Homework #3

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February 16, 2023

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1. For which positive integer values of α and β can we guarantee that $K_{\alpha,\beta}$ is Eulerian. Give a detailed proof of your claim.

   Solution: The positive integer values for which the above statement holds true are $\alpha = 2m \ \forall m \in \mathbb{Z}^+$ and $\beta = 2n \ \forall n \in \mathbb{Z}^+$ i.e. α and β are even.

   Proof: Assume we have $K_{\alpha,\beta}$. This means that we are assuming that the graph is a **complete bipartite graph and therefore connected**.

   Side proof of a complete bipartite graph being connected: We pick a vertex v in the graph. For the graph to be connected, there must be a path between v and any other vertex u. I claim I can construct such a path. There are two cases to consider:

   Case 1: u is not in the same partite set as v Then, as the graph is complete and bipartite, u is adjacent to v and the path between is precisely that edge and those two vertices.

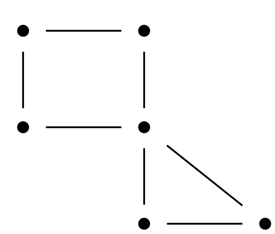
   Case 2: u is in the same partite set as v. Then, to construct the path, we pick any edge incident to v. This edge will have its other endpoint at a vertex u' in the opposite partite set, so we make it the first edge in the path (with v being the first vertex) and u' the second vertex. Then u' will be adjacent to u from our assumption of a complete bipartite graph, and therefore we make that edge the second edge of the path and end on u.

   Main proof continued: So our assumptions so far are that the graph is connected (and complete bipartite). From **Euler's graph theorem** we know that the additional assumption we have to make to ensure that the graph is Eulerian is that every vertex has even degree.

   Now, fix a $v \in$ the β partite set. From the Eulerian graph theorem, we know that v has even degree. Additionally, from the fact that the graph is complete and bipartite, v is adjacent to all and only the vertices in the α partite set. Therefore, the degree of v is the number of vertices in the α partite set. $\therefore$ α must be even. The exact same reasoning can be applied for a $v \in$ the α partite set to give us β must be even. ■

2. (a) Give a detail proof or provide a counterexample to the claim that every Eulerian simple graph with an even number of vertices has an even number of edges.

   Counter example:



   Reasoning: The above graph has 6 vertices and 7 edges, and is Eulerian as we can construct an Eulerian circuit, say starting from the bottom right vertex, moving to the top of the triangle directly, going all around the square, and then taking the final path left over to return to the bottom left vertex.

(b) Give a detail proof or provide a counterexample to the claim that every Eulerian simple graph with maximum degree < 3 and an even number of vertices has an even number of edges.

Proof: From the handshaking lemma we know that

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

Now, as a degree is a whole number, the claim above is for an Eulerian graph with max degree ≤ 2. However, as an Eulerian graph is also connected, this means that every vertex has at least degree one. However, as the graph is Eulerian, every vertex has even degree, and hence min degree ≥ 2, therefore $\forall v \in V(G)$, $2 \leq \deg(v) \leq 2$ hence deg $(v) = 2$. Therefore, our equation becomes

$$\sum_{1}^{2n} 2 = 2|E(G)|$$

for some $n \in \mathbb{Z}^+$ as we have an even number of vertices, all of degree 2. This means that we get

$$4n = 2|E(G)| \implies 2n = |E(G)|$$

for some $n \in \mathbb{Z}^+$, which is precisely the statement that we have an even number of edges. ■

3. Let $\mathbf{A} \in \{0,1\}^{n \times n}$ be the adjacency matrix of an undirected simple graph G on n vertices. What is the maximum number of non-zero entries in $\mathbf{A}$ if every walk in G contains only odd cycles.

Solution: For $n = 2k + 1$ for some $k \in \mathbb{Z}^+$ (i.e. n is odd) the maximum number of non-zero entries is

$$\frac{3(n-1)}{2}$$

. For $n = 2m$ for some $m \in \mathbb{Z}^+$ (i.e. n is even) the maximum number of non-zero entries is

$$\frac{3(n-1)}{2} + 1$$

.

Reasoning: Finding the maximum number of non-zero entries in $\mathbf{A}$ is equivalent to finding the maximum number of edges. The statement can be reframed as finding the maximum number of edges if G contains only odd cycles.

Now, if G contains only odd cycles, i.e. no even cycles, I claim it must be edge-disjoint.

Proof of above claim (edge-disjoint lemma: Taking the contrapositive form of the above statement, we get if two cycles C_1 and C_2 that are not edge disjoint, then G has an even cycle. There are then two cases to consider:

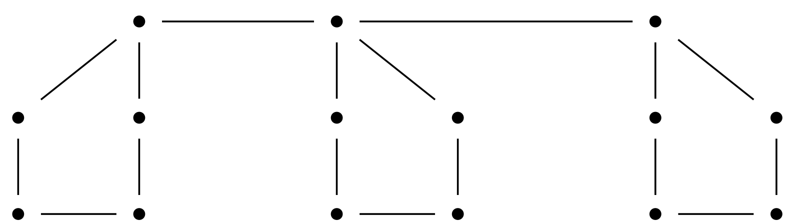
Case 1: Assume C_1 or C_2 is even. Then, G has an even cycle, which is precisely C_1 or C_2.

Case 2: C_1 and C_2 are both odd. Let S be a maximal path of shared edges between C_1 and C_2 starting at a vertex u and ending at a vertex v Then, there are two subcases to consider:

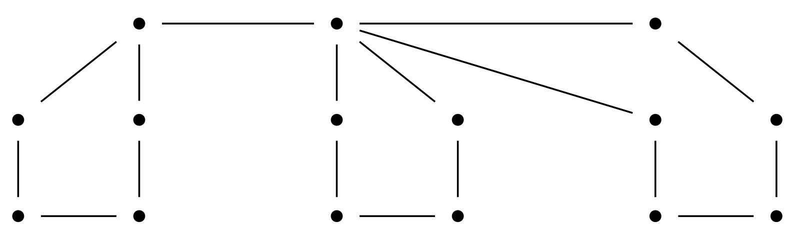
Case (2) a: Assume S has odd length. Then, the path $P_1 = C_1 \setminus S$ must be even and the path $P_2 = C_2 \setminus S$ must be even. Then, we can construct an even cycle of the form u, P_1, v, P_2, u which must be even, as it is the sum of two even paths.

Case (2) b: Assume S has even length. Then, the path $P_1 = C_1 \setminus S$ must be odd and the path $P_2 = C_2 \setminus S$ must be odd. Then, we can construct an even cycle of the form u, P_1, v, P_2, u which must be even, as it is the sum of two odd paths.

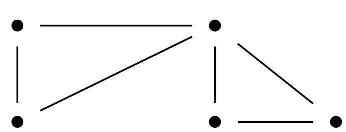
Main reasoning part continued: As we know the graph must be edge-disjoint and contain odd cycles, and additionally that we must maximize the number of edges, the graph must be connected. Then, the odd-cycles can either be connected at a vertex or by edges. I claim that connecting the odd cycles via edges is not maximizing the number of edges. I will show this by considering a specific case of odd cycles not connected by vertices:



We can use the same number of edges to get the graph:



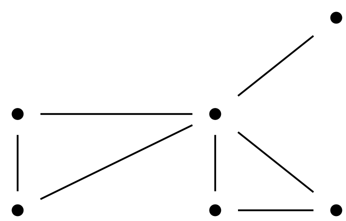
Which clearly does not maximize the number of edges, but we cannot increase the number, as that will construct even cycles. Hence, we must ensure all the cycles are connected by a set of vertices. For simplicity, we will bind them to a single vertex. Now, I claim that the maximum number of edges for a graph with only odd cycles would be to break those odd cycles down into triangles. i.e. in the form:



This comes from the fact that if we had another decomposition of the graph into odd cycles i.e. mk where m is the number of edges in an odd cycle and k is the number of odd cycles of that size, that:

$$\frac{3k(m-1)}{2} \geq mk$$

i.e., for every decomposition of the graph into odd cycles, we can increase the number of edges further by decomposing those odd cycles into triangle. If the number of vertices is odd, then the single vertex connected to all odd cycles graph will be exactly all triangles. In the even case, we add one edge in order to account for the extra vertex i.e.:



■

4. (a) What is the maximum number of edges in a connected graph which contains no cycle. Can the graph be Eulerian. Give a detailed proof of your claim.

Solution: The maximum number of edges is $n-1$ where $n = |V(G)|$. The graph cannot be Eulerian.

Proof: The graph above is a tree, which means that it is a connected forest (a forest is a simple undirected graph that is a acyclic). Then, we know from Listing's lemma (1861) on a forest that

$$0 = |E(G)| - |V(G)| + \text{no. of connected components} \implies |E(G)| = |V(G)| - \text{no. of connected components}$$

. The number of connected components for a connected graph is exactly one, hence the equation becomes:

$$|E(G)| = |V(G)| - 1$$

$n = |V(G)|$ the number of edges can only be $n-1$, and hence the maximum number of edges is $n-1$. Additionally, the graph is not Eulerian, as there exists no cycle, and therefore no circuit, and therefore no Eulerian circuit. ■

(b) What is the maximum number of edges in a connected graph which contains exactly one cycle. Can the graph be Eulerian. Give a detailed proof of your claim.

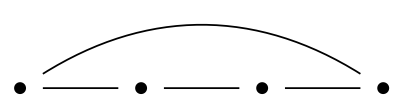
Solution: The maximum number of edges is $|V(G)|$ and the graph can indeed be Eulerian.

Proof: Let us assume we have a graph with the maximum number of edges, and the graph is connected with exactly one cycle. We can reduce the number of edges in the graph by 1 by removing an edge. The edge choice is arbitrary, so I choose to remove a cycle edge. The graph remains connected in this case.

Side proof that removing a cycle edge in a connected graph keeps the graph connected: Let us assume that we have a connected graph G. Then $\exists$ a path between every $u, v \in V(G)$. Additionally, as we have a cycle, in other words, there exists a $u \in V(G)$ s.t. there exists a non-empty trail with only the first and last vertex repeating. In other words, in the cycle edge sequence, there must exist two paths by which u is connected to every vertex in the cycle sequence. Hence, removing a single cycle edge just allows for the alternative path to be taken for the connection to be maintained (as the graph is connected outside the cycle edges).

Note: The above proof is just a side note. The fact that the graph remains connected comes from the cut-edge theorem

Main proof continued: Then, from part (a) we know that the number of edges is $|V(G)| - 1$, so adding the removed edge back gives us the number of edges as $|V(G)|$. The graph can indeed be Eulerian. I shall show this by construction:



The graph is Eulerian, as we have an Eulerian circuit starting at the leftmost vertex, going through the upper edge such that it is adjacent to the rightmost vertex, and then following the connected path back to the leftmost vertex. ■

5. Let $\mathbf{A} \in \{0,1\}^{n \times n}$ be the adjacency matrix of an undirected simple graph G on n vertices.

   Prove that the coefficient of x^{n-1} in the characteristic polynomial $\det(x\mathbf{I}_n - \mathbf{A})$ must be zero

   Proof: Let us first consider the matrix formed by $(x\mathbf{I}_n - \mathbf{A})$:

   Let $1 \leq i, j \leq n$. Then the entries of the matrix are

$$q_{ij} = \begin{cases} x & \text{if } i = j \\ -1 & \text{if } i \text{ adjacent to } j \\ 0 & \text{if otherwise } i \text{ not adjacent to } j \end{cases} \tag{1}$$

   Then the determinant of the matrix is:

$$\sum_{\sigma \in S_n} \{\prod_{i=1}^{n} \operatorname{sgn}(\sigma) q_{i\sigma(i)}\}$$

   where S_n is the set of permutations of i from 1 to n and σ is an element of the set.

   From this definition of the determinant, we can first see that the coefficient of x^n can only be provided in the trivial permutations case i.e. $\sigma(i) = i \ \forall i$ as that multiplies all the x's in the diagonal. Hence, the coefficient of x^n is 1.

   Now, let us say we want to find the coefficient of x^{n-1}. This means that we identify the number of places that x^{n-1} shows up in the sum above and its associated coefficients and sum up all those coefficients. So, let us first begin by constructing an x^{n-1} product in the case above. We know that the trivial permutation does not generate the product, as it only considers elements along the diagonal. So, let us consider any other permutation. Consider a permutation $\sigma(i) \neq i \forall i$. Then, $\exists$ at least one element k s.t. $\sigma(k) \neq k$. As a permutation is a bijection, this means that there must be some other element k_1 s.t. $\sigma(k_1) \neq k_1$. $\therefore$ this means that the minimum permutation (not counting the trivial permutation) on i must permute at least two elements in $1 \leq i \leq n$, and therefore, the next product element that is possible is x^{n-2}. In other words, as x^{n-1} is not possible in this sum, for any σ its coefficient must be zero. ■

(b) Prove that the absolute value of the coefficient of x^{n-2} in the characteristic polynomial $\det(x\mathbf{I}_n - \mathbf{A})$ must be $|E(G)|$

   Proof: Continuing our reasoning from above, we now count the number of places (weighted with the coefficients) that x^{n-2} shows up in the sum. This counts the number of minimum permutations and its associated coefficients and sums them up. As above, the minimum permutation (not counting the trivial one), bijects two elements k and k_1 and it must be precisely the bijection of the form $\sigma(k) = k_1$ and $\sigma(k_1) = k$, as any other bijection to, say, an element k_2 would then involve a further bijection of the form $\sigma(k_2) = k_3$ where $k_3 \neq k, k_1$. As this permutation is of the above form, it is a swap of the two elements, in other words, the coefficient in the product for a fixed σ will be of the form:

$$\text{coefficient of } x^{n-2} \text{ in the product (not the sum)} = \begin{cases} -1 & \text{if } i \text{ adjacent to } \sigma(i) \\ 0 & \text{if otherwise } i \text{ not adjacent to } \sigma(i) \end{cases} \tag{2}$$

   Summing up those terms gives us precisely $-1|E(G)|$, which, if we take its absolute value, gives us $|E(G)|$ ■

(c) Prove that the coefficient of x^{n-3} in the characteristic polynomial det $(x\mathbf{I}_n - \mathbf{A})$ must be twice the number of triangles in G

Proof: We can again apply the same reasoning as above. To identify the coefficient of x^{n-3} we count the number of places that x^{n-3} shows up in the sum, weighted with its coefficients in the product. In order to identify this, I claim that the associated permutation set that we are looking at is precisely all the permutation of three elements of i. There are two cases that occur, the first being $\sigma k = k_1$, $\sigma(k_1) = k_2$ $\sigma k_2 = k$ and $\sigma k = k_2$, $\sigma(k_1) = k$ $\sigma k_2 = k_1$ for arbitrary k, k_1 and k_2. Each of these consider the same set of 3 edges for the same vertex elements, all connected (by definition of the adjacency matrix) as they are precisely the off diagonal elements, and hence it counts, for a fixed triplet of vertices, twice the number of triangles for those vertices. Summing over all permutations counts all triplets of vertices, and therefore $-2 *$ number of triangles in G. The absolute value of this gives us the claim. ■