# HW 3

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## Problem 1

**Problem 4.4.a** The simplified motor model can be written as

$$\dot{\theta} = \omega$$
$$\dot{\omega} = u + w_1,$$

where $\theta$ is the angular position, $\omega$ is the angular velocity of the shaft, $u$ is the control input and $w_1$ is the acceleration noise. The state-space model can be written in the matrix form as follows:

$$\begin{aligned}\underbrace{\begin{bmatrix}\dot{\theta}\\ \dot{\omega}\end{bmatrix}}_{\dot{x}} &= \underbrace{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\theta\\ \omega\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix}0\\ w_1\end{bmatrix}}_{w}\\ y &= \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C} x + v,\end{aligned} \tag{1}$$

where the scalar $v$ is the measurement noise. We know that the solution for the states, considering $\Delta t = t_{k+1} - t_k$, and $u(t_k) = u_k$ constant in $[t_k, t_{k+1}]$, in the discretized model is:

$$x_{k+1} = \underbrace{e^{A\Delta t}}_{F_k} x_k + \underbrace{\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} B \mathrm{d}\tau\, u_k}_{G_k} + \underbrace{\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} w(\tau) \mathrm{d}\tau}_{W_k}.$$

Now, eigen values of $A$ are $0, 0$ and corresponding eigen vectors are $[1\ 0]^T, [0\ 1]^T$. Therefore, we can form a matrix $Q$ taking the eigen vectors and compute $e^{A\Delta t}$ as follows:

$$e^{A\Delta t} = Q e^{\hat{A}\Delta t} Q^{-1},$$

where $Q = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I$, $\hat{A}$ is the jordan form of $A$, which in this case is same as $A$. Therefore,

$$\begin{aligned}F_k &= e^{A\Delta t}\\ &= Q e^{\hat{A}\Delta t} Q^{-1}\\ &= e^{\hat{A}\Delta t}\\ &= \begin{bmatrix}1 & \Delta t\\ 0 & 1\end{bmatrix}.\end{aligned}$$

Also,

$$\begin{aligned}G_k &= F_k \int_0^{\Delta t} e^{-A\Delta \tau} \mathrm{d}\tau B\\ &\approx F_k \left[I\Delta t - \frac{(\Delta t)^2}{2} A\right] B\\ &= F_k \left(\begin{bmatrix}\Delta t & 0\\ 0 & \Delta t\end{bmatrix} - \begin{bmatrix}0 & \frac{(\Delta t)^2}{2}\\ 0 & 0\end{bmatrix}\right) B\end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta t & -\frac{(\Delta t)^2}{2} \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \Delta t & \frac{(\Delta t)^2}{2} \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(\Delta t)^2}{2} \\ \Delta t \end{bmatrix}.
\end{aligned}
$$

Now, from the discretized model $x_{k+1} = F_k x_k + G_k + W_k$, we can see that,

$$
\begin{aligned}
x_{k+2} &= F_{k+1} \left( F_k x_k + G_k u_k + W_k \right) + W_{k+1} \\
&= (F_k)^2 x_k + F_k G_k u_k + F_k W_k + W_{k+1}.
\end{aligned}
$$

Similarly, we can see that the multiple state transition matrix is given by,

$$
\begin{aligned}
F_{k+i} &= (F_k)^i \\
&= e^{iA\Delta t} \\
&= \begin{bmatrix} 1 & i\Delta t \\ 0 & 1 \end{bmatrix}.
\end{aligned}
$$

**4.4.b** We know that,

$$
P_k = F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1}.
$$

Therefore, for a fixed noise covariance, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,

$$
\begin{aligned}
P_1 &= F_0 P_0 F_0^T + Q \\
&= e^{A\Delta t} P_0 e^{A^T \Delta t} + Q.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
P_2 &= F_1 P_1 F_1^T + Q \\
&= e^{A\Delta t} \left( e^{A\Delta t} P_0 e^{A^T \Delta t} + Q \right) e^{A^T \Delta t} + Q \\
&= e^{2A\Delta t} P_0 e^{2A^T \Delta t} + e^{A\Delta t} Q e^{A^T \Delta t} + Q \\
&= e^{2A\Delta t} P_0 e^{2A^T \Delta t} + \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} + Q \\
&= e^{2A\Delta t} P_0 e^{2A^T \Delta t} + Q + Q \\
&= e^{2A\Delta t} P_0 e^{2A^T \Delta t} + 2Q.
\end{aligned}
$$

Similarly, we can proceed further and derive that,

$$
\begin{aligned}
P_k &= e^{kA\Delta t} P_0 e^{kA^T \Delta t} + kQ \\
&= \begin{bmatrix} 1 & k\Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ k\Delta t & 1 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} k+1 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

## Problem 2

The dynamic system in this problem is given by,

$$\begin{aligned}\dot{x} &= w\\ \mathbb{E}[w] &= 0\\ \mathbb{E}[w(t)w(\tau)] &= Q_c\delta(t-\tau)\\ Q_c &= 1.\end{aligned}$$

**2.a** The dynamic system equation can be written as:

$$\dot{x} = 0x + 0u + w.$$

Therefore, $A = 0$ and $B = 0$.

**2.b** $F$ and $G$ matrices of the discretized system can be derived as:

$$\begin{aligned}F &= e^{AT}\\ &= 1,\end{aligned}$$

where $T$ is the sampling interval equal to 1 sec. Also,

$$\begin{aligned}G &= e^{AT}\int_0^T e^{-A\alpha}\mathrm{d}\alpha B\\ &= 0.\end{aligned}$$

**2.c** The covariance of discrete process noise is:

$$\begin{aligned}Q &= \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k} e^{A(t_k-\tau)}\mathbb{E}[w(\tau)w(\alpha)]e^{A^T(t_k-\alpha)}\mathrm{d}\tau\mathrm{d}\alpha\\ &= \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k} Q_c\delta(\tau-\alpha)\mathrm{d}\tau\mathrm{d}\alpha\\ &= \int_{t_{k-1}}^{t_k} Q_c\mathrm{d}\tau\\ &= Q_c(t_k - t_{k-1})\\ &= Q_cT\\ &= T\\ &= 1.\end{aligned}$$

**2.d** The plot from the Monte-Carlo simulation is shown in Fig. 1. Ensemble standard deviation at $t_k = 5, 25$ and 50 are $2.044, 4.99$ and 6.49 respectively.

**2.e** The plot from the Monte-Carlo simulation is shown in Fig. 2. Values of $\sigma$ at $t_k = 5$ and 10 are 2.23 and 3.16 respectively.

**2.f** If I do not have any historical state measurements in hand, I would go for covariance analysis because it gives the theoretical state estimate standard deviation while to prove that Monte-Carlo simulation gives an accurate prediction of covariance, I need to run the simulation for many time, ideally infinite times. Usually, we go for Monte-Carlo where it is not possible to calculate the quantity theoretically. However, If I have historical state measurements in hand, I can check for any outliers or modeling error by doing Monte-Carlo analysis and compare that with the historical measurements.

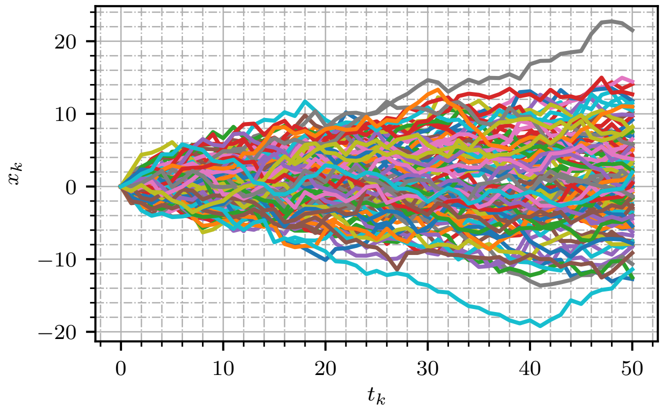


Figure 1: Q2.d: Monte-Carlo simulation of the dynamic process

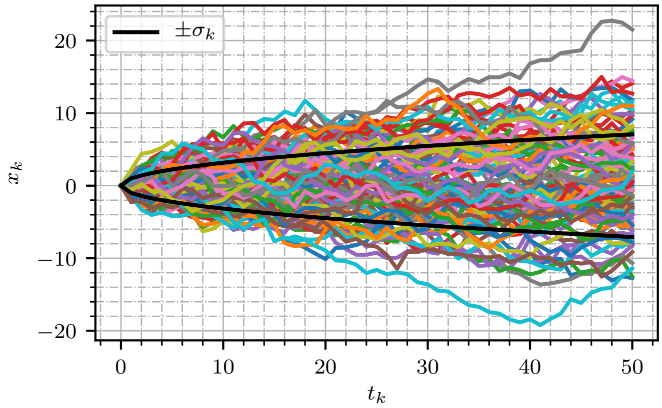


Figure 2: Q2.e: Monte-Carlo simulation along with plot from covariance analysis

## Problem 3

**Problem 5.9.a** The system equations are:

$$\begin{aligned} x_{k+1} &= x_k \\ y_k &= x_k + v_k \\ v_k &\sim (0, R). \end{aligned}$$

We are also given that $\mathbb{E}[x_0^2] = 1$ *i.e.* $P_1^- = 1$. From the system equation, we can see that,

$$\begin{aligned} F &= 1 \\ G &= 0 \\ H &= 1 \\ Q &= 0. \end{aligned}$$

Therefore, the Kalman gain is:

$$\begin{aligned} K_k &= P_k^- H^T (H P_k^- H^T + R)^{-1} \\ &= \frac{P_k^-}{R + P_k^-}. \end{aligned}$$

$$\begin{aligned} P_k^- &= F P_{k-1}^+ F^T + Q \\ &= P_{k-1}^+ \end{aligned}$$

$$
\begin{aligned}
&= (I - K_{k-1}H)P_{k-1}^-(I - K_{k-1}H)^T + K_{k-1}RK_{k-1}^T \\
&= (1 - K_{k-1})^2 P_{k-1}^- + RK_{k-1}^2 \\
&= \left(1 - \frac{P_{k-1}^-}{R + P_{k-1}^-}\right)^2 P_{k-1}^- + R\left(\frac{P_{k-1}^-}{R + P_{k-1}^-}\right)^2 \\
&= \left(\frac{R}{R + P_{k-1}^-}\right)^2 P_{k-1}^- + R\left(\frac{P_{k-1}^-}{R + P_{k-1}^-}\right)^2 \\
&= \frac{RP_{k-1}^-}{R + P_{k-1}^-}. \qquad (2)
\end{aligned}
$$

Proceeding further, we can see that,

$$
\begin{aligned}
P_k^- &= \frac{RP_{k-1}^-}{R + P_{k-1}^-} \\
&= \frac{R\frac{RP_{k-2}^-}{R+P_{k-2}^-}}{R + \frac{RP_{k-2}^-}{R+P_{k-2}^-}} \\
&= \frac{R^2 P_{k-2}^-}{R^2 + 2RP_{k-2}^-} \\
&= \frac{RP_{k-2}^-}{R + 2P_{k-2}^-}.
\end{aligned}
$$

Proceeding further till $P_1^-$, we get that,

$$
\begin{aligned}
P_k^- &= \frac{RP_1^-}{R + (k-1)P_1^-} \\
&= \frac{R}{R + k - 1}.
\end{aligned}
$$

Now, the steady state value of $P_k^-$ is:

$$
\begin{aligned}
\lim_{k\to\infty} P_k^- &= \lim_{k\to\infty} \frac{R}{R + k - 1} \\
&= 0.
\end{aligned}
$$

**5.9.b** If the actual system equation is:

$$x_{k+1} = x_k + w_k,$$

where $w_k \sim (0, Q)$ then, if we use the filter equation derived in part (a), we would get

$$
\begin{aligned}
P_k^- &= FP_{k-1}^+F^T + Q \\
&= P_{k-1}^+ + Q \\
&= \frac{RP_{k-1}^-}{R + P_{k-1}^-} + Q \qquad \text{[from (2)]}.
\end{aligned}
$$

Thus we can see that at each step $Q$ would get added which we will not count for if we use the filter equation derived in part (a). In this case, the steady state value would be infinite.

**(c)** The solution is consistent with my understanding. If there is no process noise, then the steady state value of $P_k^-$ becomes 0 which will in turn make $K_k = \frac{P_k^-}{R+P_k^-} = 0$. Therefore, measurements $y_k$ will be completely ignored. This happens because in steady state, $R$ will be infinitely large compared to $Q$, and hence filter will ignore the measurements.

Similarly, the answer in part (b) we can see that as $Q > 0$, $P_k^- = P_{k-1}^+ + Q$ will always be larger than $P_{k-1}^+$. Therefore, when $P_k^-$ converges, it converges to a larger value, which in limiting case goes to infinity.

## Problem 4

The dynamic model of the population system is:

$$\begin{aligned} p_{k+1} &= 0.5p_k + 2f_k \\ f_{k+1} &= f_k + w_f \\ y_k &= p_k + v_k, \end{aligned}$$

where $w_f \sim \mathcal{N}(0, 10)$ and $v_k \sim \mathcal{N}(0, 10)$. In matrix form,

$$\begin{bmatrix} p_{k+1} \\ f_{k+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_k \\ f_k \end{bmatrix} + \begin{bmatrix} 0 \\ w_k \end{bmatrix}$$
$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_k \\ f_k \end{bmatrix} + v_k.$$

Therefore, in this problem $F = \begin{bmatrix} 0.5 & 2 \\ 0 & 1 \end{bmatrix}$, $G = 0$, $Q = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}$, $R = 10$ and $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

**4.a** Fig. 3 shows the true and estimated population for 10 time steps. Fig. 4 shows the true and estimated food supply for 10 time steps. Fig. 5 shows the standard deviation (both $\pm$) of population and food supply estimates for 10 time steps. Fig. 6 shows the Kalman gains for population and food supply estimates for 10 time steps.

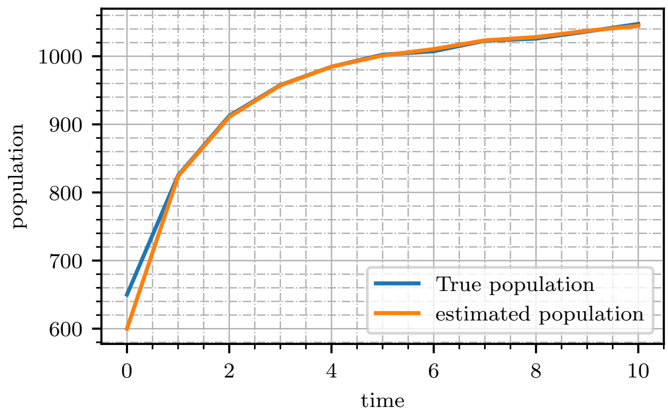


Figure 3: Q4.a: True and estimated population for 10 time steps

**4.b** Fig. 7 shows the standard deviation (both $\pm$) of population and food supply estimates for 10 time steps along with the steady state theoretical values (computed using covariance analysis). From Fig. 7, we can see that the steady state theoretical values of standard deviations of the population and food supply estimates do not closely match with the estimated standard deviations of the same. This happens because 10 time steps are not enough for the Kalman filter to converge closely to the theoretical value computed from covarinace analysis. We can see the improvement in 4.c.

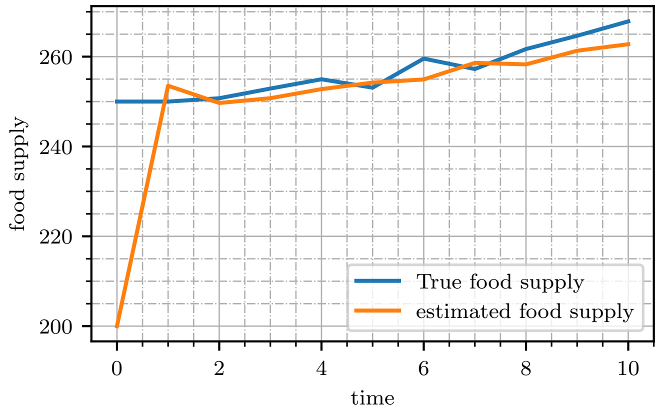


Figure 4: Q4.a: True and estimated food supply for 10 time steps

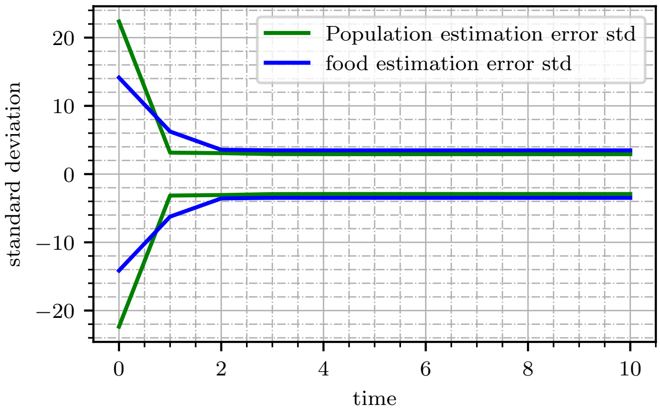


Figure 5: Q4.a: Standard deviation of estimated population and food supply for 10 time steps

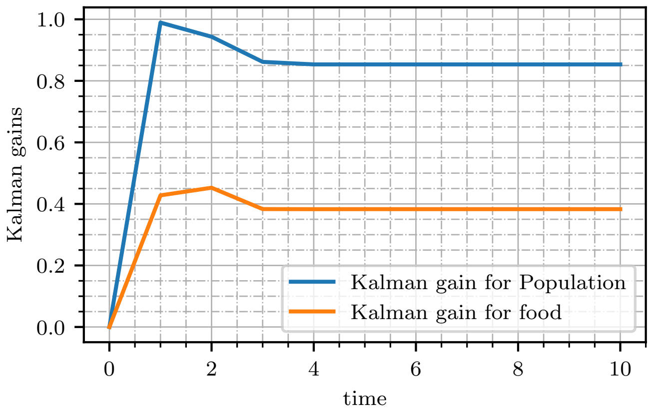


Figure 6: Q4.a: Kalman gains for population and food supply for 10 time steps

**4.c** Fig. 8 shows the standard deviation (both $\pm$) of population and food supply estimates for 1000 time steps along with the steady state theoretical values (computed using covariance analysis). From Fig. 8, we can see that the estimated standard deviations of the population and food supply converge to the steady state theoretical values as time goes by. Therefore, we can see that 1000 time steps really improves the performance of the Kalman filter.

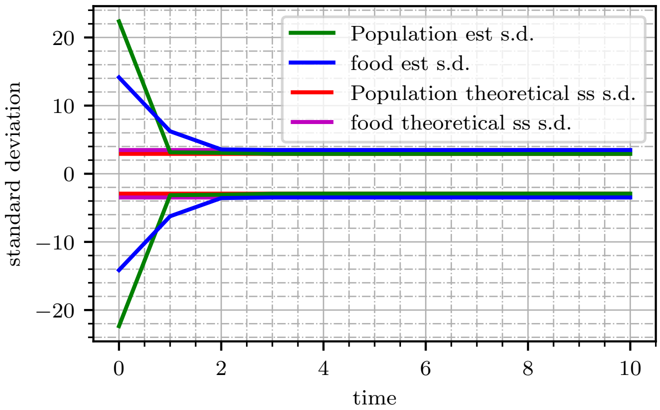


Figure 7: Q4.b: Standard deviations of population and food supply estimates for 10 time steps along with steady state(ss) theoretical values

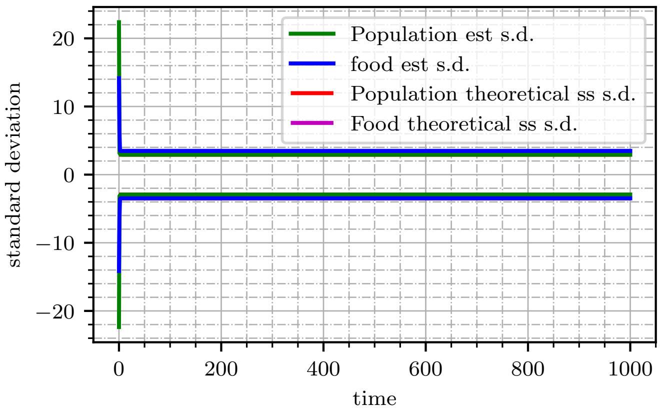


Figure 8: Q4.c: Standard deviations of population and food supply estimates for 1000 time steps along with steady state(ss) theoretical values

### Problem 5

**5.a** Fig. 9 shows the Forward and smoothed a posteriori covariance of population error estimation for 10 time steps. Fig. 10 shows the Forward and smoothed a posteriori covariance of food supply error estimation

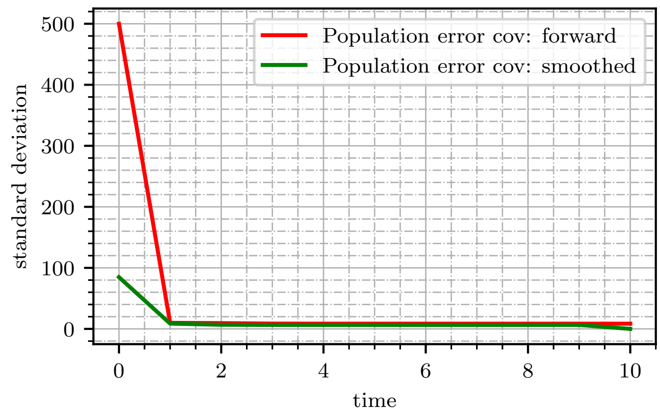


Figure 9: Q5.a: Forward and smoothed a posteriori covariance of population error estimation for 10 time steps

for 10 time steps.

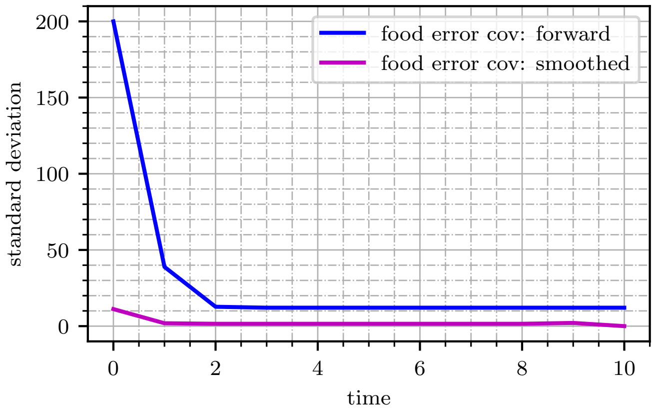


Figure 10: Q5.a: Forward and smoothed a posteriori covariance of population error estimation for 10 time steps

**5.b** Percentage improvement in the population estimation error variance due to smoothing at initial time is 490.09%.

Percentage improvement in the food supply estimation error variance due to smoothing at initial time is 1687.64%.

Improvement in food supply estimation at initial time is more because we do not measure food supply as we do for population, in which case measurement noise plays a role. Measurement noise degrades the estimate of the smoother in case of population. This is the reason of higher improvement percentage of food supply error estimation covariance at the initial time.

**5.c** Table 1 shows the numerical and theoretical state estimation error covariance for both the states. Thus from Table 1 we can see that both the numerical and theoretical estimates are close to each other,

Table 1: Q5: Numerical and theoretical initial state estimatiom error covariance

| State | Numerical | Theoretical from part(b) |
|---|---|---|
| Population | 84.73 | 84.73 |
| Food | 11.18 | 11.18 |

therefore, we can see one of the advantages of using smoother.