Q3: Belief propagation

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1 Q3.i

Initially the likelihood of W and H will be $(1, 1)$ as both of them are leaf node with no children. Therefore, initial condition of the belief propagation algorithm would be:

Table 1: Initial condition

X	$BEL(X)$	$\pi(X)$	$\lambda(X)$
R	-	(0.2,0.8)	-
S	-	(0.1,0.9)	-
W	-	-	(1,1)
H	-	-	(1,1)

As in the first step, W and H will pass $(1, 1)$ to their parents,

$$\lambda(R) = \lambda(W)\lambda(H) = (1, 1).$$

Similarly, $\lambda(S)$ will be $(1, 1)$. Now, we can compute $BEL(R)$ and $BEL(S)$ as follows:

$$BEL(R) = \lambda(R).\pi(R) = (0.2, 0.8)$$
$$BEL(S) = \lambda(S).\pi(S) = (0.1, 0.9).$$

Therefore, we arrive at the following:

Table 2: Step 1

X	$BEL(X)$	$\pi(X)$	$\lambda(X)$
R	(0.2,0.8)	(0.2,0.8)	(1,1)
S	(0.1,0.9)	(0.1,0.9)	(1,1)
W	-	-	(1,1)
H	-	-	(1,1)

Once, we calculate the belief of the parents, it can be propagated to the children i.e. W and H to update their priors. Then, we have the following:

$$\begin{aligned}
\pi_W(R) &= \frac{BEL(R)}{\lambda_W(R)} \\
&= \frac{(0.2, 0.8)}{(1, 1)} \\
&= (0.2, 0.8).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\pi_W = (&P(W = T|R = T) * 0.2P(W = T|R = F) * 0.8, \\
&P(W = F|R = T) * 0.2 + P(W = F|R = F) * 0.8) \\
&= ((1)(0.2) + (0.2)(0.8), (0)(0.2) + (0.8)(0.8)) \\
&= (0.36, 0.64).
\end{aligned}$$

and,

$$\begin{aligned}
\pi_H(R) &= \frac{BEL(R)}{\lambda_H(R)} \\
&= (0.2, 0.8) \\
\pi_H(S) &= \frac{BEL(S)}{\lambda_H(S)} \\
&= (0.1, 0.9).
\end{aligned}$$

Hence,

$$\begin{aligned}
\pi_H = (&P(H = T|R = T, S = T) * 0.2 * 0.1 + P(H = T|R = F, S = T) * 0.8 * 0.1 \\
&+ P(H = T|R = T, S = F) * 0.2 * 0.9 + P(H = T|R = F, S = F) * 0.8 * 0.9, \\
&P(H = F|R = T, S = T) * 0.2 * 0.1 + P(H = F|R = F, S = T) * 0.8 * 0.1, \\
&+ P(H = F|R = T, S = F) * 0.2 * 0.9 + P(H = F|R = F, S = F) * 0.8 * 0.9) \\
&= (0.02 + 0.072 + 0.18, 0.008 + 0.72) \\
&= (0.272, 0.728).
\end{aligned}$$

Therefore, after this step we arrive at the following:

Table 3: Step 2

X	$BEL(X)$	$\pi(X)$	$\lambda(X)$
R	(0.2,0.8)	(0.2,0.8)	(1,1)
S	(0.1,0.9)	(0.1,0.9)	(1,1)
W	-	(0.36,0.64)	(1,1)
H	-	(0.272,0.728)	(1,1)

Next, we can calculate the belief of W and H as follows:

$$BEL(W) = ((0.36)(1), (0.64)(1)) = (0.36, 0.64)$$
$$BEL(H) = ((0.272)(1), (0.728)(1)) = (0.272, 0.728).$$

Thus we can update the table as follows:

Table 4: Step 3

X	$BEL(X)$	$\pi(X)$	$\lambda(X)$
R	(0.2,0.8)	(0.2,0.8)	(1,1)
S	(0.1,0.9)	(0.1,0.9)	(1,1)
W	(0.36,0.64)	(0.36,0.64)	(1,1)
H	(0.272,0.728)	(0.272,0.728)	(1,1)

Therefore, the belief that Watson's grass is wet is 0.36.

2 Q3.ii

As calculated previously, the belief that Holmes's grass is wet is 0.272.

3 Q3.iii

If Holmes observes that his grass is wet, we need to update the table as follows:

Table 5: Step 3

X	$BEL(X)$	$\pi(X)$	$\lambda(X)$
R	(0.2,0.8)	(0.2,0.8)	(1,1)
S	(0.1,0.9)	(0.1,0.9)	(1,1)
W	(0.36,0.64)	(0.36,0.64)	(1,1)
H	(1,0)	(0.272,0.728)	(1,0)

This message has to be passed on to R and S. Using the formula

$$\lambda_X(U_i) = \sum_X \lambda(X) \sum_{U_k, k \neq i} P(X|U_1, \ldots, U_k) \prod_{k \neq i} \pi_X(U_k),$$

and following the same procedure as shown in Q3.i, we can compute the following table:

Table 6: Step 4

X	$BEL(X)$	$\pi(X)$	$\lambda(X)$
R	(0.74,0.26)	(0.2,0.8)	(1,0.09)
S	(0.338,0.662)	(0.1,0.9)	(0.92,0.2)
W	(0.788,0.212)	(0.788,0.212)	(1,1)
H	(1,0)	(0.272,0.728)	(1,0)

Therefore, the belief that Watson's grass is wet is 0.788.

4 Q3.iii

Now, if Holmes observes that Watson's grass is also wet as well as his own, then we need to update the belief of node W as well and the table is shown below:

Table 7: Step 4

X	$BEL(X)$	$\pi(X)$	$\lambda(X)$
R	(0.74,0.26)	(0.2,0.8)	(1,0.09)
S	(0.338,0.662)	(0.1,0.9)	(0.92,0.2)
W	(1,0)	(0.788,0.212)	(1,0)
H	(1,0)	(0.272,0.728)	(1,0)

Then, this message has to be passed on to R and we need to update the prior $\pi(R)$. We can use the following formulas for these:

$$\lambda_X(U_i) = \sum_X \lambda(X) \sum_{U_k, k \neq i} P(X|U_1, \ldots, U_k) \prod_{k \neq i} \pi_X(U_k),$$

$$\pi(X) = \sum_{U_1, \ldots, U_k} P(X|U_1, \ldots, U_k) \prod_{i=1}^{n} \pi_X(U_i).$$

Then we can compute the beliefs using $BEL(X) = \lambda(X)\pi(X)$ as we did in Q3.i. We can arrive at the following:

$$BEL(s) = (0.16, 0.84).$$

Therefore, the belief that the sprinkler was on is 0.16.