HW 4

Arnab Dey
Student ID: 5563169
Email: dey00011@umn.edu

Problem 1.i

The swing equations and simplified turbine-governor model are given by:

$$\begin{aligned}
\dot{\delta}_g &= \omega_g - \omega_s \\
M_g \dot{\omega}_g &= P_g^m - P_g^e \\
\tau_g \dot{P}_g^m &= P_g^r - P_g^m - \frac{1}{R_g \omega_s}(\omega_g - \omega_s).
\end{aligned} \tag{1}$$

The generator reference power input is given by:

$$P_g^r = P_g^* + \alpha_g \left(\xi - \sum_{j \in \mathcal{G}} P_j^* \right), \tag{2}$$

where α_g is the AGC participation factor and $\sum_{g \in \mathcal{G}} \alpha_g = 1$. ξ is the AGC state whose evolution is given by:

$$\begin{aligned}
\dot{\xi} &= -\xi - \text{ACE} + \sum_{g \in \mathcal{G}} P_g^e \\
\text{ACE} &= b \left((1/G) \sum_{g \in \mathcal{G}} (w_g) - w_s \right),
\end{aligned}$$

and $b > 0$ and there are a total of G generators in the area. Economic dispatch problem is formulated as follows:

$$\begin{aligned}
&\min_{P_g, g \in \mathcal{G}} \sum_{g \in \mathcal{G}} C_g(P_g) \\
&\text{s.t.} \ \sum_{g \in \mathcal{G}} P_g = P_{load} + P_{loss}(P_{\mathcal{G}}).
\end{aligned} \tag{3}$$

It is given in the question that after a load change in the system, the new load is given by $\overline{P}_{load} = P_{load} + \Delta P_{load}$ and let us denote the corresponding changes in the loss as $\overline{P}_{loss} = P_{loss} + \Delta P_{loss}$. In steady state, after the load change, we get the following:

$$\begin{aligned}
M_g \dot{\overline{\omega}}_g = 0 &\implies \overline{P}_g^m = \overline{P}_g^e \\
\tau_g \dot{\overline{P}}_g^m = 0 &\implies \overline{P}_g^r - \overline{P}_g^m - \frac{1}{R_g \omega_s}(\overline{\omega}_g - \omega_s) \\
\dot{\overline{\xi}} = 0 &\implies \overline{\xi} = -\frac{b}{G}(\overline{\omega}_g - \omega_s) + \sum_{g \in \mathcal{G}} \overline{P}_g^e.
\end{aligned} \tag{4}$$

From 2, summing over all generators in the area, we get,

$$\begin{aligned}
&\sum_{j \in \mathcal{G}} \overline{P}_j^r = \sum_{j \in \mathcal{G}} P_j^* + \sum_{j \in \mathcal{G}} (\alpha_g \overline{\xi}) - \sum_{j \in \mathcal{G}} \alpha_g \sum_{j \in \mathcal{G}} P_j^* \\
&\sum_{j \in \mathcal{G}} \overline{P}_j^r = \overline{\xi},
\end{aligned}$$

as $\sum_{j\in\mathcal{G}} \alpha_j = 1$. Therefore,

$$\begin{aligned}
&\sum_{j\in\mathcal{G}} \overline{P}^r_j = -\frac{b}{G}(\overline{\omega}_g - \omega_s) + \sum_{j\in\mathcal{G}} \overline{P}^e_g \\
\implies &\sum_{j\in\mathcal{G}} \overline{P}^r_j - \sum_{j\in\mathcal{G}} \overline{P}^e_g = -\frac{b}{G}(\overline{\omega}_g - \omega_s) \\
\implies &\sum_{j\in\mathcal{G}} \overline{P}^r_j - \sum_{j\in\mathcal{G}} \overline{P}^m_g = -\frac{b}{G}(\overline{\omega}_g - \omega_s) \\
\implies &\frac{1}{R_g\omega_s}(\overline{\omega}_g - \omega_s) + \frac{b}{G}(\overline{\omega}_g - \omega_s) = 0 \\
\implies &\overline{\omega}_g = \omega_s.
\end{aligned}$$

Therefore, in steady-state, $\overline{\mathrm{ACE}} = 0$. Thus,

$$\bar{\xi} = \sum_{j\in\mathcal{G}} \overline{P}^e_g = \overline{P}_{load} + \overline{P}_{loss} = P_{load} + \Delta P_{load} + P_{loss} + \Delta P_{loss}.$$

Now, as P^*_g is the outcome of economic dispatch optimization problem, it has to satisfy the constraint, $\sum_{j\in\mathcal{G}} P^*_g = P_{load} + P_{loss}$. Hence, from 4,

$$\begin{aligned}
\overline{P}^m_g &= \overline{P}^r_g \\
&= P^*_g + \alpha_g \left(\bar{\xi} - \sum_{j\in\mathcal{G}} P^*_j \right) \\
&= P^*_g + \alpha_g \left(P_{load} + \Delta P_{load} + P_{loss} + \Delta P_{loss} - P_{load} - P_{loss} \right) \\
&= P^*_g + \alpha_g \left(\Delta P_{load} + \Delta P_{loss} \right).
\end{aligned}$$

Problem 1.ii

From 4,

$$\begin{aligned}
\overline{P}^e_g &= \overline{P}^m_g \\
&= P^*_g + \alpha_g \left(\Delta P_{load} + \Delta P_{loss} \right).
\end{aligned}$$

Problem 1.iii

Using Lagrange multiplier λ and KKT conditions for the given economic dispatch optimization problem, we get the dual objective as follows:

$$\left(\sum_{g\in\mathcal{G}} C_g(P_g) \right) + \lambda \left(P_{load} + P_{loss}(P_{\mathcal{G}}) - \sum_{g\in\mathcal{G}} P_g \right)$$

Denoting the optimal variables as P^*_g and λ^*, we get,

$$\frac{\partial}{\partial P_g} \left(\sum_{g\in\mathcal{G}} C_g(P_g) \right) + \frac{\partial}{\partial P_g} \lambda \left(P_{load} + P_{loss}(P_{\mathcal{G}}) - \sum_{g\in\mathcal{G}} P_g \right) = 0.$$

Carrying out the derivative, and denoting as the optimal values, we get:

$$\begin{aligned}
&C'(P^*_g) - \lambda^* \left(1 - \frac{\partial}{\partial P_g} P_{loss}(P_{\mathcal{G}^*}) \right) = 0 \\
\implies &C'(P^*_g) - \frac{\lambda^*}{\Lambda^*_g} = 0, \qquad (5)
\end{aligned}$$

where, $\Lambda^*_g = \left(1 - \frac{\partial}{\partial P_g} P_{loss}(P_{\mathcal{G}^*}) \right)^{-1}$.

Problem 1.iv

AGC participation factor is given by:

$$\alpha_g = \frac{(C''_g(P^*_g))^{-1}}{\sum_{j\in\mathcal{G}}(C''_j(P^*_j))^{-1}}.$$

It is given that,

$$\overline{\Lambda}^*_g C'_g(\overline{P}^*_g) - \Lambda^*_g C'_g(P^*_g) = (\overline{P}^*_g - P^*_g)C''_g(P^*_g). \tag{6}$$

Now, from 5 and 6, we get,

$$\begin{aligned} \overline{\lambda}^* - \lambda^* &= (\overline{P}^*_g - P^*_g)C''_g(P^*_g) \\ \implies \frac{\overline{\lambda}^* - \lambda^*}{C''_g(P^*_g)} &= (\overline{P}^*_g - P^*_g). \end{aligned}$$

where $\overline{\lambda}^*$ is the optimal value of Lagrange multiplier of the optimization problem solution after the load change. Summing over all the generators, we get:

$$(\overline{\lambda}^* - \lambda^*)\sum_{j\in\mathcal{G}}(C''_j(P^*_j))^{-1} = \sum_{j\in\mathcal{G}}(\overline{P}^*_j - P^*_j).$$

From the derivation in part (ii),

$$\begin{aligned} \overline{P}^e_g &= P^*_g + \alpha_g\left(\Delta P_{load} + \Delta P_{loss}\right) \\ &= P^*_g + \alpha_g\left(\sum_{j\in\mathcal{G}}\overline{P}^*_j - \sum_{j\in\mathcal{G}}P^*_j\right) \\ &= P^*_g + \frac{(C''_g(P^*_g))^{-1}}{\sum_{j\in\mathcal{G}}(C''_j(P^*_j))^{-1}}\left(\sum_{j\in\mathcal{G}}\overline{P}^*_j - \sum_{j\in\mathcal{G}}P^*_j\right) \\ &= P^*_g + \frac{(C''_g(P^*_g))^{-1}}{\sum_{j\in\mathcal{G}}(C''_j(P^*_j))^{-1}}\sum_{j\in\mathcal{G}}(\overline{P}^*_j - P^*_j) \\ &= P^*_g + \frac{(C''_g(P^*_g))^{-1}}{\sum_{j\in\mathcal{G}}(C''_j(P^*_j))^{-1}}(\overline{\lambda}^* - \lambda^*)\sum_{j\in\mathcal{G}}(C''_j(P^*_j))^{-1} \\ &= P^*_g + (C''_g(P^*_g))^{-1}(\overline{\lambda}^* - \lambda^*) \\ &= P^*_g + (C''_g(P^*_g))^{-1}(\overline{P}^*_g - P^*_g)C''_g(P^*_g) \\ &= P^*_g + \overline{P}^*_g - P^*_g \\ &= \overline{P}^*_g. \end{aligned}$$

Therefore, under the given conditions [A1]-[A2], the setpoints as derived from economic dispatch solution becomes exactly equal to th electrical output power of the generators.

Problem 1.v

Convex cost function.

Problem 2.i

I am not repeating the notations from the homework to conserve space. We know that complex power injection at i^{th} bus is given by:

$$S_i = V_i I^*_i.$$

Now, we know that:

$$
\begin{aligned}
\begin{bmatrix} I \\ I_{N+1} \end{bmatrix} &= \begin{bmatrix} Y & \overline{Y} \\ \overline{Y}^T & y \end{bmatrix} \begin{bmatrix} V \\ V_0 e^{j\theta_0} \end{bmatrix} \\
&= \begin{bmatrix} YV + \overline{Y} V_0 e^{j\theta_0} \\ \overline{Y}^T V + y V_0 e^{j\theta_0} \end{bmatrix}.
\end{aligned} \tag{7}
$$

In matrix notation, therefore, we can write the following:

$$S = V \circ I^*,$$

where $\circ$ operator denotes Hadamard product. Therefore from 7,

$$
\begin{aligned}
S &= V \circ (YV + \overline{Y} V_0 e^{j\theta_0})^* \\
&= V \circ (Y^* V^* + \overline{Y}^* V_0 e^{-j\theta_0}) \\
&= \operatorname{diag}(V)(Y^* V^* + \overline{Y}^* V_0 e^{-j\theta_0}),
\end{aligned}
$$

as for any two vectors $x, y \in \mathbb{R}^N$, $x \circ y = \operatorname{diag}(x) y$.

Problem 2.ii

From the previous part, we have

$$S = V \circ (Y^* V^* + \overline{Y}^* V_0 e^{-j\theta_0}).$$

If we express $V = V^{nom} + \Delta V$, and choose $V^{nom} = -Y^{-1} \overline{Y} V_0 e^{j\theta_0}$, then

$$
\begin{aligned}
& S = V \circ (Y^* V^* + \overline{Y}^* V_0 e^{-j\theta_0}) \\
\implies & S^* = V^* \circ (Y^* V^* + \overline{Y}^* V_0 e^{-j\theta_0})^* \\
\implies & S^* = V^* \circ (YV + \overline{Y} V_0 e^{j\theta_0}) \\
\implies & S^* = V^* \circ (Y(V^{nom} + \Delta V) + \overline{Y} V_0 e^{j\theta_0}) \\
\implies & S^* = V^* \circ (YV^{nom} + Y\Delta V + \overline{Y} V_0 e^{j\theta_0}) \\
\implies & S^* = V^* \circ (Y(-Y^{-1} \overline{Y} V_0 e^{j\theta_0}) + Y\Delta V + \overline{Y} V_0 e^{j\theta_0}) \\
\implies & S^* = V^* \circ (-\overline{Y} V_0 e^{j\theta_0} + Y\Delta V + \overline{Y} V_0 e^{j\theta_0}) \\
\implies & S^* = V^* \circ (Y\Delta V) \\
\implies & S^* = (V^{nom} + \Delta V)^* \circ (Y\Delta V) \\
\implies & S^* = (V^{nom})^* \circ Y\Delta V + \Delta V^* \circ Y\Delta V.
\end{aligned}
$$

If we neglect second-order terms i.e. $\Delta V^* \circ Y\Delta V$,

$$
\begin{aligned}
S^* &= (V^{nom})^* \circ Y\Delta V \\
&= \operatorname{diag}((V^{nom})^*) Y\Delta V.
\end{aligned}
$$

Therefore ΔV can be solved from the above linear equation.

Problem 2.iii

As given in the homework,

$$
\begin{aligned}
K &= \operatorname{diag}(V^{nom}) Y^* \\
J &= \begin{bmatrix} \operatorname{Re}(K) & \operatorname{Im}(K) \\ \operatorname{Im}(K) & -\operatorname{Re}(K) \end{bmatrix}.
\end{aligned}
$$

From the previous part, we got,

$$\begin{aligned}
&S^* = (V^{nom})^* \circ Y\Delta V \\
\implies &S = (V^{nom}) \circ Y^*\Delta V^* \\
\implies &S = \mathrm{diag}(V^{nom})Y^*\Delta V^* \\
\implies &S = K\Delta V^* \\
\implies &P + jQ = (\mathrm{Re}(K) + j\mathrm{Im}(K))(\Delta V_{re} - j\Delta V_{im}) \\
\implies &P + jQ = (\mathrm{Re}(K)\Delta V_{re} + \mathrm{Im}(K)\Delta V_{im}) + j(\mathrm{Im}(K)\Delta V_{re} - \mathrm{Re}(K)\Delta V_{im}).
\end{aligned}$$

Equating real and imaginary parts and using matrix notations, we get

$$\begin{aligned}
\begin{bmatrix} P \\ Q \end{bmatrix} &= \begin{bmatrix} \mathrm{Re}(K)\Delta V_{re} + \mathrm{Im}(K)\Delta V_{im} \\ \mathrm{Im}(K)\Delta V_{re} - \mathrm{Re}(K)\Delta V_{im} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Re}(K) & \mathrm{Im}(K) \\ \mathrm{Im}(K) & -\mathrm{Re}(K) \end{bmatrix} \begin{bmatrix} \Delta V_{im} \\ \Delta V_{re} \end{bmatrix} \\
&= J \begin{bmatrix} \Delta V_{im} \\ \Delta V_{re} \end{bmatrix}.
\end{aligned}$$

Therefore, $[\Delta V_{im}\ \Delta V_{re}]^T$ can be solved from the following linear equations:

$$\begin{bmatrix} \Delta V_{im} \\ \Delta V_{re} \end{bmatrix} = J^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}.$$

Problem 2.iv

Let us denote $\eta = re^{j\alpha}$ and it is given that $|\eta| = r << 1$. Now,

$$\begin{aligned}
|1 + \eta| &= |1 + r\cos\alpha + jr\sin\alpha| \\
&= \sqrt{(1 + r\cos\alpha)^2 + (r\sin\alpha)^2} \\
&= \sqrt{1 + 2r\cos\alpha + r^2\cos^2\alpha + r^2\sin^2\alpha}.
\end{aligned}$$

Now, $0 \leq \sin^2\alpha \leq 1$ and $0 \leq r^2 << 1$. Therefore, $0 \leq r^2\sin^2\alpha << 1$ and thus, $1+r^2\sin^2\alpha \approx 1$. Therefore,

$$\begin{aligned}
|1 + \eta| &= \sqrt{1 + 2r\cos\alpha + r^2\cos^2\alpha + r^2\sin^2\alpha} \\
&\approx \sqrt{1 + 2r\cos\alpha + r^2\cos^2\alpha} \\
&= \sqrt{(1 + r\cos\alpha)^2} \\
&= 1 + r\cos\alpha \\
&= 1 + \mathrm{Re}(\eta).
\end{aligned}$$

Now,

$$\angle(1 + \eta) = \arctan\left(\frac{r\sin\alpha}{1 + r\cos\alpha}\right).$$

Now, $-1 \leq \cos\alpha \leq 1$ and $0 \leq r << 1$. Therefore, $1 + r\cos\alpha \approx 1$. Similarly, $-1 \leq \sin\alpha \leq 1$, and thus $r\sin\alpha$ is also very small. Therefore,

$$\begin{aligned}
\angle(1 + \eta) &= \arctan\left(\frac{r\sin\alpha}{1 + r\cos\alpha}\right) \\
&\approx \arctan(r\sin\alpha) \\
&\approx r\sin\alpha \qquad \text{[using small angle approximation of arctan]} \\
&= r\sin\alpha \\
&= \mathrm{Im}(\eta).
\end{aligned}$$

Problem 2.v

It is given that,

$$\begin{aligned} V &= V^{nom} + \Delta V \\ &= V^{nom} \circ \left(\mathbb{1} + (V^{nom})^{\circ -1} \circ \Delta V\right), \end{aligned}$$

where, $(V^{nom})^{\circ-1}$ is the Hadamard inverse of V^{nom}. Now,

$$|V| = |V^{nom}| \circ |\mathbb{1} + (V^{nom})^{\circ -1} \circ \Delta V|.$$

As, $|\Delta V| << \mathbb{1}$, $|(V^{nom})^{\circ-1}\Delta V| << \mathbb{1}$, considering $|V^{nom}| >> |\Delta V|$. Therefore, we can apply the result we obtained in part iv as follows:

$$\begin{aligned} |V| &\approx |V^{nom}| \circ \left(\mathbb{1} + \mathrm{Re}((V^{nom})^{\circ -1} \circ \Delta V)\right) \\ &= |V^{nom}| + |V^{nom}| \circ \mathrm{Re}((V^{nom})^{\circ -1} \circ \Delta V) \\ &= |V^{nom}| + |V^{nom}| \circ \mathrm{Re}(|V^{nom}|^{\circ -1} \circ (\cos\theta^{nom} - j\sin\theta^{nom}) \circ (\Delta V_{re} + j\Delta V_{im})) \\ &= |V^{nom}| + |V^{nom}| \circ \mathrm{Re}(|V^{nom}|^{\circ -1} \circ \cos\theta^{nom} \circ \Delta V_{re} + |V^{nom}|^{\circ -1} \circ \sin\theta^{nom} \circ \Delta V_{im} \\ &+ j|V^{nom}|^{\circ -1} \circ \cos\theta^{nom} \Delta V_{im} - j|V^{nom}|^{\circ -1} \circ \sin\theta^{nom} \circ \Delta V_{re}) \\ &= |V^{nom}| + |V^{nom}| \circ (|V^{nom}|^{\circ -1} \circ \cos\theta^{nom} \circ \Delta V_{re} + |V^{nom}|^{\circ -1} \circ \sin\theta^{nom} \circ \Delta V_{im}) \\ &= |V^{nom}| + \cos\theta^{nom} \circ \Delta V_{re} + \sin\theta^{nom} \circ \Delta V_{im} \\ &= |V^{nom}| + [\mathrm{diag}(\cos\theta^{nom}) \;\; \mathrm{diag}(\sin\theta^{nom})] \begin{bmatrix} \Delta V_{re} \\ \Delta V_{im} \end{bmatrix} \\ &= |V^{nom}| + [\mathrm{diag}(\cos\theta^{nom}) \;\; \mathrm{diag}(\sin\theta^{nom})] J^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}. \end{aligned}$$

Problem 2.vi

Similarly,

$$\begin{aligned} \angle V = \theta &= \angle(V^{nom} \circ \left(\mathbb{1} + (V^{nom})^{\circ -1} \circ \Delta V\right)) \\ &= \angle V^{nom} + \angle(\left(\mathbb{1} + (V^{nom})^{\circ -1} \circ \Delta V\right)) \\ &= \theta^{nom} + \angle(\left(\mathbb{1} + (V^{nom})^{\circ -1} \circ \Delta V\right)) \\ &\approx \theta^{nom} + \mathrm{Im}((V^{nom})^{\circ -1} \circ \Delta V) \\ &= \theta^{nom} + \mathrm{Im}(|V^{nom}|^{\circ -1} \circ (\cos\theta^{nom} - j\sin\theta^{nom}) \circ (\Delta V_{re} + j\Delta V_{im})) \\ &= \theta^{nom} + \mathrm{Im}(|V^{nom}|^{\circ -1} \circ \cos\theta^{nom} \circ \Delta V_{re} + |V^{nom}|^{\circ -1} \circ \sin\theta^{nom} \circ \Delta V_{im} \\ &+ j|V^{nom}|^{\circ -1} \circ \cos\theta^{nom} \Delta V_{im} - j|V^{nom}|^{\circ -1} \circ \sin\theta^{nom} \circ \Delta V_{re}) \\ &= \theta^{nom} + |V^{nom}|^{\circ -1} \circ \cos\theta^{nom} \circ \Delta V_{im} - |V^{nom}|^{\circ -1} \circ \sin\theta^{nom} \circ \Delta V_{re} \\ &= \theta^{nom} + |V^{nom}|^{\circ -1} \circ (\cos\theta^{nom} \circ \Delta V_{im} - \sin\theta^{nom} \circ \Delta V_{re}) \\ &= \theta^{nom} + |V^{nom}|^{\circ -1} \circ [-\mathrm{diag}(\sin\theta^{nom}) \;\; \mathrm{diag}(\cos\theta^{nom})] \begin{bmatrix} \Delta V_{re} \\ \Delta V_{im} \end{bmatrix} \\ &= \theta^{nom} + \mathrm{diag}(|V^{nom}|)^{-1} [-\mathrm{diag}(\sin\theta^{nom}) \;\; \mathrm{diag}(\cos\theta^{nom})] \begin{bmatrix} \Delta V_{re} \\ \Delta V_{im} \end{bmatrix} \\ &= \theta^{nom} + \mathrm{diag}(|V^{nom}|)^{-1} [-\mathrm{diag}(\sin\theta^{nom}) \;\; \mathrm{diag}(\cos\theta^{nom})] J^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}. \end{aligned}$$

Problem 2.vii