HW 4

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Problem 1.i

The swing equations and simplified turbine-governor model are given by:

$$\dot{\delta}_g = \omega_g - \omega_s$$
$$M_g\dot{\omega}_g = P_g^m - P_g^e$$
$$\tau_g\dot{P}_g^m = P_g^r - P_g^m - \frac{1}{R_g\omega_s}(\omega_g - \omega_s). \tag{1}$$

The generator reference power input is given by:

$$P_g^r = P_g^* + \alpha_g\left(\xi - \sum_{j\in\mathcal{G}}P_j^*\right), \tag{2}$$

where α_g is the AGC participation factor and $\sum_{g\in\mathcal{G}}\alpha_g = 1$. ξ is the AGC state whose evolution is given by:

$$\dot{\xi} = -\xi - \text{ACE} + \sum_{g\in\mathcal{G}}P_g^e$$

$$\text{ACE} = b\left((1/G)\sum_{g\in\mathcal{G}}(w_g) - w_s\right),$$

and $b > 0$ and there are a total of G generators in the area. Economic dispatch problem is formulated as follows:

$$\min_{P_g,g\in\mathcal{G}}\sum_{g\in\mathcal{G}}C_g(P_g)$$
$$\text{s.t. } \sum_{g\in\mathcal{G}}P_g = P_{load} + P_{loss}(P_{\mathcal{G}}). \tag{3}$$

It is given in the question that after a load change in the system, the new load is given by $\overline{P}_{load} = P_{load} + \Delta P_{load}$ and let us denote the corresponding changes in the loss as $\overline{P}_{loss} = P_{loss} + \Delta P_{loss}$. In steady state, after the load change, we get the following:

$$M_g\dot{\overline{\omega}}_g = 0 \implies \overline{P}_g^m = \overline{P}_g^e$$
$$\tau_g\dot{\overline{P}}_g^m = 0 \implies \overline{P}_g^r - \overline{P}_g^m - \frac{1}{R_g\omega_s}(\overline{\omega}_g - \omega_s)$$
$$\dot{\overline{\xi}} = 0 \implies \overline{\xi} = -\frac{b}{G}(\overline{\omega}_g - \omega_s) + \sum_{g\in\mathcal{G}}\overline{P}_g^e. \tag{4}$$

From 2, summing over all generators in the area, we get,

$$\sum_{j\in\mathcal{G}}\overline{P}_j^r = \sum_{j\in\mathcal{G}}P_j^* + \sum_{j\in\mathcal{G}}(\alpha_g\overline{\xi}) - \sum_{j\in\mathcal{G}}\alpha_g\sum_{j\in\mathcal{G}}P_j^*$$
$$\sum_{j\in\mathcal{G}}\overline{P}_j^r = \overline{\xi},$$

as $\sum_{j \in \mathcal{G}} \alpha_j = 1$. Therefore,

$$\sum_{j \in \mathcal{G}} \overline{P}^r_j = -\frac{b}{G}(\overline{\omega}_g - \omega_s) + \sum_{j \in \mathcal{G}} \overline{P}^e_g$$

$$\implies \sum_{j \in \mathcal{G}} \overline{P}^r_j - \sum_{j \in \mathcal{G}} \overline{P}^e_g = -\frac{b}{G}(\overline{\omega}_g - \omega_s)$$

$$\implies \sum_{j \in \mathcal{G}} \overline{P}^r_j - \sum_{j \in \mathcal{G}} \overline{P}^m_g = -\frac{b}{G}(\overline{\omega}_g - \omega_s)$$

$$\implies \frac{1}{R_g \omega_s}(\overline{\omega}_g - \omega_s) + \frac{b}{G}(\overline{\omega}_g - \omega_s) = 0$$

$$\implies \overline{\omega}_g = \omega_s.$$

Therefore, in steady-state, $\overline{\text{ACE}} = 0$. Thus,

$$\overline{\xi} = \sum_{j \in \mathcal{G}} \overline{P}^e_g = \overline{P}_{load} + \overline{P}_{loss} = P_{load} + \Delta P_{load} + P_{loss} + \Delta P_{loss}.$$

Now, as P^*_g is the outcome of economic dispatch optimization problem, it has to satisfy the constraint, $\sum_{j \in \mathcal{G}} P^*_g = P_{load} + P_{loss}$. Hence, from 4,

$$\overline{P}^m_g = \overline{P}^r_g$$

$$= P^*_g + \alpha_g \left(\overline{\xi} - \sum_{j \in \mathcal{G}} P^*_j \right)$$

$$= P^*_g + \alpha_g \left(P_{load} + \Delta P_{load} + P_{loss} + \Delta P_{loss} - P_{load} - P_{loss} \right)$$

$$= P^*_g + \alpha_g \left(\Delta P_{load} + \Delta P_{loss} \right).$$

Problem 1.ii

From 4,

$$\overline{P}^e_g = \overline{P}^m_g$$

$$= P^*_g + \alpha_g \left(\Delta P_{load} + \Delta P_{loss} \right).$$

Problem 1.iii

Using Lagrange multiplier λ and KKT conditions for the given economic dispatch optimization problem, we get the dual objective as follows:

$$\left(\sum_{g \in \mathcal{G}} C_g(P_g) \right) + \lambda \left(P_{load} + P_{loss}(P_{\mathcal{G}}) - \sum_{g \in \mathcal{G}} P_g \right)$$

Denoting the optimal variables as P^*_g and λ^*, we get,

$$\frac{\partial}{\partial P_g} \left(\sum_{g \in \mathcal{G}} C_g(P_g) \right) + \frac{\partial}{\partial P_g} \lambda \left(P_{load} + P_{loss}(P_{\mathcal{G}}) - \sum_{g \in \mathcal{G}} P_g \right) = 0.$$

Carrying out the derivative, we get:

$$C'_g(P^*_g) - \lambda^* \left(1 - \frac{\partial}{\partial P_g} P_{loss}(P_{\mathcal{G}^*}) \right) = 0$$

$$\implies C'_g(P^*_g) - \frac{\lambda^*}{\Lambda^*_g} = 0, \tag{5}$$

where, $\Lambda^*_g = \left(1 - \frac{\partial}{\partial P_g} P_{loss}(P_{\mathcal{G}^*}) \right)^{-1}$.

Problem 1.iv

AGC participation factor is given by:

$$\alpha_g = \frac{(C_g''(P_g^*))^{-1}}{\sum_{j\in\mathcal{G}}(C_j''(P_j^*))^{-1}}.$$

It is given that,

$$\overline{\Lambda}_g^* C_g'(\overline{P}_g^*) - \Lambda_g^* C_g'(P_g^*) = (\overline{P}_g^* - P_g^*)C_g''(P_g^*). \tag{6}$$

Now, from 5 and 6, we get,

$$\overline{\lambda}^* - \lambda^* = (\overline{P}_g^* - P_g^*)C_g''(P_g^*)$$

$$\implies \frac{\overline{\lambda}^* - \lambda^*}{C_g''(P_g^*)} = (\overline{P}_g^* - P_g^*).$$

where $\overline{\lambda}^*$ is the optimal value of Lagrange multiplier of the optimization problem solution after the load change. Summing over all the generators, we get:

$$(\overline{\lambda}^* - \lambda^*)\sum_{j\in\mathcal{G}}(C_j''(P_j^*))^{-1} = \sum_{j\in\mathcal{G}}(\overline{P}_j^* - P_j^*).$$

From the derivation in part (ii),

$$\overline{P}_g^e = P_g^* + \alpha_g\left(\Delta P_{load} + \Delta P_{loss}\right)$$

$$= P_g^* + \alpha_g\left(\sum_{j\in\mathcal{G}}\overline{P}_j^* - \sum_{j\in\mathcal{G}}P_j^*\right)$$

$$= P_g^* + \frac{(C_g''(P_g^*))^{-1}}{\sum_{j\in\mathcal{G}}(C_j''(P_j^*))^{-1}}\left(\sum_{j\in\mathcal{G}}\overline{P}_j^* - \sum_{j\in\mathcal{G}}P_j^*\right)$$

$$= P_g^* + \frac{(C_g''(P_g^*))^{-1}}{\sum_{j\in\mathcal{G}}(C_j''(P_j^*))^{-1}}\sum_{j\in\mathcal{G}}(\overline{P}_j^* - P_j^*)$$

$$= P_g^* + \frac{(C_g''(P_g^*))^{-1}}{\sum_{j\in\mathcal{G}}(C_j''(P_j^*))^{-1}}(\overline{\lambda}^* - \lambda^*)\sum_{j\in\mathcal{G}}(C_j''(P_j^*))^{-1}$$

$$= P_g^* + (C_g''(P_g^*))^{-1}(\overline{\lambda}^* - \lambda^*)$$

$$= P_g^* + (C_g''(P_g^*))^{-1}(\overline{P}_g^* - P_g^*)C_g''(P_g^*)$$

$$= P_g^* + \overline{P}_g^* - P_g^*$$

$$= \overline{P}_g^*.$$

Therefore, under the given conditions [A1]-[A2], the setpoints as derived from economic dispatch solution becomes exactly equal to th electrical output power of the generators. This helps to minimize the impact of inaccurate loss penalty factor consideration.

Problem 1.v

Convex cost function with convex feasible set would satisfy [A1]. In lossless networks, [A2] would get automatically satisfied.

Problem 2.i

I am not repeating the notations from the homework to conserve space. We know that complex power injection at i^{th} bus is given by:

$$S_i = V_i I_i^*.$$

Now, we know that:

$$\begin{bmatrix} I \\ I_{N+1} \end{bmatrix} = \begin{bmatrix} Y & \overline{Y} \\ \overline{Y}^T & y \end{bmatrix} \begin{bmatrix} V \\ V_0 e^{j\theta_0} \end{bmatrix}$$

$$= \begin{bmatrix} YV + \overline{Y}V_0 e^{j\theta_0} \\ \overline{Y}^T V + y V_0 e^{j\theta_0} \end{bmatrix}. \tag{7}$$

In matrix notation, therefore, we can write the following:

$$S = V \circ I^*,$$

where $\circ$ operator denotes Hadamard product. Therefore from 7,

$$\begin{aligned} S &= V \circ (YV + \overline{Y}V_0 e^{j\theta_0})^* \\ &= V \circ (Y^* V^* + \overline{Y}^* V_0 e^{-j\theta_0}) \\ &= \mathrm{diag}(V)(Y^* V^* + \overline{Y}^* V_0 e^{-j\theta_0}), \end{aligned}$$

as for any two vectors $x, y \in \mathbb{R}^N$, $x \circ y = \mathrm{diag}(x)y$.

Problem 2.ii

From the previous part, we have

$$S = V \circ (Y^* V^* + \overline{Y}^* V_0 e^{-j\theta_0}).$$

If we express $V = V^{nom} + \Delta V$, and choose $V^{nom} = -Y^{-1}\overline{Y}V_0 e^{j\theta_0}$, then

$$\begin{aligned} S &= V \circ (Y^* V^* + \overline{Y}^* V_0 e^{-j\theta_0}) \\ \Longrightarrow S^* &= V^* \circ (Y^* V^* + \overline{Y}^* V_0 e^{-j\theta_0})^* \\ \Longrightarrow S^* &= V^* \circ (YV + \overline{Y}V_0 e^{j\theta_0}) \\ \Longrightarrow S^* &= V^* \circ (Y(V^{nom} + \Delta V) + \overline{Y}V_0 e^{j\theta_0}) \\ \Longrightarrow S^* &= V^* \circ (YV^{nom} + Y\Delta V + \overline{Y}V_0 e^{j\theta_0}) \\ \Longrightarrow S^* &= V^* \circ (Y(-Y^{-1}\overline{Y}V_0 e^{j\theta_0}) + Y\Delta V + \overline{Y}V_0 e^{j\theta_0}) \\ \Longrightarrow S^* &= V^* \circ (-\overline{Y}V_0 e^{j\theta_0} + Y\Delta V + \overline{Y}V_0 e^{j\theta_0}) \\ \Longrightarrow S^* &= V^* \circ (Y\Delta V) \\ \Longrightarrow S^* &= (V^{nom} + \Delta V)^* \circ (Y\Delta V) \\ \Longrightarrow S^* &= (V^{nom})^* \circ Y\Delta V + \Delta V^* \circ Y\Delta V. \end{aligned}$$

If we neglect second-order terms i.e. $\Delta V^* \circ Y\Delta V$,

$$\begin{aligned} S^* &= (V^{nom})^* \circ Y\Delta V \\ &= \mathrm{diag}((V^{nom})^*)Y\Delta V. \end{aligned}$$

Therefore ΔV can be solved from the above linear equation.

Problem 2.iii

As given in the homework,

$$K = \text{diag}(V^{nom})Y^*$$

$$J = \begin{bmatrix} \text{Re}(K) & \text{Im}(K) \\ \text{Im}(K) & -\text{Re}(K) \end{bmatrix}.$$

From the previous part, we got,

$$S^* = (V^{nom})^* \circ Y \Delta V$$
$$\implies S = (V^{nom}) \circ Y^* \Delta V^*$$
$$\implies S = \text{diag}(V^{nom})Y^* \Delta V^*$$
$$\implies S = K \Delta V^*$$
$$\implies P + jQ = (\text{Re}(K) + j\text{Im}(K))(\Delta V_{re} - j\Delta V_{im})$$
$$\implies P + jQ = (\text{Re}(K)\Delta V_{re} + \text{Im}(K)\Delta V_{im}) + j(\text{Im}(K)\Delta V_{re} - \text{Re}(K)\Delta V_{im}).$$

Equating real and imaginary parts and using matrix notations, we get

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} \text{Re}(K)\Delta V_{re} + \text{Im}(K)\Delta V_{im} \\ \text{Im}(K)\Delta V_{re} - \text{Re}(K)\Delta V_{im} \end{bmatrix}$$
$$= \begin{bmatrix} \text{Re}(K) & \text{Im}(K) \\ \text{Im}(K) & -\text{Re}(K) \end{bmatrix} \begin{bmatrix} \Delta V_{im} \\ \Delta V_{re} \end{bmatrix}$$
$$= J \begin{bmatrix} \Delta V_{im} \\ \Delta V_{re} \end{bmatrix}.$$

Therefore, $[\Delta V_{im} \ \Delta V_{re}]^T$ can be solved from the following linear equations:

$$\begin{bmatrix} \Delta V_{im} \\ \Delta V_{re} \end{bmatrix} = J^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}.$$

Problem 2.iv

Let us denote $\eta = re^{j\alpha}$ and it is given that $|\eta| = r << 1$. Now,

$$|1 + \eta| = |1 + r\cos\alpha + jr\sin\alpha|$$
$$= \sqrt{(1 + r\cos\alpha)^2 + (r\sin\alpha)^2}$$
$$= \sqrt{1 + 2r\cos\alpha + r^2\cos^2\alpha + r^2\sin^2\alpha}.$$

Now, $0 \leq \sin^2\alpha \leq 1$ and $0 \leq r^2 << 1$. Therefore, $0 \leq r^2\sin^2\alpha << 1$ and thus, $1 + r^2\sin^2\alpha \approx 1$. Therefore,

$$|1 + \eta| = \sqrt{1 + 2r\cos\alpha + r^2\cos^2\alpha + r^2\sin^2\alpha}$$
$$\approx \sqrt{1 + 2r\cos\alpha + r^2\cos^2\alpha}$$
$$= \sqrt{(1 + r\cos\alpha)^2}$$
$$= 1 + r\cos\alpha$$
$$= 1 + \text{Re}(\eta).$$

Now,

$$\angle(1 + \eta) = \arctan\left(\frac{r\sin\alpha}{1 + r\cos\alpha}\right).$$

Now, $-1 \leq \cos\alpha \leq 1$ and $0 \leq r << 1$. Therefore, $1 + r\cos\alpha \approx 1$. Similarly, $-1 \leq \sin\alpha \leq 1$, and thus $r\sin\alpha$ is also very small. Therefore,

$$
\begin{aligned}
\angle(1 + \eta) &= \arctan\left(\frac{r\sin\alpha}{1 + r\cos\alpha}\right) \\
&\approx \arctan(r\sin\alpha) \\
&\approx r\sin\alpha \qquad \text{[using small angle approximation of arctan]} \\
&= r\sin\alpha \\
&= \text{Im}(\eta).
\end{aligned}
$$

Problem 2.v

It is given that,

$$
\begin{aligned}
V &= V^{nom} + \Delta V \\
&= V^{nom} \circ \left(\mathbb{1} + (V^{nom})^{\circ-1} \circ \Delta V\right),
\end{aligned}
$$

where, $(V^{nom})^{\circ-1}$ is the Hadamard inverse of V^{nom}. Now,

$$
|V| = |V^{nom}| \circ |\mathbb{1} + (V^{nom})^{\circ-1} \circ \Delta V|.
$$

As, $|\Delta V| << \mathbb{1}$, $|(V^{nom})^{\circ-1}\Delta V| << \mathbb{1}$, considering $|V^{nom}| >> |\Delta V|$. Therefore, we can apply the result we obtained in part iv as follows:

$$
\begin{aligned}
|V| &\approx |V^{nom}| \circ \left(\mathbb{1} + \text{Re}((V^{nom})^{\circ-1} \circ \Delta V)\right) \\
&= |V^{nom}| + |V^{nom}| \circ \text{Re}((V^{nom})^{\circ-1} \circ \Delta V) \\
&= |V^{nom}| + |V^{nom}| \circ \text{Re}(|V^{nom}|^{\circ-1} \circ (\cos\theta^{nom} - j\sin\theta^{nom}) \circ (\Delta V_{re} + j\Delta V_{im})) \\
&= |V^{nom}| + |V^{nom}| \circ \text{Re}(|V^{nom}|^{\circ-1} \circ \cos\theta^{nom} \circ \Delta V_{re} + |V^{nom}|^{\circ-1} \circ \sin\theta^{nom} \circ \Delta V_{im} \\
&\quad + j|V^{nom}|^{\circ-1} \circ \cos\theta^{nom}\Delta V_{im} - j|V^{nom}|^{\circ-1} \circ \sin\theta^{nom} \circ \Delta V_{re}) \\
&= |V^{nom}| + |V^{nom}| \circ (|V^{nom}|^{\circ-1} \circ \cos\theta^{nom} \circ \Delta V_{re} + |V^{nom}|^{\circ-1} \circ \sin\theta^{nom} \circ \Delta V_{im}) \\
&= |V^{nom}| + \cos\theta^{nom} \circ \Delta V_{re} + \sin\theta^{nom} \circ \Delta V_{im} \\
&= |V^{nom}| + [\text{diag}(\cos\theta^{nom}) \quad \text{diag}(\sin\theta^{nom})]\begin{bmatrix} \Delta V_{re} \\ \Delta V_{im} \end{bmatrix} \\
&= |V^{nom}| + [\text{diag}(\cos\theta^{nom}) \quad \text{diag}(\sin\theta^{nom})]J^{-1}\begin{bmatrix} P \\ Q \end{bmatrix}.
\end{aligned}
$$

Problem 2.vi

Similarly,

$$
\begin{aligned}
\angle V = \theta &= \angle(V^{nom} \circ \left(\mathbb{1} + (V^{nom})^{\circ-1} \circ \Delta V\right)) \\
&= \angle V^{nom} + \angle\left((\mathbb{1} + (V^{nom})^{\circ-1} \circ \Delta V)\right) \\
&= \theta^{nom} + \angle\left((\mathbb{1} + (V^{nom})^{\circ-1} \circ \Delta V)\right) \\
&\approx \theta^{nom} + \text{Im}((V^{nom})^{\circ-1} \circ \Delta V) \\
&= \theta^{nom} + \text{Im}(|V^{nom}|^{\circ-1} \circ (\cos\theta^{nom} - j\sin\theta^{nom}) \circ (\Delta V_{re} + j\Delta V_{im})) \\
&= \theta^{nom} + \text{Im}(|V^{nom}|^{\circ-1} \circ \cos\theta^{nom} \circ \Delta V_{re} + |V^{nom}|^{\circ-1} \circ \sin\theta^{nom} \circ \Delta V_{im} \\
&\quad + j|V^{nom}|^{\circ-1} \circ \cos\theta^{nom}\Delta V_{im} - j|V^{nom}|^{\circ-1} \circ \sin\theta^{nom} \circ \Delta V_{re}) \\
&= \theta^{nom} + |V^{nom}|^{\circ-1} \circ \cos\theta^{nom} \circ \Delta V_{im} - |V^{nom}|^{\circ-1} \circ \sin\theta^{nom} \circ \Delta V_{re} \\
&= \theta^{nom} + |V^{nom}|^{\circ-1} \circ (\cos\theta^{nom} \circ \Delta V_{im} - \sin\theta^{nom} \circ \Delta V_{re})
\end{aligned}
$$

$$= \theta^{nom} + |V^{nom}|^{\circ-1} \circ [-\mathrm{diag}(\sin\theta^{nom}) \quad \mathrm{diag}(\cos\theta^{nom})] \begin{bmatrix} \Delta V_{re} \\ \Delta V_{im} \end{bmatrix}$$

$$= \theta^{nom} + \mathrm{diag}(|V^{nom}|)^{-1}[-\mathrm{diag}(\sin\theta^{nom}) \quad \mathrm{diag}(\cos\theta^{nom})] \begin{bmatrix} \Delta V_{re} \\ \Delta V_{im} \end{bmatrix}$$

$$= \theta^{nom} + \mathrm{diag}(|V^{nom}|)^{-1}[-\mathrm{diag}(\sin\theta^{nom}) \quad \mathrm{diag}(\cos\theta^{nom})]J^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}.$$

Problem 2.vii

The code and approximation result is in the attached code ('q4_vii_script.m').