Homework #2

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1. (a) Prove or disprove (by providing a counter example) the claim that for any simple graph G, the graph and its complement have the same automorphism group.

   Proof: It holds true for the automorphism group of a graph G, that $\exists$ a bijection on the vertices of G, which we will call ϕ which satisfies the following properties, for any pair of vertices u, v in the graph:

$$\phi(u), \phi(v) \in E(G) \Leftrightarrow u, v \in E(G)$$

which is equivalent to saying:

$$\phi(u), \phi(v) \notin E(G) \Leftrightarrow u, v \notin E(G)$$

which is equivalent to saying:

$$\phi(u), \phi(v) \in E(\overline{G}) \Leftrightarrow u, v \in E(\overline{G})$$

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   (b) Express in terms of the size of Aut(G) the number of distinct labeled graphs isomorphic to a given graph G also called the number of conjugacy classes. Derive from your count bounds the number of unlabeled graphs on n vertices.

   Solution (No. of distinct labeled graphs):

$$\frac{n!}{\text{Aut}(G)}$$

   Reasoning: For a given graph G there will be $n!$ ways in which to associate a graph isomorphic to it, by merely counting the number of permutations on n vertices. However, as we care about distinct labelings, we are not going to count the labelings that preserve the exact same edge-vertex relationships, i.e. the automorphism group of G, hence we divide by it to get our answer.

   No. of unlabeled graphs on n vertices:

   Let k be the No. of unlabeled graphs on n vertices. Then:

$$\frac{2^{\binom{n}{2}}}{n!} \leq k \leq 2^{\binom{n}{2}}$$

where:

$$k = \frac{2^{\binom{n}{2}}}{n!}\text{Aut}(G)$$

Reasoning: The number of possible labeled graphs on n vertices is $2^{\binom{n}{2}}$ as there either is an edge between two vertices or not. As we want to remove labels, we are treating graphs that are isomorphic to themselves as the same graph, hence we divide that number by the number of isomorphisms, which is $\frac{n!}{\text{Aut}(G)}$. As $\text{Aut}(G)$ ranges from 1 to $n!$, we get the range for k as written above.

2. Provide a sharp upper bound for the number of connected components in a simple graph with n vertices and k edges.

   Solution (Sharp Upper bound):

$$\text{ceil}\left(n - \left(\frac{-1 + \sqrt{1 + 8k}}{2}\right)\right)$$

   i.e. The number of connected components will always be less than or equal to this number for n vertices and k edges.

   Reasoning: The maximum possible connected components for an n vertex graph is n. To look at the maximum possible connected components with k edges, for every edge we add, we try and add it to vertices that are already connected until we produce a clique. We can call the size of the clique l where l is a positive integer. The following edge will then be added to another vertex outside the clique, until we can reform the clique. The number of edges in a clique can be thought of as the number of edges in a complete graph, and hence, by handshake lemme it will be $\binom{l}{2}$ $\therefore$ we can state that

$$\frac{l(l-1)}{2} < k \leq \frac{l(l+1)}{2}$$

   The strictly greater than statement of the left side comes from the fact that we are counting upto, but not including the extra vertex being a part of the l clique, and by symmetry, that means we can apply the strict case to the left side, and the non-strict case to the right. Additionally, for a graph the number of vertices in a connected component being i will affected the number of connected components with the form

$$n - (i - 1)$$

   . Now, from our idea of cliques $i = l$ or $l + 1$. As we care about the upper bound we can fix $i = l + 1$. Then, we get the number of connected components equal to $n - l$. Solving for l from the right half of our inequality gives us the solution.

3. Prove or disprove (by providing a counter example) the claim for a simple graph G

$$\sum_{v \in V(G)} \binom{\text{degree}(v)}{2} > \binom{n}{2} \implies C_4 \subset G$$

   Proof: From the statement, we can assume that

$$\sum_{v \in V(G)} \binom{\text{degree}(v)}{2} > \binom{n}{2}$$

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Let $\binom{n}{2}$ represent the number of two vertex combinations possible in an n vertex set. Additionally, let

$$\binom{\text{degree}(v)}{2}$$

represent the number of two vertex combinations (we can all them a, b adjacent to v with $v \in V(G)$.

This implies that

$$\sum_{v \in V(G)} \binom{\text{degree}(v)}{2}$$

is the number of two vertex combinations adjacent to all the vertices $v \in V(G)$. From our assumption, we know that this is greater than the number of two vertex combinations on n vertices. Therefore, **by pigeonhole**, there will exist at least two vertices $v, u \in V(G)$ s.t. they will be adjacent to the same a, b. This is sufficient for there to be a $C_4 \in G$. ■

4. A simple connected undirected graph is regular if every vertex of the graph has the same degree. Let A denote the adjacency matrix of a simple undirected graph G. Determine in terms of the eigenvalues of A a necessary and sufficient condition for G to be both bipartite and regular. Give a detailed proof of your claim.

Solution: A graph G is bipartite and regular iff ΔG and -ΔG are eigenvalues of $\mathbf{A}$.

Proof: **(Only If case)** Assume G is bipartite and regular.

Regular case: If G is regular, then there are $\Delta(G)$ 1's in every row and column of $\mathbf{A}$. Then

$$\mathbf{A} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \Delta(G) \\ \Delta(G) \\ \vdots \\ \Delta(G) \end{bmatrix} = \Delta(G) \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

and $\therefore$ $\Delta(G)$ is an eigenvalue of $\mathbf{A}$.

Bipartite case: If G is bipartite, then it can split into two disjoint vertex sets with M and N with m and n being the number of vertices in each set respectively. Then, $\mathbf{A}$ can be written as:

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

Where the top right and bottom left part of the matrix are $m \times m$ and $n \times n$ respectively and the top left and bottom right are $m \times n$ and $n \times m$ respectively. Now, I pick an eigenvalue λ which, we can set to be $\Delta(G)$, and any associated eigenvector, call it $(\mathbf{v}, \mathbf{w})$. By definition of the eigenvector and its associated eigenvalue, this means that:

$$\mathbf{A} \cdot (\mathbf{v}, \mathbf{w}) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} B \cdot \mathbf{w} \\ B^T \cdot \mathbf{v} \end{bmatrix} = \begin{bmatrix} \lambda \cdot \mathbf{v} \\ \lambda \cdot \mathbf{w} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}$$

Then, we can construct another eigenvector $(\mathbf{v}, -\mathbf{w})$ which gives:

$$\mathbf{A} \cdot (\mathbf{v}, -\mathbf{w}) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} \\ -\mathbf{w} \end{bmatrix} = \begin{bmatrix} -B \cdot \mathbf{w} \\ B^T \cdot \mathbf{v} \end{bmatrix} = \begin{bmatrix} -\lambda \cdot \mathbf{v} \\ \lambda \cdot \mathbf{w} \end{bmatrix} = -\lambda \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}$$

i.e. for every λ, $-\lambda$ is also an eigenvalue.

If case (Both regular and bipartite): Assume that $\Delta(G)$ and $-\Delta(G)$ are both eigenvalues of $\mathbf{A}$.

Now, let us pick an eigenvector $\mathbf{v}$ associated with our eigenvalue $\Delta(G)$ and let v_k be the largest component of $\mathbf{v}$, where $1 \leq k \leq n$ for our $n \times n$ adjacency matrix. We rescale our eigenvectors so that $v_k = 1$ (i.e. divide by v_k). Then, we have:

$$|\Delta(G)| = |\Delta(G) \cdot v_k| = \sum_{i=1}^{n} |a_{ki} v_i| \leq \sum_{i=1}^{n} a_{ki} |v_i| \leq \sum_{i=1}^{n} a_{ki} |v_k| = \deg(k) \leq \Delta(G)$$

From the above equality we get that

$$|\Delta(G)| = \sum_{i=1}^{n} |a_{ki} v_i| = \Delta(G)$$

$\forall i$. Therefore every $|v_i| = 1$ and we can let $k = i$, which makes k arbitrary. This means that every vertex has degree $\Delta(G)$ and hence the graph is regular.

The above equality also gives us $\sum_{i=1}^{n} a_{ki} |v_k| = \sum_{i=1}^{n} a_{ki} |v_i|$ which means that $|v_i| = |v_k|$. Now, I claim I can construct a bipartite set, and $\therefore$ this shows that one exists. The two sets I create are $\{i | v_i < 0\}$ and $\{j | v_j < 0\}$ for $1 \leq i, j \leq n$. To show that the two sets are bipartite it is sufficient to show that every v_i is only adjacent to every v_j. This comes from:

$$\Delta(G) \cdot v_j = \sum_{i=1}^{n} a_{kj} v_j = -\sum_{i=1}^{n} a_{ki} v_i = -\sum_{i \text{ adjacent to } j} v_i$$

which is precisely the statement that the degree of any vertex in one of the sets is precisely equal to the number of vertices it is adjacent to in the other set. ■

5. (a) Let G, H denote two undirected graphs whose adjacency matrices are $\mathbf{A}_G, \mathbf{A}_H \in \{0,1\}^{n \times n}$. Give a detailed proof that

$$G \simeq H$$

iff $\exists$ a solution matrix $\mathbf{P} \in \mathbb{C}^{n \times n}$ to the following system of equations

$$\begin{cases} P(A_H)^k P^* = (A_G)^k, & k \in \{0,1\} \\ P^{\circ^2} = P \end{cases}$$

Proof (Only If case): Let us assume that

$$G \simeq H$$

This means precisely that $\exists$ a bijection ϕ. $\forall u \in V(G)$ s.t. if $\{u, v\} \in E(G)$ then $\{\phi(u), \phi(v)\} \in E(G)$. The following proof will be constructive (i.e., I will construct a $\mathbf{P}$ that satisfies the constraints of the system of equations above, which automatically will prove its existence.)

Let us begin by considering the statement involving the Hadamard product i.e.

$$P^{\circ^2} = P$$

. From the definition of the Hadamard product, if $\mathbf{P} \circ \mathbf{P} = \mathbf{P}$, then this means that for $p_{ij} \in \mathbf{P}$, $p_{ij}^2 = p_{ij}$ for $1 \leq i, j \leq n$. $\therefore$ we can conclude that $p_{ij} = \{0, 1\}$, i.e. every element of $\mathbf{P}$ is either 0 or 1.

Now, let us consider the first constraint, specifically let us consider the case when $k = 0$. Then, we get $\mathbf{PP}^* = \mathbf{I}$. We can use this to demonstrate the condition on the rows and columns this provides. Let $c_{ij} \in \mathbf{I}_{n \times n}$ for $1 \leq i, j \leq n$. Then, $p_{ij} = \delta_{ij}$ where δ_{ij} is the **Kronecker delta function** (This is by definition of an identity matrix). The definition of the Kronecker delta function is as follows:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

By matrix multiplication rules in summation form, this means that

$$\delta_{ij} = \sum_{t=1}^{n} a_{it} b_{tj}$$

where $a_{it} \in \mathbf{P}$ and $b_{tj} \in \mathbf{P}^*$. From the additional conclusion of the Hadamard product constraint (that $\mathbf{P}$ only contains 1's and 0's) this means that every row and column of P and P^* has exactly 1 1, and both must be $n \times n$ matrices.

Now, let us pick our $\mathbf{P}$ that satisfies these constraints. Let $\mathbf{P}$ be defined such that $p_{ij} = \delta_{\phi(i)j}$ for $1 \leq i, j \leq n$. Then, our proof is complete.

If case: The if case is essentially the same as the only if case run backward. We assume that there is a matrix solution to the system of equations, and that the solution is precisely $\mathbf{P}$ defined such that $p_{ij} = \delta_{\phi(i)j}$ for $1 \leq i, j \leq n$ with ϕ being some bijection on i. Then, the proof is complete, as the bijection will be precisely ϕ for the system of equation solution form shown. ■

(b) Prove that if $\exists$ a solution matrix $\mathbf{P} \in \mathbb{C}^{n \times n}$ to the matrix system of equations above then $\exists$ a solution matrix $\mathbf{Q} \in \mathbb{C}^{n \times n}$ to the following matrix system of equations

$$\begin{cases} Q(A_G)^k Q^* = (A_H)^k, & k \in \{0, 1\} \\ Q^{\circ 2} = Q \end{cases}$$

Proof: We know from Proposition 1 in class, that if the relaxation of the conditions has a solution i.e.

$$P(A_H)^k P^* = (A_G)^k, k \in \{0, 1\}$$

has a solution, then A_H and A_G have the same eigenvalues. Using this, let D be the diagonal matrix of the eigenvalues of A_G (equivalently of A_H). As A_H and A_G are symmetric matrices (due to the assumption of G and H being undirected graphs), we know that they have a spectral decomposition.

Then, let $A_G = UDU^*$ with $I = UU^*$. This is precisely letting

$$(A_G)^k = U(D)^k U^*, k \in \{0, 1\}$$

for matrices U, U^*.

Similarly, we let

$$(A_H)^k = V(D)^k V^*, k \in \{0, 1\}$$

Now, we let $Q = VU^*$ (this is just writing a matrix as the product of two other matrices). Then $Q^* = UV^*$ (by conjugate transposition of matrix product rules). This gives us:

$$Q(A_G)^k Q^* = VU^*U(D)^k U^*UV^*, k \in \{0, 1\}$$

$$\implies Q(A_G)^k Q^* = VI(D)^k IV^*, k \in \{0, 1\}$$

$$\implies Q(A_G)^k Q^* = V(D)^k V^*, k \in \{0, 1\}$$

$$\implies Q(A_G)^k Q^* = A_H, k \in \{0, 1\}$$

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(c) Let

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, A_H = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

How many distinct matrix solutions $\mathbf{P} \in \mathbb{C}^{n \times n}$ are there in the system

$$\begin{cases} P(A_H)^k P^* = (A_G)^k, & k \in \{0, 1\} \\ P^{\circ 2} = P \end{cases}$$

How many distinct solutions are there to the relaxation problem

$$P(A_H)^k P^* = (A_G)^k, k \in \{0, 1\}$$

Solution: For the original strict case, there are 0 distinct matrix solutions. For the relaxation problem there are 2^7 distinct matrix solutions.

Reasoning: The original strict case has no solutions due to the fact that the two graphs will not be isomorphic. This is clear as we can see that ∃ as row and column in A_H with 51's present, which directly corresponds to a vertex with degree 5. However, no such row or column exists in A_G, i.e. there are vertices with different degree, which means that no bijection can be produced between the two graphs that preserves the number of edges adjacent to a vertex for all the vertices between the graphs i.e., the two graphs are not isomorphic, and from part (a), we can conclude that therefore no matrix solution exists to the strict case.

Relaxation case reasoning: From the relaxed solution, we know we can let $P = VU^*$, with V^* representing the eigenvectors associated with the solutions, if we find the unique eigenvectors associated with a matrix solution, we find that solution. We know that a positive set of eigenvectors corresponding to an eigenvalue provide a solution, and the negative form corresponds to another P matrix. Hence, each eigenvalue provides 2 solutions. For our 6×6 matrices, we get 2^6 solutions. Computing eigenvalues, we see that one of the eigenvalues has multiplicity 2, hence the total number of solutions is 2^7.