


Figure 5.9

A typical section of the beam is shown in Fig. 5.9, extending from a station just to the left of point i to a station just to the left of point $i + 1$. The length of the section is l_i and it has a bending stiffness modulus EI_i. Force $\omega^2 m_i w_i$ at point i arises from the inertia of the vibrating mass. For this example, rotatory inertia is neglected.

Equations relating deflections w, slopes ψ, bending moments M, and shears V at the two ends of the section are derived from the beam equations and are given by

$$V = \frac{dM}{dx}$$

$$M = EI\frac{d\psi}{dx}$$

$$\psi = \frac{dw}{dx}$$

From equilibrium requirements the two equations below are written

$$V_{i+1} = V_i + \omega^2 m_i w_i \tag{5.26}$$

$$M_{i+1} = l_i V_i + M_i + \omega^2 m_i l_i w_i \tag{5.27}$$

By successive integration the slope and deflection relationships are derived, which lead to the next two equations.

$$\psi_{i+1} = \frac{l_i^2}{2EI_i} V_i + \frac{l_i}{EI_i} M_i + \psi_i + \frac{\omega^2 m_i l_i^2}{2EI_i} w_i \tag{5.28}$$

$$w_{i+1} = \frac{l_i^3}{6EI_i} V_i + \frac{l_i^2}{2EI_i} M_i + l_i \psi_i + \left(1 + \frac{\omega^2 m_i l_i^3}{6EI_i}\right) w_i \tag{5.29}$$

The four equations (5.26) through (5.29), are involved in the computation of V, M, ψ, and w at each point starting at one end of the beam and proceeding point by point to the other end. To begin the computation the boundary conditions must be known. If we start at the left end with point 1 the two boundary conditions at that end are inserted into the equations. At the completion of the computations, the two boundary conditions at the right end must be satisfied and this provides the criterion for determining the natural frequencies. Again, we shall consider a specific example by choosing a simply-supported beam for which the boundary conditions are

$$\left.\begin{aligned} w_1 &= M_1 = 0 \\ w_n &= M_n = 0 \end{aligned}\right\} \tag{5.30}$$

For point 1, Eqs. (5.26) to (5.29) are obtained by substituting the boundary values at that point.

$$\left.\begin{aligned} V_2 &= V_1 \\ M_2 &= l_1 V_1 \\ \psi_2 &= \frac{l_1^2}{2EI_1} V_1 + \psi_1 \\ w_2 &= \frac{l_1^3}{6EI_1} V_1 + l_1 \psi_1 \end{aligned}\right\}$$

It is seen that the quantities V_1 and ψ_1 are undetermined, although one of them could be, if desired, selected arbitrarily. We shall, however, carry them along in the computations. To continue, for example, the equations at point 2 become

$$V_3 = \left(1 + \frac{\omega^2 m_2 l_1^3}{6EI_1}\right) V_1 + \omega^2 m_2 l_1 \psi_1$$

$$M_3 = \left(l_1 + l_2 + \frac{\omega^2 m_2 l_1^3 l_2}{6EI_1}\right) V_1 + \omega^2 m_2 l_1 l_2 \psi_1$$

$$\psi_3 = \left(\frac{l_1^2}{2EI_1} + \frac{l_1 l_2}{EI_2} + \frac{l_2^2}{2EI_2} + \frac{\omega^2 m_2 l_1^2 l_2^2}{12E^2 I_1 I_2}\right) V_1 + \left(1 + \frac{\omega^2 m_2 l_1 l_2^2}{2EI_2}\right)\psi_1$$

$$w_3 = \left(\frac{l_1^3}{6EI_1} + \frac{l_1^2 l_2}{2EI_1} + \frac{l_1 l_2^2}{2EI_2} + \frac{l_2^3}{6EI_2} + \frac{\omega^2 m_2 l_1^2 l_2^3}{36E^2 I_1 I_2}\right) V_1 + \left(l_1 + l_2 + \frac{\omega^2 m_2 l_1 l_2^3}{6EI_2}\right)\psi_1$$

It is evident that the four quantities may be determined at all points in a similar way and that they will be expressed as linear combinations of V_1 and ψ_1. In this example we are particularly concerned with satisfying the boundary values on w_n and M_n. From our computations we will obtain

$$M_n = AV_1 + B\psi_1$$
$$w_n = CV_1 + D\psi_1$$

where A, B, C, D are constants which will depend upon the constants of the system and the trial frequency. To satisfy the boundary conditions these quantities must be such that

$$AV_1 + B\psi_1 = 0$$
$$CV_1 + D\psi_1 = 0$$

For nonzero values of V_1 and ψ_1, the determinant of their coefficients in the above equations must vanish, or

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0 \tag{5.31}$$

The value of this determinant is plotted against trial frequencies ω in a manner similar to Fig. 5.7 in order to determine natural frequencies. Once these frequencies are found, they may be inserted into Eqs. (5.26) to (5.29) and the computations repeated to determine values of w at all points along with values of ψ, M, and V, if desired. Thus, the mode shapes are found.

5.7 Transfer Matrices

When the equations of the Holzer and Myklestad-Thomson methods are expressed in matrix form the methodology takes on added clarity and regularity, and the computations are even more readily handled by digital computers, since they involve the basic operations of matrix multiplication and inversion. Expression of these equations in matrix form gives rise to the concept of the transfer matrix which has wider application than implied in the methods of the last two sections. The formulation of transfer matrices and their use in a wide variety of structural and mechanical problems has been studied extensively by Pestel and Leckie.[23] These applications include structural systems such as frames and built-up shell structures. In this chapter we shall be content to discuss the concept and use of transfer matrices in connection with elemental beams in bending and/or torsion.

First, let us consider the torsional vibration problem of the last section and rewrite Eqs. (5.24) and (5.23) as follows

$$\theta_{i+1} = \left(1 - \frac{\omega^2 I_i}{k_i}\right)\theta_i + \frac{1}{k_i} M_{T_i}$$

$$M_{T_{i+1}} = -\omega^2 I_i \theta_i + M_{T_i}$$

In matrix form these equations appear as

$$\begin{Bmatrix} \theta \\ M_T \end{Bmatrix}_{i+1} = \begin{bmatrix} \left(1 - \dfrac{\omega^2 I_i}{k_i}\right) & \dfrac{1}{k_i} \\ -\omega^2 I_i & 1 \end{bmatrix} \begin{Bmatrix} \theta \\ M_T \end{Bmatrix}_i \tag{5.32}$$

Similarly, Eqs. (5.26) through (5.29), expressing the relationships in flexural vibration of a beam, may be put in the matrix form

$$\begin{Bmatrix} w \\ \psi \\ M \\ V \end{Bmatrix}_{i+1} = \begin{bmatrix} \left(1 + \dfrac{\omega^2 m_i l_i^3}{6EI_i}\right) & l_i & \dfrac{l_i^2}{2EI_i} & \dfrac{l_i^3}{6EI_i} \\ \dfrac{\omega^2 m_i l_i^2}{2EI_i} & 1 & \dfrac{l_i}{EI_i} & \dfrac{l_i^2}{2EI_i} \\ \omega^2 m_i l_i & 0 & 1 & l_i \\ \omega^2 m_i & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} w \\ \psi \\ M \\ V \end{Bmatrix}_i \tag{5.33}$$

Eqs. (5.32) and (5.33) have the same form despite the fact that the orders of the matrices differ. This form may be expressed in the matrix equation

$$\{Z\}_{i+1} = [U]_i \{Z\}_i \tag{5.34}$$

The column matrices $\{Z\}_{i+1}$ and $\{Z\}_i$ are called *state vectors* because they contain state variables which give information concerning the force-deflection state of the structure at each station. For the torsion problem, these state variables are θ and M_T, and in the bending problem they are w, ψ, M, and V. In other types of structures the state variables may be different, but so long as they are linearly dependent the state vectors at adjacent stations may be related through Eq. (5.34). The square matrix $[U]_i$ is called the *transfer matrix* at station i. From the two examples already carried out we see that its elements depend only upon the properties of the structure at station i and the frequency of vibration.

Equation (5.34) may be used to solve the boundary value problems of the last two sections and other similar problems in the following way. Again, we lump the properties of the beam at n points and number the points starting with 1 at the left end and progressing to point n at the right end as shown in Fig. 5.10. The boundary or

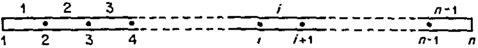


Figure 5.10

end conditions are known at the two end points. Starting at point 1 we may write the following equations relating the state vectors at successive stations.

$$\{Z\}_2 = [U]_1\{Z\}_1$$
$$\{Z\}_3 = [U]_2\{Z\}_2 = [U]_2[U]_1\{Z\}_1$$
$$\{Z\}_4 = [U]_3\{Z\}_3 = [U]_3[U]_2[U]_1\{Z\}_1$$

Finally, reaching the right end, the state vectors at the two end points are found to be related by the equation

$$\begin{aligned}\{Z\}_n &= [U]_{n-1}[U]_{n-2}\ldots[U]_2[U]_1\{Z\}_1 \\ &= [P]\{Z\}_1\end{aligned} \tag{5.35}$$

The matrix $[P]$ is an overall transfer matrix formed by taking the products of all the intermediate transfer matrices in the order indicated. It relates the vectors at the two end points at which the boundary conditions are known. These conditions fix the values of some of the elements in each of these state vectors. Hence, Eq. (5.35) imposes constraints on the elements of the matrix $[P]$ which can be satisfied only by certain values of the frequency ω. These values are the natural frequencies of vibration.

Again, we consider the torsional vibration problem in which the state vectors and transfer matrices are given in Eq. (5.32) and whose end conditions are specified in Eq. (5.25). Equation (5.35) for this problem appears as

$$\begin{Bmatrix} 0 \\ M_T \end{Bmatrix}_n = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{Bmatrix} \theta \\ 0 \end{Bmatrix}_1 \tag{5.36}$$

The elements p_{ij} of the product matrix $[P]$ depend on trial values of ω which are inserted into the transfer matrices $[U]_{n-1}\ldots[U]_1$ before multiplication. If we write the first equation of the pair included in Eq. (5.36) we obtain

$$0 = p_{11}\theta_1$$

Since θ_1, the amplitude of vibration at the free end of the beam is different from zero, it follows that this equation is satisfied only by the vanishing of the element p_{11}. Thus

$$p_{11} = 0 \tag{5.37}$$

This is the frequency equation for this problem through which the natural frequencies $\omega_1, \omega_2, \ldots, \omega_n$ are determined. Values of p_{11}, corresponding to trial values of ω, may be plotted as in Fig. 5.7 and the intercept values of ω determined.

For other end conditions the frequency equations result from the vanishing of the other elements of the product matrix. Since there are four elements, four frequency equations exist corresponding to four possible end conditions, as discussed in Section 5.5.

The procedure for determining the frequencies in flexural vibrations of the beam is similar to that discussed above but is complicated somewhat by the greater number of required boundary constraints. Equation (5.35) takes the following form in flexure.

$$\begin{Bmatrix} w \\ \psi \\ M \\ V \end{Bmatrix}_n = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \begin{Bmatrix} w \\ \psi \\ M \\ V \end{Bmatrix}_1 \tag{5.38}$$

For the simply-supported beam considered in Section 5.6 whose end conditions are given by Eq. (5.30), the last equation takes the specific form

$$\begin{Bmatrix} 0 \\ \psi \\ 0 \\ V \end{Bmatrix}_n = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{31} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \begin{Bmatrix} 0 \\ \psi \\ 0 \\ V \end{Bmatrix}_1 \tag{5.39}$$

The first and third equations of the set are

$$0 = p_{12}\psi_1 + p_{14}V_1$$

$$0 = p_{32}\psi_1 + p_{34}V_1$$

Since ψ_1 and V_1 are not zero unless the beam is in a state of rest equilibrium, the determinant of their coefficients in the two equations must vanish. Thus

$$\begin{vmatrix} p_{12} & p_{14} \\ p_{32} & p_{34} \end{vmatrix} = 0 \tag{5.40}$$

This is the frequency equation for this problem and the set of frequencies are to be found for which it is satisfied.

For other end conditions the frequency equation will require that other 2×2 subdeterminants of the product matrix $[P]$ vanish. Consider, for example, a beam clamped at station 1 and free at station n. For this beam Eq. (5.38) is

$$\begin{Bmatrix} w \\ \psi \\ 0 \\ 0 \end{Bmatrix}_n = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ M \\ V \end{Bmatrix}_1 \tag{5.41}$$

Following the reasoning used in the case of the simply-supported (hinged-hinged) beam, the frequency equation in this case is

$$\begin{vmatrix} p_{33} & p_{34} \\ p_{43} & p_{44} \end{vmatrix} = 0 \tag{5.42}$$

In the 4×4 product matrix for beams in flexure, 36 such subdeterminants exist. Therefore, it might be presumed that these correspond to 36 different sets of end conditions. This is not true, however, because some of the subdeterminants correspond to combinations of end constraints which are physically incompatible. We return to the discussion of Section 5.4 and point out that the vanishing of transverse shear and transverse displacement simultaneously, or of bending moment and rotation simultaneously, cannot occur physically and that these combinations do not ensure that work done on the beams by the end constraints is zero. Therefore, these combinations must be ruled out. This means that subdeterminants formed from rows 1 and 4 or rows 2 and 3, and from columns 1 and 4 or columns 2 and 3, are ruled out. There remain only 16 subdeterminants that correspond to physically possible boundary conditions. Following the reasoning used in Section 5.4 it can be seen that sixteen different combinations of end conditions are indeed possible, only ten of which are different when the beam is uniform or symmetrical with respect to a plane normal to it through its midpoint.

Finally, we turn to the problem of determining the natural modes of vibration by use of transfer matrices. Turning to the discussion leading to Eq. (5.35) we observe that the state vector at any intermediate station on the structure, say the rth, is given in terms of the state vector at station 1 by the equation

$$\{Z\}_r = [U]_{r-1}[U]_{r-2} \ldots [U]_2[U]_1\{Z\}_1 \tag{5.43}$$

For the desired mode, the corresponding frequency is inserted into the transfer matrices and the partial products found. Thus, we may determine the complete state of the structure at station r in terms of the state variables at station 1. In the torsion problem the only state variable at station 1 is θ_1. [See Eq. (5.36).] This quantity may be arbitrarily assigned. Hence, the torsional deflections and torsional moments are found at all stations. In the beam flexure problem two state variables at station 1 are undetermined such as ψ_1 and V_1 in the hinged-hinged beam. [See Eq. (5.39).] These two quantities are related, however, by either the first or third equations of that set. For this example it is noted that

$$\frac{V_1}{\psi_1} = -\frac{p_{12}}{p_{14}} = -\frac{p_{32}}{p_{34}}$$

where the p's are computed using the correct frequency for the mode to be determined. One of these two quantities, say ψ_1, may be assigned arbitrarily and it follows that corresponding state variables at all stations can be determined.

5.8 Construction of Transfer Matrices

In addition to the treatment of natural vibrations, transfer matrices may be used to determine the response of structures to both static and dynamic forces. The forces may be distributed or concentrated and may include reactions at intermediate supports and forces at structural connections. These forces may be included in the transfer matrices so that these matrices can be thought of as being constructed from two kinds of information—that concerned with properties of the structure and that concerned with the applied loads or forces. For this reason it is convenient to consider the transfer matrix as a product of two constituent matrices: a *field matrix*, concerned with properties of the structure in a region or "field" between points of force application, and a *point matrix* concerned with the force or reaction at a point.

At present we are concerned with natural vibrations. Hence, we shall consider this matter in context with the problems of the last section. First, considering the point matrix associated with point i, we relate state vectors at stations immediately to the left and to the right of that point by the equation*

$$\{Z\}_i^R = [U_P]_i\{Z\}_i^L \tag{5.44}$$

where the square matrix $[U_P]_i$ is the point matrix at point i. Next, we relate state vectors to the right of point i, and to the left of point $i+1$, through a field matrix associated with the region i between those two points. Thus

$$\{Z\}_{i+1}^L = [U_F]_i\{Z\}_i^R \tag{5.45}$$

Combining the two equations, we obtain

$$\{Z\}_{i+1}^L = [U_F]_i[U_P]_i\{Z\}_i^L \tag{5.46}$$

* Superscripts R and L designate right and left, respectively.

Comparing with Eq. (5.34) it is seen that the transfer matrix $[U]_i$ is the product of the field and point matrices as follows.

$$[U]_i = [U_F]_i [U_P]_i \tag{5.47}$$

In Eq. (5.34) it was implied that the two state vectors were associated with sections immediately to the left of their respective stations.

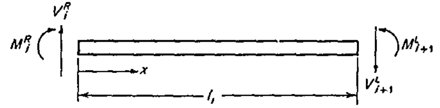


Figure 5.11

Next, let us consider the construction of the field matrix for section i of a beam in flexure. The section is shown in Fig. 5.11. The shears at the two ends of the section are related by

$$V_{i+1}^L = V_i^R \tag{5.48}$$

The bending moment M_x at any point at distance x from the left end, is given by the equation

$$M_x = V_i^R x + M_i^R$$

By successive integration the slope and deflection are obtained.

$$\psi_x = \frac{x^2}{2EI_i} V_i^R + \frac{x}{EI_i} M_i^R + \psi_i^R$$

$$w_x = \frac{x^3}{6EI_i} V_i^R + \frac{x^2}{2EI_i} M_i^R + x\psi_i^R + w_i^R$$

From these equations the following relationships are determined among the state variables to the right of station i and the left of station $i + 1$.

$$\left.\begin{aligned} M_{i+1}^L &= l_i V_i^R + M_i^R \\ \psi_{i+1}^L &= \frac{l_i^2}{2EI_i} V_i^R + \frac{l_i}{EI_i} M_i^R + \psi_i^R \\ w_{i+1}^L &= \frac{l_i^3}{6EI_i} V_i^R + \frac{l_i^2}{2EI_i} M_i^R + l_i\psi_i^R + w_i^R \end{aligned}\right\} \tag{5.49}$$

Equations (5.48) and (5.49) are expressed in matrix form and from this matrix equation the field matrix, as defined by Eq. (5.45), is seen to be

$$[U_F]_i = \begin{bmatrix} 1 & l_i & \frac{l_i^2}{2EI_i} & \frac{l_i^3}{6EI_i} \\ 0 & 1 & \frac{l_i}{EI_i} & \frac{l_i^2}{2EI_i} \\ 0 & 0 & 1 & l_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.50}$$

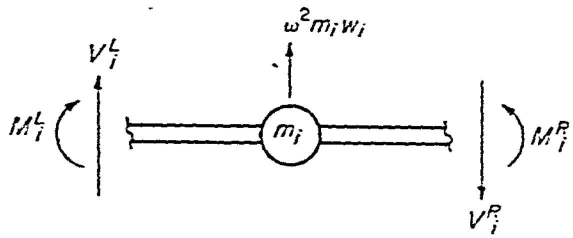


Figure 5.12

Next, the point matrix $[U_P]_i$ is determined by reference to Fig. 5.12. In this example the point force is the inertial force $\omega^2 m_i w_i$ associated with the lumped mass at point i. The equations of force and moment equilibrium are

$$\left.\begin{aligned} V_i^R &= V_i^L + \omega^2 m_i w_i \\ M_i^R &= M_i^L \end{aligned}\right\} \tag{5.51}$$

Assuming that we have deflection and slope continuity at point i, the point matrix defined by Eq. (5.44) is given by

$$[U_P]_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega^2 m_i & 0 & 0 & 1 \end{bmatrix} \tag{5.52}$$

It is readily verified that multiplication of the field and point matrices according to Eq. (5.47), results in the transfer matrix of Eq. (5.33).

The effect of rotatory inertia is easily handled by inclusion of the appropriate angular inertia term in the point matrix. If the radius of gyration of the mass m_i in Fig. 5.12 is ρ_i, the inertial moment would be $\omega^2 m_i \rho_i^2 \psi_i$ acting in a counterclockwise direction. The equation of moment equilibrium becomes

$$M_i^R = M_i^L - \omega^2 m_i \rho_i^2 \psi_i$$

The corresponding point matrix is given be

$$[U_P]_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\omega^2 m_i \rho_i^2 & 1 & 0 \\ \omega^2 m_i & 0 & 0 & 1 \end{bmatrix} \tag{5.53}$$

Other techniques are available for the construction of transfer matrices in problems of the type discussed here, as well as other problems of greater difficulty. A further discussion of these techniques may be found in Reference 23.

5.9 The Delta Matrix

In using the transfer matrix technique to determine natural modes and frequencies of beams, numerical difficulties are encountered in dealing with the higher modes. These difficulties arise in computing the 2×2 determinants involved in the frequency equations, such as in Eqs. (5.40) and (5.42), because in higher modes the determinant is a difference of two large numbers. Hence, it is necessary to carry a large number of significant figures in the computation in order to locate the eigenvalues accurately. The number of decimal digits necessary can exceed the capacity of most high-speed digital computers. A technique has been developed by which these numerical difficulties are circumvented through the use of the *delta matrix*.[24] In this section we shall discuss briefly the origin of the numerical difficulty and the development and use of the delta matrix.

If we confine our study to uniform, homogeneous beams we find it possible to determine an "overall" transfer matrix by use of Eqs. (5.13), (5.15), (5.16), and (5.17) which relates the state vectors at the two ends of the beam. Examination of this matrix reveals the source and extent of the numerical problem, and it may logically be inferred that the same general problem arises also in dealing with nonuniform beams. If we determine the constants C_1, C_2, C_3, and C_4 in the above mentioned equations, in terms of the state vector $\{Z\}_1$ at the left end of the beam where $x = 0$, we arrive at the following equation which relates the state vectors at the two ends.

$$\begin{Bmatrix} w \\ \psi \\ M \\ V \end{Bmatrix}_{\substack{n \\ x=l}} = \frac{1}{2} \begin{bmatrix} (\cosh\beta l + \cos\beta l) & \dfrac{\sinh\beta l + \sin\beta l}{\beta} & \dfrac{\cosh\beta l - \cos\beta l}{\alpha\beta^2} & \dfrac{\sinh\beta l - \sin\beta l}{\alpha\beta^3} \\ \beta(\sinh\beta l - \sin\beta l) & (\cosh\beta l + \cos\beta l) & \dfrac{\sinh\beta l + \sin\beta l}{\alpha\beta} & \dfrac{\cosh\beta l - \cos\beta l}{\alpha\beta^2} \\ \alpha\beta^2(\cosh\beta l - \cos\beta l) & \alpha\beta(\sinh\beta l - \sin\beta l) & (\cosh\beta l + \cos\beta l) & \dfrac{\sinh\beta l + \sin\beta l}{\beta} \\ \alpha\beta^3(\sinh\beta l + \sin\beta l) & \alpha\beta^2(\cosh\beta l - \cos\beta l) & \beta(\sinh\beta l - \sin\beta l) & (\cosh\beta l + \cos\beta l) \end{bmatrix} \begin{Bmatrix} w \\ \psi \\ M \\ V \end{Bmatrix}_{\substack{1 \\ x=0}} \tag{5.54}$$

where $\alpha = EI$

The square matrix is the overall transfer matrix for a uniform beam with distributed mass and stiffness properties, and corresponds to the product matrix $[P]$ of Eq. (5.35) obtained in dealing with the lumped parameter beam.

Although the argument to be developed applies irrespective of boundary conditions, it will be advantageous for clarity to select a particular set of boundary conditions. Let us choose the clamped-free beam and note that the frequency equation requires that the following determinant vanish.

$$\begin{vmatrix} (\cosh \beta l + \cos \beta l) & \dfrac{\sinh \beta l + \sin \beta l}{\beta} \\ \beta(\sinh \beta l - \sin \beta l) & (\cosh \beta l + \cos \beta l) \end{vmatrix} = 0 \qquad (5.55)$$

When expanded this leads to the frequency Eq. (5.19) already explored.

$$\cosh \beta l \cos \beta l + 1 = 0 \qquad (5.19)$$

We observe that the location of the eigenvalues depends upon the periodicity of $\cos \beta l$. However, in the transfer-matrix analysis the elements of the determinant of Eq. (5.55) are numerical so that the contribution of the $\sin \beta l$ and $\cos \beta l$ terms cannot be isolated. For higher values of βl the numerical values of these terms are very small compared with the terms $\sinh \beta l$ and $\cosh \beta l$. For example, the eigenvalue pertaining to the fifth mode ($n = 5$) is approximately

$$(\beta l)_5 \approx \frac{2 \times 5 - 1}{2} \pi = 14.1372 \qquad \text{(See Table 5.1.)}$$

For this eigenvalue we have

$$\sinh (\beta l)_5 \approx \cosh (\beta l)_5 \approx \tfrac{1}{2} e^{(\beta l)_5} \approx 0.66 \times 10^6$$

By comparison, the sine and cosine terms cannot exceed unity. Therefore, the numbers must be carried out to at least six significant digits for the influence of the sine and cosine terms to be felt at all. For accuracy a greater number of digits should be used. Obviously, this situation becomes rapidly more critical for the sixth and higher modes. By contrast, the calculations for the first eigenvalue present no problem because we are then concerned with a value of βl equal to 1.875, and the corresponding values of $\sinh \beta l$ and $\cosh \beta l$ are of the same order of magnitude as the values of $\sin \beta l$ and $\cos \beta l$.

So much for the explanation of the numerical difficulty. Now, let us view Eq. (5.54) in a slightly different way. Let us divide the

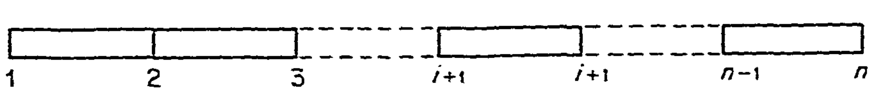


Figure 5.13

uniform beam *into* $n-1$ sections, as shown *in* Fig. 5.13, without lumping the masses at the station points.

Equation (5.54) may be viewed as relating state vectors at two adjacent stations, say $i+1$ and i, by *a transfer matrix if the* length l in that matrix is set equal to l_i, the length of the ith section. If we now divide the beam into ten equal sections, and seek the fifth eigenvalue, the value βl_i in the transfer matrix is equal to

$$\beta l_i = \frac{(\beta l)_5}{10} = 1.41372$$

If we were to compute the values of the subdeterminants of the transfer matrix for this case, there would be no difficulty in doing so because, *again, the* $\sinh \beta l_i$ *and* $\cosh \beta l_i$ *terms are of the same order* of magnitude as the $\sin \beta l_i$ and $\cos \beta l_i$ terms. This leads to the idea that if, in some way, we can compute and use the subdeterminants of the *separate transfer matrices* in Eq. (5.35) rather than multiplying these matrices first, then computing the appropriate subdeterminant of the product matrix, we will eliminate the numerical difficulty previously discussed, provided we divide the beam into a sufficiently large number of sections. This is precisely what is done through the use of delta matrices.

A delta matrix $[U^\Delta]$ is a square matrix formed from a corresponding transfer matrix $[U]$ in such a way that each of its elements is equal to a corresponding 2×2 subdeterminant in the transfer matrix. We have already noted that 36 subdeterminants exist in the 4×4 transfer matrix. Hence, the delta matrix has 36 elements and is *of order* 6×6. The *transformation is accomplished* through the use of the following lexicon which identifies the number of the row or column in the delta matrix with the corresponding pairs of rows or columns in the transfer matrix.

TABLE 5.2 DELTA MATRIX LEXICON

Row or Column in Delta Matrix	1	2	3	4	5	6
Row or Column Pair in Transfer Matrix	12	13	14	23	24	34

If we define the transfer matrix as

$$[U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix} \tag{5.56}$$

and the delta matrix as

$$[U^{\Delta}] = \begin{bmatrix} u_{11}^{\Delta} & u_{12}^{\Delta} & u_{13}^{\Delta} & u_{14}^{\Delta} & u_{15}^{\Delta} & u_{16}^{\Delta} \\ u_{21}^{\Delta} & u_{22}^{\Delta} & u_{23}^{\Delta} & u_{24}^{\Delta} & u_{25}^{\Delta} & u_{26}^{\Delta} \\ u_{31}^{\Delta} & u_{32}^{\Delta} & u_{33}^{\Delta} & u_{34}^{\Delta} & u_{35}^{\Delta} & u_{36}^{\Delta} \\ u_{41}^{\Delta} & u_{42}^{\Delta} & u_{43}^{\Delta} & u_{44}^{\Delta} & u_{45}^{\Delta} & u_{46}^{\Delta} \\ u_{51}^{\Delta} & u_{52}^{\Delta} & u_{53}^{\Delta} & u_{54}^{\Delta} & u_{55}^{\Delta} & u_{56}^{\Delta} \\ u_{61}^{\Delta} & u_{62}^{\Delta} & u_{63}^{\Delta} & u_{64}^{\Delta} & u_{65}^{\Delta} & u_{66}^{\Delta} \end{bmatrix} \tag{5.57}$$

we may then illustrate the use of the lexicon by the examples

$$u_{11}^{\Delta} = \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix}$$

$$u_{24}^{\Delta} = \begin{vmatrix} u_{12} & u_{13} \\ u_{32} & u_{33} \end{vmatrix}$$

$$u_{36}^{\Delta} = \begin{vmatrix} u_{13} & u_{14} \\ u_{43} & u_{44} \end{vmatrix}$$

In using the delta matrix technique, a delta matrix corresponding to each transfer matrix of Eq. (5.35) is constructed according to the above transformation. The product of these delta matrices is then taken, and must be equal to the delta matrix corresponding to the product matrix $[P]$. Thus, to prove that this technique is valid it must be proven that if

$$[U]_{n-1}[U]_{n-2}\ldots[U]_2[U]_1 = [P]$$

then

$$[U^{\Delta}]_{n-1}[U^{\Delta}]_{n-2}\ldots[U^{\Delta}]_2[U^{\Delta}]_1 = [P^{\Delta}]$$

where $[P^{\Delta}]$ is the delta matrix constructed from $[P]$. Because of its length the proof will not be given here. It may be found in Reference 24.

Since each element of the matrix $[P^{\Delta}]$ corresponds to a subdeterminant of $[P]$, each one may be identified in terms of a particular set of boundary conditions. As discussed in Section 5.7, only 16 of these elements correspond to physically real boundary conditions. From our previous discussion of boundary conditions, as related to Eq. (5.38), we have seen that the boundary conditions on the left end of the beam determine the columns in $[P]$ from which the subdeterminant is derived. Also, the boundary conditions on the right end determine the rows. Therefore, we can identify four columns in the delta matrix $[P^{\Delta}]$ with left-end boundary conditions and four rows of $[P^{\Delta}]$ with right-end conditions. These rows and columns are identified as follows.

$$
\begin{array}{c}
\textit{Left End} \\
\begin{array}{cccccccc}
\text{Free} & \text{Guided} & & & \text{Hinged} & \text{Clamped} & & \\
\left[\begin{array}{c} p_{11}^{\Delta} \\ p_{21}^{\Delta} \\ p_{31}^{\Delta} \\ p_{41}^{\Delta} \\ p_{51}^{\Delta} \\ p_{61}^{\Delta} \end{array}\right. &
\begin{array}{c} p_{12}^{\Delta} \\ p_{22}^{\Delta} \\ p_{32}^{\Delta} \\ p_{42}^{\Delta} \\ p_{52}^{\Delta} \\ p_{62}^{\Delta} \end{array} &
\begin{array}{c} p_{13}^{\Delta} \\ p_{23}^{\Delta} \\ p_{33}^{\Delta} \\ p_{43}^{\Delta} \\ p_{53}^{\Delta} \\ p_{63}^{\Delta} \end{array} &
\begin{array}{c} p_{14}^{\Delta} \\ p_{24}^{\Delta} \\ p_{34}^{\Delta} \\ p_{44}^{\Delta} \\ p_{54}^{\Delta} \\ p_{64}^{\Delta} \end{array} &
\begin{array}{c} p_{15}^{\Delta} \\ p_{25}^{\Delta} \\ p_{35}^{\Delta} \\ p_{45}^{\Delta} \\ p_{55}^{\Delta} \\ p_{65}^{\Delta} \end{array} &
\left.\begin{array}{c} p_{16}^{\Delta} \\ p_{26}^{\Delta} \\ p_{36}^{\Delta} \\ p_{46}^{\Delta} \\ p_{56}^{\Delta} \\ p_{66}^{\Delta} \end{array}\right] &
\begin{array}{l} \text{Clamped} \\ \text{Hinged} \\ \\ \\ \text{Guided} \\ \text{Free} \end{array} &
\textit{Right End}
\end{array}
\end{array}
$$

Thus, the frequency equation, in terms of the delta matrix of a beam with left end free and right end hinged, is given by

$$p_{21}^{\Delta} = 0$$

The remaining fifteen cases can be deduced from this illustration.

5.10 Coupled Bending-Torsion Modes of a Beam

In this section we shall consider the treatment of coupled bending-torsion vibrations of a beam where the coupling is introduced because the locus of the mass centers does not coincide with the structural axis (i.e., the locus of the shear centers) of the beam. Let us consider

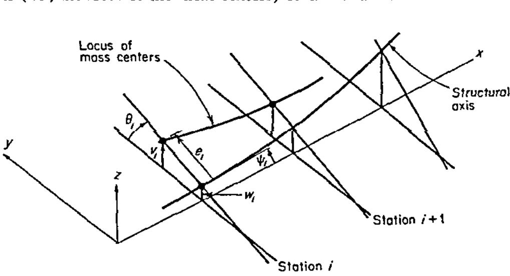


Figure 5.14

the beam shown in Fig. 5.14. The mass is lumped at n stations along the length. The center of mass at each station is located at distance e from the structural axis or shear center of the section, e being considered positive in the positive y direction. Each section perpendicular to the beam axis is considered to be rigid so that a straight line joining the shear center with the center of mass remains a straight line throughout the motion. Then, the deflections at the center of mass and at the shear center are connected by the equation

$$v_i = w_i + e_i\theta_i \tag{5.58}$$

where

v_i = deflection at the center of mass of station i
w_i = deflection at shear center of station i
θ_i = torsional deflection at station i

The state vectors will include six elements in this problem; four associated with bending and two with torsion. The transfer matrix will then be of order 6 × 6. The easiest way to derive the transfer matrix is to consider the field and point matrices separately, then multiply as in Eq. (5.47). Coupling in this problem is dynamic; therefore, the coupling terms appear in the point matrix and not in the field matrix. Thus, the latter may be easily constructed using the field matrices for the uncoupled bending and torsion problems. This matrix has the form

$$\begin{bmatrix} w \\ \psi \\ M \\ V \\ \theta \\ M_T \end{bmatrix}_{i}^{L} = \left[\begin{array}{cccc|cc} 1 & l_i & \dfrac{l_i^2}{2EI_i} & \dfrac{l_i^3}{6EI_i} & 0 & 0 \\ 0 & 1 & \dfrac{l_i}{EI_i} & \dfrac{l_i^2}{2EI_i} & 0 & 0 \\ 0 & 0 & 1 & l_i & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & \dfrac{l_i}{GJ_i} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} w \\ \psi \\ M \\ V \\ \theta \\ M_T \end{bmatrix}_{i-1}^{R} \tag{5.59}$$

The field matrix is partitioned to indicate the uncoupled field matrices for bending and torsion, the former being located in the upper-left corner and the latter in the lower-right corner. The former matrix is given in Eq. (5.50) and the latter may be easily derived from the discussion in Section 5.8.

To construct the point matrix, we again note that dynamic coupling exists because the centers of mass do not coincide with the shear centers. If we write the equations of equilibrium of vertical forces and of torsional moments, the nature of the coupling becomes clear. These equations are

$$V_i^R = V_i^L + \omega^2 m_i v_i \tag{5.60}$$

$$M_{T_i}^R = M_{T_i}^L - \omega^2 m_i e_i v_i - \omega^2 m_i \rho_i^2 \theta_i \tag{5.61}$$

where

ρ_i = radius of gyration of mass at station i about an axis through the center of mass.

Substituting Eq. (5.58) these equations become

$$V_i^R = V_i^L + \omega^2 m_i w_i + \omega^2 m_i e_i \theta_i \tag{5.62}$$

$$M_{T_i}^R = M_{T_i}^L - \omega^2 m_i e_i w_i - \omega^2 m_i (\rho_i^2 + e_i^2)\theta_i \tag{5.63}$$

In this form these equations can be used to construct the point matrix. Again, note that rotatory inertia is not included in this matrix.

$$\begin{Bmatrix} w \\ \psi \\ M \\ V \\ \theta \\ M_T \end{Bmatrix}_i^R = \left[\begin{array}{cccc:cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \omega^2 m_i & 0 & 0 & 1 & \omega^2 m_i e_i & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 \\ -\omega^2 m_i e_i & 0 & 0 & 0 & -\omega^2 m_i(\rho_i^2 + e_i^2) & 1 \end{array}\right] \begin{Bmatrix} w \\ \psi \\ M \\ V \\ \theta \\ M_T \end{Bmatrix}_i^L \tag{5.64}$$

Multiplying the field and point matrices, as in Eq. (5.47), yields the transfer matrix

$$[U]_i = \left[\begin{array}{cccc:cc} \left(1 + \dfrac{\omega^2 m_i l_i^3}{6EI_i}\right) & l_i & \dfrac{l_i^2}{2EI_i} & \dfrac{l_i^3}{6EI_i} & \dfrac{\omega^2 m_i e_i l_i^3}{6EI_i} & 0 \\ \dfrac{\omega^2 m_i l_i^2}{2EI_i} & 1 & \dfrac{l_i}{EI_i} & \dfrac{l_i^2}{2EI_i} & \dfrac{\omega^2 m_i e_i l_i^2}{2EI_i} & 0 \\ \omega^2 m_i l_i & 0 & 1 & l_i & \omega^2 m_i e_i l_i & 0 \\ \omega^2 m_i & 0 & 0 & 1 & \omega^2 m_i e_i & 0 \\ \hdashline -\dfrac{\omega^2 m_i e_i l_i}{GJ_i} & 0 & 0 & 0 & \left(1 - \dfrac{\omega^2 m_i(\rho_i^2 + e_i^2) l_i}{GJ_i}\right) & \dfrac{l_i}{GJ_i} \\ -\omega^2 m_i e_i & 0 & 0 & 0 & -\omega^2 m_i(\rho_i^2 + e_i^2) & 1 \end{array}\right] \tag{5.65}$$

Here the submatrices in the upper-left and lower-right corners may be compared with the transfer matrices for uncoupled bending and torsion, as given in Eqs. (5.33) and (5.32). Note that the coupling elements in the remaining two submatrices depend upon the distance e_i and all vanish if that distance goes to zero. In that case the problem degenerates into separate problems in uncoupled bending and torsion.

To proceed with the solution of the problem, the transfer matrices are multiplied as in Eq. (5.35) to determine the product matrix $[P]$. In this problem, $[P]$ is of order 6×6 and Eq. (5.35) takes the specific form

$$\begin{Bmatrix} w \\ \psi \\ M \\ V \\ \theta \\ M_T \end{Bmatrix}_n = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} \end{bmatrix} \begin{Bmatrix} w \\ \psi \\ M \\ V \\ \theta \\ M_T \end{Bmatrix}_1 \tag{5.66}$$

The frequency equation requires that a subdeterminant of third order vanish. For example, let us consider a beam that has clamped-free ends in bending and in torsion. The boundary conditions are given by

$$\begin{aligned} w_1 = \psi_1 = \theta_1 = 0 \\ M_n = V_n = M_{T_n} = 0 \end{aligned} \tag{5.67}$$

In this case the frequency equation becomes

$$\begin{vmatrix} p_{33} & p_{34} & p_{35} \\ p_{43} & p_{44} & p_{45} \\ p_{63} & p_{64} & p_{65} \end{vmatrix} = 0 \tag{5.68}$$

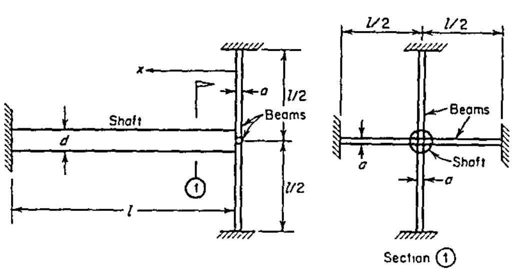
Once the natural frequencies are determined, the natural modes are obtained by determining all the intermediate state vectors by the same procedure as described previously for uncoupled vibrations.

The delta matrix technique is applicable in this case of coupled vibrations as well as in the uncoupled bending problem considered in Section 5.9. It can be shown that such a delta matrix is of order 20×20. Hence, the labor of constructing and multiplying these matrices is very much greater than in the bending case.

PROBLEMS

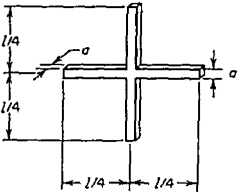
1. Using the differential equation method, derive the frequency equation and show how the mode shapes can be computed for the torsional free vibration of the shaft shown in the figure for Problem 1 on the next page. The shaft is fixed at one end and is connected rigidly to two beams at

its other end so that the end of the shaft and the beams undergo equal rotation at the junction $x = 0$. Neglect the mass of the two beams. The mass density of the shaft is ρ.



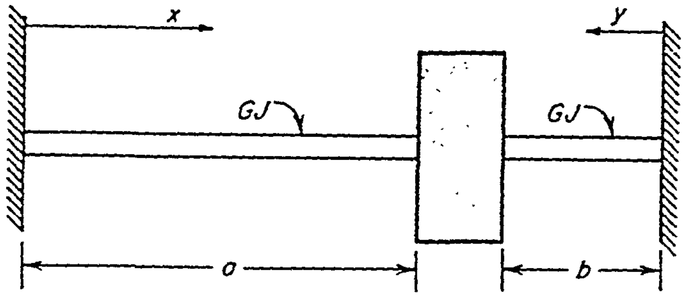
Problem 1

2. Repeat Problem 1, accounting for the mass of the beams. As a crude approximation for this purpose consider the mass of the beams in the form of a rigid cruciform rotating with the end of the shaft. The mass density of the beams is ρ and the cruciform dimensions are as shown.



Problem 2

3. Derive the frequency equation for the shafts with a rigid body of mass moment of inertia I_{II} attached so that $\theta(x) = \theta(y)$ for $x = a$ and $y = b$. The mass moment of inertia of the shafts is I per unit length. Show also how the mode shapes can be computed. Use the differential equation method.

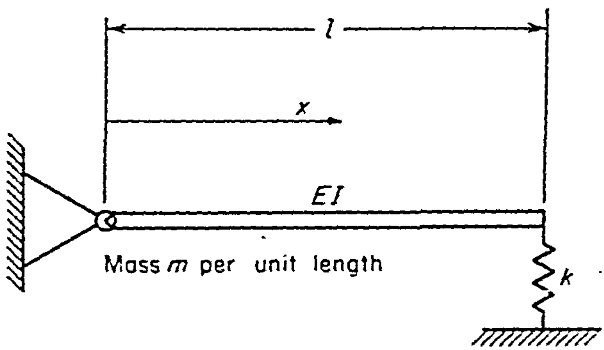


Problem 3

4. Compute the first three frequencies and plot the corresponding mode shapes for a fixed-fixed shaft with torsional rigidity GJ and mass moment of inertia I per unit length. Use the differential equation method.

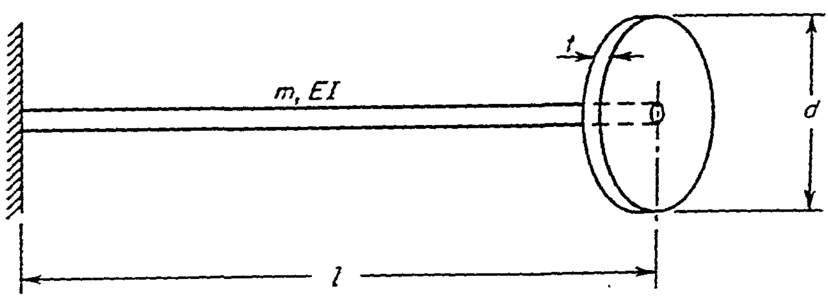
5. Find the natural frequencies and modes of vibration for beam 5 of Table 5.1. Compare your results with those of beam 6 of Table 5.1 and discuss.

6. Derive the frequency equation for the uniform beam shown. The beam is pinned at $x = 0$ and supported by a spring of stiffness k at $x = l$.



Problem 6

7. Derive the frequency equation for the uniform cantilever beam with a rigid disk attached at the free end. The beam has mass m per unit length and bending rigidity EI. The disk has a mass density ρ, thickness t, and diameter d.



Problem 7

8. Find the natural frequencies and mode shapes for beam 3 of Table 5.1.

9. Find the natural frequencies and mode shapes for beam 8 of Table 5.1. Compare the results with those of Problem 8 and discuss.

10. Using the Holzer method, compute the natural frequencies and modes in torsional vibration for the structure shown in the figure for Problem 16, Chapter 2.

11. Show that the constants C_1, C_2, C_3, C_4 in Eqs. (5.13), (5.15), (5.16), (5.17) for a uniform slender beam with distributed mass, can be expressed in terms of the state vector $\{Z\}_i$ at the left end of the beam, where $x = 0$, by the equation

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & \frac{1}{\alpha\beta^2} & 0 \\ 0 & \frac{1}{\beta} & 0 & \frac{1}{\alpha\beta^3} \\ 1 & 0 & \frac{-1}{\alpha\beta^2} & 0 \\ 0 & \frac{1}{\beta} & 0 & \frac{-1}{\alpha\beta^3} \end{bmatrix} \begin{Bmatrix} w \\ \psi \\ M \\ V \end{Bmatrix}_{x=0}$$

where
$$\alpha = EI$$
$$\beta^4 = \frac{\omega^2 m}{EI}$$
$$\psi = \frac{dw}{dx}$$
$$M = EI\frac{d^2w}{dx^2}$$
$$V = EI\frac{d^3w}{dx^3}$$

Use this result to verify Eq. (5.54).

12. Show that the solution to the differential equation for the longitudinal vibration of a uniform bar with distributed mass, is given by

$$u(x) = C_1 \cos\sqrt{\frac{\omega^2 m}{EA}}\,x + C_2 \sin\sqrt{\frac{\omega^2 m}{EA}}\,x$$

where $u(x)$ is the longitudinal displacement
m is the mass of the bar per unit length
A is the area of the cross-section
E is the modulus of elasticity
C_1, C_2 are constants determined from the boundary conditions

Using the longitudinal force-deflection relation for a bar

$$N(x) = EA\frac{\partial u}{\partial x} \qquad (N \text{ designates the longitudinal force.})$$

and the preceding equation for $u(x)$, solve for C_1 and C_2 in terms of $\begin{Bmatrix} u \\ N \end{Bmatrix}_{x=0}$ and write the vector $\begin{Bmatrix} u \\ N \end{Bmatrix}_{x=l}$ in terms of $\begin{Bmatrix} u \\ N \end{Bmatrix}_{x=0}$.

13. Use the results of Problem 12 and Eq. (5.54) to construct the 6×6 field matrix $[U_F]$ for a uniform slender beam with distributed mass. Write the result in the form

$$\{Z\}_{\substack{2 \\ x=l}} = [U_F]\{Z\}_{\substack{1 \\ x=0}}$$

where

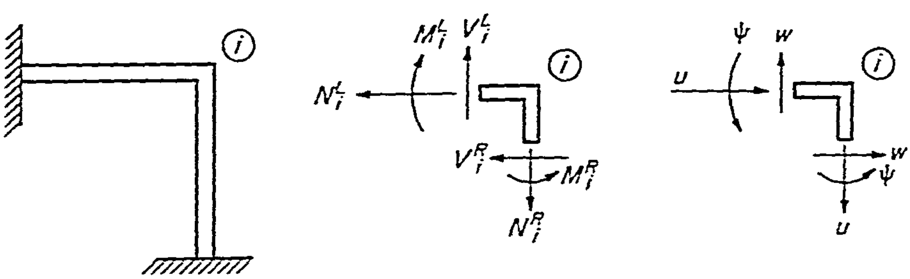
$$\{Z\} = \begin{Bmatrix} u \\ w \\ \psi \\ M \\ V \\ N \end{Bmatrix}$$

14. Construct the point matrix at corner i of two perpendicular beams with distributed mass. Use the force displacement coordinates as shown and write the result in the form

$$\{Z\}_i^R = [U_P]_i\{Z\}_i^L$$

where

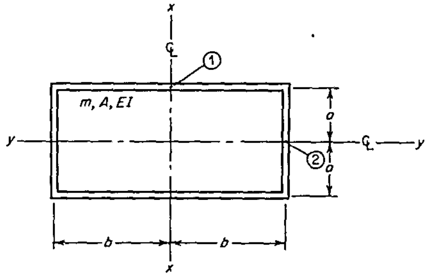
$$\{Z\} = \begin{Bmatrix} u \\ w \\ \psi \\ M \\ V \\ N \end{Bmatrix}$$



Problem 14

15. Using the results of Problems 13 and 14, construct the transfer matrix relating the state vector $\{Z\}_2$ at ② to the state vector $\{Z\}_1$ at ① in the

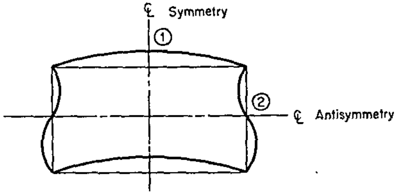
closed symmetric ring frame shown. The beams have a distributed mass m per unit length, a cross-sectional area A, and bending rigidity EI.



Problem 15

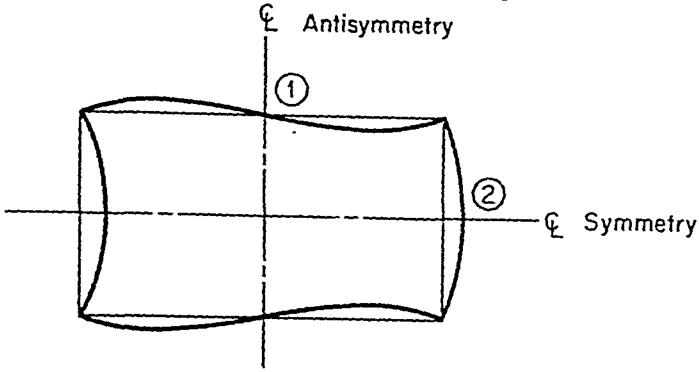
16. Consider the ring of Problem 15 to undergo free vibrations in the plane of the paper in symmetric-antisymmetric modes in which the deformed shapes are symmetric about the x-x axis and antisymmetric about the y-y axis. One such deformed configuration in a symmetric-antisymmetric mode, is shown in the illustration. The mass for all beams is m per unit length, the cross-sectional area is A, and the bending rigidity is EI. Construct the frequency determinant for this symmetric-antisymmetric free vibration. Taking advantage of symmetry, only one quarter of the frame need be considered in formulating the frequency determinant.

Hint: Use your results from Problem 15 and introduce the appropriate boundary conditions at ① and ②.



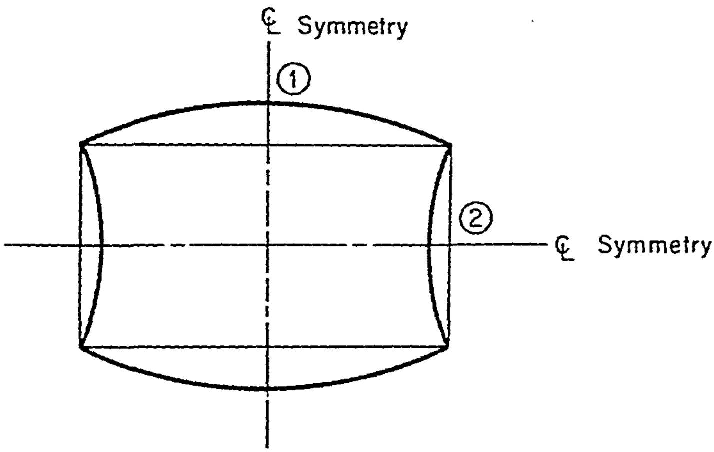
Problem 16. A symmetric-antisymmetric mode of vibration.

17. Repeat Problem 16 for free vibrations in antisymmetric-symmetric modes.



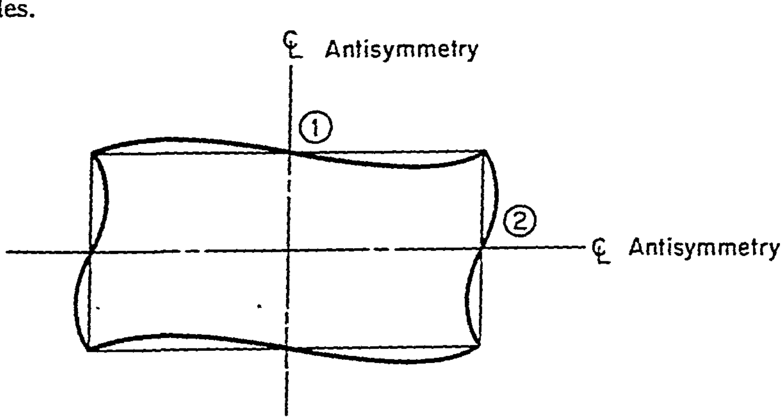
Problem 17. An antisymmetric-symmetric mode of vibration.

18. Repeat Problem 16 for free vibrations in symmetric-symmetric modes.



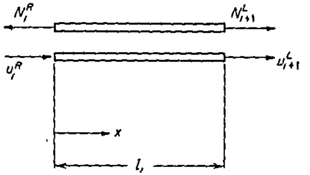
Problem 18. A symmetric-symmetric mode of vibration.

19. Repeat Problem 16 for free vibrations in antisymmetric-antisymmetric modes.



Problem 19. An antisymmetric-antisymmetric mode of vibration.

20. Using the following field matrix relating the state vector of longitudinal force and displacement at stations $(i+1)$ and (i) of a massless beam,



$$\begin{Bmatrix} u \\ N \end{Bmatrix}_{i+1}^{R} = \begin{bmatrix} 1 & \dfrac{l_i}{EA} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u \\ N \end{Bmatrix}_{i}^{L}$$

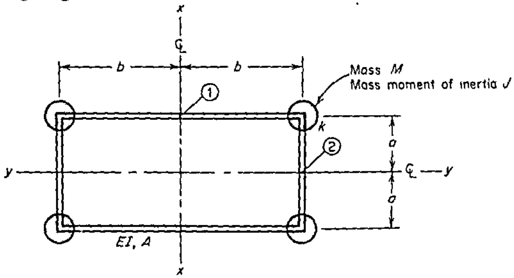
extend the matrix of Eq. (5.50) to a 6×6 field matrix $[U_F]_i$ relating the state vector $\{Z\}_{i+1}^R$ to $\{Z\}_i^L$ through

$$\{Z\}_{i+1}^R = [U_F]_i \{Z\}_i^L$$

where

$$\{Z\} = \begin{Bmatrix} u \\ w \\ \psi \\ M \\ V \\ N \end{Bmatrix}$$

The symmetric ring frame shown has rigid masses at its corners and is considered to be constructed of massless uniform beams of cross-sectional area A and bending rigidity EI. Construct the point transfer matrix at ⓚ using the same force and displacement coordinates as shown in the illustration for Problem 14. Construct the overall transfer matrix from ① to ②.



Problem 20

21. Using the results of Problem 20 and introducing the appropriate boundary conditions, construct the frequency determinant and the frequency equation for the ring of Problem 20 undergoing free vibrations in the plane of the paper in symmetric-antisymmetric modes (symmetric displacement configurations about the x-x axis and antisymmetric displacement configurations about the y-y axis).

22. Repeat Problem 21 for free vibrations in antisymmetric-symmetric modes.

23. Repeat Problem 21 for free vibrations in symmetric-symmetric modes.

24. Repeat Problem 21 for free vibrations in antisymmetric-antisymmetric modes.

CHAPTER 6

Integral Equation Methods

6.1 Formulation of the Integral Equation

It was shown in Chapter 1 how the deflection of an elastic structure could be determined at every point in terms of the influence function. Eq. (1.64) gives the relationship for a one-dimensional structure.

$$u(x) = \int_{\xi=0}^{l} a(x, \xi) f(\xi) \, d\xi$$

The natural modes for natural vibrations may be found from this equation by expressing the distributed load $f(\xi)$, which in this case is entirely inertial, in terms of the distributed mass as

$$f(\xi) = \omega^2 m(\xi) \, u(\xi) \tag{6.1}$$

where $m(\xi)$ is the mass (or mass moment of inertia) per unit of length. Then the equation for the natural modes is

$$u(x) = \omega^2 \int_0^l a(x, \xi) \, m(\xi) \, u(\xi) \, d\xi \tag{6.2}$$

This is an integral equation identified as Fredholm's equation of the second kind.[3] The function $u(x)$ to be determined appears in the integrand as well as in the left member of the equation. The function

$$K(x, \xi) = a(x, \xi) \, m(\xi) \tag{6.3}$$

is called the *kernel;* and the influence function $a(x, \xi)$ is known in mathematics as the *Green's function* and was discussed in Chapter 1.

Inasmuch as the influence function satisfies the boundary conditions, it follows that the integral equation of the form (6.2) defines the boundary value problem completely. The boundary conditions are specified implicitly by the equation through the inclusion of the influence functions. In this connection it will be instructive to consider a specific example in which the integral equation is derived directly from the differential equation and a specified set of boundary conditions. The differential equation for the natural modes of a uniform shaft in torsion is

$$\frac{d^2\theta}{dx^2} + \lambda\theta = 0 \qquad \text{(See Eq. 5.2.)} \tag{6.4}$$

where $\lambda = \omega^2 I/GJ$ for this example.

Let us complete the definition of the problem by specifying the boundary conditions

$$\left.\begin{aligned} \frac{d\theta}{dx} &= 0 \qquad \text{at } x = 0 \\ \theta &= 0 \qquad \text{at } x = l \end{aligned}\right\} \tag{6.5}$$

These conditions apply to a member free at $x = 0$ and fixed at $x = l$. Equation (6.4) is integrated to obtain

$$\frac{d\theta}{dx} = -\lambda \int_0^x \theta(\xi)\, d\xi + C_1 \tag{6.6}$$

where C_1 is the value of $d\theta/dx$ at $x = 0$ and, hence, is equal to zero, according to the first boundary condition. Next, we integrate again to obtain $\theta(x)$.

$$\theta(x) = -\lambda \int_0^x (x - \xi)\, \theta(\xi)\, d\xi + C_2 \tag{6.7}$$

where the second constant $C_2 = \theta(0)$ is not given as a specified boundary condition. Verification of the above form for the integral can best be had by differentiating with respect to x to obtain Eq. (6.6). To evaluate the constant C_2 we shall have to write Eq. (6.7) for $x = l$. Thus, we obtain

$$\theta(l) = -\lambda \int_0^l (l - \xi)\, \theta(\xi)\, d\xi + C_2$$

from which C_2 is obtained by substituting the second boundary condition of Eq. (6.5).

$$C_2 = \lambda \int_0^l (l - \xi)\, \theta(\xi)\, d\xi \tag{6.8}$$

Substituting C_1 into Eq. (6.7) and writing the integral in two parts, we obtain

$$\theta(x) = -\lambda \int_0^x (x - \xi)\, \theta(\xi)\, d\xi + \lambda \int_0^x (l - \xi)\, \theta(\xi)\, d\xi + \lambda \int_x^l (l - \xi)\, \theta(\xi)\, d\xi$$

$$= \lambda \int_0^x (l - x)\, \theta(\xi)\, d\xi + \lambda \int_x^l (l - \xi)\, \theta(\xi)\, d\xi \qquad (6.9)$$

If we define a kernel $K(x, \xi)$ as

$$\begin{aligned} K(x, \xi) &= l - \xi \quad \text{for} \quad 0 < x < \xi \\ &= l - x \quad \text{for} \quad \xi < x < l \end{aligned} \qquad (6.10)$$

we may write Eq. (6.9) in the form

$$\theta(x) = \lambda \int_0^l K(x, \xi)\, \theta(\xi)\, d\xi \qquad (6.11)$$

By the method illustrated in Chapter 1, an influence function may be derived for the uniform shaft in torsion. It is found to be

$$\begin{aligned} a(x, \xi) &= \frac{1}{GJ}(l - \xi) \quad \text{for} \quad 0 < x < \xi \\ &= \frac{1}{GJ}(l - x) \quad \text{for} \quad \xi < x < l \end{aligned} \qquad (6.12)$$

Comparing with Eq. (6.10) it is seen that

$$K(x, \xi) = GJ\, a(x, \xi) \qquad (6.13)$$

Thus, we may write Eqs. (6.4) and (6.5) as

$$\theta(x) = \omega^2 \int_0^l a(x, \xi)\, I(\xi)\, \theta(\xi)\, d\xi \qquad (6.14)$$

This completes our identification of the differential equation form with the form (6.2). Again, in this example the Green's function $G(x, \xi)$ is equivalent to the influence function $a(x, \xi)$. It is pertinent to note once again that the boundary conditions are satisfied through the absorption of the constants of integration into the Eq. (6.14).

6.2 Orthogonality of the Natural Modes

Solution of Eq. (6.2) yields the natural mode function $u(x)$ and the natural frequency ω. The equation may be written specifically for each of the modes. For the rth mode

$$u_r(x) = \omega_r^2 \int_0^l a(x, \xi)\, m(\xi)\, u_r(\xi)\, d\xi \qquad (6.15)$$

For the sth mode

$$u_s(x) = \omega_s^2 \int_0^l a(x, \xi)\, m(\xi)\, u_s(\xi)\, d\xi \tag{6.16}$$

The natural mode functions so obtained are orthogonal with respect to the function $m(x)$, sometimes called the *weighting function.* The orthogonality condition is

$$\left.\begin{aligned} \int_0^l u_r(x)\, m(x)\, u_s(x)\, dx &= 0 && \text{for} \quad r \neq s \\ &= \text{a constant} && \text{for} \quad r = s \end{aligned}\right\} \tag{6.17}$$

The proof of orthogonality in the case of continuous functions is analogous to that for the case in which modal columns define mode amplitudes at discrete points in the structure. In the latter case, orthogonality of the natural modes was considered in Section 3.4. The proof of the orthogonality condition expressed by Eq. (6.17) proceeds by multiplying Eq. (6.15) by $m(x)\, u_s(x)$ and integrating over the interval $0, l$.

$$\begin{aligned} &\int_0^l u_r(x)\, m(x)\, u_s(x)\, dx \\ &\qquad = \omega_r^2 \int_0^l m(x)\, u_s(x) \left\{ \int_0^l a(x, \xi)\, m(\xi)\, u_r(\xi)\, d\xi \right\} dx \end{aligned} \tag{6.18}$$

We may reverse the order of integration in the right-hand member and write

$$\begin{aligned} &\int_0^l u_r(x)\, m(x)\, u_s(x)\, dx \\ &\qquad = \omega_r^2 \int_0^l m(\xi)\, u_r(\xi) \left\{ \int_0^l a(x, \xi)\, m(x)\, u_s(x)\, dx \right\} d\xi \end{aligned} \tag{6.19}$$

Now, inasmuch as the integrals in both members of the above equations are definite we are free to interchange the variables of integration x, ξ. In addition, we make use of the property of symmetry of the influence functions in order to substitute $a(x, \xi)$ for $a(\xi, x)$. Then we obtain

$$\begin{aligned} &\int_0^l u_r(x)\, m(x)\, u_s(x)\, dx \\ &\qquad = \omega_r^2 \int_0^l m(x)\, u_r(x) \left\{ \int_0^l a(x, \xi)\, m(\xi)\, u_s(\xi)\, d\xi \right\} dx \end{aligned} \tag{6.20}$$

Next, multiply both sides of Eq. (6.16) by $m(x) u_r(x)$ and integrate over $0, l$ to obtain

$$\begin{aligned} &\int_0^l u_s(x)\, m(x)\, u_r(x)\, dx \\ &\qquad = \omega_s^2 \int_0^l m(x)\, u_r(x) \left\{ \int_0^l a(x, \xi)\, m(\xi)\, u_s(\xi)\, d\xi \right\} dx \end{aligned} \tag{6.21}$$

Equations (6.20) and (6.21) are alike in both members except for the squares of the frequencies ω_r and ω_s. In general ω_r and ω_s are different.

$$\omega_r^2 \neq \omega_s^2 \tag{6.22}$$

Therefore, for $r \neq s$ the equalities given by Eqs. (6.20) and (6.21) can hold only by virtue of the orthogonality condition of Eq. (6.17). For $r = s$ Eqs. (6.20) and (6.21) may hold, even though the integrals are different from zero. We note that orthogonality in the above case implies also that

$$\int_0^l m(x)\, u_r(x) \left\{ \int_0^l a(x, \xi)\, m(\xi)\, u_s(\xi)\, d\xi \right\} dx = 0 \qquad \text{for} \quad r \neq s \tag{6.23}$$

6.3 Approximate Solutions

Exact solution of integral equations of the form (6.2) is possible only in cases in which the kernel takes on certain specific forms. In problems involving structures, these forms generally do not appear. Therefore, as in the solution of differential equations, approximation methods are used. These methods are considered in the remaining sections of this chapter.

We shall first briefly consider a simple and straightforward method in which the integral is approximated by a finite sum. The interval of integration $0, l$ is divided into an arbitrary number, say $n - 1$, of sub-intervals which need not be equal. The stations so established are numbered $1, 2, 3, \ldots, i, j, \ldots, n$, and the influence functions are defined in terms of these stations; e. g., $a(x_i, x_j)$ is the deflection at station $i (x = x_i)$ caused by a unit force applied at station $j (x = x_j)$. Thus at station i the integral equation is approximated by

$$u(x_i) \approx \omega^2 \sum_{j=1}^{n} a(x_i, x_j)\, m(x_j)\, \Delta x_j\, u(x_j) \tag{6.24}$$

The quantity $m(x_j)\Delta x_j$ is the weighted mass at station j and may be obtained by using any of the approximate formulas for numerical integration. If we adopt the more convenient notation in which

$$u(x_i) = u_i$$
$$a(x_i, x_j) = a_{ij}$$
$$m(x_j)\,\Delta x_j = m_j$$

Eq. (6.24) appears in the familiar form

$$u_i = \omega^2 \sum_{j=1}^{n} a_{ij} m_j u_j \tag{6.25}$$

The set of equations obtained by writing (6.25) for each station appears in matrix form as

$$\{u\} = \omega^2 [a][m]\{u\} \tag{6.26}$$

This is the same set of equations as obtained in Section 3.2, except here they are defined for a lumped structure in which $\{u\}$ represents the displacements of the individual lumped masses. It is readily seen that the approximation of the integral in Eq. (6.2) by the finite sum of Eq. (6.25) is equivalent to the approximation of the distributed structure by a lumped structure.

6.4 Approximation of Natural Modes by Superposition of Functions

An approximate method which proceeds in a manner similar to that of the Rayleigh-Ritz method makes use of the superposition of chosen functions with undetermined coefficients, thus

$$u(x) \approx \sum_{k=1}^{n} C_k \phi_k(x) \tag{6.27}$$

The functions $\phi(x)$ are chosen, as in the Rayleigh-Ritz method, to approximate as well as possible the expected natural modes. In general, they satisfy the boundary conditions so that the approximation to $u(x)$ will satisfy those conditions. If we insert Eq. (6.27) into Eq. (6.2), we obtain

$$\begin{aligned}\sum_{k=1}^{n} C_k\phi_k(x) &= \omega^2 \int_0^l a(x, \xi)\, m(\xi) \left(\sum_{k=1}^{n} C_k\phi_k(\xi)\right) d\xi \\ &= \omega^2 \sum_{k=1}^{n} C_k \int_0^l a(x, \xi)\, m(\xi)\, \phi_k(\xi)\, d\xi\end{aligned} \tag{6.28}$$

It is now required that we determine $C_1, C_2, \ldots, C_k, \ldots, C_n$ to within an arbitary multiplicative constant such that the approximation (6.27) be as nearly exact as possible. This will depend upon the method we use for the determimation of the constants. Two methods by which they may be found are discussed in the two sections following.

6.5 Method of Collocation

In this procedure we establish stations defined by $x = x_i (i = 1, 2, \ldots, n)$, and require that Eq. (6.28) hold for each of these stations. The equation for the ith station is

$$\sum_{k=1}^{n} C_k\phi_k(x_i) = \omega^2 \sum_{k=1}^{n} C_k \int_0^l a(x_i, \xi)\, m(\xi)\, \phi_k(\xi)\, d\xi \tag{6.29}$$

The functions $\phi_k(x)$ are defined only at the station points and we may, therefore, adopt the following abbreviated notation

$$\phi_k(x_i) = \phi_{ik} \tag{6.30}$$

It will be convenient to define a set of functions $\psi_k(x)$ as

$$\psi_k(x) = \int_0^l a(x, \xi)\, m(\xi)\, \phi_k(\xi)\, d\xi \tag{6.31}$$

In the collocation method[10], these functions will also be defined only at the station points and, following the notation of Eq. (6.30), we write

$$\psi_{ik} = \int_0^l a(x_i, \xi)\, m(\xi)\, \phi_k(\xi)\, d\xi \tag{6.32}$$

Equation (6.29) pertaining to the ith station takes the abbreviated form

$$\sum_{k=1}^{n} \phi_{ik} C_k = \omega^2 \sum_{k=1}^{n} \psi_{ik} C_k \tag{6.33}$$

The set of n equations takes the matrix form

$$[\phi]\{C\} = \omega^2 [\psi]\{C\} \tag{6.34}$$

A significant feature of Eq. (6.34) may be brought to light by approximating the integral in Eq. (6.32) with a finite sum given by

$$\begin{aligned} \psi_{ik} &= \sum_{j=1}^{n} a(x_i, \xi_j)\, m(\xi_j)\, \phi_k(\xi_j) \\ &= \sum_{j=1}^{n} a_{ij} m_j \phi_{jk} \end{aligned} \tag{6.35}$$

where m_j is a weighted mass assigned to each station. Using Eq. (6.35), matrix $[\psi]$ in Eq. (6.34) is approximated by

$$[\psi] = [a][m][\phi] \tag{6.36}$$

When this is inserted into Eq. (6.34) we obtain

$$[\phi]\{C\} = \omega^2 [a][m][\phi]\{C\} \tag{6.37}$$

Now, note that if we use Eq. (6.27) to define a set of displacements $u(x_i) = u_i$ at the station points, these displacements are given in matrix form by

$$\{u\} = [\phi]\{C\} \tag{6.38}$$

Substitution of Eq. (6.38) into Eq. (6.37) yields the same equation as (6.26) showing that the collocation method leads to an eigenvalue problem whose solution yields natural modes defined at the station points. However, we wish to consider Eq. (6.37) in a different light by observing that if the matrix $[\phi]$ is equal to the identity matrix $[I]$, then that equation will converge on a set of modal columns $\{C\}$ which will define a set of natural modes through Eq. (6.27). If we place this constraint on the matrix $[\phi]$, its elements must have the following property.

$$\left.\begin{aligned}\phi_{ik} &= 1 \qquad \text{for } k = i \\ &= 0 \qquad \text{for } k \neq i\end{aligned}\right\} \tag{6.39}$$

Evaluation of Functions $\phi_i(x)$ by the Method of Station Functions.

The function $\phi_i(x)$ satisfying conditions (6.39) at predetermined stations. can be selected on the basis of experience or generated more systematically by the method of station functions.

In devising the method of station functions, Rauscher[25] considered those functions, which we shall designate $\phi_k(x)$ $(k = 1, 2, 3, \ldots, n)$, to be continuous, but required that they satisfy the property (6.39) at n stations $(i = 1, 2, \ldots, n)$ thus

$$\left.\begin{aligned}\phi_k(x_i) &= 1 \qquad \text{for} \quad i = k \\ &= 0 \qquad \text{for} \quad i \neq k\end{aligned}\right\} \tag{6.40}$$

Functions which satisfy Eqs. (6.40) are called *interpolation functions.* In the method of station functions, they are continuous and their values at intermediate points between stations remain to be determined.

Station functions are essentially deflection functions which are determined by applying to the structure a set of n forces, each of which is associated with one of the stations. These may be point forces but, in practice, they are usually distributed in order to simulate a distributed inertial loading in natural vibrations. In Reference 25, it is suggested that triangular distributions be employed, as

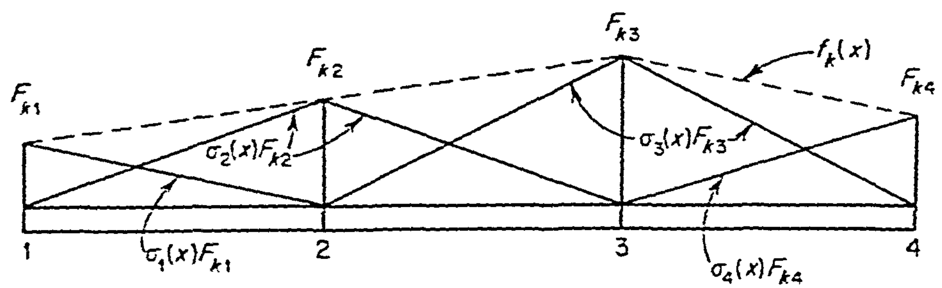


Figure 6.1

shown in Fig. 6.1, where four stations are indicated. The distributed force $f_k(x)$, obtained by summing the triangular distributions, is given by the equation

$$f_k(x) = \sum_{j=1}^{n} F_{kj}\,\sigma_j(x) \tag{6.41}$$

where the $\sigma_j(x)$ are the distribution functions with unit amplitude at the apex of the triangles. The kth station function $\phi_k(x)$ is the deflection caused by this distributed force and is computed, using the equation

$$\phi_k(x) = \int_0^l a(x, \xi) m(\xi) f_k(\xi) \, d\xi \tag{6.42}$$

The functions thus determined, satisfy boundary conditions through the use of influence functions $a(x, \xi)$. The distributed mass $m(\xi)$, which may include concentrated masses at discrete stations ξ_i, appears in the integrand in order to make the total force function more nearly simulate the distributed inertial force in natural vibrations, inasmuch as this force is dependent upon the distribution of mass. We now substitute Eq. (6.41) into (6.42) and interchange the order of integration and summation to give the following equation for the kth station function.

$$\phi_k(x) = \sum_{j=1}^{n} F_{kj} \int_0^l a(x, \xi) m(\xi) \sigma_j(\xi) \, d\xi \tag{6.43}$$

The force amplitudes F_{kj} are not arbitrary but will be determined from the requirement that the station functions be interpolation functions, i. e., that Eq. (6.40) be satisfied at the station points. To show how this requirement is satisfied, let us define functions $g_j(x)$ as

$$g_j(x) = \int_0^l a(x, \xi) m(\xi) \sigma_j(\xi) \, d\xi \tag{6.44}$$

At the station points Eq. (6.43) becomes

$$\phi_k(x_i) = \sum_{j=1}^{n} F_{kj} g_j(x_i)$$

In abbreviated notation this equation is

$$\phi_{ik} = \sum_{j=1}^{n} g_{ij} F_{kj} \tag{6.45}$$

This relationship leads to the matrix equation

$$[\phi] = [g][F]^T \tag{6.46}$$

where the elements of the matrices are the values of the corresponding functions for $k = 1, 2, 3, \ldots, n$ computed at the n station points; $i = 1, 2, 3, \ldots, n$. Since the matrix $[\phi]$ must be the identity matrix, it follows from Eq. (6.46) that the matrix $[F]$ must be given by

$$[F]^T = [g]^{-1} \tag{6.47}$$

Thus, the elements F_{kj} are determined, and by use of Eq. (6.43) the station functions are found. Once the coefficients C_k corresponding to each natural mode are determined, the modes may be found using Eq. (6.27).

To determine the C_k, turn to Eq. (6.29) and substitute Eqs. (6.43) and (6.44).

$$\begin{aligned}\sum_{k=1}^{n} C_k\phi_k(x_i) &= \omega^2 \sum_{k=1}^{n} C_k \int_0^l a(x_i, \xi)m(\xi) \sum_{j=1}^{n} F_{kj}g_j(\xi)\,d\xi \\ &= \omega^2 \sum_{k=1}^{n} C_k \sum_{j=1}^{n} F_{kj} \int_0^l a(x_i, \xi)m(\xi)g_j(\xi)\,d\xi\end{aligned} \tag{6.48}$$

We define a function $h_j(x)$ by the equation

$$h_j(x) = \int_0^l a(x, \xi)m(\xi)g_j(\xi)\,d\xi \tag{6.49}$$

The integral in the right side of Eq. (6.48) is then $h_j(x_i) = h_{ij}$. Substituting into Eq. (6.48) we obtain

$$\sum_{k=1}^{n} \phi_{ik} C_k = \omega^2 \sum_{j=1}^{n} \sum_{k=1}^{n} h_{ij}F_{kj}C_k \tag{6.50}$$

The set of equations in matrix form is

$$[\phi]\{C\} = \omega^2[h][F]^T\{C\} \tag{6.51}$$

Remember that matrix $[\phi]$ is the identity matrix and this substitution, together with the substitution of Eq. (6.47), yields matrix equation

$$\{C\} = \omega^2[h][g]^{-1}\{C\} \tag{6.52}$$

From this equation the natural frequencies and the modal columns $\{C^{(1)}\}, \{C^{(2)}\}, \ldots, \{C^{(n)}\}$ are found.

From Eqs. (6.44) and (6.49), it is seen that the basic load distribution functions $\sigma_j(x)$ are very important in determining the accuracy of the station function method. Although the triangular distributions shown in Fig. 6.1 are suggested, the method is not restricted to them. Experience has shown that excellent results are obtained with this method using only the same number of stations as the number of natural modes desired.[11]

6.6 Galerkin Method as Applied to Integral Equations

We regard Eq. (6.2) as one whose solution yields the natural modes $u(x)$ exactly, and Eq. (6.27) as an approximation to the natural modes in terms of selected functions $\phi(x)$. The approximation error is written as

$$\epsilon(x) = \sum_{k=1}^{n} C_k\phi_k(x) - \omega^2 \int_0^l a(x, \xi)m(\xi)u(\xi)\,d\xi \tag{6.53}$$

We substitute Eq. (6.27) into the integrand in the above expression and write

$$\begin{aligned}\epsilon(x) &= \sum_{k=1}^{n} C_k\phi_k(x) - \omega^2 \int_0^l a(x,\xi)m(\xi)\sum_{k=1}^{n} C_k\phi_k(\xi)\,d\xi \\ &= \sum_{k=1}^{n} C_k\left\{\phi_k(x) - \omega^2\int_0^l a(x,\xi)m(\xi)\phi_k(\xi)\,d\xi\right\}\end{aligned} \tag{6.54}$$

The Galerkin method as applied to this equation is based on a requirement that the function $\epsilon(x)$ be orthogonal to each of the selected functions[3] $\phi(x)$, thus

$$\int_0^l m(x)\phi_j(x)\epsilon(x)\,dx = 0 \qquad \text{for} \quad j = 1, 2, 3, \ldots, n\,. \tag{6.55}$$

Substituting Eq. (6.54) into (6.55) we obtain

$$\int_0^l m(x)\phi_j(x)\sum_{k=1}^{n} C_k\left\{\phi_k(x) - \omega^2\int_0^l a(x,\xi)m(\xi)\phi_k(\xi)\,d\xi\right\}dx=0$$

This may be put into the form

$$\begin{aligned}&\sum_{k=1}^{n} C_k\int_0^l m(x)\phi_j(x)\phi_k(x)\,dx \\ &\quad - \omega^2\sum_{k=1}^{n} C_k\int_0^l m(x)\phi_j(x)\left[\int_0^l a(x,\xi)m(\xi)\phi_k(\xi)\,d\xi\right]dx = 0\end{aligned} \tag{6.56}$$

Note that the integral in the first term is the generalized mass

$$m_{jk} = \int_0^l m(x)\phi_j(x)\phi_k(x)\,dx \tag{6.57}$$

The double integral in the second term yields a number G_{jk} for each pair of functions chosen.

$$G_{jk} = \int_0^l m(x)\phi_j(x)\left[\int_0^l a(x,\xi)m(\xi)\phi_k(\xi)\,d\xi\right]dx \tag{6.58}$$

Using the above notation, Eq. (6.56) appears as

$$\sum_{k=1}^{n} m_{jk}C_k - \omega^2\sum_{k=1}^{n} G_{jk}C_k = 0 \tag{6.59}$$

In matrix form the set of equations obtained by setting $j = 1, 2, \ldots, n$, becomes

$$[m]\{C\} = \omega^2[G]\{C\} \tag{6.60}$$

The solution to this equation will converge on the natural modes and frequencies. It is apparent that the solution satisfies the boundary conditions if the functions $\phi(x)$ are chosen to satisfy them. Orthogonality of the modes is assured if $[G]$ is symmetric (it is apparent upon inspection that $[m]$ is symmetric). To demonstrate symmetry of $[G]$ we derive G_{kj} from Eq. (6.58).

$$G_{kj} = \int_0^l m(x)\phi_k(x)\left[\int_0^l a(x,\xi)m(\xi)\phi_j(\xi)\,d\xi\right]dx$$

Reverse the order of integration to obtain

$$G_{kj} = \int_0^l m(\xi)\phi_j(\xi)\left[\int_0^l a(x,\xi)m(x)\phi_k(x)\,dx\right]d\xi$$

Now, if we interchange the variables of integration and, in addition, write $a(x,\xi) = a(\xi,x)$ from the known symmetry of the influence functions, we obtain

$$G_{kj} = \int_0^l m(x)\phi_j(x)\left[\int_0^l a(x,\xi)m(\xi)\phi_k(\xi)\,d\xi\right]dx = G_{jk} \tag{6.61}$$

Equation (6.60) is identical with Eq. (4.111) which was derived using the energy methods of Chapter 4. Comparing Eq. (6.58) with Eq. (4.109), shows that the quantities G_{ij} are the same. Hence, we may say that the Galerkin method leads to the same results as the Rayleigh-Ritz method when influence functions are used in the latter.

6.7 Solution by Iteration (Stodola Method[26])

The integral Eq. (6.2) may be solved by iteration to yield the natural modes of a structure. The procedure begins with the selection of a trial function $u^{(0)}(x)$ which, for rapid convergence, should approximate the first mode as closely as possible. This function is inserted into the integrand of Eq. (6.2) and the integration is performed to yield a new function, say $u^{(1)}(x)$. This function is, in turn, inserted in the equation to yield $u^{(2)}(x)$. The functions obtained in this manner will converge, after a sufficient number of iterations, to the first mode which we shall designate $\Phi_1(x)$.

To demonstrate convergence we shall suppose that the trial function $u^{(0)}(x)$ is such that it may be represented by superposition of the natural modes $\Phi(x)$, in an infinite series

$$u^{(0)}(x) = \sum_{i=1}^{\infty} C_i\Phi_i(x) \tag{6.62}$$

where the C_i are constants. This means that the trial function should satisfy the boundary conditions and should have the general characteristics of the first normal mode at least. Insertion of $u^{(0)}(x)$ into Eq. (6.2) yields a new function $u^{(1)}(x)$, thus

$$\begin{aligned} u^{(1)}(x) &= \lambda_1 \int_0^l a(x,\xi)m(\xi)u^{(0)}(\xi)\,d\xi \\ &= \lambda_1 \int_0^l a(x,\xi)\,m(\xi)\sum_{i=1}^{\infty} C_i\Phi_i(\xi)\,d\xi \\ &= \lambda_1 \sum_{i=1}^{\infty} C_i \int_0^l a(x,\xi)m(\xi)\Phi_i(\xi)\,d\xi \end{aligned} \tag{6.63}$$

In this equation suppose that $u^{(0)}(x)$ is normalized and that, upon integration, we normalize $u^{(1)}(x)$. The integral and the normalized function $u^{(1)}(x)$ are then equal to within a multiplicative constant which we have called λ_1. Now we recall that the integral in the right-hand member of Eq. (6.63) yields the ith natural mode, (See Eq. 6.2), thus

$$\Phi_i(x) = \omega_i^2 \int_0^l a(x, \xi) m(\xi) \Phi_i(\xi) \, d\xi \tag{6.64}$$

Then, we may write Eq. (6.63) in the form

$$u^{(1)}(x) = \lambda_1 \sum_{i=1}^{\infty} \frac{C_i}{\omega_i^2} \Phi_i(x) \tag{6.65}$$

To proceed with the next iteration we operate on $u^{(1)}(x)$ according to Eq. (6.2) to give a new function $u^{(2)}(x)$.

$$u^{(2)}(x) = \lambda_2 \int_0^l a(x, \xi) m(\xi) u^{(1)}(\xi) \, d\xi \tag{6.66}$$

Insert Eq. (6.65) into the integrand to obtain (ignoring the normalizing constant λ_1)

$$\begin{aligned} u^{(2)}(x) &= \lambda_2 \int_0^l a(x, \xi) \, m(\xi) \sum_{i=1}^{\infty} \frac{C_i}{\omega_i^2} \Phi_i(\xi) \, d\xi \\ &= \lambda_2 \sum_{i=1}^{\infty} \frac{C_i}{\omega_i^2} \int_0^l a(x, \xi) m(\xi) \Phi_i(\xi) \, d\xi \\ &= \lambda_2 \sum_{i=1}^{\infty} \frac{C_i}{\omega_i^4} \Phi_i(x) \end{aligned} \tag{6.67}$$

In a similar manner, a third iteration yields

$$u^{(3)}(x) = \lambda_3 \sum_{i=1}^{\infty} \frac{C_i}{\omega_i^6} \Phi_i(x) \tag{6.68}$$

and n iterations result in

$$u^{(n)}(x) = \lambda_n \sum_{i=1}^{\infty} \frac{C_i}{\omega_j^{2n}} \Phi_i(x) \tag{6.69}$$

According to the conventional system of ordering the natural modes, the frequencies are ordered as

$$\omega_1 < \omega_2 < \omega_3 <, \ldots, < \omega_n \tag{6.70}$$

The rate of convergence depends upon how widely the frequencies are separated. The number of iterations n must be sufficient so that

$$\frac{1}{\omega_1^{2n}} \gg \frac{1}{\omega_2^{2n}} \gg \frac{1}{\omega_3^{2n}} \cdots$$

Depending upon the accuracy desired, n may be sufficiently great that within the desired approximation

$$\sum_{i=1}^{\infty} \frac{C_i}{\omega_i^{2n}} \Phi_i(x) = \frac{C_1}{\omega_1^{2n}} \Phi_1(x) \tag{6.71}$$

In this case, Eq. (6.69) becomes

$$u^{(n)}(x) = \lambda_n \frac{C_1}{\omega_1^{2n}} \Phi_1(x) \tag{6.72}$$

Therefore, the function obtained after n iterations has the form of the first normal mode $\Phi_1(x)$. After normalizing, $u^{(n)}(x)$ is equal to $\Phi_1(x)$. If one more iteration is carried out by using this normalized function then

$$\begin{aligned} u^{(n+1)}(x) &= \lambda_{n+1} \int_0^l a(x, \xi) m(\xi) \Phi_1(\xi)\, d\xi \\ &= \lambda_{n+1} \frac{\Phi_1(x)}{\omega_1^2} \end{aligned} \tag{6.73}$$

Since $u^{(n+1)}(x)$ also has the form of $\Phi_1(x)$, the normalizing constant λ_{n+1} is equal to ω_1^2. Thus after n iterations, the functional form repeats with each iteration and the multiplicative constant converges to the square of the first mode frequency.

Having obtained the first normal mode it is frequently necessary to determine higher modes also. Any arbitrary function with which the iteration is started and which may be represented by Eq. (6.62) will converge on the first mode provided it contains the first mode as a component, i.e., the constant C_1 in Eq. (6.62) is different from zero. It is possible to determine a "sweeping" procedure by which the first mode component may be suppressed so that the resulting function will converge, by iteration, on the second mode. To derive the equations by which sweeping is accomplished we again consider a trial function which could be represented by Eq. (6.62). Since we have already found the first mode, the orthogonality condition is used to determine the constant C_1. We write

$$\int_0^l m(x)\Phi_1(x)u^{(0)}(x)\, dx = \sum_{i=1}^{\infty} C_i \int_0^l m(x)\Phi_1(x)\Phi_i(x)\, dx \tag{6.74}$$

Because the normal modes are orthogonal, the right-hand member of this equation reduces to

$$C_1 \int_0^l m(x)\Phi_1^2(x)\, dx \tag{6.75}$$

Therefore, C_1 may be determined by

$$C_1 = \frac{\int_0^l m(x)\Phi_1(x)u^{(0)}(x)\, dx}{\int_0^l m(x)\Phi_1^2(x)\, dx} \tag{6.76}$$

Again, referring to Eq. (6.62), if we subtract from the trial function

$u^{(0)}(x)$ the first mode component which is $C_1\Phi_1(x)$ we obtain a new trial function from which the first mode has been "swept", i.e., a function constrained so that it will not converge to the first mode. Thus, the "swept" function is given by

$$u^{(0)}(x) - \frac{\int_0^l m(x)\Phi_1(x)u^{(0)}(x)\,dx}{\int_0^l m(x)\Phi_1^2(x)\,dx}\Phi_1(x) \tag{6.77}$$

Note that when this function is used in the iteration process, each iteration produces a new function which may include a first mode component. Therefore, it is generally necessary to carry out the sweeping process after each iteration so that convergence will proceed to the second mode.

Having obtained the first and second modes, the trial function may be swept so that further iteration will converge to the third mode. The constant C_2 may be determined by requiring that the second mode be orthogonal to the trial function. Thus

$$C_2 = \frac{\int_0^l m(x)\Phi_2(x)u^{(0)}(x)\,dx}{\int_0^l m(x)\Phi_2^2(x)\,dx} \tag{6.78}$$

The required trial function will be swept to suppress both the first and second modes by carrying out the subtraction to follow.

$$u^{(0)}(x) - C_1\Phi_1(x) - C_2\Phi_2(x) \tag{6.79}$$

It is seen from the foregoing discussion that iteration may be carried out to yield all the higher modes. In practice the method becomes very laborious for higher modes because selection of satisfactory trial functions is more difficult and sweeping is more complicated.

The integrations involved in the process described here may be performed by approximate methods of numerical integration. If this is done and if the equations are written in matrix form, the iteration procedure described here is very much the same as the method of matrix iteration described in Chapter 3.

As applied to beams, the results obtained by carrying out the integration

$$\int_0^l a(x,\xi)m(\xi)u(\xi)\,d\xi$$

are completely equivalent to that obtained by carrying out the following steps based on the elementary beam theory.

1. Determine an inertia load distribution $m(\xi)\omega^2u(\xi)$ for $\omega = 1$.
2. Integrate the inertia loading successively to obtain the shear,

bending moment, slope, and deflection functions. Constants of integration are evaluated to satisfy the boundary conditions.

The iteration process may be carried out by determining from the initial trial function improved deflection functions using the above integration procedure. These integrations may be performed graphically or numerically if the functional relationships are not known analytically.

6.8 Formulation of the Eigenvalue Problem for Unconstrained or Partially Constrained Structures by Use of Flexibility Influence Coefficients

It was shown in Section 6.3 that the natural modes of structures whose properties are approximated by lumping at a finite number of stations, may be determined by use of their flexibility matrix. This method of analysis was used in Chapter 3 where it was applied specifically to structures subjected to external constraints. However, as was shown in Chapter 1, Section 9, the flexibility matrix does not exist for structures that are unconstrained, or only partially constrained. Despite this fact, it is possible to construct a flexibility matrix for the structure by placing fictitious constraints on it. These constraints are allowed to move in a manner simulating the rigid-body motion associated with an unconstrained or partially constrained structure as discussed in Chapter 4, Section 10. Using the flexibility matrix so derived, we can form a dynamical matrix $[D]$, and the iteration process applied to $[D]$ will converge to the lowest mode first as is often desired. The basic difference between the method of analysis treated here, and the method discussed in Chapter 4, Section 10 is that here we generate a flexibility matrix in the process, whereas

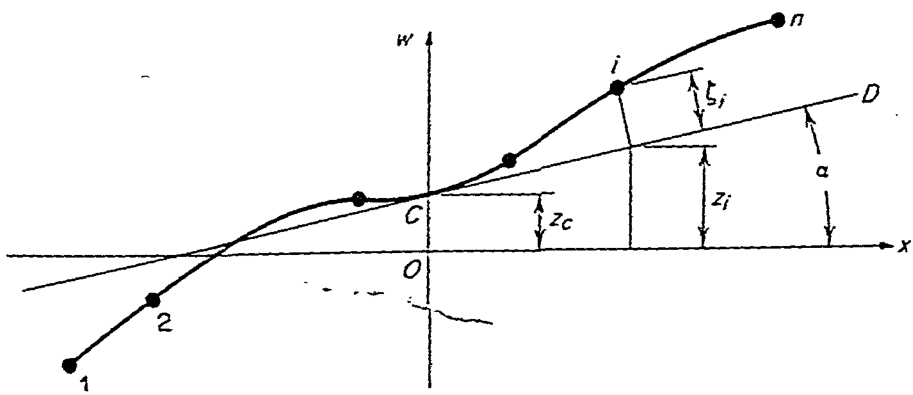


Figure 6.2

in Section 4.10, a nonsingular stiffness matrix was developed from which a flexibility matrix can be obtained by inversion. To demonstrate the method of this section we apply it to a free-free beam.

In Fig. 6.2 a free-free beam is shown with its mass lumped at n stations along its length.* It is considered to be executing vibration in a natural mode about an axis x from which the displacements w_i are measured. The motion is considered in two parts: rigid-body translation and rotation to axis CD which is tangent to the axis of the strained beam at point C, and deflection from axis CD to the final strained configuration. The amplitudes in the two motions are related through momentum considerations. For small amplitudes, the total displacement at the ith station is given by

$$w_i = z_i + \zeta_i \tag{6.80}$$

where

z_i = displacement corresponding to the rigid-body motion

ζ_i = displacement corresponding to deflection of the beam from a straight line (line CD in Fig. 6.2)

The displacement z_i may be expressed in terms of the displacement z_c of the reference point C and the angular displacement α of the line CD. Thus

$$z_i = z_c + \alpha x_i \tag{6.81}$$

where x_i is the x coordinate of the ith station.

The following equations are obtained by setting the linear and angular momenta equal to zero (note that the angular momentum is taken with reference to the point C).

$$\left.\begin{aligned} \sum_{i=1}^{n} m_i \dot{w}_i &= 0 \\ \sum_{i=1}^{n} m_i x_i \dot{w}_i &= 0 \end{aligned}\right\} \tag{6.82}$$

The motion at every point is harmonic, with frequency ω. Hence,

$$\dot{w}_i = i\omega w_i \tag{6.83}$$

where $i = \sqrt{-1}$. It follows that Eqs. (6.82) may be expressed in terms of the displacements.

$$\left.\begin{aligned} \sum_{i=1}^{n} m_i w_i &= 0 \\ \sum_{i=1}^{n} m_i x_i w_i &= 0 \end{aligned}\right\} \tag{6.84}$$

*The procedure that follows can be applied equally well to a distributed mass system with influence functions replacing the influence coefficients associated with a lumped mass system.

Substituting Eqs. (6.80) and (6.81), we write Eqs. (6.84) as

$$\left.\begin{aligned} z_c \sum_{i=1}^{n} m_i + \alpha \sum_{i=1}^{n} m_i x_i + \sum_{i=1}^{n} m_i \zeta_i = 0 \\ z_c \sum_{i=1}^{n} m_i x_i + \alpha \sum_{i=1}^{n} m_i x_i^2 + \sum_{i=1}^{n} m_i x_i \zeta_i = 0 \end{aligned}\right\} \tag{6.85}$$

At this point we simplify the analysis by taking the reference point C at the center of mass of the undeformed beam. It follows that

$$\sum_{i=1}^{n} m_i x_i = 0 \tag{6.86}$$

We also make the following definitions.

$$\sum_{i=1}^{n} m_i = m, \qquad \text{the total mass of the beam} \tag{6.87}$$

$$\sum_{i=1}^{n} m_i x_i^2 = m\bar{x}^2, \qquad \text{the mass moment of inertia relative to the center of mass.} \tag{6.88}$$

Equations (6.85) are used to specify the rigid-body motion variables z_c and α in terms of the deflection amplitudes. Using Eqs. (6.86), (6.87), and (6.88) in Eq. (6.85), we write

$$z_c = -\frac{1}{m} \sum_{i=1}^{n} m_i \zeta_i \tag{6.89}$$

$$\alpha = -\frac{1}{m\bar{x}^2} \sum_{i=1}^{n} m_i x_i \zeta_i \tag{6.90}$$

Since the deflections ζ_i are measured from the tangent CD, the point C may be considered as a point at which the beam is constrained for the purpose of computing these deflections. The two segments of the beam to the right and to the left of point C may be regarded as cantilevers whose influence coefficients are used to relate deflections and forces through the relation

$$\zeta_i = \sum_j a_{ij} F_j \tag{6.91}$$

where the summation over j is taken only over the right segment or the left one depending upon the position of station i. It can be seen that the flexibility matrix $[a]$ for the entire beam has the form

$$[a] = \left[\begin{array}{c|c} [a]_{11} & 0 \\ \hline 0 & [a]_{22} \end{array}\right] \tag{6.92}$$

where the submatrix $[a]_{11}$ relates to the left-hand segment and $[a]_{22}$ to the right-hand one. The force F_j is inertial and is dependent upon the total amplitude at that point

$$F_j = \omega^2 m_j w_j \tag{6.93}$$

Substitution of Eq. (6.93) into (6.91) yields an equation related to station i. There are n such equations which have the matrix form

$$\{\zeta\} = \omega^2 [a][m]\{w\} \tag{6.94}$$

In this equation the matrix $[a]$ is that which is shown in partitioned form in Eq. (6.92).

By substituting Eqs. (6.81), (6.89), and (6.90) into Eq. (6.80), the total displacement w_i may be expressed in terms of the deflection ζ_i by the relation

$$w_i = \zeta_i - \frac{1}{m}\sum_{j=1}^{n} m_j\zeta_j - \frac{x_i}{m\bar{x}^2}\sum_{j=1}^{n} m_j x_j \zeta_j \tag{6.95}$$

This set of equations is written in the form

$$\{w\} = [N]\{\zeta\} \tag{6.96}$$

where the matrix $[N]$ is given by

$$[N] = [I] - \frac{1}{m}[\bar{m}] - \frac{1}{m\bar{x}^2}[x][\bar{m}][x] \tag{6.97}$$

In this equation $[I]$ is the identity matrix. Matrix $[\bar{m}]$ is an $n \times n$ matrix having identical rows as follows

$$[\bar{m}] = \begin{bmatrix} m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_n \\ \vdots & \vdots & & \vdots \\ m_1 & m_2 & \cdots & m_n \end{bmatrix} \tag{6.98}$$

Matrix $[x]$ is a diagonal matrix of the coordinates $x_1, x_2, \ldots, x_n$

$$[x] = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix} \tag{6.99}$$

Now, substitute Eq. (6.96) into Eq. (6.94) to yield the following equation in the eigenvectors $\{\zeta\}$.

$$\{\zeta\} = \omega^2 [a][m][N]\{\zeta\} \tag{6.100}$$

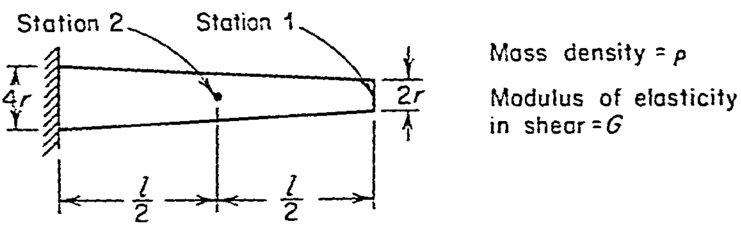
This equation may be used to determine the natural frequencies and the natural modes in terms of the ζ's. The modal columns for total displacements may then be found from Eq. (6.96). The matrix obtained from the product $[m][N]$ may be regarded as a mass matrix modified to permit its use with the quasi-flexibility matrix. It is a symmetric matrix, as may be determined by forming the products of $[m]$ with each of the component matrices of $[N]$ as given by Eq. (6.97), and noting that each of these products is symmetric. It follows that the modal columns $\{\xi\}$ are orthogonal with respect to the weighting matrix $[m][N]$. Therefore, if the modal columns are found by the matrix iteration procedure, the sweeping matrices may be determined using this orthogonality criterion in the same manner as demonstrated in Chapter 3.

PROBLEMS

1. Derive the appropriate influence functions and formulate the integral equation for a uniform cantilevered beam. Neglect the effects of shear and rotatory inertia. The beam has mass m per unit length, cross-section area A, and bending rigidity EI.

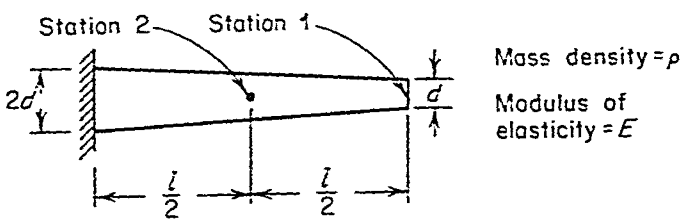
2. Show that when the integral in Eq. (6.15) is approximated by a summation such as given by Eq. (6.24), then the proof of the orthogonality of Section 6.2 reduces to the same proof given in Chapter 3, Section 4, for a lumped mass system.

3. Find the first two natural modes and frequencies of a cantilevered, tapered, circular shaft using station functions. Construct the station functions from the deflected shape due to concentrated torques at the selected stations. Use two stations as shown. Evaluate the station functions numerically at six equally spaced points along the shaft and use this information in your computations.



Problem 3

4. Find the first two natural modes and frequencies of the nonuniform cantilevered beam using station functions. Construct the station functions from the deflected shape due to concentrated loads at the selected stations. Use



Problem 4

two stations as shown. Evaluate the station functions numerically at six equally spaced points along the beam and use this information in your computations.

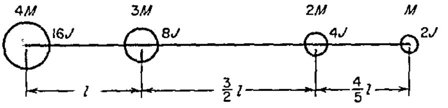
5. Repeat Problem 3 using a triangular distribution for the torque to generate the station functions. Compare the results with those of Problem 3.

6. Repeat Problem 4 using a triangular distribution for the load to generate the station functions. Compare the results with those of Problem 4.

7. Compare the Rayleigh-Ritz and Galerkin methods of solution applied to the first two modes of the nonuniform cantilever beam of Problem 4.

8. *Determine the first two natural modes of a uniform cantilever beam by the Stodola iteration method.*

9. The uniform free-free beam shown has flexural rigidity EI and is considered massless. The attached concentrated masses $m, 2m, 3m,$ and $4m$ have mass moments of inertia $2J, 4J, 8J,$ and $16J$ about axes perpendicular to the plane of the paper and going through their respective mass centers. Using the method of Section 6.8, formulate the eigenvalue problem for free vibration of this system in the plane of the paper.



Problem 9

CHAPTER 7

Damping

7.1 Introduction

It has been shown in earlier chapters that with a knowledge of the mass distribution and stiffness properties of a structural system, the natural modes of vibration can be computed. This may be done, for instance, through the solution of Eq. (3.7).

$$[D]\{q\} = \frac{1}{\omega^2}\{q\} \tag{3.7}$$

This matrix equation represents a homogeneous set of linear equations. They are derived by considering the vibrations to persist once being initiated without the application of external forces. In reality such a vibration without decrease in amplitude is never realized. The presence of damping forces causes the dissipation of energy that progressively reduces the amplitude of vibration and ultimately stops the motion when all energy initially stored in the system has been dissipated. Consequently the continuous exchange between potential and kinetic energy with the total energy of the system held at a constant level is valid only for an ideal conservative system. In a nonconservative system in which damping forces are present energy is dissipated (lost) from the system.

If the energy of a nonconservative system is to be maintained at a constant level, an external source must supply energy to the system at a rate equal to the rate of energy dissipation.

Figure 7.1 shows a plot of energy per cycle vs. amplitude of a single-degree-of-freedom system. Curve I represents the energy input

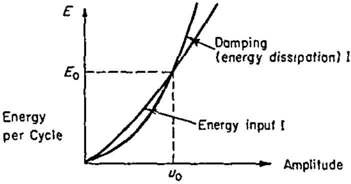


Figure 7.1

from an external source; curve II represents the energy dissipated. It is seen from this figure that for amplitudes smaller than u_0 the energy input per cycle is larger than the energy dissipated and, consequently, the amplitude will increase. For amplitudes larger than u_0 the energy dissipated is larger and amplitudes will decrease. At amplitude u_0 the energy of the system is maintained at a constant level with vibration persisting at a constant peak amplitude.

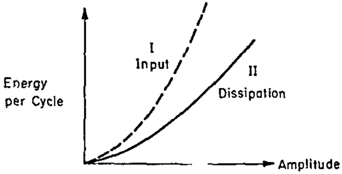


Figure 7.2

In Fig. 7.2 the energy input always exceeds the energy dissipated, consequently no state of equilibrium is reached. Amplitudes will progressively increase to structural failure.

7.2 Nature of Damping

While mass and stiffness are inherent characteristics of the system, damping may not be classified as such from the outset. Damping forces may depend upon the vibrating system as well as on elements exterior to it. The formulation of expressions for the damping forces poses a difficult problem that still requires extensive research. The nature of damping is customarily described as one of the following:

(a) Structural damping
(b) Viscous damping
(c) Coulomb damping
(d) Negative damping

Structural damping is due to internal friction within the material or at connections between elements of a structural system. The resulting damping forces are a function of the strain, or deflections, in the structure. For an elastic system* the jth structural damping force F_{D_j} is proportional in magnitude to the internal elastic force F_{E_j}, and opposite in direction to the velocity vector $\dot{u}_j$. This relation is expressed by

$$F_{D_j} = igF_{Ej}$$

where g is a constant and i is the unit imaginary number.

Viscous damping results when a system vibrates in a fluid (air, oil, etc.). Shock absorbers, hydraulic dashpots, and sliding of a body on a lubricated surface are some examples where viscous damping may be encountered. In viscous damping the jth damping force is expressed as

$$F_{D_j} = c_j\dot{u}_j$$

in which the constant c_j characterizes the jth damping mechanism. The amplitude of free vibration with viscous damping decays exponentially, as is shown in Fig. 7.3.

Coulomb damping, or dry friction, results from the motion of a body on a dry surface. The resulting damping force is nearly constant. It depends upon the normal pressure N between the moving body and the surface on which it moves, and the coefficient of kinetic friction μ.

$$F_D = \mu N$$

The amplitude of free vibration with Coulomb damping decays linearly[15] as shown in Fig. 7.4.

*A system vibrating so that the induced stresses are within the elastic range.

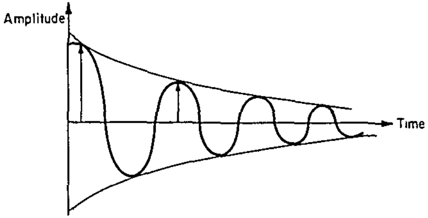


Figure 7.3. Amplitude vs. time in free vibration with viscous damping.

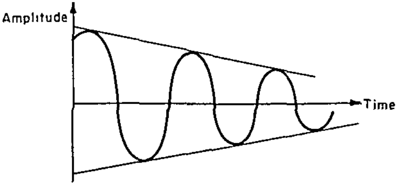


Figure 7.4. Amplitude vs. time in free vibration with coulomb damping.

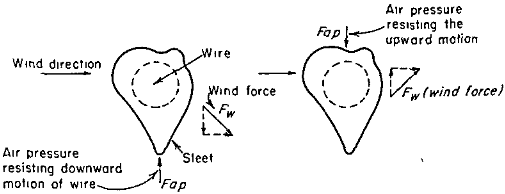


Figure 7.5

Negative damping results when the nature of the damping is such that instead of dissipating energy from the vibrating system, energy is added to it. As an example of negative damping consider the cross-section of a transmission line wire (Fig. 7.5). At temperatures around 32°F sleet forms around the wire cross-section as shown in the figure. With the wind blowing to the right, the elongated shape of wire cross-section and sleet causes the aerodynamic force F_w on the wire to act in a direction different from the wind direction. During the downward motion of the wire the air pressure from below resists the motion with force $Fa.p.$ ($Fa.p.$ = Force of air pressure). However, if the downward component of F_w is larger than $Fa.p.$, the net result is a force in the direction of motion with energy added to the vibrating transmission line wire. Similar reasoning follows during the upward motion of the wire with energy added to the system when the upward component of F_w is larger than $Fa.p.$ A discussion on the flow of air around the elongated wire cross-section causing the wind force to be directed as shown in Fig. 7.5 is found in Reference 27. When negative damping occurs, amplitudes increase progressively as shown in Fig. 7.6.

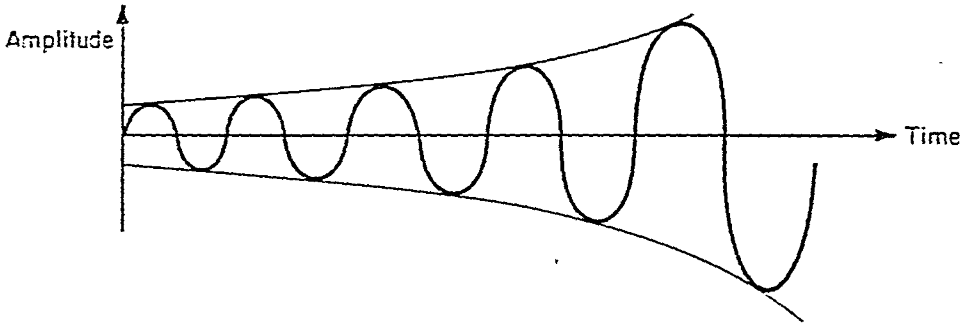


Figure 7.6. Amplitude vs. time in free vibration with negative damping.

Damping forces encountered in vibrating systems are not necessarily linear functions of velocity or displacement of the moving body. Thus, for instance, experiments show that the resistance of air to bodies moving through it at high speed is approximately proportional to the square of the velocity. In most actual physical systems it is very difficult to derive the expression for the damping forces. The complexity involved in the solution of the differential equations of motion depends very much on the expression for the damping forces. For the case of structural or viscous damping, the differential equations are not difficult to handle. It is customary (and practical), therefore, to replace the damping forces in a system by an equivalent viscous damping causing the same amount of energy dissipation[28]. This book will discuss the formulation and solution of problems in which linear structural or viscous damping is present.

7.3 Lagrange's Equations with Damping

From Chapter 2, Eq. (2.116), the ith of Lagrange's equations in generalized coordinates $q_i (i = 1, 2, \ldots, n)$ is expressed by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} - Q_{D_i} = Q_{A_i} \tag{7.1}$$

The term Q_{D_i} represents the ith generalized damping force. The expression for Q_{D_i} in terms of damping forces $F_{D_j} (j = 1, 2, \ldots, m)$ in the constrained $u_j (j = 1, 2, \ldots, m)$ coordinate system can be derived from the principle of virtual work. The virtual work δW_D done by damping forces F_{D_j} along virtual displacements δu_j in the u coordinate system is given by

$$\delta W_D = \sum_j F_{D_j} \delta u_j \tag{7.2}$$

We now apply a coordinate transformation

$$u_j = u_j(q_1, q_2, \ldots, q_n)$$
$$j = 1, 2, \ldots, m$$

to take us from constrained coordinates u to generalized coordinates q. Then

$$\delta u_j = \sum_{i=1}^{n} \frac{\partial u_j}{\partial q_i} \delta q_i \tag{7.3}$$

and Eq. (7.2) becomes

$$\delta W_D = \sum_j F_{D_j} \sum_i \frac{\partial u_j}{\partial q_i} \delta q_i$$

Interchanging the order of summation and rearranging, we write

$$\delta W_D = \sum_i \delta q_i \sum_j F_{D_j} \frac{\partial u_j}{\partial q_i} \tag{7.4}$$

The virtual work δW_D can also be expressed as the sum of the work done by the generalized damping forces Q_{D_i} along their corresponding virtual displacements δq_i

$$\delta W_D = \sum_i Q_{D_i} \delta q_i \tag{7.5}$$

Comparing Eqs. (7.4) and (7.5) we write

$$Q_{D_i} = \sum_j F_{D_j} \frac{\partial u_j}{\partial q_i} \qquad i = 1, 2, \ldots, n \tag{7.6}$$

Proceeding in a similar manner, the ith generalized applied force Q_{A_i} on the right-hand side of Eq. (7.1) can be expressed in the form

$$Q_{A_i} = \sum_j F_{A_j} \frac{\partial u_j}{\partial q_i} \qquad i = 1, 2, \ldots, n \tag{7.7}$$

in which $F_{A_j}(j = 1, 2, \ldots, m)$ are the applied forces in the constrained $u_j(j = 1, 2, \ldots, m)$ coordinate system.

When the transformation from coordinates u to coordinates q is linear, namely the u's are linear functions of the q's, we write

$$\{u\} = [C]\{q\}$$

in which

$$C_{jk} = \frac{\partial u_j}{\partial q_k} \qquad j = 1, 2, \ldots, m \qquad k = 1, 2, \ldots, n$$

Using these results the generalized damping forces and the generalized applied forces expressed by Eqs. (7.6) and (7.7), respectively, take the form

$$\{Q\}_D = [C]^T\{F\}_D \tag{7.8}$$

$$\{Q\}_A = [C]^T\{F\}_A \tag{7.9}$$

These results agree with the same expressions derived in Chapter 2, Eq. (2.132).

7.4 Lagrange's Equations with Structural Damping

For the special case of structural damping, the damping forces F_{D_j} are proportional in magnitude to the elastic forces F_{E_j} and opposite in direction to the velocities $\dot{u}_j$ in the u coordinate system.

$$F_{D_j} = igF_{E_j} \tag{7.10}$$

Here, consider g to have the same value for all points of the structure. Substituting Eq. (7.10) in Eq. (7.6) we obtain

$$Q_{D_i} = ig \sum_j F_{E_j} \frac{\partial u_j}{\partial q_i} \tag{7.11}$$

Proceeding in a manner similar to the derivation of Eq. (7.6) we can show that the ith generalized elastic force Q_{E_i} is given by

$$Q_{E_i} = \sum_j F_{E_j} \frac{\partial u_j}{\partial q_i} \tag{7.12}$$

Using this result in Eq. (7.11) we write for the ith generalized structural damping force

$$Q_{D_i} = igQ_{E_i}$$

When the elastic forces Q_{E_i} are derived from a potential U, namely the strain energy of the system, we have from Chapter 2, Eq. (2.115)

$$Q_{E_i} = -\frac{\partial U}{\partial q_i}$$

and consequently

$$Q_{D_i} = -ig\frac{\partial U}{\partial q_i}$$

Using the last expression in Eq. (7.1) we conclude that for a system in which the energy dissipated is due to structural damping, the ith Lagrange equation in generalized coordinates q takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + (1 + ig)\frac{\partial U}{\partial q_i} = Q_{Ai} \qquad i = 1, 2, \ldots, n \qquad (7.13)$$

When the kinetic energy T is a function of the generalized velocities $\dot{q}_j (j = 1, 2, \ldots, n)$ only then $\partial T / \partial q_i = 0$, and the differential equations of motion (7.13) take the matrix form

$$[m]\{\ddot{q}\} + (1 + ig)[k]\{q\} = \{Q\}_A \qquad (7.14)$$

7.5 Viscous Damping Forces

It is customary to represent viscous damping by a dashpot as shown in Fig. 7.7. The damping force F_{Di} is related linearly to the relative velocity of the piston with respect to the cylinder containing the viscous fluid.

$$\text{Damping Force} = c_i[\dot{u}_i(\text{piston}) - \dot{u}_i(\text{cylinder})]$$

In much the same way as the spring constant k_i characterizes an elastic spring, c_i is a positive constant characterizing the dashpot. Thus, c_i represents the damping force resulting from a unit velocity difference between piston and cylinder in Fig. 7.7. Dashpots can be attached to a structural model in two basic ways: between discrete mass points or from a fixed reference to any mass point. In Fig. 7.8 damping forces from c_1, c_4, c_5, c_6 depend on the absolute velocity of the masses to which these dashpots are connected. This may, for instance, represent the damping from air through which the structure vibrates. Damping forces from c_2 and c_3 depend on the relative velocity between masses. These forces may be representative of internal friction which depends on relative velocity.

To demonstrate the analogy between the spring coefficients k_i and the damping coefficients c_i, consider the building of Fig. 7.8. k_1, k_2, and k_3 represent the stiffness coefficients for the columns of the first,

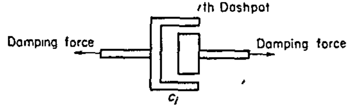


Figure 7.7. Representation of ith viscous damping mechanism.

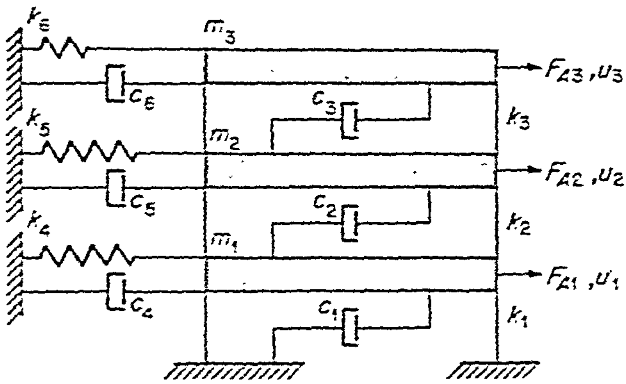


Figure 7.8

second, and third story, respectively. The mass of the building is assumed concentrated at the floor levels. The equations of motion for masses m_1, m_2, and m_3 take the form

$$m_1\ddot{u}_1 = -k_1u_1 - k_4u_1 - c_1\dot{u}_1 - c_4\dot{u}_1 - k_2(u_1 - u_2) - c_2(\dot{u}_1 - \dot{u}_2) + F_{A_1}$$
$$= -(k_1 + k_4)\,u_1 - k_2\,(u_1 - u_2) - (c_1 + c_4)\dot{u}_1 - c_2(\dot{u}_1 - \dot{u}_2) + F_{A_1}$$
$$m_2\ddot{u}_2 = -k_5u_2 - k_2(u_2 - u_1) - k_3(u_2 - u_3) - c_5\dot{u}_2 - c_2(\dot{u}_2 - \dot{u}_1)$$
$$- c_3(\dot{u}_2 - \dot{u}_3) + F_{A_2}$$
$$m_3\ddot{u}_3 = -k_6u_3 - k_3(u_3 - u_2) - c_6\dot{u}_3 - c_3(\dot{u}_3 - \dot{u}_2) + F_{A_3}$$

Rearranging terms and using a matrix form, the equations of motion in the u coordinate system become

$$[m]_u\{\ddot{u}\} + [c]_u\{\dot{u}\} + [k]_u\{u\} = \{F\}_A \tag{7.15}$$

in which

$$[m_u] = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix}, \qquad \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \qquad \{F\}_A = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_A$$

$$[c]_u = \begin{bmatrix} c_1 + c_2 + c_4 & -c_2 & 0 \\ -c_2 & c_2 + c_3 + c_5 & -c_3 \\ 0 & -c_3 & c_3 + c_6 \end{bmatrix}$$

$$[k]_u = \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & 0 \\ -k_2 & k_2 + k_3 + k_5 & -k_3 \\ 0 & -k_3 & k_3 + k_6 \end{bmatrix}$$

The elements c_{ij} of the damping matrix $[c]$ have a physical significance analogous to elements k_{ij} of the stiffness matrix $[k]$. Specifically, c_{ij} is equal to the external force required at mass i in direction

u_i, to produce a unit velocity at mass j in direction u_j with velocities at all other masses (stations) zero. It follows by analogy to the elements of the stiffness matrix $[k]$ that

$$c_{ij} = c_{ji}$$

This property of symmetry (reciprocity relation) holds for any generalized coordinate system as demonstrated by the following.

7.6 Reciprocity Relation for Damping Coefficients c_{ij}

In Fig. 7.8 the masses m_1, m_2, m_3 are vibrating with velocities $\dot{u}_1, \dot{u}_2, \dot{u}_3$. The external applied force F_{A_i}, in direction u_i, required at mass m_i to produce these velocities is given by

$$F_{A_i} = \sum_j c_{ij}\dot{u}_j \tag{7.16}$$

This follows from the physical significance of the c_{ij} terms as defined earlier. For simplicity assume all spring forces to be zero. Then, to satisfy conditions of equilibrium, the external force F_{A_i} must be equilibrated by the internal force F_{D_i} imposed by the internal damping at mass i in the ith coordinate. Thus,

$$F_{A_i} + F_{D_i} = 0$$

or

$$F_{D_i} = -F_{A_i}$$

Substituting for F_{A_i} from Eq. (7.16)

$$F_{D_i} = -\sum_j c_{ij}\dot{u}_j \tag{7.17}$$

The virtual work δW_D done by damping forces F_{D_i} along the corresponding virtual displacements δu_i in the u coordinate system then becomes

$$\delta W_D = \sum_i F_{D_i}\delta u_i = -\sum_i \delta u_i \sum_j c_{ij}\dot{u}_j \tag{7.18}$$

Applying a coordinate transformation

$$u_j = u_j(q_1, q_2, \ldots, q_n)$$

and substituting

$$\delta u_i = \sum_k \frac{\partial u_i}{\partial q_k}\delta q_k \qquad \dot{u}_j = \sum_l \frac{\delta u_j}{\partial q_l}\dot{q}_l$$

Eq. (7.18) becomes

$$\delta W_D = -\sum_i \sum_k \frac{\partial u_i}{\partial q_k}\delta q_k \sum_j c_{ij} \sum_l \frac{\partial u_j}{\partial q_l}\dot{q}_l$$

Rearranging

$$\delta W_D = -\sum_k \delta q_k \sum_l \dot{q}_l \sum_i \sum_j c_{ij} \frac{\partial u_i}{\partial q_k} \frac{\partial u_j}{\partial q_l} \tag{7.19}$$

In terms of the generalized damping forces Q_{D_k} the virtual work δW_D may be written

$$\delta W_D = \sum_k Q_{D_k} \delta q_k \tag{7.20}$$

From Eqs. (7.19) and (7.20) by direct comparison

$$Q_{D_k} = -\sum_l \dot{q}_l \sum_i \sum_j c_{ij} \frac{\partial u_i}{\partial q_k} \frac{\partial u_j}{\partial q_l} \tag{7.21}$$

Using Eq. (7.17) the generalized damping force Q_{D_k} may also be expressed as

$$Q_{D_k} = -\sum_l c_{kl} \dot{q}_l \tag{7.22}$$

in which the terms c_{kl} are the generalized damping coefficients in generalized coordintes q. Comparing Eqs. (7.21) and (7.22) we write

$$c_{kl} = \sum_i \sum_j c_{ij} \frac{\partial u_i}{\partial q_k} \frac{\partial u_j}{\partial q_l} \tag{7.23}$$

From Eq. (7.23) it is deduced that the generalized damping coefficients also possess the property of symmetry

$$c_{kl} = c_{lk}$$

In matrix form Eq. (7.17) becomes

$$\{F\}_D = -[c]_z \{\dot{u}\} \tag{7.24}$$

Applying the linear coordinate transformation

$$\{u\} = [C]\{q\}$$

we write

$$\{\dot{u}\} = [C]\{\dot{q}\} \tag{7.25}$$

Substituting Eqs. (7.24) and (7.25) into Eq. (7.8), the generalized viscous damping forces become

$$\{Q\}_D = -[C]^T [c]_z [C]\{\dot{q}\} \tag{7.26}$$

Equation (7.22) has the matrix form

$$\{Q\}_D = -[c]_q \{\dot{q}\} \tag{7.27}$$

in which $[c]_q$ represents the generalized damping matrix in the q coordinate system. Comparing Eqs. (7.26) and (7.27) we obtain

$$[c]_q = [C]^T [c]_z [C] \tag{7.28}$$

This is a restatement of Eq. (7.23) in matrix form accounting for all the generalized damping coefficients c_{kl}.

From the last three relations we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = (m_1 + m_2)\,\ddot{q}_1 + m_2\ddot{q}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m_2\ddot{q}_1 + (m_2 + m_3)\,\ddot{q}_2 + m_3\ddot{q}_3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) = m_3\ddot{q}_2 + m_3\ddot{q}_3$$

The strain energy U is a function of the displacements $u_i (i = 1, 2, 3)$ only

$$U = U(u_i)$$

and the coordinate transformation is

$$u_i = u_i(q_j)$$

We can therefore write

$$\frac{\partial U}{\partial q_j} = \sum_i \frac{\partial U}{\partial u_i}\frac{\partial u_i}{\partial q_j}$$

in which, for a linear transformation,

$$\frac{\partial u_i}{\partial q_j} = C_{ij}$$

or

$$\left[\frac{\partial u_i}{\partial q_j}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

For $j = 1$ we have

$$\frac{\partial U}{\partial q_1} = (k_1 + k_4)u_1 + (k_3 + k_5)u_2 - k_3 u_3$$

Substituting for u_i from $\{u\} = [C]\{q\}$ we have

$$\frac{\partial U}{\partial q_1} = (k_1 + k_3 + k_4 + k_5)q_1 + k_5 q_2 k_3 q_3$$

Proceeding in a similar manner we obtain

$$\frac{\partial U}{\partial q_2} = k_5 q_1 + (k_2 + k_5 + k_6)q_2 + k_6 q_3$$

$$\frac{\partial U}{\partial q_3} = -k_3 q_1 + k_6 q_2 + (k_3 + k_6)q_3$$

To express the generalized damping forces Q_{D_i} use Eq. (7.6).

$$Q_{D_i} = \sum_j F_{D_j}\frac{\partial u_j}{\partial q_i}$$

For $i = 1$ we have

$$Q_{D_1} = -(c_1 + c_4)\dot{u}_1 - (c_3 + c_5)\dot{u}_2 + c_3\dot{u}_3$$

Substituting for $\dot{u}_i$ in terms of $\dot{q}_i$ we write

$$Q_{D_1} = -(c_1 + c_3 + c_4 + c_5)\dot{q}_1 - c_5\dot{q}_2 + c_3\dot{q}_3$$

Proceeding in a similar manner

$$Q_{D_2} = -c_5\dot{q}_1 - (c_2 + c_5 + c_6)\dot{q}_2 - c_6\dot{q}_3$$

$$Q_{D_3} = c_3\dot{q}_1 - c_6\dot{q}_2 - (c_3 + c_6)\dot{q}_3$$

The generalized applied forces Q_{A_i} are expressed in terms of forces F_{A_i} in coordinates u by Eq. (7.7). Using this equation for the forces applied to the structure of Fig. 7.8, we obtain

$$Q_{A_1} = F_{A_1} + F_{A_2}$$

$$Q_{A_2} = F_{A_2} + F_{A_3}$$

$$Q_{A_3} = F_{A_3}$$

Using all relevant results in Lagrange's equations we obtain the following set of three differential equations of motion in the q coordinate system.

$$[m]_q\{\ddot{q}\} + [c]_q\{\dot{q}\} + [k]_q\{q\} = \{Q\}_A$$

in which

$$[m]_q = \begin{bmatrix} m_1 + m_2 & m_2 & 0 \\ m_2 & m_2 + m_3 & m_3 \\ 0 & m_3 & m_3 \end{bmatrix}$$

$$[c]_q = \begin{bmatrix} c_1 + c_3 + c_4 + c_5 & c_5 & -c_3 \\ c_5 & c_2 + c_5 + c_6 & c_6 \\ -c_3 & c_6 & c_3 + c_6 \end{bmatrix}$$

$$[k]_q = \begin{bmatrix} k_1 + k_3 + k_4 + k_5 & k_5 & -k_3 \\ k_5 & k_2 + k_5 + k_6 & k_6 \\ -k_3 & k_6 & k_3 + k_6 \end{bmatrix}$$

$$\{Q\}_A = \begin{Bmatrix} F_1 + F_2 \\ F_2 + F_3 \\ F_3 \end{Bmatrix}_A$$

To formulate the equations of motion by applying Eq. (7.29) directly, first derive the mass, stiffness, and damping matrices as well as the applied force vector in coordinates u. These are respectively

$$[m]_u = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix}$$

$$[k]_u = \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & 0 \\ -k_2 & k_2 + k_3 + k_5 & -k_3 \\ 0 & -k_3 & k_3 + k_6 \end{bmatrix}$$

$$[c]_u = \begin{bmatrix} c_1 + c_2 + c_4 & -c_2 & 0 \\ -c_2 & c_2 + c_3 + c_5 & -c_3 \\ 0 & -c_3 & c_3 + c_6 \end{bmatrix}$$

$$\{F\}_A = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_A$$

Following the linear coordinate transformation

$$\{u\} = [C]\{q\}$$

we compute the generalized mass, stiffness, damping matrices, and the vector of generalized applied forces from the relations

$$[m]_q = [C]^T[m]_u[C]$$

$$[k]_q = [C]^T[k]_u[C]$$

$$[c]_q = [C]^T[c]_u[C]$$

and

$$\{Q\}_A = [C]^T\{F\}_A$$

respectively. Using the last relations in Eq. (7.29) we obtain the same set of differential equations of motion in coordinates q as were obtained by the use of Lagrange's equations. We leave the verification of these results as an exercise to the reader.

7.8 The Dissipation Function

In analogy with the strain energy function

$$U = \tfrac{1}{2} \sum_i \sum_j k_{ij} u_i u_j$$

we define a dissipation function[29] R in the u coordinate system by

$$R = \tfrac{1}{2} \sum_i \sum_j c_{ij} \dot{u}_i \dot{u}_j \tag{7.30}$$

To extend the analogy we shall derive Lagrange's equations for a system in which viscous damping is present through the use of the dissipation function. We apply a coordinate transformation

$$u = u(q_1, q_2, \ldots, q_n)$$

and substitute

$$\dot{u}_i = \sum_k \frac{\partial u_i}{\partial q_k} \dot{q}_k, \qquad \dot{u}_j = \sum_l \frac{\partial u_j}{\partial q_l} \dot{q}_l$$

in Eq. (7.30) to obtain

$$R = \frac{1}{2} \sum_i \sum_j c_{ij} \sum_k \frac{\partial u_i}{\partial q_k} \dot{q}_k \sum_l \frac{\partial u_j}{\partial q_l} \dot{q}_l$$

Rearranging and changing order of summation, we write

$$R = \frac{1}{2} \sum_k \sum_l c_{kl} \dot{q}_k \dot{q}_l \tag{7.31}$$

in which the generalized damping coefficients c_{kl} are given by Eq. (7.23). The rth generalized damping force Q_{D_r} may be obtained from the dissipation function [Eq. (7.31)] by differentiating R with respect to the rth generalized velocity $\dot{q}_r$

$$\frac{\partial R}{\partial \dot{q}_r} = \sum_l c_{rl} \dot{q}_l \tag{7.32}$$

Comparing Eqs. (7.22) and (7.32) we have

$$Q_{D_r} = -\frac{\partial R}{\partial \dot{q}_r} \tag{7.33}$$

The rth Lagrange equation can then be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial U}{\partial q_r} + \frac{\partial R}{\partial \dot{q}_r} = Q_{A_r} \tag{7.34}$$

Thus the dissipation function R corresponds to the strain energy function U in much the same way as the terms c_{ij} correspond to the terms k_{ij}. The dissipation function R represents one half the time rate of energy dissipation through viscous damping, or

$$R = \frac{1}{2} \dot{W}_D \tag{7.35}$$

To prove this relation, recall that

$$\delta W_D = -\sum_k Q_{D_k} \delta q_k{}^{\dagger}$$

represents the work done by the damping forces or the energy dissipated when the system undergoes virtual displacements δq_k. Hence, the time rate of energy dissipation is given by

$$\dot{W}_D = -\sum_k Q_{D_k} \dot{q}_k \tag{7.36}$$

Substituting for Q_{D_k} from Eq. (7.22)

$$\dot{W}_D = \sum_k \sum_l c_{kl} \dot{q}_k \dot{q}_l \tag{7.37}$$

Comparing Eqs. (7.31) and (7.37) establishes the validity of relation (7.35).

†The minus sign is due to the fact that the Q_{D_k} terms represent the damping forces acting on the system masses and are therefore directed opposite to virtual displacements δq_k of the same masses.

7.9 The Concept of Critical Viscous Damping

In free vibration of a single degree of freedom system in which viscous damping is present, the amplitude decreases with time as was shown in Fig. 7.3. As the damping coefficient c increases, damped free vibration continues to take place with increasing period T as long as $c < 2\sqrt{km}$. The constant $2\sqrt{km}$ is termed the critical damping of the system. For damping coefficient $c \geq 2\sqrt{km}$ no vibration takes place and the curve describing displacement $u(t)$ of the mass m may cross the line $u(t) = 0$ only once or not cross it at all, depending on the initial conditions. We will derive the foregoing results in the following discussion.

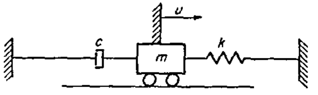


Figure 7.9

The differential equation of motion for the mass, spring, and viscous damper of Fig. 7.9, undergoing free vibration (no external loads are applied) is given by

$$m\ddot{u} + c\dot{u} + ku = 0 \tag{7.38}$$

A solution of the form

$$u = Ce^{\lambda t} \tag{7.39}$$

will satisfy the differential equation. Substituting u and its first and second time derivatives as obtained from Eq. (7.39) into Eq. (7.38) we obtain

$$(m\lambda^2 + c\lambda + k)Ce^{\lambda t} = 0$$

For a nontrivial solution $u = Ce^{\lambda t} \neq 0$, we have

$$m\lambda^2 + c\lambda + k = 0 \tag{7.40}$$

Equation (7.40) is the characteristic equation of the system. (See Chapter 3, Section 3.3.) The roots of the characteristic equation are

$$\left.\begin{aligned} \lambda_1 &= -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \\ \lambda_2 &= -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \end{aligned}\right\} \tag{7.41}$$

Equation (7.39) will be satisfied by the particular solutions

$$u = C_1 e^{\lambda_1 t}$$

or

$$u = C_2 e^{\lambda_2 t}$$

Consequently the sum of the two solutions will also satisfy Eq. (7.39). We therefore write the general solution in the form

$$u = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{7.42}$$

in which the arbitrary constants C_1 and C_2 are determined from the initial conditions. From Eqs. (7.41) and (7.42), we make the following observations.

(a) When $\left(\frac{c}{2m}\right)^2 \geq \frac{k}{m}$

or

$$c \geq 2\sqrt{km}$$

the roots λ_1 and λ_2 are real and negative; consequently the displacement $u(t)$ decays as a function of time with no vibration taking place.

(b) When $\left(\frac{c}{2m}\right)^2 < \frac{k}{m}$

or

$$c < 2\sqrt{km}$$

then

$$\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = +\sqrt{(-1)\left[\frac{k}{m} - \left(\frac{c}{2m}\right)^2\right]} = i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

and the roots λ_1, λ_2 form a complex conjugate pair. The imaginary part of the roots represents the oscillatory part of the motion with a damped free vibration frequency given by

$$\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

The real part of λ_1 and λ_2 represents the exponential decay of the amplitudes of vibration as shown in Fig. 7.3. Using Eqs. (7.41) and (7.42) we write

$$u = C_1 e^{-\frac{c}{2m}t} e^{i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t} + C_2 e^{-\frac{c}{2m}t} e^{-i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t}$$

Substituting the relations (Euler's formula)

$$e^{i\phi} = \cos\phi + i\sin\phi$$

and

$$e^{-i\phi} = \cos\phi - i\sin\phi$$

we write for the displacement

$$u = A_1 e^{-\frac{c}{2m}t} \cos\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t + A_2 e^{-\frac{c}{2m}t} \sin\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t \tag{7.43}$$

in which

$$A_1 = C_1 + C_2$$

and

$$A_2 = (C_1 - C_2)i$$

For free vibration with zero damping, Eq. (7.43) reduces to

$$u = A_1 \cos \omega t + A_2 \sin \omega t$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

The maxima of displacement $u(t)$ in damped free vibration occur at equal time intervals equal to the period T, where

$$T = \frac{2\pi}{\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}}$$

This can be verified by differentiating Eq. (7.43) with respect to time t. The ratio between consecutive maxima of u can be shown to be a constant given by

$$e^{\frac{c}{2m}T}$$

The natural logarithm of this constant, $(c/2m)\,T$, is called the *logarithmic decrement.*

7.10 Methods for Solving Vibration Problems

The differential equations of motion for a system in which structural damping or viscous damping are present are given, respectively, by

$$[m]\{\ddot{q}\} + (1 + ig)[k]\{q\} = \{Q\}_A \quad \text{(See Eq. 7.14)}$$

and

$$[m]\{\ddot{q}\} + [c]\{\dot{q}\} + [k]\{q\} = \{Q\}_A \quad \text{(See Eq. 7.29)}$$

These two equations represent the general formulation of a wide range of problems in dynamics of structures. The applied forces $\{Q\}_A$ may be

1. Zero
2. Harmonic

3. Periodic
4. Aperiodic
5. Random

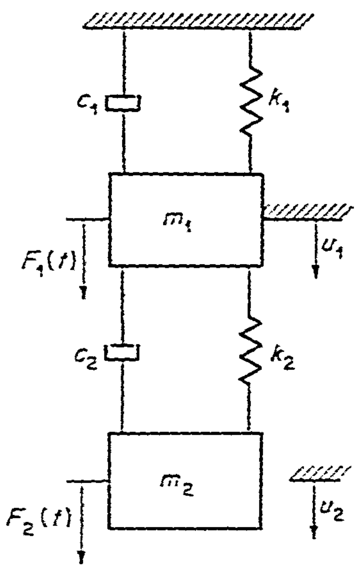
Each type of forcing function may be imposed on a structural system in which

1. damping is present or
2. considered insignificant

Two methods of solution to such vibration problems are introduced in Chapters 8 and 10. The problem of random excitations is dealt with in Chapter 11.

PROBLEMS

1. Formulate the differential equations of motion in the u coordinates for the system shown. The u's are measured from the equilibrium position.



Problem 1

2. Formulate the equation of motion for the system of Problem 1 in coordinates q where

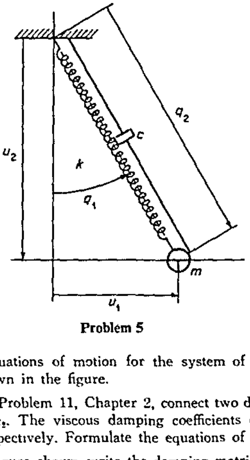
$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

Use your results from Problem 1.

3. Repeat Problem 2, using Lagrange's equations. Start with expressions for the strain energy U, the kinetic energy T, and the dissipation function R.
4. The system of Problem 1 is submerged in a viscous fluid with viscous damping coefficient c. Formulate the equations of motion in the q coordinate system given the transformation

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

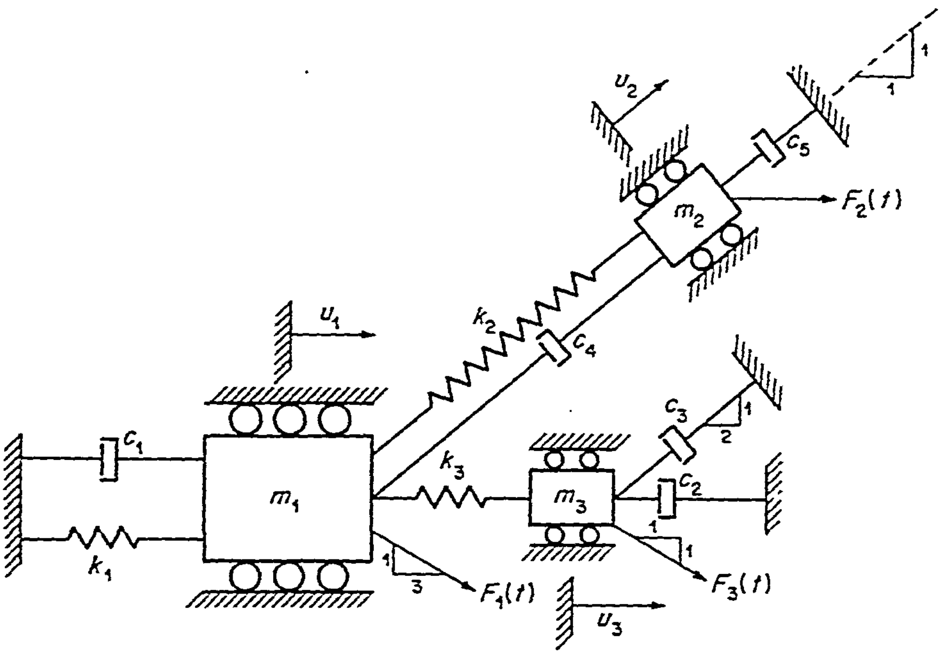
5. The mass particle m shown is suspended by a spring of free length l and stiffness k. The dashpot connecting mass m to the point of suspension has a coefficient of viscous damping c. The entire system is submerged in a viscous fluid with a viscous damping coefficient c_1. Formulate the equations of motion in coordinates u. Use Lagrange's equations as expressed by Eq. (7.34).



Problem 5

6. Formulate the equations of motion for the system of Problem 5 in the q coordinates shown in the figure.
7. In the figure for Problem 11, Chapter 2, connect two dashpots in parallel with springs k_1, k_2. The viscous damping coefficients of these dampers are c_1 and c_2, respectively. Formulate the equations of motion.
8. Referring to the figure shown, write the damping matrix and the applied force vector in generalized coordinates q, given

$$\{u\} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \{q\}$$



Problem 8

9. Write the equations of motion in coordinates q for the system of Problem 8.

10. The mass of Fig. 7.9 is initially displaced an amount $u(0)$ from its rest position and then released. Plot approximately the displacement $u(t)$ (measured from the rest position) as a function of time.

$$k = 40 \text{ lb/in}$$
$$m = 20 \text{ lb sec}^2/\text{ft.}$$
$$c = 2\sqrt{km}$$

11. Repeat Problem 10 for the following initial conditions:

$$u(o) = 0$$
$$\dot{u}(o) \neq 0$$

12. How can the logarithmic decrement, $(c/2m)T$, be used in an experiment designed to measure the viscous damping coefficient c?

CHAPTER 8

Normal Mode Method

8.1 Introduction

The *normal mode method* is characterized by the fact that the differential equations of motion are decoupled when the displacements are expressed in terms of the normal modes. Therefore, in a system having n degrees of freedom, we may deal with n independent differential equations rather than with a system of n simultaneous differential equations. Thus, the algebra required in the solution is considerably reduced. For convenience we approach the development of these differential equations by means of the Lagrange equations. For a system in which the damping forces are derived from a dissipation function R, the rth Lagrange equation of the set of n is given by (see Eq. 7.34, Chapter 7)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\eta}_r}\right) - \frac{\partial T}{\partial \eta_r} + \frac{\partial U}{\partial \eta_r} + \frac{\partial R}{\partial \dot{\eta}_r} = N_r \tag{8.1}$$

Here the coordinates η_r $(r = 1, 2, 3, \ldots, n)$ are normal coordinates and are a special set of generalized coordinates defined by

$$w(x,t) = \sum_{i=1}^{n} \Phi_i(x)\eta_i(t) \tag{8.2}$$

where $w(x,t)$ is the deflection of point x at time t and $\Phi_i(x)$ is the displacement configuration of the system vibrating in its ith natural mode. Eq. (8.2) then expresses the deflection at any point in the structure in terms of the normal modes of the structure and the normal coordinates. The force N_r of Eq. (8.1) denotes the rth generalized external force in the normal coordinate system.

It will now be shown how the introduction of the coordinate transformation, given by Eq. (8.2), into Lagrange's Eq. (8.1) will result in a decoupled equation of motion, or a differential equation in terms of a single normal coordinate η_r. In the first part of this chapter, the decoupled equations of motion will be derived for a system with distributed mass, stiffness, and damping. The external forces acting on the system will be considered when both distributed and discrete. The normal mode method as applied to lumped parameter systems will be treated next, including a formulation of the problem in matrix form. The last part of the chapter will be devoted to the solution of the decoupled differential equations followed by methods of computing internal forces due to dynamic loads.

8.2 Distributed Parameter System

For clarity of presentation we shall develop the method for a slender uniform beam with viscous damping. The application of the method to other structures will be apparent. Thus, for the slender beam the kinetic and strain energies are

$$T = \tfrac{1}{2}\int m\dot{w}^2\,dx \tag{8.3}$$

$$U = \tfrac{1}{2}\int EIw''^2\,dx \tag{8.4}$$

and the dissipation function is

$$R = \tfrac{1}{2}\int c\dot{w}^2\,dx \tag{8.5}$$

in which c is the damping per unit length of beam. The integrals of Eqs. (8.3), (8.4), (8.5) are taken over the length of the beam.

From Eq. (8.2) we may write

$$\dot{w}(x,t) = \sum_{i=1}^{n} \Phi_i(x)\dot{\eta}_i(t) \tag{8.6}$$

$$w''(x,t) = \sum_{i=1}^{n} \Phi_i''(x)\eta_i(t) \tag{8.7}$$

Substituting these equations into Eqs. (8.3), (8.4), and (8.5) we may express T, U, and R in terms of the normal coordinates η. However, it will be more convenient to proceed immediately to the determination of the partial derivatives

$$\frac{\partial T}{\partial \dot{\eta}_r}, \ \frac{\partial T}{\partial \eta_r}, \ \frac{\partial U}{\partial \eta_r}, \ \frac{\partial R}{\partial \dot{\eta}_r}$$

to be inserted in the Lagrange equation (8.1). Thus, from Eq. (8.3)

$$\frac{\partial T}{\partial \dot{\eta}_r} = \int m\dot{w}\frac{\partial \dot{w}}{\partial \dot{\eta}_r}\,dx = \int m\Phi_r(x)\left[\sum_{i=1}^{n}\Phi_i(x)\,\dot{\eta}_i\right]dx \tag{8.8}$$

The order of integration and summation may be interchanged to give

$$\frac{\partial T}{\partial \dot{\eta}_r} = \sum_{i=1}^{n}\dot{\eta}_i\int m\Phi_r(x)\Phi_i(x)\,dx \tag{8.9}$$

Inasmuch as $\Phi_r(x)$ and $\Phi_i(x)$ are normal mode shape functions, the integral in Eq. (8.9) is zero except when $i = r$ according to the condition of orthogonality. (See Problem 7, Chapter 3.) Thus

$$\begin{aligned}\int m\Phi_r(x)\Phi_i(x)\,dx &= 0, && \text{for } i \neq r\\ &= \int m\Phi_r^2(x)\,dx = M_r, && \text{for } i = r\end{aligned} \tag{8.10}$$

Therefore, Eq. (8.9) takes the simple form

$$\frac{\partial T}{\partial \dot{\eta}_r} = M_r\dot{\eta}_r \tag{8.11}$$

Let the viscous damping coefficient c be expressed by

$$c = 2\beta m \tag{8.12}$$

where β is a constant. The choice of the factor 2 is for convenience in the solution of the resulting decoupled differential equation [See Eq. (8.81).] Using relation (8.12) Eq. (8.5) becomes

$$R = 2\beta\tfrac{1}{2}\int m\dot{w}^2\,dx = 2\beta T$$

consequently

$$\frac{\partial R}{\partial \dot{\eta}_r} = 2\beta\frac{\partial T}{\partial \dot{\eta}_r}$$

or using Eq. (8.11)

$$\frac{\partial R}{\partial \dot{\eta}_r} = 2\beta M_r\dot{\eta}_r \tag{8.13}$$

It should be kept in mind that relation (8.13) was possible only because the damping distribution parameter c was selected proportional to the mass distribution parameter m. We shall return to discuss this matter in dealing with lumped parameter systems.

To find $\partial U/\partial \eta_r$, differentiate Eq. (8.4) with respect to η_r and substitute Eq. (8.7).

$$\begin{aligned}\frac{\partial U}{\partial \eta_r} &= \int EIw'' \frac{\partial w''}{\partial \eta_r}\,dx \\ &= \int EI\Phi_r''(x)\left[\sum_{i=1}^{n} \Phi_i''(x)\eta_i\right] dx\end{aligned} \tag{8.14}$$

Again, interchanging the order of integration and summation, we obtain

$$\frac{\partial U}{\partial \eta_r} = \sum_{i=1}^{n} \eta_i \int EI\Phi_r''(x)\Phi_i''(x)\,dx \tag{8.15}$$

From the condition of orthogonality, the integral of Eq. (8.15) must vanish for all i except for $i = r$. (See Problem 7, Chapter 3.) Thus

$$\begin{aligned}\int EI\Phi_r''(x)\Phi_i''(x)\,dx &= 0 && \text{for } i \neq r \\ &= \omega_r^2 \int m\Phi_r^2(x)\,dx = \omega_r^2 M_r, && \text{for } i = r\end{aligned} \tag{8.16}$$

Hence, Eq. (8.15) takes on the simple form

$$\frac{\partial U}{\partial \eta_r} = M_r\omega_r^2\eta_r \tag{8.17}$$

The term $\partial T/\partial \eta_r$ is readily seen to be zero inasmuch as the velocity $\dot{w}$ is a function of $\dot{\eta}$ only, according to Eq. (8.6). Therefore, $\partial \dot{w}/\partial \eta$ is zero, and consequently

$$\frac{\partial T}{\partial \eta_r} = \int m\dot{w}\frac{\partial \dot{w}}{\partial \eta_r}\,dx = 0 \tag{8.18}$$

Using Eqs. (8.11), (8.13), (8.17), and (8.18), the rth Lagrange equation (8.1) becomes

$$M_r\ddot{\eta}_r + 2\beta M_r\dot{\eta}_r + M_r\omega_r^2\eta_r = N_r$$

or dividing through by M_r

$$\ddot{\eta}_r + 2\beta\dot{\eta}_r + \omega_r^2\eta_r = \frac{N_r}{M_r} \tag{8.19}$$

Equation (8.19) is the rth decoupled equation of motion. It remains to determine the generalized forces N_r in terms of the force system applied to the structure. This will be done separately for distributed forces and discrete forces.

8.3 Distributed Force System

Consider a distributed force having intensity $p(x, t)$ at a point given by x and at time t. The virtual work done by this force on virtual displacement $\delta w(x, t)$ is given by

$$\delta W = \int p(x, t)\,\delta w\,dx \tag{8.20}$$

We wish to determine the virtual displacement $\delta w(x,t)$ in terms of virtual displacements of the coordinates η. From Eq. (8.2) we may write

$$\delta w(x,t) = \sum_{i=1}^{n} \Phi_i(x)\delta\eta_i \tag{8.21}$$

Substitution into Eq. (8.20) yields

$$\begin{aligned} \delta W &= \int p(x,t)\left(\sum_{i=1}^{n} \Phi_i(x)\delta\eta_i\right) dx \\ &= \sum_{i=1}^{n} \delta\eta_i \int p(x,t)\Phi_i(x)\,dx \end{aligned} \tag{8.22}$$

In terms of the generalized forces N_i, the work done by these forces on virtual displacements $\delta\eta_i$ is

$$\delta W = \sum_{i=1}^{n} N_i\,\delta\eta_i \tag{8.23}$$

Comparing Eqs. (8.22) and (8.23) we write

$$N_r = \int p(x,t)\Phi_r(x)\,dx \tag{8.24}$$

If the function $p(x,t)$ is separable in x and t, then

$$p(x,t) = \frac{P_0}{l}p(x)f(t) \tag{8.25}$$

where $p(x)$ is the force distribution function and $f(t)$ is the time dependence. P_0 may be considered as the maximum value of the integrated force* and this quantity together with the length l are introduced in order to make the function $p(x)$ dimensionless. In this special case, then, the rth generalized force is

$$N_r = \frac{P_0}{l}f(t)\int p(x)\Phi_r(x)\,dx \tag{8.26}$$

The integral in this equation divided by l is called the *participation factor* and may be thought of as a measure of the extent to which the rth normal mode participates in synthesizing the total load on the structure. It is denoted by Γ_r, thus

$$\Gamma_r = \frac{1}{l}\int p(x)\Phi_r(x)\,dx \tag{8.27}$$

and

$$N_r = P_0\Gamma_r f(t) \tag{8.28}$$

Using Eq. (8.28) in (8.19), we write the decoupled differential equation for the rth mode in the form

$$\ddot{\eta}_r + 2\beta\dot{\eta}_r + \omega_r^2\eta_r = \frac{P_0\Gamma_r}{M_r}f(t) \tag{8.29}$$

*A value for P_0 may be selected in a number of other ways.

Consider a more general expression for the distributed load function

$$p(x,t) = \frac{P_0}{l}\sum_{j=1}^{m} p_j(x) f_j(t) \tag{8.30}$$

Here we consider the load to be expressible by m components each of which is separable in x and t. In this case Eq. (8.24) becomes

$$\begin{aligned} N_r &= \frac{P_0}{l}\int \Phi_r(x)\left(\sum_{j=1}^{m} p_j(x) f_j(t)\right) dx \\ &= \frac{P_0}{l}\sum_{j=1}^{m} f_j(t)\int p_j(x)\Phi_r(x)\, dx \end{aligned} \tag{8.31}$$

Here each quantity given by the integral divided by l may be considered as the rth participation factor, Γ_{rj}, for the jth load component; that is

$$\Gamma_{rj} = \frac{1}{l}\int p_j(x)\Phi_r(x)\, dx \tag{8.32}$$

In this case the rth decoupled differential equation is

$$\ddot{\eta}_r + 2\beta\dot{\eta}_r + \omega_r^2\eta_r = \frac{P_0}{M_r}\sum_{j=1}^{m}\Gamma_{rj} f_j(t) \tag{8.33}$$

8.4 Discrete Force System

We now consider a set of m discrete point forces $F(x_j, t)$ $(j = 1, 2, \ldots, m)$ applied to the structure. The virtual work equation is

$$\delta W = \sum_{j=1}^{m} F(x_j, t)\, \delta w(x_j, t) \tag{8.34}$$

But $\delta w(x_j, t)$ is given in terms of the normal coordinates by

$$\delta w(x_j, t) = \sum_{i=1}^{n} \Phi_i(x_j)\, \delta\eta_i$$

In which $\Phi_i(x_j)$ is the value of $\Phi_i(x)$ at x_j. Hence, the virtual work may be expressed as

$$\delta W = \sum_{j=1}^{m} F(x_j, t)\sum_{i=1}^{n}\Phi_i(x_j)\, \delta\eta_i \tag{8.35}$$

Interchange the order of summation and write this equation in the form

$$\delta W = \sum_{i=1}^{n}\delta\eta_i\sum_{j=1}^{m} F(x_j, t)\Phi_i(x_j) \tag{8.36}$$

Comparing Eq. (8.36) with the expression for the virtual work in terms of the generalized forces N_i

$$\delta W = \sum_{i=1}^{n} N_i\, \delta\eta_i \qquad \text{[see Eq. (8.23)]}$$

it follows that

$$N_r = \sum_{j=1}^{m} F(x_j, t)\Phi_r(x_j)$$

If $F(x_j, t)$ is separable in x and t, then

$$F(x_j, t) = P_0\, p(x_j) f(t) \tag{8.37}$$

in which the force P_0 may be regarded as the peak value of $F(x_j, t)$,* $p(x_j)$ is a dimensionless force distribution function, and $f(t)$ expresses the time dependence. In this special case the rth generalized force becomes

$$N_r = P_0 f(t) \sum_{j=1}^{m} p(x_j)\Phi_r(x_j) \tag{8.38}$$

The participation factor Γ_r is defined in this case by the summed quantity

$$\Gamma_r = \sum_{i=1}^{m} p(x_j)\Phi_r(x_j) \tag{8.39}$$

and consequently

$$N_r = P_0 \Gamma_r f(t) \tag{8.40}$$

This is the same expression for N_r as was obtained for the distributed force system [Eq. (8.28)]. It follows, then, that the rth decoupled differential equation for the discrete force system will have the same form as that for the distributed force system, namely Eq. (8.29). As in the case of distributed forces, greater generality may be considered by writing

$$F(x_j, t) = P_0 \sum_k p_k(x_j) f_k(t) \tag{8.41}$$

In this case the rth decoupled differential equation will take the same form as that of Eq. (8.33) for the corresponding case of a distributed load system,

$$\ddot{\eta}_r + 2\beta\dot{\eta}_r + \omega_r^2\eta_r = \frac{P_0}{M_r} \sum_k \Gamma_{rk} f_k(t) \tag{8.42}$$

8.5 Evaluation of the Generalized Mass M_r

At this point it is well to discuss briefly the calculation of M_r, the generalized mass corresponding to the rth normal mode. From Eq. (8.10) this mass is found to be

$$M_r = \int m\Phi_r^2(x)\, dx \tag{8.43}$$

Now, suppose that the normal mode function $\Phi_r(x)$ is determined

*Where $F(x_j, t)$ may be any one of the forces.

by superposition of a number, say n, of arbitrary functions $\phi(x)$ by means of a procedure such as the Rayleigh-Ritz or Galerkin method. Thus

$$\Phi_r(x) = \sum_{i=1}^{n} \phi_i(x) q_i^{(r)} \tag{8.44}$$

where the $q^{(r)}$ are generalized coordinates belonging to the rth modal column. Then we may determine $\Phi_r^2(x)$ as

$$\Phi_r^2(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_i(x) \phi_j(x) q_i^{(r)} q_j^{(r)}$$

If we insert this expression into Eq. (8.43) we obtain

$$M_r = \int m \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \phi_i(x) \phi_j(x) q_i^{(r)} q_j^{(r)} \right) dx$$

Proceeding with the integration before summing we write

$$\begin{aligned} M_r &= \sum_{i=1}^{n} \sum_{j=1}^{n} q_i^{(r)} q_j^{(r)} \int m \phi_i(x) \phi_j(x)\, dx \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} q_i^{(r)} q_j^{(r)} \end{aligned} \tag{8.45}$$

where the integral denoted by m_{ij} is the generalized mass corresponding to the functions ϕ_i and ϕ_j. Equation (8.45) may be given as a triple matrix product which is a convenient form for calculations

$$M_r = \{q^{(r)}\}^T [m] \{q^{(r)}\} \tag{8.46}$$

Here the elements of the matrix $[m]$ are the m_{ij} of Eq. (8.45), where

$$m_{ij} = \int m \phi_i(x) \phi_j(x)\, dx \tag{8.47}$$

8.6 The Normal Mode Method Applied to Lumped Parameter Systems

We now apply the normal mode method to a system composed of a finite number of connected, discrete, point masses. For systems with such a "lumping" approximation the applied forces will be discrete point forces. Again, using the Lagrangian approach, the kinetic energy, strain energy, and the dissipation function are written in terms of the velocities and displacements at the mass points of the system.

$$T = \tfrac{1}{2} \sum_i m_i \dot{w}_i^2 \tag{8.48}$$

$$U = \tfrac{1}{2} \sum_i \sum_j k_{ij} w_i w_j \tag{8.49}$$

$$R = \tfrac{1}{2} \sum_i \sum_j c_{ij} \dot{w}_i \dot{w}_j \tag{8.50}$$

Here, m_i is the mass lumped at the ith point of the system; k_{ij} is the stiffness influence coefficient associating the forces and displacements at the points i and j; and the ij damping coefficient element c_{ij} is analogous to k_{ij} as discussed in Chapter 7.

In a manner similar to that of Eq. (8.2) the displacements w_i are expressed in terms of normal mode numbers Φ_{ij} and generalized coordinates η.

$$w_i = \sum_{j=1}^{n} \Phi_{ij}\eta_j \tag{8.51}$$

The normal mode number Φ_{ij} is the deflection on coordinate i in the normal mode j. By differentiating Eq. (8.51) with respect to time, the velocities $\dot{w}_i$ are obtained.

$$\dot{w}_i = \sum_{j=1}^{n} \Phi_{ij}\dot{\eta}_j \tag{8.52}$$

For the leading term in the rth Lagrange equation the following partial derivative is taken.

$$\frac{\partial T}{\partial \dot{\eta}_r} = \sum_{i=1}^{n} m_i \dot{w}_i \frac{\partial \dot{w}_i}{\partial \dot{\eta}_r}$$

Substituting for $\dot{w}_i$ and $\partial \dot{w}_i/\partial \dot{\eta}_r$ from Eq. (8.52)

$$\frac{\partial T}{\partial \dot{\eta}_r} = \sum_{i=1}^{n} m_i \Phi_{ir} \sum_{j=1}^{n} \Phi_{ij}\dot{\eta}_j$$

which, upon interchanging order of summation gives

$$\frac{\partial T}{\partial \dot{\eta}_r} = \sum_{j=1}^{n} \dot{\eta}_j \sum_{i=1}^{n} m_i \Phi_{ir}\Phi_{ij} \tag{8.53}$$

Now, define the generalized mass M_{rj} by

$$M_{rj} = \sum_{i=1}^{n} m_i \Phi_{ir}\Phi_{ij} \tag{8.54}$$

Orthogonality of the normal modes [see Eq. (3.34) and Problem 6 of Chapter 3] dictates that $M_{rj} = 0$ for $j \neq r$. For $j = r$, the generalized mass M_r for the rth normal mode is

$$M_r = \sum_{i=1}^{n} m_i \Phi_{ir}^2 \tag{8.55}$$

Using Eqs. (8.54) and (8.55) in (8.53) we write

$$\frac{\partial T}{\partial \dot{\eta}_r} = \sum_{j=1}^{n} M_{rj}\dot{\eta}_j = M_r\dot{\eta}_r \tag{8.56}$$

This result is identical to that of Eq. (8.11) for distributed systems.

Following the reasoning leading to Eq. (8.18)

$$\frac{\partial T}{\partial \eta_r} = 0$$

To evaluate $\partial U/\partial \eta_r$, substitute for w_i and w_j from Eq. (8.51) in Eq. (8.49).

$$U = \tfrac{1}{2} \sum_i \sum_j k_{ij} \sum_k \Phi_{ik}\eta_k \sum_l \Phi_{jl}\eta_l$$

Differentiating with respect to η_r and rearranging,

$$\frac{\partial U}{\partial \eta_r} = \sum_k K_{kr}\eta_k \tag{8.57}$$

in which

$$K_{kr} = \sum_i \sum_j k_{ij}\Phi_{ik}\Phi_{jr} \tag{8.58}$$

From the orthogonality relation [see Eq. (3.35) and Problem 6 of Chapter 3] it follows that

$$\begin{aligned} K_{kr} &= \omega_r^2 M_r, \qquad \text{for } k = r \\ &= 0 \qquad\qquad \text{for } k \neq r \end{aligned} \tag{8.59}$$

and Eq. (8.57) becomes

$$\frac{\partial U}{\partial \eta_r} = \omega_r^2 M_r \eta_r \tag{8.60}$$

The evaluation of $\partial R/\partial \dot{\eta}_r$ requires some discussion. In the introductory part of this chapter it was emphasized that the normal mode method owes its simplicity to the fact that the differential equations of motion are decoupled when this method is used. In the absence of damping forces, this holds true. However, when damping forces are present, the damping coefficients must be proportional to either the mass or stiffness constants of the system in order for the differential equations of motion to become decoupled when the method of this chapter is applied.* Thus in the case of the distributed parameter system we introduced the proportionality

$$c = 2\beta m \qquad \text{[see Eq. (8.12)]}$$

For the lumped parameter system under discussion, let us first consider the damping coefficients c_{ij} to be proportional to the corresponding stiffness coefficients k_{ij}; that is

$$c_{ij} = \alpha k_{ij} \tag{8.61}$$

for all i and j.

We substitute for $\dot{w}_i$ and $\dot{w}_j$ from Eq. (8.52) into Eq. (8.50) to obtain

$$R = \tfrac{1}{2} \sum_i \sum_j c_{ij} \sum_k \Phi_{ik}\dot{\eta}_k \sum_l \Phi_{jl}\dot{\eta}_l$$

Differentiating with respect to $\dot{\eta}_r$, using relation (8.61) and rearranging, we write

*The extension of the normal mode method to the case of nonproportional damping is treated in Chapter 9.

$$\frac{\partial R}{\partial \dot{\eta}_r} = \alpha \sum_k \dot{\eta}_k \sum_i \sum_j k_{ij}\Phi_{ik}\Phi_{jr}$$

or

$$\frac{\partial R}{\partial \dot{\eta}_r} = \alpha \sum_k K_{kr}\dot{\eta}_k \tag{8.62}$$

in which K_{kr} is defined by Eq. (8.58). Using relations (8.59) in (8.62) we have

$$\frac{\partial R}{\partial \dot{\eta}_r} = \alpha\omega_r^2 M_r \dot{\eta}_r \tag{8.63}$$

If the damping coefficients are linearly related to both the mass and stiffness coefficients, then Eqs. (8.13) and (8.63) are combined to give the more general expression

$$\frac{\partial R}{\partial \dot{\eta}_r} = (2\beta + \alpha\omega_r^2)M_r\dot{\eta}_r$$

$$= 2\zeta_r\omega_r M_r \dot{\eta}_r$$

in which ζ_r is the damping factor in the rth mode. (See Chapter 9, Sections 1 and 2.)

Inasmuch as the forces $F(x_j, t)$ acting on the lumped system are discrete, the generalized force for the rth normal mode is given by

$$N_r = P_0 \Gamma_r f(t)$$

in which $F(x_j, t)$ is considered separable in x and t; and P_0 and Γ_r are defined from Eqs. (8.37) and (8.39), respectively.

Substituting Eqs. (8.56), (8.18), (8.60), and (8.63) into the rth Lagrangian equation [Eq. (8.1)], the rth decoupled differential equation of motion becomes

$$\ddot{\eta}_r + \alpha\omega_r^2\dot{\eta}_r + \omega_r^2\eta_r = \frac{P_0}{M_r}\Gamma_r f(t)$$

or

$$\ddot{\eta}_r + 2\beta\dot{\eta}_r + \omega_r^2\eta_r = \frac{P_0}{M_r}\Gamma_r f(t) \tag{8.64}$$

in which $2\beta = \alpha\omega_r^2$. Equation (8.64) is identical to Eq. (8.29), derived for the distributed parameter system.

When the dissipation function of Eq. (8.50) has the form

$$R = \tfrac{1}{2}\sum c_i \dot{v}_i^2$$

and the damping coefficients c_i are proportional to the mass constants m_i

$$c_i = 2\beta m_i$$

then

$$\frac{\partial R}{\partial \dot{\eta}_r} = 2\beta\frac{\partial T}{\partial \dot{\eta}_r} = 2\beta M_r \dot{\eta}_r$$

and the resulting rth decoupled differential equation of motion again takes the form of Eq. (8.64).

8.7 Matrix Forms

The results of the last section will now be derived and presented in matrix form. Consider an n degree-of-freedom lumped parameter system with mass matrix $[m_i]$, stiffness matrix $[k_{ij}]$, damping matrix $[c_{ij}]$, and column matrix of external forces $\{F(x_j, t)\}$. m_i, k_{ij}, c_{ij}, and F_i are expressed in the w coordinate system.

For $F(x_j, t)$ separable in x and t

$$\{F(x_j, t)\} = P_0\{p(x_j)\}f(t)$$

The differential equations of motion in the w coordinate system take the matrix form

$$[m]\{\ddot{w}\} + [c]\{\dot{w}\} + [k]\{w\} = P_0\{p(x_j)\}f(t) \tag{8.65}$$

Apply a coordinate transformation

$$\{w\} = [\Phi]\{\eta\} \tag{8.66a}$$

in which each column of Φ is a modal column* of the system and $\{\eta\}$ represents the normal coordinates.

From Eq. (8.66a)

$$\{\dot{w}\} = [\Phi]\{\dot{\eta}\} \tag{8.66b}$$

and

$$\{\ddot{w}\} = [\Phi]\{\ddot{\eta}\} \tag{8.66c}$$

Substituting Eqs. (8.66) into (8.65) and premultiplying by the transpose of $[\Phi]$, Eq. (8.65) becomes

$$[\Phi]^T[m][\Phi]\{\ddot{\eta}\} + [\Phi]^T[c][\Phi]\{\dot{\eta}\} + [\Phi]^T[k][\Phi]\{\eta\}$$
$$= P_0[\Phi]^T\{p(x_j)\}f(t) \tag{8.67}$$

From the orthogonality relations Eqs. (3.34) and (3.35) it follows that

$$[\Phi]^T[m][\Phi] = [M_r] \tag{8.68}$$

$$[\Phi]^T[k][\Phi] = [\omega_r^2][M_r] = [K_r] \tag{8.69}$$

(See also Problem 6, Chapter 3.) Comparing the triple matrix product

$$[\Phi]^T[c][\Phi] \tag{8.70}$$

with Eqs. (8.68) and (8.69) it becomes apparent that this triple product will result in a diagonal matrix only when the damping matrix $[c]$

*An element Φ_{ij} represents the displacement at i in the jth normal mode.

is proportional to either the mass matrix $[m]$ or the stiffness matrix $[k]$, that is

$$[c] = 2\beta[m] \tag{8.71}$$

or

$$[c] = \alpha[k] \tag{8.72}$$

Using relation (8.71) in (8.70) and comparing with (8.68)

$$[\Phi]^T[c][\Phi] = 2\beta[M_r] \tag{8.73}$$

Using relation (8.72) in (8.70) and comparing with (8.69), we obtain

$$[\Phi]^T[c][\Phi] = \alpha[\omega_r^2][M_r] \tag{8.74}$$

The substitution of relations (8.68), (8.69) and (8.73) in Eq. (8.67) results in a set of n decoupled differential equations of motion

$$[M_r]\{\ddot{\eta}\} + 2\beta[M_r]\{\dot{\eta}\} + [\omega_r^2][M_r]\{\eta\} = [\Phi]^T\{p(x_i)\}P_0 f(t) \tag{8.75}$$

The rth of Eqs. (8.75) has the form

$$M_r\ddot{\eta}_r + 2\beta M_r\dot{\eta}_r + \omega_r^2 M_r\eta_r = P_0\{\Phi^{(r)}\}^T\{p(x_i)\}f(t) \tag{8.76}$$

Note that $\{\Phi^{(r)}\}^T\{p(x_i)\}$ is a restatement of the right side of Eq. (8.39) representing the participation factor Γ_r. Hence, the right side of Eq. (8.76) takes the form of Eq. (8.40)

$$P_0\{\Phi^{(r)}\}^T\{p(x_i)\}f(t) = P_0\Gamma_r f(t)$$

Dividing through by M_r in Eq. (8.76), and using the last relation, we write

$$\ddot{\eta}_r + 2\beta\dot{\eta}_r + \omega_r^2\eta_r = \frac{P_0}{M_r}\Gamma_r f(t) \tag{8.77}$$

If proportionality relation (8.72) is used in place of (8.71), the rth decoupled differential equation of motion is identical to Eq. (8.77) with β defined from the relation

$$2\beta = \alpha\omega_r^2$$

8.8 The Solution of the Decoupled Differential Equations of Motion

In the preceding sections it was shown that the rth decoupled differential equation of motion has the same form for both the distributed parameter system [Eq. (8.29)] and the lumped parameter system [Eq. (8.64)]. Consequently, this equation includes the case of a combination of the two systems as well.

$$\ddot{\eta}_r + 2\beta\dot{\eta}_r + \omega_r^2\eta_r = \frac{P_0}{M_r}\Gamma_r f(t) \tag{8.29}$$

Note that this differential equation for the rth mode is independent of those for all other modes. Therefore, it may be integrated directly to yield the normal displacement $\eta_r(t)$. This may be repeated independently for all n equations, to solve for the n normal displacements $\eta_i(t)$ $i = 1, 2, \ldots, n$.

We recall that for a somewhat more general expression of the external loads given by Eqs. (8.30) and (8.41), the resulting decoupled differential equations of motion took the form of Eqs. (8.33) and (8.42), respectively. These equations differed from Eq. (8.29) in the expressions on the right-hand side of the equations. Instead of a single term in Eq. (8.29) a sum of terms

$$\frac{P_0}{M_r} \sum_j \Gamma_{rj} f_j(t)$$

appeared. From the theory of differential equations we know that the solution for $\eta_r(t)$ in Eqs. (8.33) and (8.42) is obtained by summing the solutions resulting from each of the terms on the right considered separately. Therefore, we may concentrate our attention on the solution of Eq. (8.29) which contains only one such term on the right. This solution can then easily be extended to cover the more general case of Eqs. (8.33) and (8.42).

Solution by Laplace Transforms. A general form of solution to Eq. (8.29) is obtained readily by the use of the Laplace Transform.[30,31] We make the definitions

$$\mathcal{L} f(t) = \int_0^\infty e^{-st} f(t)\, dt = f(s) \tag{8.78a}$$

$$\mathcal{L}\, \eta_r(t) = \eta_r(s) \tag{8.78b}$$

It follows that the Laplace transforms of $\dot{\eta}_r(t)$ and $\ddot{\eta}_r(t)$ are given by

$$\mathcal{L}\, \dot{\eta}_r(t) = s\eta_r(s) - \eta_r(0) \tag{8.78c}$$

$$\mathcal{L}\, \ddot{\eta}_r(t) = s^2\eta_r(s) - s\eta_r(0) - \dot{\eta}_r(0) \tag{8.78d}$$

in which $\eta_r(0)$ and $\dot{\eta}_r(0)$ are, respectively, the initial $(t = 0)$ displacement and velocity of the coordinate η_r. In the following development t will be measured from the time of application of the force system. The displacements $w(x, t)$ will be measured from $w(x, 0) = 0$. Therefore

$$\eta_r(0) = \dot{\eta}_r(0) = 0 \qquad r = 1, 2, 3, \ldots, n \tag{8.79}$$

Using relations (8.78b), (8.78c), and (8.78d) and inserting the initial conditions prescribed by Eq. (8.79), the Laplace transform of Eq. (8.29) may be written in the form

$$\eta_r(s)\,(s^2 + 2\beta s + \omega_r^2) = \frac{P_0 \Gamma_r}{M_r} f(s)$$

or multiplying and dividing the right-hand side by ω_r^2, we have

$$\eta_r(s) = \frac{P_0 \Gamma_r}{\omega_r^2 M_r} H(s) f(s) \tag{8.80}$$

in which

$$H(s) = \frac{\omega_r^2}{s^2 + 2\beta s + \omega_r^2} = \frac{\omega_r^2}{(s+\beta)^2 + (\omega_r^2 - \beta^2)} \tag{8.81}$$

The displacement $\eta_r(t)$ is given by the inverse transform of Eq. (8.80), or

$$\eta_r(t) = \frac{P_0 \Gamma_r}{\omega_r^2 M_r} \mathcal{L}^{-1}\{H(s) f(s)\} \tag{8.82}$$

By use of the convolution integral the inverse transform of Eq. (8.82) is expressed in the general form

$$\mathcal{L}^{-1}\{H(s) f(s)\} = \int_0^t h(t-\tau) f(\tau)\, d\tau \tag{8.83}$$

For $H(s)$ as expressed by Eq. (8.81) we have*

$$h(t-\tau) = \mathcal{L}^{-1} H(s) = \frac{\omega_r^2}{\sqrt{\omega_r^2 - \beta^2}} e^{-\beta(t-\tau)} \sin \sqrt{\omega_r^2 - \beta^2}(t-\tau) \tag{8.84}$$

Introducing Eqs. (8.83) and (8.84) into (8.82) the solution for the rth normal coordinate $\eta_r(t)$ takes the form

$$\eta_r(t) = \frac{P_0 \Gamma_r}{\omega_r^2 M_r} D_r(t) \tag{8.85}$$

in which

$$\begin{aligned} D_r(t) &= \int_0^t h(t-\tau) f(\tau)\, d\tau \\ &= \int_0^t \frac{\omega_r^2}{\sqrt{\omega_r^2 - \beta^2}} e^{-\beta(t-\tau)} \sin [\sqrt{\omega_r^2 - \beta^2}(t-\tau)] f(\tau)\, d\tau \end{aligned} \tag{8.86}$$

$D_r(t)$ is called the *dynamic load factor.* It is a dimensionless quantity which is a function of time. From Eq. (8.86) $D_r(t)$ is seen to depend upon the force time function $f(t)$, upon the natural frequency of the structure in the rth mode ω_r, and upon the damping coefficient β.

In the absence of damping forces $\beta = 0$ and the dynamic load factor [Eq. (8.86)] takes the form

$$D_r(t) = \int_0^t \omega_r \sin \omega_r(t-\tau) f(\tau)\, d\tau \tag{8.87}$$

For this special case it is easily shown that for a suddenly applied load, for which $f(t)$ is the unit step function,

$$D_r(t) = 1 - \cos \omega_r t$$

in which case the dynamic load factor has a maximum value of 2. For a statically applied load the value of $D_r(t)$ is unity. Hence, $D_r(t)$ represents the magnification of the response η_r in the rth natural

*See Reference 31, p. 342.

mode resulting from the loads being applied dynamically as compared with the same loads applied statically.

The total deflection of the structure, $w(x,t)$, is obtained by inserting Eq. (8.85) into (8.2). Thus

$$w(x,t) = P_0 \sum_{i=1}^{n} \frac{\Gamma_i}{\omega_i^2 M_i} \Phi_i(x) D_i(t) \tag{8.88}$$

We note that when the excitation $f(\tau)$ is obtained from experiments and cannot be expressed analytically, the integral of Eq. (8.83) may be evaluated by numerical methods.

The response $D_r(t)$ expressed by the convolution integral in Eq. (8.83) has a physical interpretation. The function $h(t-\tau)$ represents the response, $D_r(t)$, at time t, due to a unit impulse exciting the system $(t-\tau)$ units of time earlier. For an impulse $f(\tau)\Delta\tau$ exciting the system the response is

$$h(t-\tau)f(\tau)\Delta\tau$$

The exciting force function $f(\tau)$ can then be divided into impulses $f(\tau)\Delta\tau$ (see Fig. 8.1), and since the system is linear, the overall response $D_r(t)$ can be computed by a superposition of the responses to the individual impulses and taking the limit of the resulting sum as $\Delta\tau \to 0$.

$$\begin{aligned} D_r(t) &= \lim_{\Delta t \to 0} \sum_{\tau=0}^{t} h(t-\tau) f(\tau)\Delta\tau \\ &= \int_0^t h(t-\tau) f(\tau)\, d\tau \end{aligned}$$

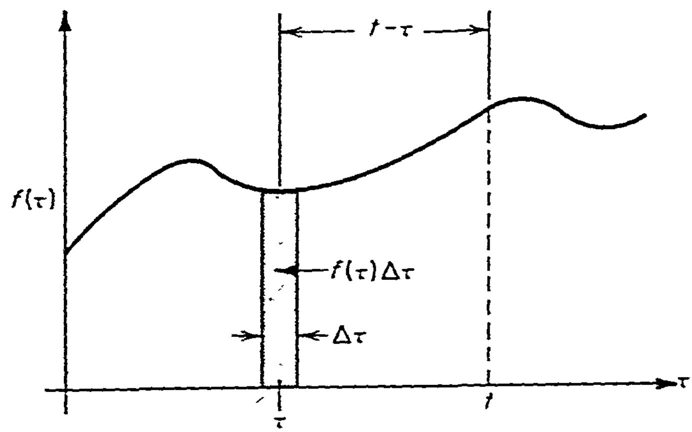


Figure 8.1

Response at time t to a unit impulse applied at time $\tau (t > \tau)$
$= h(t-\tau)$

Response at time t to an impulse $f(\tau)\Delta\tau$ applied at time $\tau (t > \tau)$
$= h(t-\tau)f(\tau)\Delta\tau$

Over all response at time t $= \int_0^t h(t-\tau)f(\tau)\,d\tau$

8.9 Illustrative Example

As an example of the normal mode method, the response of the uniform beam shown in Fig. 8.2 will be determined. The beam is

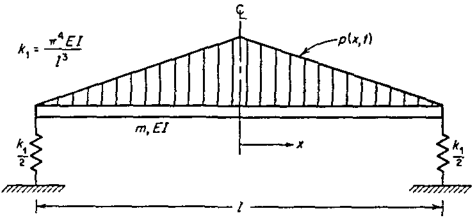


Figure 8.2

supported by two springs of stiffness $k_1/2 = \pi^4 EI/2l^3$ at its ends, and is so mounted that its ends are free to rotate about an axis perpendicular to the plane of the figure and can translate vertically. Lateral motion is prevented. The beam has mass m per unit length and stiffness EI. Damping is considered small and is disregarded in the calculations. For simplicity we consider a symmetric load as shown. The load distribution function $p(x)$ does not change with time and is given by

$$p(x) = 1 - \frac{2x}{l} \quad \text{for} \quad 0 \leq x \leq \frac{l}{2} \tag{8.69}$$

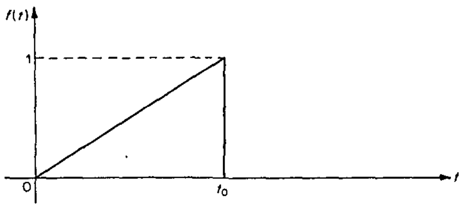


Figure 8.3

The time-dependent function $f(t)$ is shown in Fig. 8.3: that is, the load increases linearly from zero at $t = 0$ to a maximum at $t = t_0$, then drops abruptly and is zero for $t > t_0$. The maximum value of $f(t)$, i.e., $f(t_0)$, is taken as unity. If P_0 is taken as the total load on the entire beam at $t = t_0$, then the load function is

$$p(x, t) = \frac{2P_0}{l}\left(1 - \frac{2x}{l}\right)f(t) \qquad 0 \leq x \leq \frac{l}{2} \tag{8.90}$$

Since the applied load system is symmetric, we will determine the first two symmetric normal modes of the system. This we can do by using the Rayleigh-Ritz method and selecting mode functions which do not admit antisymmetrical motion. To insure reasonable accuracy for the second normal mode select three mode functions. The first mode function is selected to permit vertical motion of the beam as a rigid body.

$$\phi_1(x) = 1$$

For the second and third functions we choose the first two symmetrical normal modes of a uniform simply-supported beam with ends constrained against translation.

$$\phi_2(x) = \cos\frac{\pi x}{l}$$

$$\phi_3(x) = \cos\frac{3\pi x}{l}$$

The displacement at any point x on the beam is then given by

$$w(x) = \sum_{j=1}^{3} \phi_j(x) q_j$$

where the q's are generalized coordinates. Using the three selected functions, ϕ_j $(j = 1, 2, 3)$, we obtain the following generalized mass and stiffness matrices

$$[m] = ml\begin{bmatrix} 1 & \frac{2}{\pi} & -\frac{2}{3\pi} \\ \frac{2}{\pi} & \frac{1}{2} & 0 \\ -\frac{2}{3\pi} & 0 & \frac{1}{2} \end{bmatrix}$$

$$[k] = k_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{81}{2} \end{bmatrix}$$

Solution of the resulting equations of motion in the q coordinates *yields the following frequencies, modal columns, and corresponding* normal mode functions.

$$\left.\begin{aligned} \{q^{(1)}\} &= \begin{Bmatrix} 1.000 \\ 1.414 \\ 0 \end{Bmatrix} \\ \Phi_1(x) &= \sum_{j=1}^{3} \phi_j(x) q_j^{(1)} = 1 + 1.414 \cos \frac{\pi x}{l} \\ \omega_1 &= 0.725 \frac{\pi^2}{l^2} \sqrt{\frac{EI}{m}} \end{aligned}\right\} \text{First symmetrical mode}$$

$$\left.\begin{aligned} \{q^{(2)}\} &= \begin{Bmatrix} 1.000 \\ -1.450 \\ -0.045 \end{Bmatrix} \\ \Phi_2(x) &= \sum_{j=1}^{3} \phi_j(x) q_j^{(2)} \\ &= 1 - 1.45 \cos \frac{\pi x}{l} - 0.045 \cos \frac{3\pi x}{l} \\ \omega_2 &= 2.86 \frac{\pi^2}{l^2} \sqrt{\frac{EI}{m}} \end{aligned}\right\} \text{Second symmetrical mode}$$

Using this information we next calculate the masses M_1 and M_2 from Eq. (8.46)

$$M_1 = [1 \quad 1.414 \quad 0]ml \begin{bmatrix} 1 & 0.637 & -0.212 \\ 0.637 & 0.500 & 0 \\ -0.212 & 0 & 0.500 \end{bmatrix} \begin{Bmatrix} 1 \\ 1.414 \\ 0 \end{Bmatrix}$$

$$= 3.801\ ml$$

$$M_2 = [1 \quad -1.45 \quad -0.045]ml \begin{bmatrix} 1 & 0.637 & -0.212 \\ 0.637 & 0.500 & 0 \\ -0.212 & 0 & 0.500 \end{bmatrix} \begin{Bmatrix} 1 \\ -1.45 \\ -0.045 \end{Bmatrix}$$

$$= 0.224\ ml$$

The participation factors are calculated from Eq. (8.27) as

$$\Gamma_r = \frac{1}{l} \int p(x) \Phi_r(x)\, dx$$

$$= \frac{1}{l} \int p(x) \sum_j \phi_j q_j^{(r)}\, dx$$

$$\Gamma_r = \frac{1}{l} \sum_j q_j^{(r)} \int p(x) \phi_j(x)\, dx$$

Thus

$$\Gamma_1 = \frac{2}{l} \sum_{j=1}^{3} q_j^{(1)} \int_0^{l/2} \left(1 - \frac{2x}{l}\right) \phi_j(x)\, dx$$

$$\Gamma_2 = \frac{2}{l} \sum_{j=1}^{3} q_j^{(2)} \int_0^{l/2} \left(1 - \frac{2x}{l}\right) \phi_j(x)\, dx$$

The three integrals required are evaluated

$$\int_0^{l/2} \left(1 - \frac{2x}{l}\right) dx = 0.250l$$

$$\int_0^{l/2} \left(1 - \frac{2x}{l}\right) \cos \frac{\pi x}{l}\, dx = \frac{2}{\pi^2} l = 0.203l$$

$$\int_0^{l/2} \left(1 - \frac{2x}{l}\right) \cos \frac{3\pi x}{l}\, dx = \frac{2}{9\pi^2} l = 0.0225l$$

Using these values, together with the appropriate values of q, the participation factors are found.

$$\Gamma_1 = 2(0.25 + 1.414 \times 0.203 + 0) = 1.074$$

$$\Gamma_2 = 2(0.25 - 1.45 \times 0.203 - 0.045 \times 0.0225) = -0.091$$

Next, the dynamic load factors corresponding to the function $f(t)$ must be determined. This may be done conveniently, in this problem, by the insertion of $f(t)$ in Eq. (8.87). We obtain:

For $t \leqq t_0$

$$\begin{aligned} D_1(t) &= \frac{\omega_1}{t_0} \int_0^t \tau \sin \omega_1 (t - \tau)\, d\tau \\ &= \frac{t}{t_0} - \frac{1}{\omega_1 t_0} \sin \omega_1 t \\ D_2(t) &= \frac{t}{t_0} - \frac{1}{\omega_2 t_0} \sin \omega_2 t \end{aligned}$$

For $t \geqq t_0$

$$D_1(t) = \frac{1}{\omega_1 t_0} \{\sin \omega_1 (t - t_0) - \sin \omega_1 t\} + \cos \omega_1 (t - t_0)$$

$$D_2(t) = \frac{1}{\omega_2 t_0} \{\sin \omega_2 (t - t_0) - \sin \omega_2 t\} + \cos \omega_2 (t - t_0)$$

Each of these dynamic load factors, $D_i(t)$ $(i = 1, 2)$, will appear somewhat as shown in Fig. 8.4. The function has a zero slope at $t = 0$ and also at $\omega_i t = 2\pi, 4\pi, 6\pi, \ldots$, for $t < t_0$. Hence in this region the slope is never negative. The function and its first derivative are continuous at $t = t_0$. For $t > t_0$ it oscillates about the zero axis and, since zero damping was postulated, the oscillations will continue with frequency ω_i without attenuation. It follows that the maximum dynamic load factor will occur at $t \geqq t_0$. For example, it can be shown that a maximum dynamic load factor corresponding to a value of $t_0 = 1.3\pi/\omega_i$ is approximately equal to 1.26. The maximum load factor for this case is reached at $t = 1.34\pi/\omega_i > 1.3\pi/\omega_i = t_0$ which verifies the statement above.

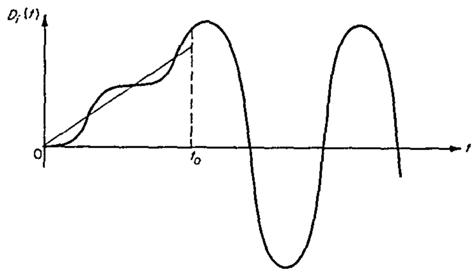


Figure 8.4

The normal coordinates $\eta_1(t)$ and $\eta_2(t)$ are evaluated as

$$\eta_1(t) = \frac{2 \times 1.074 P_0 D_1(t)}{3.801 ml \times (0.725)^2 \frac{\pi^4}{l^4}\frac{EI}{m}} = \frac{1.078}{\pi^4}\frac{P_0 l^3}{EI} D_1(t) \tag{8.91}$$

$$\eta_2(t) \doteq \frac{-2 \times 0.091 P_0 D_2(t)}{0.224 ml \times (2.86)^2 \frac{\pi^4}{l^4}\frac{EI}{m}} = -\frac{0.099}{\pi^4}\frac{P_0 l^3}{EI} D_2(t) \tag{8.92}$$

Finally, the response amplitude $w(x, t)$ is found by inserting these coordinates and the normal mode functions into Eq. (8.2).

$$\begin{aligned} w(x,t) &= \Phi_1(x)\eta_1(t) + \Phi_2(x)\eta_2(t) \\ &= \frac{1.078}{\pi^4}\frac{P_0 l^3}{EI} D_1(t)\left(1 + 1.414\cos\frac{\pi x}{l}\right) \\ &\quad - \frac{0.099}{\pi^4}\frac{P_0 l^3}{EI} D_2(t)\left(1 - 1.45\cos\frac{\pi x}{l} - 0.045\cos\frac{3\pi x}{l}\right) \\ &= \frac{P_0 l^3}{\pi^4 EI}\left\{\left(1.078 + 1.525\cos\frac{\pi x}{l}\right) D_1(t)\right. \\ &\quad \left. - \left(0.099 - 0.144\cos\frac{\pi x}{l} - 0.0045\cos\frac{3\pi x}{l}\right) D_2(t)\right\} \end{aligned} \tag{8.93}$$

The deflection at the center of the beam where $x = 0$ is

$$w(0,t) = \frac{P_0 l^3}{\pi^4 EI}\{2.603 D_1(t) + 0.050 D_2(t)\} \tag{8.94}$$

The above result is, of course, approximate inasmuch as it results from the superposition of a finite number (in this case 2) of normal modes. However, it can be seen that the second normal mode contributes little to the total deflection. It is of interest to compare this result with the exact solution for the same load applied as a static load. The static deflection thus calculated is

$$w(0) = 2.62 \frac{P_0 l^3}{\pi^4 EI} \tag{8.95}$$

If the dynamic load factors $D_1(t)$ and $D_2(t)$ were both unity, the approximate result from Eq. (8.94) would be

$$w(0) = 2.653 \frac{P_0 l^3}{\pi^4 EI} \tag{8.96}$$

The accuracy in this case is quite satisfactory for most practical applications.

8.10 Internal Forces or Stresses Due to Dynamic Loads

In the preceding section we illustrated by an example the computations for the displacement configuration $w(x, t)$ of a system excited by dynamic loads. When the displacement configuration of the system at any instant of time t is known, we can compute the corresponding internal forces* at that time t by the methods of Chapter 1. Because this chapter deals with normal coordinates, we shall use these coordinates to discuss computations for internal forces. However, any desired set of coordinates can be used to compute the internal forces corresponding to a known displacement configuration. In this section we shall discuss two methods for computing internal forces due to dynamic loads.[12]

Method 1

In this method we use the displacements of the system to compute the internal forces. We recall that using normal coordinates the displacement of a point x at time t is given by Eq. (8.2).

$$w(x, t) = \sum_{i=1}^{n} \Phi_i(x)\eta_i(t)$$

In this equation $w(x, t)$ and $\Phi_i(x)$ may be continuous functions of x or may be defined at discrete points $x_j(1, 2, \ldots, m)$ on the system. In either case, to each normal mode $\Phi_i(x)$ there corresponds an internal stress pattern $P_i(x)$ where P_i may represent stresses resulting from internal moment, shear, torque, and direct load. When the ith normal mode is amplified by the value of the ith normal coordinate $\eta_i(t)$, the corresponding stress pattern $P_i(x)$ is also amplified by the same

*We shall continue to make reference to internal forces with the understanding that these forces can be used to compute the internal stresses.

amount. Using superposition we write for the total internal stress pattern $P(x,t)$ at time t

$$P(x,t) = \sum_{i=1}^{n} P_i(x)\eta_i(t) \tag{8.97}$$

in which the normal coordinates $\eta_i(t)$ are computed by the method of this chapter.

In computing the internal forces $P_i(x)$ corresponding to normal modes $\Phi_i(x)$ we distinguish two cases. In the first case the forces $P_i(x)$ are computed directly from the displacement at point x in the ith mode. Such is the case when we compute the spring forces in the system of Fig. 8.2 in the example of the last section. In the second case the internal forces $P_i(x)$ are computed from spatial derivatives of the ith normal mode function $\Phi_i(x)$. For example, in the beam of Fig. 8.2 of the last section, the bending moment $M_i(x)$ and the shear $V_i(x)$ corresponding to the ith normal mode are given, respectively, by

$$M_i(x) = EI\frac{d^2\Phi_i(x)}{dx^2} \tag{8.98}$$

and

$$V_i(x) = EI\frac{d^3\Phi_i(x)}{dx^3} \tag{8.99}$$

These expressions are perfectly acceptable when the mode shapes $\Phi_i(x)$ are the correct mode shape functions. In practice, however, these mode shapes are obtained only approximately by methods such as the Rayleigh-Ritz method. We know that when we use this method, for instance, the computed natural frequencies and corresponding mode shapes are not very sensitive to the particular choice of functions $\phi_i(x)$. However, the second derivatives $\Phi_i''(x)$ of the resulting mode shapes may differ greatly (as will the internal forces that are a function of these derivatives) depending upon the selected functions. This will be demonstrated for a simply supported uniform beam (Fig. 8.5).

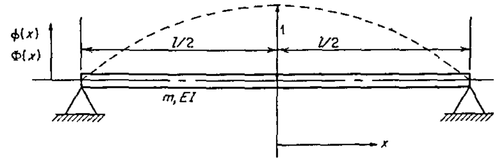


Figure 8.5

If we select a parabola

$$\phi(x) = \frac{4}{l^2}\left(\frac{l^2}{4} - x^2\right)$$

to approximate the first mode shape of the beam, we obtain by use of Rayleigh's method (Eq. 4.9, Chapter 4),

$$\omega_1 = 10.67\sqrt{\frac{EI}{ml^4}}$$

This result compares very well with the exact first natural frequency given by

$$\omega_1 = \pi^2\sqrt{\frac{EI}{ml^4}}$$

The displacement at $x = l/4$ is given by

$$\phi\left(\frac{l}{4}\right) = 0.75$$

This again is reasonably close to the true displacement at this point in the exact first mode shape given by

$$\Phi(x) = \cos\frac{\pi x}{l}$$

$$\Phi\left(\frac{l}{4}\right) = \cos\frac{\pi}{4} = 0.707$$

However, the second and third derivatives with respect to x of the selected and true mode shapes are, respectively,

$$\phi''(x) = -\frac{8}{l^2} \qquad \phi'''(x) = 0$$

$$\Phi''(x) = -\left(\frac{\pi}{l}\right)^2\cos\frac{\pi x}{l} \qquad \Phi'''(x) = \left(\frac{\pi}{l}\right)^3\sin\frac{\pi x}{l}$$

Here the difference is apparent. Using $\phi''(x)$ and $\phi'''(x)$ in Eqs. (8.98) and (8.99) respectively, we obtain a constant moment throughout the beam, and a zero shear at all points along the beam, whereas the true moment and shear corresponding to the first mode displacement configuration are

$$M(x) = -EI\left(\frac{\pi}{l}\right)^2\cos\frac{\pi x}{l}$$

$$V(x) = EI\left(\frac{\pi}{l}\right)^3\sin\frac{\pi x}{l}$$

To minimize the errors associated with computations for the internal forces $P_i(x)$ when the normal modes $\Phi_i(x)$ are approximate, we can proceed as follows. To each normal mode $\Phi_i(x)$ there corresponds an external load system, which, when applied to the structure, will cause it to deform in its ith normal mode with an amplitude of unity $[\eta_i(t) = 1]$. For the ith mode this loading is given by the following inertial forces:

load intensity at $x = m\omega_i^2\Phi_i(x)$ for a distributed mass system
load intensity at $x_j = m_j\omega_i^2\Phi_i(x_j)$ for a lumped mass system

$$(8.100)$$

Using these inertial loads, we can compute the corresponding internal forces $P_i(x)$ by the methods in Chapter 1. Note, however, that the inertial forces above are a function of the modal displacements and not of their spatial derivatives.

If we apply the foregoing procedure to the beam of Fig. 8.5, and compute the inertial forces associated with the function of the parabola $\phi(x)$, then calculate the bending moment and shear along the beam, we find good agreement with the true results corresponding to $\Phi(x) = \cos(\pi x)/l$. This is explained by the fact that the inertial forces as computed by Eq. (8.100) depend on the modal displacements and not on their spatial derivatives. In addition the shear and moment are obtained respectively by integrating these inertial forces and their moments with respect to a point. This tends to reduce the errors that may result from discrepancies in the load intensities computed from Eq. (8.100) when the modes $\Phi_i(x)$ are approximate as compared with the true load intensities corresponding to the true modes.

Method 2

In this method the internal forces are computed in two parts. First, we apply the external forces $F(x, t)$ statically and compute the internal forces under the assumption that all points on the system have zero velocity and zero acceleration. Then we add the internal forces associated with the velocity and acceleration of the system. To develop method 2, we refer to the rth decoupled differential equation of motion in normal coordinates given by

$$\ddot{\eta}_r + 2\beta\dot{\eta}_r + \omega_r^2\eta_r = \frac{P_0}{M_r}\Gamma_r f(t) \qquad \text{(See Eq. 8.29.)}$$

This equation can be written in the form

$$\eta_r = \frac{P_0\Gamma_r}{\omega_r^2 M_r} f(t) - \frac{2\beta}{\omega_r^2}\dot{\eta}_r - \frac{\ddot{\eta}_r}{\omega_r^2} \tag{8.101}$$

or

$$\eta_r = \eta_{r,\mathrm{I}} + \eta_{r,\mathrm{II}} \tag{8.102}$$

where

$$\eta_{r,\mathrm{I}} = \frac{P_0\Gamma_r}{\omega_r^2 M_r} f(t)$$

and

$$\eta_{r,\mathrm{II}} = -\frac{2\beta}{\omega_r^2}\dot{\eta}_r - \frac{\ddot{\eta}_r}{\omega_r^2}$$

$$t_0 = \tfrac{1}{2}T_1$$

where t_0 is the time at which the peak load $p(x, t)$ occurs (see Fig. 8.3), and T_1 is the period of the fundamental mode

$$T_1 = \frac{2\pi}{\omega_1}$$

Using methods 1 and 2 of the last section, we wish to compute the forces in the springs on which the beam is mounted. For brevity we seek these forces only for values of $t \leq t_0$.

From Section 8.9 we have for the first two modes

$$\Phi_1(x) = 1 + 1.414 \cos \frac{\pi x}{l}$$

$$\omega_1 = 0.725 \frac{\pi^2}{l^2} \sqrt{\frac{EI}{m}}$$

$$T_1 = \frac{2\pi}{\omega_1}$$

$$\Phi_2(x) = 1 - 1.45 \cos \frac{\pi x}{l} - 0.045 \cos \frac{3\pi x}{l}$$

$$\omega_2 = 2.86 \frac{\pi^2}{l^2} \sqrt{\frac{EI}{m}}$$

$$T_2 = \frac{2\pi}{\omega_2}$$

The total inertial loads corresponding to the first two modes $\Phi_1(x)$ and $\Phi_2(x)$ are respectively given by

$$\left.\begin{aligned}
Q_1 &= 2 \int_0^{l/2} \omega_1^2 m \Phi_1(x)\, dx \\
&= 2\omega_1^2 m \int_0^{l/2} \left(1 + 1.414 \cos \frac{\pi x}{l}\right) dx \\
&= 1.90 \omega_1^2 m l \\
\text{and} & \\
Q_2 &= 2 \int_0^{l/2} \omega_2^2 m \Phi_2(x)\, dx \\
&= 2\omega_2^2 m \int_0^{l/2} \left(1 - 1.45 \cos \frac{\pi x}{l} - 0.045 \cos \frac{3\pi x}{l}\right) dx \\
&= 0.0856 \omega_2^2 m l
\end{aligned}\right\} \quad (8.105)$$

Substituting

$$t_0 = \frac{1}{2} T_1 = \frac{1}{2} \frac{2\pi}{\omega_1} = \frac{\pi}{\omega_1}$$

in the expressions for the dynamic load factors $D_1(t)$ and $D_2(t)$

derived in Section 8.9 (for $t < t_0$), we write $D_1(t)$, $D_2(t)$ and their second time derivatives in the form

$$\left.\begin{aligned} D_1(t) &= \frac{\omega_1 t}{\pi} - \frac{1}{\pi}\sin\omega_1 t \\ \ddot{D}_1(t) &= \frac{\omega_1^2}{\pi}\sin\omega_1 t \\ D_2(t) &= \frac{\omega_1 t}{\pi} - \frac{\omega_1}{\omega_2\pi}\sin\omega_2 t \\ \ddot{D}_2(t) &= \frac{\omega_1\omega_2}{\pi}\sin\omega_2 t \end{aligned}\right\} \tag{8.106}$$

Substituting from Eq. (8.106) in Eqs. (8.91) and (8.92), we write for the normal coordinates $\eta_1(t)$; $\eta_2(t)$ and their second time derivatives

$$\left.\begin{aligned} \eta_1(t) &= \frac{1.078}{\pi^4}\frac{P_0 l^3}{EI}\left(\frac{\omega_1 t}{\pi} - \frac{1}{\pi}\sin\omega_1 t\right) \\ \ddot{\eta}_1(t) &= \frac{1.078}{\pi^4}\frac{P_0 l^3}{EI}\frac{\omega_1^2}{\pi}\sin\omega_1 t \\ \eta_2(t) &= \frac{-0.099}{\pi^4}\frac{P_0 l^3}{EI}\left(\frac{\omega_1 t}{\pi} - \frac{\omega_1}{\omega_2\pi}\sin\omega_2 t\right) \\ \ddot{\eta}_2(t) &= \frac{-0.099}{\pi^4}\frac{P_0 l^3}{EI}\frac{\omega_1\omega_2}{\pi}\sin\omega_2 t \end{aligned}\right\} \tag{8.107}$$

Using Eqs. (8.105) and (8.107) we can compute the spring force as a function of time for $t < t_0$. To be more specific let us compute the spring force at $t = \frac{1}{2}t_0 = \pi/2\omega_1$.

Solution by Method 1

From symmetry of the modes $\Phi_1(x)$ and $\Phi_2(x)$, it follows that half the inertial loads Q_1 and Q_2 is reacted by each spring. Hence the total spring force $P(\pm l/2, t)$ is given by

$$P\left(\pm\frac{l}{2}, t\right) = \sum_{i=1}^{2}\frac{Q_i}{2}\eta_i(t) \quad \text{(See Eq. 8.97)} \tag{8.108}$$

Substituting for Q_1 and Q_2 from Eq. (8.105) and for $\eta_1(t)$ and $\eta_2(t)$ from Eq. (8.107) and setting $t = \frac{1}{2}t_0$, we obtain

$$P\left(\pm\frac{l}{2}, \frac{t_0}{2}\right) = 0.0801P_0$$

where P_0 is the total load on the beam of Fig. 8.2 at $t = t_0$. Alternately, we compute the spring forces from the spring displacements to obtain

$$\begin{aligned} P\left(\pm\frac{l}{2}, t\right) &= \frac{k_1}{2}w\left(\pm\frac{l}{2}, t\right) \\ &= \frac{k_1}{2}\sum_{i=1}^{2}\Phi_i\left(\pm\frac{l}{2}\right)\eta_i(t) \end{aligned} \tag{8.109}$$

where $k_1/2$ is the spring stiffness

$$\left(k_1 = \pi^4 \frac{EI}{l^3}\right) \quad \text{and} \quad \Phi_1\left(\pm\frac{l}{2}\right) = \Phi_2\left(\pm\frac{l}{2}\right) = 1$$

For $t = \frac{1}{2}t_0$ we have

$$P\left(\pm\frac{l}{2}, \frac{t_0}{2}\right) = \frac{k_1}{2}\sum_{i=1}^{2}\eta_i\left(\frac{t_0}{2}\right)$$
$$= 0.073P_0$$

Solution by Method 2

We apply Eq. (8.104) and write

$$P\left(\pm\frac{l}{2}, t\right) = P_{\text{I}}\left(\pm\frac{l}{2}, t\right) + P_{\text{II}}\left(\pm\frac{l}{2}, t\right) \tag{8.110}$$

The spring force, $P_{\text{I}}(\pm l/2, t)$, due to the total external load $p(x, t)$ applied statically is given by

$$P_{\text{I}}\left(\pm\frac{l}{2}, t\right) = \frac{1}{2}P_0\frac{t}{t_0}$$

The spring force, $P_{\text{II}}(\pm l/2, t)$, due to the acceleration is computed from Eq. (8.103)

$$P_{\text{II}}\left(\pm\frac{l}{2}, t\right) = \sum_{i=1}^{2} P_i\eta_{i,\text{II}}(t)$$

where

$$P_i = \frac{Q_i}{2} \quad (Q_i \text{ is given by Eq. 8.105.})$$

and

$$\eta_{i,\text{II}}(t) = -\frac{\ddot{\eta}_i(t)}{\omega_i^2}$$ (See bottom of Eq. 8.102 and recall that in the present problem the damping is zero, hence, $\beta = 0$.)

Substituting these results in Eq. (8.110), write

$$P\left(\pm\frac{l}{2}, t\right) = \frac{1}{2}P_0\frac{t}{t_0} - \sum_{i=1}^{2}\frac{Q_i}{2}\frac{\ddot{\eta}_i(t)}{\omega_i^2} \tag{8.111}$$

where $\ddot{\eta}_i(t)$ $(i = 1, 2)$ are given by Eq. (8.107). For $t = \frac{1}{2}t_0$, we obtain from Eq. (8.111)

$$P\left(\pm\frac{l}{2}, t\right) = 0.0788P_0$$

We compare, below, the spring forces computed by methods 1 and 2 at times $t = \frac{1}{2}t_0$ and $t = t_0$ for three different values of t_0; $t_0 = \frac{1}{2}T_1$, $t_0 = T_1$, and $t_0 = T_2$.

applied statically to the beam will result in a displacement proportional to the rth mode shape.

3. Derive the mass and stiffness matrices in coordinates q for the system of Fig. 8.2 in Section 8.9. Use the same three functions $\phi_i(x)$ as those used in Section 8.9 and compare with the given results.

4. Verify the natural modes and frequencies computed for the system of Fig. 8.2 in Section 8.9.

5. Using the results of Section 8.9, compute the spring forces in the system of Fig. 8.2 at times

$$\left.\begin{aligned} t &= \tfrac{1}{2}t_0 \\ t &= t_0 \\ t &= 2t_0 \end{aligned}\right\} \qquad t_0 = \tfrac{3}{2}T_1$$

6. Consider the uniform beam in the system of Fig. 8.2, Section 8.9, to be a 12 WF 27 of length $l = 24$ ft. A 12 WF 27 has a moment of inertia $I = 204.1$ in^4 and a weight of 27 lb/ft. The modulus of elasticity is $E = 30 \times 10^6$ psi. Using the results of Section 8.9 compute the shear and the moment at $x = \pm l/4$ and $x \pm l/2$ at time $t = \frac{3}{4}t_0$. Set $t_0 = 2T_1$ where T_1 is the fundamental period of the beam-spring system. Use method 1 of Section 8.10. Compute the shear forces at $x = \pm l/2$ from the inertial loads.

7. Repeat Problem 6 using method 2 of Section 8.10. Compare the results with those of Problem 6.

8. In Problem 6 compute the spring force from the simple relation

$$\begin{aligned} \text{Spring force} &= \text{stiffness} \times \text{displacement} \\ &= \left(\frac{k_1}{2}\right) w\left(\frac{l}{2}, t\right) \end{aligned}$$

Compare the result with the shear at $x = \pm l/2$ computed in Problems 6 and 7.

9. Should there be any difference in the internal forces computed by methods 1 and 2 of Section 8.10 if the normal mode shapes $\Phi_i(x)$ $(i = 1, 2, \ldots, n)$ are the true mode shapes and all n of them are considered in the analysis. Explain.

10. The system of Fig. 8.2 with the load as shown is submerged in a viscous fluid. The coefficient of viscous damping is given by

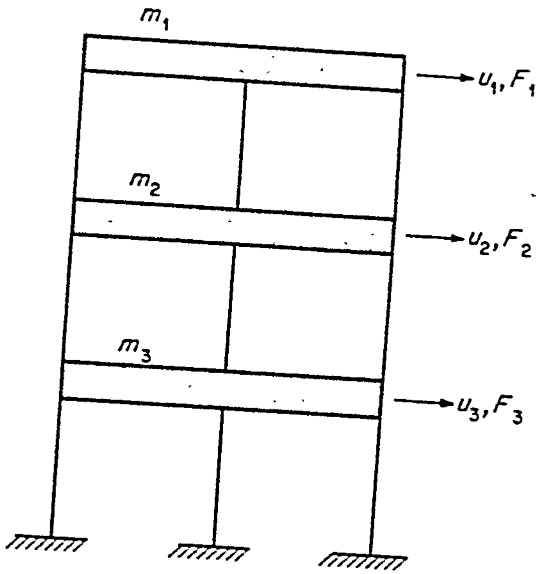
$$c = 0.3\sqrt{\frac{EIm}{l^3}}$$

Using the results of Section 8.9 formulate the first two decoupled differential equations of motion of the system. Write the expressions for normal coordinates $\eta_1(t)$ and $\eta_2(t)$ and compare with Eqs. (8.91) and (8.92).

11. The three-story building shown has the following mass and stiffness matrices in the u coordinate system.

$$[m]_u = 38.4\,\frac{\text{kip-sec}^2}{\text{ft}}\begin{bmatrix} 1.5 & & \\ & 1 & \\ & & 1.5 \end{bmatrix}$$

$$[k]_u = 10^3\,\frac{\text{kip}}{\text{ft}}\begin{bmatrix} 42 & -42 & 0 \\ -42 & 100 & -58 \\ 0 & -58 & 126 \end{bmatrix}$$



Problem 11

The normal modes and frequencies of the building are given by

$$\{\Phi^{(1)}\} = \begin{Bmatrix} 2.86 \\ 1.95 \\ 1.00 \end{Bmatrix} \qquad \omega_1 = 15.1 \text{ rad/sec}$$

$$\{\Phi^{(2)}\} = \begin{Bmatrix} -0.657 \\ 0.725 \\ 1.000 \end{Bmatrix} \qquad \omega_2 = 38.5 \text{ rad/sec}$$

$$\{\Phi^{(3)}\} = \begin{Bmatrix} 0.387 \\ -1.61 \\ 1.00 \end{Bmatrix} \qquad \omega_3 = 61.7 \text{ rad/sec}$$

Consider a dynamic load vector $\{F(x_j)\}$ applied to the mass of the building, where

$$\{F(x_j)\} = 28 \text{ kips}\begin{Bmatrix} 3 \\ 2 \\ 1 \end{Bmatrix} f(t)$$

and

$$f(t) = 1.0 \qquad \text{for } t > 0$$
$$= 0 \qquad \text{for } t \leq 0$$

Find the expressions for the displacements u_j ($j = 1, 2, 3$) of the building floors as a function of time t. Plot the displacement configuration of the building at time $t = \frac{1}{2}T_1$ where $T_1 = 2\pi/\omega_1$.

12. In Problem 11, compute the shear at each story of the building at time

$$t = \tfrac{1}{8}T_1$$
$$t = \tfrac{1}{4}T_1$$
$$t = \tfrac{1}{2}T_1$$

where $T_1 = 2\pi/\omega_1$. Compare your results with the shears resulting from the static application of the loads.

13. Repeat Problem 11 for

$$f(t) = 1 - \frac{t}{t_0}$$

and

$$t_0 = T_1 = \frac{2\pi}{\omega_1}$$

14. The building of Problem 11 has a damping matrix in coordinates u given by

$$[c]_u = 0.05\,[k]_u$$

Find the expressions for the floor displacements $u_j(t)$ ($j = 1, 2, 3$) due to a dynamic load vector

$$\{F(x_j)\} = 20 \text{ kips} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} f(t)$$

where $f(t) = t$.

15. *In Problem 13 compute the story shears at time* $t = \frac{1}{2}T_1$. Use methods 1 and 2 of Section 8.10 and compare the results.

16. *An earthquake causes a horizontal acceleration $\ddot{y} = \sin \Omega t$ at the foundation of the building in Problem 1, Chapter 3. Formulate the decoupled differential equations of motion and find the expressions for the displacements $u_j(t)$ of the building floors.* Set $\Omega = \frac{1}{2}\omega_1$ and compute the story shear at time $t = \frac{1}{2}T_1 = \pi/\omega_1$ where ω_1 and ω_2 are the natural modes of the system vibrating in the yz plane.

17. The portal frame shown is constructed of a 12 WF 27 beam ($I = 204.1$ in^4) and 8 WF 31 columns ($I = 109.7$ in^4). The beam and columns are rigidly connected so that the 90° angle is maintained. The mass of the floor supported by the frame is lumped at the corners as shown with mass

$$M = \frac{40{,}000}{32.2} \; \frac{\text{lb-sec}^2}{\text{ft}}$$

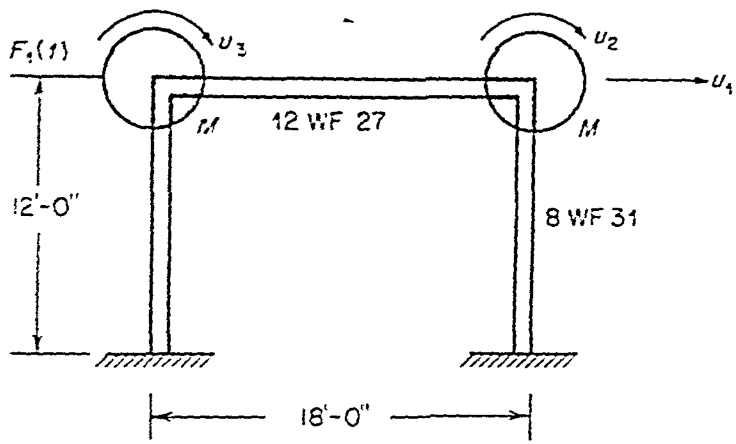
The mass moment of inertia of each mass about an axis perpendicular to the plane of the frame and going through the column beam intersection, is given by

$$J = \frac{120{,}000}{32.2}\ \text{lb-ft-sec}^2$$

Considering bending energy only and neglecting the mass of the beam and columns, find the expressions for the response $u_j(t)$ $j = 1, 2, 3$ due to a load $F_1(t)$ applied laterally at the top of the frame as shown.

$$F_1(t) = t \qquad \text{for } 0 < t < t_0$$
$$= 0 \qquad \text{for} \qquad t > t_0$$
$$t_0 = T_1$$

where T_1 is the fundamental period of the system.



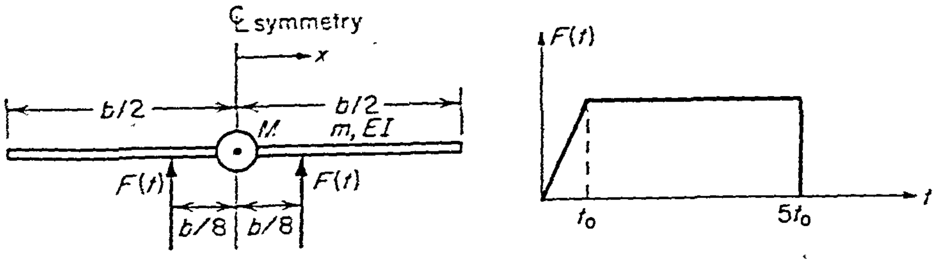
Problem 17

18. In Problem 17 draw the shear and bending moment diagrams for the frame members at time

$$t = \tfrac{1}{2}t_0$$

Plot the bending moment at the fixed column bases as a function of time. When is this moment a maximum?

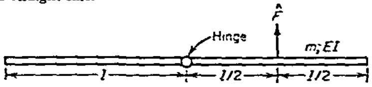
19. The figure shown represents schematically the cross-section of an airplane fuselage and a projection of the wings. The mass M of the fuselage is considered lumped at $x = 0$ while the mass of the wings is uniformly



Problem 19

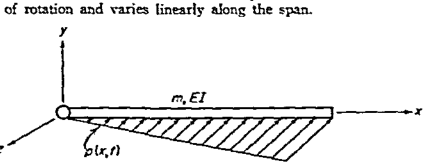
distributed between $x = 0$ and $x = \pm b/2$ with mass m per unit length. The stiffness of the wings is constant (EI). At time $t = 0$ the wheels of the airplane contact the ground with a vertical sinking velocity v_s. The landing gear forces $F(t)$, which are considered to be equal for both wings, increase linearly with time from $t = 0$ to a maximum force at $t = t_s$; then remain constant until the velocity of the wing at the point of landing gear attachment becomes zero. Find the response of the airplane as a function of time.

20. Two uniform beams are linked together by a frictionless hinge. The beams are free to move in the plane of the figure. Using the normal-mode method find the response to an impulsive force $\hat{F}$, applied laterally at the center of one of the beams. The beams are initially at rest and lie on a straight line.



Problem 20

21. A uniform elastic beam is free at one end and hinged at the other, so it can rotate about the z axis. The beam rotates with constant angular velocity Ω about the y axis. Find the response of the beam to a suddenly applied distributed force $p(x, t)$ in the xz plane which is zero at the axis of rotation and varies linearly along the span.



Problem 21

CHAPTER 9

Nonproportional Damping

9.1 Discussion and Definition of Nonproportional Damping

In the last chapter it was shown that the equations of motion for a linear system could be uncoupled by means of a transformation in which they are expressed in the normal coordinates of the system. This transformation and the normal coordinates are discussed in Chapter 3, Section 3.7, where it is shown that the transformation matrix $[\gamma]$ is derived from the eigenvectors of the undamped system. However, this transformation was shown to be successful in uncoupling the equations of motion for a linearly damped system provided the damping matrix $[c]$ as shown in Eq. (8.65) is proportional either to the mass matrix $[m]$, to the stiffness matrix $[k]$, or to a linear combination of the two.* For example, such a proportional damping matrix may be written as

$$[c] = 2\beta[m] + \alpha[k] \tag{9.1}$$

where β and α are real constants defined by Eqs. (8.71) and (8.72).

If, on the other hand, the damping matrix is not proportional in the above sense then the transformation $[\gamma]$ will no longer uncouple the equations of motion. It is possible, however, to extend the methods

* See Chapter 8, Section 8.6. Reference 50 shows that this transformation also uncouples equations of motion in which the damping matrix has other special characteristics.

of Chapter 8 and to construct a transformation which will uncouple the equations of motion even though the damping is nonproportional. The method is restricted to linear damping. It requires at the outset the solution of the homogeneous equations to give the free-vibration response of the system. From this solution one may construct the uncoupling transformation and proceed to the solution of the nonhomogeneous equations for the forced response. These matters are discussed in this chapter, but before proceeding to them, it will be helpful to compare the free vibrations of a system with proportional damping to one with nonproportional damping.

9.2 Comparison of Free Vibrations with Proportional and Nonproportional Damping

Proportional Damping. The linear equations of motion of an n-degree-of-freedom system in matrix form expressed in generalized coordinates $q_1, q_2, \ldots, q_n$ are written as

$$[m]\{\ddot{q}\} + [c]\{\dot{q}\} + [k]\{q\} = \{Q\} \tag{9.2}$$

This equation applies to a linear system whether or not matrix $[c]$ is proportional to $[m]$ or $[k]$. If it is proportional as expressed by Eq. (9.1) then, following the method of Chapter 8, the rth uncoupled equation expressed in terms of normal coordinates may be written as

$$\ddot{\eta}_r + 2\zeta_r\omega_r\dot{\eta}_r + \omega_r^2\eta_r = \frac{N_r}{M_r} \tag{9.3}$$

where η_r = the rth component of the normal coordinate vector $\{\eta\}$ defined by the transformation

$\{q\} = [\gamma]\ \{\eta\}$ [see Eq. (3.70)]

$\zeta_r = \dfrac{\beta}{\omega_r} + \dfrac{1}{2}\alpha\omega_r$, a damping factor corresponding to the rth mode

Except for the use of damping factor ζ_r, instead of damping coefficient β, this equation is identical to Eq. (8.19) and other equations in Chapter 8. This equation applies to free vibrations of a system with proportional damping if the right-hand member is set equal to zero. If motion of the system is started by applying an initial displacement $\eta_r(0)$ at time $t = 0$, the equation may be solved to give the modal displacement $\eta_r(t)$ at time t in terms of the initial displacement.

$$\eta_r(t) = \eta_r(0)\,\frac{e^{-\zeta_r\omega_r t}}{\sqrt{1-\zeta_r^2}}\cos\left(\sqrt{1-\zeta_r^2}\,\omega_r t - \psi_r\right) \tag{9.4}$$

where

$\sqrt{1-\zeta_r^2}\,\omega_r =$ the frequency of damped free vibration

$\psi_r =$ a phase angle given by

$$\tan\psi_r = \frac{\zeta_r}{\sqrt{1-\zeta_r^2}} \tag{9.5}$$

The modal velocity is obtained by differentiating Eq. (9.4) with respect to time.*

$$\dot{\eta}_r(t) = \eta_r(0)\,\frac{\omega_r e^{-\zeta_r\omega_r t}}{\sqrt{1-\zeta_r^2}}\cos\left(\sqrt{1-\zeta_r^2}\,\omega_r t + \frac{\pi}{2}\right) \tag{9.6}$$

To illustrate phase relationships, the above modal displacement and velocity are shown as rotating vectors on the complex plane in Fig. 9.1. It can be seen that the velocity vector leads the displacement vector by the angle $\psi_r + \pi/2$. Both vectors diminish in amplitude

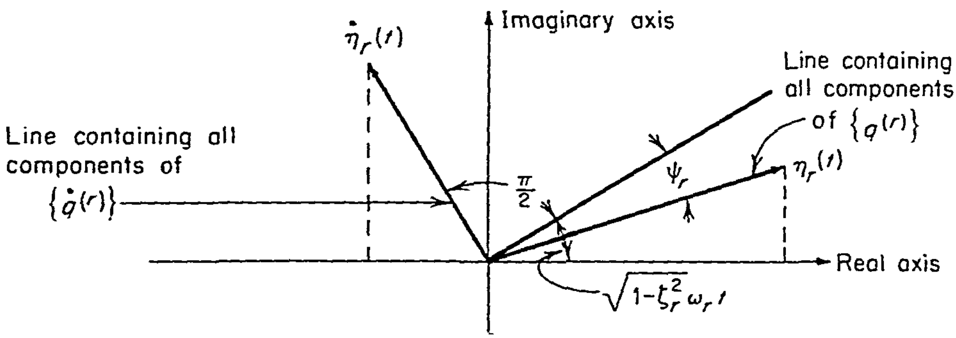


Figure 9.1

exponentially with time. From the transformation given by Eq. (3.70) it can be seen that the vector $\{q^{(r)}\}$ containing displacement components corresponding to the rth mode is given by

$$\{q^{(r)}\} = \{\gamma^{(r)}\}\,\eta_r(t)$$

where $\{\gamma^{(r)}\}$ is the rth column of matrix $[\gamma]$. Since $\{\gamma^{(r)}\}$ is a column of real numbers which define the shape of the rth natural mode of the undamped system, it follows that the components of $\{q^{(r)}\}$ also correspond to the same mode shape. Therefore, all these components have the same phase as $\eta_r(t)$, or will be 180° out of phase depending upon the sign of the corresponding component in $\{\gamma^{(r)}\}$. In Fig. 9.1 the vector $\eta_r(t)$ represents, therefore, the line of action of all of the components of $\{q^{(r)}\}$. Similarly, the vector $\dot{\eta}_r(t)$ represents the line of action of all of the components of the velocity vector $\{\dot{q}^{(r)}\}$. If

* Equations (9.5) and (9.6) are valid for a system with less than critical damping. (See Chapter 7, Section 7.9, for the concept of critical damping.)

a damped free vibration is initiated in one of the natural modes of the undamped system it will continue with the *mode shape* unchanged but with an exponential decay in amplitude at all points in the system at the same rate. Thus, in appearance this vibration is very much like the vibration of the undamped system except that the motion diminishes in amplitude until the system comes to rest.

Nonproportional Damping. In contrast, the motion of a system with nonproportional damping is quite different. As will be shown in this chapter, it is possible to construct a transformation relating coordinates $\{q\}$ to a new set of coordinates $\{z\}$ in which the equations of motion are uncoupled.

$$\{q\} = [\Delta]\{z\} \tag{9.7}$$

Again, one may refer to an rth uncoupled mode in which the displacements $\{q^{(r)}\}$ are given by

$$\{q^{(r)}\} = \{\Delta^{(r)}\}z_r(t) \tag{9.8}$$

where

$$\{\Delta^{(r)}\} = \text{the } r\text{th column of } [\Delta].$$

The function $z_r(t)$ is an exponentially damped cosine function (for a system with less than critical damping) for free vibration started by an initial displacement $z_r(0)$. However, in this case, the transformation matrix $[\Delta]$ is complex, hence, the components of $\{\Delta^{(r)}\}$ are complex numbers, each differing in phase as well as in amplitude. Consequently, the components of $\{q^{(r)}\}$ differ in phase as well as in amplitude. Plotted as vectors on the phase plane as in Fig. 9.1, all components rotate at the same velocity β_r and all decay in amplitude at the same rate. However, each has a different phase angle in general. This is illustrated by showing two typical components $q_i^{(r)}(t)$ and $q_j^{(r)}(t)$ on the phase plane of Fig. 9.2. Here, the two components $q_i^{(r)}$ and $q_j^{(r)}$ have phase angles θ_i and θ_j, respectively. In each case the

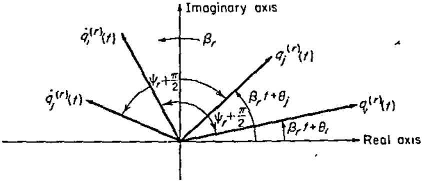


Figure 9.2

velocity components $\dot{q}_i^{(r)}$ and $\dot{q}_j^{(r)}$ lead their respective displacement components by the same phase angle $\psi_r + (\pi/2)$.

To illustrate the effect of the above noted change in phase on the motion of a system, consider the simple two-mass structure of Fig. 9.3. Figure 9.3(a) shows the structure and identifies the two

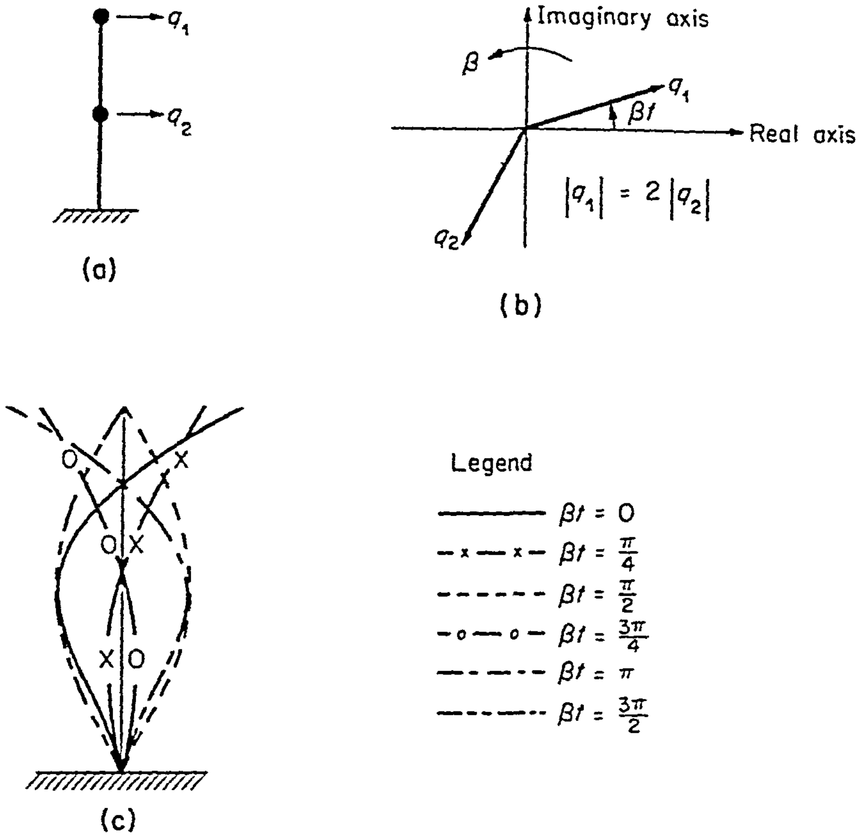


Figure 9.3

coordinates q_1 and q_2 which describe the motion at the two-mass points. Figure 9.3(b) shows a postulated phase relationship which may exist between the two displacements. Displacement q_1 leads q_2 by 135°. Figure 9.3(c) shows the displacements of the structure at several instants in time as given by the designated values of βt. It is seen that the motion is not characterized by the existence of a fixed node as would be the case if the two vectors were 180° apart, i.e., if the system were undamped or if the damping were proportional. In this illustration, the amplitudes are not shown to decrease with the motion as would actually be the case. This is done in order to simplify the picture and clarify the basic difference between the two cases to be distinguished.

In summary, a system with proportional linear viscous damping may be made to vibrate freely in a set of uncoupled modes which resemble, in shape, the normal modes of the undamped system, with amplitudes diminishing exponentially with time and uniformly over the system. These modes are distinguished by a definite spatial distribution of stationary nodal points or lines. In contrast, a system with nonproportional linear viscous damping may also be made to vibrate freely in a set of uncoupled "modes" in which all points in the system undergo exponentially damped motion at the same frequency, but at differing phase angles. In these modes, the nodes (if they may be termed as such) are not stationary.

9.3 The Equations of Motion for Nonproportional Damping

We have seen that each component of any eigenvector $\{q^{(r)}\}$ for an undamped system or for one in which damping is proportional is distinguished from other components by amplitude only, the phases being equal or 180° apart as determined by sign. It is useful to think of the n equations of motion (for an n-degree-of-freedom system) as a set of equations whose solution yields the set of n amplitudes for any given mode. This solution for an undamped system was discussed in Chapter 3, Section 3.3. For a system with nonproportional damping each component of eigenvector $\{q^{(r)}\}$ is distinguished not only by amplitude but also by phase; thus, two bits of information are required to determine each one. It follows that $2n$ equations are required to determine all components of an n-degree-of-freedom system in each mode. Therefore, to the n equations of motion (9.2), must be added another n equations giving a system of $2n$ equations to be solved in the case of nonproportional damping. An ingenious method of analysis has been developed[33] in which the additional n equations are supplied in a most interesting way. These equations are given by the following matrix identity.

$$[m]\{\dot{q}\} - [m]\{\dot{q}\} = \{0\} \tag{9.9}$$

Equations (9.2) and (9.9) are combined to give the following matrix equation of order $2n$

$$\left[\begin{array}{c|c} [0] & [m] \\ \hline [m] & [c] \end{array}\right] \left\{\begin{array}{c} \{\ddot{q}\} \\ \hline \{\dot{q}\} \end{array}\right\} + \left[\begin{array}{c|c} -[m] & [0] \\ \hline [0] & [k] \end{array}\right] \left\{\begin{array}{c} \{\dot{q}\} \\ \hline \{q\} \end{array}\right\} = \left\{\begin{array}{c} \{0\} \\ \hline \{Q\} \end{array}\right\} \tag{9.10}$$

This equation, often referred to as the "reduced" form of Eq. (9.2), is written as

$$[A]\{\dot{y}\} + [B]\{y\} = \{Y\} \tag{9.11}$$

where

$$[A] = \left[\begin{array}{c|c} [0] & [m] \\ \hline [m] & [c] \end{array}\right]$$

$$[B] = \left[\begin{array}{c|c} -[m] & [0] \\ \hline [0] & [k] \end{array}\right]$$

$$\{y\} = \left\{\begin{array}{c} \{\dot{q}\} \\ \hline \{q\} \end{array}\right\}$$

$$\{Y\} = \left\{\begin{array}{c} \{0\} \\ \hline \{Q\} \end{array}\right\}$$

The great advantage of this formulation lies in the fact that the matrices $[A]$ and $[B]$, both of order $2n$, are real and symmetric. Therefore, to solve Eq. (9.11) techniques very similar to those used in the treatment of undamped systems may be employed.

9.4 Solution of the Homogeneous Equation

In this section we consider the homogeneous equation obtained by setting the right of Eq. (9.11) equal to zero.

$$[A]\{\dot{y}\} + [B]\{y\} = \{0\} \tag{9.12}$$

Solutions to this linear equation will be found in which the displacements and velocities have the form e^{pt}, hence

$$\{\dot{y}\} = p\{y\} \tag{9.13}$$

or

$$\{\ddot{q}\} = p\{\dot{q}\}$$
$$\{\dot{q}\} = p\{q\}$$

Equation (9.12) is then written as follows in terms of the unknown number p and unknown vector $\{y\}$.

$$p[A]\{y\} = -[B]\{y\} \tag{9.14}$$

In general $[B]$ will have an inverse except under conditions for which the stiffness matrix $[k]$ is singular as discussed in Chapter 1, Section 1.9. If the stiffness matrix is singular as will be the case if the system is unconstrained with respect to one or more rigid-body displacements, then such rigid-body modes must be removed from the system. An example of this treatment is given in Chapter 4, Section 4.10. If Eq. (9.14) is premultiplied by the inverse of $[B]$ it may be written in the form

$$[D]\{y\} = \lambda\{y\} \tag{9.15}$$

where

$$[D] = -[B]^{-1}[A]$$

$$\lambda = \frac{1}{p}$$

It is noted immediately that this equation is identical in form to Eq. (3.7) for an undamped system where $[D]$ is again called a dynamical matrix. However, in the present case this matrix is of order $2n$ and important differences exist in the solutions of the two equations. We note that the inverse of $[B]$ has the form

$$[B]^{-1} = \left[\begin{array}{c|c} -[m]^{-1} & [0] \\ \hline [0] & [k]^{-1} \end{array}\right]$$

Hence, the dynamical matrix may be expressed in partitioned form leading to the following partitioned form of Eq (9.15).

$$\left[\begin{array}{c|c} [0] & [I] \\ \hline -[k]^{-1}[m] & -[k]^{-1}[c] \end{array}\right] \begin{Bmatrix} \{\dot{q}\} \\ \hline \{q\} \end{Bmatrix} = \frac{1}{p} \begin{Bmatrix} \{\dot{q}\} \\ \hline \{q\} \end{Bmatrix} \tag{9.16}$$

where

$[I]$ = the identity matrix of order n

$[k]^{-1}[m]$ = the dynamical matrix of order n for the undamped system

Following the procedure of Chapter 3, Section 3.3, Eq. (9.15) may be solved by expressing it in the form

$$[L(\lambda)]\{y\} = \{0\} \tag{9.17}$$

where

$$[L(\lambda)] = [D] - \lambda[I]$$

For a nontrivial solution of this equation, the secular determinant must vanish, or

$$|L(\lambda)| = 0 \tag{9.18}$$

This leads to a set of $2n$ roots or eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$. For a stable system, each of these roots will either be real and negative (for a critically damped or overdamped mode) or complex with a negative real part (for an underdamped mode).* If there are complex eigenvalues they will occur in conjugate pairs. For example, if the kth and lth eigenvalues are complex conjugates, they may be written as

$$\begin{aligned} \lambda_k &= \mu_k + i\nu_k \\ \lambda_l &= \bar{\lambda}_k = \mu_k - i\nu_k \end{aligned} \tag{9.19}$$

* Overdamped and underdamped mode refer, respectively, to a mode with damping above and below the critical damping

where μ_k and ν_k are the real and imaginary parts, respectively, and i is the unit imaginary number.

Corresponding to each eigenvalue λ_k there exists an eigenvector $\{y^{(k)}\}$ having $2n$ components. There are $2n$ of these eigenvectors, $k = 1, 2, 3, \ldots, 2n$. As in Chapter 3, these eigenvectors may be shown to be proportional to the columns of the adjoint matrix of $[L(\lambda)]$, for those eigenvectors corresponding to distinct eigenvalues. Let $[J_{ji}(\lambda)]$ be the adjoint matrix of $[L(\lambda)]$. Then

$$\{y^{(k)}\} = \epsilon_k\{J_{ji}(\lambda_k)\} = \epsilon_k\{J_{ji}^{(k)}\} \tag{9.20}$$

where

$\{J_{ji}^{(k)}\}$ = any column of the adjoint matrix $[J_{ji}(\lambda_k)]$ obtained by inserting the eigenvalue λ_k into $[J_{ji}(\lambda)]$

ϵ_k = a constant of proportionality.

For complex eigenvalues it can be seen that the form of the complex matrix $[L(\lambda)]$ is such that complex numbers occur only on the principal diagonal. The cofactors of this matrix have the property that complex conjugate cofactors result from the insertion into $[L(\lambda)]$ of complex conjugate eigenvalues. It follows that for a pair of conjugate complex eigenvalues as given by Eq. (9.19), the corresponding eigenvectors are complex conjugates, thus

$$\{y^{(l)}\} = \{\bar{y}^{(k)}\} \tag{9.21}$$

Therefore, if all eigenvalues of a system are complex, in which case they occur in conjugate complex pairs, all eigenvectors will be complex and will also occur in conjugate pairs.

Let us consider a complex eigenvector $\{y^{(k)}\}$ corresponding to complex eigenvalue

$$\lambda_k = \mu_k + i\nu_k$$

These satisfy Eqs. (9.15) or the alternate form (9.16) from which it is seen that any one velocity component of vector $\{y^{(k)}\}$ is related to the corresponding displacement component by

$$\dot{q}_i^{(k)} = p_k q_i^{(k)} \tag{9.22}$$

where

$$p_k = \frac{1}{\lambda_k} = \alpha_k + i\beta_k \tag{9.23}$$

Note that

$$\left.\begin{aligned} \alpha_k &= \frac{\mu_k}{\mu_k^2 + \nu_k^2} \\ \beta_k &= -\frac{\nu_k}{\mu_k^2 + \nu_k^2} \end{aligned}\right\} \tag{9.24}$$

Since the solution has the time dependence

$$e^{p_k t} = e^{\alpha_k t}\, e^{i\beta_k t}$$

we see that in the kth mode α_k will be negative for a damped system, and β_k represents the frequency of damped free vibration. If the displacements q_i ($i = 1, 2, 3, \ldots, n$) are plotted as rotating vectors on the complex plane as in Fig. 9.2, their angular velocity is β_k. From Eq. (9.22) it can be seen that the velocities lead the corresponding displacements by phase angle ψ_k which is the argument of vector p_k when p_k is expressed in the polar form

$$p_k = (\alpha_k^2 + \beta_k^2)^{1/2} e^{i\psi_k} \tag{9.25}$$

where

$$\tan \psi_k = \frac{\beta_k}{\alpha_k} \tag{9.26}$$

Examples will show that, in general, the various displacement components q_i have differing phase angles θ_i as shown in Fig. 9.2. We recall that solution of the eigenvalue problem yields relative, not absolute, amplitudes of displacement, hence, only the relative magnitudes of the complex displacements are determined. Similarly, the absolute phase angles θ_i are not determined, but rather, the differences in phase angles. This can be seen by considering Eq. (9.20) if we note that the constant of proportionality ϵ_k is a complex number. Hence, normalization of a complex eigenvector $\{y^{(k)}\}$ consists of not only scaling all magnitudes proportionally but of rotating all components through the same angle in the complex plane.

9.5 Orthogonality of the Uncoupled Modes

The eigenvectors of a damped system are orthogonal just as are those for an undamped system and the proof of orthogonality may proceed in the same manner as shown in Chapter 3. Consider the rth and sth eigenvectors $\{y^{(r)}\}$ and $\{y^{(s)}\}$, both of which satisfy Eq. (9.14). First, we write that equation for the rth mode and premultiply both sides by the transposed vector $\{y^{(s)}\}^T$, thus

$$p_r\{y^{(s)}\}^T[A]\{y^{(r)}\} = -\{y^{(s)}\}^T[B]\{y^{(r)}\}$$

Using the reversal law for transposed matrix products and recalling that $[A]$ and $[B]$ are symmetrical matrices, we transpose both sides as

$$p_r\{y^{(r)}\}^T[A]\{y^{(s)}\} = -\{y^{(r)}\}^T[B]\{y^{(s)}\} \tag{9.27}$$

Next, write Eq. (9.14) for the sth mode and premultiply by $\{y^{(r)}\}^T$.

$$p_s\{y^{(r)}\}^T[A]\{y^{(s)}\} = -\{y^{(r)}\}^T[B]\{y^{(s)}\} \tag{9.28}$$

If Eq. (9.28) is subtracted from (9.27) the result is

$$(p_r - p_s)\{y^{(r)}\}^T[A]\{y^{(s)}\} = 0$$

If the eigenvalues p_r and p_s are different, the following orthogonality property relates the two eigenvectors

$$\{y^{(r)}\}^T[A]\{y^{(s)}\} = 0 \tag{9.29}$$

From either Eq. (9.27) or (9.28) it follows that these vectors are also orthogonal with respect to matrix $[B]$.

$$\{y^{(r)}\}^T[B]\{y^{(s)}\} = 0 \tag{9.30}$$

For underdamped modes in which the eigenvalues are complex, the above orthogonality criteria relate two modes having different frequencies of vibration. Furthermore, it is emphasized that the relationship applies also to two conjugate complex eigenvectors which are associated with a single mode, for the only requirement that Eqs. (9.29) and (9.30) hold is that the two conjugate complex eigenvalues associated with that mode be different. The fact that they are different is seen as follows. If

$$p_r = \alpha_r + i\beta_r$$

and

$$p_s = \alpha_r - i\beta_r$$

then

$$p_r - p_s = (\alpha_r + i\beta_r) - (\alpha_r - i\beta_r) = 2i\beta_r \neq 0$$

For underdamped systems it is useful to derive the orthogonality relationships in terms of the real and imaginary parts of the eigenvectors. We let

$$\begin{aligned} \{y^{(r)}\} &= \{\xi^{(r)}\} + i\{\eta^{(r)}\} \\ \{y^{(s)}\} &= \{\xi^{(s)}\} + i\{\eta^{(s)}\} \end{aligned} \tag{9.31}$$

Equation (9.29) may be written as

$$(\{\xi^{(r)}\}^T + i\{\eta^{(r)}\}^T)[A](\{\xi^{(s)}\} + i\{\eta^{(s)}\}) = 0 \tag{9.32}$$

By expanding the products and noting that the real and imaginary parts must vanish separately, the two following equations are obtained.

$$\left.\begin{aligned} \{\xi^{(r)}\}^T[A]\{\xi^{(s)}\} - \{\eta^{(r)}\}^T[A]\{\eta^{(s)}\} = 0 \\ \{\xi^{(r)}\}^T[A]\{\eta^{(s)}\} + \{\eta^{(r)}\}^T[A]\{\xi^{(s)}\} = 0 \end{aligned}\right\} \tag{9.33}$$

Next we consider an eigenvector $\{y^{(s)}\}$ that is orthogonal not only to $\{y^{(r)}\}$ which results in Eq. (9.33), but also to the conjugate eigenvector $\{\bar{y}^{(r)}\}$. Thus,

$$(\{\xi^{(r)}\}^T - i\{\eta^{(r)}\}^T)[A](\{\xi^{(s)}\} + i\{\eta^{(s)}\}) = 0$$

This separates into the two equations

$$\left.\begin{aligned} \{\xi^{(r)}\}^T[A]\{\xi^{(s)}\} + \{\eta^{(r)}\}^T[A]\{\eta^{(s)}\} = 0 \\ \{\xi^{(r)}\}^T[A]\{\eta^{(s)}\} - \{\eta^{(r)}\}^T[A]\{\xi^{(s)}\} = 0 \end{aligned}\right\} \tag{9.34}$$

Considering the first equation of (9.33) together with the first one of (9.34) leads us to the following results.

$$\left.\begin{aligned}\{\xi^{(r)}\}^T[A]\{\xi^{(s)}\} = 0\\ \{\eta^{(r)}\}^T[A]\{\eta^{(s)}\} = 0\end{aligned}\right\} \tag{9.35}$$

The second equations of (9.33) and (9.34) yield

$$\left.\begin{aligned}\{\xi^{(r)}\}^T[A]\{\eta^{(s)}\} = 0\\ \{\eta^{(r)}\}^T[A]\{\xi^{(s)}\} = 0\end{aligned}\right\} \tag{9.36}$$

One additional relationship is useful. It is derived from Eq. (9.33) considering $\{y^{(s)}\}$ as the conjugate of $\{y^{(r)}\}$, i.e.,

$$\{\xi^{(s)}\} = \{\xi^{(r)}\}$$

and

$$\{\eta^{(s)}\} = -\{\eta^{(r)}\}.$$

In this case the first equation of (9.33) yields the following result.

$$\{\xi^{(r)}\}^T[A]\{\xi^{(r)}\} = -\{\eta^{(r)}\}^T[A]\{\eta^{(r)}\} \tag{9.37}$$

Eqs. (9.35), (9.36), and (9.37) express orthogonality of the eigenvectors in terms of their real and imaginary parts. These relationships are developed with matrix $[A]$ as the weighting matrix starting with Eq. (9.29). It is easy to see that if we start with Eq. (9.30) we can show that the same relationships hold with matrix $[B]$ as the weighting matrix. These orthogonality relationships will be useful in considering a matrix iteration procedure for determining the system eigenvalues and eigenvectors. They will also be useful in the solution of the nonhomogeneous Eq. (9.11) to be considered in determining the response of a system to time-dependent forces.

9.6 Determination of Complex Eigenvalues and Eigenvectors by Matrix Iteration

The reduced homogeneous equation (9.15) may be solved to yield the eigenvalues and eigenvectors of the damped system by a method of matrix iteration similar to that described in Chapter 3 for undamped systems. For complex eigenvectors, convergence is not so readily recognized, however, and additional study has to be given to the test for convergence of both the complex eigenvectors and complex eigenvalues.

It might be supposed that the iteration procedure would start with the use of a trial vector $\{y\}$ that is complex, in which case one would have to select a vector having components arbitrarily distributed in phase as well as in magnitude. Normalization at each iteration would then require normalization of the amplitudes and, in addition, rotation of the group of complex components in the complex plane as discussed in Section 9.4. Such a procedure would be complicated and a simpler one can be shown.[11] It will be shown that if a real trial vector is used, resulting in successive vectors which are also real, the iteration procedure nevertheless converges to complex eigenvectors and eigenvalues for systems which are less than critically damped.

If the real trial vector $\{\xi\}$, is considered to be the real part of a complex vector $\{y\}$,

$$\{y\} = \{\xi\} + i\{\eta\} \tag{9.38}$$

then the iteration proceeds as follows. From Eq. (9.15)

$$\left.\begin{aligned} [D]\,{}^{1}\{\xi\} &= {}^{2}\{\xi\} \\ [D]\,{}^{2}\{\xi\} &= {}^{3}\{\xi\} \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ [D]\,{}^{n}\{\xi\} &= {}^{n+1}\{\xi\} \end{aligned}\right\} \tag{9.39}$$

where the superscript above and to the left of the vector represents the iteration number. We suppose that the number n is large enough so that convergence to within a satisfactorily small error is achieved. Examination of vectors obtained by successive iterations after convergence will show no apparent evidence of convergence. That this is true is clearly shown by examining the effect of one iteration on the actual complex vector. Thus, after convergence, we have

$$[D]\,{}^{(n)}\{y\} = \lambda\,{}^{(n)}\{y\} = {}^{(n+1)}\{y\} \tag{9.40}$$

Since $\lambda = 1/p$ the last equality can be expressed as

$${}^{(n)}\{y\} = p\,{}^{(n+1)}\{y\} \tag{9.41}$$

This shows that each iteration has the effect of diminishing the amplitude of each component of the vector by dividing by the magnitude of the eigenvalue p. Also, it has the effect of decreasing the phase angle of each component by the phase angle ψ of p where p is expressed in polar form as in Eq. (9.25). In particular, rotation of the group of complex components in the complex plane through a constant angle ψ with each iteration, has the effect of changing the real parts of these components in a way that would seem to be entirely without order. The fact that the successive vectors ${}^{n}\{\xi\}$, ${}^{n-1}\{\xi\}$, ${}^{n-2}\{\xi\}$, etc., obtained after convergence will yield complete information concerning the eigenvalue and eigenvector will now be shown.

Let us write Eq. (9.40) in terms of the real and imaginary parts of $\{y\}$ and λ by substituting Eqs. (9.19) and (9.31).

$$[D]({}^n\{\xi\} + i\,{}^n\{\eta\}) = (\mu + i\nu)({}^n\{\xi\} + i\,{}^n\{\eta\})$$

By separating the real and imaginary parts of this equation which must be separately equal, the two following real equations are obtained.

$$[D]\,{}^n\{\xi\} = \mu\,{}^n\{\xi\} - \nu\,{}^n\{\eta\} = {}^{n+1}\{\xi\} \tag{9.42}$$

$$[D]\,{}^n\{\eta\} = \nu\,{}^n\{\xi\} + \mu\,{}^n\{\eta\} = {}^{n+1}\{\eta\} \tag{9.43}$$

The last equality in (9.42) comes from Eq. (9.39) while that in (9.43) follows from similar reasoning applied to iteration on the imaginary part of $\{y\}$. From these equations we solve for vector ${}^n\{\eta\}$

$${}^n\{\eta\} = \frac{\mu}{\nu}\,{}^n\{\xi\} - \frac{1}{\nu}\,{}^{n+1}\{\xi\} \tag{9.44}$$

$${}^n\{\eta\} = \frac{1}{\mu}\,{}^{n+1}\{\eta\} - \frac{\nu}{\mu}\,{}^n\{\xi\} \tag{9.45}$$

Eliminating ${}^n\{\eta\}$ from these two equations we find the following relationship.

$$\left(\frac{\mu}{\nu} + \frac{\nu}{\mu}\right){}^n\{\xi\} - \frac{1}{\nu}\,{}^{n+1}\{\xi\} - \frac{1}{\mu}\,{}^{n+1}\{\eta\} = \{0\} \tag{9.46}$$

Now, by replacing n with $n + 1$, Eq. (9.44) may be written to give an expression for the vector ${}^{n+1}\{\eta\}$

$${}^{n+1}\{\eta\} = \frac{\mu}{\nu}\,{}^{n+1}\{\xi\} - \frac{1}{\nu}\,{}^{n+2}\{\xi\}$$

Substituting this into Eq. (9.46) leads to the following equation among successive columns $\{\xi\}$

$$(\mu^2 + \nu^2)\,{}^n\{\xi\} - 2\mu\,{}^{n+1}\{\xi\} + {}^{n+2}\{\xi\} = \{0\} \tag{9.47}$$

Again, by replacing n with $n + 1$ in this equation, we see that the following also holds.

$$(\mu^2 + \nu^2)\,{}^{n+1}\{\xi\} - 2\mu\,{}^{n+2}\{\xi\} + {}^{n+3}\{\xi\} = \{0\} \tag{9.48}$$

Since Eqs. (9.47) and (9.48) relate the successive vectors ${}^n\{\xi\}, \ldots, {}^{n+3}\{\xi\}$, they also relate the separate components of these vectors in the same way. Thus, for the ith components of these vectors we may write two corresponding equations

$$\left.\begin{aligned} (\mu^2 + \nu^2)\,{}^n\xi_i - 2\mu\,{}^{n+1}\xi_i + {}^{n+2}\xi_i = 0 \\ (\mu^2 + \nu^2)\,{}^{n+1}\xi_i - 2\mu\,{}^{n+2}\xi_i + {}^{n+3}\xi_i = 0 \end{aligned}\right\} \tag{9.49}$$

Eliminating the second terms from these equations leads to the following equation which gives the square of the amplitude of the eigenvalue λ.

$$(\mu^2 + \nu^2) = \frac{{}^{(n+1)}\xi_i\,{}^{(n+3)}\xi_i - {}^{(n+2)}\xi_i^2}{{}^{(n)}\xi_i\,{}^{(n+2)}\xi_i - {}^{(n+1)}\xi_i^2} \tag{9.50}$$

By eliminating the first term in Eqs. (9.49), we find the real part of eigenvalue λ.

$$2\mu = \frac{{}^{(n)}\xi_i \, {}^{(n+3)}\xi_i - {}^{(n+1)}\xi_i \, {}^{(n+2)}\xi_i}{{}^{(n)}\xi_i \, {}^{(n+2)}\xi_i - {}^{(n+1)}\xi_i^2} \tag{9.51}$$

Thus, we see that it is possible to obtain the eigenvalue λ, together with associated eigenvalue p through Eq. (9.24), by using successive columns $\{\xi\}$ obtained by iteration on Eq. (9.15) with a trial column of real numbers as carried out in Eq. (9.39). At each step in that iteration the associated computations given by Eqs. (9.50) and (9.51) are carried out. The iteration process continues until these latter computations yield repeating eigenvalues λ. At this point the real part of the eigenvector $\{y\}$ is given by the column ${}^{(n)}\{\xi\}$. To obtain the imaginary part of the eigenvector return to Eq. (9.44) and insert the values of μ and ν as determined above.

9.7 Sweeping

By following a proof for convergence similar to that given in Chapter 3, Section 3.5, it can be shown that iteration on Eq. (9.15) as described in the foregoing section will converge to the mode having the eigenvalue λ with the largest absolute value. To converge by iteration to the mode having the next largest eigenvalue it will be necessary to remove the first mode by a sweeping procedure similar to that for undamped systems. In terms of complex eigenvectors this procedure is based on the requirement that the complex trial vector be orthogonal to the first eigenvector $\{y^{(1)}\}$ and to its complex conjugate $\{\bar{y}^{(1)}\}$. Thus, two equations are written

$$\begin{aligned} \{y^{(1)}\}^T[A]\{y\} &= 0 \\ \{\bar{y}^{(1)}\}^T[A]\{y\} &= 0 \end{aligned} \tag{9.52}$$

Following the discussion of Section 9.5 these orthogonality statements can be written in terms of the real and imaginary parts, leading to the following equations which are comparable to Eqs. (9.35) and (9.36).

$$\left.\begin{aligned} \{\xi^{(1)}\}^T[A]\{\xi\} &= 0 \\ \{\eta^{(1)}\}^T[A]\{\eta\} &= 0 \\ \{\xi^{(1)}\}^T[A]\{\eta\} &= 0 \\ \{\eta^{(1)}\}^T[A]\{\xi\} &= 0 \end{aligned}\right\} \tag{9.53}$$

where $\{\xi\}$ and $\{\eta\}$ are the real and imaginary parts, respectively, of the trial vector $\{y\}$. Since our iteration procedure discussed in Section 9.6 involves only the real part of the vector, the first and last equations of (9.53) are used. This is a pair of real equations which ensure that iteration on a trial vector $\{\xi\}$, so constrained, will converge to the second mode. The numerical procedure involved in constructing the sweeping matrix $[S]$ is exactly like that shown in Chapter 3. It will be seen that the use of these two equations will result in a sweeping matrix which, when premultiplied by the dynamical matrix $[D]$, will result in a constrained dynamical matrix in which two null rows and columns occur. Thus, each sweeping process sweeps out two rows and columns in the dynamical matrix. Since this matrix is of order $2n$, for a system with n degrees of freedom, only n of these sweeping processes are required to yield all $2n$ eigenvectors of the system. Since these vectors occur in conjugate complex pairs this is not surprising.

As suggested above, the first and fourth equations of (9.53) may be used to construct the sweeping matrix. The fourth equation requires calculation of the imaginary part of the eigenvector which may be done through Eq. (9.44). This step may be avoided, if desired, by use of an alternate pair of orthogonality equations. To construct this alternate pair of equations write the first and fourth equations of (9.53) using the real and imaginary parts of the first eigenvector as obtained after n iterations.

$$\left.\begin{aligned} {}^{(n)}\{\xi^{(1)}\}^T[A]\{\xi\} &= 0 \\ {}^{(n)}\{\eta^{(1)}\}^T[A]\{\xi\} &= 0 \end{aligned}\right\} \tag{9.54}$$

From Eq. (9.44) we may determine the imaginary part

$$ {}^{(n)}\{\eta^{(1)}\} = \frac{\mu_1}{\nu_1}\,{}^{(n)}\{\xi^{(1)}\} - \frac{1}{\nu_1}\,{}^{(n+1)}\{\xi^{(1)}\}$$

Substituting this into the second equation of (9.54), we obtain the following equation.

$$\frac{\mu_1}{\nu_1}\,{}^{(n)}\{\xi^{(1)}\}^T[A]\{\xi\} - \frac{1}{\nu_1}\,{}^{(n+1)}\{\xi^{(1)}\}^T[A]\{\xi\} = 0$$

Since the first term in this equation is zero by virtue of the first equation of (9.54), it follows that the second term is also zero. Thus, the two orthogonality equations referred to are

$$\left.\begin{aligned} {}^{(n)}\{\xi^{(1)}\}^T[A]\{\xi\} &= 0 \\ {}^{(n+1)}\{\xi^{(1)}\}^T[A]\{\xi\} &= 0 \end{aligned}\right\} \tag{9.55}$$

Hence, two vectors $\{\xi^{(1)}\}$ obtained in successive iterations after convergence may be used to construct the sweeping matrix.

9.8 Uncoupling the Nonhomogeneous Equations

In a manner similar to that described in Chapter 8, the eigenvectors which define the modes of free vibration of the system may be used to construct a transformation of coordinates in which the equations of motion are uncoupled.[34] In this chapter that transformation is given by

$$\{y\} = [\Delta]\{z\} \tag{9.56}$$

The transformation matrix $[\Delta]$ is constructed column-by-column using the $2n$ eigenvectors; $\{y^{(1)}\}, \{\bar{y}^{(1)}\}, \ldots, \{y^{(r)}\}, \{\bar{y}^{(r)}\}, \ldots, \{y^{(n)}\}, \{\bar{y}^{(n)}\}$. Thus, the matrix is of order $2n$. Where complex eigenvectors exist, both the vector and its conjugate are used.

$$\begin{aligned}[\Delta] &= [\{y^{(1)}\}\{\bar{y}^{(1)}\}\ldots\{y^{(r)}\}\{\bar{y}^{(r)}\}\ldots\{y^{(n)}\}\{\bar{y}^{(n)}\}] \\ &= [\{\Delta^{(1)}\}\{\Delta^{(2)}\}\ldots\ldots\ldots\ldots\ldots\ldots\{\Delta^{(2n)}\}]\end{aligned} \tag{9.57}$$

When this transformation is applied to the nonhomogeneous equation (9.11), the following result is obtained.

$$[\Delta]^T[A][\Delta]\{\dot{z}\} + [\Delta]^T[B][\Delta]\{z\} = [\Delta]^T\{Y\} \tag{9.58}$$

The triple matrix products are computed and the resulting equation is written as

$$[\mathcal{A}]\{\dot{z}\} + [\mathcal{B}]\{z\} = \{Z\} \tag{9.59}$$

where

$$[\mathcal{A}] = [\Delta]^T[A][\Delta]$$

$$[\mathcal{B}] = [\Delta]^T[B][\Delta]$$

$$\{Z\} = [\Delta]^T\{Y\}$$

The matrices $[\mathcal{A}]$ and $[\mathcal{B}]$ are diagonal as a result of the orthogonality conditions expressed by Eqs. (9.29) and (9.30). This is shown by the following. The elements $\mathcal{A}_{rs}$ and $\mathcal{B}_{rs}$ can be written in the form

$$\mathcal{A}_{rs} = \{\Delta^{(r)}\}^T[A]\{\Delta^{(s)}\}$$

$$\mathcal{B}_{rs} = \{\Delta^{(r)}\}^T[B]\{\Delta^{(s)}\}$$

Since the columns $\{\Delta\}$ can be identified with the eigenvectors $\{y\}$, according to Eq. (9.57), we may write these equations as

$$\mathcal{A}_{rs} = \{y^{(r)}\}^T[A]\{y^{(s)}\}$$

$$\mathcal{B}_{rs} = \{y^{(r)}\}^T[B]\{y^{(s)}\}$$

Both of these coefficients are zero for $s \neq r$ [See Eqs. (9.29) and (9.30)] and have the following values for $s = r$.

$$\begin{aligned}\mathcal{A}_r &= \{y^{(r)}\}^T[A]\{y^{(r)}\} \\ \mathcal{B}_r &= \{y^{(r)}\}^T[B]\{y^{(r)}\}\end{aligned} \tag{9.60}$$

If in Eq. (9.27) we set $s = r$, the following relationship results

$$\mathcal{B}_r = -p_r\mathcal{A}_r \tag{9.61}$$

The rth equation of the uncoupled set (9.59) can now be written as

$$\mathcal{A}_r\dot{z}_r + \mathcal{B}_r z_r = Z_r$$

or, substituting (9.61) this becomes

$$\dot{z}_r - p_r z_r = \frac{Z_r}{\mathcal{A}_r} \tag{9.62}$$

where

$$Z_r = \{y^{(r)}\}^T\{Y\}$$

If $\{y^{(r)}\}$ is complex and therefore has a complex conjugate $\{\bar{y}^{(r)}\}$, then a companion equation to (9.62) exists. It is not difficult to show that the following is true.

$$\bar{\mathcal{A}}_r = \{\bar{y}^{(r)}\}^T[A]\{\bar{y}^{(r)}\}$$
$$\bar{\mathcal{B}}_r = \{\bar{y}^{(r)}\}^T[B]\{\bar{y}^{(r)}\}$$

where $\bar{\mathcal{A}}_r$ and $\bar{\mathcal{B}}_r$ are complex conjugates of $\mathcal{A}_r$ and $\mathcal{B}_r$, respectively. Also it may be shown that

$$\bar{\mathcal{B}}_r = -\bar{p}_r\bar{\mathcal{A}}_r$$

Similarly, it is seen that the following is true.

$$\bar{Z}_r = \{\bar{y}^{(r)}\}^T\{Y\}$$

From these results the companion equation to Eq. (9.62) may be written as

$$\dot{z}_r^* - \bar{p}_r z_r^* = \frac{\bar{Z}_r}{\bar{\mathcal{A}}_r} \tag{9.63}$$

Here we use the star to distinguish the variable z_r associated with the companion equation. Equations (9.62) and (9.63) are both solved to determine the response in the rth uncoupled mode.

9.9 Solution of the Uncoupled Equations

Solutions to Eqs. (9.62) and (9.63) may be written in general form by use of Laplace transforms. Use the following transforms (See Chapter 8, Section 8.8).

$$\mathcal{L}z_r(t) = \int_0^\infty e^{-st}z_r(t)\,dt = z_r(s)$$
$$\mathcal{L}\dot{z}_r(t) = sz_r(s) - z_r(t = 0)$$
$$\mathcal{L}Z_r(t) = Z_r(s)$$

We shall consider the initial displacement $z_r\,(t = 0)$ to be zero, hence, the Laplace transform of Eq. (9.62) is

$$z_r(s) = \frac{1}{\mathcal{A}_r}\frac{Z_r(s)}{s - p_r} \tag{9.64}$$

The inverse transform can be written immediately in terms of the convolution integral as

$$z_r(t) = \frac{1}{\mathcal{A}_r} \int_0^t e^{p_r(t-\tau)} Z_r(\tau)\, d\tau \tag{9.65}$$

In a similar manner the solution to the companion equation (9.63) may be written

$$z_r^*(t) = \frac{1}{\bar{\mathcal{A}}_r} \int_0^t e^{\bar{p}_r(t-\tau)} \bar{Z}_r(\tau)\, d\tau \tag{9.66}$$

From the form of the two foregoing solutions, it can be verified that one is the complex conjugate of the other, therefore,

$$z_r^*(t) = \bar{z}_r(t) \tag{9.67}$$

Having now the solution which gives the response of the rth uncoupled mode to the time-dependent force $Z_r(t)$, we may proceed through the coordinate transformations to find the response to the generalized forces $\{Q(t)\}$ in terms of the generalized displacement vector $\{q(t)\}$. The transformation Eq. (9.56) may, for this purpose, be written in the convenient form

$$\{y(t)\} = \sum_{r=1}^{n} (\{y^{(r)}\} z_r(t) + \{\bar{y}^{(r)}\} \bar{z}_r(t))$$

It can be seen that $\{y(t)\}$ is real and may be written as twice the real part of the complex number on the right side of this equation. Thus, we may write

$$\{y(t)\} = 2 \sum_{r=1}^{n} \mathcal{R}\{y^{(r)}\} z_r(t) \tag{9.68}$$

where $\mathcal{R}$ stands for "the real part of." Next, we use the transformation connected with Eq. (9.11) relating $\{y\}$ and $\{q\}$, which is rewritten for convenience.

$$\{y\} = \begin{Bmatrix} \{\dot{q}\} \\ \hline \{q\} \end{Bmatrix} \tag{9.69}$$

Applying Eq. (9.13) we see that the rth eigenvector may be written in the form

$$\{y^{(r)}\} = \begin{Bmatrix} p_r\{q^{(r)}\} \\ \hline \{q^{(r)}\} \end{Bmatrix} \tag{9.70}$$

If we substitute Eqs. (9.69) and (9.70) into (9.68), we can write the displacement part of the response vector $\{y\}$ in terms of eigenvector $\{q^{(r)}\}$ which is identical to the lower half of the eigenvector $\{y^{(r)}\}$.

$$\{q(t)\} = 2 \sum_{r=1}^{n} \mathcal{R}\{q^{(r)}\} z_r(t) \tag{9.71}$$

Substituting Eq. (9.65) the solution has the form

$$\{q(t)\} = 2 \sum_{r=1}^{n} \mathcal{R} \frac{\{q^{(r)}\}}{\mathcal{A}_r} \int_0^t e^{p_r(t-\tau)} Z_r(\tau)\, d\tau \tag{9.72}$$

The force Z_r is expressed in terms of the force vector $\{Q\}$ by use of the definition associated with Eqs. (9.62)

$$Z_r = \begin{Bmatrix} p_r\{q^{(r)}\} \\ \{q^{(r)}\} \end{Bmatrix}^T \begin{Bmatrix} \{0\} \\ \{Q\} \end{Bmatrix} = \{q^{(r)}\}^T\{Q\}$$

$$= \sum_{k=1}^{n} q_k^{(r)} Q_k(\tau) \tag{9.73}$$

Substituting this into Eq. (9.72) gives

$$\{q(t)\} = 2 \sum_{r=1}^{n} \mathcal{R} \frac{\{q^{(r)}\}}{\mathcal{A}_r} \sum_{k=1}^{n} q_k^{(r)} \int_0^t e^{p_r(t-\tau)} Q_k(\tau)\, d\tau \tag{9.74}$$

To carry the solution further it will be convenient to introduce the following phase angles. Let

$$\mathcal{A}_r = |\mathcal{A}_r| e^{i\Theta^{(r)}}$$

$$q_j^{(r)} = |q_j^{(r)}| e^{i\theta_j^{(r)}}$$

$$q_k^{(r)} = |q_k^{(r)}| e^{i\theta_k^{(r)}}$$

Using these expressions and Eq. (9.23) the jth component of vector $\{q(t)\}$ in Eq. (9.74) may be expressed in the following way.

$$q_j(t) = 2 \sum_{r=1}^{n} \mathcal{R} \frac{|q_j^{(r)}|}{|\mathcal{A}_r|} e^{i\theta_j^{(r)}} e^{-i\Theta^{(r)}} \sum_{k=1}^{n} |q_k^{(r)}| e^{i\theta_k^{(r)}} \int_0^t e^{\alpha_r(t-\tau)} e^{i\beta_r(t-\tau)} Q_k(\tau)\, d\tau$$

Collecting the phase angles and expressing only the real part of the complex result we obtain

$$q_j(t) =$$

$$2 \sum_{r=1}^{n} \frac{|q_j^{(r)}|}{|\mathcal{A}_r|} \sum_{k=1}^{n} |q_k^{(r)}| \int_0^t e^{\alpha_r(t-\tau)} \cos\,[\beta_r(t-\tau) - \Theta^{(r)} + \theta_j^{(r)} + \theta_k^{(r)}]\, Q_k(\tau)\, d\tau \tag{9.75}$$

This equation expresses, in real form, the displacement response to the time-dependent generalized forces.

9.10 Example

An example is included in this concluding section of the chapter to illustrate the methods already discussed. Because of the lengthy calculations involved in a complete solution, only portions of the solution are shown. A simple frame structure is selected in which two columns are bridged by a single girder. Damping in the columns is considered to be very small and is neglected, while that in the girder is great enough to be significant. Only two degrees of freedom are considered.

The mass, damping, and stiffness matrices to be included in Eq. (9.2) are given below without showing the calculations leading to them.

$$[m] = ml\begin{bmatrix} 1.38181 & 0.00972 \\ 0.00972 & 0.01808 \end{bmatrix}$$

$$[c] = \frac{\zeta}{l}\sqrt{mEI}\begin{bmatrix} 1.79676 & 0.59890 \\ 0.59890 & 0.19963 \end{bmatrix}$$

$$[k] = \frac{EI}{l^3}\begin{bmatrix} 5.45451 & -0.000085 \\ -0.000085 & 1.52735 \end{bmatrix}$$

where

m = mass per unit length of girder (used as reference)
l = length of the girder (used as reference)
EI = bending modulus of the girder (used as reference)
ζ = damping factor

The generalized force is given as

$$\{Q(t)\} = \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix}$$

where $F(t)$ = a time-dependent force applied to the frame horizontally at the level of the girder.

It is convenient to express displacements, velocities, and accelerations in consistent dimensions by using a time constant T defined as

$$\frac{1}{T} = \sqrt{\frac{EI}{ml^4}}$$

Using this constant and taking the damping factor ζ to be 0.1, Eq. (9.2) appears as follows for this example.

$$\begin{bmatrix} 1.38181 & 0.00972 \\ 0.00972 & 0.01808 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 0.17968 & 0.05989 \\ 0.05989 & 0.01996 \end{bmatrix}\begin{Bmatrix} \dot{q}_1/T \\ \dot{q}_2/T \end{Bmatrix}$$
$$+ \begin{bmatrix} 5.45451 & -0.000085 \\ -0.000085 & 1.52735 \end{bmatrix}\begin{Bmatrix} q_1/T^2 \\ q_2/T^2 \end{Bmatrix} = \frac{F(t)}{ml}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

We next form the reduced equation (9.11) in which

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} \dot{q}_1/T \\ \dot{q}_2/T \\ q_1/T^2 \\ q_2/T^2 \end{Bmatrix}$$

Matrices $[A]$ and $[B]$ are given below.

$$[A] = \left[\begin{array}{cc|cc} 0 & 0 & 1.38181 & 0.00972 \\ 0 & 0 & 0.00972 & 0.01808 \\ \hline 1.38181 & 0.00972 & 0.17968 & 0.05989 \\ 0.00972 & 0.01808 & 0.05989 & 0.01996 \end{array}\right]$$

$$[B] = \left[\begin{array}{cc|cc} -1.38181 & -0.00972 & 0 & 0 \\ -0.00972 & -0.01808 & 0 & 0 \\ \hline 0 & 0 & 5.45451 & -0.000085 \\ 0 & 0 & -0.000085 & 1.52735 \end{array}\right]$$

First, the homogeneous equation is solved by putting it in the form of Eq. (9.15). The dynamical matrix is

$$[D] = \left[\begin{array}{cc|cc} 0 & 0 & 1.00000 & 0 \\ 0 & 0 & 0 & 1.00000 \\ \hline -0.25333 & -0.00178 & -0.03294 & -0.01098 \\ -0.00638 & -0.01184 & -0.03921 & -0.01307 \end{array}\right]$$

Iteration on the first mode follows the procedure of Section 9.6 and the successive columns representing the real vector $^n\{\xi\}$ are tabulated below, followed by a tabulation of the calculations required by Eqs. (9.50) and (9.51). In applying Eqs. (9.50) and (9.51), the third component ξ_3 of $^n\{\xi\}$ is used, but any other component could be selected for this purpose.

$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
0	100	−3.294	−25.18144	1.67595	6.31318	−0 63536	−1 57556
0	0	−3.921	−0.45759	1.06079	0 08650	−0.27192	−0 01284
100	−3.294	−25.18144	1.67595	6.31318	−0 63536	−1.57556	0.21348
0	−3.921	−0.45759	1.06079	0.08650	−0.27192	−0 01284	0 06922

①	②	③	④	⑤	⑥
n	ξ_3	$^n\xi_3\,^{n+2}\xi_3 - {}^{n+1}\xi_3^2$	③$^{n+1}$/③^n $= \mu^2 + \nu^2$ (See Eq. 9 50)	$^n\xi_3\,^{n+3}\xi_3 - {}^{n+1}\xi_3\,^{n+2}\xi_3$	⑤/③ $= 2\mu$ (See Eq 9 51.)
1	100	−2528.99444	0 252917	84 6473366	−0.033471
2	−3 294	−639 62550	0 252935	21.407219	−0.033468
3	−25.18144	−161.78377	0 252937	5 4187057	−0 0334935
4	1 67595	−40.92107	0.252938	1.3705823	−0 0334933
5	6.31318	−10.35048	0 252938	0.3466899	−0 033495
6	−0 63536	−2.61803			
7	−1.57556				
8	0 21348				

From these results we find the real and imaginary parts μ_1 and ν_1, respectively, of the eigenvalue λ_1.

$$\mu_1 = -0.016747$$
$$\nu_1 = \pm 0.50265$$

From Eq. (9.24) we may find the real and imaginary parts of p_1, and write

$$p_1 \frac{t}{T} = (-0.0662 \pm 1.987\,i)\sqrt{\frac{EI}{ml^4}}\,t$$

The two iterated columns under $n = 5$ and $n = 6$ are used in subsequent calculations. In fact, they are used at this point to construct a sweeping matrix $[S]$ so that we may next converge by iteration on the second mode. Without showing all the details of the calculation these columns are used in Eqs. (9.55) to find the two following equations.

$$8.72446\,\xi_1 + 0.062928\,\xi_2 + 3.46569\,\xi_3 + 0.41529\quad \xi_4 = 0$$
$$-0.88059\,\xi_1 - 0.011092\,\xi_2 + 8.59401\,\xi_3 + 0.019449\,\xi_4 = 0$$

Solving for ξ_1 and ξ_3 in terms of ξ_2 and ξ_4, we are led to the following sweeping matrix.

$$[S] = \begin{bmatrix} 0 & -0.0074234 & 0 & -0.0448751 \\ 0 & 1.00000 & 0 & 0 \\ 0 & 0.0005300 & 0 & -0.0068612 \\ 0 & 0 & 0 & 1.00000 \end{bmatrix}$$

The constrained dynamical matrix is formed by taking the product $[D]\,[S]$. Iteration on this matrix will converge to the second mode. The second iteration is not shown but the results are

$$\mu_2 = -0.00625735$$
$$\nu_2 = \pm 0.108509$$
$$p_2 \frac{t}{T} = (-0.52968 \pm 9.185\,i)\sqrt{\frac{EI}{ml^4}}\,t$$

$${}^5\{\xi^{(2)}\} = \begin{Bmatrix} -0.008194 \\ 0.293722 \\ -0.000788 \\ 0.134031 \end{Bmatrix} \quad \text{and} \quad {}^6\{\xi^{(2)}\} = \begin{Bmatrix} -0.000764 \\ 0.134031 \\ 0.000107 \\ -0.005147 \end{Bmatrix}$$

At this point we have all the information necessary to determine the imaginary parts of the two eigenvectors $\{y^{(1)}\}$ and $\{y^{(2)}\}$. Using Eq. (9.44) we find them to be

$${}^5\{\eta^{(1)}\} = \pm \begin{Bmatrix} -12.61563 \\ -0.20743 \\ 1.05368 \\ 0.53809 \end{Bmatrix} \quad \text{and} \quad {}^5\{\eta^{(2)}\} = \pm \begin{Bmatrix} 0.007513 \\ -1.252144 \\ -0.000938 \\ 0.039707 \end{Bmatrix}$$

Therefore, the two complex eigenvectors selected from column 5 of the iteration on each of the two modes are

$$\{y^{(1)}\} = \begin{Bmatrix} 1.67595 \\ 1.06079 \\ 6.31318 \\ 0.08650 \end{Bmatrix} \pm i \begin{Bmatrix} -12.61563 \\ -0.20743 \\ 1.05368 \\ 0.53809 \end{Bmatrix}$$

$$\{y^{(2)}\} = \begin{Bmatrix} -0.008194 \\ 0.293722 \\ -0.000788 \\ 0.134031 \end{Bmatrix} \pm i \begin{Bmatrix} 0.007513 \\ -1.252144 \\ -0.000938 \\ 0.039707 \end{Bmatrix}$$

In order to use these eigenvectors in the response equations (9.75), it is necessary to represent them in polar form. They are given as

$$\begin{aligned}
y_1^{(1)} &= 12.72647 \ \underline{/277^\circ 34'} = \dot{q}_1^{(1)}/T \\
y_2^{(1)} &= 1.08088 \ \underline{/348^\circ 56'} = \dot{q}_2^{(1)}/T \\
y_3^{(1)} &= 6.40051 \ \underline{/\ \ 9^\circ 29'} = q_1^{(1)}/T^2 \\
y_4^{(1)} &= 0.54500 \ \underline{/\ 80^\circ 52'} = q_2^{(1)}/T^2
\end{aligned}$$

$$\begin{aligned}
y_1^{(2)} &= 0.011117 \ \underline{/137^\circ 29'} = \dot{q}_1^{(2)}/T \\
y_2^{(2)} &= 1.286133 \ \underline{/283^\circ 12'} = \dot{q}_2^{(2)}/T \\
y_3^{(2)} &= 0.001225 \ \underline{/229^\circ 57'} = q_1^{(2)}/T^2 \\
y_4^{(2)} &= 0.139789 \ \underline{/\ 16^\circ 30'} = q_2^{(2)}/T^2
\end{aligned}$$

The next step is to compute the two numbers $\mathcal{a}_1$ and $\mathcal{a}_2$ from Eq. (9.60). This computation is straightforward and results in the following values for this example.

$$\begin{aligned}
\mathcal{a}_1 &= 73.20744 - 212.40479\, i \\
&= 224.6667 \ \underline{/289^\circ 1'} \\
\mathcal{a}_2 &= 0.0035234 - 0.005421\, i \\
&= 0.0064654 \ \underline{/303^\circ 1'}
\end{aligned}$$

All information is now available to substitute into Eq. (9.75) and thus, to write the response equations.

$$\begin{aligned}
q_1(t) = \frac{2l^3}{EI}\Bigg\{0.18234 \int_0^{t/T} e^{-0.06620(t-\tau)/T} \cos\left[1.987\left(\frac{t-\tau}{T}\right) - 270^\circ\ 3'\right] F(\tau)\frac{d\tau}{T} \\
+ 0.000232 \int_0^{t/T} e^{-0.52968(t-\tau)/T} \cos\left[9.185\left(\frac{t-\tau}{T}\right) + 156^\circ 53'\right] F(\tau)\frac{d\tau}{T}\Bigg\}
\end{aligned}$$

$$q_2(t) = \frac{2l^3}{EI}\left\{0.01553 \int_0^{t/T} e^{-0.00620(t-\tau)/T} \cos\left[1.987\left(\frac{t-\tau}{T}\right) - 198°40'\right] F(\tau)\frac{d\tau}{T}\right.$$
$$\left. + 0.02649 \int_0^{t/T} e^{-0.52968(t-\tau)/T} \cos\left[9.185\left(\frac{t-\tau}{T}\right) - 56°34'\right] F(\tau)\frac{d\tau}{T}\right\}$$

In reviewing Eq. (9.75) in the light of this example, it is seen that all q's are divided by the square of the time constant T. Also, in developing the equations of motion, we divided through by the constant ml so that the following quantity appears as a factor on the right side of the above equations.

$$\frac{T^2}{ml} = \frac{1}{ml} \cdot \frac{ml^4}{EI} = \frac{l^3}{EI}$$

CHAPTER 10

Frequency Response Method

10.1 Introduction

In Chapters 8 and 9 we demonstrated how modes characterizing a dynamical system can be used to calculate its response to an excitation. The development in Chapter 8 was limited to systems with proportional damping in which the damping matrix $[c]$ is proportional to either the stiffness or mass matrix

$$[c] = 2\beta[m]$$

$$[c] = \alpha[k]$$

while Chapter 9 dealt with nonproportional damping.

In this chapter we shall develop an alternate method of calculating the response of a system to exciting disturbances. We label this alternate method as the *frequency response method* because we characterize the system by its response as related to the frequency of a simple harmonic exciting force function. This frequency response characteristic is used to compute the response to periodic as well as nonperiodic excitations. In Chapter 11 we shall make use of the frequency response concept to study the response of a system to a random excitation. In order to fix ideas we shall treat first a single-degree-of-freedom system.

10.2 The Frequency Response Method Applied to a Single-Degree-of-Freedom System

Response of a System With Viscous Damping. The single-degree-of-freedom system of Fig. 10.1 is subjected to a force excitation $F(t)$. The differential equation of motion of the system is

$$m\ddot{u} + c\dot{u} + ku = F(t) \quad (10.1)$$

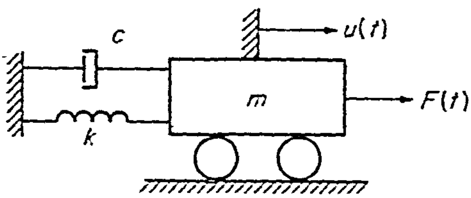


Figure 10.1

For convenience and consistency of the development let

$$F(t) = kf(t) \tag{10.2}$$

so that when $F(t)$ is applied statically to the system of Fig. 10.1 the resulting displacement becomes

$$u = \frac{F(t)}{k} = f(t)$$

Let us also define a dimensionless parameter ζ by

$$\zeta = \frac{c}{2\sqrt{km}} \tag{10.3}$$

where c is the viscous damping coefficient of the system and $2\sqrt{km}$ is the value of the critical damping for the same system (see Section 7.9, Chapter 7). Substituting Eq. (10.2) into (10.1) and dividing through by m we have

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = \frac{k}{m}f(t)$$

If we designate the undamped natural frequency of the system by ω

$$\omega = \sqrt{\frac{k}{m}}$$

we have from relation (10.3)

$$\frac{c}{m} = 2\zeta\omega$$

Using the last two relations the differential equation of motion written above becomes

$$\ddot{u} + 2\zeta\omega\dot{u} + \omega^2 u = \omega^2 f(t) \tag{10.3a}$$

The Laplace transform of this equation is

$$s^2u(s) - su(0) - \dot{u}(0) + 2\zeta\omega su(s) - 2\zeta\omega u(0) + \omega^2 u(s) = \omega^2 f(s)$$

where $u(0)$ and $\dot{u}(0)$ are, respectively, the initial displacement and velocity of the system. Rearranging we write

$$u(s)(s^2 + 2\zeta\omega s + \omega^2) = \omega^2 f(s) + (s + 2\zeta\omega)u(0) + \dot{u}(0)$$

or

$$u(s) = \frac{1}{Z(s)}\omega^2 f(s) + \frac{1}{Z(s)}(s + 2\zeta\omega)u(0) + \frac{1}{Z(s)}\dot{u}(0) \qquad (10.4)$$

where $Z(s)$ is the system *impedance* defined by

$$Z(s) = s^2 + 2\zeta\omega s + \omega^2 \qquad (10.5)$$

The response $u(t)$ is computed by finding the inverse transform of the three terms on the right-hand side of Eq. (10.4). In the following we will develop the inverse transform of the first term containing the exciting force function. The steps leading to the response to the initial displacement and velocity will be apparent. Setting in Eq. (10.4)

$$u(0) = \dot{u}(0) = 0$$

we have

$$u(s) = \frac{1}{Z(s)}\omega^2 f(s) \qquad (10.6)$$

In the present development we are interested in finding the response to a simple harmonic excitation, consequently, we write for the force function

$$f(t) = e^{i\Omega t}$$

The Laplace transform of this time function is given by

$$f(s) = \frac{1}{s - i\Omega}$$

hence, Eq. (10.6) becomes

$$u(s) = \frac{\omega^2}{Z(s)(s - i\Omega)} \qquad (10.7)$$

The inverse transform of Eq. (10.7) can be obtained by the convolution integral (see Chapter 8, Section 8). However, to develop the concepts of this chapter we shall use the alternate approach of partial fractions. In this approach we write the right-hand side of Eq. (10.7) as a sum of partial fractions,[12] find their inverse transform and sum to obtain $u(t)$. Before proceeding this way we factor the impedance in the form

$$Z(s) = (s - s_1)(s - s_2)$$

This can be accomplished by finding the roots s_1, s_2 of the equation

$$Z(s) = 0$$

Alternately, we can find the factors $(s - s_1)$ and $(s - s_2)$ by completing the square in $Z(s)$ and extracting the roots. Adding and subtracting $(\zeta\omega)^2$ to $Z(s)$ as expressed by Eq. (10.5) we write

$$\begin{aligned} Z(s) &= s^2 + 2\zeta\omega s + (\zeta\omega)^2 + \omega^2 - (\zeta\omega)^2 \\ &= (s + \zeta\omega)^2 + \omega^2(1 - \zeta^2) \\ &= [s + \zeta\omega + i\omega(1 - \zeta^2)^{1/2}][s + \zeta\omega - i\omega(1 - \zeta^2)^{1/2}] \end{aligned}$$

or

$$Z(s) = (s - s_1)(s - s_2)$$

where

$$s_1 = -\zeta\omega - i\omega(1 - \zeta^2)^{1/2}$$

and

$$s_2 = -\zeta\omega + i\omega(1 - \zeta^2)^{1/2} \tag{10.8}$$

We can now expand the right-hand side of Eq. (10.7) as a sum of partial fractions, each partial fraction having in its denominator one of the factors of $Z(s)(s - i\Omega)$.

$$\frac{\omega^2}{(s - s_1)(s - s_2)(s - s_3)} = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \frac{A_3}{s - s_3} \tag{10.9}$$

where s_1 and s_2 are given by Eq. (10.8) and $s_3 = i\Omega$. The constants A_1, A_2, A_3 in this expansion are determined as follows. Multiply Eq. (10.9) by the denominator of the partial fraction containing the constant which is being computed, then set this denominator equal to zero and evaluate the constant.* We demonstrate this procedure by computing A_1, A_2 and A_3 of Eq. (10.9).

To compute A_1 multiply Eq. (10.9) by $(s - s_1)$

$$\frac{\omega^2}{(s - s_2)(s - s_3)} = A_1 + \frac{A_2}{s - s_2}(s - s_1) + \frac{A_3}{s - s_3}(s - s_1)$$

then set

$$s - s_1 = 0$$

or

$$s = s_1$$

and solve for A_1 to yield

$$\begin{aligned} A_1 &= \frac{\omega^2}{(s_1 - s_2)(s_1 - s_3)} \\ &= \frac{\omega^2}{-i2\omega(1 - \zeta^2)^{1/2}\{-\zeta\omega - i[\omega(1 - \zeta^2)^{1/2} + \Omega]\}} \end{aligned}$$

We recall that s_1, s_2, s_3 are the roots of the denominator on the right-hand side of Eq. (10.7) when it is set equal to zero

$$Z(s)(s - i\Omega) = 0$$

*The procedure is somewhat different when multiple roots s_i exist (see Reference 31).

The roots of this equation are also referred to as the poles of $u(s)$ in Eq. (10.7).

To obtain A_2 we multiply Eq. (10.9) by $(s - s_2)$ and then set it equal to zero (which determines a pole at $s = -\zeta\omega + i\omega(1 - \zeta^2)^{1/2}$) and solve for A_2

$$A_2 = \frac{\omega^2}{i2\omega(1 - \zeta^2)^{1/2}\{-\zeta\omega + i[\omega(1 - \zeta^2)^{1/2} - \Omega]\}}$$

For A_3 we multiply Eq. (10.9) by $(s - s_3)$ then set

$$s - i\Omega = 0$$

which determines a pole at

$$s = i\Omega$$

and write

$$A_3 = \frac{\omega^2}{Z(i\Omega)} = \frac{\omega^2}{\omega^2 + i2\zeta\omega\Omega - \Omega^2} \tag{10.10}$$

With A_1, A_2, A_3 evaluated, we write the inverse transform for each term on the right-hand side of Eq. (10.9) from the relation

$$\mathcal{L}^{-1}\left(\frac{A_j}{s - s_j}\right) = A_j e^{s_j t}$$

Each term $A_j e^{s_j t}$ is also referred to as the residue at the pole $s = s_j$. The response $u(t)$ is computed by summing the residues at all the poles s_j

$$\begin{aligned} u(t) &= \sum_{j=1}^{3} A_j e^{s_j t} \\ &= A_1 e^{-\zeta\omega t} e^{-i\omega(1-\zeta^2)^{1/2} t} + A_2 e^{-\zeta\omega t} e^{i\omega(1-\zeta^2)^{1/2} t} + A_3 e^{i\Omega t} \end{aligned} \tag{10.11}$$

Transient and Steady-State Response. The first two terms on the right-hand side of Eq. (10.11) represent the *transient* part of the response. The transient response, for damping below the critical damping of the system, is an oscillation with frequency $\omega\sqrt{1 - \zeta^2}$ and an amplitude which decreases with time as is apparent from the term $e^{-\zeta\omega t}$. This was discussed in Chapter 7, Section 9. The response to the initial displacement $u(0)$ and initial velocity $\dot{u}(0)$ is also transient.* This can be verified by setting $f(s) = 0$, $u(0) \neq 0$, $\dot{u}(0) \neq 0$ in Eq. (10.4) and computing the response by following the steps that led to Eq. (10.11). The transient portion of the response disappears with time for all practical purposes. The term

$$A_3 e^{i\Omega t}$$

on the right-hand side of Eq. (10.11) represents the *steady state* portion of the response. The steady-state vibration has the frequency

*See Eq. (7.43), Chapter 7.

Ω of the exciting force function $f(t)$ and it is maintained at a constant amplitude A_3 as long as $f(t)$ persists.

The Complex Frequency Response $H(\Omega)$. We now define the *complex frequency response* $H(\Omega)$ of the system of Fig. 10.1 such that for an excitation $F(t) = kf(t)$ in which $f(t)$ is a simple harmonic function

$$f(t) = e^{i\Omega t}$$

the steady-state response of the system is given by

$$u(t) = H(\Omega)f(t) \tag{10.12}$$

From Eq. (10.11) this response is given by

$$u(t) = A_3 f(t)$$

hence

$$H(\Omega) = A_3$$

Substituting for A_3 from Eq. (10.10) we write

$$H(\Omega) = \frac{\omega^2}{Z(i\Omega)} = \frac{1}{1 - \left(\frac{\Omega}{\omega}\right)^2 + i2\zeta\frac{\Omega}{\omega}} \tag{10.13}$$

Equation (10.13) establishes a relation between the complex frequency response $H(\Omega)$ and the system impedance $Z(s)$ so that either concept can be used in the frequency response method. We recall from the early part of this section that $f(t)$ is the displacement of the system when $F(t)$ is applied statically. Then from Eq. (10.12), $H(\Omega)$ can be thought of as a *magnification factor* representing the complex ratio of the steady-state dynamic displacement to the displacement which results when the exciting force is applied statically.

The frequency response $H(\Omega)$ contains also information regarding the phase lag of the response with respect to the exciting force function. We can write $H(\Omega)$ as the vector sum of a real part $R(\Omega)$, and an imaginary part $I(\Omega)$.

$$H(\Omega) = R(\Omega) + iI(\Omega)$$

or

$$H(\Omega) = |H(\Omega)|e^{i\psi} \tag{10.14}$$

in which $|H(\Omega)|$ and ψ designate, respectively, the amplitude and phase of the response $u(t)$ of the system of Fig. 10.1 when it is excited by the simple harmonic forcing function $F(t) = ke^{i\Omega t}$. Hence, substituting Eq. (10.14) into (10.12) we have

$$u(t) = |H(\Omega)|e^{i(\Omega t + \psi)} \tag{10.15}$$

$|H(\Omega)|$ and ψ are computed from expression (10.13). Multiplying both the numerator and the denominator of this expression by $1 - (\Omega/\omega)^2 - i2\zeta(\Omega/\omega)$ we have

$$H(\Omega) = \frac{1}{\left[1 - \left(\frac{\Omega}{\omega}\right)^2\right]^2 + \left(2\zeta\frac{\Omega}{\omega}\right)^2}\left[1 - \left(\frac{\Omega}{\omega}\right)^2 - i2\zeta\frac{\Omega}{\omega}\right]$$

from which the magnitude of $H(\Omega)$ is found to be

$$|H(\Omega)| = \left\{\left[1 - \left(\frac{\Omega}{\omega}\right)^2\right]^2 + \left(2\zeta\frac{\Omega}{\omega}\right)^2\right\}^{-1/2} \tag{10.16a}$$

and the phase angle

$$\psi = \tan^{-1} - \frac{2\zeta\frac{\Omega}{\omega}}{1 - \left(\frac{\Omega}{\omega}\right)^2} \tag{10.16b}$$

Response of a System With Structural Damping. When the viscous damping in Fig. 10.1 is replaced by structural damping with coefficient g, the differential equation of motion becomes

$$\ddot{u} + \omega^2(1 + ig)u = \omega^2 f(t) \tag{10.17}$$

and the complex frequency response is written as

$$H(\Omega) = \frac{1}{1 - \left(\frac{\Omega}{\omega}\right)^2 + ig} \tag{10.18}$$

For an exciting force function

$$f(t) = e^{i\Omega t}$$

the steady-state response $u(t)$ is given by

$$\begin{aligned} u(t) &= H(\Omega)f(t) \\ &= |H(\Omega)|e^{i\psi}e^{i\Omega t} \\ &= |H(\Omega)|e^{i(\Omega t + \psi)} \end{aligned}$$

where

$$|H(\Omega)| = \left\{\left[1 - \left(\frac{\Omega}{\omega}\right)^2\right]^2 + g^2\right\}^{-1/2} \tag{10.19a}$$

and

$$\psi = \tan^{-1} - \frac{g}{1 - \left(\frac{\Omega}{\omega}\right)^2} \tag{10.19b}$$

Equation (10.17) requires some discussion in the light of the concepts of transient and steady-state response introduced earlier when a system with viscous damping was considered. The formulation of equations of motion with structural damping was discussed in Section 7.4 of Chapter 7. There we stated that the structural damping force for a system with structural damping, is valid only for the steady-state response of the system.

Response to Nonharmonic Forces. When the forcing function $F(t)$ in Fig. 10.1 is periodic but not harmonic it may be resolved into a set of harmonic components by the use of the Fourier series technique. The response of the system is then computed by summing its responses to the harmonic components of the exciting force. When the exciting force function $F(t)$ is nonperiodic it may be synthesized from harmonic components whose frequencies form a continuous spectrum. This synthesis is accomplished by the use of the Fourier integral. We shall return to discuss these techniques in more detail later in the chapter in connection with multi-degree-of-freedom systems.

10.3 The Frequency Response Method Applied to Multi-Degree-of-Freedom Systems

We now extend the concepts of the last section to a multi-degree-of-freedom system. Considering a system with viscous damping which is subjected to externally applied exciting force functions, we can write the equations of motion in generalized coordinates q_i $(i = 1, 2, \ldots, n)$ in the form†

$$[m]\{\ddot{q}\} + [c]\{\dot{q}\} + [k]\{q\} = \{Q\} \tag{10.21}$$

[See Equation (7.29), Chapter 7]

If we allow the externally applied generalized forces $\{Q\}$ to be time dependent in an arbitrary manner we cannot postulate a priori any conclusion concerning the time dependence of the q's. To be explicit we shall admit force functions $Q(t)$ which are single valued, continuous (they may be sectionally continuous), and of exponential order. Then we may apply the Laplace transform technique to the solution of Eqs. (10.21) and write

$$\begin{aligned} s^2[m]\{q(s)\} &+ s[c]\{q(s)\} + [k]\{q(s)\} \\ &= \{Q(s)\} + s[m]\{q(0)\} + [m]\{\dot{q}(0)\} + [c]\{q(0)\} \end{aligned} \tag{10.22}$$

where $q(0)$ and $\dot{q}(0)$ are, respectively, the displacement and velocity at time $t = 0$.

We now define an *impedance matrix* $[Z(s)]$ by

$$[Z(s)] = s^2[m] + s[c] + [k] \tag{10.23}$$

This is analogous to the impedance of a single d.o.f. system defined

†Note that we do not postulate proportionality of the damping matrix to the stiffness or mass matrix.

by Eq. (10.5). However, note that in Eq. (10.5) m has been divided out so that s^2 is free of m while here $[m]$ multiplies s^2. This is done here for convenience to avoid carrying $[m]^{-1}$ in many terms. Using Eq. (10.23) in (10.22) we write

$$[Z(s)]\{q(s)\} = \{Q(s)\} + (s[m] + [c])\{q(0)\} + [m]\{\dot{q}(0)\} \qquad (10.24)$$

The solution of Eq. (10.24) in transform is obtained by premultiplying the equation by the inverse of the impedance matrix.

$$\begin{aligned}\{q(s)\} = [Z(s)]^{-1}\{Q(s)\} + [Z(s)]^{-1}(s[m] + [c])\{q(0)\} \\ + [Z(s)]^{-1}[m]\{\dot{q}(0)\}\end{aligned} \qquad (10.25)$$

The response $\{q(t)\}$ is evaluated by finding the inverse transforms of the three terms on the right-hand side of Eq. (10.25). The first term on the right involves the forcing functions while the second and third terms contain the initial displacements and velocities, respectively. For simplicity and clarity we shall investigate the response of an undamped system, in which case we substitute $[c] = [0]$ in Eq. (10.25) and write

$$\{q(s)\} = [Z(s)]^{-1}\{Q(s)\} + s[Z(s)]^{-1}[m]\{q(0)\} + [Z(s)]^{-1}[m]\{\dot{q}(0)\} \qquad (10.26)$$

where

$$[Z(s)] = s^2[m] + [k] \qquad (10.27)$$

Our primary purpose in this chapter is to find the inverse transform of the first term involving the force functions in Eq. (10.26) or (10.25). However, before proceeding to that investigation it is instructive to determine the response of an undamped structure [Eq. (10.26)] to initial displacements and velocities.

10.4 Response of an Undamped System to Initial Displacements

When the externally applied loads and the initial velocities of the undamped system are set equal to zero, Eq. (10.26) reduces to

$$\{q(s)\} = s[Z(s)]^{-1}[m]\{q(0)\} \qquad (10.28)$$

The inverse transform of this equation yields the response $\{q(t)\}$ of the undamped structure following release from initial displacements $\{q(0)\}$. The inverse of the impedance matrix $[Z(s)]$ can be written as

$$[Z(s)]^{-1} = \frac{1}{|Z(s)|}[C(s)] \qquad (10.29)$$

where $|Z(s)|$ is the determinant of $[Z(s)]$ and $[C(s)]$ is its adjoint matrix. Using (10.29), Eq. (10.28) becomes

$$\{q(s)\} = \frac{1}{|Z(s)|} s[C(s)][m]\{q(0)\} \tag{10.30}$$

We now proceed to compute the *inverse transform of Eq. (10.30) by* expanding the right-hand side as a sum of partial fractions, each partial fraction having as its denominator one of the factors of $|Z(s)|$ when it is factored in the form

$$|Z(s)| = A(s - s_1)(s - s_2)\ldots(s - s_r)\ldots(s - s_{2n}) \tag{10.31}$$

A is a constant determined by the elements of the mass matrix which multiply s^2 in the impedance matrix [see Eqs. (10.23) and (10.27)]. The subscript n designates the number of degrees of freedom of the vibrating system. The reason for the $2n$ factors will become apparent later. We recall that in the single d.o.f. system treated in Section 10.2, $Z(s)$ had two factors.

Using Eq. (10.31) we expand the right-hand side of Eq. (10.30) in the form

$$\begin{aligned}\frac{1}{|Z(s)|} s[C(s)][m]\{q(0)\} \\ = \frac{\{A_1\}}{s - s_1} + \frac{\{A_2\}}{s - s_2} + \cdots + \frac{\{A_r\}}{s - s_r} + \cdots + \frac{\{A_{2n}\}}{s - s_{2n}}\end{aligned} \tag{10.32}$$

Each column of constants $\{A_r\}$ is computed by multiplying Eq. (10.32) by its denominator $(s - s_r)$, then setting $(s - s_r) = 0$, as was demonstrated in Section 10.2. The response $\{q(t)\}$ is obtained by summing the inverse transforms of the partial fractions on the right-hand side of Eq. (10.32) (or what is also referred to as summing the residues at the poles s_r of $\{q(s)\}$) where

$$\mathcal{L}^{-1}\left(\frac{\{A_r\}}{s - s_r}\right) = \{A_r\}e^{s_r t}$$

Hence

$$\{q(t)\} = \sum_{r=1}^{2n} \{A_r\}e^{s_r t} \tag{10.33}$$

We shall now evaluate the poles s_r and the columns $\{A_r\}$ which appear on the right-hand side of Eq. (10.33). The poles s_r are computed from Eq. (10.31) when it is set equal to zero

$$|Z(s)| = A(s - s_1)(s - s_2)\ldots(s - s_{2n}) = 0 \tag{10.34}$$

We recall* that for an undamped system the equations of motion for normal mode vibrations are given by

$$-\omega^2[m]\{q\} + [k]\{q\} = \{0\}$$

or

$$(-\omega^2[m] + [k])\{q\} = \{0\} \tag{10.35}$$

*See Eq. (3.5), Chapter 3.

In view of relation (10.27), Eq. (10.35) may be written in the form

$$[Z(\pm i\omega)]\{q\} = \{0\} \tag{10.36}$$

where $[Z(\pm i\omega)]$ is the impedance matrix with

$$s^2 = -\omega^2$$

or

$$s = \pm i\omega$$

Following the discussion in Sections 3.2 and 3.3 of Chapter 3, the natural frequencies of the system are computed from the frequency equation

$$\begin{aligned}|Z(\pm i\omega)| &= A(s^2 + \omega_1^2)(s^2 + \omega_2^2)\dots(s^2 + \omega_r^2)\dots(s^2 + \omega_n^2)\\ &= A(s - i\omega_1)(s + i\omega_1)(s - i\omega_2)(s + i\omega_2)\dots(s - i\omega_n)(s + i\omega_n) = 0\end{aligned} \tag{10.37}$$

Comparing Eqs. (10.34) and (10.37) we have

$$\begin{aligned} s_1 &= i\omega_1 \\ s_2 &= -i\omega_1 \\ &\vdots \\ s_{2n-1} &= i\omega_n \\ s_{2n} &= -i\omega_n \end{aligned}$$

This explains the $2n$ factors in Eq. (10.31) since the solution to the eigenvalue problem as formulated by Eq. (10.35) yields n eigenvalues each of which is the square of a natural frequency.

Using Eq. (10.37) we write Eq. (10.32) in the form

$$\begin{aligned}&\frac{1}{A(s - i\omega_1)(s + i\omega_1)\dots(s - i\omega_n)(s + i\omega_n)}s[C(s)][m]\{q(0)\}\\ &\quad = \frac{\{A_1\}}{s - i\omega_1} + \frac{\{A_2\}}{s + i\omega_1} + \dots + \frac{\{A_{2r-1}\}}{s - i\omega_r} + \frac{\{A_{2r}\}}{s + i\omega_r} + \dots\\ &\qquad + \frac{\{A_{2n-1}\}}{s - i\omega_n} + \frac{\{A_{2n}\}}{s + i\omega_n}\end{aligned} \tag{10.38}$$

Multiplying Eq. (10.38) by $s - i\omega_1$ and setting $s = i\omega_1$, we obtain $\{A_1\}$. Similarly $\{A_2\}$ is computed. Using the values of $\{A_1\}$ and $\{A_2\}$ we can evaluate the inverse transform of the first two terms on the right of Eq. (10.38)

$$\begin{aligned}&\mathcal{L}^{-1}\left(\frac{\{A_1\}}{s - i\omega_1} + \frac{\{A_2\}}{s + i\omega_1}\right)\\ &\quad = \frac{\left[\dfrac{i\omega_1 e^{i\omega_1 t}}{2i\omega_1} + \dfrac{i\omega_1 e^{-i\omega_1 t}}{2i\omega_1}\right]}{A(-\omega_1^2 + \omega_2^2)(-\omega_1^2 + \omega_3^2)\dots(-\omega_1^2 + \omega_n^2)}[C(\pm i\omega_1)][m]\{q(0)\}\end{aligned}$$

system excited by harmonic forces. However, the steps leading to the response of a damped system will be apparent. Using complex notation we let

$$Q_j(t) = Q_{0j} f_j(t) \tag{10.46}$$

where

$$f_j(t) = e^{i(\Omega_j t - \psi_j)}$$

$Q_j(t)$ is the jth element of column $\{Q(t)\}$, and Q_{0j} is its amplitude. The angular frequency of the jth harmonic force is Ω_j and the phase angle ψ_j is introduced to allow the several forces in the set to differ in phase. The Laplace transform of $Q_j(t)$ is

$$Q_j(s) = Q_{0j} f_j(s) = \frac{Q_{0j} e^{-i\psi_j}}{s - i\Omega_j} \tag{10.47}$$

Setting the initial displacements and velocities equal to zero in Eq. (10.26), we have for the transform of the response to external forces

$$\{q(s)\} = [Z(s)]^{-1}\{Q(s)\} = \frac{1}{|Z(s)|}[C(s)]\{Q(s)\} \tag{10.48}$$

It will be convenient to consider each of the force components separately, therefore Eq. (10.48) will be written in the following form where $\{C(s)^{(j)}\}$ is the jth column of the matrix $[C(s)]$

$$\begin{aligned}\{q(s)\} &= \frac{1}{|Z(s)|}\{C(s)^{(1)}\}Q_1(s) + \frac{1}{|Z(s)|}\{C(s)^{(2)}\}Q_2(s) + \cdots \\ &\quad + \frac{1}{|Z(s)|}\{C(s)^{(n)}\}Q_n(s) \\ &= \sum_{j=1}^{n} \frac{1}{|Z(s)|}\{C(s)^{(j)}\}\frac{Q_{0j}e^{-i\psi_j}}{s - i\Omega_j}\end{aligned} \tag{10.49}$$

It is seen that poles exist at $s = i\Omega_j$ and at $s = \pm i\omega_r$ $(r = 1, 2, \ldots, n)$ where ω_r is the natural frequency of the rth mode. Therefore, the complete solution will contain harmonic components at each of the natural frequencies of the system as well as harmonic components at each frequency Ω_j. If damping is present (as is always the case in physical structures), the vibration components at the natural frequencies will decrease with time and, for practical purposes, disappear. These are the *transient* vibrations discussed in Section 10.2. On the other hand those components having the frequencies of the impressed forces will persist, each component maintaining a constant amplitude. These vibrations are the *steady-state* vibrations (also discussed in Section 10.2). We shall now determine the amplitude and phase of each of the steady-state components. This is accomplished by determining only the residues at $s = i\Omega_j$ and summing. Thus

$$\{q(t)\} = \sum_{j=1}^{n} \frac{Q_{0j}}{|Z(i\Omega_j)|}\{C(i\Omega_j)^{(j)}\}e^{i(\Omega_j t - \psi_j)} \tag{10.50}$$

Inasmuch as the elements of $|Z(s)|$ for an undamped system [Eq. (10.27)] contain s to the second power only, both $|Z(s)|$ and the elements of $[C(s)]$ contain even powers of s. Therefore, $|Z(i\Omega_j)|$ and the elements of $\{C(i\Omega_j)^{(j)}\}$ are real and the force and response components will be either in phase or 180° out of phase. For example, the kth component of $\{q(t)\}$ resulting from the application of the jth force only is

$$q_k(t) = \frac{Q_{0j}}{|Z(i\Omega_j)|} C_{kj}(i\Omega_j) e^{i(\Omega_j t - \epsilon_j)} \tag{10.51}$$

Since the coefficient in the right-hand member of this equation is real, then the response component will be either in phase or 180 degrees out of phase with the force component $Q_j(t)$ according to the sign of the real coefficient

$$\frac{C_{kj}(i\Omega_j)}{|Z(i\Omega_j)|}$$

In practice it is often convenient to define a *dynamic magnification factor* $(MF)_{kj}$. This is the ratio of the maximum dynamic amplitude $q_k(t)_{max}$ of a point in the system when the system is subjected to a harmonic force $Q_j(t)$ of amplitude Q_{0j} to the deflection of that point when the force Q_{0j} is applied very slowly as a static force. The deflection q_k caused by the application of a static force Q_{0j} is

$$q_k(\text{static}) = a_{kj} Q_{0j} \tag{10.52}$$

where a_{kj} is the deflection influence coefficient. Then the dynamic magnification factor is given by

$$(MF)_{kj} = \frac{q_k(t)_{max}}{q_k(\text{static})} = \frac{C_{kj}(i\Omega_j)}{|Z(i\Omega_j)| a_{kj}} \tag{10.53}$$

It is seen from Eq. (10.53) that the response amplitude of an undamped structure subjected to harmonic force excitation tends to infinity as the impedance $|Z(i\Omega_j)|$ vanishes. This occurs when the exciting frequency Ω_j coincides with one of the natural frequencies ω_r of the structure. This can be seen from Eq. (10.37) by substituting

$$s = i\Omega_j = i\omega_r$$

in which case the rth factor becomes

$$(-\omega_r^2 + \omega_r^2) = 0$$

and hence $|Z(i\Omega_j)| = 0$.

When this condition takes place the structure is said to be in a state of *resonance*. In actual structures the presence of damping limits the amplitudes at resonance to finite values as will be shown in a later section of this chapter for a structure in which structural damping is present.

The foregoing method will be applied to the problem of a cantilever beam to which two harmonic forces are applied. It is assumed that no damping exists. We point out, however, that Eqs. (10.48, 10.49, 10.50 and 10.53) of this section are also valid for a system in which damping is present, except that the elements of $[Z(i\Omega_r)]$ and $[C(i\Omega_r)]$ are no longer real numbers. (See Eq. 10.23.)

10.7 Illustrative Example: Steady-State Vibration of a Cantilever Beam

The beam to be considered is shown in Fig. 10.3. For convenience it is considered to be uniform and the mass will be lumped at four

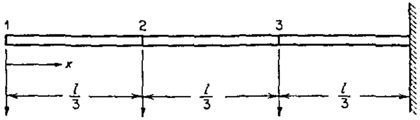


Figure 10.3

equidistant points along its length. Using the trapezoidal rule for lumping the mass, we have

$$m_1 = m_4 = \tfrac{1}{6}M$$
$$m_2 = m_3 = \tfrac{1}{3}M$$

where M is the total mass of the beam. Thus the mass matrix is

$$[m] = \frac{M}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Deflection influence coefficients are computed and the following flexibility matrix results

$$[a] = \frac{l^3}{162\,EI}\begin{bmatrix} 50 & 28 & 8 \\ 28 & 16 & 5 \\ 8 & 5 & 2 \end{bmatrix}$$

By matrix inversion we obtain

$$[k] = \frac{81\,EI}{13\,l^3}\begin{bmatrix} 7 & -16 & 12 \\ -16 & 44 & -46 \\ 12 & -46 & 80 \end{bmatrix}$$

For convenience a parameter μ is defined as

$$\mu = \frac{486\,EI}{13\,Ml^3}$$

Then, in terms of μ, the impedance matrix becomes

$$[Z(s)] = \frac{M}{6}\begin{bmatrix} s^2 + 7\mu & -16\mu & 12\mu \\ -16\mu & 2s^2 + 44\mu & -46\mu \\ 12\mu & -46\mu & 2s^2 + 80\mu \end{bmatrix}$$

Let us consider two forces $Q_1(t)$ and $Q_2(t)$ to be applied to the beam at points 1 and 2, respectively. These forces are harmonic.

$$Q_1(t) = Q_{01}e^{i\Omega_1 t}$$
$$Q_2(t) = Q_{02}e^{i(\Omega_2 t - \epsilon_2)}$$

If we designate the amplitude by $q(t)$ then the Laplace transforms of the responses are given by

$$\{q(s)\} = [Z(s)]^{-1}\{Q(s)\}$$
$$= \frac{1}{|Z(s)|}[C(s)]\{Q(s)\}$$
$$= \sum_{j=1}^{2} \frac{1}{|Z(s)|}\{C(s)^{(j)}\}Q_j(s)$$

The steady-state amplitudes $q(t)$ are, from Eq. (10.50),

$$\{q(t)\} = \sum_{j=1}^{2} \frac{Q_{0j}}{|Z(i\Omega_j)|}\{C(i\Omega_j)^{(j)}\}e^{i(\Omega_j t - \epsilon_j)}$$

For the present problem let us suppose that the force frequencies are given by

$$\Omega_1^2 = \tfrac{1}{8}\mu \qquad \Omega_2^2 = \tfrac{1}{3}\mu$$

It is pertinent to point out that these two frequencies bracket the lowest natural frequency of the beam; the first being about 65% of ω_1 and the second about 5% greater than ω_1.

Next, we consider each of the two forces in turn and calculate the impedance and appropriate cofactors C_{ij} for each. For the force $Q_1(t)$

$$s^2 = -\Omega_1^2 = -\tfrac{1}{8}\mu$$

$$[Z(i\Omega_1)] = \frac{M}{6}\mu\begin{bmatrix} 6.875 & -16 & 12 \\ -16 & 43.75 & -46 \\ 12 & -46 & 79.75 \end{bmatrix}$$

$$\{C(i\Omega_1)^{(1)}\} = \frac{M^2}{6^2}\mu^2\begin{Bmatrix} 1373.06 \\ 724 \\ 211 \end{Bmatrix}$$

$$|Z(i\Omega_1)| = 387.80\,\frac{M^3}{6^3}\mu^3$$

For the force $Q_2(t)$

$$s^2 = -\Omega_2^2 = -\tfrac{1}{3}\mu$$

$$[Z(i\Omega_2)] = \frac{M}{6}\mu \begin{bmatrix} 6.667 & -16 & 12 \\ -16 & 43.333 & -46 \\ 12 & -46 & 79.333 \end{bmatrix}$$

$$\{C(i\Omega_2)^{(2)}\} = \frac{M^2}{6^2}\mu^2 \begin{Bmatrix} 717.33 \\ 384.91 \\ 114.68 \end{Bmatrix}$$

$$|Z(i\Omega_2)| = -73.13\frac{M^3}{6^3}\mu^3$$

The resulting amplitudes become

For force $Q_1(t)$

$$\{q(t)^{(1)}\} = \frac{6Q_{01}}{387.80\,M\mu} \begin{Bmatrix} 1373.06 \\ 724 \\ 211 \end{Bmatrix} e^{i\Omega_1 t}$$

$$= \begin{Bmatrix} 0.568 \\ 0.300 \\ 0.087 \end{Bmatrix} \frac{Q_{01}l^3}{EI} e^{i\Omega_1 t}$$

For force $Q_2(t)$

$$\{q(t)^{(2)}\} = -\frac{6Q_{02}}{73.13\,M\mu} \begin{Bmatrix} 717.33 \\ 384.91 \\ 114.68 \end{Bmatrix} e^{i(\Omega_2 t - \phi_2)}$$

$$= -\begin{Bmatrix} 1.57 \\ 0.84 \\ 0.25 \end{Bmatrix} \frac{Q_{02}l^3}{EI} e^{i(\Omega_2 t - \phi_2)}$$

Using Eq. (10.53) we obtain for the magnification factors

$$\{MF^{(1)}\} = \begin{Bmatrix} 1.70 \\ 1.73 \\ 1.76 \end{Bmatrix}$$

$$\{MF^{(2)}\} = \begin{Bmatrix} -9.1 \\ -8.5 \\ -8.1 \end{Bmatrix}$$

Note that for each force at the frequencies investigated the magnification factors are nearly the same for each point on the beam. This means that the modes of vibration excited by these forces are not greatly different from the deflection modes that are produced by static forces at the same points. This same result will not be obtained from excitation at higher frequencies. It is instructive to compute the amplitudes and magnification factors resulting from force $Q_1(t)$, for example, when its frequency is increased to a value near the second natural frequency of the beam. Let

$$\Omega_1^2 = 10\mu$$

This value of Ω_1 is about 2% higher than the second natural frequency. The following computations are made.

$$[Z(i\Omega_1)] = \frac{M}{6}\mu\begin{bmatrix} -3 & -16 & 12 \\ -16 & 24 & -46 \\ 12 & -46 & 60 \end{bmatrix}$$

$$\{C(i\Omega_1)^{(1)}\} = \frac{M^2}{6^2}\mu^2\begin{Bmatrix} -676 \\ 408 \\ 448 \end{Bmatrix}$$

$$|Z(i\Omega_1)| = 876\,\frac{M^3}{6^3}\mu^3$$

The amplitudes are

$$\{q(t)^{(1)}\} = \begin{Bmatrix} -0.124 \\ 0.075 \\ 0.082 \end{Bmatrix}\frac{Q_{01}l^3}{EI}e^{i\Omega_1 t}$$

The magnification factors are

$$\{MF^{(1)}\} = \begin{Bmatrix} -0.372 \\ 0.434 \\ 1.66 \end{Bmatrix}$$

These results are shown in Fig. 10.4. In this case it is seen that the mode of vibration excited resembles the second normal mode.

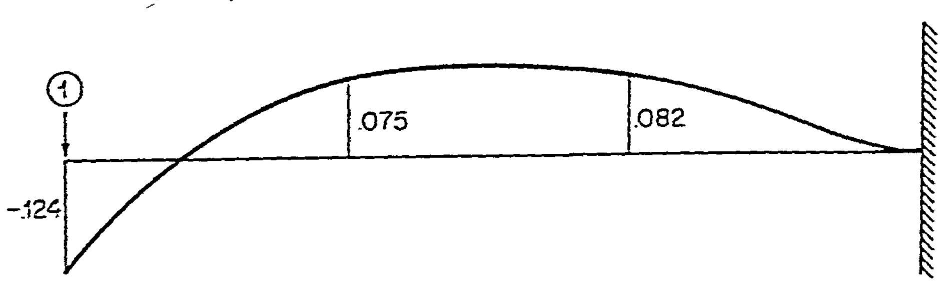


Figure 10.4 Amplitudes excited by force applied at 1 with a frequency approximately equal to the second natural frequency of the beam.

10.8 Influence of Structural Damping on Forced Vibrations

In this section we shall consider the influence of structural damping on the amplitude and phase of steady-state forced vibration. While *the concept of structural damping is hypothetical and is not completely verified by experiments,* it is useful in analyzing structures *under near-resonance conditions where even small amounts of damping* may produce important effects on the amplitude and phase of the response.

From Eq. (7.14), Chapter 7, we have for forced vibrations with structural damping

$$[m]\{\ddot{q}\} + (1 + ig)[k]\{q\} = \{Q\} \tag{10.54}$$

In accordance with the discussion at the end of Section 10.2 this *formulation is valid since we are interested in the steady-state response* in which case the displacements and velocities are 90° out of phase. The number g in Eq. (10.54) is small for most structures of engineering interest and ordinarily does not exceed 0.05. Experimental data are meager on this subject. Setting the initial displacements and velocities equal to zero, the Laplace transform of Eq. (10.54) becomes

$$(s^2[m] + (1 + ig)[k])\{q(s)\} = \{Q(s)\} \tag{10.55}$$

We define the impedance matrix by

$$[Z(s)] = s^2[m] + (1 + ig)[k] \tag{10.56}$$

Substituting (10.56) into Eq. (10.55) and premultiplying by $[Z(s)]^{-1}$ *we obtain the transform of the response*

$$\{q(s)\} = [Z(s)]^{-1}\{Q(s)\} = \frac{1}{|Z(s)|}[C(s)]\{Q(s)\} \tag{10.57}$$

Equation (10.57) is identical to Eq. (10.48) which is also valid for a system with viscous damping. Hence, the steps leading to Eq. (10.50) remain valid and, therefore, that equation remains applicable to the damped system.* The elements of the impedance matrix are now complex numbers and the elements of the column $\{C(i\Omega_j)^{(j)}\}$ are also complex. This brings about a change of phase between force and response. These effects are most pronounced at resonance.

Let us return now to the beam problem of the last section and determine the effect of structural damping. We shall take g equal to 0.05. The only change in the impedance matrix is the inclusion of the factor $(1 + ig)$ in every element, thus

*See Problem 5, Chapter 10.

$$[Z(s)] = \frac{M}{6}\begin{bmatrix} s^2 + 7\mu(1+ig) & -16\mu(1+ig) & 12\mu(1+ig) \\ -16\mu(1+ig) & 2s^2 + 44\mu(1+ig) & -46\mu(1+ig) \\ 12\mu(1+ig) & -46\mu(1+ig) & 2s^2 + 80\mu(1+ig) \end{bmatrix}$$

We again consider the two harmonic forces

$$Q_1(t) = Q_{01}e^{i\Omega_1 t}, \qquad \Omega_1^2 = \tfrac{1}{8}\mu$$

and

$$Q_2(t) = Q_{02}e^{i(\Omega_2 t - \epsilon_2)}, \qquad \Omega_2^2 = \tfrac{1}{3}\mu$$

Computations follow the previous example, thus

$$[Z(i\Omega_1)] = \frac{M}{6}\mu\begin{bmatrix} (6.875 + 0.35i) & -(16 + 0.80i) & (12 + 0.60i) \\ -(16 + 0.80i) & (43.75 + 2.20i) & -(46 + 2.30i) \\ (12 + 0.60i) & -(46 + 2.30i) & (79.75 + 4.0i) \end{bmatrix}$$

$$\{C(i\Omega_1)^{(1)}\} = \frac{M^2}{6^2}\mu^2\begin{Bmatrix} 1369.55 + 138.85i \\ 722.18 + 72.6i \\ 210.48 + 20.95i \end{Bmatrix}$$

$$|Z(i\Omega_1)| = \frac{M^3}{6^3}\mu^3(383.48 + 72.28i)$$

For the second force

$$[Z(i\Omega_2)] = \frac{M}{6}\mu\begin{bmatrix} (6.667 + 0.35i) & -(16 + 0.80i) & (12 + 0.60i) \\ -(16 + 0.80i) & (43.333 + 2.20i) & -(46 + 2.30i) \\ (12 + 0.60i) & -(46 + 2.30i) & (79.333 + 4.0i) \end{bmatrix}$$

$$\{C(i\Omega_2)^{(2)}\} = \frac{M^2}{6^2}\mu^2\begin{Bmatrix} 715.51 + 72.27i \\ 383.85 + 40.03i \\ 114.34 + 12.23i \end{Bmatrix}$$

$$|Z(i\Omega_2)| = \frac{M^3}{6^3}\mu^3(-76.60 + 24.85i)$$

The amplitudes are again computed from Eq. (10.50).

For force $Q_1(t)$

$$\{q(t)^{(1)}\} = \frac{6Q_{01}}{M\mu(383.48 + 72.28i)}\begin{Bmatrix} 1369.55 + 138.85i \\ 722.18 + 72.6i \\ 210.48 + 20.95i \end{Bmatrix} e^{i\Omega_1 t}$$

The complex numbers are changed from rectangular form to polar form (see Figure 10.5).

$$383.48 + 72.28i = 390.0e^{10°40' i}$$

or

$$= 390.0\underline{|10°40'}$$

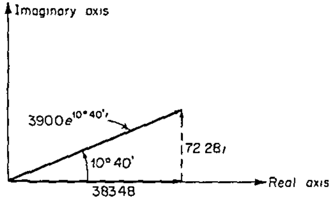


Figure 10.5

Then the following amplitudes and phase angles result.

$$q_1^{(1)}(t) = 0.564 \frac{Q_{01} l^3}{EI} e^{i(\Omega_1 t - 4°52')}$$

$$q_2^{(1)}(t) = 0.298 \frac{Q_{01} l^3}{EI} e^{i(\Omega_1 t - 4°56')}$$

$$q_3^{(1)}(t) = 0.087 \frac{Q_{01} l^3}{EI} e^{i(\Omega_1 t - 5°)}$$

For convenience the following notation is adopted to designate the amplitude and phase angle of the response

$$\{q^{(1)}\} = \frac{Q_{01} l^3}{EI} \begin{Bmatrix} 0.564 \underline{|-4°52'} \\ 0.298 \underline{|-4°56'} \\ 0.087 \underline{|-5°} \end{Bmatrix}$$

For force $Q_2(t)$ the results are

$$\{q^{(2)}\} = \frac{6 Q_{02}}{M\mu(-76.60 + 24.85i)} \begin{Bmatrix} 715.51 + 72.27i \\ 383.85 + 40.03i \\ 114.34 + 12.23i \end{Bmatrix} e^{i(\Omega_2 t - \psi_2)}$$

Using the foregoing notation, this leads to

$$\{q^{(2)}\} = \frac{Q_{02} l^3}{EI} \begin{Bmatrix} 1.43 \underline{|-156°16' - \psi_2} \\ 0.773 \underline{|-156°5' - \psi_2} \\ 0.229 \underline{|-155°55' - \psi_2} \end{Bmatrix}$$

It will be noted that in this example damping has the effect of decreasing amplitudes slightly but the effect on phase angle is appreciable, especially in the case of the response to $Q_2(t)$ whose frequency is very near resonance.

10.9 Response to Nonharmonic Periodic Forces: Fourier Series

When a generalized forcing function $Q_j(t)$ exciting a system is periodic but not harmonic it can be resolved into a set of harmonic components by use of the Fourier series technique. The response to the jth generalized force is then computed by adding the responses to its harmonic components. For the steady state part of the response we use Eq. (10.50). We now develop this procedure.

Resolution of a Periodic Force into Harmonic Components. Let

$$Q_j(t) = Q_{0j} f_j(t)$$

be a periodic, but not harmonic, force function with period $2T$, namely

$$Q_j(t) = Q_j(t + 2T)$$

or

$$Q_{0j} f_j(t) = Q_{0j} f_j(t + 2T)$$

in which Q_{0j} is a constant. Expand $f_j(t)$ by a Fourier series[9] and write

$$f_j(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos \Omega_k t + b_k \sin \Omega_k t) \tag{10.58}$$

in which

$$\left.\begin{aligned} a_k &= \frac{1}{T} \int_{-T}^{T} f_j(t) \cos \Omega_k t \, dt \\ b_k &= \frac{1}{T} \int_{-T}^{T} f_j(t) \sin \Omega_k t \, dt \\ \Omega_k &= k\left(\frac{2\pi}{2T}\right) = k\left(\frac{\text{radians per cycle}}{\text{period of force function}}\right) \\ k &= 0, 1, 2, \ldots, \infty \end{aligned}\right\} \tag{10.59}$$

The series expressed by Eq. (10.58) can be written more compactly in a complex exponential form. To derive this form we write each pair of terms in the sum of Eq. (10.58) as

$$\begin{aligned} a_k \cos \Omega_k t + b_k \sin \Omega_k t \\ = \tfrac{1}{2}(a_k - i b_k) e^{i\Omega_k t} + \tfrac{1}{2}(a_k + i b_k) e^{-i\Omega_k t} \end{aligned} \tag{10.60}$$

This can be verified by using Euler's formula

$$e^{\pm i\Omega t} = \cos \Omega t \pm i \sin \Omega t$$

Substituting for a_k and b_k from relations (10.59) we can write

$$a_k - i b_k = \frac{1}{T} \int_{-T}^{T} f_j(t) (\cos \Omega_k t - i \sin \Omega_k t) \, dt$$

Using Euler's formula this becomes

$$a_k - ib_k = \frac{1}{T} F_j(\Omega_k) \tag{10.61}$$

where

$$F_j(\Omega_k) = \int_{-T}^{T} f_j(t) e^{-i\Omega_k t}\, dt$$

Similarly

$$a_k + ib_k = \frac{1}{T} F_j(-\Omega_k) \tag{10.62}$$

where

$$F_j(-\Omega_k) = \int_{-T}^{T} f_j(t) e^{i\Omega_k t}\, dt$$

Substituting from (10.61) and (10.62) into Eq. (10.60) we have

$$a_k \cos \Omega_k t + b_k \sin \Omega_k t = \frac{1}{2T} [F_j(\Omega_k) e^{i\Omega_k t} + F_j(-\Omega_k) e^{-i\Omega_k t}] \tag{10.63}$$

From the expression for $F_j(\Omega_k)$ written below Eq. (10.61), or from $F_j(-\Omega_k)$ written below Eq. (10.62) we can write for $k = 0$ [in which case $\Omega_k = 0$ as is seen from the third of Eqs. (10.59)]

$$F_j(0) = \int_{-T}^{T} f_j(t)\, dt$$

or dividing both sides by $2T$

$$\frac{1}{2T} F_j(0) = \frac{1}{2T} \int_{-T}^{T} f_j(t)\, dt$$

Comparing this result with the first of Eqs. (10.59) when it is written for $k = 0$, we conclude that

$$\frac{1}{2T} F_j(0) = \frac{a_0}{2}$$

or

$$\frac{1}{2T} F_j(\Omega_k) e^{i\Omega_k t}\Big|_{k=0} = \frac{a_0}{2} \tag{10.64}$$

Using Eqs. (10.63) and (10.64) in Eq. (10.58) we write

$$\begin{aligned} f_j(t) &= \frac{1}{2T} F_j(\Omega_k) e^{i\Omega_k t}\Big|_{k=0} + \frac{1}{2T} \sum_{k=1}^{\infty} [F_j(\Omega_k) e^{i\Omega_k t} + F_j(-\Omega_k) e^{-i\Omega_k t}] \\ &= \frac{1}{2T} \sum_{k=0}^{\infty} F_j(\Omega_k) e^{i\Omega_k t} + \frac{1}{2T} \sum_{k=1}^{\infty} F_j(-\Omega_k) e^{-i\Omega_k t} \end{aligned} \tag{10.65}$$

If we recall that $\Omega_k = k(2\pi/2T)$ [See Eq. (10.59)] then extending k to include negative integers we can write for the second summation on the right-hand side of Eq. (10.65)

$$\frac{1}{2T} \sum_{k=1}^{\infty} F_j(-\Omega_k) e^{-i\Omega_k t} = \frac{1}{2T} \sum_{k=-\infty}^{-1} F_j(\Omega_k) e^{i\Omega_k t}$$

and Eq. (10.65) becomes

$$f_j(t) = \frac{1}{2T} \sum_{k=-\infty}^{\infty} F_j(\Omega_k) e^{i\Omega_k t} \tag{10.66}$$

where $F_j(\Omega_k)$ is given by the expression below Eq. (10.61) and is repeated here for completeness

$$F_j(\Omega_k) = \int_{-T}^{T} f_j(t) e^{-i\Omega_k t}\, dt \tag{10.67}$$

Equation (10.66) expresses the periodic. but not necessarily harmonic, function $f_j(t)$ as a linear combination of an infinite number of harmonic functions of time written in a complex exponential form. In practice, satisfactory accuracy is obtained when only a finite number. of the harmonic components of $f_j(t)$ are considered in the analysis.

Response of a System Excited by Periodic Nonharmonic Forces. To find the response of a system excited by a periodic nonharmonic force

$$Q_j(t) = Q_{0j} f_j(t)$$

we take the Laplace transform of Eq. (10.66) to get $f_j(s)$ and write

$$Q_j(s) = Q_{0j} f_j(s) = \frac{Q_{0j}}{2T_j} \sum_{k=-\infty}^{\infty} \frac{F_j(\Omega_k)}{s - i\Omega_k} \tag{10.68}$$

The subscript j in $2T_j$ designates that this is the period of the jth generalized force since other forces of the set $\{Q(t)\}$ may be periodic nonharmonic with different periods. Substituting for $Q_j(s)$ from Eq. (10.68) into Eq. (10.48) the transform of the response to the jth periodic nonharmonic force becomes

$$\{q(s)^{(j)}\} = \frac{Q_{0j}}{2T_j} \frac{1}{|Z(s)|} \{C(s)^{(j)}\} \sum_{k=-\infty}^{\infty} \frac{F_j(\Omega_k)}{s - i\Omega_k} \tag{10.69}$$

To compute only the steady-state part of the response we evaluate the residues at the poles

$$s = i\Omega_k$$

and then sum these residues. Following this procedure we write for the steady-state response to the periodic nonharmonic force $Q_j(t)$

$$\{q(t)^{(j)}\} = \frac{Q_{0j}}{2T_j} \sum_{k=-\infty}^{\infty} \frac{F_j(\Omega_k)}{|Z(i\Omega_k)|} \{C(i\Omega_k)^{(j)}\} e^{i\Omega_k t} \tag{10.70}$$

As we discussed earlier, sufficient accuracy is usually obtained by considering a finite number of harmonic components in Eq. (10.70).

When a system is subjected to n generalized forces $Q_j(t)$, $(j = 1, 2, \ldots, n)$, which are periodic but not necessarily harmonic, the total response is obtained by adding their individual contributions to the response. These individual contributions are given by Eq. (10.70) for each force $Q_j(t)$. The total steady-state response to forces $Q_j(t)$, $(j = 1, 2, \ldots, n)$, is obtained from Eq. (10.70) by summing over j.

10.10 Response to Nonperiodic Forces: Fourier Integral

In the last section we demonstrated how to calculate the response of a system to periodic nonharmonic force functions. This was done by synthesizing the force function from its Fourier frequency components. These frequency components as well as the corresponding frequencies were separate and distinct and considered together comprised a *discontinuous* or discrete frequency spectrum. The response to the force was synthesized by summing the responses to these discrete frequency components.

The method of the last section can be extended to forces which are nonperiodic. We can think of a nonperiodic force function as a periodic one having a period $2T$ of infinite duration $(T \to \infty)$, and synthesize the nonperiodic force function from harmonic components whose frequencies form a *continuous* spectrum. This synthesis is accomplished by the use of the Fourier integral which may be regarded as an extension of the Fourier series to the case of forces having a continuous spectrum of frequency components. The response is obtained by integrating the response of the system to the continuous spectrum of the frequency components of the exciting force function. We shall demonstrate this by extending the development of the last section.

Before proceeding, however, it is instructive to point out that we can obtain an approximate solution for the response to a nonperiodic excitation by directly applying the method of the last section. For this purpose we replace the nonperiodic function $f_i(t)$ (Fig. 10.6) by

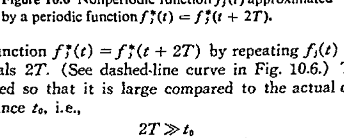


Figure 10.6 Nonperiodic function $f_i(t)$ approximated by a periodic function $f_i^*(t) = f_i^*(t + 2T)$.

a periodic function $f_i^*(t) = f_i^*(t + 2T)$ by repeating $f_i(t)$ at periodic time intervals $2T$. (See dashed-line curve in Fig. 10.6.) The period $2T$ is selected so that it is large compared to the actual duration of the disturbance t_0, i.e.,

$$2T \gg t_0$$

Using the periodic function $f_j^*(t)$ we can calculate the approximate response by the method of the last section. This approximation will improve by increasing the selected period $2T$. The exact solution is obtained when we let $2T \to \infty$. We shall now proceed to develop the steps leading to the exact solution by first showing how the Fourier integral is used to represent a nonperiodic force function $f_j(t)$.

The Fourier Integral. Let us return to Eqs. (10.66) and (10.67) which we rewrite here for convenience

$$f_j(t) = \frac{1}{2T} \sum_{k=-\infty}^{\infty} F_j(\Omega_k) e^{i\Omega_k t} \tag{10.66}$$

$$F_j(\Omega_k) = \int_{-T}^{T} f_j(t) e^{-i\Omega_k t} \, dt \tag{10.67}$$

We recall from the third of Eqs. (10.59) that the frequencies Ω_k are given by

$$\Omega_k = k\left(\frac{2\pi}{2T}\right)$$

where k takes on all integer values. Because k is an integer, the smallest frequency increment $\Delta\Omega$ separating any two frequencies Ω_k and Ω_{k+1} is given by

$$\begin{aligned} \Delta\Omega &= \frac{2\pi}{2T}[(k+1) - k] \\ &= \frac{\pi}{T} \end{aligned} \tag{10.71}$$

This may be rewritten in the form

$$\frac{1}{T} = \frac{1}{\pi}\Delta\Omega \tag{10.72}$$

Substituting Eq. (10.72) into Eq. (10.66) we have

$$f_j(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} F_j(\Omega_k) e^{i\Omega_k t} \, \Delta\Omega \tag{10.73}$$

When the function $f_j(t)$ is nonperiodic we can think of it as a periodic function with a period of infinite duration and write

$$2T \to \infty$$

From this it follows that the frequency increment $\Delta\Omega$ separating any two frequencies approaches zero

$$\Delta\Omega \to 0 \qquad \text{[See Eq. (10.71).]}$$

and consequently the frequency Ω becomes a continuous variable. The operation of summation (Σ) in Eq. (10.73) then approaches an operation of integration with $d\Omega$ replacing $\Delta\Omega$, and the continuous variable Ω replaces the discrete frequency Ω_k. Hence, we write

$$\begin{aligned} f_j(t) &= \lim_{\Delta\Omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} F_j(\Omega_k) e^{i\Omega_k t} \, \Delta\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_j(\Omega) e^{i\Omega t} \, d\Omega \end{aligned} \tag{10.74}$$

Similarly Eq. (10.67) becomes at the limit $2T \to \infty$

$$F_j(\Omega_k) = \lim_{T \to \infty} \int_{-T}^{T} f_j(t) e^{-i\Omega_k t} \, dt$$

or

$$F_j(\Omega) = \int_{-\infty}^{\infty} f_j(t) e^{-i\Omega t} \, dt \tag{10.75}$$

$f_j(t)$ and $F_j(\Omega)$ of Eqs. (10.74) and (10.75), respectively, form a transform pair. $F_j(\Omega)$ is the Fourier transform (F.T.) of $f_j(t)$, and $f_j(t)$ is the inverse F.T. of $F_j(\Omega)$. For our present purpose we are concerned with Eq. (10.75) from which we can obtain $F_j(\Omega)$ given the force function $f_j(t)$. Note, however, that the integral in Eq. (10.75) is not always convergent.

Response to Nonperiodic Forces. Having obtained a means of representing the nonperiodic force, we now wish to determine the response function. Refer to Eq. (10.70) which expresses the steady-state response to the jth periodic force $Q_j(t)$. This steady-state solution was obtained from Eq. (10.69) by summing the residues at the poles

$$s = i\Omega_k$$

and does not contain the residues at the poles which exist at the roots of the characteristic equation

$$|Z(s)| = 0$$

namely the transient part of the response. When the jth forcing function is not periodic there is no steady-state response because the entire response is transient. It turns out, however, that when we apply to the steady-state solution of Eq. (10.70) the limiting process that led to the Fourier integral [Eq. (10.74)] we obtain the entire response (transient) to the nonperiodic force. As we shall point out again later, in going from a summation of time functions to an integral in the frequency domain, we essentially return to evaluate the residues at the poles of

$$|Z(s)| = 0$$

Following this discussion we substitute Eq. (10.72) into Eq. (10.70) and write the steady-state response to the jth periodic force $Q_j(t) = Q_{0j} f_j(t)$

$$\{q(t)^{(j)}\} = \sum_{k=-\infty}^{\infty} \frac{Q_{0j}}{2\pi} \frac{F_j(\Omega_k)}{|Z(i\Omega_k)|} \{C(i\Omega_k)^{(j)}\} e^{i\Omega_k t} \, \Delta\Omega \tag{10.76}$$

For a nonperiodic force $Q_j(t)$ we apply to Eq. (10.76) the same limiting process that was applied to Eq. (10.73) and write

$$\begin{aligned}\{q(t)^{(j)}\} &= \lim_{\Delta\Omega\to 0} \sum_{k=-\infty}^{\infty} \frac{Q_{0j}}{2\pi} \frac{F_j(\Omega_k)}{|Z(i\Omega_k)|} \{C(i\Omega_k)^{(j)}\} e^{i\Omega_k t} \Delta\Omega \\ &= \frac{Q_{0j}}{2\pi} \int_{-\infty}^{\infty} \frac{F_j(\Omega)}{|Z(i\Omega)|} \{C(i\Omega)^{(j)}\} e^{i\Omega t} \, d\Omega\end{aligned} \tag{10.77}$$

in which $F_j(\Omega)$ is computed from Eq. (10.75). The response to a number of nonperiodic forces $Q_j(t)$ is obtained by adding their individual contributions to the total response [summing Eq. (10.77) over j].

The integral of Eq. (10.77) can be computed by applying the residue theorem[9] which results in summing the residues at the poles given by the roots of

$$|Z(i\Omega)| = 0$$

and multiplying the sum by $2\pi i$. This is equivalent to summing the residues at the poles given by the roots

$$|Z(s)| = 0.$$

We shall demonstrate this by computing the response to a unit impulse. For simplicity we consider a single-degree-of-freedom system.

10.11 Example: Response of a Single-Degree-of-Freedom System to a Unit Impulse

Let the system of Fig. 10.1 be excited by an impulse $F(t) = k\delta(t)$ at time $t = 0$, where k is the spring constant in Fig. 10.1 and $\delta(t)$ is the unit impulse defined by

$$\begin{aligned}\delta(t) &= 0, \qquad \text{for } t \neq 0 \\ \int \delta(t)\, dt &= 1\end{aligned} \tag{10.78}$$

We shall compute the response of the system, first using the method of Section 10.2 and then by using Eq. (10.77) of the last section.

1. *Using the Method of Section 10.2.*

Following the steps of Section 10.2 the response in transform is given by Eq. (10.6)

$$u(s) = \frac{1}{Z(s)} \omega^2 f(s) \tag{10.6}$$

In our present problem $f(s)$ is the Laplace transform of the unit impulse $\delta(t)$. Applying Eq. (8.78a), Chapter 8, we have

$$f(s) = \mathcal{L}\,\delta(t) = \int_0^{\infty} e^{-st} \delta(t)\, dt = 1$$

This result follows from Eq. (10.78) which defines the unit impulse. As a result of this, Eq. (10.6) becomes

$$u(s) = \frac{1}{Z(s)}\omega^2$$

The response $u(t)$ is obtained by summing the residues at the poles which exist at the roots of the characteristic equation

$$|Z(s)| = (s - s_1)(s - s_2) = 0$$

Thus

$$u(s) = \frac{\omega^2}{Z(s)} = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2}$$

$$A_1 = \left.\frac{\omega^2}{s - s_2}\right|_{\substack{s - s_1 = 0 \\ \text{or } s = s_1}} = \frac{\omega^2}{s_1 - s_2} = \frac{\omega^2}{-i2\omega(1 - \zeta^2)^{1/2}}$$

$$A_2 = \left.\frac{\omega^2}{s - s_1}\right|_{\substack{s - s_2 = 0 \\ \text{or } s = s_2}} = \frac{\omega^2}{s_2 - s_1} = \frac{\omega^2}{i2\omega(1 - \zeta^2)^{1/2}}$$

The response is

$$u(t) = \sum_{j=1}^{2} A_j e^{s_j t} \tag{10.79}$$

where A_1, A_2 are given above and s_1, s_2 are given by Eq. (10.8). Substituting these values into Eq. (10.79) we have

$$u(t) = \frac{i\omega}{2(1 - \zeta^2)^{1/2}} e^{-\zeta\omega t}\left[e^{-i\omega(1-\zeta^2)^{1/2}t} - e^{i\omega(1-\zeta^2)^{1/2}t}\right]$$

Using Euler's formula this reduces to

$$u(t) = \frac{\omega}{(1 - \zeta^2)^{1/2}} e^{-\zeta\omega t} \sin \omega(1 - \zeta^2)^{1/2} t \tag{10.80}$$

2. *Using Eq. (10.77).*

Now let us solve the same problem by applying Eq. (10.77) of the last section. For a single-degree-of-freedom system the terms appearing in Eq. (10.77) take on the following values.

$$\{C(i\Omega)\} = 1$$

Referring to Eq. (10.23) to recall how the impedance matrix $[Z(s)]$ was defined we have

$$\begin{aligned} |Z(s)| &= ms^2 + cs + k \\ &= m\left(s^2 + \frac{c}{m}s + \frac{k}{m}\right) \\ &= m(s^2 + 2\zeta\omega s + \omega^2) \end{aligned}$$

Hence,

$$|Z(i\Omega)| = m(-\Omega^2 + i2\zeta\omega\Omega + \omega^2)$$

In addition since the exciting force is given by

$$Q(t) = F(t) = k\delta(t)$$

it follows that

$$Q_0 = k$$

(We drop the subscript j since a single force is involved.)

For the (F.T.) $F_j(\Omega)$ of $\delta(t)$ we apply Eq. (10.75) in which the unit impulse $\delta(t)$ replaces $f_j(t)$, hence,

$$F(\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\Omega t}\, dt = 1$$

This follows again from relations (10.78) as we demonstrated for the Laplace transform of $\delta(t)$. Using these relations, Eq. (10.77) becomes

$$\begin{aligned} u(t) = q(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{k}{mZ(i\Omega)} e^{i\Omega t}\, d\Omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^2}{Z(i\Omega)} e^{i\Omega t}\, d\Omega \end{aligned} \tag{10.81}$$

where $Z(i\Omega) = -\Omega^2 + i2\zeta\omega\Omega + \omega^2$.

Using the residue theorem we compute the integral of Eq. (10.81) as follows. Using partial fractions we write

$$\frac{1}{2\pi}\frac{\omega^2}{Z(i\Omega)} e^{i\Omega t} = \frac{\alpha_1}{\Omega - i\zeta\omega + \omega(1-\zeta^2)^{1/2}} + \frac{\alpha_2}{\Omega - i\zeta\omega - \omega(1-\zeta^2)^{1/2}}$$

from which we compute the residues α_1 and α_2 at the poles

$$\Omega = i\zeta\omega - \omega(1-\zeta^2)^{1/2}$$

and

$$\Omega = i\zeta\omega + \omega(1-\zeta^2)^{1/2}$$

respectively. This yields

$$\alpha_1 = \frac{1}{2\pi}\frac{\omega}{2(1-\zeta^2)^{1/2}} e^{-\zeta\omega t - i\omega(1-\zeta^2)^{1/2}t}$$

$$\alpha_2 = -\frac{1}{2\pi}\frac{\omega}{2(1-\zeta^2)^{1/2}} e^{-\zeta\omega t + i\omega(1-\zeta^2)^{1/2}t}$$

The response $u(t)$ is obtained by multiplying the sum of the residues by $2\pi i$

$$u(t) = 2\pi i(\alpha_1 + \alpha_2)$$

Substituting for α_1 and α_2 from above and using Euler's formula this reduces to

$$u(t) = \frac{\omega}{(1-\zeta^2)^{1/2}} e^{-\zeta\omega t} \sin \omega(1-\zeta^2)^{1/2} t$$

which is the same result as obtained in Eq. (10.80). Hence, the process of summing residues at the poles given by $Z(s) = 0$ in the procedure of Section 10.2 led to the same result as evaluating the integral of Eq. (10.81) which represents the application of Eq. (10.77). This confirms the fact that Eq. (10.77) yields the complete response although we arrived at it by applying a limiting process to the steady-state response to a periodic force.

10.12 Relation Between the Complex Frequency Response $H(\Omega)$ and the Response to a Unit Impulse $h(t)$

Let us designate by $h(t)$ the response at time t of the system of Fig. 10.1 when it is subjected to a unit impulse at time $t = 0$. The response $h(t)$ is given by Eq. (10.81), repeated here for convenience

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\omega^2}{Z(i\Omega)} e^{i\Omega t}\, d\Omega$$

The expression $\omega^2/Z(i\Omega)$ is equal to the complex frequency response of the system $H(\Omega)$ as can be verified from Eq. (10.13). Consequently, the expression for the response to a unit impulse takes the form

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\Omega) e^{i\Omega t}\, d\Omega \tag{10.82}$$

If we compare this equation with Eq. (10.74) we conclude that the complex frequency response $H(\Omega)$ is the Fourier transform of the response to a unit impulse $h(t)$. Using Eq. (10.75) we can write for $H(\Omega)$

$$H(\Omega) = \int_{-\infty}^{\infty} h(t) e^{-i\Omega t}\, dt \tag{10.83}$$

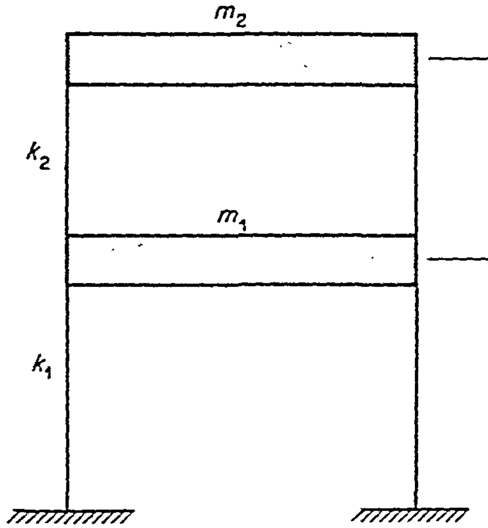
Equations (10.82) and (10.83) state that $h(t)$ and $H(\Omega)$ form a Fourier transform pair.

PROBLEMS

1. Using the method of Section 10.2, derive the expression for the response of the system of Fig. 10.1 due to an initial displacement $u(0)$ and initial velocity $\dot{u}(0)$. Compare your result with Eq. (7.43) of Chapter 7 after evaluating the coefficients A_1 and A_2 in this equation.
2. Verify Eqs. (10.16) and (10.19).
3. Starting with Eq. (10.44), derive Eq. (10.45).
4. Show that for a single-degree-of-freedom system with viscous damping, Eq. (10.53) reduces to the complex frequency response expressed by (10.13).
5. Starting with Eq. (10.25) and operating on it as we operated on Eq. (10.26) in Section 10.6, show that Eqs. (10.49), (10.50), and (10.53) are valid for a system in which viscous damping is present.

6. Derive the expression for the complete response of a multi-degree-of-freedom system in which viscous damping is present and which is subjected to harmonic forces of the form of Eq. (10.46). The initial displacements and velocities are zero.

7. The system shown is excited by a harmonic force $Q_1(t) = \cos \Omega t$ and $Q_2(t) = \sin \Omega t$. Plot the response $q_1(t)$ of mass m_1 as a function of the frequency Ω of the exciting forces.



$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$k_1 = 1000$ lb/ft
$k_2 = 2000$ lb/ft

$m_1 = 1$ lb-sec^2/ft
$m_2 = 2$ lb-sec^2/ft

Problem 7

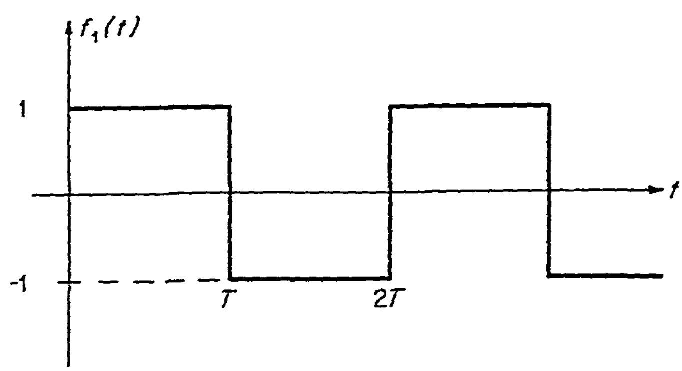
8. The system of Problem 7 is excited by a periodic forcing function

$$Q_1(t) = Q_{01} f_1(t)$$

where

$$Q_{01} = 200 \text{ lb}$$

and $f_1(t)$ is the square wave shown in the figure. The period $2T$ of the exciting force is the same as the fundamental period of the system. Find

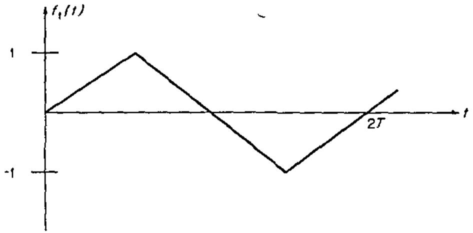


Problem 8

the expressions for the response of the system. Use the first three terms in the Fourier series expansion of the forcing function. Compute the displacements $q_1(t)$, $q_2(t)$ at times

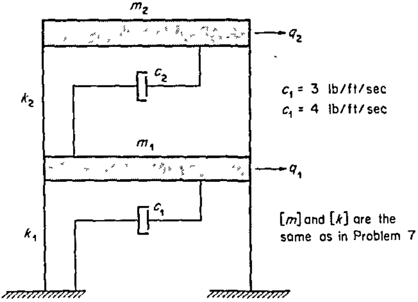
$$t = \tfrac{1}{4}T, \tfrac{1}{2}T, T, \tfrac{3}{2}T, 2T, 3T$$

9. Repeat Problem 8 replacing $f_1(t)$ by the periodic function shown.



Problem 9

10. The system shown is excited by a force $Q_1(t) = \cos \Omega t$. Plot the steady-state response $q_2(t)$ of mass m_2 as a function of the frequency of the exciting force.



Problem 10

11. The system of Problem 10 is excited by periodic forcing functions

$$Q_1(t) = Q_{01} f_1(t)$$

and

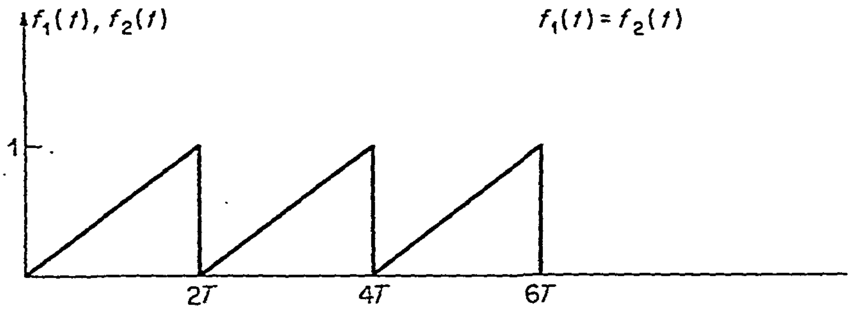
$$Q_2(t) = Q_{02} f_2(t)$$

where

$$Q_{01} = 100 \text{ lb}$$

$$Q_{02} = 200 \text{ lb}$$

and $f_1(t)$ and $f_2(t)$ are given by the saw tooth wave shown. The period of the forcing functions is equal to the second period of the undamped system. Find the expression for the total response of the system. Use the first two terms of the Fourier series expansion. Compute the displacements $q_1(t)$ and $q_2(t)$ at times $t = \frac{1}{8}T, \frac{1}{4}T, \frac{1}{2}T, T, \frac{3}{2}T, 2T$.



Problem 11

12. In Problem 11, plot the transient response and the steady-state response separately as a function of time.

13. The system of Problem 10 is excited at $t = 0$ by forces

$$Q_1(t) = k_1 \delta(t)$$

$$Q_2(t) = k_2 \delta(t)$$

where $\delta(t)$ is the unit impulse. Find the expression for the response of the system.

14. The system of Problem 10 is subjected to the following initial conditions

$$q_1(0) = q_2(0)$$

$$\dot{q}_1(0) = v_1$$

$$\dot{q}_2(0) = v_2$$

Find the expression for the response of the system.

15. Compare the results of Problems 13 and 14 and discuss.

CHAPTER 11

Response to Random Excitation

11.1 *Introduction*

We can characterize the response studies of the preceding chapters by *stating that these studies were concerned with the response of a* linear system excited by ***deterministic*** forcing functions. By deterministic we mean forcing functions which are known. These may be periodic or nonperiodic, analytically expressible or plotted from data, but in all cases they are uniquely established. As long as the excitations are deterministic and the system characteristics are known we find that the response is also deterministic.

In physical reality, however, we often find that the forcing functions are not known, and at best they are only predicted. The predictions are based on experience, or when experience is meager, on the results of experiments. A predicted excitation is no longer a *deterministic one. In fact there is no way for us to guarantee* that this predicted excitation will actually take place. At best we can hope that the actual excitation will not deviate much from the predicted one. It is also apparent that under such circumstances we cannot evaluate the response any better than we can predict the excitation. The predictions of excitations and associated responses are not aided by statements such as, "The response will not exceed the value of 2 inches by very much." One analyst's "very much" may differ greatly from that of another. To avoid such statements we characterize nondeterministic or random excitations as well as the corresponding response through the use of statistical theory. Using the tools of statistics we will be in a position to make statements such as, "The probability that the response will exceed the value of 2 inches is 0.001." The lower the probability, the less likely it is to have a response that exceeds 2 inches.

In this chapter we shall develop the tools that will enable us to make such probability statements with reference to the response of a linear system subjected to a random excitation such as an earthquake, blast, gust, etc. To do this we must first introduce some statistical concepts.

11.2 Sample Space, Random Variable, Probability

In the last section we made use of the terms random and probability. It is appropriate to throw a little more light on the significance of these terms. We speak of probabilities and random variables in relation to a conceptual experiment, referred to as *a sample space*. For example, all possible outcomes that may result from a single toss of three coins constitutes a sample space. This sample space has 2^3 points, each corresponding to one possible outcome. If we designate a coin falling with head up by H and with tail up by T we have the following sample space for our experiment with the three coins.

TABLE 11.1 SAMPLE SPACE CORRESPONDING TO THE TOSS OF THREE COINS

Coin number	1	2	3
	H	H	H
	H	T	H
	H	H	T
	H	T	T
	T	H	H
	T	T	H
	T	H	T
	T	T	T

In our experiment we shall be careful to exclude any extraneous effects which might influence the outcome. Hence, we will choose coins which are not biased so that one side is more likely to show up than the other. And we shall perform the experiment in a certain and the variance represents the *mean square* value of x. Using relations (11.1b) and (11.2) the expected value of a constant α is equal to

$$E(\alpha) = \alpha \int_{-\infty}^{\infty} f(x)\, dx = \alpha$$

11.5 Two Random Variables

Consider two continuous random variables x, y such as the coordinates of hits on a target. The joint probability density function $f(x, y)$ of x ond y is defined by the following properties.

$$f(x,y) \geq 0 \quad \begin{matrix} -\infty < x < \infty \\ -\infty < y < \infty \end{matrix} \qquad \text{(a)}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1 \qquad \text{(b)} \qquad (11.5)$$

$$Pr\begin{pmatrix} x_a < x < x_b \\ y_c < y < y_d \end{pmatrix} = \int_{x_a}^{x_b} \int_{y_c}^{y_d} f(x,y)\, dy\, dx \qquad \text{(c)}$$

$Pr\begin{pmatrix} x_a < x < x_b \\ y_c < y < y_d \end{pmatrix}$ denotes the probability that an observation (for instance, the coordinates x and y of a hit on a target) has values x and y within the prescribed limits. According to (11.5a) and (11.5c) this probability is represented by the volume bounded by the planes $x = x_a$, $x = x_b$, $y = y_c$, $y = y_d$, $f(x, y) = 0$ and the surface $f(x, y)$ (see Fig. 11.2).

The probability density function of x only, or the so called *mar-*

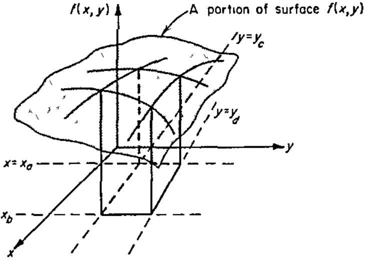


Figure 11.2

ginal distribution $g(x)$ of x, is obtained from the joint probability density function $f(x, y)$ by integrating over y.

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \tag{11.6}$$

Similarly the marginal distribution $h(y)$ of y is given by

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx \tag{11.7}$$

Extending the concept defined by relation (11.2), we define the expected value of a function $q(x, y)$ as

$$E[q(x, y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(x, y) f(x, y)\, dydx \tag{11.8}$$

The expected value of the product $(x - \mu_x)(y - \mu_y)$ is called the *covariance* of x and y and is denoted by cov (x, y). Thus

$$\begin{aligned} \operatorname{cov}(x, y) &= E[(x - \mu_x)(y - \mu_y)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_x)(y - \mu_y) f(x, y)\, dydx \end{aligned} \tag{11.9}$$

where

$$\mu_x = \int_{-\infty}^{\infty} xg(x)\, dx$$

and

$$\mu_y = \int_{-\infty}^{\infty} yh(y)\, dy$$

Two continuous random variables x and y are said to be statistically independent when

$$f(x, y) = g(x) \cdot h(y) \tag{11.10}$$

in which $g(x)$ and $h(y)$ are the marginal distributions given by Eqs. (11.6) and (11.7). We will now show that when x and y are statistically independent $\operatorname{cov}(x, y) = 0$. Substituting relation (11.10) into (11.9) we have

$$\begin{aligned} \operatorname{cov}(x, y) &= \int_{-\infty}^{\infty} (x - \mu_x) g(x) dx \int_{-\infty}^{\infty} (y - \mu_y) h(y) dy \\ &= E[(x - \mu_x)] \cdot E[(y - \mu_y)] \\ &= [E(x) - \mu_x][E(y) - \mu_y] \\ &= 0 \end{aligned}$$

11.6 Linear Combinations of Random Variables

Let $x_i (i = 1, 2, 3, \ldots, n)$ be random variables with means μ_i, variances σ_i^2 and covariances $\operatorname{cov}(x_i, x_j)$. We define a new random variable w as a linear of combination of variables x_i

$$w = \sum_{i=1}^{n} a_i x_i$$

where the a_i are constants. Applying relations (11.3) and (11.4) we obtain, respectively, for the mean μ_w and the variance σ_w^2 of w

$$\begin{aligned} \mu_w &= E(w) = E(\sum a_i x_i) \\ &= \sum_i a_i E(x_i) \\ &= \sum_i a_i \mu_i \end{aligned} \tag{11.11}$$

and

$$\begin{aligned} \sigma_w^2 = E[(w - E(w))^2] &= E[(\sum_i a_i x_i - \sum_i a_i \mu_i)^2] \\ &= E[(\sum_i a_i (x_i - \mu_i))^2] \\ &= E[\sum_i \sum_j a_i a_j (x_i - \mu_i)(x_j - \mu_j)] \end{aligned}$$

When random variables x_i are statistically independent of x_j for all $i \neq j$

$$E[(x_i - \mu_i)(x_j - \mu_j)] = \text{cov}\,(x_i, x_j) = 0 \quad \text{for } i \neq j$$

Then

$$\sigma_w^2 = \sum_i a_i^2 E(x_i - \mu_i)^2 = \sum_i a_i^2 \sigma_i^2 \tag{11.12}$$

11.7 Random Process

A collection of records ${}^k\{x(t)\}$ $(k = 1, 2, 3, \ldots, n)$ of a random variable x (such as earthquake acceleration records obtained from a seismograph, see Fig. 11.3) represents a *random process* when these records can be characterized by statistical properties. Superscript k in ${}^k\{x(t)\}$ denotes the record number and $-T < t < T$ $(T \to \infty)$.

If the expected value (average) of $x(t_1)$ for a fixed time t_1 (considering all records)

$$E[x(t_1)] = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} {}^k x(t_1) \tag{11.13}$$

is independent of t, or $E[x(t_1)] = E[x(t_1 + t)]$ for all t, then the random process is called *stationary*. If, in addition to the process being stationary, each record is statistically equivalent to any other record, so that $E[x(t_1)]$ in Eq. (11.13) can be replaced by a time average of a single representative record $x(t)$,

$$E(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt \tag{11.14a}$$

then the stationary process is *ergodic*. For many applications the

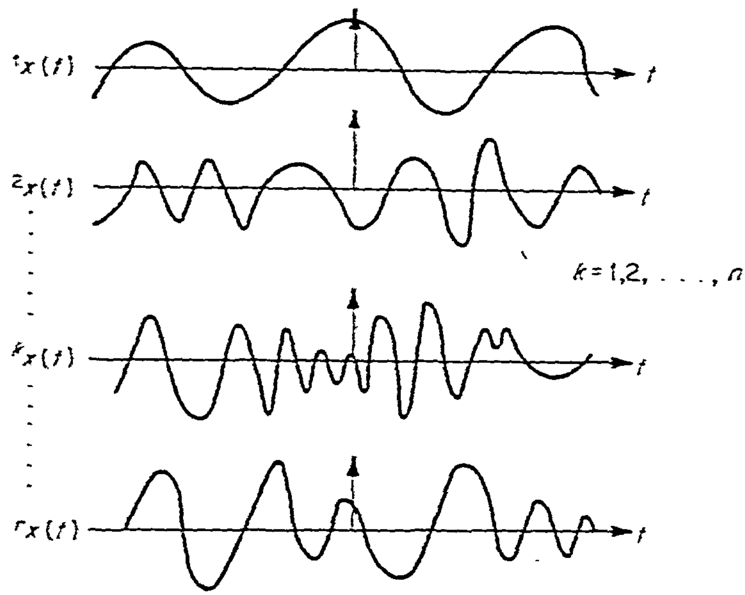


Figure 11.3 Records of a random process (for instance, earthquake records obtained from a seismograph).

ergodic assumption is reasonable. The variance σ_x^2 of $x(t)$ representing an ergodic process is given by

$$E[x - E(x)]^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [x - E(x)]^2 \, dt$$

For the special case where $E(x) = 0$ the variance σ_x^2 of x becomes its mean square value $\overline{x^2(t)}$ given by

$$\overline{x^2(t)} = E[x^2(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t) \, dt \qquad (11.14b)$$

11.8 The Role of Statistics in Response Studies

In order to attach physical meaning (in the structural dynamics sense) to a random variable $y(t)$ representing a random process, let $y(t)$ represent the lateral displacement at the top of a one story building (Fig. 11.4a) due to a random excitation such as resulting from an earthquake (this random excitation process will be assumed to be ergodic). Let us suppose that sensitive equipment is mounted on the top of the building and in order to secure the adequate functioning of this equipment we must limit the displacements $|y(t)|$ to a specified value y_k. However, when the exciting forces are random, it is conceivable that y_k may be exceeded. Consequently,

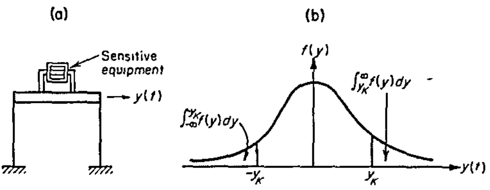


Figure 11.4 (a) Sensitive equipment mounted on a one story building. (b) Probability density function of the response $y(t)$ of the building in (a).

the best we can hope for is to make the probability very small that y_k will be exceeded, or

$$Pr(|y(t)| > y_k) = \int_{y_k}^{\infty} f(y)\,dy + \int_{-\infty}^{-y_k} f(y)\,dy = \epsilon \qquad (11.15)$$

in which ϵ is a very small positive number. See Fig. 11.4 b.

But how is the probability density function $f(y)$ to be obtained so that $Pr(|y(t)| > y_k)$ may be evaluated from Eq. (11.15)? Indeed we may build a model of the structure, subject it to a random excitation simulating an earthquake and measure deformations $y(t)$ at small time intervals Δt. Using these data the number n_i of observations (displacement measurements) in any interval

$$y_i - (\Delta y/2) < y < y_i + (\Delta y/2)$$

are counted. If the total number of observations is n, then the relative frequency of occurrence of observations in an interval $y_i - (\Delta y/2) < y < y_i + (\Delta y/2)$ is given by $f(y_i) = (n_i/n)(1/\Delta y)$ (where $1/\Delta y$ is introduced so that $\sum_i f(y_i)\Delta y_i = 1$). The procedure outlined will, at best, yield a discrete frequency distribution for the sample of recorded measurements. But the distribution may vary from sample to sample. The distribution function $f(y)$ in Eq. (11.15) represents the entire population of random variable $y(t)$ and not just one or more samples of observations. If we are unable, therefore, to establish the distribution function $f(y)$ how can we then assess the probability of exceeding a specified response y_k? Two very powerful statistical theorems come to our aid, *the Chebychev inequality*, and the *central limit theorem*.

11.9 Chebychev's Inequality

The Chebychev inequality* states that for a random variable y from a population with mean μ and variance σ^2

$$Pr(|y-\mu| > k\sigma) < \frac{1}{k^2} \qquad (11.16)$$

in which k is an arbitrary positive constant. Stated in words, inequality (11.16) reads: The probability that random variable y deviates from its population mean by more than k standard deviations† (σ) is smaller than $1/k^2$, where k is an arbitrary positive constant. There is no mention of the probability density function $f(y)$ of y. Consequently, it is possible to set an upper limit to the probability of exceeding a specified y_k without knowledge of $f(y)$. Substituting $y_k = k\sigma$ into Eq. (11.16) we write

$$Pr(|y-\mu| > y_k) < \frac{\sigma^2}{y_k^2}$$

and for the special case when $\mu = 0$ this becomes

$$Pr(|y| > y_k) < \frac{\sigma^2}{y_k^2} \qquad (11.17)$$

It still remains to establish σ^2 and μ. If these parameters are not known, they may be estimated [See Eqs. (11.29 and 11.30)].

The Chebychev inequality is too broad for most practical applications; that is, the upper limit $1/k^2$ of Eq. (11.16) is too high. The probability statement of Eq. (11.16) can be improved when more is known regarding the probability density function of the random variable. A step in this direction is offered by the *central limit theorem.*

11.10 The Central Limit Theorem

Let $\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_n$ be independent variables having the same distribution with means μ_i and finite variances σ_i^2. It can be shown[38] that as $n \rightarrow \infty$ the distribution of a new random variable y given by

$$y = \sum_i \alpha_i$$

approaches the *normal distribution function* (also referred to as the *Gaussian distribution function*) with mean

*See Reference 38 for proof (in this reference the name is spelled as Tchebysheff).
†The *Standard deviation* is the square root of the variance σ^2.

$$\mu_y = \sum_i \mu_i \qquad \text{(a)}$$

and variance (11.18)

$$\sigma_y^2 = \sum_i \sigma_i^2 \qquad \text{(b)}$$

This is a statement of the *Central Limit Theorem.*

The *normal distribution* function of a random variable y mentioned here is given by

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-(y-\mu_y)^2/2\sigma_y^2} \tag{11.19}$$

in which μ_y and σ_y^2 are, respectively, the mean and variance of y. The normal distribution function is completely characterized by the two parameters μ and σ^2. The central limit theorem explains why the normal distribution function is very important and so widely used.

11.11 The Central Limit Theorem Applied to an Ergodic Process

Let $y(t)$ be a representative record of an ergodic process, and for convenience let us set the mean value of $y(t)$ equal to zero*

$$E[y(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)\, dt = 0$$

We expand $y(t)$ by a Fourier series and write it as a linear combination of random variables $\alpha_k (k = 1, 2, \ldots, \infty)$

$$y(t) = \sum_{k=1}^{\infty} \alpha_k(t) \tag{11.20}$$

in which

$$\alpha_k(t) = a_k \sin k\,\Omega t + b_k \cos k\Omega t$$

The mean μ_k and variance σ_k^2 of the kth variable α_k are given, respectively, by

$$\mu_k = E[\alpha_k(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \alpha_k(t)\, dt = 0$$

and

$$\sigma_k^2 = E[\alpha_k - E(\alpha_k)]^2 = E(\alpha_k^2) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \alpha_k^2(t)\, dt$$

Using Eqs. (11.11) and (11.12) derived for the mean and variance of a linear combination of random variables, and assuming the α_k to be statistically independent it follows that

$$\mu_y = \sum_k \mu_k = 0 \tag{11.21}$$

*This can be accomplished by a coordinate translation.

and

$$\sigma_y^2 = \sum_k \sigma_k^2 = \sum_k \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \alpha_k^2(t)\,dt \tag{11.22}$$

It will now be shown that σ_y^2 of Eq. (11.22) is the mean square $\overline{y^2(t)}$. Using Eq. (11.14b) we write

$$\overline{y^2(t)} = E[y^2(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y^2(t)\,dt \tag{11.23}$$

Substituting for $y(t)$ from Eq. (11.20) into (11.23) we have

$$\overline{y^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_k \sum_l \alpha_k(t)\alpha_l(t)\,dt \tag{11.24}$$

But

$$\begin{aligned} \int_{-T}^{T} \alpha_k(t)\alpha_l(t)\,dt &= 0 && \text{for } k \neq l \\ &= \int_{-T}^{T} \alpha_k^2(t)\,dt && \text{for } k = l \end{aligned}$$

This follows from the orthogonality of the sine and cosine terms contained in α_k. Consequently, Eq. (11.24) becomes

$$\overline{y^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_k \alpha_k^2(t)\,dt \tag{11.25}$$

Interchanging the order of summation and integration in Eq. (11.25) we write

$$\overline{y^2(t)} = \sum_k \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \alpha_k^2(t)\,dt \tag{11.26}$$

We now compare Eqs. (11.22) and (11.26) and conclude that

$$\sigma_y^2 = \sum_k \sigma_k^2 = \overline{y^2(t)} \tag{11.27}$$

Applying the central limit theorem to random variable $y(t)$ as expressed by Eq. (11.20) we can state that the probability density function of this variable approaches a normal distribution with mean

$$\mu_y = \sum_k \alpha_k \qquad \text{[See Eq. (11.18a)]} \tag{11.28}$$

and variance

$$\sigma_y^2 = \sum_k \sigma_k^2 \qquad \text{[See Eq (11.18b)]} \tag{11.29}$$

Substituting relations (11.21) and (11.27) into (11.28) and (11.29), respectively, we write

$$\mu_y = 0$$
$$\sigma_y^2 = \overline{y^2(t)}$$

Estimates of σ_y^2 and μ_y. We note, however, that to evaluate $\overline{y^2(t)}$ from Eq. (11.23) or Eq. (11.25) we need a record of infinite duration

($T \to \infty$). Obviously we cannot accomplish this. In practice we will approximate $\overline{y^2(t)}$ from a record of finite duration which constitutes a sample of the population. The value of $\overline{y^2(t)}$ thus computed yields an estimate $\hat{\sigma}_y^2$ of the true variance σ_y^2 of the population

$$\hat{\sigma}_y^2 = \frac{1}{2T}\int_{-T}^{T} y^2(t)\,dt \tag{11.30}$$

Similarly for an estimate $\hat{\mu}_y$ of the mean value μ_y of $y(t)$

$$\hat{\mu}_y = \frac{1}{2T}\int_{-T}^{T} y(t)\,dt \tag{11.31}$$

It can be shown[38] by statistical sampling theory that $\hat{\sigma}_y^2$ and $\hat{\mu}_y$ from Eqs. (11.30) and (11.31) have the properties of "good" estimators, that is, estimators whose distribution is concentrated near the true value of the parameter being estimated. (See Fig. 11.5.)

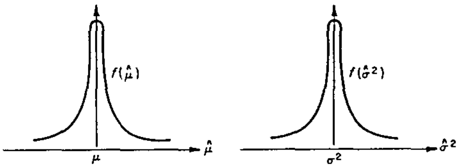


Figure 11.5 Distributions of good estimators.

With $\hat{\sigma}_y^2 \approx \overline{y^2(t)} = \sigma_y^2$ computed and the distribution of $y(t)$ given, approximately, by the normal distribution function

$$f(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-(1/2)[y-E(y)]^2/\sigma_y^2}$$

it is possible to evaluate probability statements such as

$$Pr(y > y_k) = \int_{y_k}^{\infty} f(y)\,dy$$

Most standard books in statistics have $Pr(|\xi| > \xi_k)$ tabulated for a wide range of values ξ. The variable ξ is known as the *standardized normal variable* and is linearly related to the original random variable y through the relation

$$\xi = \frac{y - E(y)}{\sigma_y} \tag{11.32}$$

so that

$$\mu_\xi = E(\xi) = 0$$

and

$$\sigma_\xi^2 = E[\xi - E(\xi)]^2 = 1$$

Mean Square of Excitation and Response. So far it has been shown that the mean square value $\overline{y^2(t)}$ of a random variable $y(t)$ in a random process is of interest since it enables us to make probability statements with reference to the variable. For instance, if $y(t)$ represents the response of the system of Fig. 11.4a when it is excited by a random forcing function, then we can make statements with reference to the probability of exceeding a specified response provided $\overline{y^2(t)}$ is given. This may be done either by using the Chebychev inequality [Eq. (11.17)] or, in view of the central limit theorem, by assuming the distribution of $y(t)$ to be approximately normal [Eq. (11.19)], with variance $\sigma_y^2 \approx \hat{\sigma}_y^2$ and mean $\mu_y \approx \hat{\mu}_y$ as given by Eqs. (11.30) and (11.31), respectively. But to evaluate the mean square response $\overline{y^2(t)}$ we must conduct an experiment to collect records of the response to simulated random excitations and compute $\overline{y^2(t)}$. This would mean a new experiment for each new linear system subjected to the same random excitation. A simpler procedure is to characterize the random excitation $f(t)$ through its mean square value $\overline{f^2(t)}$, characterize the linear system through its complex frequency response $H(\Omega)$, and then find a relation that expresses the mean square response $\overline{y^2(t)}$ as a function of $\overline{f^2(t)}$ and $H(\Omega)$. Such a functional relationship can be derived but first we must introduce the *Parseval theorem* and the concept of *Power Spectral Density Function* (*p.s.d.f.*). Before proceeding to develop these concepts note that for a linear system, when the excitation $f(t)$ is approximately normally distributed with zero mean and variance $\sigma_f^2 \approx \overline{f^2(t)}$, the response $y(t)$ is also approximately normally distributed with zero mean and variance $\sigma_y^2 \approx \overline{y^2(t)}$. This follows from the fact that the response of the linear system can be thought of as a linear function of the excitation. Consequently, when the excitation is normal the response is also normal, and once $\overline{y^2(t)}$ is computed we can make probability statements regarding the response.

11.12 The Parseval Theorem

Let $y_1(t), Y_1(\Omega)$ and $y_2(t), Y_2(\Omega)$ be two Fourier transform pairs. $y_1(t)$ and $y_2(t)$ are real functions each representative of an ergodic process. If we use the Fourier transform $Y_2(\Omega)$ to express $y_2(t)$ [see Eq. (10.74), Chapter 10] the product $y_1(t)\, y_2(t)$ may be written as

$$y_1(t)\, y_2(t) = y_1(t) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} Y_2(\Omega)\, e^{i\Omega t}\, d\Omega \right\} \tag{11.33}$$

Integrating Eq. (11.33) in the range $-\infty < t < \infty$ we write

$$\int_{-\infty}^{\infty} y_1(t)\, y_2(t)\, dt = \int_{-\infty}^{\infty} y_1(t) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} Y_2(\Omega)\, e^{i\Omega t}\, d\Omega \right\} dt$$

If we change the order of integration we have

$$\begin{aligned} \int_{-\infty}^{\infty} y_1(t)\, y_2(t)\, dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y_2(\Omega) \left\{ \int_{-\infty}^{\infty} y_1(t)\, e^{i\Omega t} dt \right\} d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y_2(\Omega)\, \overline{Y_1(\Omega)}\, d\Omega \end{aligned} \tag{11.34}$$

in which

$$\overline{Y_1(\Omega)} = \int_{-\infty}^{\infty} y_1(t)\, e^{i\Omega t} dt = Y_1(-\Omega)$$

$\overline{Y_1(\Omega)}$ is the complex conjugate of $Y_1(\Omega)$.

Equation (11.34) is the statement of Parseval's theorem. For the special case where $y_1(t) = y_2(t) = y(t)$ Eq. (11.34) becomes

$$\begin{aligned} \int_{-\infty}^{\infty} y^2(t)\, dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\Omega)\, \overline{Y(\Omega)}\, d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\Omega)|^2\, d\Omega \end{aligned} \tag{11.35}$$

The expression $|Y(\Omega)|^2$ in Eq. (11.35) is an even function of Ω,

$$|Y(\Omega)|^2 = |Y(-\Omega)|^2$$

consequently, Eq. (11.35) may be written as

$$\int_{-\infty}^{\infty} y^2(t)\, dt = \frac{1}{\pi} \int_0^{\infty} |Y(\Omega)|^2\, d\Omega \tag{11.36}$$

In view of Eq. (11.36) the mean square value $\overline{y^2(t)}$ of $y(t)$ becomes

$$\overline{y^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y^2(t)\, dt = \lim_{T\to\infty} \frac{1}{2\pi} \int_0^{T} \frac{|Y(\Omega)|^2}{T}\, d\Omega \tag{11.37}$$

11.13 The Power Spectral Density Function (p. s. d. f.)

We define a new function $y(\Omega)$ from the relation

$$y(\Omega) = \frac{|Y(\Omega)|^2}{T} \tag{11.38}$$

The function $y(\Omega)$ is referred to as the *power spectral density function* of $y(t)$. Substituting relation (11.38) into Eq. (11.37) the mean square $\overline{y^2(t)}$ may be expressed in terms of the power spectral density function in the form

$$\overline{y^2(t)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y^2(t)\,dt = \frac{1}{2\pi}\int_0^\infty y(\Omega)\,d\Omega \tag{11.39}$$

From Eq. (11.38) it is seen that the power spectral density function $y(\Omega)$ is a function of the square of the magnitude of the amplitude function $Y(\Omega)$ in the Fourier integral expansion of $y(t)$. To attach more physical significance to the concept of the p. s. d. f., $y(\Omega)$, we refer to Figs. 11.6 and 11.7 in which we show how the mean square, $\overline{y^2(t)}$, of a variable, $y(t)$, is derived from the time function, $y(t)$, and from its p. s. d. f., $y(\Omega)$, respectively. In Fig. 11.7 the quantity $(1/2\pi)y(\Omega_k)\Delta\Omega$ represents the contribution $\Delta\overline{y_k^2(t)}$ to $\overline{y^2(t)}$ of components with frequencies in the interval

$$\Omega_k - \Delta\Omega/2 < \Omega < \Omega_k + \Delta\Omega/2$$

in the Fourier integral expansion of $y(t)$.

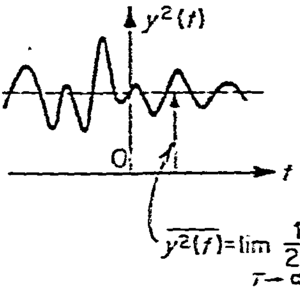


Figure 11.6 Mean square value $\overline{y^2(t)}$ as derived from $y(t)$.

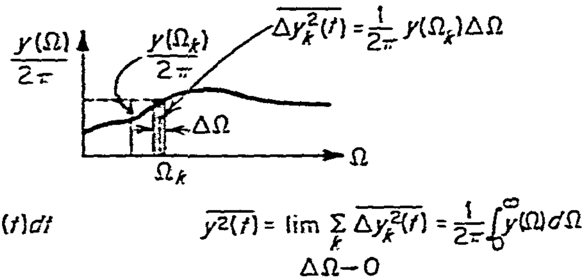


Figure 11.7 Mean square value $\overline{y^2(t)}$ as derived from p.s.d.f. $y(\Omega)$.

In terms of physical reality, the spectral density $y(\Omega_k)$ at any particular frequency Ω_k may be regarded as the average power

$$\frac{1}{2T}\int_{-T}^{T} y^2(t)\,dt$$

passing when a random signal $y(t)$ is filtered by a narrow band pass filter centered at Ω_k. Methods for calculating the power spectral density function of an excitation will be discussed in Section 11.15.

11.14 Response of a Single-Degree-of-Freedom Linear System Subjected to a Random Excitation

The tools developed thus far will now be used to derive the mean square response of a single-degree-of-freedom linear system subjected to a random excitation.

p.s.d.f. $y(\Omega)$ of the Response in Terms of p.s.d.f. $f(\Omega)$ of the Excitation. Let us consider again the single-degree-of-freedom system of Fig. 10.1 and subject it to a forcing function $f(\tau)$. For consistency with the notation used earlier in this chapter, we designate the response by $y(t)$. Using the convolution integral discussed in Section 8 of Chapter 8, and given by Eq. (8.83) we write for the response

$$y(t) = \int_0^t h(t-\tau) f(\tau)\, d\tau \tag{11.40}$$

in which

$$\begin{aligned} h(t-\tau) &= 0 \quad \text{for } t-\tau < 0 \quad \text{or } t < \tau \\ &\neq 0 \quad \text{for } t-\tau > 0 \quad \text{or } t > \tau \\ f(\tau) &= 0 \quad \text{for } \tau < 0 \\ &\neq 0 \quad \text{for } \tau > 0 \end{aligned} \tag{11.41}$$

For completeness we repeat here the physical interpretation to Eq. (11.40) as was done at the end of Section 8.8, Chapter 8. In Eq. (11.40) $h(t-\tau)$ represents the response at time t due to a unit impulse exciting the system $(t-\tau)$ units of time earlier. For an impulse $f(\tau)\,d\tau$ the response at time t is

$$h(t-\tau) f(\tau)\, d\tau$$

Superposing the responses due to all impulses $f(\tau)\,d\tau$ which comprise the exciting force function $f(\tau)$ we obtain the integral of Eq. (11.40). (See Fig. 8.1, Chapter 8.)

In view of the range of the functions $h(t-\tau)$ and $f(\tau)$ as expressed by Eq. (11.41), the limits of integration in Eq. (11.40) may be extended from $-\infty$ to $+\infty$ with no contribution to $y(t)$ from the integrand in the added range. Consequently, Eq. (11.40) can be written as

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau) f(\tau)\, d\tau \tag{11.42}$$

Applying a Fourier transformation to Eq. (11.42) we write

$$\int_{-\infty}^{\infty} y(t)\, e^{-i\Omega t}\, dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(t-\tau) f(\tau)\, d\tau \right] e^{-i\Omega t}\, dt$$

Changing the order of integration we have

$$Y(\Omega) = \int_{-\infty}^{\infty} f(\tau) \int_{-\infty}^{\infty} h(t-\tau)\, e^{-i\Omega t}\, dt\, d\tau \tag{11.43}$$

In Eq. (11.43) the inner integration is with respect to t.

We introduce now the following change of variable

$$\begin{aligned} \eta &= t - \tau \\ d\eta &= dt \end{aligned} \tag{11.44}$$

The limits of integration in Eq. (11.43) remain unchanged as a result of this change in variable because

$$\begin{array}{ll} \text{as } t \to \infty & \eta \to \infty \\ \text{and as } t \to -\infty & \eta \to -\infty \end{array}$$

Introducing the change of variable into Eq. (11.43) we obtain

$$Y(\Omega) = \int_{-\infty}^{\infty} f(\tau)\, e^{-i\Omega\tau}\, d\tau \int_{-\infty}^{\infty} h(\eta)\, e^{-i\Omega\eta}\, d\eta \tag{11.45}$$

The first and second integrals on the right-hand side of Eq. (11.45) represent the Fourier transform $F(\Omega)$ and $H(\Omega)$ of $f(\tau)$ and $h(\eta)$, respectively. Hence,

$$Y(\Omega) = F(\Omega)\, H(\Omega)$$

or (11.46)

$$Y(\Omega) = H(\Omega)\, F(\Omega)$$

$H(\Omega)$ is the Fourier transform of the response $h(\eta)$ to a unit impulse. Then according to Section 10.12, Chapter 10, it is also the complex frequency response of the system. This justifies our use of the symbol $H(\Omega)$ in the above development. If we multiply each side of Eq. (11.46) by its complex conjugate and divide by T we have

$$\frac{Y(\Omega)\,\overline{Y(\Omega)}}{T} = \frac{1}{T} H(\Omega)\, F(\Omega)\overline{H(\Omega)\, F(\Omega)}$$

or

$$\frac{|Y(\Omega)|^2}{T} = |H(\Omega)|^2 \frac{|F(\Omega)|^2}{T} \tag{11.47}$$

In this equation

$$\frac{|Y(\Omega)|^2}{T} = y(\Omega) \quad \text{and} \quad \frac{|F(\Omega)|^2}{T} = f(\Omega)$$

are, respectively, the power spectral density function of response $y(t)$ and excitation $f(t)$. Using these relations in Eq. (11.47) the following simple expression relates $y(\Omega)$ to $f(\Omega)$ through the complex frequency response $H(\Omega)$ of the system

$$y(\Omega) = |H(\Omega)|^2 f(\Omega) \tag{11.48}$$

Table 11.2 summarizes the expressions for the response of a single-degree-of-freedom linear system in the time and frequency domain.

and

$$\psi(\tau) = \frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\Omega)\, e^{i\Omega\tau}\, d\Omega \tag{11.54}$$

Since $f(\Omega)$ is an even function of Ω, Eq. (11.54) can be written in the form

$$\psi(\tau) = \frac{1}{2\pi}\int_{0}^{\infty} f(\Omega)\, e^{i\Omega\tau}\, d\tau$$

We note that for the special case when $\tau = 0$, Eq. (11.54) becomes

$$\psi(0) = \frac{1}{2\pi}\int_{0}^{\infty} f(\Omega)\, d\Omega = \overline{f^2(t)} \qquad \text{[see Eq. (11.39)]} \tag{11.55}$$

The p.s.d. function of a random process represented by $f(t)$ may also be obtained in an experimental way. The random function $f(t)$ is fed into a spectrum analyzer. The analyzer transmits components of $f(t)$ within a narrow frequency range $\Delta\Omega$ centered at Ω_k. The output $\Delta\overline{f^2(t)}$, representing a fraction of the mean square value of $f(t)$, is recorded on a mean square meter after the values of $f(t)$ are squared and averaged. The intensity $f(\Omega_k)$ of the p.s.d.f. at frequency Ω_k is then obtained as

$$\frac{1}{2\pi} f(\Omega_k)\, \Delta\Omega = \Delta\overline{f^2(t)}$$

$$f(\Omega_k) = \frac{\Delta\overline{f^2(t)}}{\Delta\Omega}\, 2\pi \qquad \text{[See Fig. 11.8]}$$

Varying Ω_k, we obtain values of the p.s.d.f. $f(\Omega_k)$ at closely spaced discrete frequencies from which the p.s.d.f. $f(\Omega)$ is approximated.

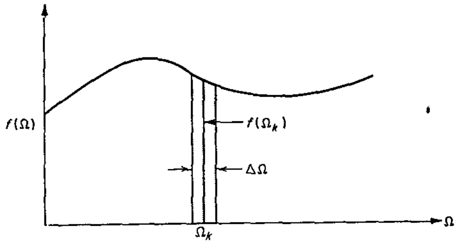


Figure 11.8

11.16 Response of Lightly Damped Single-Degree-of-Freedom Systems Subjected to a Random Excitation

Consider the single-degree-of-freedom system of Fig. 10.1 excited by a constant power spectral density function $f(\Omega) = f$ which is referred to as *white noise* (see Fig. 11.9). Substituting f for $f(\Omega)$ in Eq. (11.50) we have for the mean square response $\overline{y^2(t)}$ of the system

$$\overline{y^2(t)} = \frac{1}{2\pi} f \int_0^\infty |H(\Omega)|^2 \, d\Omega \tag{11.56}$$

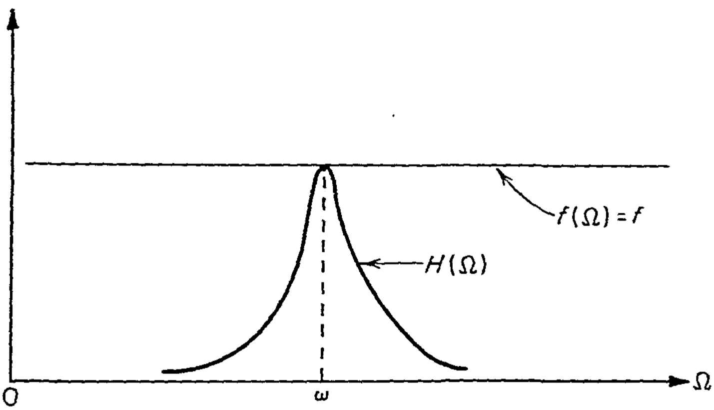


Figure 11.9

Replacing $|H(\Omega)|$ by its expression from Eq. (10.16a), Chapter 10, we write

$$\overline{y^2(t)} = \frac{f}{2\pi} \int_0^\infty \frac{1}{\left[1 - \left(\frac{\Omega}{\omega}\right)^2\right]^2 + 4\xi^2 \frac{\Omega^2}{\omega^2}} \, d\Omega \tag{11.57}$$

As in Chapter 10, ω in Eq. (11.57) designates the natural frequency of the single-degree-of-freedom system. For light damping in which $\xi \ll 1$ and $\sqrt{1 - \xi^2} \approx 1$, the integral of Eq. (11.57) is evaluated by the method of residues to yield

$$\overline{y^2(t)} = \frac{f}{2\pi} \int_0^\infty \frac{1}{\left[1 - \left(\frac{\Omega}{\omega}\right)^2\right]^2 + 4\xi^2 \frac{\Omega^2}{\omega^2}} \, d\Omega = \frac{f}{2\pi}\left(\frac{\pi}{4}\frac{\omega}{\xi}\right) = \frac{f\omega}{8\xi} \tag{11.58}$$

When viscous damping is replaced by structural damping the expression for the complex frequency response from Eq. (10.19a) is substituted in Eq. (11.56). Integrating the resulting expression under *the assumption of $g \ll 1$ the mean square response to white noise* takes the form

$$\overline{y^2(t)} = \frac{f}{2\pi}\int_0^\infty \frac{1}{\left[1-\frac{\Omega^2}{\omega^2}\right]^2 + g^2}\,d\Omega = \frac{f}{2\pi}\left(\frac{\pi}{2}\frac{\omega}{g}\right) = \frac{f\omega}{4g} \qquad (11.59)$$

The value of the integral

$$\int_0^\infty |H(\Omega)|^2\,d\Omega$$

in Eq. (11.56) is equal to

$$\frac{\pi}{4}\frac{\omega}{\zeta}$$

for viscous damping [see Eq. (11.58)] *and to*

$$\frac{\pi}{2}\frac{\omega}{g}$$

for structural damping [see Eq. (11.59)]. These two results are rewritten in a more convenient form for future reference as

$$\int_0^\infty |H(\Omega)|^2\,d\Omega = \frac{\pi}{2}\left(\frac{1}{2\zeta}\right)^2(2\zeta\omega) \qquad (11.60)$$

for a single-degree-of-freedom system with viscous damping, and

$$\int_0^\infty |H(\Omega)|^2\,d\Omega = \frac{\pi}{2}\left(\frac{1}{g}\right)^2(g\omega) \qquad (11.61)$$

for a single-degree-of-freedom with structural damping.

From Eqs. (10.16a) and (10.19a) it follows that the peak value of $|H(\Omega)|$ occurs at the natural frequency of the system, ω, and is given by

$$\left.\begin{aligned} |H(\omega)| &= \frac{1}{2\zeta}, \text{ for viscous damping} \\ &= \frac{1}{g}, \text{ for structural damping} \end{aligned}\right\} \qquad (11.62)$$

The frequencies Ω_1, Ω_2, (Fig. 11.10), for which

$$|H(\Omega_1)|^2 = |H(\Omega_2)|^2 = \tfrac{1}{2}|H(\omega)|^2 \qquad (11.63)$$

are called the *half power points*. The frequency difference $(\Omega_2 - \Omega_1)$ is referred to as the *bandwidth* of the magnitude of the complex frequency response function $|H(\Omega)|$. It can be established from Eqs. (10.16a), (10.19a), (11.62), and (11.63) that the bandwidth $(\Omega_2 - \Omega_1)$ is given by

$$\left.\begin{aligned} &2\zeta\omega \text{ for viscous damping} \\ \text{and by}\quad & \\ &g\omega \text{ for structural damping} \end{aligned}\right\} \qquad (11.64)$$

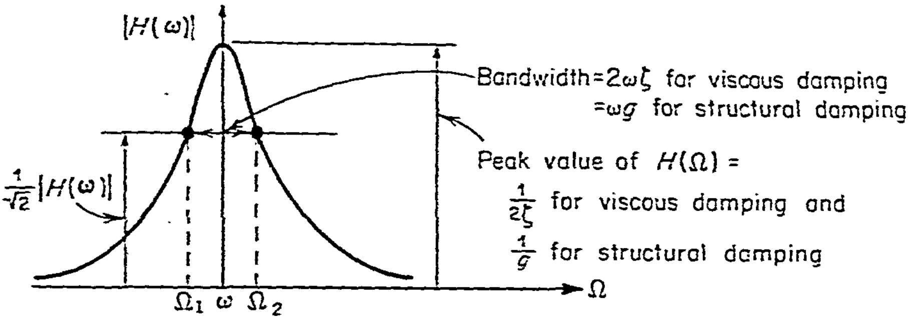


Figure 11.10 A plot of $|H(\Omega)|$ vs. Ω for a single-degree-of-freedom system.

If we identify the terms on the right-hand side of Eqs. (11.60) and (11.61) by referring to Eqs. (11.62) and (11.64) we can write Eqs. (11.60) and (11.61) for a lightly damped single d.o.f. system in the general form

$$\int_0^\infty |H(\Omega)|^2\, d\Omega = \frac{\pi}{2}\,(\text{peak value of } |H(\Omega)|)^2\ (\text{bandwidth}) \tag{11.65}$$

This equation is very useful in computing the response to a random excitation. For a lightly damped system the contribution to the response is small except in the vicinity of the natural frequency. Consequently, the mean square response may be approximated by considering $f(\Omega)$ a constant equal to $f(\omega)$ (see Fig. 11.11), and writing

$$\overline{y^2(t)} = \frac{1}{2\pi} f(\omega) \int_0^\infty |H(\Omega)|^2\, d\Omega \tag{11.66}$$

in which the integral is evaluated from Eq. (11.65) and results, of course, in the same expressions as (11.58) or (11.59) depending on whether viscous or structural damping is present.

A lightly damped system excited by a nearly constant psd function

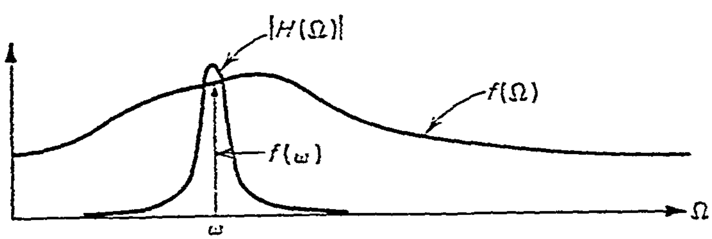


Figure 11.11 A lightly damped system excited by a nearly constant p.s.d. function.

11.17 The Use of Random Vibration Studies as a Tool in the Design and Analysis of a System

In this section we shall apply our studies in the preceding sections to the design of a single-degree-of-freedom system subjected to a random excitation. The analysis of the system will serve as a link in the design process as will be seen in the following.

Let us consider again the single-degree-of-freedom structure of Fig. 11.4a. We subject this structure to a random excitation $f(t)$ with a power spectral density function $f(\Omega)$ (see Fig. 11.12). The excitation

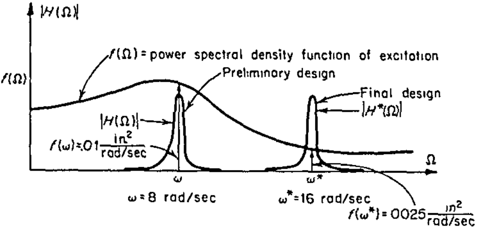


Figure 11.12

is assumed to be normal (Gaussian) with zero mean and variance $\sigma_f^2 \approx \overline{f^2(t)}$. This is a reasonable assumption for many practical applications in view of the central limit theorem. Because of the sensitive equipment mounted on the structure the lateral displacement $y(t)$ must not exceed 3 inches if the equipment is to function properly. Since we deal with a random phenomenon we cannot guarantee that this displacement will not be exceeded. However, we can attempt to make the *probability* of exceeding this response very small.

Evaluation of a Preliminary Design. Let us assume that a preliminary design resulted in a structure with a natural frequency of $\omega = 8$ rad/sec and a coefficient of structural damping $g = 0.02$. The plot of $|H(\Omega)|$ vs. Ω for the system is shown in Fig. 11.12. We can approximate the mean square response of this structure by considering $f(\Omega)$ to have a constant value of $f(\omega)$ and then applying Eq. (11.66). Thus

$$\overline{y^2(t)} = \frac{1}{2\pi} f(\omega) \int_0^\infty |H(\Omega)|^2 \, d\Omega$$

more $f(y)$ is spread. This is also seen in Fig. 11.13 where we plotted two normal distribution functions having the same zero mean (for convenience) but different variances. We recall from Eq. (11.1b) that the area under each of the normal distribution functions in Fig. 11.13 is unity.

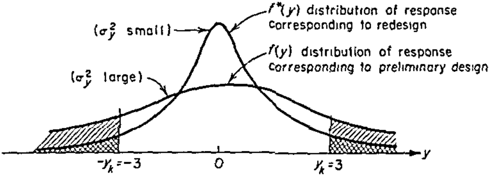


Figure 11.13 The total area marked by ////// which lies between $f(y)$ and the y axis and extends to ∞ and $-\infty$ represents $Pr(|y(t)| > y_k)$ for the preliminary design. The total area marked by ▩ which lies between $f^*(y)$ and the y axis and extends to ∞ and $-\infty$ represents $Pr(|y(t)| > y_k)$ for the redesign.

We also observe from Fig. 11.13 that when σ_y^2 is smaller, the probability of exceeding a specified response y_k is smaller. Returning now to our design problem we note from Eq. (11.66) that the mean square response $\overline{y^2(t)}$ gets smaller as $f(\omega)$ gets smaller. A re-examination of Fig. 11.12 shows that if we redesign our system with a natural frequency $\omega^* > \omega$, for instance $\omega^* = 2\omega$, then the value of the p.s.d.f. $f(\Omega)$ at ω^* is much smaller than at ω, or

$$f(\omega^*) \ll f(\omega)$$

From Eq. (11.59) we have then

$$\begin{aligned}\overline{y^2(t)} &= \frac{f(\omega^*)\,\omega^*}{4g} \\ &= \frac{0.0025 \times 16}{4 \times 0.02} \\ &= 0.5 \text{ in.}^2 < 1\end{aligned}$$

where $f(\omega^*)$ and ω^* are obtained from Fig. 11.12. Since $\overline{y^2(t)}$ serves as an estimate of the variance σ_y^2 of the response $y(t)$ then σ_y^2, in our redesign, is smaller than the corresponding variance in the preliminary design. Consequently, we expect the probability of exceeding the response of 3 inches to be smaller. We shall now evaluate this probability.

The mean of the response is zero as before and the variance is estimated as

$$\sigma_y^2 \approx \overline{y^2(t)} = 0.5 \text{ in.}^2$$

Assuming a normal distribution we have from Eq. (11.19)

$$f(y) = \frac{1}{\sqrt{2\pi\, 0.5}}\, e^{-\frac{y^2}{2\times 0.5}} \tag{11.70}$$

Substituting Eq. (11.70) into Eq. (11.67) we obtain

$$Pr(|y(t)| > 3) = 0.000022 < 0.0027$$

A more direct approach to computing this probability can be employed by first transforming random variable y to a standardized normal variable ξ and then using the tables for the normal distribution. This was discussed in Section 11 of this chapter. From Eq. (11.32) we write in the present case

$$\xi = \frac{y - \mu_y}{\sigma_y} = \frac{y - 0}{\sqrt{0.5}} = \frac{y}{\sqrt{0.5}}$$

and

$$\mu_\xi = E(\xi) = 0$$

$$\sigma_\xi^2 = E(\xi - \mu_\xi)^2 = 1$$

hence

$$f(\xi) = \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2}$$

Also, when

$$y = \pm 3 \qquad \xi = \pm\frac{3}{\sqrt{0.5}} = \pm 4.25$$

In terms of random variable ξ we write Eq. (11.67) in the form

$$\begin{aligned} Pr(|y(t)| > 3) &= Pr(|\xi| > 4.25) \\ &= \int_{-\infty}^{-4.25} \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2}\, d\xi + \int_{4.25}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2}\, d\xi \\ &= 0.000022 \end{aligned}$$

where the integrals are obtained from normal distribution tables.[39] Hence, our redesign, based on our knowledge of the power spectral density function $f(\Omega)$ of the excitation, resulted in more than a 100 fold reduction in the probability of exceeding the specified critical response of 3 inches.

11.18 Random Excitations Applied to Multi-Degree-of-Freedom Systems

For simplicity and clarity in the presentation we shall discuss an n degree-of-freedom system with proportional damping. From the normal mode method of Chapter 8 we recall that the equations of motion for such a system become decoupled when normal coordinates η_r $(r = 1, 2, \ldots, n)$ are used and the rth decoupled equation has the form

$$\ddot{\eta}_r + 2\zeta_r\omega_r\dot{\eta}_r + \omega_r^2\eta_r = \frac{P_0\Gamma_r}{M_r}f(t) \qquad (11.71)$$

$$r = 1, 2, \ldots, n$$

Equation (11.71) is identical to Eq. (8.29) in Chapter 8 except that the constant β of Eq. (8.29) is replaced by $\zeta_r\omega_r$ in Eq. (11.71). This is done for consistency with the development in Chapter 10 and the earlier sections of this chapter. The left-hand side of Eq. (11.71) for any r can be thought of as representing a single-degree-of-freedom system with natural frequency ω_r and a fraction of critical damping given by ζ_r (compare with Eq. 10.3a, Chapter 10).

If we substitute for $f(t)$ in Eq. (11.71) a simple harmonic forcing function

$$f(t) = e^{i\Omega t}$$

then from Section 2, Chapter 10, the steady-state solution of Eq. (11.71) is given by

$$\eta_r(t) = \frac{P_0\Gamma_r}{\omega_r^2 M_r}\frac{1}{1 - \left(\dfrac{\Omega}{\omega_r}\right)^2 + i2\zeta_r\dfrac{\Omega}{\omega_r}}f(t)$$

$$= \frac{P_0\Gamma_r}{\omega_r^2 M_r}H_r(\Omega)f(t) \qquad (11.72)$$

$$r = 1, 2, \ldots, n$$

where

$$H_r(\Omega) = \frac{1}{1 - \left(\dfrac{\Omega}{\omega_r}\right)^2 + i2\zeta_r\dfrac{\Omega}{\omega_r}}$$

The response $y(x, t)$ of point x on the system at time t can be expressed in terms of the normal modes $\Phi_r(x)$ and the normal coordinates $\eta_r(t)$ by applying Eq. (8.2) of Chapter 8

$$y(x, t) = \sum_{r=1}^{n} \Phi_r(x)\,\eta_r(t) \qquad (11.73)$$

We note again that y is used in this chapter to denote displacement. The mean square of the response $y(x, t)$ is written as

$$\overline{y^2(x, t)} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} y^2(x, t)\,dt \qquad (11.74)$$

Substituting in Eq. (11.74) from Eq. (11.73) and then from Eq. (11.72) we have

$$\overline{y^2(x,t)} = \lim_{T\to\infty} \sum_{r=1}^{n} \sum_{s=1}^{n} \Phi_r(x)\Phi_s(x) \frac{1}{2T} \int_{-T}^{T} \eta_r(t)\eta_s(t)\,dt$$

$$= \sum_{r=1}^{n} \sum_{s=1}^{n} \Phi_r(x)\Phi_s(x) \frac{P_0^2 \Gamma_r \Gamma_s}{\omega_r^2 \omega_s^2 M_r M_s} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} H_r(\Omega) H_s(\Omega) f^2(t)\,dt \tag{11.75}$$

From Eq. (10.14), Chapter 10, we can write

$$H_r(\Omega) = |H_r(\Omega)|\, e^{i\psi_r}$$

$$r = 1, 2, \ldots, n$$

where

$$|H_r(\Omega)| = \left\{\left[1 - \left(\frac{\Omega}{\omega_r}\right)^2\right]^2 + \left(2\zeta_r \frac{\Omega}{\omega_r}\right)^2\right\}^{-1/2} \quad \text{[See Eq. (10.16a)]}$$

and ψ_r is the phase angle given by

$$\psi_r = \tan^{-1} \frac{2\zeta_r \dfrac{\Omega}{\omega_r}}{1 - \left(\dfrac{\Omega}{\omega_r}\right)^2} \quad \text{[See Eq. (10.16b), Chapter 10].}$$

If as an approximation[40] we disregard phase relations which will tend to result in a higher mean square value in Eq. (11.75) then the integral on the right-hand side of this equation can be written as

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} H_r(\Omega) H_s(\Omega) f^2(t)\,dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |H_r(\Omega)|\,|H_s(\Omega)| f^2(t)\,dt \tag{11.76}$$

When the forcing function $f(t)$ is a representative record of an ergodic random process, we can transform the limiting process of Eq. (11.76) from the time domain to the frequency domain because the function $f(t)$ is then represented by frequency components in a continuous frequency spectrum $0 < \Omega < \infty$. Applying Eq. (11.39) we write for the mean square value $\overline{f^2(t)}$ of $f(t)$

$$\overline{f^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f^2(t)\,dt = \frac{1}{2\pi} \int_0^\infty f(\Omega)\,d\Omega$$

Using this transformation from the time to the frequency domain in Eq. (11.76) we write

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |H_r(\Omega)|\,|H_s(\Omega)|\, f^2(t)\,dt$$
$$= \frac{1}{2\pi} \int_0^\infty |H_r(\Omega)|\,|H_s(\Omega)|\, f(\Omega)\,d\Omega \tag{11.77}$$

Substituting from Eq. (11.77) into Eq. (11.75) we write for the mean square response of an n degree-of-freedom system excited by a random forcing function

$$\overline{y^2(x,t)} = \sum_{r=1}^{n}\sum_{s=1}^{n} \Phi_r(x)\Phi_s(x)\frac{P_0^2\Gamma_r\Gamma_s}{\omega_r^2\omega_s^2 M_r M_s}\frac{1}{2\pi}\int_0^\infty |H_r(\Omega)||H_s(\Omega)|f(\Omega)\,d\Omega \tag{11.78}$$

We note here that in going from a simple harmonic force to a random forcing function that is represented in the frequency domain by components in a continuous frequency spectrum, the steady-state response transforms into the transient response. (See discussion in Section 10, Chapter 10.)

11.19 An Approximate Solution to the Random Vibration of Multi-Degree-of-Freedom Systems

The solution of Eq. (11.78) of the last section can be simplified if we introduce another approximation which is discussed in the following. Consider Fig. 11.14 showing the magnification factors

$$|H_r(\Omega)| = \left[\left(1 - \frac{\Omega^2}{\omega_r^2}\right)^2 + 4\zeta_r^2\frac{\Omega^2}{\omega_r^2}\right]^{-1/2}$$

$$r = 1, 2, \ldots, n$$

for a lightly damped multi-degree-of-freedom system. The magnification factors $|H_r(\Omega)|$ have regions of pronounced peaks in the neighborhood of the corresponding natural frequencies ω_r. The products $|H_r(\Omega)||H_s(\Omega)|$ for $r \neq s$ are seen to be small in comparison with the same products for $r = s$. In addition, in Eq. (11.78), terms with $r \neq s$ may be negative as well as positive depending upon the sign of the product $\Phi_r(x)\Phi_s(x)\Gamma_r\Gamma_s$ while terms with $r = s$ are always positive. The contribution of cross product terms $(r \neq s)$ to the mean square response [Eq. (11.78)] will therefore be small. Hence,

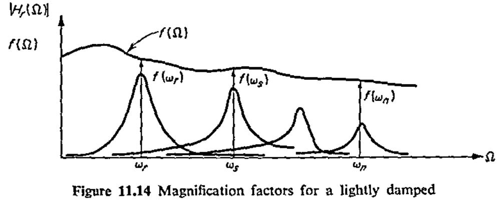


Figure 11.14 Magnification factors for a lightly damped multi-degree-of-freedom system.

we approximate $\overline{y^2(t)}$ from Eq. (11.78) by disregarding the cross product terms corresponding to $r \neq s$ and write

$$\overline{y^2(x,t)} = \sum_{r=1}^{n} \Phi_r^2(x) \frac{P_c^2\Gamma_r^2}{\omega_r^4 M_r^2} \frac{1}{2\pi} \int_0^\infty |H_r(\Omega)|^2 f(\Omega)\, d\Omega \qquad (11.79)$$

We note that the mean square response expressed by Eq. (11.79) was arrived at by introducing two approximations. First, we disregarded the phase relationship in Eq. (11.75). Then, we disregarded the cross product terms in the resulting Eq. (11.78). However, Eq. (11.79) may not necessarily result in a conservative evaluation of $\overline{y^2(x,t)}$ although it represents a reasonable approximation for lightly damped systems.

The integrals of Eq. (11.79) can be approximated by replacing $f(\Omega)$ by its discrete values $f(\omega_r)$ at the natural frequencies ω_r (see Fig. 11.14) and using Eq. (11.58) when viscous damping is present. Proceeding this way Eq. (11.79) is approximated by the following relation

$$\overline{y^2(x,t)} = \sum_{r=1}^{n} \Phi_r^2(x) \frac{P_c^2\Gamma_r^2}{\omega_r^4 M_r^2} \frac{f(\omega_r)\,\omega_r}{8\zeta_r} \qquad (11.80)$$

For a multi-degree-of-freedom system with structural damping, the decoupled equations of motion have the form

$$\ddot{\eta}_r(t) + (1 + ig)\,\omega_r^2 \eta_r(t) = \frac{P_c\Gamma_r}{M_r} f(t) \qquad (11.81)$$

and Eq. (11.79) still applies; except

$$|H_r(\Omega)|^2 = \left[\left(1 - \frac{\Omega^2}{\omega_r^2}\right)^2 + g^2\right]^{-1}$$

If we use the results of Eq. (11.59) in Eq. (11.79) and replace $f(\Omega)$ with discrete values $f(\omega_r)$, we obtain the following approximate value for the mean square response

$$\overline{y^2(x,t)} = \sum_{r=1}^{n} \Phi_r^2(x) \frac{P_c^2\Gamma_r^2}{\omega_r^4 M_r^2} \frac{f(\omega_r)\,\omega_r}{4g} \qquad (11.82)$$

Using the result of Eq. (11.80) or (11.82) we have an estimate to the variance σ_y^2 at a point x on the system. Introducing the approximation of a normal distribution we can then make probability statements with reference to the response exceeding a specified value y_t at some point x. We can also make statements regarding the relationship between the responses at two points x_i, x_j or at more than two points. This requires the evaluation of the joint distribution function of the responses at the points of interest and is not treated here.*

*Some aspects of this problem are treated in Reference 41.

11.20 Example: The Response of a Three Story Building Excited by a Random Ground Acceleration

We shall now apply the results of the last section to evaluate the mean square response at the floor levels in a three story building excited by a random ground acceleration. The building is shown in Fig. 11.15. This is the same building as the one in Problem 11,

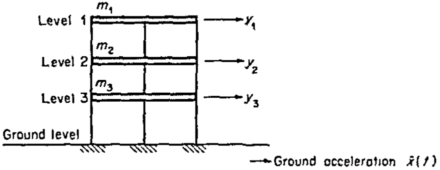


Figure 11.15

Chapter 8. The properties are repeated here for convenience. The mass and stiffness matrices in the y coordinates are

$$[m] = 38.4\,\frac{\text{kip}\cdot\text{sec}^2}{\text{ft}}\begin{bmatrix} 1.5 & & \\ & 1.0 & \\ & & 1.5 \end{bmatrix}$$

$$[k] = 10^3\,\frac{\text{kip}}{\text{ft}}\begin{bmatrix} 42 & -42 & 0 \\ -42 & 100 & -58 \\ 0 & -58 & 126 \end{bmatrix}$$

The normal modes and frequencies of the building are given by

$$\{\Phi^{(1)}\} = \begin{Bmatrix} 2.86 \\ 1.95 \\ 1.00 \end{Bmatrix} \qquad \omega_1 = 15.1 \text{ rad/sec}$$

$$\{\Phi^{(2)}\} = \begin{Bmatrix} -0.657 \\ 0.725 \\ 1.000 \end{Bmatrix} \qquad \omega_2 = 38.5 \text{ rad/sec}$$

$$\{\Phi^{(3)}\} = \begin{Bmatrix} 0.387 \\ -1.610 \\ 1.000 \end{Bmatrix} \qquad \omega_3 = 61.7 \text{ rad/sec}$$

The foundation of the building is subjected to a random ground acceleration $\ddot{x}(t)$ in the lateral direction as shown in Fig. 11.15. The power spectral density function $f(\Omega)$ of this excitation is considered constant*

$$f(\Omega) = 0.12 \frac{\text{ft}^2}{\text{sec}^4} \Big/ \frac{\text{rad}}{\text{sec}}$$

The distribution of $\ddot{x}(t)$ is approximately normal with zero mean and variance $\overline{\ddot{x}^2(t)}$. We are interested in computing the mean square of the response at each floor level so that we can make statements regarding the probability of exceeding some specified response.

In Fig. 11.15, y_i $(i = 1, 2, 3)$ denotes the absolute displacements of the floors. The ground or foundation displacement is denoted by x. Consequently, the floor displacements relative to the foundation are given by $(y_i - x)$ $(i = 1, 2, 3)$. Considering structural damping with a coefficient g $(g = 0.01)$ to be present, the equations of motion are

$$[m]\{\ddot{y}\} + (1 + ig)\,[k]\{y - x\} = \{0\}$$

We introduce a new variable

$$z_i = y_i - x$$

Hence

$$\ddot{y}_i = \ddot{z}_i + \ddot{x}$$

and the equations of motion become

$$[m]\{\ddot{z}\} + (1 + ig)[k]\{z\} = -[m]\{\ddot{x}\} \tag{11.83}$$

Since $[m]$ is a diagonal matrix and

$$\{\ddot{x}\} = \ddot{x}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

the right-hand side of Eq. (11.83) can be written as

$$-\ddot{x}\{m\}$$

where $\{m\}$ is a column matrix containing the three masses m_1, m_2, m_3. Eq. (11.83) then becomes

$$[m]\{\ddot{z}\} + (1 + ig)\,[k]\{z\} = -\ddot{x}\{m\} \tag{11.84}$$

We now express the relative floor displacements z_i in terms of the normal modes Φ and normal coordinates η_r $(r = 1, 2, 3)$

$$z_i(t) = \sum_r \Phi_{ir}\eta_r(t)$$

*This is a reasonable assumption according to Reference 42.

or

and

$$\{z\} = [\Phi]\{\eta\} \qquad \text{(a)}$$
$$\{\ddot{z}\} = [\Phi]\{\ddot{\eta}\} \qquad \text{(b)} \tag{11.85}$$

Substituting Eq. (11.85) into Eq. (11.84) and premultiplying the resulting equation by $[\Phi]^T$ we obtain a set of decoupled differential equations in the normal coordinates η_r. [See Chapter 8, Section 8.7, in particular Eqs. (8.68) and (8.69).]

$$[M_r]\{\ddot{\eta}\} + (1 + ig)[\omega_r^2][M_r]\{\eta\} = -\ddot{x}[\Phi]^T\{m\} \tag{11.86}$$

where

$$[M_r] = [\Phi]^T[m][\Phi] \tag{11.87}$$

The rth of Eq. (11.86) has the form

$$\ddot{\eta}_r + (1 + ig)\,\omega_r^2\eta_r = \frac{\Gamma_r}{M_r}f(t) \tag{11.88}$$

in which

$$\Gamma_r = \{\Phi^{(r)}\}^T\{m\} \tag{11.89}$$

and

$$f(t) = \ddot{x}(t)$$

Equation (11.88) is identical to Eq. (11.81) with $P_0 = 1$, hence, we can apply Eq. (11.82) to compute the mean square of the response

$$\overline{z_i^2(t)} = \sum_{r=1}^{3} \Phi_{ir}^2 \frac{\Gamma_r^2}{\omega_r^4 M_r^2}\frac{f(\Omega)\,\omega_r}{4g} = \frac{f(\Omega)}{4g}\sum_{r=1}^{3}\Phi_{ir}^2\frac{\Gamma_r^2}{\omega_r^3 M_r^2} \tag{11.90}$$

where $i = 1, 2, 3$ represents the three levels of the building in Fig. 11.15. In order to evaluate the mean squares $\overline{z_i^2(t)}$ from Eq. (11.90) we compute M_r and Γ_r from Eqs. (11.87) and (11.89), respectively, and construct the following table.

r	$\frac{\Gamma_r}{M_r}$	$\frac{\Gamma_r^2}{M_r^2}$	$\frac{1}{\omega_r^3}\frac{\Gamma_r^2}{M_r^2}$	$\frac{\Phi_{1r}^2}{\omega_r^3}\frac{\Gamma_r^2}{M_r^2}\cdot 10^3$	$\frac{\Phi_{2r}^2}{\omega_r^3}\frac{\Gamma_r^2}{M_r^2}\cdot 10^3$	$\frac{\Phi_{3r}^2}{\omega_r^3}\frac{\Gamma_r^2}{M_r^2}\cdot 10^3$
1	$\frac{7.74}{17.6}$	0.194	$0.056\cdot 10^{-3}$	0.459	0.213	0.056
2	$\frac{1.24}{2.7}$	0.21	$0.037\cdot 10^{-4}$	0.00159	0.0192	0.0037
3	$\frac{0.47}{4.3}$	0.012	$0.0051\cdot 10^{-5}$	0.00000765	0.000133	0.000051
			$10^3\sum_{r=1}^{3}\frac{\Phi_{ir}^2\Gamma_r^2}{\omega_r^3 M_r^2} =$	0.4606	0.2323	0.0597

Using these results and the constants $f(\Omega) = 0.12$ (ft^2/sec^4)/(rad/sec) and $g = 0.01$ in Eq. (11.90) we obtain

$$\sigma_1^2 = \overline{z_1^2(t)} = 1.3818 \cdot 10^{-3}\ \text{ft}^2$$
$$\sigma_2^2 = \overline{z_2^2(t)} = 0.6969 \cdot 10^{-3}\ \text{ft}^2$$
$$\sigma_3^2 = \overline{z_3^2(t)} = 0.1791 \cdot 10^{-3}\ \text{ft}^2$$

Because the excitation is approximately normally distributed with zero mean, it follows that the response of levels 1, 2, 3 is also approximately normally distributed with zero means and variances given by the respective mean square values $\overline{z_i^2(t)}$. With the $\overline{z_i^2(t)}$ available, we can make statements regarding the probability of exceeding a specified response with respect to the ground at any of the three levels. For instance, the probabilities that the displacement at level i relative to the ground will exceed the root mean square response $\sigma_i = \sqrt{\overline{z_i^2(t)}}$, $2\sigma_i$ and $3\sigma_i$ are given, respectively, by

$$Pr(|z_i(t)| > \sqrt{\overline{z_i^2(t)}}) = 0.3173$$
$$Pr(|z_i(t)| > 2\sqrt{\overline{z_i^2(t)}}) = 0.0455$$
$$Pr(|z_i(t)| > 3\sqrt{\overline{z_i^2(t)}}) = 0.0027$$

11.21 The "Apparent Frequency" of Response of a Lightly Damped Single d.o.f. System

So far in this chapter we have discussed the distribution of an instantaneous response $y(t)$ to a random excitation, and have made statements regarding the probability of exceeding a specified response at a point in any instant of time. Rice[43] deduced a number of interesting properties pertaining to the *entire record* of the response. Such values as the "apparent frequency," which is one-half of the expected number of times n_0 that the line $y(t) = 0$ is crossed by the response record in a unit of time, and the Rayleigh distribution, which is the probability distribution of the response envelope, are of great importance in fatigue studies.[44] In the following we discuss briefly the "apparent frequency." The Rayleigh distribution is discussed in the next section.

Apparent Frequency. Let $y(t)$ be a normally distributed response record with a power spectral density function $y(\Omega)$. Starting with the joint distribution function $f(y, dy/dt)$ an expression for the number n_0 of zero crossings per unit of time can be derived.[45] This expression has the form

$$n_0 = \frac{1}{\pi}\left[\frac{\int_0^\infty \Omega^2 y(\Omega)\, d\Omega}{\int_0^\infty y(\Omega)\, d\Omega}\right]^{1/2} \tag{11.91}$$

The expected number of times n_Y that $y(t)$ passes through a particular value $y(t) = Y$ per unit of time is given by[45]

$$n_Y = n_0 e^{-Y^2/2\sigma_y^2} \tag{11.92}$$

where

$$\sigma_y^2 = E[y^2(t)] = \frac{1}{2\pi}\int_0^\infty y(\Omega)\,d\Omega$$

As should be expected, it follows from Eq. (11.92) that the maximum number of crossings per unit time occur at y = 0 where $n_{Y=0} = n_0$, and as $y(t) = Y$ gets larger, the expected number of crossings per unit of time decreases (see Fig. 11.16). Because there are two zero crossings for each cycle (see Fig. 11.16) it follows that the "apparent frequency" of the response $y(t)$ is given by $n_0/2$.

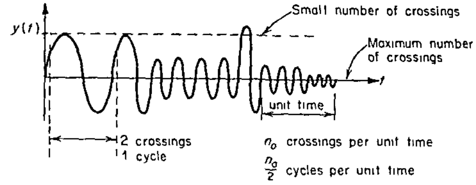


Fig. 11.16

A lightly damped linear mechanical system (narrow band), subjected to a random excitation with a nearly constant p.s.d. function, is expected to respond predominately with frequencies in the neighborhood of the natural frequency of the system. This will be demonstrated by using Eq. (11.91). Consider a single-degree-of-freedom system with viscous damping subjected to a random excitation with a nearly constant power spectral density function $f(\Omega) \approx f_0(\Omega)$; then the two integrals of Eq. (11.91) may be evaluated by the method of residues to yield

$$\int_0^\infty y(\Omega)\,d\Omega = f_0(\Omega)\int_0^\infty |H(\Omega)|^2\,d\Omega = f_0(\Omega)\,\frac{\pi}{2}\,\frac{\omega}{2\zeta}$$

[See Eqs. (11.49 and 11.60)]. Similarly

$$\int_0^\infty \Omega^2 y(\Omega)\,d\Omega = f_0(\Omega)\int_0^\infty \Omega^2|H(\Omega)|^2\,d\Omega = f_0(\Omega)\,\frac{\pi}{2}\,\frac{\omega^3}{2\zeta}$$

Substituting the above values of the integrals into Eq. (11.91) we have

$$n_0 = \frac{\omega}{\ }$$

and the corresponding "apparent frequency" of the response becomes

$$f_0 = \frac{n_0}{2} = \frac{\omega}{2\pi} \tag{11.93}$$

f_0 is the natural frequency of the system in cycles per second as expected.

11.22 The Rayleigh Probability Density Function

In the light of the preceding discussion it appears reasonable to express the response of a lightly damped (narrow band) single-degree-of-freedom linear system excited by a wide band random excitation (of nearly constant p.s.d. function) as

$$y(t) = Y(t) \sin[\omega t + \psi(t)] \quad \text{(See Fig. 11.17)} \tag{11.94}$$

in which ω is the natural frequency of the system,

$$Y(t) > 0 \qquad \text{and} \qquad 0 < \psi(t) < 2\pi$$

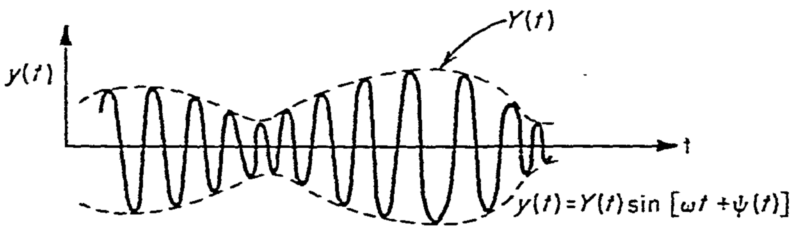


Figure 11.17 The response of a lightly damped (narrow band) linear system excited by a nearly constant wide band excitation.

The amplitude $Y(t)$ and phase $\psi(t)$ are independent random variables. The joint distribution function of Y and ψ has the form[45]

$$f(Y, \psi) = \frac{Y}{2\pi\sigma^2} e^{-Y^2/2\sigma^2} \tag{11.95}$$

where

$$\sigma^2 = E[y^2(t)]$$

The Marginal Distribution of the Amplitude Y. Using Eq. (11.6) the marginal distribution function of Y is obtained by integrating Eq. (11.95) over the range of ψ

$$\begin{aligned} g(Y) &= \int_0^{2\pi} f(Y, \psi)\, d\psi \\ &= \frac{Y}{\sigma^2} e^{-Y^2/2\sigma^2}, \qquad Y > 0 \end{aligned} \tag{11.96}$$

This function is known as the *Rayleigh probability density function*, and is plotted in Fig. 11.18. The probability that the envelope $Y(t)$ will exceed a specified value Y_a is given by

$$Pr(Y > Y_a) = \int_{Y_a}^{\infty} g(Y)\,dY \qquad \text{(See Fig. 11.18)} \qquad (11.97)$$

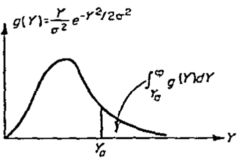


Figure 11.18 The Rayleigh probability density function.

From Eq. (11.3) the expected value (mean) of the envelope Y is given by

$$E(Y) = \int_0^{\infty} Yg(Y)\,dY$$

The Marginal Distribution of the Phase Angle ψ. Because Y and ψ are independent it follows from Eq. (11.10) that

$$f(Y, \psi) = g(Y)\,h(\psi) \qquad (11.98)$$

Using Eqs. (11.95) and (11.96) in Eq. (11.98) the marginal distribution $h(\psi)$ of ψ becomes

$$h(\psi) = \frac{1}{2\pi}, \quad 0 < \psi < 2\pi \qquad (11.99)$$

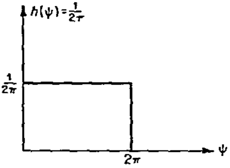


Figure 11.19

This equation is plotted in Fig. 11.19 and indicates that the phase angle ψ has a uniform distribution. From Fig. 11.19 we note that the expected value of the phase angle ψ is π. This can be verified by applying Eq. (11.3)

$$\begin{aligned} E(\psi) &= \int_0^{2\pi} \psi h(\psi)\,d\psi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \psi\,d\psi \\ &= \frac{1}{2\pi}\frac{1}{2}\psi^2\Big|_0^{2\pi} \\ &= \pi \end{aligned}$$

PROBLEMS

1. What is the probability of getting at least four heads in a toss of five coins?
2. Given that a random variable y is normally distributed with mean μ_y

and variance σ_y^2, show that the standardized normal variable ξ defined by Eq. (11.32)

$$\xi = \frac{y - \mu_y}{\sigma_y}$$

has a mean

$$\mu_\xi = 0$$

and variance

$$\sigma_\xi^2 = 1$$

3. In deriving Eq. (11.78) the phase relations between the responses in the normal modes were disregarded. Discuss why this should result in a higher probability of a specified response being exceeded at some point on the structure.

4. The building of Fig. 11.15 is excited by a random ground acceleration $f(t) = \ddot{x}(t)$ which is approximately normally distributed with zero mean and variance $\overline{f^2(t)}$. The power spectral density function of $f(t)$ given by

$$f(\Omega) = \frac{1}{1 + \Omega^2} \qquad 0 < \Omega < \infty$$

Find the mean square displacements $\overline{z_i^2(t)}$ of each of the levels with respect to the ground, and evaluate the following probabilities

$$Pr\left(|z_i(t)| > 2\sqrt{\overline{z_i^2(t)}}\right)$$

and

$$Pr\left(|z_i(t)| > 3\sqrt{\overline{z_i^2(t)}}\right)$$

$$i = 1, 2, 3$$

5. Repeat Problem 4 for a ground acceleration $f(t) = \ddot{x}(t)$ with a power spectral density function given by

$$f(\Omega) = \frac{1}{1 + (20 - \Omega)^2} \qquad 0 < \Omega < \infty$$

6. In the illustrative example of Section 9, Chapter 8, let the time dependence function $f(t)$ of the load be a representative record of an ergodic process with a power spectral density function given by

$$f(\Omega) = \alpha \quad \text{for} \quad 0.5\frac{\pi^2}{l^2}\sqrt{\frac{EI}{m}} < \Omega < 4.0\frac{\pi^2}{l^2}\sqrt{\frac{EI}{m}}$$

$$= 0 \quad \text{elsewhere}$$

$$\alpha \text{ is a constant}$$

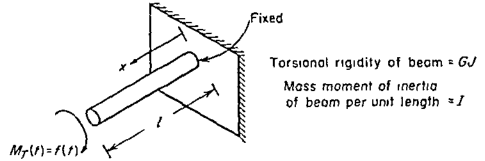
$f(t)$ is approximately normally distributed with zero mean and variance $\overline{f^2(t)}$. Find the expression for the mean square value of the response at the center of the beam. Consider structural damping to be present with a coefficient $g = 0.01$.

7. In Problem 6 find the expression for the mean square value of the force in the two end springs.

8. Derive the expression for the mean square torsional response $\overline{\theta^2(x,t)}$ of a uniform cantilevered beam excited by a random torque $M_T(t) = f(t)$ at its free end. $M_T(t)$ is approximately normally distributed with zero mean and variance $\overline{M_T^2(t)}$. The power spectral density function of $M_T(t)$ is given by

$$f(\Omega) = \overline{M_T^2(t)}\, e^{-\Omega^2}$$

Consider structural damping to be present with a coefficient $g = 0.01$. (Use the relevant results of Section 2, Chapter 5.)



Problem 8

9. In Problem 8 find the expression for the mean square internal torque $\overline{M_T^2(x,t)}$ at any point x along the beam. From Section 2, Chapter 5, we recall that

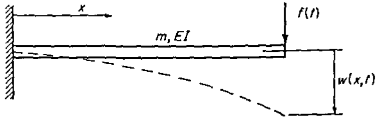
$$M_T(x,t) = GJ\frac{\partial \theta}{\partial x}$$

10. A uniform cantilever beam is excited by a random load $F(t) = f(t)$ which is approximately normally distributed with zero mean and variance $\overline{F^2(t)}$. The power spectral density function of $f(t)$ is given by

$$f(\Omega) = \overline{F^2(t)} \qquad \Omega_\alpha < \Omega < \Omega_\beta$$
$$= 0 \qquad \text{elsewhere}$$

$$\Omega_\alpha = \frac{\omega_3 + \omega_4}{2}$$

$$\Omega_\beta = \frac{\omega_9 + \omega_{10}}{2}$$



Problem 10

ω_3, ω_4, ω_9, and ω_{10} are, respectively, the 3rd, 4th, 9th, and 10th natural frequencies of the cantilever beam. Find the expression for the mean square response $\overline{w^2(x,t)}$ of the beam. Consider structural damping to be present with $g = 0.01$. (Use the relevent results from Section 3 of Chapter 5.)

11. In Problem 10, derive the expression for the mean square bending moment $\overline{M^2(x,t)}$ at any point

$$M(x,t) = EI\frac{\partial^2 w(x,t)}{\partial x^2}$$

12. In Problem 10, derive the expression for the mean square shear $\overline{V^2(x,t)}$ and the mean square rotation $\overline{\psi^2(x,t)}$ at any point x along the beam.

CHAPTER 12

Use Of Computers in Dynamics Of Structures

12.1 Introduction

The advent of the digital computer in the 1950's made it possible to solve, in reasonable time, problems that were considered prohibitive earlier. The computer also enables us to study the behavior of complex structures and acquire more information regarding their characteristics. The reader who attempted the solution of some problems in the preceding chapters will welcome the computer as a tool that will relieve him of lengthy numerical calculations. The engineer and scientist who wishes to use a computer must be aware of two fundamental requirements:

1. He must understand his problem thoroughly and be able to formulate it in a way that will exploit the capabilities of the computer.
2. He must be able to communicate with the computer so that he can instruct it to perform the operations leading to the solution of a problem.

The formulation of the problem is very important. Here the analyst can display a mastery in the field of his specialty, because while the computer will execute the computations the analyst must devote most of his time preparing the problem for the computer by formulating it correctly and wisely. The first impression that the computer may create is that there is no limit to the size and the complexity of the problem that can be handled by it in reasonable time. This leads, for instance, to the formulation of problems that call for the inversion of matrices of very high orders. This operation is not only costly in computer time, but often no meaningful results can be obtained even when the most efficient computers are used. To overcome such difficulties it is often desirable to simplify the mathematical model representing the "real world" problem so that meaningful answers can be obtained in reasonable time (and hence, reasonable cost). To do this the analyst must know how much accuracy in the computed results is sacrificed in going from a complex to a simpler model. This knowledge can be gained through the use of the computer by studying the particular problem or class of problems under consideration, and comparing computed results for models with varying degrees of simplification. The problem treated in this chapter is devoted to such a study.

With respect to communication with the computer a great deal of progress has been made. Special languages have been developed to make it easier to communicate with the computer and instruct it to perform all the operations leading to the solution of a problem. The "vocabulary" in these languages is remote from the computer language but is very similar to the English language and the language of mathematics. In one such language† the words

```
DO
READ
PRINT
```

instruct the computer to do, read, and print; and an equation such as‡

```
Y = A + B - C * D/5.0
```

instructs the computer to carry out all operations on the right-hand side of the equation and assign the result to the variable Y on the left-hand side. As another example of the power and conciseness of these special languages, we write the instructions to the computer to execute the matrix multiplication [A][B] and assign the result to matrix [C] (matrix [A] is of order N × M and [B] is of order M × N)

```
      DO 5 I = 1, N
      DO 5 J = 1, N
      C(I, J) = 0.0
      DO 5 K = 1, M
5     C(I, J) = C(I, J) + A(I, K) * B(K, J)
```

†IBM FORTRAN
‡The asterisk(*) and slash(/) designate multiplication and division, respectively.

To show how the computer is used to solve a problem in dynamics and how we can gain more knowledge about the dynamic behavior of structures so that the model used in the analysis can be simplified, we shall study the effect of joint rotation on the natural modes and frequencies of a tall, framed building. The computer language used in this study is the IBM FORTRAN.† There are other comparable computer languages; the use of FORTRAN here is a matter of convenience.

12.2 *Assumptions in the Computations of the Natural Modes and Frequencies of a Tall, Framed Building*‡

The dynamic analysis of a tall building frame requires extensive calculations involving the inversion of high-order matrices. Since a framed building consists, normally, of a number of such frames, the computational efforts increase greatly when the entire building is analyzed. As an example let us consider a 30-story building consisting, in the north-south direction, of three different frame types with eight columns in each. To find the natural modes and frequencies of this building, vibrating in the north-south direction, three matrices of order 480×480 will have to be inverted in the course of the computations when the following assumptions are introduced:

1. The mass of the building is lumped at the floor levels.
2. The beams of the building frames are infinitely stiff axially.

Using the IBM 709 computer and the matrix inversion method employed in this study (Jordan's method), the inversion of a matrix of order 75 is accomplished in approximately $4\frac{1}{2}$ minutes. The matrix inversion time is approximately proportional to the cube of the matrix order. Consequently, the inversion of three 480×480 matrices will require approximately 59 *hours* of computer time, quite a costly operation. On the other hand, we will show later that, when further simplifications are introduced in the model analyzed, the inversion of three 480×480 matrices is reduced to computations requiring the inversion of a *single* 30×30 matrix which requires *less than one minute* in computer time. This shows why it is desirable to simplify the model analyzed and reduce the amount of computations, provided that the corresponding sacrifice in accuracy of the computed results is acceptably small.

†Contraction of FORmula TRANslator.

‡The study in this chapter is taken from Reference 46.

One simplification, mentioned earlier, deals with the mass distribution. The mass of the building is assumed to be lumped at the floor levels with three degrees of freedom of motion per floor: translation in two horizontal directions, and a rotation about a vertical axis. In the event that the foundation is not rigid enough to fix the building against rotation at the base, it may also rotate about two horizontal axes.

Additional assumptions can be introduced to simplify the spring characteristics of the building frames. Experience has shown that for buildings with a height to width ratio (aspect ratio) of five or less, the axial deformation in all the frame members can be disregarded with a small sacrifice in the accuracy of the computed natural modes and frequencies of the building. In the following we undertake to study the effect of further assumptions regarding the rotations of the frame joints (beam column intersections). To assess the accuracy of the computed results when such assumptions are made, we shall compute the natural modes and frequencies of a multi-story building for the following four cases:

Case a. No joint rotation takes place.

Case b. All joints within a floor (for all frames) undergo an equal rotation.

Case c. All joints of a given type frame within a floor undergo an equal rotation.

Case d. No restriction is placed on joint rotation.

In all the above cases the frame members are assumed infinitely stiff in their axial direction and the building mass is lumped at the floor levels.

12.3 Method of Analysis

Lateral Stiffness Matrix of a Frame.

Let us idealize the building of Fig. 12.1 by lumping the mass at the floor levels in a manner similar to that shown in Fig. 2.2 of Chapter 2, and assuming that the beams and columns are infinitely stiff in their axial direction. Using the coordinates shown in Fig. 12.1(b) we can relate the force vector $\{Q\}$ of any frame to the corresponding displacement vector $\{q\}$ through the frame stiffness matrix $[k]$

$$\{Q\} = [k]\{q\} \tag{12.1}$$

We partition the above matrix equation to separate the floor displacements and the joint rotations (see Fig. 12.1b) and write

$$\begin{Bmatrix} \{F\} \\ \hline \{M\} \end{Bmatrix} = \left[\begin{array}{c|c} [k]_{11} & [k]_{12} \\ \hline [k]_{21} & [k]_{22} \end{array}\right] \begin{Bmatrix} \{u\} \\ \hline \{\theta\} \end{Bmatrix} \tag{12.2}$$

where $\{F\}$ and $\{u\}$ are, respectively, the vectors of horizontal loads and displacements, and $\{M\}$ and $\{\theta\}$ are, respectively, the vectors of joint moments and rotations.

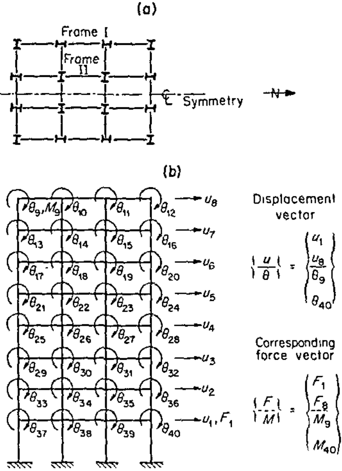


Figure 12.1 A symmetrical tall building. (a) Typical floor plan showing only rigid frames. (b) Schematic elevation of frame I or II showing force and displacement coordinates.

For free undamped vibration of the building (Fig. 12.1), in the north-south direction, only horizontal inertial forces result and all inertial moments at the joints are zero; consequently, for any frame $\{M\} = \{0\}$. We proceed now to express $\{F\}$ in terms of $\{u\}$ in Eq. (12.2) by following the steps used in reducing Eq. (1.49) in Section 1.5, and obtain

$$\{F\} = [k]^*\{u\} \tag{12.3}$$

in which $[k]^*$ is the reduced lateral stiffness matrix of a frame and is given by

$$[k]^* = [k]_{11} - [k]_{12}[k]_{22}^{-1}[k]_{21} \tag{12.4}$$

[See also Eq. (1.53), Section 1.5.]

Lateral Stiffness Matrix of a Building.

A relation similar to Eq. (12.3) can be written for each frame. Thus, for frames I and II in Fig. 12.1 we write

$$\begin{aligned} \{F\}_{\mathrm{I}} &= [k]_{\mathrm{I}}^*\{u\}_{\mathrm{I}} \\ \{F\}_{\mathrm{II}} &= [k]_{\mathrm{II}}^*\{u\}_{\mathrm{II}} \end{aligned} \tag{12.5}$$

For the symmetrical building of Fig. 12.1, undergoing free vibration in the north-south direction with the floor slabs assumed infinitely stiff in their own plane

$$\{u\}_{\mathrm{I}} = \{u\}_{\mathrm{II}} = \{u\} \tag{12.6}$$

where $\{u\}$ is the vector of floor displacements in the north-south direction. Since there are two frames of each type, the vector of lateral forces, $\{F\}$, acting on the building in the north-south direction is given by

$$\{F\} = 2(\{F\}_{\mathrm{I}} + \{F\}_{\mathrm{II}}) \tag{12.7}$$

Substituting for $\{F\}_{\mathrm{I}}$ and $\{F\}_{\mathrm{II}}$ from Eqs. (12.5) and using relation (12.6), Eq. (12.7) becomes

$$\{F\} = [K]^*\{u\} \tag{12.8}$$

in which $[K]^*$ is the reduced stiffness matrix for the building and is given by

$$[K]^* = 2([k]_{\mathrm{I}}^* + [k]_{\mathrm{II}}^*) \tag{12.9}$$

The Eigenvalue Problem.

The equations of motion for free undamped vibration of the building can be written from Eq. (3.8), Chapter 3, in the form

$$\left(D - \frac{1}{\omega^2}[I]\right)\{u\} = \{0\} \tag{12.10}$$

where

$$[D] = [K]^{*-1}[m]$$

$[K]^*$ is given by Eq. (12.9), $[m]$ is the mass matrix of the building, and $[I]$ is the identity matrix. If Eq. (3.10) of Chapter 3 is used, we can write

$$([D]^{-1} - \omega^2[I])\,\{u\} = \{0\} \tag{12.11}$$

where

$$[D]^{-1} = [m]^{-1}[K]^*$$

The nontrivial solution of the eigenvalue problem represented by Eqs. (12.10) or (12.11) yields natural modes and frequencies of the building (see Section 3.3, Chapter 3).

12.4 Assumptions Regarding Joint Rotation

No Restriction on Joint Rotation (*Case d*).

Consider the stiffness matrix $[K]^*$ as expressed by Eq. (12.9). This matrix is equal to twice the sum of the reduced stiffness matrices $[k]_I^*$ and $[k]_{II}^*$ of frames I and II, respectively. Each of these reduced matrices is computed from Eq. (12.4) in which $[k]_{11}$ is a square matrix of order 8 equal to the number of degrees-of-freedom in horizontal translation in the north-south direction (see Fig. 12.1). $[k]_{22}$ is a square matrix of order 32 equal to the number of degrees-of-freedom in joint rotations of each frame. $[k]_{12}$ and $[k]_{21}$, the coupling matrices, are of order 8×32 and 32×8, respectively. The computations leading to the reduced stiffness matrix $[K]^*$ for the building of Fig. 12.1 will, therefore, require the inversion of two matrices of order 32 when no restriction is placed on joint rotation (Case *d*).

No Joint Rotation Takes Place (*Case a*)

When this assumption is introduced, we set $\{\theta\} = \{0\}$ in Eq. (12.2) and recalling Eq. (12.6) we write for frames I and II

$$\{F\}_I = [k]_{11}^I\{u\}$$

$$\{F\}_{II} = [k]_{11}^{II}\{u\}$$

The reduced stiffness matrix $[K]^*$ of the building is obtained directly from the stiffness elements of frames I and II.

$$[K]^* = 2([k]_{11}^I + [k]_{11}^{II})$$

All Joints of a Given Type Frame Within a Floor Undergo an Equal Rotation (*Case c*).

Under this assumption we set for any frame in Fig. 12.1,

$$\begin{aligned} \theta_9^* &= \theta_i \qquad \text{for } i = 9, 10, 11, 12 \\ \theta_{10}^* &= \theta_i \qquad \text{for } i = 13, 14, 15, 16 \\ &\;\vdots \\ \theta_{16}^* &= \theta_i \qquad \text{for } i = 37, 38, 39, 40 \end{aligned} \tag{12.12}$$

Using these relations in Eq. (12.2) we can reduce the stiffness matrix $[k]$ for frames I and II from 40×40 to 16×16 and write

$$\underset{16 \times 1}{\left\{\begin{array}{c}\{F\} \\ \hline \{M^*\}\end{array}\right\}} = \underset{16 \times 16}{\left[\begin{array}{c|c}[k]_{11} & [k^*]_{12} \\ \hline [k^*]_{21} & [k^*]_{22}\end{array}\right]} \underset{16 \times 1}{\left\{\begin{array}{c}\{u\} \\ \hline \{\theta^*\}\end{array}\right\}} \tag{12.13}$$

where

$$\begin{aligned}
F_1 &= \sum_{j=1}^{8} k_{1,j}u_j + \theta_9^* \sum_{j=9}^{12} k_{1,j} + \theta_{10}^* \sum_{j=13}^{16} k_{1,j} + \cdots + \theta_{16}^* \sum_{j=37}^{40} k_{1,j} \\
F_2 &= \sum_{j=1}^{8} k_{2,j}u_j + \theta_9^* \sum_{j=9}^{12} k_{2,j} + \theta_{10}^* \sum_{j=13}^{16} k_{2,j} + \cdots + \theta_{16}^* \sum_{j=37}^{40} k_{2,j} \\
&\vdots \\
F_8 &= \sum_{j=1}^{8} k_{8,j}u_j + \theta_9^* \sum_{j=9}^{12} k_{8,j} + \theta_{10}^* \sum_{j=13}^{16} k_{8,j} + \cdots + \theta_{16}^* \sum_{j=37}^{40} k_{8,j}
\end{aligned} \tag{12.14a}$$

and

$$\begin{aligned}
M_1^* &= \sum_{i=9}^{12} M_i \\
&= u_1 \sum_{i=9}^{12} k_{i,1} + u_2 \sum_{i=9}^{12} k_{i,2} + \cdots + u_8 \sum_{i=9}^{12} k_{i,8} \\
&\quad + \theta_9^* \sum_{i=9}^{12} \sum_{j=9}^{12} k_{ij} + \theta_{10}^* \sum_{i=9}^{12} \sum_{j=13}^{16} k_{ij} + \cdots + \theta_{16}^* \sum_{i=9}^{12} \sum_{j=37}^{40} k_{ij} \\
M_2^* &= \sum_{i=13}^{16} M_i \\
&= u_1 \sum_{i=13}^{16} k_{i,1} + u_2 \sum_{i=13}^{16} k_{i,2} + \cdots + u_8 \sum_{i=13}^{16} k_{i,8} \\
&\quad + \theta_9^* \sum_{i=13}^{16} \sum_{j=9}^{12} k_{ij} + \theta_{10}^* \sum_{i=13}^{16} \sum_{j=13}^{16} k_{ij} + \cdots + \theta_{16}^* \sum_{i=13}^{16} \sum_{j=37}^{40} k_{ij} \\
&\vdots \\
M_8^* &= \sum_{i=37}^{40} M_i \\
&= u_1 \sum_{i=37}^{40} k_{i,1} + u_2 \sum_{i=37}^{40} k_{i,2} + \cdots + u_8 \sum_{i=37}^{40} k_{i,8} \\
&\quad + \theta_9^* \sum_{i=37}^{40} \sum_{j=9}^{12} k_{ij} + \theta_{10}^* \sum_{i=37}^{40} \sum_{j=13}^{16} k_{ij} + \cdots + \theta_{16}^* \sum_{i=37}^{40} \sum_{j=37}^{40} k_{ij}
\end{aligned} \tag{12.14b}$$

The elements of $[k^*]_{12}, [k^*]_{21}$ and $[k^*]_{22}$ in Eq. (12.13) are identified from Eqs. (12.14) and are obtained by adding blocks of stiffness elements k_{ij} in the original stiffness matrix of Eq. (12.2). Submatrix $[k]_{11}$ remains unchanged. With reference to frames I and II of Fig. 12.1, submatrix $[k^*]_{22}$ of Eq. (12.13) is of order 8×8, hence, two matrices of order 8 (one for each frame) must be inverted in computing the stiffness matrix $[K]^*$ for the building in Case *c*. (See Eqs. 12.4 and 12.9.)

All Joints Within a Floor (for all frames) Undergo an Equal Rotation (Case b).

For this case we again write Eq. (12.13) for frames I and II

$$\begin{Bmatrix} \{F\} \\ \hline \{M^*\} \end{Bmatrix}_{\text{I}} = \left[\begin{array}{c|c} [k]^{\text{I}}_{11} & [k^*]^{\text{I}}_{12} \\ \hline [k^*]^{\text{I}}_{21} & [k^*]^{\text{I}}_{22} \end{array}\right] \begin{Bmatrix} \{u\} \\ \hline \{\theta^*\} \end{Bmatrix}_{\text{I}} \tag{12.15}$$

$$\begin{Bmatrix} \{F\} \\ \hline \{M^*\} \end{Bmatrix}_{\text{II}} = \left[\begin{array}{c|c} [k]^{\text{II}}_{11} & [k^*]^{\text{II}}_{12} \\ \hline [k^*]^{\text{II}}_{21} & [k^*]^{\text{II}}_{22} \end{array}\right] \begin{Bmatrix} \{u\} \\ \hline \{\theta^*\} \end{Bmatrix}_{\text{II}} \tag{12.16}$$

But under the present assumption we have

$$\begin{Bmatrix} \{u\} \\ \hline \{\theta^*\} \end{Bmatrix}_{\text{I}} = \begin{Bmatrix} \{u\} \\ \hline \{\theta^*\} \end{Bmatrix}_{\text{II}} = \begin{Bmatrix} \{u\} \\ \hline \{\theta^*\} \end{Bmatrix}$$

consequently, Eqs. (12.15) and (12.16) can be added and multiplied by two to obtain for the entire building

$$\begin{Bmatrix} \{F\} \\ \hline \{M^*\} \end{Bmatrix} = \left[\begin{array}{c|c} [\rho]_{11} & [\rho]_{12} \\ \hline [\rho]_{21} & [\rho]_{22} \end{array}\right] \begin{Bmatrix} \{u\} \\ \hline \{\theta^*\} \end{Bmatrix} \tag{12.17}$$

where

$$\begin{Bmatrix} \{F\} \\ \hline \{M^*\} \end{Bmatrix} = 2\begin{Bmatrix} \{F\} \\ \hline \{M^*\} \end{Bmatrix}_{\text{I}} + 2\begin{Bmatrix} \{F\} \\ \hline \{M^*\} \end{Bmatrix}_{\text{II}}$$

and

$$\left[\begin{array}{c|c} [\rho]_{11} & [\rho]_{12} \\ \hline [\rho]_{21} & [\rho]_{22} \end{array}\right] = 2\left[\begin{array}{c|c} [k]^{\text{I}}_{11} + [k]^{\text{II}}_{11} & [k^*]^{\text{I}}_{12} + [k^*]^{\text{II}}_{12} \\ \hline [k^*]^{\text{I}}_{21} + [k^*]^{\text{II}}_{21} & [k^*]^{\text{I}}_{22} + [k^*]^{\text{II}}_{22} \end{array}\right]$$

Because no inertial moments act at the joints, we set $\{M^*\} = \{0\}$ in Eq. (12.17) and obtain

$$\{F\} = [K]^*\{u\}$$

in which $[K]^*$ is the reduced stiffness matrix for the building and is given by

$$[K]^* = [\rho]_{11} - [\rho]_{12}[\rho]_{22}^{-1}[\rho]_{21}$$

Since $[\rho]_{22}$ is of order 8×8, only one matrix (of order equal to the number of floors) must be inverted in order to compute $[K]^*$ for Case *b*.

12.5 The Effect of Joint Rotation on the Natural Modes and Frequencies of a 19-Story Building

We shall now assess the merits of the assumptions regarding joint rotation by computing the natural modes and frequencies of the 19-story building shown in Figs. 12.2 and 12.3. The building is a rigid

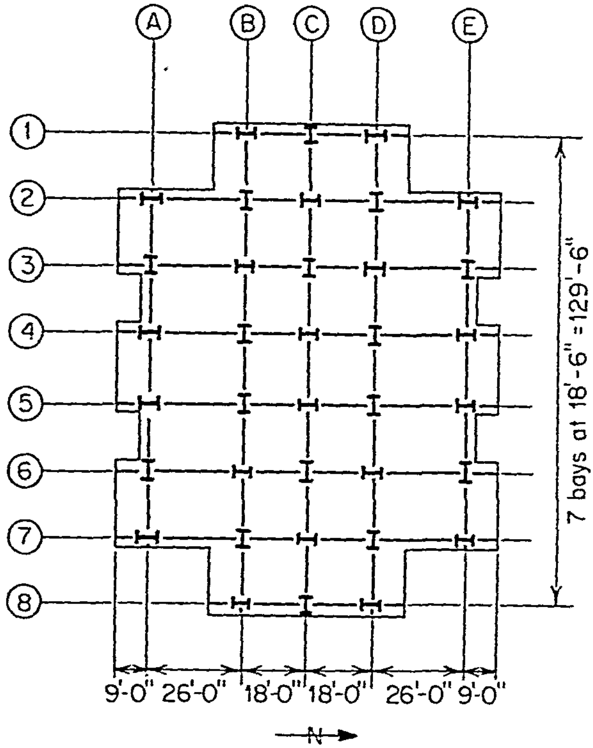


Figure 12.2 Floor plan of a multi-story framed structure.

frame steel structure. The column and girder sections are indicated in Fig. 12.3. The structural properties of the framed members are tabulated in the *Steel Construction Manual* of the American Institute of Steel Construction.[47] The floors are constructed of 6-inch concrete slabs and are considered infinitely rigid in their own plane. Exterior walls and interior partitions are lightweight construction. The columns and girders are assumed infinitely rigid in their axial direction. It is also assumed that the building rests on rigid ground, the columns are fixed against rotation at the first floor level, and no foundation settlement occurs. The mass of the building is lumped at the floor levels.

The weight w_r of a typical story is 1,235,200 pounds. The mass lumped at a typical floor (3rd through 19th) is given therefore by

$$m_r = \frac{w_r}{g} = 38.4 \frac{\text{kip-sec}^2}{\text{ft}}$$

$$r = 3, 4, 5, \ldots, 19$$

where $g = 32.2 \text{ ft/sec}^2$ is the acceleration due to gravity.

The mass of the 2nd floor m_2 which includes the mezzanine, and

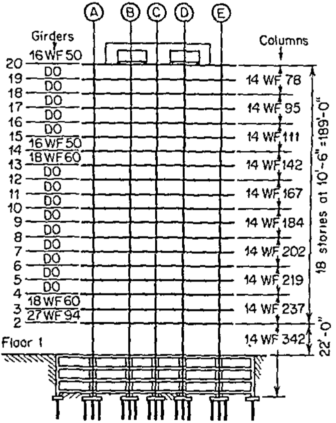


Figure 12.3 Schematic elevation showing column and girder sections of a multi-story framed structure.

the mass of the 20th floor m_{20} which includes the mechanical equipment, are each equivalent to 1.5 times the typical floor mass.

$$m_2 = m_{20} = 1.5\, m_{\text{typical floor}}$$

The mass matrix $[m]$ of the building becomes then

$$[m] = m_{\text{typical floor}} \begin{bmatrix} 1.5 & & & & \text{elements not shown are zero} \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & & 1.5 \end{bmatrix} \tag{12.18}$$

The dynamic analysis is carried out in the north-south direction. In this direction, the building consists of eight frames (Fig. 12.2) of three different types. The frames on lines 1 and 8 are of one type, the frames on lines 3 and 6 are of a second type, and the frames on lines 2, 4, 5, and 7 are of a third type.

The natural modes and frequencies of the building in the north-south direction are computed for the following cases:

Case a. No joint rotation takes place.

Case b. All joints within a floor (for all frames) undergo an equal rotation.

Case c. All joints of a given type frame within a floor undergo an equal rotation.

Case d. No restriction is placed on joint rotation.

Using Eq. (12.11) the mass matrix $[m]$ given by Eq. (12.18) must be inverted in each case. This is, of course, a trivial matter since $[m]$ is a diagonal matrix. No additional matrix inversion is required in the computations for Case *a*. A matrix of order 19 is inverted in the computations of $[K]^*$ for Case *b*. Three matrices (one for each frame type) of order 19 are inverted in the computations of $[K]^*$ for Case *c*. Two matrices of order 95 and one of order 57 are inverted in the computations of $[K]^*$ for Case *d*. This is required since the interior type frames contain 95 joints each, and the exterior type frames contain 57 joints each. Only joints free to rotate are counted.

In all four cases, the natural modes and corresponding frequencies are obtained through the solution of an eigenvalue problem of order 19, equal to the number of degrees-of-freedom of motion of the building in the north-south direction. All computations were done with the aid of an IBM 709 computer. The computation times were

2.39 minutes for Case *a*
2.64 minutes for Case *b*
2.69 minutes for Case *c*
22.84 minutes for Case *d*

Results and Conclusion.

The first ten mode shapes and frequencies of the building computed for Cases *a*, *b*, *c*, and *d* are compared in Figs. 12.4 through 12.13. The mode shapes of Cases *b*, *c*, and *d* are so close that they are plotted as a single curve in the figures. Frequencies and frequency ratios are compared in Table 12.1.

Mode shapes for Case *a* differ from those of Case *d*. The difference is most pronounced in the first, second, and third modes. This is expected, since the effect of joint rotation diminishes in the higher modes; it is conceivable that the joints in a given floor may not rotate at all in a high mode.

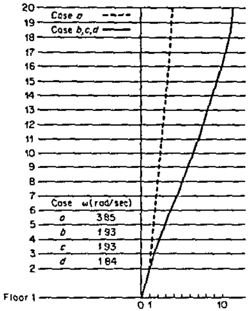


Figure 12.4 First mode shapes for the structure of Figs. 12.2, 12.3.

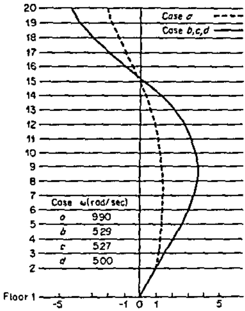


Figure 12.5 Second mode shapes for the structure of Figs. 12.2, 12.3.

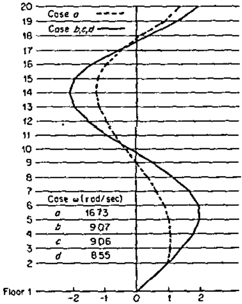


Figure 12.6 Third shapes for the structure of Figs. 12.2, 12.3.

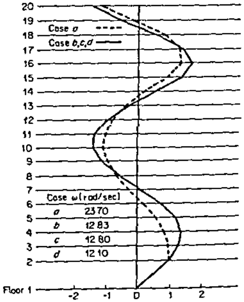


Figure 12.7 Fourth mode shapes for the structure of Figs. 12.2, 12.3.

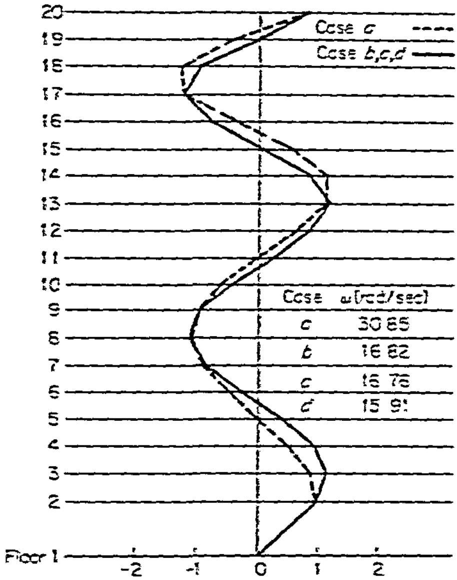


Figure 12.8 Fifth mode shapes for the structure of Figs. 12.2, 12.3.

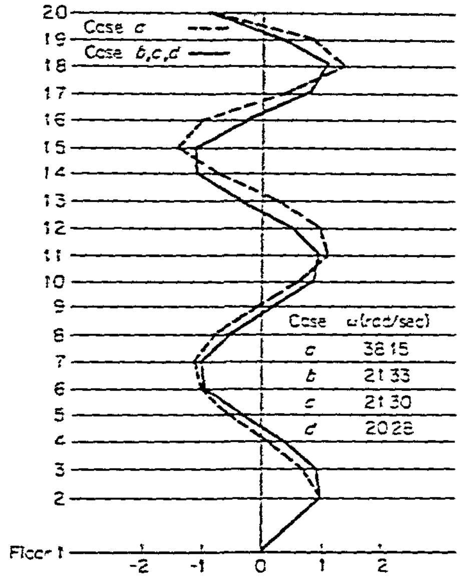


Figure 12.9 Sixth mode shapes for the structure of Figs. 12.2, 12.3.

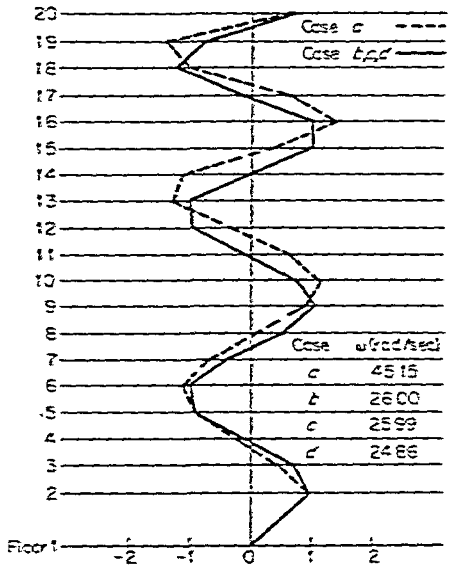


Figure 12.10 Seventh mode shapes for the structure of Figs. 12.2, 12.3.

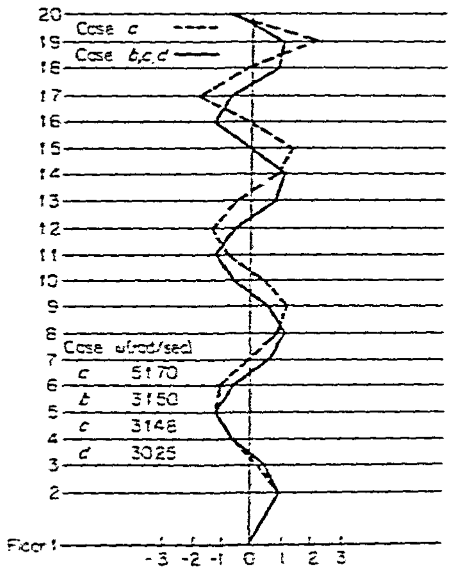


Figure 12.11 Eighth mode shapes for the structure of Figs. 12.2, 12.3.

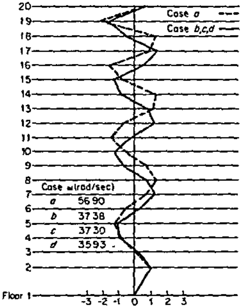


Figure 12.12 Ninth mode shapes for the structure of Figs. 12.2, 12.3.

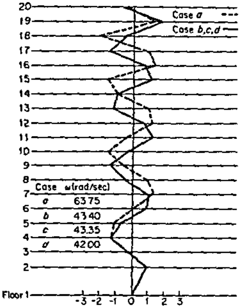


Figure 12.13 Tenth mode shapes for the structure of Figs. 12.2, 12.3.

The frequencies for Case *a* differ from the corresponding frequencies of Case *d*. The deviation is largest in the first natural frequency (109%); deviations become smaller for the higher frequencies as the effect of joint rotation diminishes.

The frequencies for Cases *b* and *c* are the same for all practical purposes. This is explained by the fact that the largest difference in joint behavior is between exterior and interior joints. Once this difference is reconciled by assuming an equal rotation for the joints within a floor, it makes little difference whether this rotation is considered an average joint rotation per floor for a single frame type, or for all frames.

TABLE 12.1 FREQUENCIES AND FREQUENCY RATIOS FOR THE STRUCTURE OF FIGS. 12.2 AND 12.3

Mode Number	Frequencies (Radians per Second)				Frequency Ratios		
	ω_a Case *a*	ω_b Case *b*	ω_c Case *c*	ω_d Case *d*	$\frac{\omega_a}{\omega_d}$	$\frac{\omega_b}{\omega_d}$	$\frac{\omega_c}{\omega_d}$
1	3.85	1.93	1.93	1.84	2.09	1.05	1.05
2	9.90	5.29	5.27	5.00	1.98	1.06	1.05
3	16.73	9.07	9.06	8.55	1.96	1.06	1.06
4	23.70	12.83	12.80	12.10	1.96	1.06	1.06
5	30.85	16.82	16.7[illegible]	15.91	1.94	1.06	1.05
6	38.15	21.33	21.30	20.28	1.88	1.05	1.05
7	45.15	26.00	25.99	24.86	1.82	1.05	1.05
8	51.70	31.50	31.48	30.25	1.71	1.04	1.04
9	56.90	37.38	37.30	35.93	1.58	1.04	1.04
10	63.75	43.50	43.35	42.00	1.52	1.03	1.03

The deviations of the frequencies computed for Cases *b* and *c* from those computed for *d* are in the order of 3-6% for the first ten modes, with the deviation decreasing for the higher frequencies.

The frequencies computed for Case *a* are consistently largest in value; those computed for Case *d* are smallest in value, and frequencies of Case *b* and *c* lie between these two extremes. This is expected, since frequency increases with stiffness. In Case *a* the assumption of rigid joints attributes a high stiffness to the structure. Consequently, high frequencies are obtained. In Cases *b*, *c*, and *d* the assumption of rigid joints is relaxed progressively and, consequently, the computed frequencies are reduced in value progressively from Case *a* to *b* to *c* to *d* (See Table 12.1). All mode shapes and frequencies were computed to an accuracy of six significant figures.

In comparing the results of Case *a* with those of Case *d*, we conclude that the assumption of no joint rotation results in mode shapes and frequencies with a discrepancy in the order of 100% for the lower frequencies. Cases *b* and *c* yield very satisfactory results, with only a 5-6% discrepancy for the low modes. However, Case *b* requires less computations than Case *c* and the results obtained for these two cases are practically the same. Consequently, we conclude that when the assumption of equal joint rotation for all joints within a floor is introduced in the dynamic analysis of a building similar to the one considered here, computer time is greatly reduced (a reduction of about 900% in the example in this chapter) with a small sacrifice in accuracy of the computed natural modes and frequencies.

This conclusion could only be reached through the use of the computer which enabled us to assess the accuracy of simplified models (Cases *a*, *b*, *c*) by comparing them with a more accurate model (Case *d*). As we recall, the computations for Case *d* required the inversion of matrices of order 95, a task that is prohibitive without a computer. In the following section we shall give a partial description of the computer program that was used to compute the results of this section.

12.6 The Computer Program

To reduce the details in the following description let us suppose that the reduced stiffness matrices for Cases *a*, *b*, *c*, and *d* have been synthesized earlier by the method of Chapter 1 using a digital computer. With this information available, we proceed to develop a computer program for the dynamic analysis of the last section. For clarity we shall discuss in detail only the computer program for Case *a*.

Symbols.

Using the Fortran programming language, variables can be designated by 1 to 6 alphabetic or numerical characters of which the first character is alphabetic but not I, J, K, L, M, or N when the variable is not an integer. The first character must be I, J, K, L, M, or N for integer variables.† Using these rules let us choose the following designations

FMASS(I, J) for the IJth element of the mass matrix
STF(I, J) for the IJth element of the building stiffness matrix $[K]^*$
C(K) for the Kth eigenvalue.

Flow Chart.

The following steps are involved in the computer solution for the natural modes and frequencies in Case a [Eq. (12.11) is used].

1. The data, i.e., the elements FMASS(I, J) of the mass matrix, and the elements STF(I, J) of the stiffness matrix are read into the computer from an input media such as punched cards, paper tape, or magnetic tape.
2. The mass matrix $[m]$ is inverted.
3. The product $[m]^{-1}[K]^*$ is evaluated.
4. The eigenvalues and eigenvectors are computed.
5. The eigenvalues and eigenvectors are printed.

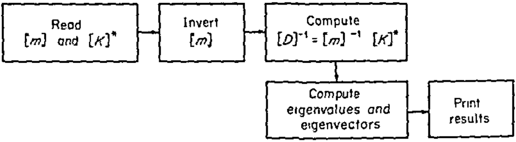


Figure 12.14 Flow chart for case a.

The five steps outlined above can be described diagrammatically by a *flow chart* (see Fig. 12.14) which indicates the sequence of operations. The flow chart shown describes blocks of operations; each block could be further expanded to describe by additional flow

†We suggest that the reader consult Reference 48 for more detail after reading this section.

charts more detailed steps of the various operations. It is very desirable to draw a flow chart of the operations before writing a computer program.

General Description of the Program.

Table 12.2 constitutes the main *program* (i.e., all instructions to the computer in sequential order), which instructs the computer to execute in detail the steps outlined in the flow chart and compute the natural modes and frequencies for the building of Figs. 12.2 and 12.3 for Case *a*. The main program of Table 12.2 calls for subprograms (referred to as *subroutines*) to perform specific operations. Such subroutines in Table 12.2 are matrix multiplication designated by MATMPY, and matrix inversion designated by MATINV. The program for subroutine MATMPY is given in Table 12.3. The subroutines will be discussed later.

Each line in the main program (Table 12.2) and its subroutines (for instance, Table 12.3) is called a *Fortran statement*. Assuming that punched cards are used as an input media, then each statement will be punched on one card. Cards are punched by depressing keys corresponding to the letters, numbers, and other symbols which appear in the Fortran statements. The keyboard on the punching equipment resembles that of a typewriter.

From Tables 12.2 and 12.3 it is seen that the Fortran language is a special language. Thus, an asterisk designates a multiplication, a slash designates a division; commas appear in certain positions in the statement, etc. The first step then is to check the program for errors in the use of the Fortran language, such as omissions of commas, parentheses, etc. This is done first by the programmer himself. Then the deck of Fortran cards is checked automatically (precompiled) by the computer to detect any errors in the statements insofar as the Fortran language is concerned. Special typed signals will indicate the general character of errors, if any, so that they may be corrected.

As soon as the Fortran deck has been precompiled with no error detected, this deck of cards which includes the main program followed by the subroutines is placed in the computer for *compilation* (translation to the computer language). As a result of the compilation, a new deck of cards is punched automatically. In our case this new deck will contain the program of Table 12.2 and its subroutines in a language that the computer understands. This machine language deck of cards is referred to as the *object program*, while the program written in the Fortran language is known as the *source program*.

TABLE 12.2 MAIN PROGRAM FOR COMPUTING THE NATURAL MODES AND FREQUENCIES OF THE BUILDING IN FIGS. 12.2, 12.3
CASE a (NO JOINT ROTATION)

STATEMENT NUMBER	FORTRAN STATEMENT
1	DIMENSION STF(19, 19), FMASS(19, 19), RY(19, 19), RZ(19, 19), C(19), V(19)
2	DO 6 J = 1, 19
3	DO 6 I = 1, 19
4	C(I) = 0.0
5	STF(I, J) = 0.0
6	FMASS(I, J) = 0.0
C 7	READ PRINCIPAL DIAGONAL OF FMASS AND STF
8	READ 28, (FMASS(I, I), I = 1, 19)
9	READ 28, (STF(I, I), I = 1, 19)
C 10	READ FIRST DIAGONAL BELOW PRINCIPAL FOR STF
11	READ 28, (STF(I, I-1), I = 2, 19)
C 12	COPY FIRST DIAGONAL ABOVE PRINCIPAL FOR STF
13	DO 14 I = 2, 19
14	STF (I-1, I) = STF(I, I-1)
15	CALL MATINV (FMASS, 19, B, 0, DETERM)
16	CALL MATMPY (FMASS, STF, RY)
17	DO 19 J = 1, 19
18	DO 19 I = 1, 19
19	RZ(I, J) = RY(I, J)
20	CALL EIGN(19, RZ, B, C, 1)
21	DO 27 K = 1, 19
22	DO 24 J = 1, 19
23	DO 24 I = 1, 19
24	RZ(I, J) = RY(I,J)
25	ALPHA = C(K)
26	CALL VCTR (RZ, ALPHA, 19, V)
27	PRINT 29, C(K), (V(I), I = 1,19)
28	FORMAT (F8.3, 4F 11.3)
29	FORMAT (1HO, E16.6/(7E16 6))
30	CALL EXIT
31	END

At this point the object program can be tested with a simple problem to which the solution is known. All required data called for in the program, namely FMASS(I, J) and STF(I, J), is punched on cards in a form prescribed in the program. These data cards are arranged in an order called for by the program and placed in the back of the object program. The entire deck of cards (program and data) is placed in the computer. As soon as the start button is depressed, the object program is read into the computer memory. After the last card of the object program is read in, the computer begins to follow the instruction dictated by the program (Table 12.2) in sequential order. Note that the data has not been read yet by the computer. Only after encountering an instruction READ will the computer start reading the data from the data cards. After all computations are completed, the desired answers will be typed out following a PRINT statement referring to these answers. Once the object program has been tested with a problem and proved satisfactory, it may be kept in a library of programs and used as often as desired with new data.

From the preceding description, it is realized that the programmer need be concerned only with the Fortran language. To learn some of the "vocabulary" of this language, let us study some of the statements of Table 12.2. The number preceding each statement is optional. A number is required only for those statements which are referred to by other statements in the program as will become apparent. We numbered all statements in Table 12.2 for ease of reference. A letter C on the left of a statement indicates that the statement is a comment used by the programmer to identify the operations of the program. A comment is not executed by the computer; it is ignored.

The "Vocabulary" of Table 12.2.

Statement 1 causes the computer to allocate proper space to all subscripted variables appearing in the program. Thus, the computer will prepare room for all the data, FMASS(I, J) and STF(I, J) which is read into the computer, and for all subscripted quantities generated in the course of the computations such as RY(I, J), RZ(I, J), C(K) and V(I).

Statement 2, DO 6 J = 1, 19, causes the computer to assign to J the value of 1 and execute all following statements down to, and including statement 6. Then J is incremented by one and assigned the value of 2 and all statements down to, and including statement 6 are executed for J = 2. Only after this operation has been executed 19 times, first with J = 1, then with J = 2, J = 3, and finally with J = 19 will the computer proceed to statement 7. (Statement 7 is a comment and hence, is ignored by the computer.)

TABLE 12.3 SUBROUTINE MATRIX MULTIPLICATION (MATMPY)

STATEMENT NUMBER	FORTRAN STATEMENT
	SUBROUTINE MATMPY (A, B, D)
	DIMENSION A(19, 19), B(19, 19), D(19, 19)
	DO 38 I = 1, 19
	DO 38 J = 1, 19
	D(I, J) = 0.0
	DO 38 K = 1, 19
38	D(I, J) = D(I, J) + A(I, K)*B(K, J)
	RETURN
	END

will be assigned to a space in computer memory allocated to matrix D. In our program (see Table 12.2, statement 16) A, B, and D are, respectively, FMASS, STF, and RY. Consequently, statement 16 will cause the matrix product [FMASS] [STF] to be assigned to matrix [RY]. Note, however, from the flow chart (Fig. 12.14) that we need the product $[FMASS]^{-1}$ [STF]. Actually this is what is accomplished by statement 16 as will be explained by the following. In statement 15 we call for a subroutine of matrix inversion (MATINV) to invert FMASS. The way this subroutine is written, the elements of FMASS are replaced by the corresponding elements of its inverse.‡ Consequently, following statement 15, the space in computer memory designated by FMASS has the inverse of the mass matrix, hence, from here on FMASS(I, J) represents the IJth element of $[m]^{-1}$. Therefore, statement 16 actually executes the product $[m]^{-1}[K]^{*}$.

The statement RETURN at the end of a subroutine, such as that of Table 12.3, causes computer control to return to the main program (Table 12.2); to the statement following the CALL for the particular subroutine. In our particular case, following the execution of statement 16, control will return to statement 17 and the execution of the program will continue.

Subroutines similar to MATMPY executing different operations were written for

MATINV	matrix inversion
EIGN	eigenvalue evaluation
VCTR	eigenvector evaluation.

Statements 17, 18, 19, 22, 23, 24 cause the elements of matrix RY to be assigned to matrix RZ. Thus RY and RZ are identical matrices in two different locations in computer memory. This step was required because the execution of subroutines EIGN and VCTR destroys the matrix on which it operates (in our case *RZ*). Consequently, RZ is repeatedly reconstructed from RY so that all eigenvalues and eigenvectors of RY can be evaluated.

Statements 21 to 27 inclusive. The execution of these statements for each value of K results in computations for eigenvalue C(K) and corresponding eigenvector V(I) ($I = 1, 2, \ldots, 19$), and the printing of these results in a FORMAT specified by statement 29.

Statement 29. In this statement, 1HO specifies a carriage control in the printing so that a line will be skipped and a new line started prior to printing each eigenvalue and its corresponding eigenvector.

‡This saves space in the computer memory when the original matrix is not used any more following its inversion.

E16.6 indicates that C(K) (the eigenvalue) of statement 27 will be printed in floating point (this is designated by the letter E) with 16 characters, 6 of which follow the decimal. /7E16.6 indicates that V(I) [the eigenvector corresponding to C(K)] which follows C(K) in statement 27 will be printed below C(K) with 7 numbers to a line in the same form as C(K) (E16.6).

Statement 30, CALL EXIT causes the computer control to be transferred to the next program which is to be executed.

Statement 31, END is ignored in the computations. It is only necessary in the compilation (translation to machine language) stage to tell the computer where to stop compiling.

An Exercise for the Reader.

While the above is not intended to be a course in Fortran, we suggest that the reader not familiar with this language refer to Table 12.3 and simulate the computer by executing all the instructions. For this purpose the reader should construct two simple matrices A and B, each of order 3×3, for instance, then replace the upper limit of 19 by 3 for I, J, and K in Table 12.3 and execute the steps leading to the matrix product AB which will be stored in a space designated by D. Note that statement 38 in Table 12.3

$$D(I, J) = D(I, J) + A(I, K) * B(K, J)$$

must be interpreted as follows: Look in the "compartment" or space on paper (corresponding to the space in memory) which was allocated to D(I, J) and record the number you see there on a piece of paper, then add to it the product of A(I, K) * B(K, J) and record the result in the space allocated to D(I, J) (shown on the left of the equal sign) replacing the former value. Hence, the equal sign does not have the conventional meaning. For instance, let the box shown in Fig. 12.15 be reserved for D(1, 2) in which the value of 25 is recorded. Now if we execute statement 38 for I = 1, J = 2 and K = 3, we have D(1, 2) = D(1, 2) + A(1, 3) * B(3, 2). Suppose A(1, 3) = 6 and B(3, 2) = 5, then, performing the instructions on the right of the equal sign in statement 38, we look into box D(1, 2) and record 25 to which we add the product $6 \times 5 = 30$. The result, 55, is then recorded in location D(1, 2) which is specified on the left of the equal sign replacing the 25. Note that the numbers

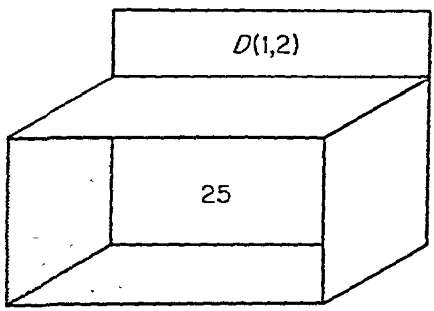


Figure 12.15 Conceptual model of a location D(1, 2) in a computer memory.

6 and 5 are read from compartments labeled A(1, 3) and B(3, 2), respectively, just as the 25 is read from a compartment labeled D(1, 2).

Following this exercise with Table 12.3, we suggest that the reader who has not done so already, familiarize himself with Fortran or a comparable computer language by resorting to an appropriate reference.

Flow Chart for Case d.

The flow chart for Case *d* in which we relax all assumptions

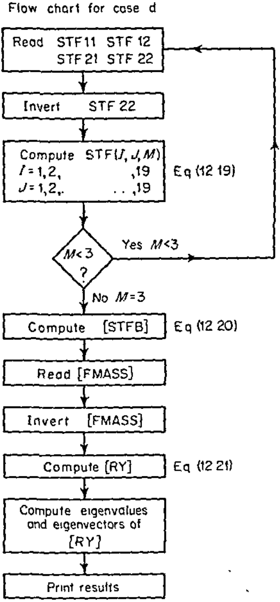


Figure 12.16 Flow chart for case *d*.

regarding joint rotations is shown in Fig. 12.16. Considering again that the stiffness matrices of the individual building frames have been synthesized earlier, the data consists of four matrices: Mass matrix $[m]$, and 3 stiffness matrices

$$\left[\begin{array}{c|c} \text{STF 11} & \text{STF 12} \\ \hline \text{STF 21} & \text{STF 22} \end{array}\right];$$

one for frame type 1 (frames on lines 1 and 8 of Fig. 12.2) of order 76×76, one for frame type 2 (frames on lines 2, 4, 5 and 7) of order 114×114, one for frame type 3 (frames on lines 3 and 6) of order 114×114. In the above matrix, submatrix STF 11 represents the elements of the stiffness matrix corresponding to floor displacements, STF22 represents the elements corresponding to the rotation of the frame joints, and STF12 and STF21 represent the coupling terms.

The reduced lateral stiffness matrix of each frame type is computed from Eq. (12.4) and is given by

$$[\text{STF}] = [\text{STF11}] - [\text{STF12}][\text{STF22}]^{-1}[\text{STF21}] \tag{12.19}$$

In order to save room in computer memory, the following procedure is used. First the sub-matrices STF11, STF12, STF21, and STF22 corresponding to frame type 1 are read into the computer memory and the reduced stiffness matrix is computed. Then STF11, STF12, STF21, and STF22 for frame type 2 are read into the same location in computer memory replacing the corresponding values for frame type 1, and the reduced stiffness matrix is computed. Finally, the data for frame type 3 is read, replacing the corresponding data for frame type 2, and the reduced stiffness matrix for frame type 3 is computed. To distinguish between the three reduced stiffness matrices in subsequent computations, we designate the elements of these matrices by

$$\text{STF(I, J, M)}$$

where M = 1, 2, 3 for frame types 1, 2 and 3, respectively. The upper limit of I and J is 19 for all frame types.

The elements of the reduced stiffness matrix, [STFB], for the entire building in the north-south direction are given by

$$\begin{gathered}\text{STFB(I, J)} = 2.0 * \text{STF(I, J, 1)} + 4.0 * \text{STF(I, J, 2)} + 2.0 * \text{STF(I, J, 3)} \\ \text{I} = 1, 2, \ldots, 19 \\ \text{J} = 1, 2, \ldots, 19\end{gathered} \tag{12.20}$$

The coefficients 2.0, 4.0, 2.0 on the right-hand side of the equation represent the number of frames in types 1, 2, and 3, respectively.

Following the computations for [STFB], the mass matrix [FMASS] is read into the computer and is subsequently inverted. After the product

$$[RY] = [FMASS]^{-1} [STFB] \tag{12.21}$$

is evaluated, the eigenvalues and eigenvectors of [RY] are computed and printed in the same way as was done for Case *a*.

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