HW 4

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Problem 1.a

It is given that

$$\hat{y} = \sin\phi.$$

If $\phi \sim U[0,\pi]$ then pdf of ϕ is given by

$$f_\Phi(\phi) = \frac{1}{\pi}.$$

Now, when $\phi \in [0, \frac{\pi}{2}]$,

$$\begin{aligned} f_{\hat{Y}}(\hat{y}) &= \frac{f_\Phi(\phi)}{|\frac{\partial \hat{y}}{\partial \phi}|} \\ &= \frac{\frac{1}{\pi}}{\cos\phi}. \end{aligned}$$

Therefore, $\cos\phi = \frac{1}{\pi f_{\hat{Y}}(\hat{y})}$. Hence,

$$\begin{aligned} &\cos^2\phi + \sin^2\phi = 1 \\ \implies &\frac{1}{\pi^2 f_{\hat{Y}^2(\hat{y})}} + \hat{y}^2 = 1 \\ \implies &\frac{1}{f_{\hat{Y}(\hat{y})}} = \pi\sqrt{1-\hat{y}^2} \\ \implies &f_{\hat{Y}(\hat{y})} = \frac{1}{\pi\sqrt{1-\hat{y}^2}} \end{aligned}$$

We can find a similar solution when $\phi \in [\frac{\pi}{2}, \pi]$, Therefore,

$$f_{\hat{Y}}(\hat{y}) = \begin{cases} \frac{2}{\pi\sqrt{1-\hat{y}^2}}, & \text{if } y \in [0,1] \\ 0, & \text{otherwise} . \end{cases}$$

Problem 1.b

$$\begin{aligned} E[\hat{Y}] &= \int_{-\infty}^{\infty} \hat{y} f_{\hat{Y}}(\hat{y}) \mathrm{d}\hat{y} \\ &= \int_0^1 \frac{2\hat{y}}{\pi\sqrt{1-\hat{y}^2}} \mathrm{d}\hat{y}. \end{aligned}$$

Let $k^2 = 1 - \hat{y}^2$. Then, $\hat{y}\mathrm{d}\hat{y} = -k\mathrm{d}k$. So,

$$\begin{aligned} E[\hat{Y}] &= \frac{2}{\pi}\int_0^1 \mathrm{d}k \\ &= \frac{2}{\pi} \end{aligned}$$

$$= 0.6366.$$

Now,

$$\begin{aligned}
E[\hat{Y}^2] &= \int_{-\infty}^{\infty} \hat{y}^2 f_{\hat{Y}}(\hat{y})\mathrm{d}\hat{y} \\
&= \frac{2}{\pi}\int_0^1 \frac{\hat{y}^2}{\sqrt{1-\hat{y}^2}}\mathrm{d}\hat{y}.
\end{aligned}$$

Let $\hat{y} = \sin k$, then $\mathrm{d}\hat{y} = \cos k \mathrm{d}k$. Therefore,

$$\begin{aligned}
\int \frac{\hat{y}^2}{\sqrt{1-\hat{y}^2}}\mathrm{d}\hat{y} &= \int \frac{\sin^2 k}{\cos k}\cos k \mathrm{d}k \\
&= \int \frac{1}{2}(1-\cos(2k))\mathrm{d}k \\
&= \frac{k}{2} - \frac{1}{4}\int 2\cos(2k)\mathrm{d}k + C_1 \\
&= \frac{k}{2} - \frac{1}{4}\sin(2k) + C_2 \\
&= \frac{k}{2} - \frac{1}{2}\sin k \cos k + C_2 \\
&= \frac{1}{2}(\sin^{-1}\hat{y}) - \frac{1}{2}\hat{y}\sqrt{1-\hat{y}^2} + C_2.
\end{aligned}$$

Considering the limits of the integral, we get,

$$E[\hat{Y}^2] = \frac{2}{\pi}\frac{1}{2}\left[\frac{\pi}{2}\right] = \frac{1}{2}.$$

Therefore,

$$\begin{aligned}
Var(\hat{Y}) &= E[\hat{Y}^2] - (E[\hat{Y}])^2 \\
&= \frac{1}{2} - \frac{4}{\pi^2} \\
&= \frac{\pi^2 - 8}{2\pi^2} \\
&= 0.0947.
\end{aligned}$$

Problem 1.c

From the Monte Carlo simulation I get the following:

$$\begin{aligned}
E[\hat{Y}_{mc}] &= 0.635 \\
\mathrm{Var}(\hat{Y}_{mc}) &= 0.096.
\end{aligned}$$

Fig. 1 shows the plot of analytically computed PDF of $\hat{Y}$ and PDF histogram of $\hat{Y}$ from Monte Carlo simulation. The figure reveals that both analytical and Monte Carlo results are similar. Also, the mean and variance of $\hat{Y}$ from Monte Carlo are close enough to analytical results.

Problem 1.d

I have used Monte Carlo simulation for this part. When $\phi \sim \mathcal{N}(0,1)$, from Monte Carlo simulation, I get the following:

$$E[\hat{Y}_{mc}] = 0.007$$

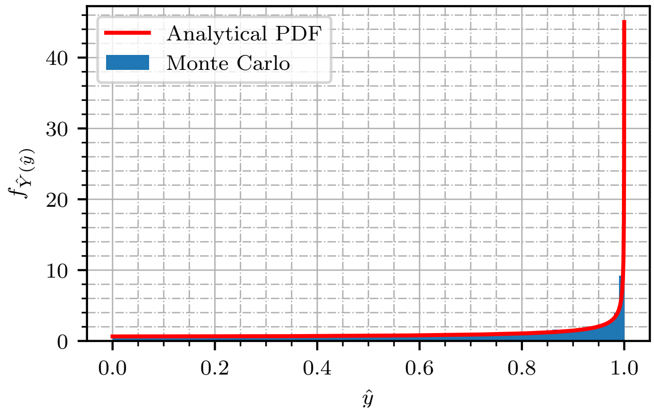

Figure 1: Q1.c: Analytical vs. Monte Carlo PDF of $\hat{Y}$ when $\phi \sim U[0, \pi]$

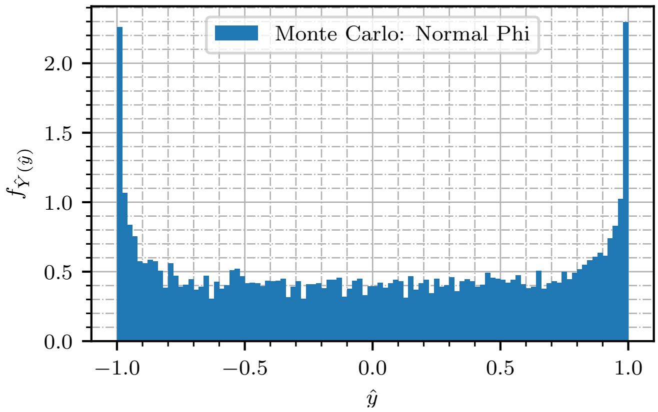

Figure 2: Q1.c: Monte Carlo PDF of $\hat{Y}$ when $\phi \sim \mathcal{N}(0, 1)$

$$\mathrm{Var}(\hat{Y}_{mc}) = 0.432.$$

The histogram based PDF of $\hat{Y}$ is shown in Fig. 2, when $\phi \sim \mathcal{N}(0, 1)$

I used Monte Carlo approach for this problem as calculating the PDF of $\hat{Y}$ under the non-linear transformation when $\phi \sim \mathcal{N}(0, 1)$ is much harder and even deducing the mean of variance of it is much harder compared to Monte Carlo simulation. This is an example of how non-linear transformation of random variables result into very difficult analytical expression.

Problem 2.a

It is given that

$$y_k = \sqrt{x}(1 + v_k)$$
$$v_k \sim \mathcal{N}(0, R).$$

Therefore,

$$E[v_k] = 0$$
$$E[v_k^2] = R$$
$$E[v_k^3] = 0$$
$$E[v_k^4] = 3R^2.$$

If we set $\hat{x}_k = y_k^2$, then the mean of the estimation error:

$$E[x - \hat{x}_k] = E[x - x(1 + v_k)^2]$$

$$
\begin{aligned}
&= xE[1-(1+v_k)^2] \\
&= xE[1-1-2v_k-v_k^2] \\
&= x(-2E[v_k]-E[v_k^2]) \\
&= -xR.
\end{aligned}
$$

The variance of the estimation error is:

$$
\begin{aligned}
E[(x-\hat{x}_k)(x-\hat{x}_k)^T] &= E[(x-\hat{x}_k)^2] \\
&= E[(x-x(1+v_k)^2)^2] \\
&= E[x^2(1-(1+v_k)^2)^2] \\
&= E[x^2(1-1-2v_k-v_k^2)^2] \\
&= E[x^2v_k^2(2+v_k)^2] \\
&= E[x^2v_k^2(4+v_k^2+4v_k)] \\
&= E[4x^2v_k^2+x^2v_k^4+4x^2v_k^3] \\
&= 4x^2R+3x^2R^2.
\end{aligned}
$$

Problem 2.b

If we set

$$
\hat{x}_k = \frac{1}{k}\sum_{i=1}^{k} y_i^2,
$$

then the mean of estimation error becomes

$$
\begin{aligned}
E[x-\hat{x}_k] &= E\left[x-\frac{1}{k}\sum_{i=1}^{k} y_i^2\right] \\
&= x-\frac{1}{k}\sum_{i=1}^{k} E[y_i^2] \\
&= x-\frac{1}{k}\sum_{i=1}^{k} E[x(1+v_i)^2] \\
&= x-\frac{1}{k}\sum_{i=1}^{k} xE[1+2v_i+v_i^2] \\
&= x-\frac{1}{k}\sum_{i=1}^{k} x(1+R) \\
&= x-x(1+R) \\
&= -xR.
\end{aligned}
$$

To compute the variance of estimation error, we need to find out $E[(x-\hat{x}_k)^2]$. Now,

$$
\begin{aligned}
E[(x-\hat{x}_k)^2] &= E[x^2+\hat{x}_k^2-2x\hat{x}_k] \\
&= x^2-2xE[\hat{x}_k]+E[\hat{x}_k^2] \\
&= x^2-2x(x(1+R))+E[\hat{x}_k^2] \\
&= x^2-2x^2(1+R)+E[\hat{x}_k^2] \qquad (1)
\end{aligned}
$$

Now,

$$
E[\hat{x}_k^2] = E\left[\left(\frac{1}{k}\sum_{i=1}^{k} y_i^2\right)^2\right]
$$

$$
\begin{aligned}
&= E\left[\left(\frac{1}{k}\sum_{i=1}^{k} x(1+v_i)^2\right)^2\right] \\
&= E\left[\frac{x^2}{k^2}\left(\sum_{i=1}^{k}(1+v_i)^2\right)^2\right] \\
&= E\left[\frac{x^2}{k^2}\left(\sum_{i=1}^{k}(1+v_i)^4 + 2\sum_{i=1}^{k}\sum_{j=i,j\neq i}^{k}(1+v_i)^2(1+v_j)^2\right)\right] \\
&= \frac{x^2}{k^2}\sum_{i=1}^{k} E\left[(1+v_i)^4\right] + \frac{2x^2}{k^2}\sum_{i=1}^{k}\sum_{j=i,j\neq i}^{k} E\left[(1+v_i)^2(1+v_j)^2\right] \\
&= \frac{x^2}{k^2}\sum_{i=1}^{k} E\left[1+4v_i+6v_i^2+4v_i^3+v_i^4\right] \\
&+ \frac{2x^2}{k^2}\sum_{i=1}^{k}\sum_{j=i,j\neq i}^{k} E\left[1+2v_j+2v_i+4v_iv_j+2v_iv_j^2+2v_i^2v_j+v_i^2+v_j^2+v_i^2v_j^2\right] \\
&= \frac{x^2}{k^2}\sum_{i=1}^{k}(1+6R+3R^2) + \frac{2x^2}{k^2}\sum_{i=1}^{k}\sum_{j=i,j\neq i}^{k}(1+R+R+R^2) \\
&= \frac{x^2}{k^2}k(1+6R+3R^2) + \frac{2x^2}{k^2}\sum_{i=1}^{k}\sum_{j=i,j\neq i}^{k}(1+2R+R^2) \\
&= \frac{x^2}{k^2}\left(k+6kR+3kR^2+2\binom{k}{2}(1+2R+R^2)\right) \\
&= \frac{x^2}{k^2}\left(k+6kR+3kR^2+k(k-1)(1+2R+R^2)\right)
\end{aligned}
$$

Therefore, from (1),

$$
E[(x-\hat{x}_k)^2] = x^2 - 2x^2(1+R) + \frac{x^2}{k^2}\left(k+6kR+3kR^2+k(k-1)(1+2R+R^2)\right).
$$

To prove that we get the same value as derived in part (a), put $k=1$ above. Then, we get, $E[(x-\hat{x}_k)^2] = x^2 - 2x^2(1+R) + x^2(1+6R+3R^2) = -2x^2R + 6x^2R + 3x^2R^2 = 4x^2R + 3x^2R^2$. Also,

$$
\begin{aligned}
\lim_{k\to\infty} E[(x-\hat{x}_k)^2] &= x^2 - 2x^2(1+R) + x^2(1+2R+R^2) \\
&= x^2R^2.
\end{aligned}
$$

Problem 3

x is uniformly distributed on $[-1,1]$. Therefore, the PDF of x is:

$$
f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } x\in[-1,1] \\ 0, & \text{otherwise}. \end{cases}
$$

Therefore, the expected value of $y = e^x$ is given by

$$
\begin{aligned}
\bar{y} = E[y] &= \int_{-\infty}^{\infty} e^x f_X(x)\mathrm{d}x \\
&= \frac{1}{2}\int_{-1}^{1} e^x \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left[e - \frac{1}{e}\right] \\
&= 1.175.
\end{aligned}
$$

The mean of x is $\bar{x} = \frac{1}{2}(1-1) = 0$ and the covariance of x is:

$$
\begin{aligned}
P &= E[xx^T] \\
&= E[x^2] \\
&= \int_{-1}^{1} x^2 f_X(x)\mathrm{d}x \\
&= \frac{1}{2}\int_{-1}^{1} x^2 \mathrm{d}x \\
&= \frac{1}{3}.
\end{aligned}
$$

First, we form two sigma points,

$$
\begin{aligned}
x^{(1)} &= \sqrt{P} = \frac{1}{\sqrt{3}} \\
x^{(2)} &= -\sqrt{P} = -\frac{1}{\sqrt{3}}.
\end{aligned}
$$

Then we calculate the non-linear transformation of the sigma points,

$$
\begin{aligned}
y^{(1)} &= e^{x^{(1)}} = 1.781 \\
y^{(2)} &= e^{x^{(2)}} = 0.5614.
\end{aligned}
$$

Therefore, the approximated mean of y is given by,

$$
\begin{aligned}
\bar{y}_u &= \frac{1}{2}\sum_{i=1}^{2} y^{(i)} \\
&= \frac{1}{2}[1.781 + 0.5614] \\
&= 1.1712.
\end{aligned}
$$

Problem 4.a

Maximum likelihood estimate of x is given by $\text{argmax}_x pdf(x)$. The pdf of x is given by

$$
pdf(x) = \begin{cases} 1 - \frac{x}{2}, & \text{if } x \in [0,2] \\ 0, & \text{otherwise .} \end{cases}
$$

We can see that $pdf(x)$ takes the maximum value when $x = 0$. Therefore, the MLE of x is, $\hat{x}_{MLE} = 0$.

Problem 4.b

The min-max estimate is given by $\text{argmin}_{\hat{x}}(\max|x - \hat{x}|)$. Therefore, as pdf is 0 if $x \notin [0,2]$, the min-max estimate of x is given by $\frac{1}{2}(0+2) = 1$.

Problem 4.c

$$
E[(x-\hat{x})^2] = \int_{-\infty}^{\infty} (x - \hat{x})^2 p(x)\mathrm{d}x
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} x^2 p(x)\mathrm{d}x - 2\hat{x}\int_{-\infty}^{\infty} xp(x)\mathrm{d}x + \hat{x}^2\int_{-\infty}^{\infty} p(x)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} x^2 p(x)\mathrm{d}x - 2\hat{x}\int_{-\infty}^{\infty} xp(x)\mathrm{d}x + \hat{x}^2
\end{aligned}
$$

Now, suppose $\hat{x}_{min}$ minimizes the above function. Then,

$$
\begin{aligned}
&\frac{\partial E[(x-\hat{x})^2]}{\partial \hat{x}} = 0 \\
\implies &\int_{-\infty}^{\infty} xp(x)\mathrm{d}x = \hat{x}_{min} \\
\implies &\hat{x}_{min} = E[x].
\end{aligned}
$$

Now,

$$
\begin{aligned}
E[x] &= \int_{-\infty}^{\infty} xp(x)\mathrm{d}x \\
&= \int_0^2 (1-\frac{x}{2})\mathrm{d}x \\
&= \left[x - \frac{x^2}{4}\right]_0^2 \\
&= 1.
\end{aligned}
$$

Therefore, $\hat{x}_{min} = 1$.

Problem 4.d

$$
\begin{aligned}
E[x] &= \int_{-\infty}^{\infty} xp(x)\mathrm{d}x \\
&= \int_0^2 (1-\frac{x}{2})\mathrm{d}x \\
&= \left[x - \frac{x^2}{4}\right]_0^2 \\
&= 1.
\end{aligned}
$$

Therefore, $\hat{x} = 1$.

Problem 5.a

The scalar system is given by

$$
\begin{aligned}
x_{k+1} &= x_k + w_k,\ w_k \sim U[-1,1] \\
y_k &= x_k + v_k,\ v_k \sim U[-1,1] \\
x_0 &\sim U[-1,1].
\end{aligned}
$$

Therefore, the pdf of x_1 given Y_0 is

$$
\begin{aligned}
p(x_1|Y_0) &= \int_{-\infty}^{\infty} p(x_1|x_0)p(x_0|Y_0)\mathrm{d}x_0 \\
&= \int_{-1}^{1} p(x_1|x_0)p(x_0)\mathrm{d}x_0 \qquad [\because x_0 \sim U[-1,1]] \\
&= \frac{1}{2}\int_{-1}^{1} p(x_1|x_0)\mathrm{d}x_0.
\end{aligned}
$$

Now,

$$\begin{aligned} x_1 &= x_0 + w_0 \\ \implies w_0 &= x_1 - x_0. \end{aligned}$$

and,

$$p(w_0) = p(x_1 - x_0) = p(x_1|x_0) \sim U[-1,1] = \begin{cases} \frac{1}{2}, & \text{if } -1 \le x_1 - x_0 \le 1 \\ 0, & \text{otherwise }. \end{cases}$$

Therefore, if, $2 \ge x_1 \ge 0$, then

$$\begin{aligned} p(x_1|Y_0) &= \frac{1}{2}\int_{-1+x_1}^{1} \frac{1}{2}\mathrm{d}x_0 \\ &= \frac{1}{4}(2 - x_1). \end{aligned}$$

If $-2 \le x_1 < 0$, then,

$$\begin{aligned} p(x_1|Y_0) &= \frac{1}{2}\int_{-1}^{1+x_1} \frac{1}{2}\mathrm{d}x_0 \\ &= \frac{1}{4}(x_1 + 2). \end{aligned}$$

Therefore,

$$p(x_1|Y_0) = \begin{cases} \frac{1}{4}(2 - x_1), & \text{if } 0 \le x_1 \le 2 \\ \frac{1}{4}(x_1 + 2), & \text{if } -2 \le x_1 < 0 \\ 0, & \text{otherwise }. \end{cases} \tag{2}$$

Now,

$$p(x_1|Y_1) = \frac{p(y_1|x_1)p(x_1|Y_0)}{p(y_1|Y_0)}.$$

Now,

$$p(y_1|x_1) = p(v_1) = p(y_1 - x_1) = \begin{cases} \frac{1}{2}, & \text{if } -1 \le y_1 - x_1 \le 1 \\ 0, & \text{otherwise }. \end{cases}$$

It is given that $y_1 = 1$. Therefore,

$$p(y_1|x_1) = \begin{cases} \frac{1}{2}, & \text{if } 0 \le x_1 \le 2 \\ 0, & \text{otherwise }. \end{cases}$$

Also,

$$\begin{aligned} p(y_1|Y_0) &= \int_{-\infty}^{\infty} p(y_1|x_1)p(x_1|Y_0)\mathrm{d}x_1 \\ &= \int_0^2 \frac{1}{2}p(x_1|Y_0)\mathrm{d}x_1 \\ &= \frac{1}{2}\int_0^2 \frac{1}{4}(2 - x_1)\mathrm{d}x_1 \qquad [\text{ from } (2)] \\ &= \frac{1}{8}[4 - 2] = \frac{1}{4}. \end{aligned}$$

Therefore,

$$p(x_1|Y_1) = \begin{cases} \frac{\frac{1}{2}\frac{1}{4}(2 - x_1)}{\frac{1}{4}} = 1 - \frac{x_1}{2}, & \text{if } 0 \le x_1 \le 2 \\ 0, & \text{otherwise }. \end{cases}$$

Problem 5.b

In this problem, $Q_k = E[w_k w_k^T] = \frac{1}{3}$ and $R_k = E[v_k v_k^T] = \frac{1}{3}$. Also $F = H = 1$.
We can find the Kalman filter estimate of $\hat{x}_1^+$ in the following way:

$$\begin{aligned}
\hat{x}_0^+ &= E[x_0] = 0 \qquad [\because x_0 \sim U[-1,1]] \\
P_0^+ &= E[x_0^2] = \frac{1}{3} \\
P_1^- &= F P_0^+ F^T + Q = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \\
K_1 &= P_1^- H^T (H P_1^- H^T + R)^{-1} = \frac{2}{3} \\
\hat{x}_1^- &= F \hat{x}_0^+ = 0 \\
\hat{x}_1^+ &= \hat{x}_1^- + K_1(y_1 - H\hat{x}_1^-) = 0 + \frac{2}{3}(1-0) = \frac{2}{3}.
\end{aligned}$$

Kalman filter estimate of $\hat{x}^+$ is indicative of MAP estimate of the pdf of $p(x_1|Y_1)$. We can calculate it theoretically and find that it is close enough to the Kalman filter estimate.