Homework #1

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1. (a) Let G be a simple graph of order n. Let $S(G, x_0, \ldots, x_{n-1})$ denote Sylvester's adjacency polynomial of G as discussed in class. Prove that for each $i \in V(G) = \mathbb{Z}_n$, the vertex degree of vertex i in G (the number of distinct edges incident to i in G) is equal to the degree of the variable x_i in the polynomial $S(G, x_0, \ldots, x_{n-1})$.

   Proof: We can denote the Sylvester adjacency polynomial as:

$$\prod_{i<j, i\leftrightarrow j}^{n-1} (x_j - x_i) \tag{1}$$

   Before continuing, here are a few important definitions I will use in my proof:

   Degree of a vertex- The number of edges incident on the vertex.

   Adjacent- Two vertices are adjacent iff there is an edge between them.

   $\therefore$ by definition, the only members of the factored polynomial present will be the ones adjacent to each other (as that is the condition for them to be written in that fashion in the polynomial), which, by definition is equivalent to saying that there is an edge between them.

   Intuition: Now, let us fix a particular $i \in V(G)$. Then, by definition, the total number of places it appears in the factorized form, i.e. **the degree of the polynomial**, either as an x_i or an x_j, will denote the number of vertices it is adjacent to, which by definition means that is the number of edges associated with it, which is **the degree of the vertex**.

   Precise form: Let us be a little more precise about the above statement. Let us fix $i \in V(G)$. Then, in the Sylvester adjacency polynomial, for all the places where i is associated with x_i such that it is less than j, it is, by definition equivalent to the number of edges associated with the vertices indexed greater than it. The same reasoning can be used to state that when the i is associated with a x_j, all the places it appears as the greater index value is equivalent to the number of edges associated with the vertices indexed less than it in the graph. $\therefore$ summing up those terms, as those are all and only terms associated with i in the polynomial, we get the degree of the polynomial. However, as stated earlier, the individual cases are associated with vertexes indexed greater and less than the fixed i that are adjacent to it, hence summing it up gives us the total number of edges incident on i i.e. the degree of the vertex, being equivalent to the degree of the polynomial. $\blacksquare$

   (b) Prove that the the complete graph K_n has a decomposition consisting of two copies of some graph H iff n or n - 1 is a multiple of 4

   Proof: Only if case: Let us assume K_n hase a decomposition consisting of two copies of some graph H. i.e., **this means that K_n decomposes into two n-vertex subgraphs with disjoint edge sets such that their union gives the original graph, and additional, each subgraph is isomorphic to the other, i.e. H is self-complementary**.

Then, the total number of edges will be equal to $2k$ where $k \in \mathbb{Z}$. Additionally, as we know that the total number of edges in a complete graph is $\binom{n}{2}$ we can create the equation:

$$\frac{n(n-1)}{2} = 2k \implies n(n-1) = 4k \implies n(n-1) mod 4 = 0 \tag{2}$$

Now, n as an integer is either $0, 1, 2$ or 3 mod 4.

$n = 0$ mod 4 Case: $0 \cdot 3$ mod $4 = 0$ mod 4 hence $n = 0$ is valid

$n = 1$ mod 4 Case (i.e. $n - 1$ mod $4 = 0$): $1 \cdot 0$ mod $4 = 0$ hence it is a valid case

$n = 2$ mod 4 Case: $2 \cdot 1$ mod $4 = 2$ mod 4 hence invalid.

$n = 3$ mod 4 Case: $3 \cdot 2$ mod $4 = 6$ mod 4 hence invalid.

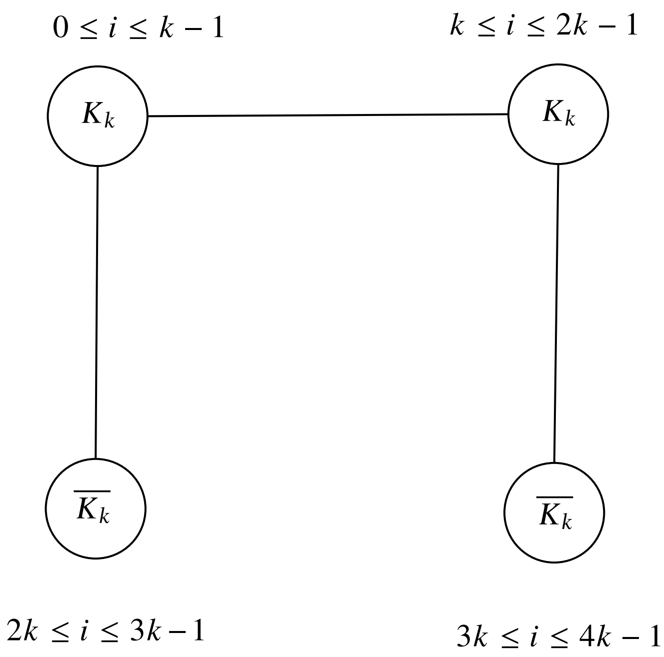
$\therefore$ the only two valid solutions are $n = 0$ mod 4 and $n = 1$ mod 4.

If Case:

Assume n is a multiple of 4 or $n - 1$ is a multiple of 4 (and that we have a graph K_n).

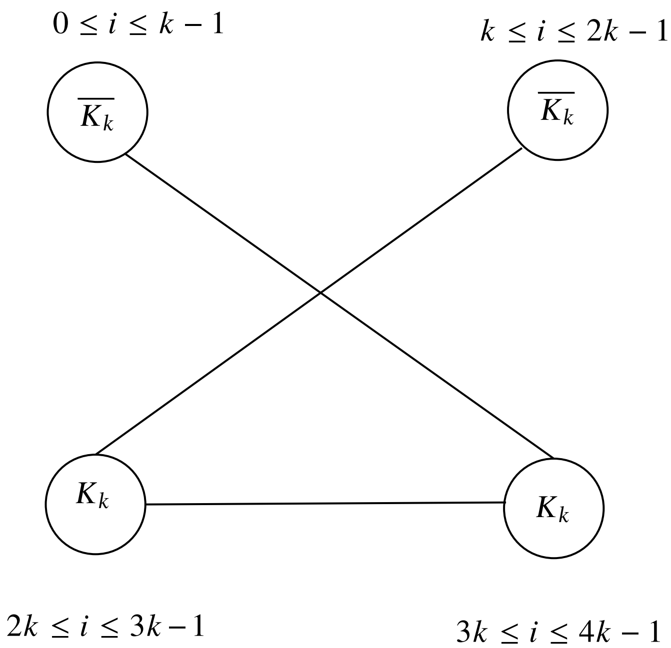
Let us consider the n mod 4 case first:

This means that $n = 4k$ for some $k \in \mathbb{Z}$ We can then look at a subgraph H of our graph K_n, which can be split into 4 vertex subsets of K_k and two of them being elements of the complements of K_k i.e. $\bar{K_k}$. The representation is illustrated below:



Note: the vertices in the complementary graph $\bar{K_k}$ are not connected to each other or anything else other than those in the complete subgraphs, as the complement of a complete graph is no edges in a vertex. Additionally, the index $i \in V(K_n)$

Now, I claim that in order to show that K_n for $n = 4k$ decomposes into two copies of H, it is sufficient to show that the above graph is isomorphic to the graph:



Now, in order to show the isomorphism, I need to construct a bijection ϕ between the two graphs (specifically between the vertex sets of the two graphs) such that of (v, u) are adjacent, then $(\phi(v), \phi(u))$ are adjacent. I claim that the mapping I wish to use is:

$$\phi(i) \begin{cases} i + 3k & \text{if } 0 \leq i \leq k-1 \\ i + k & \text{if } k \leq i \leq 2k-1 \\ i - 2k & \text{if } 2k \leq i \leq 4k-1 \end{cases} \tag{3}$$

Now, to check that the above is a bijection. This is trivially true from the fact that the mapping used is a permutation of the vertices.

Therefore, what is left to check is that if v, u are adjacent, then $(\phi(v), \phi(u))$ are adjacent. WLOG, let us fix $v, u \in V(K_n)$. Then, there are 4 cases to consider:

Assume, v, u **are adjacent** (therefore, we can ignore v, u being members of either of the $\bar{K}_k$'s)

Case 1: $0 \leq v \leq k-1$ and $0 \leq u \leq k-1$

Then $\phi(v) = v + 3k$

and $\phi(u) = u + 3k$

This means that post mapping $\phi(v)$ and $\phi(u)$ must be members of the K_k complete graph indexed from $3k$ to $4k-1$, and hence, by definition of a complete graph, must be adjacent to each other.

Case 2: $k \leq v \leq 2k-1$ and $k \leq u \leq 2k-1$

then $\phi(v) = v + k$

and $\phi(u) = u + k$

This means that $\phi(v)$ is a member of the K_k complete graph indexed from $2k$ to $3k-1$ while $\phi(u)$ is also a member of the same, hence again, by definition of complete graphs, the two are adjacent.

Case 3: $0 \leq v \leq k-1$ and $2k \leq u \leq 3k-1$

then $\phi(v) = v + 3k$

and $\phi(u) = u - 2k$

Then $\phi(v)$ is a member of the graph K_k indexed from $3k$ to $4k-1$ and $\phi(u)$ is a member of the graph $\bar{K_k}$ indexed from 0 to $k-1$, for which every vertex is adjacent to every vertex in the K_k that $\phi(v)$ is a part of, and hence the two are adjacent.

Case 4: $k \leq v \leq 2k-1$ and $3k \leq u \leq 4k-1$

then $\phi(v) = v + k$

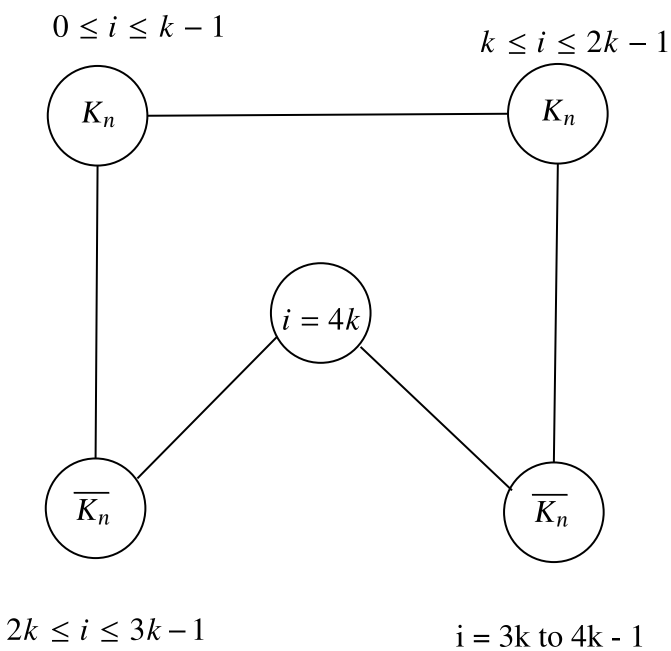
and $\phi(u) = u - 2k$

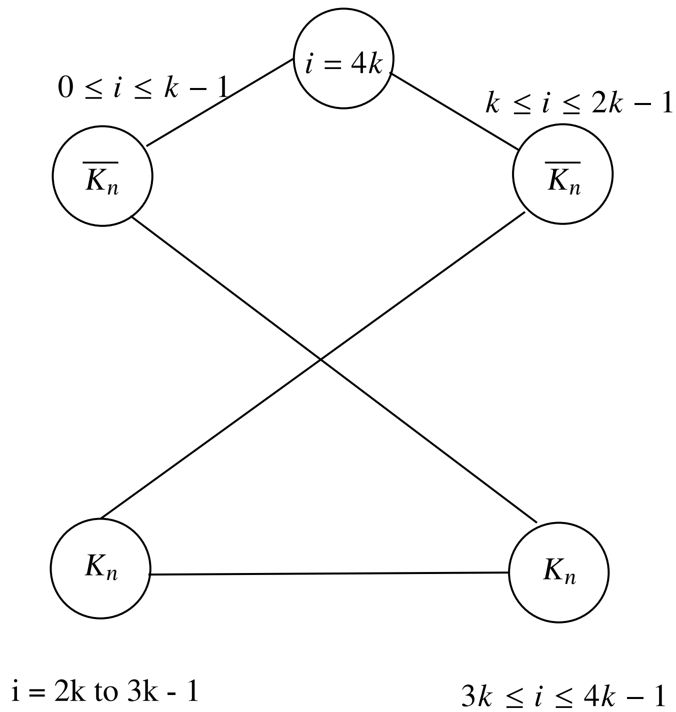
Then $\phi(v)$ is a member of the graph K_k indexed from $2k$ to $3k-1$ and $\phi(u)$ is a member of the graph $\bar{K_k}$ indexed from k to $2k-1$, for which every vertex is adjacent to every vertex in the K_k that $\phi(v)$ is a part of, and hence the two are adjacent.

Hence, we can say that the two graphs are isomorphic to each other when n is a multiple of 4.

Now, let us consider the $n-1 = 4k$ case:

By the same reasoning as above, I claim that it is sufficient to show that the below two graphs are isomorphic to each other:





Essentially, the graphs are the same as the $n = 4k$ case, with a vertex attached, so we construct the same ϕ with the $i = 4k$ vertex being mapped to itself. This is then clearly an isomorphism by the same reasoning as above. ■

2. (a) Which complete graphs are bipartite

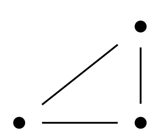
   Solution: The only complete graph that is bipartite is the K_2 graph.

   Reasoning: A bipartite graph is a graph that can be split into two independent vertex sets such that each edge has one endpoint in one set and the other endpoint in the other set. Consider any complete graph that is more than K_2. Then, any vertex subset of the graph $X \subset V(G)$ with more than one element will have all the vertices (in the subset) being pairwise adjacent, by definition of a complete graph. Therefore, they cannot be independent, and hence any complete graph that is more than K_2 is not bipartite.

   Now, let us consider the K_1 case. As the K_1 case has no edges, or even more than one vertex, it is not possible to split the graph into two distinct vertex sets, with an edge between them.

   (b) Prove that if K_n decomposes into triangles then $n-1$ or $n-3$ is a multiple of 6.

   Proof: Assume K_n decomposes into triangles. Then, each vertex in each subgraph has two edges incident to it (by definition of a triangle). This is illustrated in an example below:



   Now, WLOG let us fix a vertex $i \in V(G)$. Then, i will be a member of at least one triangle subgraph, from our assumption of completeness and the assumption that K_n decomposes into triangles. Additionally, i will either be adjacent to an edge in another subgraph or not. If it is adjacent to a vertex in the other subgraph, then it is a part of the triangle, and hence has two more edges incident to it. If it is not adjacent, then it is not part of the triangle, and no edges are incident to it from the other subgraph.

$\therefore$ as i has at least two edges associated with it, and either 0 or 2 added on from every other subgraph, it has **even degree**. We can therefore state that **every vertex in a K_n graph that decomposes into triangles has even degree.**

Now, from our assumption that K_n can be decomposed into triangles, we can state that the number of edges is equal to $3k$, where k is the number of triangles. (This comes from the definition of decomposition, in which the original graph decomposes into subgraphs, with the edge sets of each subgraph being disjoint, and hence the total number of edges will be the sum of all the edges in each subgraph. In the case of each subgraph being a triangle, the edge set total of the complete graph can be thought of as 3 times the number of triangles). Another way to think of the total number of edges in a complete graph is as $\binom{n}{2}$ i.e. $\frac{n(n-1)}{2}$ (where n is the number of vertices in the graph). Therefore, we can create the equation:

$$\frac{n(n-1)}{2} = 3k \implies n(n-1) = 6k \implies n(n-1)mod6 = 0 \tag{4}$$

We can now use the conclusion we came to that every vertex in a K_n graph has even degree, in addition to the fact every vertex in a complete graph has $n-1$ degree (from the fact that every vertex is pairwise adjacent, hence every vertex is adjacent to every other one, except it self, hence it has degree $n-1$) to state that $n-1$ is always even, and therefore n is odd. $\therefore$ our two current conditions are that $n(n-1)$ **mod** 6 **=** 0 and n **is odd**.

Now, n as an integer can be either $0, 1, 2, 3, 4$ or 5 mod 6, however, with the constraint that n is odd, we can state that $n = 1, 3, 5$ mod 6. 5 mod 6 is eliminated as $5 \cdot 4$ mod 6 is 10 mod 6, which contradicts the constraint that $n(n-1)$ mod $6 = 0$. That leaves $n = 1$ or 3 mod 6:

$n = 1$ mod 6 (i.e. $n-1$ is a multiple of 6) Case: $1 \cdot 0$ mod $6 = 0$ mod 6, hence it is valid.

$n = 3$ mod 6 (i.e. $n-3$ is a multiple of 6) Case: $3 \cdot 2$ mod $6 = 0$ mod 6, hence it is valid.

$\therefore$ We have concluded that the only two valid n's for a K_n graph that decomposes into triangles is $n = 1$ mod 6 and $n = 3$ mod 6. ■

3. What is the maximum possible degree in a SIMPLE undirected graph on n vertices? What is the minimum possible degree?

   Solution: The maximum possible degree (on a vertex) in a simple directed graph is $n-1$, where n is the number of vertices in the graph.

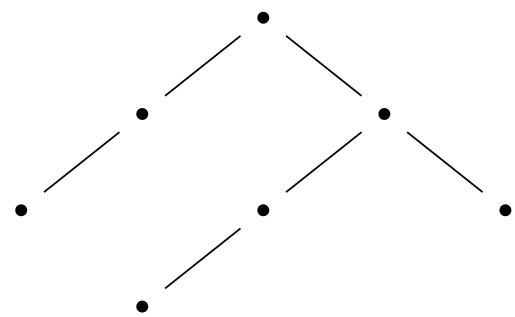
   The minimum possible degree is 0.

   Reasoning: To get the maximum possible degree, we consider an complete graph, as every vertex is adjacent to every other (and so we cannot have a greater possible degree). Then, it must be adjacent to $n-1$ vertices (everything other than itself), by definition.

   For the minimum possible degree, we just assume no edges exist, and hence every vertex has no edges incident on it, and therefore has 0 degree (which is the minimum possible degree, as the vertex cannot have any less adjacency with other vertices than no adjacency).

4. What is the minimum size of the automorphism group of a simple undirected graph having more then five vertices. Describe explicitly such a graph.

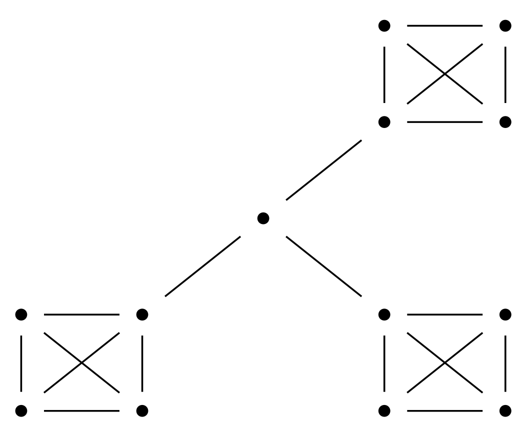
   Solution: The minimum size of the automorphism group of a simple undirected graph is 1. I have explicitly constructed such a graph below:



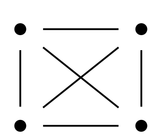
Reasoning: The minimum size of an automorphism group is 1 as every graph will always have the identity isomporphism, and hence 0 is not possible for an automorphism group size. The graph that I have constructed satisfies these requirements, as an automorphism requires not just an isomorphism, but the same edge and vertex set to be preserved post isomorphism, as it is an isomorphism from a graph to itself. $\therefore$ the above graph cannot have any bijection that preserves this other than the identity bijection, as any other mapping would involve swapping vertices with different degrees or else swapping vertices that are connected to different paths, and therefore a different edge set would be required.

5. Draw a connected graph G on 13 vertices, having no cycle of length ≥ 5 . In addition, every vertex in G has degree either 3 or 4. G has 9 cycles of length 4, G has 12 cycles of length 3, and not other cycles. Explain in words why your construction satisfies all the requirements above.

   Solution: The graph that satisfies the above requirements is:



Reasoning: Now, it is clear that the graph satisfies the requirements, if we focus on subgraphs of the graph. The important subgraph to consider is:



This graph has 4 3 cycles (from each triangle we can make for each vertex) and 3 4 cycles (the square perimeter and also two 4 cycles passing through the two cross edges) and 4 vertices. Additionally, the degree of each vertex is 3. If we take 3 disjoint graphs of this form (disjoint vertices and edges), then we will get 12 3 cycles and 9 4 cycles, and 12 vertices. If we connect any tip of each graph to a central vertex, then that vertex added on makes 13 vertices and the degree of the tip vertex of each part of the box becomes 4, while the degree of the connecting vertex will be 3, hence each vertex of the graph has degree 3 or 4 (4 for the tip vertex of each box, 3 for every other vertex). Additionally, this make the graph completely connected, and as each individual box has a maxmimum 4 cycle, a 5 cycle is not possible, as it would involve moving from one box to another, and back, which would require passing through the central vertex mulitple times.