4.7.1. Demostrar fórmula alternativa para la estimación de la segunda derivada discreta.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x')}{n!}(x-x')^{n}$$

$$\hookrightarrow f(x'\pm 2h)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x')}{n!}(x'\pm 2h-x')^{n}=\sum_{n=0}^{\infty}\frac{f^{(n)}(x')}{n!}(\pm 2h)^{n}$$

- $f(x+2h)=f(x)+2hf'(x)+2h^{2}f''(x)+\frac{4h^{3}}{3}f'''(x)+\dots$
- $f(x-2h)=f(x)-2hf'(x)+2h^{2}f''(x)-\frac{4h^{3}}{3}f'''(x)+\dots$

$$f(x+2h)+f(x-2h)=f(x)+f(x)\cancel{-2hf'(x)}\cancel{+2hf'(x)}+2h^{2}f''(x)+2h^{2}f''(x)\cancel{+\frac{4h^{3}}{3}f'''(x)}\cancel{-\frac{4h^{3}}{3}f'''(x)}\dots$$

$$f(x+2h)+f(x-2h)=2f(x)+4h^{2}f''(x)+\frac{4}{3}h^{4}f''''(x)+\dots$$

$$\hookrightarrow f''(x)=\frac{f(x+2h)-2f(x)+f(x-2h)}{4h^{2}}\underbrace{-\frac{1}{3}h^{2}f''''(x)+\dots}_{O(h^{2})}$$

$$\hookrightarrow f''(x_i)\cong\frac{f(x_{i+2})-f(x_i)+f(x_{i-2})}{4h^{2}}+O(h^{2})$$

4.7.5. Muestre que el operador $D^4 f$ está dado por:

$$D^{4}f(x_j)\cong\frac{f(x_{j+2})-4f(x_{j+1})+6f(x_j)-4f(x_{j-1})+f(x_{j-2})}{h^{4}}$$

$$f(x+h)=f(x)+hf'(x)+\frac{h^{2}}{2}f''(x)+\frac{h^{3}}{6}f'''(x)+\frac{h^{4}}{24}f^{(4)}(x)+\frac{h^{5}}{120}f^{(5)}(x)+\frac{h^{6}}{720}f^{(6)}(x)+\dots$$

$$f(x-h)=f(x)-hf'(x)+\frac{h^{2}}{2}f''(x)-\frac{h^{3}}{6}f'''(x)+\frac{h^{4}}{24}f^{(4)}(x)-\frac{h^{5}}{120}f^{(5)}(x)+\frac{h^{6}}{720}f^{(6)}(x)+\dots$$

$$f(x+2h)=f(x)+2hf'(x)+2h^{2}f''(x)+\frac{4}{3}h^{3}f'''(x)+\frac{2}{3}h^{4}f^{(4)}(x)+\frac{4h^{5}}{15}f^{(5)}(x)+\frac{4}{9}h^{6}f^{(6)}(x)+\dots$$

$$f(x-2h)=f(x)-2hf'(x)+2h^{2}f''(x)-\frac{4h^{3}}{3}f'''(x)+\frac{2h^{4}}{3}f^{(4)}(x)-\frac{4h^{5}}{15}f^{(5)}(x)+\frac{4}{9}h^{6}f^{(6)}(x)+\dots$$

$$f(x+2h)+f(x-2h)-4f(x+h)-4f(x-h)=f(x)+f(x)-4f(x)-4f(x)$$
$$+2hf'(x)-2hf'(x)+4hf'(x)-4f'(x)+2h^{2}f''(x)+2h^{2}f''(x)+2h^{2}f''(x)$$
$$+2h^{2}f''(x)+\frac{4}{3}h^{3}f'''(x)-\frac{4h^{3}}{3}f'''(x)+\frac{4h^{3}}{6}f'''(x)-\frac{4h^{3}}{6}f'''(x)+\frac{2}{3}h^{4}f^{(4)}(x)+\frac{2}{3}h^{4}f^{(4)}(x)$$
$$-\frac{h^{4}}{6}f^{(4)}(x)-\frac{h^{4}}{6}f^{(4)}(x)+\dots$$

La expresión continúa. Sin embargo al simplificar va a quedar como a continuación:

$$4f(x+h)+4f(x-h)-f(x-2h)-f(x+2h)$$
$$=6f(x)-h^4 f^{(4)}(x)+\frac{65}{360}h^6 f^{(6)}(x)+\dots$$

* note que el Signo esta cambiado a como mostrado anteriormente (moltiplicado todo Por [-1]).

$$\frac{f(x-2h)-4f(x-h)+6f(x)-4f(x+h)+f(x+2h)}{h^4}\underbrace{-\frac{65}{360}h^4 f^{(6)}(x)}_{O(h^4)}=f^{(4)}(x)$$

$$\boxed{f(x_{i+b})=f(x+bh)}$$

Por lo tanto;

$$D^4 f(x_j)\simeq\frac{f(x_{j+2})-4f(x_{j+1})+6f(x_j)-4f(x_{j-1})+f(x_{j-2})}{h^4}$$

el orden de aproximación $[O(h^k)]$ es $[O(h^4)]$ con $[k=4]$.