CHAPTER

10

INTEREST RATE RISK: DURATION ANALYSIS

Learning Objectives

- The concept of duration and its measurement.
- How duration can measure the interest rate exposure of an FI.
- The problems of applying the duration approach to real FI balance sheets.

Introduction

Duration
A measure (expressed in units of time) of interest rate elasticity of the price of an interest-bearing instrument.

In the last chapter, we introduced you to interest rate risk and showed that gap analysis is a simple, flexible, and intuitive technique for assessing interest rate risk. Gap analysis is used by virtually every FI in Canada, but it does have serious problems. One problem we did not identify is that gap analysis can be unwieldy. It does not present the analyst with a single indicator of interest rate risk; rather, it gives a whole table of numbers. If the analyst strives to avoid the problem of overaggregation in gap analysis, she will define ever smaller buckets, giving her a matrix of hundreds of numbers. **Duration** is a handy alternative tool because *it can encapsulate interest rate exposure in a single number.*

We first present the equation used to calculate duration and discuss duration's economic meaning: duration is the interest sensitivity (or interest elasticity) of an asset or liability's present value. You can also interpret duration as a sophisticated measure of average life; the duration and the average life of an instrument are identical if there is no time value of money. We show the basic arithmetic needed to calculate the duration of four instruments. We show next how duration can immunize or protect an FI against interest rate risk. Finally, we examine some problems in applying the duration measure to real-world FIs and discuss why simulation models are gaining popularity.

A General Formula for Duration

You can calculate the duration for any fixed-income security[1] using this general formula:[2]

[1]Our initial discussion assumes that the instruments for which duration is calculated bear interest at fixed rates. When considering interest rate risk, we are really interested in the time to repricing. If interest on the instrument were calculated on a floating-rate basis, you should calculate the average life to the date of repricing

$$D = \frac{\sum_{t=1}^{N} t\, CF_t \cdot DF_t}{\sum_{t=1}^{N} CF_t \cdot DF_t} = \frac{\sum_{t=1}^{N} t\, PV_t}{\sum_{t=1}^{N} PV_t} \qquad (1)$$

where

$\sum_{t=1}^{N}$ = Summation sign for addition of all terms from $t = 1$ to $t = N$

CF_t = Cash flow received on the security at end of period t
N = The last period in which the cash flow is received
DF_t = The discount factor = $\frac{1}{(1+R)^t}$, where R is the yield or current level of interest rates in the market
PV_t = The present value of the cash flow at the end of period t (which equals $CF_t \times DF_t$)

From this formula, you can see that duration is a weighted average of times, where the weighting factors are the present values of cash flows at each time, t.

The Economic Meaning of Duration

Interest Rate Elasticity
The percent change in some number (here price) caused by a percent change in interest rates.

Duration is a *direct* measure of the **interest rate** sensitivity or **elasticity** of an asset or liability. In other words, the larger the numerical value of D that is calculated for an asset or liability, the more sensitive the price of that asset or liability is to changes or shocks in interest rates.

Consider the following equation showing that the current price of a bond is equal to the present value of the coupons and principal payment on the bond, where

P = Price on the bond
C = Coupon (annual)
R = Yield to maturity
N = Number of periods to maturity
F = Face value of the bond

$$P = \frac{C}{(1+R)} + \frac{C}{(1+R)^2} + \ldots + \frac{C+F}{(1+R)^N} \qquad (2)$$

(e.g., for a loan priced over 90-day LIBOR, the average life to repricing would be 90 days). We return to this point later in this chapter.

[2]Throughout this text, we use the term "duration" as being synonymous with what is more accurately known as "Macaulay duration." The reader should, however, be aware that others may use "duration" to mean "dollar duration" (see below) or "modified duration" (see Appendix). In the following material, a number of useful examples and formulas were suggested by G. Hawawini of INSEAD. For more discussion of the duration model and a number of those examples, see G. Hawawini, "Controlling the Interest Rate Risk of Bonds: An Introduction to Duration Analysis and Immunization Strategies," *Finanzmarket and Portfolio Management* 1, 1986–87, pp. 8–18.

We want to find out how the price of the bond (P) changes when yields (R) rise. We know that bond prices fall, but we want to derive a direct measure of the size of this fall (its degree of price sensitivity).

Taking the derivative of the bond's price (P) with respect to the yield to maturity (R), we get:

$$\frac{dP}{dR} = \frac{-C}{(1+R)^2} + \frac{-2C}{(1+R)^3} + \ldots + \frac{-N(C+F)}{(1+R)^{N+1}} \tag{3}$$

By rearranging, we get:

$$\frac{dP}{dR} = -\frac{1}{(1+R)}\left[\frac{C}{(1+R)} + \frac{2C}{(1+R)^2} + \ldots + \frac{N(C+F)}{(1+R)^N}\right] \tag{4}$$

In equation (1) at the beginning of this chapter, we defined duration (D) as the weighted average time to maturity, using the present value of cash flows as weights. That is, by definition:

$$D = \frac{1 \cdot \dfrac{C}{(1+R)} + 2 \cdot \dfrac{C}{(1+R)^2} + \ldots + N \cdot \dfrac{(C+F)}{(1+R)^N}}{\dfrac{C}{(1+R)} + \dfrac{C}{(1+R)^2} + \ldots + \dfrac{(C+F)}{(1+R)^N}} \tag{5}$$

Since the denominator of the duration equation is simply the price (P) of the bond, which is equal to the present value of the cash flows on the bond, then:

$$D = \frac{1 \cdot \dfrac{C}{(1+R)} + 2 \cdot \dfrac{C}{(1+R)^2} + \ldots + N \cdot \dfrac{C+F}{(1+R)^N}}{P} \tag{6}$$

Multiplying both side of this equation by P, we get

$$P \cdot D = 1 \cdot \frac{C}{(1+R)} + 2 \cdot \frac{C}{(1+R)^2} + \ldots + N \cdot \frac{C+F}{(1+R)^N} \tag{7}$$

The term on the right side of equation (7) is the same term as that in brackets in equation (4). Substituting equation (7) into equation (4), we get:

$$\frac{dP}{dR} = -\frac{1}{(1+R)}[P \cdot D] \tag{8}$$

By cross-multiplying,

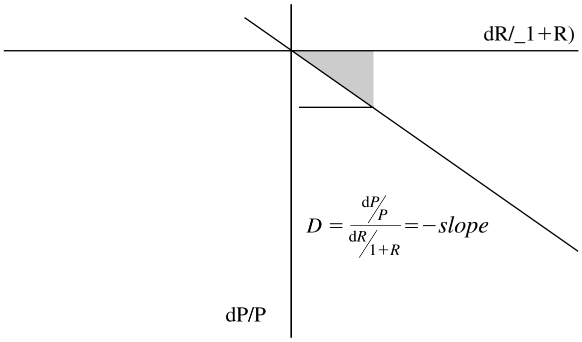
$$\frac{dP}{dR} \cdot \frac{(1+R)}{P} = -D \tag{9}$$

or, alternatively,

$$\frac{\dfrac{dP}{P}}{\dfrac{dR}{(1+R)}} = -D \tag{10}$$

FIGURE 10–1

The approximate relationship between price changes and yield changes on a bond as estimated by duration



The economic interpretation of equations (7) and (10) is that the number D is the interest elasticity or sensitivity of the security's price to small interest rate changes.[3] That is, it describes the percentage price fall of a bond (dP/P) for any given (present value) increase in required interest rates or yields (($dR/(1 + R)$)).

Equations (9) and (10) can be rearranged in another useful way for interpretation regarding interest sensitivity:

$$\frac{dP}{P} = -D\left[\frac{dR}{1 + R}\right] \tag{11}$$

Equation (11) and Figure 10–1, its graphic representation, show that for small changes in interest rates, bond prices move in an *inversely proportional* fashion according to the size of D.

Duration and Average Life

Average Life
The weighted average of times of payments on an instrument, where the weights are the amounts paid.

FI professionals frequently use the concept of average life, rather than final maturity, to summarize the length of time that a loan or bond with runoff (i.e., payments before maturity) must be funded. Let V be the **average life,** such that

$$V = \frac{\sum_{t=1}^{n} t\, CF_t}{\sum_{t=1}^{n} CF_t} \tag{13}$$

where

t = Time of each payment
CF_t = Cash flow received on the security at time t
n = Total number of payments on the security

Equation (13) looks very similar to equation (1) (the formula for duration) because average life, like duration, is a weighted average of times when payments are received. Here, however, the weights are the payments on the instrument, whereas in the case of duration they were the present value of payments on the instrument. Average life calculations are sometimes called the *method of equated time* to distinguish them from duration calculations

[3]Like all elasticities, duration is unit-less with respect to the variables in the derivative, (ie P and R). But as we demonstrated above in equation (1) it is a weighted average of times.

because they weight a dollar paid today the same as a dollar in the future. We will return to the difference between average life and duration when we discuss immunization of a portfolio later in this chapter. First, however, to help you understand how duration works in practice, we calculate five examples: an international bond, a Canada bond, a zero-coupon bond, a consol bond, and a floating-rate note (FRN).

Calculating Duration

The Duration of a Six-year International Bond

International Bond
A bond issued in a location other than the country in whose currency the bond is denominated.

International bonds (or, as they are sometimes called, Eurobonds)[4] are bonds that are issued in the offshore markets. As we mentioned in Chapter 6, the markets are offshore money and capital markets that issue and trade securities denominated in a currency other than the currency of the country where the transaction occurs. For example, a Canadian-dollar bond issued in London would be an international bond. Centred in London, the international bond market is characterized by high-quality borrowers, large principal amounts, worldwide sales, low transactions costs, and fewer regulatory constraints than onshore markets. Offshore markets are important to Canadians as the major venue for Canadian provincial debt. An international bond can be issued in any of the major convertible currencies of the world, including the Canadian dollar. International bonds form a convenient starting place for our description of duration because, by convention, international bonds have annual coupons (i.e., pay interest annually), whereas Government of Canada and corporate domestic bonds have semiannual coupons.

Suppose the annual coupon is 8 percent, the face value of the bond is \$1,000, and the current yield to maturity (R) is also 8 percent. We show the calculation of its duration in Table 10–1.

Although the final maturity of this international bond is six years, its duration is just under five years. $D = 4.99$ years. Suppose that yields rose by one basis point (one hundredth of one percent), from 8 to 8.01 percent. Then:

$$\frac{dP}{P} = -(4.99)\left[\frac{.0001}{1.08}\right]$$

$$= -.000462$$

$$\text{or} \qquad -\,0.0462\%$$

The bond price had been \$1,000, which was the present value of a six-year bond with 8-percent coupons and 8-percent yield. The duration model predicts the price of the bond would fall to \$999.54 after the increase in yield by one basis point.[5]

[4]Since its inception in the 1960s, the term *Euromarket* has been used to describe markets in securities where the place of issue differed from the currency denomination of the security. The vast majority of Euromarket issues were U.S.-dollar issues traded in Europe, primarily London. Today, although still used, the term *Eurobond* to describe international bonds is gradually falling into disuse. With the emergence of the *euro* as the currency of most of Europe, the more logical meaning of *euro*bond is simply a bond denominated in *euros.* A similar change is taking place in the term money markets, with *offshore deposit* gradually replacing the term *Eurodeposit.*

[5]That is, the price would fall by 0.0462 percent, or by \$0.462. To calculate the dollar change in value, we can rewrite the equation as $dP = (P)(dR/(1 + R)) = (\$1000)(-4.99)(0.0001/1.08) = \$0.462$.

TABLE 10–1 The Duration of a Six-year International Bond with 8-Percent Coupon and Yield

t	CF_t	DF_t	$CF_t \times DF_t$	$t \times CF_t \times DF_t$
1	80	0.9259	74.07	74.07
2	80	0.8573	68.59	137.18
3	80	0.7938	63.51	190.53
4	80	0.7350	58.80	235.20
5	80	0.6806	54.45	272.25
6	1,080	0.6302	680.58	4,083.48
			1,000.00	4,992.71

$$D = \frac{4{,}992.71}{1000} = 4.993 \text{ years}$$

The Duration of a Two-year Government of Canada Bond

Canada Bond
A bond issued by the Government of Canada.

Canada bonds (like Canadian domestic corporate bonds, U.S. treasury bonds, and U.S. corporate bonds) pay coupon interest semiannually. Suppose the annual coupon rate is 8 percent, the face value is $1,000, and the annual yield to maturity (according to the bond convention) is $R_B = 12$ percent.[6] We use the bond convention formula for discounting bonds with semiannual coupons:

$$DF_t = (1 + \tfrac{1}{2}R_B)^{-2t}$$

Thus, at $t = \frac{1}{2}$, the discount factor is $(1.06)^{-1}$, at $t = 1$ year the discount factor is $(1.06)^{-2}$, and so on. Table 10–2 calculates the duration of this bond.

A two-year Canada bond with an 8-percent coupon and a 12-percent yield has a duration of 1.88 years. A one-basis-point rise in interest rates would have the following predicted effect on its price:

$$\frac{dP}{P} = -1.88\left[\frac{.0001}{1.06}\right]$$
$$= -0.00018$$

or a 0.018 percent price fall.

The Duration of a Zero-coupon Bond

Zero-coupon Bond
A bond that does not pay any coupon interest over the life of the bond. Instead, a single payment of principal or face value is paid on maturity.

In recent years, Canadian investment dealers have created **zero-coupon bonds** by stripping individual coupons and the principal from regular Canada bonds and selling them to investors as separate securities. Elsewhere, such as in the international bond markets, corporations have issued zero-coupon bonds directly. T-bills, BAs, and commercial paper are usually issued on a discount basis and are further examples of discount securities. These instruments sell at a discount from face value on issue and pay the face value (e.g., $1,000)

[6]Note that the bond convention of semiannual compounding is not used when describing the yield of international bonds. International bond yields are quoted according to the annual compounding convention.

TABLE 10–2 The Duration of a Two-year Canada Bond with 8-Percent Coupon and 12-Percent Yield

t	CF_t	DF_t	$CF_t \times DF_t$	$t \times CF_t \times DF_t$
½	40	0.9434	37.74	18.87
1	40	0.8900	35.60	35.60
1½	40	0.8396	33.58	50.37
2	1,040	0.7921	823.78	1,647.56
			930.70	1,752.4

$$D = \frac{1{,}752.4}{930.7} = 1.88 \text{ years}$$

on maturity. The current price an investor is willing to pay for such a bond is equal to its present value, or

$$P = \frac{1{,}000}{(1 + R)^N}$$

where R is the required annually compounded yield to maturity, N is the number of periods to maturity, and P is the price. Substitute the cash flows of a zero-coupon bond into equation (1) to satisfy yourself that, because there are no intervening cash flows (i.e., coupons) between issue and maturity, the following must be true:

$$D_{zero\ coupon} = Final\ maturity$$

The duration of a discount instrument equals its final maturity. This concept can be expressed in another way. When we are calculating the durations of coupon-paying bonds and interest-bearing loans, we obtain a single number that measures interest rate sensitivity. That number is the maturity of a zero-coupon bond that would have the same interest rate sensitivity as the coupon-paying bond whose duration we calculated. Thus a six-year international bond with an 8-percent coupon and yield has the same interest rate sensitivity as a five-year zero-coupon bond. A two-year Canada bond with a 12-percent yield and an 8-percent coupon has the same interest rate sensitivity as a 1.88-year zero-coupon bond.[7]

The Duration of a Consol Bond

Consol Bond
A perpetual bond with a fixed rate of interest.

A **consol bond** is a bond that pays a fixed coupon each year. The novel feature of this bond is that it never matures; it is a perpetuity. Consol bonds have long been used in the United Kingdom, where some consol bonds issued by the government in the 1890s to finance the Boer Wars in South Africa are still outstanding. Moreover, some FIs currently issue very long maturity preferred shares whose properties are similar to consols. Consols are of theoretical interest in exploring the differences between maturity and duration. Although you

[7]Note that, in North America, yields of zero coupons are quoted according to the bond convention, i.e.

$$P = \frac{1000}{(1 + R/2)^{2N}}$$

cannot use equation (1) to calculate a consol's duration, you can easily show that its duration should be calculated as follows:[8]

$$D_c = 1 + \frac{1}{R}$$

where R is the required yield to maturity. Suppose that the market rate of interest for consols implies $R = 5$ percent. Then the duration of the consol is

$$D_c = 1 + \frac{1}{0.05} = 21 \text{ years}$$

Thus, while the maturity of the consol is infinite, its duration is finite. Moreover, as interest rates rise, the duration of the consol falls. For example, if the yield on long-term government bonds rose to 20 percent, then

$$D_c = 1 + \frac{1}{0.2} = 6 \text{ years}$$

The Duration of an FRN

FRN
Bond with variable interest rates set periodically at a spread over, typically, LIBOR, in offshore markets and BAs in Canada.

The four securities whose durations we calculated have an interest rate that is set at the time of issue and is held constant throughout the security's life: the interest rates are fixed. However, many deposits, bonds, and loans carry floating interest rates. Examples include floating-rate notes (**FRN**s), loans priced against the BA rate, prime-based loans, notice deposits, and interest payment demand deposits. We will calculate the duration of an FRN and then discuss discretionary interest rate securities: prime-based loans, notice deposits and interest-bearing demand deposits.

FRNs are bonds with variable interest rates periodically set at a spread over a floating interest rate benchmark, typically LIBOR (e.g., LIBOR + 1/2 percent). Most bonds are medium-term, annual coupon financing vehicles (e.g., seven years maturity) with no principal payable until the final maturity date. *Perpetual FRNs* are like consols in that they have no maturity date but differ from consuls because their coupons fluctuate with market rates. Perpetual FRNs, unlike consols, have been issued in recent years by Canadian issuers. What is the duration of these securities?

You will remember that, when performing gap analysis in Chapter 8, we were interested in the *time to repricing* of instruments, not the final maturity. This is because we wish to measure the sensitivity of the instrument's fair price to changes in interest rates. If the rate at which interest accrues on an instrument is reset to reflect market interest rates, clearly changes in the market interest rate will have no effect on the instrument's price. In

---

[8]To confirm that the duration of a consol is as shown, note that the price of a consol is $P_c = \frac{1}{R}$. Differentiating this relationship, $\frac{dP_c}{dR} = -\frac{1}{R^2}$. Divide the left side by P_c and the right side by $\frac{1}{R}$ for $\frac{dP_c/P_c}{dR} = -\frac{1}{R^2} \cdot R = -\frac{1}{R}$. Multiply both sides by $(1 + R)$ to get $\frac{dP_c/P_c}{dR/(1+R)} = -\frac{1+R}{R}$. Multiply both sides by -1 and rearrange terms: $-\frac{dP_c/P_c}{dR/(1+R)} = 1 + \frac{1}{R}$. As we show below, the economic definition of duration is the negative of the interest rate elasticity of price, i.e., $D = -\frac{dP/P}{dR/(1+R)}$. Therefore, $D_c = 1 + \frac{1}{R}$. See Hawawini, "Controlling the Interest Rate Risk of Bonds," note 2, for further discussion.

the same way, when calculating the duration of an FRN, we wish to calculate the duration to the time of repricing, not time to maturity.

Let us consider a perpetual FRN priced over one-year LIBOR. Two days before the beginning of each year, the reference agent for the FRN sets the coupon rate paid at the end of that year for that year only.[9] Suppose you buy the bond in the middle of the first year ($t = ½$) rather than at the beginning. You know that, as long as the credit risk of the FRN issuer does not deteriorate, the FRN's interest rate will be reset in six months so that the FRN will be valued at par. Therefore, the present value of the FRN at the time of purchase is

$$P_0 = \frac{C_1}{(1 + R)^{1/2}} + \frac{P_1}{(1 + R)^{1/2}}$$

where C_1 is a fixed cash flow of the first coupon due after purchase that was preset before the investor bought the FRN, P_0 is the current price of the FRN, and P_1 is its face value (which will equal its market price the next time the reference agent sets the rate). In short, buying this FRN is equivalent (from a duration perspective) to buying two single-payment, zero-coupon bonds, each with a maturity of six months. Because the duration of a zero-coupon bond is the same as its maturity, this FRN has

$$D = ½ \text{ year}$$

Note that this duration calculation procedure also holds for FRNs with a finite maturity and floating-rate loans priced against LIBOR, the BA rate, or any other periodic interest rate benchmark.

Durations of Discretionary Interest Rate Securities

A significant portion of FI assets and liabilities is subject to interest rate change at the discretion of the FI. Rates may be changed whenever the FI deems it appropriate. With such instruments, the duration is theoretically zero. One such instrument is a corporate loan priced at a spread above the FI's prime rate of interest. An FI's prime rate is a variable interest rate defined by the FI itself as the rate it charges to its best customers on short-term loans. Legally, the FI can change its prime rate whenever it wishes, even daily. In fact, however, most FIs refrain from adjusting prime too frequently (for example, more than once a week, even when interest rates are extremely volatile) to avoid customer loss of confidence, staff confusion, and loss of market share to competitors. Hence, the duration of a prime-based loan is several days. Posted notice deposits rates are also likely to be slightly sticky for similar reasons.

Core Deposits
Deposits that remain in a depository institution for relatively long periods.

Implicit Interest
Payment of interest in kind, such as through subsidised check-clearing services.

As we discussed in Chapter 7, a demand deposit is a **core deposit** with a relatively long maturity (i.e., time to expected withdrawal). Notwithstanding the fact that no interest is explicitly paid on demand deposits, FIs pay demand depositors an **implicit interest** in the form of fees not charged for transaction services. Because these fee and service schedules can be altered any time at the discretion of the FI, the theoretical duration of demand

[9] The FRN documentation will name the reference agent, whose sole task it is to determine the interest rate. Depending on the documentation, this may be done by referring to a Reuters page or by polling the (usually three) named reference banks at 11:00 AM London time on the rate-setting date (i.e., two days before the interest period starts) to determine the interest rate at which they are placing time deposits with leading banks for the interest period (here, one year). The reference agent averages these quotes (which are usually identical, given the depth and efficiency of the market) and publishes the determined rate.

deposits is also zero. In fact, however, fee and service schedules are revised less frequently than lending and deposit rates, so the effective duration of demand deposits exceeds that of prime loans and notice deposits. The FI revises its fee and service schedules only when it becomes evident (for example, through net demand deposit drain) that market interest rates have moved out of line with implicit interest rates.[10]

Concept Questions

1. Calculate the duration of a one-year, 8-percent-coupon, 10-percent-yield bond that pays coupons quarterly.
2. Why do you think zero-coupon bonds were introduced in the early 1980s?
3. What market interest rate would equate the duration of a zero-coupon, 10-year bond and a consol paying 7 percent in perpetuity?
4. Why is a perpetual FRN's duration so short?

Features of Duration

From the preceding examples, we derive three important features of duration relating to the maturity, yield, and coupon interest of the security being analyzed.

Duration and Maturity

Duration *increases* with the maturity of a fixed-income asset or liability, but at a *decreasing* rate:

$$\frac{\partial D}{\partial M} > 0 \qquad\qquad \frac{\partial^2 D}{\partial M^2} < 0$$

where M is the maturity of a fixed-rate instrument or the time to repricing of a floating-rate instrument. To see this, look at Figure 10–2, where we plot duration against maturity for a three-year international bond, a six-year international bond, and a consol bond using the *same yield* of 8 percent for all three and assuming an annual coupon of 8 percent on each bond.

Duration and Yield

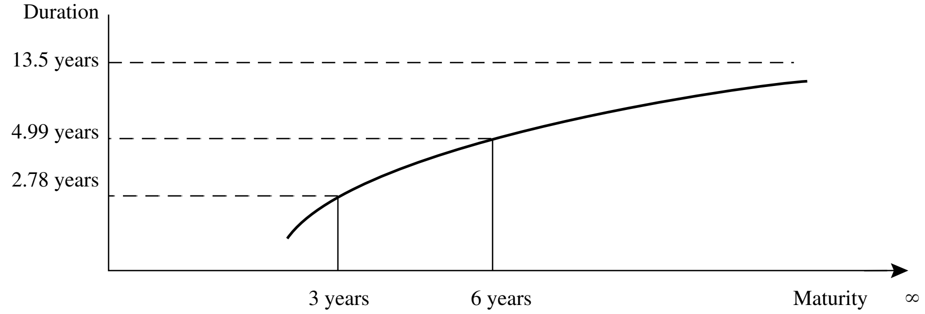
Duration decreases as yield increases:

$$\frac{\partial D}{\partial R} < 0$$

To prove this, consider the consol bond: When $R = 8$ percent, as we show in Figure 10–1, $D = 13.5$ years. If R increases to 9 percent, then $D = 12.11$ years. This makes sense intuitively because higher yields discount later cash flows more heavily and the relative importance, or weights, of those later cash flows decline when compared to earlier cash flows on an asset or liability.

[10]It may be important for FIs to analyze the sensitivity of demand deposit withdrawal to changes in the interest rate. One quantitative technique is to regress percentage change in demand deposits (the dependent variable) on interest changes and other independent variables in a time series regression. For a very sophisticated model along these lines, see "The OTS Market Value Model," OTS, Washington, DC, 1992, pp. 5.T.-1–4.

FIGURE 10–2
Duration versus maturity



Duration and Coupon Interest

The higher the coupon or promised interest payment on the security, the lower its duration:

$$\frac{\partial D}{\partial C} < 0$$

This is due to the fact that the larger the coupons or promised interest payments, the quicker cash flows are received by investors and the higher the present value weights of those cash flows in the duration calculation.[11]

Concept Questions

1. Which has the longer duration, a 30-year, 8-percent, zero-coupon bond, or an 8-percent infinite maturity consol bond?
2. Do high-coupon bonds have higher or lower durations than low coupon bonds, holding all other factors constant?
3. What is the relationship between the duration of a bond and its interest elasticity?
4. How would the formula in equation (12) on page 000 have to be modified to take into account quarterly coupon payments and monthly coupon payments?

Dollar Duration

We have been calculating simple or Macaulay duration, which was named after the economist, Frederick Robertson Macaulay, who was among the first to develop the *duration* concept.[12] The use of the word "duration" for Macaulay's concept is apt because in non-financial parlance, "duration" means a span of time and, as we have pointed out above, Macaulay duration *is* a weighted average of times. Frequently, however, practitioners prefer to express the sensitivity of a fixed income instrument to rises in the interest rate in terms of dollars lost or gained. They can do so with dollar duration. We can calculate dollar duration by referring to Equation 11 on page 000, which states that:

---

[11]For example, consider two bonds each with two years left to maturity. Each has a face value of $100, but the first bond pays a coupon of 6 percent and the second a higher coupon of 12 percent. As a result, the cash flows of the first bond will be $6 (in year 1) and $106 (in year 2). By comparison, the cash flows on the second bond will be $12 (in year 1) and $112 (in year 2). On average, a dollar of cash flow is received more quickly for bond 2 compared to bond 1.

[12]F. R. Macaulay, *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1856* (New York: NBER, 1938).

$$\frac{dP}{P} = -D \cdot \left[\frac{dR}{1+R}\right]$$

Dollar duration is simply a change in price. In the above formula, we know that dP is also a change in price and that, rearranging terms:

$$dP = -D \cdot \left[\frac{dR}{1+R}\right] \cdot P \tag{14}$$

The only problem with this formula is that, whereas duration was without units with respect to the price of the fixed income instrument and the change in interest rates (being the percent change in price for a percent change in interest), dP is not. It's in dollars. One must define the dollar amount of the fixed income instrument P and the percent change in the interest rate $dR/(1 + R)$ to which dollar duration relates. One useful convention is that dollar duration relates to a 100-basis-point change in interest rates (in other words $dR/(1+R) = 1\% = 0.01$) on a \$100 market value instrument ($P = \$100$). With this convention, and letting dollar duration $= DD$:

$$DD = -D \cdot \left[\frac{dR}{1+R}\right] \cdot P = -D \cdot [.01] \cdot 100 = -D \tag{15}$$

Here, the absolute value of dollar duration is the same as that of Macaulay duration, but the units are now in *dollars.* It is no longer an elasticity. Moreover, the analyst must always take care to confirm the convention with which dollar duration is being calculated.[13]

Immunization

In this section we will show how the measures of interest rate risk we have introduced in this chapter can be used to structure the FI's portfolio to render it "immune" to interest rate risk. First, we will return to the concept of average life to demonstrate the shortcomings of that measure of interest rate risk. Then we will discuss duration and immunization.

Matching Average Life to Reduce Interest Rate Risk

An FI could implement a strategy of matching asset and liability average lives (to maturity for fixed-rate instruments and to repricing for floating rate) to reduce interest rate risk. Yet this strategy does not eliminate all interest rate risk for an FI. The following example shows that an FI choosing to match directly the average life of its assets and liabilities does not achieve a perfect hedge, or protection for its equity holders, against interest rate risk. Consider an FI that issues a deposit with an average life of two years. To make our example simple, assume that the deposit is structured as a zero-coupon bond that pay out principal plus all accrued interest in one payment at the end of two years. Since there is only one payment, its average life is the same as its final maturity: two years. Assume that the deposit is for $\$160 \times (1.10)^2 = \$193.60$.

Suppose that the FI lends \$160 it obtains from the depositor to a corporate borrower, also at 10-percent interest. This, of course, would be unlikely, since the FI would make no

[13]In fact, the convention quoted above is often not followed. For example if duration = 2, the dollar duration for a 100-basis-point change on a \$1,000 position would be −\$20. Note also that dollar duration is frequently quoted as the absolute value, obviating the need to insert the minus sign.

TABLE 10–3 Payments of Asset with Matched Average Life

Year	*Outstanding Principal through Year*	*Interest Payment*	*Principal Payment*	*Total Payment*
1	160	16	50	66
2	110	11	50	61
3	60	6	60	66

FIGURE 10–3
Liabilities

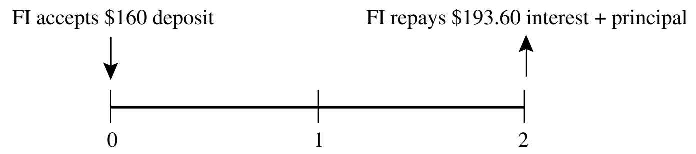
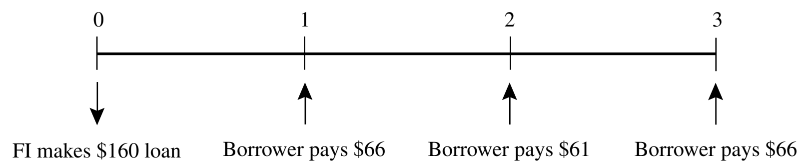


FIGURE 10–4
Assets



spread on the transaction between its cost of funds and its interest income, but the simplification helps centre attention on interest rate risk. The FI requires that interest on the outstanding balance be paid annually and that the principal be paid back in three installments: $50 at the end of the first year, $50 at the end of the second, and $60 at the end of the third. Payments on the loan are shown in Table 10–3.

You may wonder why the FI chose this principal repayment schedule. By so doing, the FI has structured its asset to have an average life of *exactly* two years.[14]

$$V_A = \frac{\sum_{t=1}^{3} tCF_t}{\sum_{t=1}^{3} CF_t} = \frac{(1)(66) + (2)(61) + (3)(66)}{66 + 61 + 66} = 2.00$$

Figures 10–3 and 10–4 detail the liability and asset cash flows associated with this transaction.

Immunizing
Fully protecting or hedging an FI's equity holders against interest rate risk.

Has the portfolio been immunized? To determine whether or not it has, assume that immediately after the loan and deposit were booked interest rates rose 1 percent to 11 percent. if the present values of the assets and liabilities fall by exactly the same amount, the position can be said to be **immunized.** Table 10–4 calculates the new present values.

[14]Note that the FI still faces liquidity risk. Asset and liability cash flows are not exactly matched. The FI faces liquidity reinvestment risk for the first interest payment and liquidity refinancing risk at the end of year 2 unless the deposit is rolled over. The objective of matching average life was, however, to avoid interest rate risk, not liquidity risk.

TABLE 10–4 a. Present Value of Liability with Matched Average Life

Year	*Cash Flow*	*Discount Factor*	*Present Value*
2	$193.6	$(1.11)^{-2} = 0.8116$	$157.13
Present value of liability after interest rate rise			$157.13

b. Present Value of Asset with Matched Average Life

Year	*Cash Flow*	*Discount Factor*	*Present Value*
1	66	$(1.11)^{-1} = 0.9009$	$ 59.46
2	61	$(1.11)^{-2} = 0.8116$	49.51
3	66	$(1.11)^{-3} = 0.7312$	48.26
Present value of asset after interest rate rise			$157.23

The matching of average life *very nearly* immunized the portfolio; the instantaneous rise in the interest rate produced a market value gain of $0.10 to the portfolio. But that dime difference shows that our hedge is not perfect.

Duration and Immunization

In the preceding example, we showed that, while matching the average life of the liabilities that fund an asset with the average life of the asset did not immunize the total position from interest rate risk, matching durations did. Next we go through an example of how duration could be used to hedge interest rate exposure for a pension fund manager. Then we return to the general question of using duration to insulate the whole balance sheet of an FI against interest rate risk.

Tables 10–5 to 10–7 show that, in this case, matching durations of liabilities and assets will achieve what matching average lives did not: a perfect hedge. First, let us calculate the duration of the liability in the example. At a 10 percent per annum rate of interest, the discount factors are show in Table 10–5.

The duration of the deposit (called D_L for duration of the liability) is

$$D_L = \frac{(2)(193.6)(0.8264)}{(193.6)(0.8264)} = 2.00$$

As we noted, the duration of a zero-coupon bond is the same as its average life, which is the same as its final maturity. But the duration of the loan (called D_A for duration of the asset) differs from its average life. Its average life is exactly two years, but its duration is only 1.93 years, as calculated below:

$$D_A = \frac{(1)(66)(0.9091) + (2)(61)(0.8263) + (3)(66)(0.7513)}{(66)(0.9091) + (61)(0.8263) + (66)(0.7513)} = 1.93$$

Because the duration of liabilities was slightly more than the duration of assets, increasing the interest rate caused the value of liabilities to fall by slightly more than the value of assets. What payment schedule will immunize the portfolio? The schedule that

TABLE 10–5 Discount Factors at 10 Percent Interest

Year	*Discount Factor*
1	$(1.10)^{-1} = 0.9091$
2	$(1.10)^{-2} = 0.8264$
3	$(1.10)^{-3} = 0.7513$

TABLE 10–6 Payment of Asset with Matched Duration

Year	*Outstanding Principal through Year*	*Interest Payment*	*Principal Payment*	*Total Payment*
1	$160	$16	$45.94	$61.94
2	114.06	11.41	45.94	56.35
3	68.12	6.81	68.12	74.93

TABLE 10–7 Present Value of Asset with Matched Duration

Year	*Cash Flow*	*Discount Factor*	*Present Value*
1	$61.94	$(1.11)^{-1} = 0.9009$	$ 55.8
2	57.35	$(1.11)^{-2} = 0.8116$	46.54
3	74.93	$(1.11)^{-3} = 0.7312$	54.79
Present value of asset after interest rate rise			$157.13

equates the *duration* of assets and liabilities, not their average lives or final maturities. In the example, if the loan is repaid in principal payments of $45.94 for the first two years and $68.12 the third year, the duration of assets will equal exactly two years. Given this schedule of principal repayments, interest payments are shown in Table 10–6.

Now, what will happen to the value of the portfolio if interest rates rise by 1 percent the moment after the deposit and loan have been booked? The deposit value will drop to $157.13 (as shown in Table 10–4), since its structure has not changed. The present value of the asset, however, will drop by a greater amount since we have extended its duration to two years to match exactly the duration of the deposit, as shown in Table 10–7.

The value of assets has dropped by exactly the same amount as the value of liabilities. The portfolio is immunized.

Duration and Immunizing Future Payments

Frequently, pension fund and life insurance company managers face the problem of structuring their asset investments so they can pay out a given cash amount to policyholders in some future period. The classic example of this is an insurance policy that pays the holder some lump sum on reaching retirement age. The risk to the life insurance company manager is that interest rates on the funds generated from investing the retiree's premiums could fall. Thus, the target or promised amount could not be met from the accumulated returns on the premiums invested. In effect, the insurance company would be forced to draw down its reserves and net worth to meet its payout commitments. (See Chapter 2 for a discussion of this risk.)

Suppose that we are in 2001 and the insurer has to make a guaranteed payment to an investor in five years (2006). For simplicity, we assume that this target guaranteed payment is $1,469, a lump-sum policy payout on retirement. Of course, realistically this payment would be much larger, but the underlying principles do not change when the payout amount is scaled down.

To immunize or protect itself against interest rate risk, the insurer needs to determine which investments would produce a cash flow of exactly $1,469 in five years, regardless of what happens to interest rates in the immediate future. The FI investing in either a five-year maturity and duration zero-coupon bond or a coupon bond with a five-year duration would produce a $1,469 cash flow in five years, no matter what happens to interest rates in the immediate future. Next, we consider the two strategies: buying five-year zero-coupon bonds and buying five-year duration coupon bonds.

Buy Five-year Maturity Zero-Coupon Bonds. Given a $1,000 face value and an 8 percent yield, and assuming annual compounding, the current price per five-year bond would be

$$P = 680.58 = \frac{1{,}000}{(1.08)^5}$$

That is, a price of $680.58 per bond. If the insurer bought 1.469 of these bonds at a total cost of $1,000 in 1996, these investments would produce exactly $1,469 on maturity in five years ($\$1{,}000 \times (1.08)^5 = \$1{,}469$). The reason is that the duration of this bond portfolio exactly matches the target horizon for the insurer's future liability to its policyholders. Intuitively, since no intervening cash flows or coupons are paid by the issuer of the zero-coupon bonds, future changes in interest rates have no reinvestment income effect. Thus, the return would be unaffected by intervening interest rate changes.

Suppose no five-year zero-coupon bonds exist. Then the portfolio manager may seek to invest in appropriate-duration coupon bonds to hedge interest rate risk. In this example, the appropriate investment would be in five-year duration coupon-bearing bonds.

Buy a Five-year Duration Coupon Bond. We demonstrated earlier in Table 10–1 that a six-year maturity international bond paying 8-percent coupons with an 8-percent yield to maturity had a duration of 4.99 years. If we buy this six-year maturity/five-year duration bond in 2001 and hold it for five years until 2006, the term exactly matches our target horizon. The cash flows generated at the end of five years will be $1,469 whether interest rates stay at 8 percent or instantaneously rise to 9 percent or fall to 7 percent. thus, buying a coupon bond whose duration exactly matches the time horizon of the insurer also immunizes the insurer against interest rate changes.

Interest Rates Remain at 8 Percent. The cash flows the insurer will receive on the bond if interest rates stay at 8 percent throughout the five years are

1. Coupons, 5 × $80	$ 400
2. Reinvestment income	69
3. Proceeds from sale of bond at end of fifth year	1,000
	$1,469

This is how to calculate each of the three components of the insurer's income from the bond investment:

1. *Coupon.* The $400 from coupons is simply the annual coupon of $80 received in each of the five years.
2. *Reinvesment income.* Because the coupons are received annually, they can be reinvested at 8 percent as they are received. Receiving annual coupons of $80 is equivalent to receiving an annuity of $80 for five years. The appropriate terminal value of receiving $1 a year for five years and reinvesting at 8 percent can be determined from the general formula:

   $$FVAF_{n,R} = \left[\frac{(1+R)^n - 1}{R}\right]$$

   In our example:

   $$FVAF_{5,\ 8\%} = \left[\frac{(1+.08)^5 - 1}{.08}\right] = 5.867$$

   Thus, the reinvestment income for $80 of coupons per year is Reinvestment income = (80 × 5.867) − 400 = 469 − 400 = 69. Note that we take away $400 since we have already counted the simple coupon income (5 × $80).
3. *Bond sale proceeds.* Calculate the proceeds from the sale by recognizing that the six-year bond has just one year left to maturity when it is sold by the insurance company at the end of the fifth year. That is,

   ↓Sell $1,080

   Year 5 Year 6

What fair market price can the insurer expect to get when selling the bond at the end of the fifth year with one year left to maturity? A buyer would be willing to pay the present value of the $1,080—final coupon plus face value—to be received at the end of the one remaining year, or

$$P_5 = \frac{1{,}080}{1.08} = \$1{,}000$$

Thus, the insurer would be able to sell the one remaining cash flow of $1,080, to be received in the bond's final year, for $1,000.

Next, we show that since this bond has a duration of five years exactly matching the insurer's target period, even if interest rates were to fall instantaneously to 7 percent or rise to 9 percent, the expected cash flows from the bond would still sum to exactly $1,469. That is, the coupons + reinvestment income + principal at the end of the fifth year would be immunized. In other words, the cash flows on the bond would be protected against interest rate changes. In the following sections are examples of rates falling from 8 to 7 percent and rising from 8 to 9 percent.

Interest Rates Fall to 7 Percent. We show in the following calculations that, the total proceeds over the five years are unchanged from when interest rates were 8 percent. to see why this occurs, consider what happens to the three parts of the cash flow when rates fall to 7 percent:

1. *Coupons.* They are unchanged, since the insurer still gets five annual coupons of \$80, which total \$400.
2. *Reinvestment income.* Again, we consider the reinvestment income by valuing the future value of an annuity of \$80 per year for five years, but here the interest rate is only 7 percent.

$$FVAF_{5,\,7\%} = \left[\frac{(1 + .07)^5 - 1}{.07}\right] = 5.751$$

   Reinvestment income = (5.751 × 80) − 400 = 60, which is \$9 less than when rates were 8 percent.

3. *Bond sale proceeds.* When the six-year maturity bond is sold at the end of the fifth year with one cash flow of \$1,080 remaining, investors will now be willing to pay more:

$$P_5 = \frac{1{,}080}{1.07} = 1{,}009$$

That is, the bond can be sold for \$9 more than when rates were 8 percent. the reason is that investors can get only 7 percent on newly issued bonds, while this older bond was issued with a higher coupon of 8 percent.

In summary, the cash flows over the five years are

1. Coupons, 5 × \$80	\$ 400
2. Reinvestment income	60
3. Bond sale proceeds	1,009
	\$1,469

By comparing reinvestment income with bond sale proceeds, you can see that the fall in rates has produced a *gain* on the bond sale proceeds of \$9. This exactly offsets the loss of reinvestment income of \$9 due to reinvesting at a lower interest rate. Thus, total cash flows remain unchanged at \$1,469.

Interest Rates Rise to 9 Percent. You can confirm that, with rising interest rates, the proceeds from the bond investment are:

1. Coupons, 5 × \$80	\$ 400
2. Reinvestment income [(5.985 × 80) − 400]	78
3. Bond sale proceeds (1,080/1.09)	991
	\$1,469

Notice that the rise in interest rates from 8 to 9 percent leaves the final terminal cash flow unaffected at \$1,469. The rise in rates has generated \$9 extra reinvestment income (\$78 − \$69), but the price at which the bond can be sold at the end of the fifth year has declined from \$1,000 to \$991—a capital loss of \$9. Thus, the gain in reinvestment income is exactly offset by the capital loss on the sale of the bond.

This example demonstrates that matching the duration of a coupon bond—or any fixed-interest rate instrument such as a loan or mortgage—to the FI's target or investment horizon *immunizes* it against instantaneous shocks to interest rates. The gains or losses on reinvestment income that result from an interest rate change are exactly offset by losses or gains from the bond proceeds on sale.

Concept Question

1. Would the FI in the proceeding example been immunized if rates had fallen to 6 percent? if they had risen to 10 percent?

Immunizing the Whole Balance Sheet of an FI

So far, we have looked at the durations of individual instrument and how we can select individual fixed-income securities to protect FIs such as life insurance companies and pension funds with certain precommitted liabilities such as future pension plan payouts. The duration model can also evaluate the overall interest rate exposure for an FI; that is, measure the *duration gap* on its balance sheet.

The Duration Gap for a Financial Institution. To estimate the overall duration gap, we first determine the duration of an FI's asset portfolio and the duration of its liability portfolio. These can be calculated as

$$D_{\mathrm{A}} = X_{1A}D_1^{\mathrm{A}} + X_{2\mathrm{A}}D_2^{\mathrm{A}} + \ldots + X_{n\mathrm{A}}D_{\mathrm{n}}^{\mathrm{A}}$$

and

$$D_{\mathrm{L}} = X_{1L}D_1^{L} + X_{2L}D_2^{L} \ldots + X_{nL}D_n^{L}$$

where

$$X_{1\mathrm{j}} + X_{2\mathrm{j}} \ldots + X_{\mathrm{nj}} = 1 \text{ and } j = A, L$$

The X-values in the equation are the proportions of each asset or liability held in the respective asset and liability portfolios. Thus, if new 30-year Canada bonds were 1 percent of a life insurer's portfolio and D_1^{A} (the duration of those bonds) was equal to 9.25 years, then $X_{1\mathrm{A}}\, D^{\mathrm{A}}{}_1 = .01(9.25) = 0.0925$. More simply, the duration of a portfolio of assets or liabilities is a market value-weighted average of the individual durations of the assets or liabilities on the FI's balance sheet.[15]

Consider an FI's simplified market value balance sheet:

Assets ($)	*Liabilities ($)*
$A = 100$	$L = 90$
	$E = 10$
100	100

[15]This derivation of an FI's duration gap closely follows G. Kaufman, "Measuring and Managing Interest Rate Risk: A Primer," Federal Reserve Bank of Chicago, *Economic Perspectives,* 1984, pp. 16–29.

From the balance sheet identity:

$$A = L + E$$

$$\text{and } \Delta A = \Delta L + \Delta E,$$

$$\text{or } \Delta E = \Delta A - \Delta L$$

That is, when interest rates change, the change in the FI's net worth is equal to the difference between the change in the market values of assets and liabilities on each side of the balance sheet since $\Delta E = \Delta A - \Delta L$, we need to determine how ΔA and ΔL—the changes in the market values of assets and liabilities on the balance sheet—are related to duration.[16]

From the duration model:

$$\frac{\Delta A}{A} = -D_{\mathrm{A}}\frac{\Delta R}{(1 + \mathrm{R})} \tag{14}$$

$$\frac{\Delta L}{L} = -D_{\mathrm{L}}\frac{\Delta R}{(1 + \mathrm{R})} \tag{15}$$

Here we have simply substituted $\Delta A/A$ or $\Delta L/L$ (the proportional change in the market values of assets or liabilities) to replace $\Delta P/P$ (the change in any single bond's price), and D_{A} or D_{L} (the duration of the FI's asset or liability portfolio) to replace D (the duration on any given bond, deposit, or loan). The term $\Delta R/(1 + R)$ reflects the shock to interest rates as before. These equations can be rewritten as:

$$\Delta A = -D_{\mathrm{A}} \cdot A \cdot \frac{\Delta R}{(1 + \mathrm{R})} \tag{16}$$

and

$$\Delta L = -\mathrm{D}_{\mathrm{L}} \cdot L \cdot \frac{\Delta R}{(1 + \mathrm{R})} \tag{17}$$

Since $\Delta E = \Delta A - \Delta L$, we can substitute these two expressions into this equation:

$$\Delta E = \left[-D_{\mathrm{A}} \cdot \mathrm{A} \cdot \frac{\Delta R}{(1 + \mathrm{R})}\right] - \left[-D_{\mathrm{L}} \cdot L \cdot \frac{\Delta R}{(1 + \mathrm{R})}\right] \tag{18}$$

Assuming that the level of rates and expected shock to interest rates are the same for both assets and liabilities:[17]

$$\Delta E = [-D_{\mathrm{A}}A + D_{\mathrm{L}}L]\,\frac{\Delta R}{(1 + \mathrm{R})} \tag{19}$$

or:

$$\Delta E = -[D_{\mathrm{A}}A - D_{\mathrm{L}}L]\,\frac{\Delta R}{(1 + \mathrm{R})} \tag{20}$$

[16]In what follows, we use the Δ (change) notation instead of *d* (derivative) notation to recognize that interest rate changes tend to be discrete rather than infinitesimally small. For example, in real-world financial markets, the smallest observed rate change is usually one basis point (1/100th of 1 percent).

[17]This assumption that the level of interest rates *R* is the same for assets and liabilities is standard in "Macaulay" duration analysis. While restrictive, this assumption can be relaxed—though the duration measure changes, as discussed later in this chapter.

To rearrange the equation in a slightly more intuitive fashion, we multiply and divide both the terms $D_A A$ and $D_L L$ by A (assets):

$$\Delta E = -\left[D_A \frac{A}{A} - D_L \frac{L}{A}\right] \cdot A \cdot \frac{\Delta R}{(1 + R)} \tag{21}$$

$$\Delta E = -[D_A - D_L k] \cdot A \cdot \frac{\Delta R}{(1 + R)} \tag{22}$$

where $k = L/A$ is a measure of the FI's leverage; that is, the amount of borrowed funds or liabilities, rather than owners' equity used to fund its asset portfolio. The effect of interest rate changes on the market value of an FI's equity or net worth (ΔE) breaks down into three effects:

1. *The leverage-adjusted duration gap* $= [D_A - D_L k]$. This gap is measured in years and reflects the degree of duration mismatch in an FI's balance sheet. Specifically, the larger this gap *in absolute terms,* the more exposed the FI is to interest rate shocks.
2. *The size of the FI.* The term A measures the size of the FI's assets. The larger the scale of the FI, the larger the dollar size of the potential net worth exposure from any given interest rate shock.
3. *The size of the interest rate shock* $= \Delta R/(1 + R)$. The larger the shock, the greater the FI's exposure.

Given this, we express the exposure of the net worth of the FI as $\Delta E = -$[Adjusted duration gap] × Asset size × Interest rate shock.

While interest rate shocks are largely external to the bank and may result from changes in monetary policy (as discussed in the first section of Chapter 10), the size of the duration gap and the size of the FI are under the control of management.

Using an example, the next section explains how a manager can use information on an FI's duration gap to restructure the balance sheet to immunize stockholders against interest rate risk.

Duration Gap Measurement and Exposure: An Example. Suppose the FI manager calculates that:

$$D_A = 5 \text{ years}$$

$$D_L = 3 \text{ years}$$

Then the manager learns from an economic forecasting unit that rates are expected to rise from 10 to 11 percent in the immediate future. That is,

$$\Delta R = 1\% = 0.01$$

$$1 + R = 1.10$$

The FI's initial balance sheet is assumed to be:

Assets ($ billions)	*Liabilities ($ billions)*
A = 100	L = 90
	E = 10
100	100

The FI's manager would calculate the potential loss to equity holders' net worth (E), if the forecast of rising rates proves true, as follows:

$$\Delta E = -(D_A - kD_L) \cdot A \cdot \frac{\Delta R}{(1 + R)}$$

$$= -(5 - (0.9)(3)) \times \$100 \text{ b.} \times \frac{0.01}{1.1} = - \$2.09 \text{ billion}$$

The FI could lose \$2.09 billion in net worth if rates rose by 1 percent. since the FI started with \$10 billion, the \$2.09 billion loss is almost 21 percent of its initial net worth. The market value balance sheet after the 1-percent rise in rates would look like this:

Assets (\$ billions)	*Liabilities (\$ billions)*
A = 95.45	L = 87.54
	E = 7.91
95.45	95.45

Even though the rise in interest rates would not push the FI into economic insolvency, where the market value of liabilities exceeds the market value of assets,[18] it reduces the FI's net worth/assets ratio from 10 (10/100) to 8.29 percent (7.91/95.45). To counter this effect, the manager might reduce the FI's adjusted duration gap. In an extreme case, the gap might be reduced to zero:

$$\Delta E = - [0] \times A \times \frac{\Delta R}{(1 + R)} = 0$$

To do this, the FI should not directly set $D_A = D_L$. That would ignore the facts that the FI's assets (A) do not equal its borrowed liabilities (L) and that k is not equal to 1. To see the importance of factoring in leverage, suppose the manager increased the duration of the FI's liabilities to five years, the same as D_A, then:

$$\Delta E = -[5 - (0.9)(5)] \times 100 \text{ b.} \times (0.01/1.1) = -\$0.45 \text{ billion}$$

The FI would still be exposed to a loss of \$0.45 billion if rates rose by 1 percent. An appropriate strategy would involve changing D_L until:

$$D_A = D_L k = 5 \text{ years}$$

$$\Delta E = -[5 - (0.9)5.55] \times \$100 \text{ b.} \times (0.01/1.1) = 0$$

The appropriate strategy would be for the FI manager to set $D_L - 5.55$ years (slightly longer than $D_A = 5$ years) to compensate for the fact that only 90 percent of assets are funded by borrowed liabilities, with the other 10 percent funded by equity. Note that the FI manager has at least three other ways to reduce the adjusted duration gap to zero:

1. *Reduce D_A*. Reduce D_A from 5 years to 2.7 years (equal to $D_L k$ or 3(0.9)) so that:

$$[D_A - kD_L] = [2.7 - (0.9)(3)] = 0$$

[18]Here we are talking about economic insolvency. The legal and regulatory definition may vary depending on what accounting rules are used. In particular, under certain conditions an FI may be closed by OSFI when the book value of its net worth is still positive. However, the true (market) value of net worth may well be negative at that time.

2. *Reduce D_A and increase D_L.* Shorten the duration of assets and lengthen the duration of liabilities at the same time. One possibility would be to *reduce* D_A to 4 years and *increase* D_L to 4.44 years such that:

$$[D_A - kD_L] = [4 - (0.9)(4.44)] = 0$$

3. *Change k and D_L.* Increase k (leverage) from 0.9 to 0.95 and increase D_L from 3 years to 5.26 years, so that:

$$[D_A - kD_L] = [5 - (0.95)(5.26)] = 0$$

Concept Question

1. Suppose $D_A = 3$ years, $D_L = 6$ years, $k = 0.8$, and $A = \$100$ billion. What is the effect on owners' net worth if $\Delta R/(1 + R)$ rises by 1 percent?

Applying the Duration Model to Real-world FI Balance Sheets

Critics of the duration model have often claimed that it is difficult to apply in real-world situations. However, as we show in the following sections, duration measures and immunization strategies are useful in most real-world situations. In fact, the duration model, together with gap analysis and simulation models (to be discussed later in this chapter and again in Chapter 11), are *the* three techniques for measuring interest rate exposure recommended by CDIC. In CDIC's words,

> ". . . Each technique provides a different perspective on interest rate risk, has distinct strengths and weaknesses, and is more effective when used in combination."[19]

We next look at some of the weaknesses of duration technique and discuss ways the FI manager can deal with them in practice. These problems include the dynamic nature of duration, the fact that duration is a linear approximation of a convex function, the parallel yield curve shift assumption of duration, imbedded options and default risk.

The Cost of Duration Matching

Critics charge that although in principle an FI manager can change D_A and D_L to better immunize the FI against interest rate risk, restructuring the balance sheet of a large, complex FI can be both time-consuming and costly. While this argument may have been true historically, the growth of purchased funds, asset securitization, and loan sales markets have considerably eased the speed and lowered the transaction costs of major balance sheet restructurings. (See Chapters 8 and 23 for a discussion of these strategies.) Moreover, an FI manager could still manage risk exposure using the duration model by employing techniques other than direct portfolio rebalancing to immunize against interest rate risk. Managers can get many of the same results of direct duration matching by taking positions in the markets for derivative securities, such as forward contracts and swaps (Chapter 20), futures (Chapter 21), and options, caps, floors, and collars (Chapter 20).[20]

[19] CDIC, Standards of Sound Business and Financial Practices: Interest Rate Rise Management, pp. 5–6.

[20] In particular, instead of directly immunizing a positive duration gap ($D_A > D_L$), an FI manager could sell futures or forwards, take the fixed-rate side of an interest rate swap, buy put options, and/or buy an interest rate cap.

The Dynamic Problem of Immunization

Immunization is an aspect of the duration model that is not well understood. Let's go back to the immunization example where an insurer sought to buy bonds providing an accumulated cash flow of $1,469 in five years no matter what happened to interest rates. We showed that buying a six-year maturity, 8-percent coupon international bond with a five-year duration would immunize the insurer against an instantaneous change in interest rates. The word *instantaneous* is very important here. This means a change in interest rates immediately after the bond is purchased. however, interest rates can change at any time over the holding period. Further, the duration of a bond changes as time passes; that is, as it approaches maturity or the target horizon date. Not only that, but duration changes at a different rate from real (calendar) time.

To see this time effect, consider the initially hedged position where the insurer bought the five-year duration (six-year maturity), 8-percent coupon international bond in 2001 to match its cash flow target of $1,469 in 2006. Suppose the FI manager doesn't think about it for a year, believing the insurance company's position to be fully hedged. After one year has passed, the manager, knowing that the target date is now only four years away, recalculates the duration of the bond. Imagine the manager's shock upon finding that the same 8-percent coupon bond with an 8-percent yield and only four years left to maturity has a duration of 4.31 years. This means the insurance company is no longer hedged; the 4.31-year duration of this bond portfolio *exceeds* the investment horizon of four years. As a result, the manager has to restructure the bond portfolio to remain immunized. One way to do this would be to sell some of the five-year bonds (4.31-year duration) and buy some bonds of shorter duration so that the overall duration of the investment portfolio is four years.

For example, suppose the insurer sold 50 percent of the five-year bond with a 4.31-year duration and invested the proceeds in zero-coupon bonds with a remaining maturity and duration of 3.69 years. Because duration and maturity are the same for zero-coupon bonds, the duration of the asset portfolio would be:

$$D_A = [4.31 \times 0.5] + [3.69 \times 0.5] = 4 \text{ years}$$

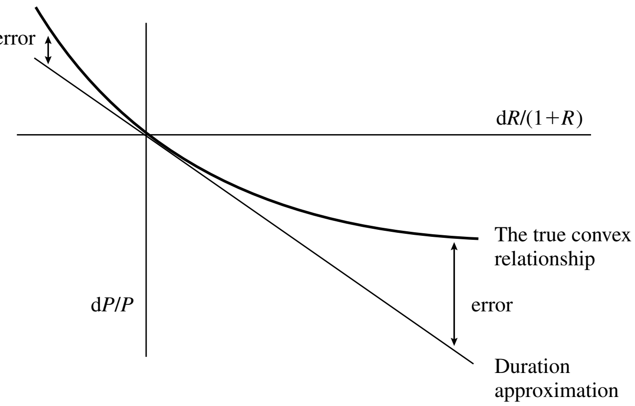
This simple example demonstrates that immunization based on duration is a dynamic strategy. In theory, it requires the portfolio manager to continuously rebalance the portfolio to make sure the duration of the investment portfolio exactly matches the investment horizon (i.e., the duration of liabilities). Because continuous rebalancing may not be easy to do and involves costly transaction fees, most portfolio managers only seek to be approximately dynamically immunized against interest rate changes by rebalancing at discrete intervals, such as quarterly or just after a relatively important change in interest rates. That is, there is a trade-off between perfect immunization and the transaction costs of maintaining an immunized balance sheet dynamically.

Large Interest Rate Changes and Convexity

Duration accurately measures the price sensitivity of fixed-income securities for small changes in interest rates of the order of one basis point. But suppose that interest rte shocks are much larger, of the order of 2 percent or 200 basis points? Then duration becomes a less accurate predictor of how much the prices of securities will change and, therefore, a less accurate measure of interest rate sensitivity. Looking at Figure 10–5, you can see the reason for this. Note first, the change in a bond's price due to yield changes according to the duration model and second, the true relationship, as calculated directly, using the exact present value calculation for bond valuation.

FIGURE 10–5

Duration versus true relationship between price changes and yield changes on a bond



The duration model predicts that the relationship between rate shocks and bond price changes will be proportional to *D* (duration. However, by precisely calculating the true change in bond prices, we find that for large interest rate increases, duration *overpredicts* the *fall* in bond prices. For large interest rate decreases, it *underpredicts* the *increase* in bond prices. That is, the duration model predicts symmetric effects for rate increases and decreases on bond prices. As Figure 10–5 shows, in actuality the *capital loss effect* for rate increases tends to be smaller than the *capital gain effect* for rate decreases. This is the result of the bond price/yield relationship exhibiting a property called **convexity** rather than *linearity,* as assumed by the basic duration model.

Convexity
The degree of curvature of the price/yield curve around some interest rate level.

Note that convexity is a desirable feature for an FI manager to capture in a portfolio of assets (but is correspondingly *undesirable* in a portfolio of liabilities). Buying bond or a portfolio of assets exhibiting a lot of convexity, or bentness, in the price/yield curve relationship is similar to buying partial interest rate risk insurance. As asset managers say, "The benter the better." Specifically, high convexity means that for equally large changes of interest rates up and down (e.g., plus or minus 2 percent), the capital gain effect of a rate decrease more than offsets the capital loss effect of a rate increase. All fixed-income assets or liabilities exhibit some convexity in their price/yield relationships.[21] We give a fuller treatment of the measurement and interpretation of convexity in Appendix 10–1.

Key assumptions of the duration model are that the yield curve or the term structure of interest rates is flat and that when rates change, the yield curve shifts in a parallel fashion. We show this in Figure 10–6.

As we discussed in Chapter 9, the yield curve can take many shapes and at best may only approximate a flat yield curve. If the yield curve is not flat, using simple duration could be a potential source of error in predicting asset and liability interest rate sensitivities. Many models can deal with this problem. These models differ according to the shapes and shocks to the yield curve that they assume.

Suppose the yield curve is not flat but shifts in such a manner that the yields on different maturity or discount bonds change in a proportional fashion.[22] Consider calculating the

[21]To be more precise, this is true of all fixed-income securities without special option features (such as callable bonds or mortgage-backed securities). A callable bond tends to exhibit negative convexity (or concavity), as do some mortgage-backed securities because of prepayments.

[22]We are interested in the yield curve on discount bonds because these yields reflect the time value of money for single payments at different maturity dates. Thus, we can use these yields as discount rates for cash flows on a security to calculate appropriate present values of its cash flows and its duration.

FIGURE 10–6

Yield curve underlying Macaulay duration

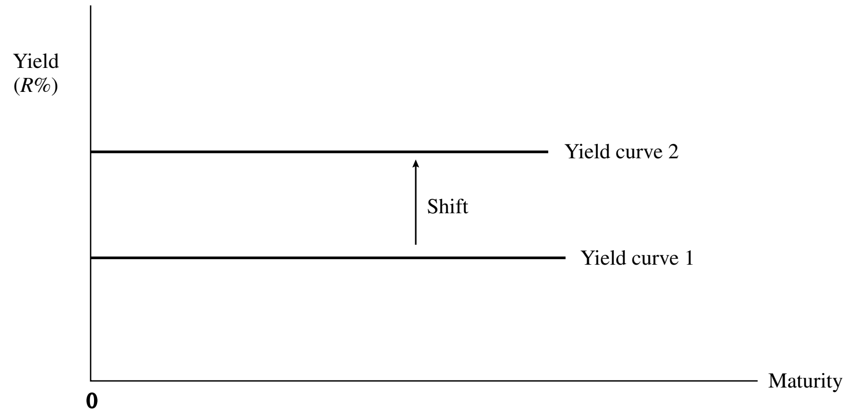
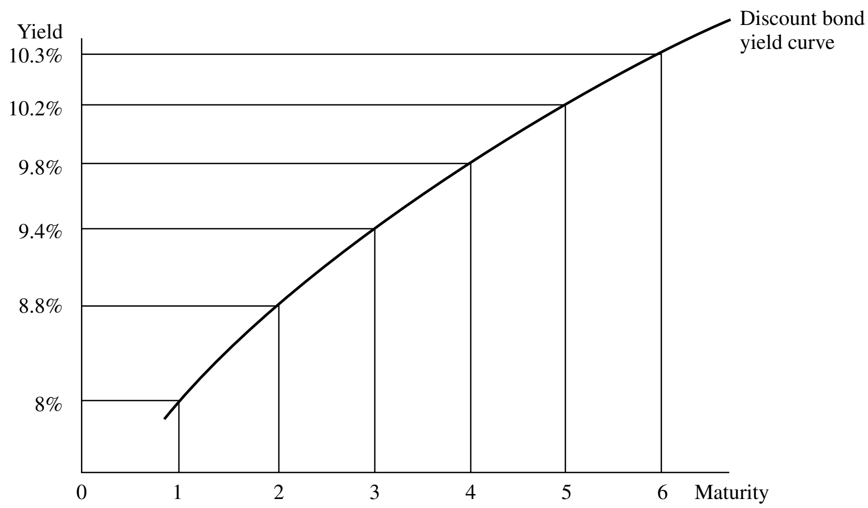


FIGURE 10–7

Nonflat yield curve



duration of the six-year international bond when the yield curve is not flat at 8 percent. Instead, the yield curve looks like that in Figure 10–7.

Suppose that the yield on one-year discount bonds rises. Assume also that the discounted changes in longer-maturity discount bond yields are just proportional to the change in the one-year discount bond yield:

$$\frac{\Delta R_1}{1 + R_1} = \frac{\Delta R_2}{1 + R_2} = \ldots = \frac{\Delta R_6}{1 + R_6}$$

This assumption is called the "parallel shift in the yield assumption" because it requires that an interest rate shock affect each point along the yield curve to the same degree. Given this quite restrictive assumption, we can prove that we can derive the appropriate duration measure of the bond—call it D^*—by discounting the coupons and principal value of

TABLE 10–8 Duration with an Upward-Sloping Yield Curve

t (years)	*CF*	*DF*	*CF* × *DF*	*CF* × *DF* × *t*
1	\$ 80	$\frac{1}{(1.08)} = 0.9259$	74.07	74.07
2	80	$\frac{1}{(1.088)^2} = 0.8448$	67.58	135.16
3	80	$\frac{1}{(1.094)^3} = 0.7637$	61.10	183.3
4	80	$\frac{1}{(1.098)^4} = 0.6880$	55.04	220.16
5	80	$\frac{1}{(1.102)^5} = 0.6153$	49.22	246.1
6	\$1,080	$\frac{1}{(1.103)^6} = 0.5553$	599.75	3,598.50
			906.76	4,457.29

$$D^* = \frac{4,457.29}{906.76} = 4.91562$$

the bond by the discount rates or yields on appropriate maturity zero-coupon bonds. Given the discount bond yield curve plotted in Figure 10–7, D^* is calculated in Table 10–8.[23]

Notice that D^* is 4.92 years, while Macaulay duration (with an assumed flat 8 percent yield curve) is 4.99 years. D^* and D differ because when the upward-sloping yield curve in Figure10–7 is taken into account, the later cash flows are discounted at higher rates than the flat yield curve assumption underlying Macaulay's measure D.

Choosing to use D^* instead of D does not change the FI manager's basic problem except for a concern with the gap between D^* on assets and leverage-weighted liabilities. That is,

$$D^*_A - kD^*_L$$

However, remember that D^* was calculated under very restrictive assumptions about the yield curve. If we change these assumptions in any way, the measure of D^* changes.[24]

To understand the restrictiveness of the "parallel shifts in the yield curve" assumption, look at Figure 10–8. We are assume that the interest rate shock strikes the yield curve identically at all points. In calculating ΔE, we multiply the whole portfolio's duration by a

[23]The measure D^* is often called Fisher-Weil duration. See Lawrence Fisher and Roman Weil, "Coping with the Risk of Interest Rate Fluctuations: Returns of Bondholders from Naïve and Optimal Strategies," *Journal of Business* 44 (1971), pp. 408–31. For more details, see Hawawini, "Controlling the Interest Rate Risk of Bonds"; and G. O. Bierwag, G. G. Kaufman, and A. Toevs, "Duration: Its Development and Use in Bond Portfolio Management," *Financial Analysts Journal* 39, 1983, pp.15–35. And James C. Van Horne *Financial Market Rates and Flows,* Prentice Hall, New Jersey, 1998.

[24]A number of authors have identified other nonstandard measures of duration for more complex yield curve shapes and shifts. See, for example, Bierwag et al., "Duration: Its Development."

FIGURE 10–8A
Parallel Shift in an Upward Sloping Yield Curve

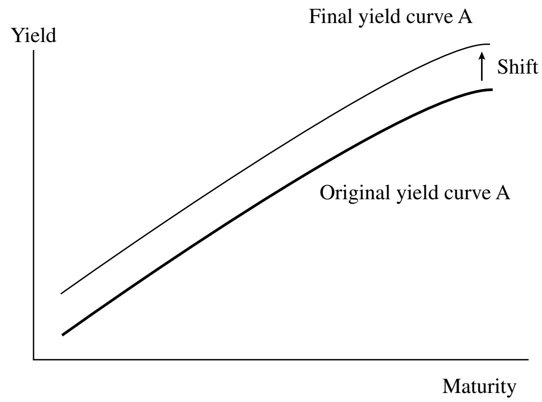
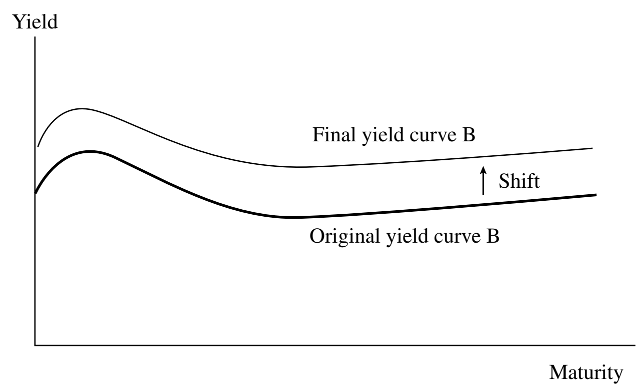


FIGURE 10–8B
Parallel Shift in a Downward Sloping Humped Yield Curve



single shock, the term $\Delta R/(1 + R)$.[25] Remember that the duration of a portfolio is itself just a weighted average of sensitivities to interest rate shocks of the present values of each cash flow. Those cash flows are spread across the yield curve.

Assume that the original yield curve is as pictured in Figure 10–8a. The D^* approximation of interest rate risk will tell you the effect on the portfolio value if the yield curve sifts to the final yield curve in 10–8a. If the original yield curve were shaped as in Figure 10–8b, the D^* approximation would tell you the effect on the portfolio value if you shifted from original yield curve B to the final yield curve B. But if the yield curve shifted from original yield curve A to final yield curve B, the duration measure of interest rate risk could give a highly erroneous measure of risk. If, for example, the FI manager were using shorter maturity assets to offset longer maturity liabilities, he would find that the value of assets falls with the interest rate rise in the short maturities while the value of liability values rise, because of the fall in long-term rates. Duration would not warn the manager of this risk from change in the shape of the yield curve.

[25]In fact, the required shift in the yield curve is not strictly parallel: $\Delta R_t/(1 + R_t)$ is being held constant for all values of t. To be a parallel shift, ΔR_t would have to be equal for all values of t. The shift is nearly parallel, however, and it is a useful phrase that emphasizes the restrictiveness of the assumption.

TABLE 10–9 Duration and Rescheduling

t (years)	*CF*	*DF*	*CF* × *DF*	$t \times CF_t \times DF$
1	$ 0.00	0.93	0.00	0.00
2	166.40	0.86	142.66	285.32
3	80.00	0.79	63.51	190.52
4	80.00	0.74	58.80	235.21
5	80.00	0.68	54.45	272.23
6	$1,080.00	0.63	680.58	4,083.50
			1,000.00	5,066.78

$$\text{Thus, } D = \frac{5{,}066.78}{1{,}000.00} = 5{,}0668$$

Deferral and Default Risk

The models and the duration calculations we have looked at assume that the issuer of bonds or the borrower of a loan pays the promised interest and principal on time and with a probability of one. That is, we assume no delay or default in the payment of cash flows. Thanks to credit risk, however, problems with principal and interest payments are common and lead to restructuring and workouts on debt contracts as bankers and bond trustees renegotiate with borrowers. Most borrowers first attempt to reschedule or recontract interest and principal payments; they default outright only as a last resort.

If we view default risk as synonymous with the rescheduling of cash flows to a later date, duration analysis can easily handle the change. Consider the six-year, 8-percent coupon, 8-percent yield international bond. Suppose the issuer gets into difficulty and cannot pay the first coupon. Instead, the borrower and the FI agree that the unpaid interest can be paid in year 2, with interest accruing on the unpaid interest at 8 percent.[26] This alleviates part of the cash flow pressure on the borrower while lengthening the duration of the bond from the bondholder's perspective (see Table 10–9). The effect of rescheduling the first interest payment is to increase duration from approximately five years to 5.07 years. Deferral risk actually contains no credit risk. As shown in Table 10–8, bondholders received the full value of the deferred bond. You can confirm this by noting that the present value of the Eurobond (in the $CF \times DF$ total) is still $1,000.

With default (rather than deferral) risk, an FI manager unsure of future cash flows might multiply the promised cash flow (CF_t) by the probability of repayment (p_t) in year t to generate $E(CF_t)$, the expected cash flows in year t.[27]

$$E(CF_t) = p_t \times CF_t$$

Chapter 13 suggests a number of ways to generate these repayment probabilities. Once the cash flows have been adjusted for default risk, a duration measure can be directly calculated in the same manner as the Macaulay formula (or D^*) except that $E(CF_t)$ replaces CF_t.

In the default risk model, as you adjust payments to be received by a probability less than one, you explicitly forecast that the issuer will never make payment on a specified cash flow with probability $(1 - p)$. The present value of the defaulted bond or loan will

[26]In fact, after a payment default, interest would likely accrue at a higher default rate of interest. We neglect this to simplify the example.

[27]The probability of repayment is between 0 and 1.

always be less than its face value. Default risk exists in all securities whose issuer is not the monetary authority of the country in whose currency the security is denominated, but it is particularly serious in high-risk loans, **speculative grade bonds,** and poor-quality sovereign issuers of nonhome currency debt.

Speculative Grade Bond
A high-yield bond that is rated Ba or lower by Moody's, BB or lower by Standard and Poor's, or is unrated by rating agencies.

The nature of payments following default has a direct impact on whether default risk lengthens or shortens duration. A reduction of the later payments of a security typically shortens duration, while a reduction in early payments with debt forgiveness (i.e., interest on the deferred coupons is never paid, so the net present value of the stream is less than the original bond's value) extends duration.[28]

Embedded Options and Contingent Claims

Calculating the durations of mortgages and mortgage-backed securities is difficult because of prepayment risk. Essentially, as the interest rate falls, mortgage holders may have the option to prepay their old mortgages and refinance with a new mortgage at lower interest rates. In the terminology of finance, fixed-rate mortgages and mortgage-backed securities contain an *embedded option.* To calculate duration, one needs to project the future cash flows on an asset. So to calculate the duration of mortgages, one needs to model the prepayment behaviour of mortgage holders. Possible ways to do this are left to Chapter 23.

When interest rates change, so do the value of off-balance-sheet derivative instruments, such as interest rate futures, options, swaps, FRAs, and caps (see Chapter 14). Market value gains and losses on these instruments also affect the net worth of the FI. The calculation of the durations of these instruments is left to Chapters 20 through 22. However, it should be noted that a fully fledged duration analysis of an FI—like the full gap analyzes we considered in Chapter 9—will take into account the durations of its derivatives portfolio as well as the durations of its on-balance-sheet assets and liabilities. This is especially so today, as FIs take greater positions in derivatives contracts.

Alternatives to Duration

When we introduced duration, we did so as an alternative to gap analysis. Notwithstanding the many shortcomings of duration, it still achieves that purpose. Gap analysis, although simple to implement and intuitive, suffers from lack of theoretical rigor and from presenting cumbersome tableaus of numbers. Duration is attractive because it sums up the whole portfolio's interest rte sensitivity in a theoretically consistent elasticity. Moreover, the theory is simple and its implementation is relatively straightforward. The duration gap can be multiplied by the expected interest rate percentage shock and total portfolio size to obtain the change in market value of the FI's portfolio. Duration analysis is a **parsimonious model** of interest rate risk.

Parsimonious Model
A model that distills a complex phenomenon into relatively few critical components.

Simulation analysis on the other hand, forgoes simplicity of calculation and instead explicitly values the entire asset and liability portfolio of the FI, security by security, as a function of various factors—an important one of which is interest rates. The factors' magnitudes of change are measured and statistically analyzed so that the analyst can determine

[28]G. O. Bierwag, G. G. Kaufman, and A. Toevs, "Durations of Non-Default Free Securities," *Financial Analysts Journal,* July/August 1988, pp. 36–46, show that under reasonable assumptions and a flat yield curve, long-maturity bonds with significant default risk can easily have durations 30 percent different from durations calculated ignoring default risk. For a discussion of a general model incorporating the term structure of interest rates as well as default risk, see Fooladi, Roberts, and Skinner, *Duration for Corporate and Provincial Bonds with Default Risk,* Administrative Sciences Association of Canada 14:1, Finance, 1993, pp. 1–12.

the likelihood of the FI undergoing various shocks. From these calculations, a single number of encapsulating the risk of the FI with respect to these factors is derived.

Simulation analysis often involves many millions of calculations to determine a large FI's risk. Such calculations, unheard of prior to the advent of computers, are today trivially simple, *once a model has been programmed and the data collected.* The benefit of simulation models is that they permit the analyst to investigate specific responses of the FI's portfolio to complex as well as simple changes in the future interest rate environment. Remember that one serious drawback of duration analysis was its inability to cope with nonparallel shifts in the yield curve. Moreover, if programmed properly, a simulation model allows the analysis of interacting market price risks—of which interest rate risk is often the most important for FIs.

A major difficulty of simulation models concerns complexity. To all but the analysts who program the simulation model, it is a black box into which assumptions are put for what-if analysis. That box will give erroneous conclusions if its internal pricing models do not reflect reality. Even if the black box is theoretically correct, its results are only as useful as the assumptions put into it. For example, even if an analyst in the late 1970s had a simulation model at his disposal, he would not have anticipated the unprecedented rise in interest rates at the beginning of the 1980s (see Chapter 9). Since history is only an imperfect guide to the future, the confidence intervals projected by simulation analysis should be used with caution.

We will return to simulation models in our discussion of market risk in the next chapter.

Summary

- Calculate duration by discounting each term in the average life calculation by the discount rate applicable to the timing of the cash flow.
- Duration is the negative of the interest rate elasticity of present value.
- Duration increases with maturity, decreases with yield, and decreases with coupon interest.
- Matching average life of assets and liabilities considerably reduces interest rate risk but is not a perfect hedge.
- Duration can be applied to a single asset or liability, a portfolio, or an entire balance sheet.
  - To immunize the equity, set the leverage adjusted duration gap to zero.
  - Equity and the capital ratio cannot be immunized simultaneously.
- Immunization requires dynamic rebalancing of the portfolio, which may be costly.
- A duration-immunized asset portfolio will appreciate more as interest rates rise than the duration model predicts because of convexity.
- Asset portfolios can be profitably structured to increase convexity.
- Duration suffers from several drawbacks.
  - If assumes a flat or parallel shift term structure.
  - Default risk is difficult to incorporate.
- Simulation models can be used effectively in conjunction with gap analysis and duration analysis to measure the interest rate risk of an FI.

Questions

1. Why would an FI wish to eliminate interest rate risk while maintaining liquidity risk? How could it do so?
2. Define duration in three separate ways, making reference to average life, elasticities, and zero-coupon bonds.
3. Why would the duration of a consol differ from that of an FRN if the two are both perpetuities?
4. Assume that an FI immunized an asset portfolio of consols by issuing zero-coupon deposits. For how long does the immunization last? Suggest a better immunization strategy.
5. Using the duration model, describe the three factors that determine the size of the change in equity from an interest rate shock.
6. What is the effect of default risk on duration?
7. Compare and contrast the advantages of gap analysis, the duration model, and simulation models in measuring interest rate risk. Why does CDIC recommend using a combination of measuring techniques?

Problems

1. Consider the following FI balance sheet

M. Match, Inc.
($ millions)

Assets		*Liabilities*	
Two-year bullet repayment loans	$175	One-year paper	$135
10-year provincial international bonds	165	Five-year note	160

Notes: All instruments are valued at par (equal to book value). The two-year loans yield 5 percent; the 10-year provincial bonds yield 9 percent; the one-year paper issue pays 4.5 percent; and the five-year notes pay 8 percent. Assume that all instruments have annual coupon payments. A bullet repayment loan is one where all principal is repaid in a single payment at loan maturity.

   a. What is the value of M. Match, Inc.'s equity?
   b. What are the average life and the duration of the FI's assets?
   c. What are the average life and the duration of the FI's liabilities?
   d. What is the difference in average lives of assets and liabilities? What is the duration gap?
   e. What does your answer to Part *d* imply about the interest rate risk exposure of M. Match, Inc.?
   f. Calculate the values of all four securities on M. Match's balance sheet if all interest rates increase by 2 percent.
   g. Using the duration formula, determine the impact on the equity of M. Match. Calculate the percentage change in the value of equity.

2. What is the price of a newly auctioned five-year Canada bond with a coupon of 7 percent and a yield of 7.05 percent? (Hint: All Canada bonds pay interest semiannually.)

3. *a.* What are all of the promised cash flows on a $1,000 one-year loan yielding 10 percent per annum that pays quarterly interest on principal that is reduced by four equal quarterly payments of $250?
   b. What is the present value of the loan if market interest rates on similar risk loans are 10 percent p.a.?
   c. What is the present value of the loan if, right after it is granted, market interest rates on loans of similar risk drop to 8 percent p.a.?

4. Calculate the duration of the loan in Question 3 assuming yields of both 10 percent and 8 percent p.a.

5. Calculate the impact of a 75-basis-point increase in interest rates on the following securities' prices. Use duration, convexity*, and the exact solution:
   a. Four-year, 6 percent annual coupon note selling at par.
   b. Four-year, 6 percent annual coupon note selling at $94.976 per $100 face value.
   c. Six-year, 3 percent annual coupon bond selling at par.
   d. Six-year, 12 percent annual coupon bond selling at par.

6. Calculate durations of the following securities:
   a. Two-year, 6-percent quarterly coupon selling at par.
   b. Three-year, 12 percent annual coupon selling at $90 per $100 face value.
   c. Four-year, 8 percent annual coupon selling at par.

7. Calculate the duration of a two-year note with $100,000 par value and an annual coupon rate of 10 percent if today's yield to maturity is 11.5 percent. What would the duration be if today's yield were 5.5 percent? (Hint: Interest is to be paid annually.)

8. *a.* Use duration to calculate the approximate price change if interest rates increase by 10 basis points for the note in Question 7.
   b. Use the mechanics of bond valuation to calculate the exact price change if interest rates increase by 10 basis points for the note in Question 7.
   c. Why are your answers for parts *a* and *b* different?

9. How would the incorporation of convexity* change your answer to Question 8*a*?

10. *a.* What is the duration of a consol bond with a required yield to maturity of 7 percent p.a.? 3 percent p.a.? 12 percent p.a.?
    b. What can you conclude from you answers to Part *a*?

11. *a.* Calculate the duration of a five-year Canada bond with a 10 percent semiannual coupon selling at par.
    b. What is the duration of the Canada bond if the yield to maturity is 14 percent paid annually?
    c. What is the duration of the bond if the yield to maturity is 16 percent paid annually?
    d. What can you conclude about the relationship between duration and yield to maturity?

12. *a.* What is the duration of a two-year Canada bond with a 10 percent semiannual coupon selling at par?
    b. What is the duration of an 11-year Canada bond with a 10 percent semiannual coupon selling at par?
    c. Use your answers to Questions 12*a* and 12*b* and Question 11 to draw conclusions about the relationship between duration and time to maturity.

13. *a.* Calculate the modified duration of the bonds in Question 12.
    b. If all interest rates increase by 10 basis points, what is the impact on the price of the two-year and 11-year Canada bonds? (That is, $\Delta R/(1 + R) = 0.001$.)

14. *a.* Calculate the semiannual payment on a $100,000, five-year maturity, 10 percent yield fully amortized loan. (A

*See discussion of convexity in Appendix 10A.

fully amortized security has no principal payment at maturity. Each coupon payment contains both an interest and a principal payment.)

b. What is the duration of the loan in Part *a*?

c. Compare the duration of the amortizing loan with the duration of the five-year Canada bond in Question 11.

15. *a.* Using the duration approximation, what is the impact of a 200-basis-point increase in annual interest rates on the price of the Canada bond in Table 10–2? Contrast your answer with the exact price using bond valuation.

b. Calculate the convexity* of the bond in Table 8–10. What is your estimate of the bond's price after the 200-basis-point increase in yields after you adjust for convexity?

c. Recompute your answers to Questions 15*a* and 15*b* assuming a 200-basis-point decrease in yields.

16. Calculate the duration gaps of FIs with the following asset and liability portfolios:

a. $250 million in assets with duration of 4.5 years. $500 million in assets with duration of 11 years. $350 million of liabilities with duration of 0.75 year; $300 million of liabilities with duration of three years.

b. $50 million in assets with duration of 0.5 year. $200 million in assets with duration of three years. $150 million of liabilities with duration of 0.75 year; $50 million of liabilities with duration of 1.5 years.

c. What is the interest rate risk exposure of the FIs in Parts *a* and *b*?

17. *a.* Calculate the duration gap of the following position: *Asset:* $1 million invested in 30-year, 10 percent semiannual coupon Canada bonds selling at par. *Liability:* $900,000 financing obtained from a two-year, 7.25 percent semiannual coupon note selling at par.

b. What is the impact on equity values if all interest rates fall 20 basis points? That is,

$$\frac{\Delta R}{1+\frac{R}{2}} = -0.002$$

18. Use the data provided for Gotbucks Bank, Inc., to answer Parts *a* through *e*.

Gotbucks Banks, Inc.
($ millions)

Assets		*Liabilities*	
Cash	$ 30	Demand and notice deposits	$ 20
Interbank deposits	20	Fixed-term retail deposits	50
Loans (floating)	105	CDs	130
Loans (fixed)	65	Equity	20
Total assets	$220	Total liabilities	$220

Notes: Currently, the interbank deposit rate is 8.5 percent. variable-rate loans are priced at 2 percent over LIBOR (currently at 11 percent). Fixed-rate loans are all five-year maturities with 12 percent interest, paid annually. Demand and notice deposits currently pay an average rate of interest of 6 percent p.a. Fixed-term deposits are priced at 8 percent, with annual interest payments and with a maturity of two years. CDs are priced over LIBOR with a final maturity of four years and currently pay 9 percent per annum.

a. What is the duration of Gotbucks Bank's (GBI) fixed-rate loan portfolio if the loans are priced at par?

b. If the average time to repricing of GBI's floating-rate loans and interbank deposits is 0.36 year, what is the duration of the bank's assets? (Note that the duration of cash is zero.)

c. What is the duration of GBI's core deposits if they are priced at par?

d. If the time to repricing of GBI's CDs is 0.40 year, what is the duration of the bank's liabilities?

e. What is GBI's duration gap? What is the bank's interest rate risk exposure? If all yields increase by 1 percent, what will be the impact on the market value of GBI's equity? (That is, $\Delta R/(1 + R) = 0.01$ for all assets and liabilities.)

19. An insurance company issued a $90 million, one-year note at 8 percent add-on annual interest (paying one coupon at the end of the year) and used the proceeds to fund a $100 million face value, two-year commercial loan at 10 percent annual interest. Immediately after these transactions were (simultaneously) undertaken, all interest rates went up 150 basis points.

a. What happened to the market value of the insurance company's loan investment? (Give a precise answer.)

b. What is the duration of the commercial loan investment when it is first issued?

c. Use the duration approximation to answer Part *a*.

d. Use the convexity adjustment* to correct your answer to Part *c*.

e. What happened to the market value of the insurance company's $90 million liability? (Give a precise answer.)

f. What is the duration of the insurance company's liability when it is first issued?

g. Use the duration approximation to answer Part *e*.

h. What is the net effect on the market value of the insurance company's equity? (Give a precise answer.)

i. How could you have used the insurance company's duration gap to estimate the answer to Part *h*?

20. Use this balance sheet information to answer the following questions:

---

*See discussion of convexity in Appendix 10A.

Balance Sheet
($ thousands)

	Duration (years)	Amount
T-bills	.5	$ 90
Discount notes	.9	55
Canada bonds	x	176
Loans	7	2,724
Deposits	1	2,092
Interbank funds borrowed	.01	238
Equity		715

Notes: Canada bonds are five-year maturities paying interest semiannually at 6 percent per annum (on the bond basis) and selling at par.

a. What is the duration of the Canada bond portfolio? (Calculate the value of x in the balance sheet.)

b. What is the duration of all the assets?

c. What is the duration of all the liabilities?

d. What is the FI's duration gap? What is the FI's interest rate risk exposure?

e. If the entire yield curve shifted upward approximately 50 basis points, what would be the impact on the FI's market value of equity? Use

$$\frac{\Delta R}{1 + R} = 0.005$$

f. If the entire yield curve shifted downward approximately 25 basis points, what would be the impact on the FI's market value of equity? Here, use

$$\frac{\Delta R}{1 + R} = -0.0025$$

Appendix 10–A Convexity

To see the importance of accounting for the effects of convexity in assessing the impact of large rate changes on an FI portfolio, consider the six-year international bond with an 8-percent coupon yield. According to Table 10–1, its duration is 4.99 years and its current price, P_0, will be \$1,000 at a yield of 8 percent:

$$P_0 = \frac{80}{(1.08)} + \frac{80}{(1.08)^2} + \frac{80}{(1.08)^3} + \frac{80}{(1.08)^4} + \frac{80}{(1.08)^5} + \frac{1{,}080}{(1.08)^6} = \$1{,}000$$

This is point *A* in the price/yield curve in Figure 10A–1.

If rates rise from 8 to 10 percent, the duration model predicts that the bond price will fall by 9.2457 percent. That is,

$$\frac{\Delta P}{P} = -4.99\left[\frac{.02}{1.08}\right] = -9.2457\%$$

or , from a price of \$1,000 to \$907.543 (see point *B* in Figure 10A–1). However, calculating the exact change in the bond's price after a rise in yield to 10 percent, we find:

$$P_1 = \frac{80}{(1.1)} + \frac{80}{(1.1)^2} + \frac{80}{(1.1)^3} + \frac{80}{(1.1)^4} + \frac{80}{(1.1)^5} + \frac{1{,}080}{(1.1)^6} = \$912.895$$

This is point *C* in Figure 10A–1. As you can see, the actual fall in price is less than the predicted fall by \$5,352. This means there is just over a 0.5 percent error using the duration model. The reason for this is the natural convexity to the price/yield curve as yields rise.

Reversing the experiment reveals the duration model would predict that the bond's price would rise by 9.2457 percent if yields fell from 8 to 6 percent, resulting in a predicted price of \$1,092.457. (See point *D* in Figure 10A–1.) By comparison, we can compute the true change in price as \$1,098.347 by estimating the present value of the bond's coupons and its face value with a 6 percent yield (see point *E* in Figure 10A–1). The duration model has underpredicted the bond price increase by \$5.89, or by over 0.5 percent of the true price increase.

An important question for the FI manager is whether a 0.5 percent error is big enough to be concerned about. This depends on the size of the interest rate change and the size of the portfolio under management. Clearly, 0.5 percent of a large number will still be a large number![29]

So far, we have established these three characteristics of convexity:

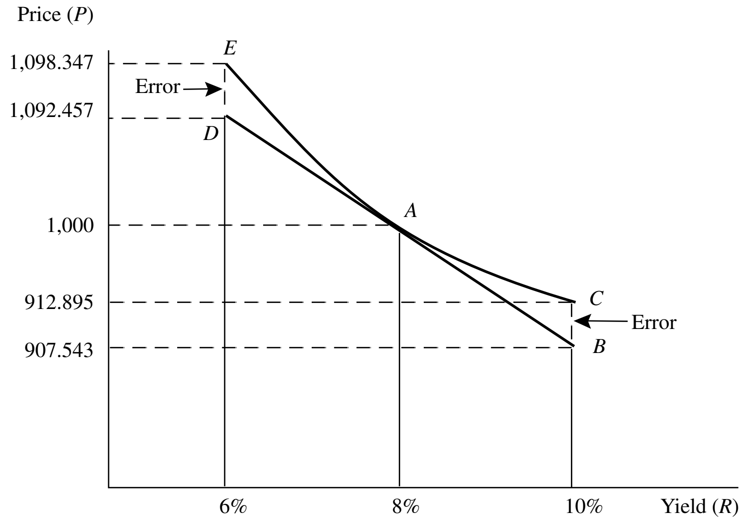
1. *Convexity in assets is desirable.* The greater the convexity of a security or a portfolio of securities, the more insurance or interest rate protection an FI manager has against rate increases, and the greater the potential gains on a portfolio of assets following interest rate falls.
2. *Convexity and duration.* The larger the interest rate changes and the more convex a fixed-income security or portfolio, the greater the error the FI manager faces in using just duration (and duration matching) to immunize exposure to interest rate shocks.
3. All fixed-income securities are convex.[30] To see this, we can take the six-year, 8-percent coupon, 8-percent yield international bond and look at two extreme price/yield scenarios. What is the price on the bon if yield falls to zero? What is its price if yield rises to some very large number, such as infinity?

[29]The effect of convexity can be seen in most U.S. treasury yield curves where the 30-year bond typically yields less than the 10-year bond. The 30-year has considerably more convexity than the 10-year, a feature for which asset managers are willing to give up yield. See Antti Ilmanen, "Convexity Bias and the Yield Curve: Understanding the yield curve part 5," Salomon Brothers Fixed Income Research, September 1995.

[30]This applies to fixed-income securities without special option features such as calls or puts.

FIGURE 10A–1

The price/yield curve for the six-year international bond



When $R = 0$:

$$P = \frac{80}{(1+0)} + \ldots + \frac{1{,}080}{(1+0)^6} = \$1{,}480$$

The price is just the simple undiscounted sum of the coupon values and the face value. Since yields can never go below zero, \$1,480 is the maximum possible price for the bond. When $R = \infty$:

$$P = \frac{80}{(1+\infty)} + \ldots + \frac{1{,}080}{(1+\infty)^n} \simeq \$1{,}480$$

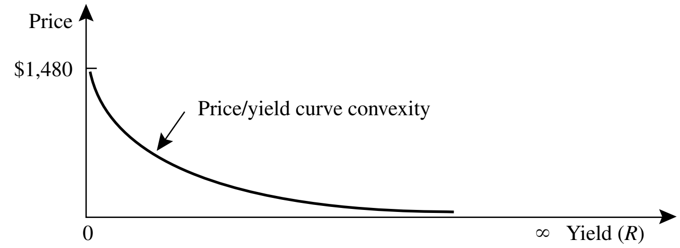
As the yield goes to infinity, the bond price falls asymptotically towards zero, but by definition a bond's price can never be negative. Thus, zero must be the minimum bond price (see Figure 10A–2).

Since convexity is a desirable feature for assets, the FI manager might ask: Can we measure convexity? And can we incorporate this measurement into the duration model to adjust for or offset the error in prediction due to its presence? The answer to both question is yes.

Theoretically speaking, duration is the slope of the price/yield curve, and convexity (or curvature) is the change in the slope of the price/yield curve. Consider the total effect of a change in interest rates on a bond's price as being broken into a number of separate effects. The precise mathematical derivation of these separate effects is based on a Taylor series expansion you probably remember from your math classes. Essentially, the first-order effect (dP/dR) of an interest rate change on the bond's price is the price/yield curve slop effect that is measured by duration. The second-order effect (d^2P/dR^2) measure the change in the slope of the price/yield curve; this is the curvature or convexity effect. There are also third, fourth, and higher-order effects from the Taylor series expansion, but for all practical purposes these can be ignored.

We have noted that overlooking the curvature of the price/yield curve may cause errors in predicting the interest sensitivity of our portfolio of assets and liabilities, especially when yields change by large amounts. We can adjust for this by explicitly recognizing the second-order effect of yield changes by measuring the change in the slope of the price/yield curve around a given point. Just as D (duration) measures the slop effect (dP/dR), we introduce a new parameter to measure the curvature effect (d^2P/dR^2) of the price/yield curve.

FIGURE 10A–2
The natural convexity of bonds



The resulting equation predicting the change in a security's price ($\Delta P/P$) is

$$\frac{\Delta P}{P} = -D\frac{\Delta R}{(1+R)} + \frac{1}{2}CX(\Delta R)^2 \tag{1}$$

or:

$$\frac{\Delta P}{P} = -MD\,\Delta R + \frac{1}{2}CX(\Delta R)^2 \tag{2}$$

Modified Duration
Macaulay duration divided by $1 + R$.

Basis Point (bp)
One one-hundredth of a percent.

The first term in equation (1) is the simple duration model that over- or underpredicts price changes for large changes in interest rates. The second term is the second-order effect of interest rate changes; that is, the convexity or curvature adjustment. In equation (2), D can be divided by $1 + R$ to produce what practitioners call **modified duration** (*MD*).[31] You can see this in equation (2). This form is more intuitive because we multiply *MD* by the simple change in R (ΔR) rather than the discounted change in R ($\Delta R/(1 + R)$). In the convexity term, the number ½ and $(\Delta R)^2$ result from the fact that the convexity effect is the second-order effect of interest rate changes, while duration is the first-order effect. The parameter *CX* reflects the degree of curvature in the price/yield curve at the current yield level; that is, the degree to which the capital gain effect exceeds the capital loss effect for an equal change in yields up or down. At best, the FI manager can only approximate the curvature effect by using a parametric measure of *CX*. Even though calculus is based on infinitesimally small changes, in financial markets the smallest change in yields normally observed is one **basis point** (bp), or a 1/100th of 1 percent change. One possible way to measure *CX* is introduced next.

So the convexity effect is the degree to which the capital gain effect more than offsets the capital loss effect for an equal increase and decrease in interest rates at the current interest rate level. In Figure 10A–3, we depict yields changing upward by one basis point (R + 0.01 percent) and downward by one basis point (R − 0.01 percent). Because convexity measures the curvature of the price/yield curve around the rate level R percent, it intuitively measures the degree to which the capital gain effect of a small yield decrease exceeds the capital loss effect of a small yield increase.[32] Definitionally, the *CX* parameter equals:

$$CX = \text{Scaling factor}\left[\begin{array}{l}\text{The capital loss}\\ \text{from a one-bp}\\ \text{rise in yield}\\ \text{(negative effect).}\end{array} + \begin{array}{l}\text{The capital}\\ \text{gain from a}\\ \text{one-bp}\\ \text{fall in yield}\\ \text{(positive effect).}\end{array}\right]$$

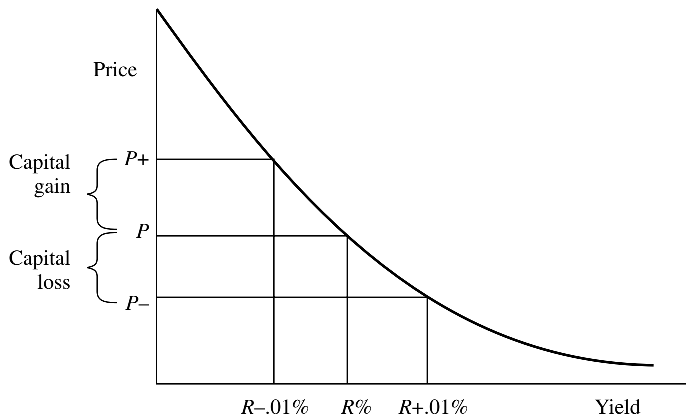
The sum of the two terms in brackets reflects the degree to which the capital gain effect exceeds the capital loss effect for a small (one-basis-point) interest rate change down and up. The scaling

[31]For bonds paying interest semiannually, modified duration is Macaulay duration divided by $1 + R/2$.

[32]We are trying to approximate, as best we can, the change in the slope of the price/yield curve at R percent. in theory, the changes are infinitesimally small (dR), but in reality, the smallest yield change normally observed is one basis point (ΔR).

FIGURE 10A–3

Convexity and the price/yield curve



factor normalizes this measure to account for a larger 1 percent change in rates. Remember, when interest rates change by a large amount, the convexity effect is important to measure. A commonly used scaling factor is 10^8 so that:[33]

$$CX = 10^8\left[\frac{\Delta P-}{P} + \frac{\Delta P+}{P}\right]$$

Calculation of CX

To calculate the convexity of the 8-percent coupon, 8-percent yield, six-year maturity international bond that had a price of $1,000:[34]

$$CX = 10^8\left[\frac{999.53785 - 1{,}000}{1{,}000} + \frac{1{,}000.46243 - 1{,}000}{1{,}000}\right]$$

Capital loss from a one-bp increase in rates + Capital loss from a one-bp decrease in rates

$$CX = 10^8[0.00000028]$$

$$CX = 28$$

This value for CX can be inserted into the bond price prediction equation with the convexity adjustment:

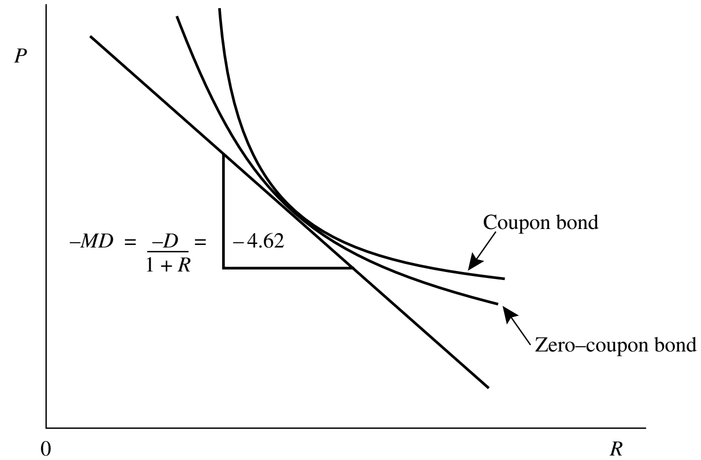
$$\frac{\Delta P}{P} = -MD\,\Delta R + \frac{1}{2}(28)\Delta R^2$$

[33]This is consistent with the effect of a 1-percent (100-basis-point) change in rates. Another common way of expressing convexity arises from dealing in percents rather than decimals. If we were dealing in percentages, the scaling factor would be 10^6 and the resulting convexity number would be 0.28 rather than 28. To be consistent, the ΔR would be expressed as 2 percent and $\Delta P/P = -4.99/1.08 \times 2 + 1/2\,(0.28)\,2^2 = -9.24 + 0.56 = -\,8.68$. The analyst must be careful as to the units in which convexity is expressed.

[34]You can easily check that $999.53785 is the price of the six-year bond when rates are 8.01 percent, and $1,000.46243 is the price of the bond when rates fall to 7.99 percent. since we are dealing in small numbers, and convexity is sensitive to the number of decimal places assumed, use at least five decimal places in calculating the capital gain or loss. In fact, the more decimal places used, the greater the accuracy of the CX measure.

FIGURE 10A–4

Convexity of a coupon versus a zero-coupon bond with the same duration



Assuming a 2-percent increase in R (from 8 to 10 percent):

$$\frac{\Delta P}{P} = -\left[\frac{4.99}{1.08}\right]0.02 + \frac{1}{2}(28)(0.02)^2$$

$$= -0.0924 + 0.0056 = -0.0868 \text{ or } -8.68\%$$

The simple duration model (the first term) predicts that a 2-percent rise in interest rates will cause the bond's price to fall by 9.24 percent. however, for large changes in yields, the duration model overpredicts the price fall. The duration model with the second-order convexity adjustment predicts a price fall of 8.68 percent; it adds back 0.56 percent due to the convexity effect. This is much closer to the true fall in the six-year, 8 percent coupon bond's price if we calculated it using 10 percent to discount the coupon and face value cash flows on the bond. The true value of the bond price fall is 8.71 percent. that is, using the convexity adjustment reduces the error between predicted value and true value to just a few basis points.[35]

In Table 10A–1 we calculate various properties of convexity, where

N = Time to maturity
R = Yield to maturity
C = Annual coupon
D = Duration
CX = Convexity

Part 1 of Table 10A–1 shows that as the bond's maturity (N) increases, so does it convexity (CX). As a result, long-term bonds have more convexity—which is a desirable asset property—than short-term bonds. This property is similar to that possessed by duration.

Part 2 of Table 10A–1 shows that coupon bonds of the same maturity (N) have less convexity than zero-coupon bonds. However, for coupon bonds and discount zero-coupon bonds of the same duration, part 3 of the table shows the coupon bond has more convexity. We depict the convexity of both in Figure 10A–4. Convexity is inversely related to yield, as part 4 of Table 10A–1 shows.

[35]In actuality, one might use the third moment of the Taylor series expansion to reduce this small error (8.71 percent versus 8.68 percent) even further. In practice, few people do this, preferring to value explicitly the portfolio. See Chapter 11.

TABLE 10A–1 Properties of Convexity (Examples)

1. Convexity Increases with Bond Maturity			2. Convexity Varies Inversely with Coupon		3. For Same Duration, Zero-coupon Bonds are Less Convex than Coupon Bonds		4. Convexity Varies Inversely with Yield	
A	*B*	*C*	*A*	*B*	*A*	*B*	*A*	*B*
$N = 6$	$N = 18$	$N = \infty$	$N = 6$	$N = 6$	$N = 6$	$N = 5$	$N = 6$	$N = 6$
$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 8\%$	$R = 0\%$
$C = 8\%$	$C = 8\%$	$C = 8\%$	$C = 8\%$	$C = 0\%$	$C = 8\%$	$C = 0\%$	$C = 8\%$	$C = 8\%$
$D = 5$	$D = 10.12$	$D = 13.5$	$D = 5$	$D = 6$	$D = 5$	$D = 5$	$D = 5$	$D = 5.19$
$CX = 28$	$CX = 130$	$CX = 312$	$CX = 28$	$CX = 36$	$CX = 28$	$CS = 25.72$	$CX = 28$	$CS = 51$

Finally before leaving convexity we might look at one important use of the concept by managers of insurance companies, pension funds, and mutual funds. Remembering that convexity is a desirable form of interest rate risk insurance, FI managers could structure an asset portfolio to maximize its desirable effects. As an example, consider a pension fund manager with a 15-year payout horizon. To immunize the risk of interest rate changes, the manager purchases bonds with a 15-year duration. Consider two alternative strategies to achieve this:

Strategy 1: Invest 100 percent of resources in a 15-year deep discount bond with an 8-percent yield.
Strategy 2:[36] Invest 50 percent in the very short-term money market (overnight loans) and 50 percent in 30-year zero-coupon bonds with an 8 percent yield.

The duration (D) and convexities (CX) of these two asset portfolios are

Strategy 1: $D = 15$, $CX = 206$
Strategy 2:[37] $D = \tfrac{1}{2}(0) + \tfrac{1}{2}(30) = 15$, $CX = \tfrac{1}{2}(0) + \tfrac{1}{2}(797) = 398.5$

Strategies 1 and 2 have the same durations, but Strategy 2 has a greater convexity. Strategy 2 is often called a barbell portfolio, as shown in Figure 10A–5 by the shaded bars.[37] Strategy 1 is the unshaded bar. To the extent that the market does not price (or fully price) convexity, the barbell strategy dominates the direct duration matching strategy.[38]

More generally, an FI manager may seek to attain greater convexity in the asset portfolio than on the liability portfolio. Both positive and negative shocks to interest rates would then have beneficial effects on the FI's net worth.

[36]The duration and convexity of one-day overnight loans are approximately zero.

[37]This is called a barbell because the weights are equally loaded at the extreme ends of the duration range or bar, as in weight-lifting.

[38]In a world in which convexity is prized, the long-term 30-year bond's price would rise to reflect the competition among buyers to include this more convex bond in their barbell asset portfolios. Thus buying bond insurance—in the form of the barbell portfolio—would involve an additional cost to the FI manager. Also, to be hedged, in both a duration and a convexity sense, the manager should not choose the convexity of the asset portfolio without seeking to match it to the convexity of its liability portfolio. For further discussion of the convexity "trap" that results when an FI mismatches its asset and liability convexities, see James H. Gilkeson and Stephen D. Smith, "The Convexity Trap: Pitfalls in Financing Mortgage Portfolios and Related Securities," Federal Reserve Bank of Atlanta, *Economic Review,* November/December 1992, pp. 14–27.

FIGURE 10A–5

Barbell strategy