Homework #4

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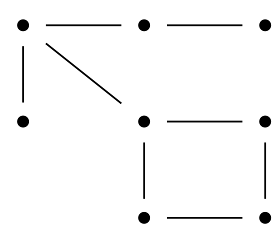
1. Determine whether the following sequences are graphical. If so construct a graph which realizes the corresponding degree sequence.

   (a) $(4, 4, 3, 2, 1)$

   Solution: The degree sequence above is not graphical. We can see that this is the case by applying the Havel-Hakimi Theorem. Using it, we know that $(4, 4, 3, 2, 1)$ is graphical iff $(3, 2, 1, 0)$ is graphical, which is itself graphical iff $(1, 0, -1)$ is graphical. However, $(1, 0, -1)$ is clearly not graphical, as there is no graph that can realize it due to the fact that the -1 term must correspond to a vertex degree, which cannot be negative. Hence, this means that the original degree sequence is not graphical. ■

   (b) $(3, 3, 2, 2, 2, 2, 2, 1, 1)$

   Solution: The above sequence is indeed graphical, and I will show this by construction:

   (c) $(7, 7, 6, 5, 4, 3, 2, 1)$

   Solution: The degree sequence above is not graphical. We can see this again by applying the Havel-Hakimi Theorem. Using it, we know that $(7, 7, 6, 5, 4, 3, 2, 1)$ is graphical iff $(6, 5, 4, 3, 2, 1, 0)$ is graphical iff $(4, 3, 2, 1, 0, -1)$ which is clearly not graphical, as there is no graph that can realize it due to the fact that the -1 term must correspond to a vertex degree, which cannot be negative. Hence, the original degree sequence is not graphical.

   (d) $(7, 6, 6, 5, 4, 3, 2, 1)$

   Solution: The above sequence is not graphical. We can see this again by applying the Havel-Hakimi Theorem. Using it, we know that $(7, 6, 6, 5, 4, 3, 2, 1)$ is graphical iff $(5, 5, 4, 3, 2, 1, 0)$ is graphical iff $(4, 3, 2, 1, 0, 0)$ is graphical iff $(2, 1, 0, -1, 0)$, which we can put in non-increasing order to get $(2, 1, 0, 0, -1)$, which is clearly not graphical as there is no graph that can realize it due to the fact that the -1 term must correspond to a vertex degree, which cannot be negative. Hence, the original degree sequence is not graphical.

2. Let G be a simple graph and $\delta(G)$ denote the minimum degree.

(a) a) Prove or provide a counterexample to the claim that G contains a path of length at least $\delta(G)$

Proof (By contradiction): We assume that G is a simple graph with min degree $\delta(G)$ with a maximum path length of $< \delta(G)$. Let us now fix the maximum path P, with $|E(P)| \leq \delta(G) - 1$ (from our assumption of max path length). This then means that the number of vertices in the path are $\leq \delta(G)$, due to the fact that it is a maximum path, it must maximally be connected with no cycles, and therefore, $|E(P)| + 1 = |V(P)|$ (from 4(a) on Homework 3). Now, WLOG, let us consider one of the endpoints $u \in V(P)$. If u is the endpoint of a maximum path, then it cannot be adjacent to any vertex outside of the vertices within that path, (else it would not be a maximum path, (which is at least a maximal path)).

However, from our min degree assumption we know that $\deg(u) \geq \delta(G)$ i.e. u must be adjacent to at least $\deg(G)$ other vertices. However, as u is not adjacent to itself, it can be adjacent to at most $\deg(G) - 1$ other vertices within the path. This then means that the remaining edge incident to u must make u adjacent to a vertex l outside the path, which contradicts our assumption of P being a maximum path. Therefore, there must exist at least one path in the graph G of length $\delta(G)$. ■

(b) If $\delta(G) \geq 2$, prove or provide a counterexample to the claim that G contains a circuit of length at least $1 + \delta(G)$.

Proof: From part (a) we now know that fixing a maximum path $P \in G$ implies that $|E(P)| \geq \delta(G) \geq 2$ in this case, which implies that $|V(P)| \geq \delta(G) + 1 \geq 3$. WLOG, we now consider an endpoint u of the graph. As u is an endpoint of a maximum path (which is indeed at least a maximal path) it cannot be adjacent to any vertex not in the path. Now, as u is the endpoint of a path, it is adjacent to the previous vertex in the path, which we shall call v. However, as deg $(u) \geq \delta(G) \geq 2$, there must exist at least one other vertex in the path which u is adjacent to which has the path $P' \in P$ s.t. $|E(P')| = \deg(u)$ between it and u, (from our maximal path assumption) which we shall call v'.

Now, I will show that this set-up allows us to construct a cycle of size $1 + \delta(G)$, which means that we have constructed a circuit of the same. Specifically, if we start with u and follow P' to v' and then follow the edge (u, v') edge, we have constructed a circuit of length $|E(P')| + 1$ and as $|E(P')| = \deg(u) \geq \delta(G)$ it must be the case that $|E(P')| \geq \delta(G)$ and $\therefore$ the circuit length is $\geq \delta(G) + 1$. ■

3. Prove that every simple graph with at least two vertices has two vertices of equal degree.

Proof (By contradiction): Assume $|V(G)| \geq 2$ and that every vertex has different degree. Then, the only degree sequence possible is of the form $(0, 1, 2, ..., |V(G)| - 1)$. However, this means that we have a vertex of 0 degree and a vertex with degree $|V(G)| - 1$ (i.e., a vertex which is adjacent to every other vertex), which is a contradiction of our assumption that every vertex has different degree. $\therefore$ the claim is proved. ■

4. Let G be an n–vertex, $K(r + 1)$–free graph with the maximum possible number of edges.

(a) Prove that if u and v are non-adjacent vertices, then the degree of u must equal the degree of v in G.

Proof (By Contradiction): Assume that G is an n–vertex, $K(r+1)$–free graph with the maximum possible number of edges, and that $(u, v) \notin E(G)$. Now, FTSOC assume that $\deg(u) \neq \deg(v)$. WLOG, we assume that $\deg(v) > \deg(u)$. Now, we create a new graph G' by deleting u from the graph, and replacing it with another copy of v, which we shall call v', we can preserve the $K(r + 1)$-free property of the graph (this comes from the claim that if we have a clique, it can only contain one copy of every vertex within it. (Proof of claim below) Additionally, v is not adjacent to itself, so neither will v' be adjacent to v).

Side proof of any clique only containing one copy of v or v': In a clique on k vertices, there are $\frac{k(k-1)}{2}$ edges. Let us fix a vertex v. Now, v will be adjacent to all the other $k-1$ vertices in the graph except itself. Now, any for v' to exist, there must be another vertex in the clique which is adjacent to all the other $k-1$ vertices (not including v). Consider an arbitrary $l \in V(G)\backslash v$. l will be adjacent to $k-1$ other vertices, inclusive of v and (as v is an arbitrary choice) $\therefore$ cannot be v'. Hence, we can state that there exists only one copy of v in any clique.

Main proof continued: We also know that replacing u with v' must mean that we are still dealing with an n-vertex graph. However, as $\deg(v) > \deg(u)$, this means that

$$|E(G')| = |E(G)| - (\deg(u) + \deg(v)) > |E(G)|$$

which contradicts our assumption that we are dealing with the maximum number of edges on the graph.

■

(b) Prove that if the vertex u is non-adjacent to both vertices v and w in G then the vertices v and w must also be non-adjacent in G.

Proof (by contradiction): Assume that G is an n–vertex, $K(r+1)$–free graph with the maximum possible number of edges, and that $(u, v) \notin E(G)$ and $(u, w) \notin E(G)$. From part (a) this implies that $\deg(u) = \deg(w) = \deg(v)$. Now, FTSOC assume that $(v, w) \in E(G)$. Now, we create a new graph G' by deleting w and v from the graph, and replacing it with another two copies of u (which have the same degree as u), which we shall call u' and u''. From the fact that any clique cannot contain more than one copy of vertex u, we know that this process conserves the $K(r+1)$–free property of this graph. However, from the fact that v and w share an edge, we get:

$$|E(G')| = |E(G)| - (\deg(v) + \ \deg(w) - 1) + 2\deg(u)$$

$$= |E(G)| - (2\deg(u) - 1) + 2\deg(u) = |E(G)| + 1 > |E(G)|$$

which contradicts our assumption that we are dealing with the maximum number of edges on the graph. ■

(c) Using the results of part a) and part b) prove that the non-adjacency relation on $V(G)$ is reflexive, symmetric and transitive. Deduce that $\overline{G}$ is a union of cliques. How many cliques are there in $\overline{G}$?

Proof of equivalence relation:

Reflexivity: It is indeed true that in G any vertex v is not-adjacent to itself, just from the definition of a simple, undirected graph, as it can have no loop edges.

Symmetry: If two vertices $(u, v) \notin E(G)$, as we are dealing with a simple undirected graph, this just means that there is no edge between them, no matter what order you pick, i.e. it is also true that $(v, u) \notin E(G)$.

Transitivity: The statement proved in part (b) was precisely that if $(u, v) \notin E(G)$ and $(u, w) \notin E(G)$ then $(v, w) \notin E(G)$, which, from our symmetry statement is precisely if $(v, u) \notin E(G)$ and $(u, w) \notin E(G)$ then $(v, w) \notin E(G)$, which is the transitive property of non-adjacency. ■

Proof that $\overline{G}$ is a union of cliques: As non-adjacency is an equivalence relation, and we assume that the graph is $K(r+1)$–free, with $0 \leq r \leq n-1$, there must exist non-adjacent vertices, and therefore $V(G)$ can be partitioned into disjoint subsets (as every equvalence relation is associated with a partition), with the members of the subsets being all and only the vertices non-adjacent to each other. Additionally, as we want to maximize the number of edges, those subsets must all have edges between each other's vertices i.e., we have a **complete multipartite graph**.

Then, we construct $\overline{G}$ where $(u, v) \in E(\overline{G})$ iff $(u, v) \notin E(G)$ i.e. all the non-adjacent vertices (i.e. the members of each partite part of the G) become adjacent to each other, while all the edges $(u, v) \in E(G)$ in between the partite sets are not present between the complements. Therefore, each partite part of G becomes a clique in $\overline{G}$ which is disjoint with all the other partite parts. $\therefore \overline{G}$ is a union of cliques. $\blacksquare$

Solution to no. of cliques in $\overline{G}$: In order to identify this, we must consider our assumption of G having the maximum number of edges. In order to maximize the number of edges in G, we must minimize the number of edges in $\overline{G}$, which, for a set of vertices that are made of disjoint cliques, the number of cliques must be maximized. However, with the additional constraint that G is $K(r+1)$–free, the number of cliques must be $< r+1$, else G would contain a $K(r+1)$, $\therefore$ **the number of cliques in $\overline{G}$ is r.**

5. Prove that every n-vertex triangle-free simple graph with the maximum number of edges must be isomorphic to $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

   Proof: From 4 (c) applied to the above statement, with $r + 1 = 3$ means that for a graph G with the above constraints, $\overline{G}$ is made up of a union of 2 disjoint cliques, which means that $\overline{\overline{G}} = G$ is a complete 2-partite graph. (i.e. a complete bipartite graph). Now, I claim that to show that the number of vertices in the two partite sets will be $\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor$ it is sufficient to show that the number of vertices in the two partite sets can differ by at most 1. This is true, as for odd vertex numbers the statement is that the two partite graphs must be split into $\frac{(n-1)}{2}$ and $\frac{(n+1)}{2}$ vertices and for even, they must be split into two $\frac{n}{2}$ vertex sets.

   Proof that the number of the two vertices in the partite sets differ by at most 1: We shall do this proof by contradiction. Assume that we have a complete bipartite graph G with the maximum number of edges, and that the number of vertices in the two partite parts differ by more than one. Let the two partite sets be A and B. WLOG, we are assuming that $|V(A)| > |V(B)| + 1$ i.e. $|V(A)| - 1 > V(B)$. Then, we can construct a G' by deleting a vertex in A and adding a vertex to B. Then:

   $$|E(G')| = |E(G)| - |V(B)| + (|V(A)| - 1) > |E(G)|$$

   which contradicts our assumption of G has the maximum number of edges. $\blacksquare$