# Homework 4

**Problem 1.** Consider an AC electric power network with synchronous generators indexed in set $\mathcal{G}$ and transmission lines indexed in set $\mathcal{E}$. For each generator $g \in \mathcal{G}$, let $\delta_g$, $\omega_g$, $P_g^{\mathrm{m}}$, $P_g^{\mathrm{e}}$, and $P_g^{\mathrm{r}}$ denote the electrical angular position, angular speed, turbine mechanical power, electrical power output, and reference power input, respectively. Dynamics of generator $g$ can be described by the swing equations augmented with a simplified turbine-governor model, as follows:

$$\dot{\delta}_g = \omega_g - \omega_{\mathrm{s}}, \tag{1}$$

$$M_g \dot{\omega}_g = P_g^{\mathrm{m}} - P_g^{\mathrm{e}}, \tag{2}$$

$$\tau_g \dot{P}_g^{\mathrm{m}} = P_g^{\mathrm{r}} - P_g^{\mathrm{m}} - \frac{1}{R_g \omega_{\mathrm{s}}}(\omega_g - \omega_{\mathrm{s}}), \tag{3}$$

where $M_g$, $\tau_g$, and $R_g$ denote its inertia constant, governor time constant, and droop constant, respectively; and $\omega_{\mathrm{s}} = 2\pi 60$ [rad/s] is the synchronous frequency of the system. The electrical power output, $P_g^{\mathrm{e}}$, is a function of bus-voltage magnitudes and phase angles as solved from the network power balance. The generator $g$ reference power input is given by

$$P_g^{\mathrm{r}} = P_g^{\star} + \alpha_g(\xi - \sum_{j \in \mathcal{G}} P_j^{\star}), \tag{4}$$

where $P_g^{\star}$ is the setpoint from economic dispatch and $\alpha_g$ is the AGC participation factor with $\sum_{g \in \mathcal{G}} \alpha_g = 1$. Moreover, $\xi$ is the AGC state whose evolution is dictated by

$$\dot{\xi} = -\xi - \mathrm{ACE} + \sum_{g \in \mathcal{G}} P_g^{\mathrm{e}}, \tag{5}$$

where ACE denotes the *area control error* that accounts for deviations in frequency from the synchronous value. For a single-area system with no tie-line flows, the area control error is formulated as $\mathrm{ACE} = b(\omega - \omega_s)$ where $b > 0$ is the area bias factor and $\omega$ the prevailing frequency of the system. (Without loss of generality, we assume $\omega$ is computed with real-time measurements as the average electrical frequency of all generators.) Let $C_g(P_g)$ denote the cost function for generator $g$, and collect $P_g$, $g \in \mathcal{G}$, in $P_{\mathcal{G}}$. We will assume the economic dispatch problem takes the following form:

$$\min_{P_g,\, g \in \mathcal{G}} \sum_{g \in \mathcal{G}} C_g(P_g) \tag{6a}$$

$$\text{s.t.} \sum_{g \in \mathcal{G}} P_g = P_{\mathrm{load}} + P_{\mathrm{loss}}(P_{\mathcal{G}}), \tag{6b}$$

where $P_{\mathrm{load}}$ is the look-ahead net load and $P_{\mathrm{loss}}(P_{\mathcal{G}})$ is the system loss modeled as a function of $P_{\mathcal{G}}$.

***Part (i).*** Let us denote values taken by variables at a new steady-state operating point by $\overline{X}$ (corresponding values at the initial operating point are denoted by $X$). Suppose the system net load $P_{\mathrm{load}}$ changes by amount $\Delta P_{\mathrm{load}}$, so that the new net load is $\overline{P}_{\mathrm{load}} = P_{\mathrm{load}} + \Delta P_{\mathrm{load}}$. In steady state after the load change, show that the dynamical system (1)–(5) converges to the following operating point:

$$\overline{\omega}_g = \omega_{\mathrm{s}}, \ \forall\, g \in \mathcal{G}, \tag{7}$$

$$\overline{P}_g^{\mathrm{m}} = P_g^{\star} + \alpha_g(\Delta P_{\mathrm{load}} + \Delta P_{\mathrm{loss}}), \ \forall\, g \in \mathcal{G}, \tag{8}$$

$$\overline{\xi} = P_{\mathrm{load}} + \Delta P_{\mathrm{load}} + \Delta P_{\mathrm{loss}}, \tag{9}$$

where $\Delta P_{\rm loss}$ is the change in system loss due to the change in operating point.

***Part (ii).*** Show that the generator $g$ steady-state electrical power output with the new net load is

$$\overline{P}_g^{\rm e} = P_g^\star + \alpha_g(\Delta P_{\rm load} + \Delta P_{\rm loss}),\ \forall\, g \in \mathcal{G}. \tag{10}$$

***Part (iii).*** Show that the Karush–Kuhn–Tucker (KKT) conditions for the economic dispatch problem (6) are given by:

$$C_g'(P_g^\star) - \frac{\lambda^\star}{\Lambda_g^\star} = 0, \forall g \in \mathcal{G}, \tag{11}$$

where $\Lambda_g^\star$ is the *loss penalty factor* for generator $g$ evaluated at $P_{\mathcal{G}}^\star$; defined as

$$\Lambda_g^\star := \left(1 - \frac{\partial P_{\rm loss}(P_{\mathcal{G}}^\star)}{\partial P_g}\right)^{-1}. \tag{12}$$

***Part (iv).*** In the lecture modules, we have seen that a good choice for the AGC participation factors, $\alpha_g, g \in \mathcal{G}$ from an economic point of view is given by

$$\alpha_g = \frac{(C_g''(P_g^\star))^{-1}}{\sum\limits_{j\in\mathcal{G}}(C_j''(P_j^\star))^{-1}},\ \forall\, g \in \mathcal{G}. \tag{13}$$

We will now explore the impact of this choice choice of participation factors on system dynamic performance. Suppose the system initially operates in steady state with net load, $P_{\rm load}$, and generator electrical power outputs corresponding to a KKT point, $P_g^\star$, $g \in \mathcal{G}$, of the economic dispatch problem (6). Note that a KKT point is a solution that satisfies the KKT conditions but may not be a global/local minimum of the optimization problem. (Every minimum/maximum is a KKT point but not vice versa.) Consider AGC action triggered by a change in net load to $\overline{P}_{\rm load} = P_{\rm load} + \Delta P_{\rm load}$. Denote by $\overline{P}_g^\star$, $g \in \mathcal{G}$, a KKT point of the economic dispatch problem solved with the new net load, $\overline{P}_{\rm load}$. Further, denote the loss penalty factors corresponding to $P_g^\star$ and $\overline{P}_g^\star$, by $\Lambda_g^\star$ and $\overline{\Lambda}_g^\star$, respectively. Consider the following assumptions to hold true

[A1] The cost function $C_g(P_g)$ is twice differentiable.

[A2] The generator-cost and system-loss functions are such that for the KKT points $P_g^\star$, $\overline{P}_g^\star$:

$$\overline{\Lambda}_g^\star C_g'(\overline{P}_g^\star) - \Lambda_g^\star C_g'(P_g^\star) = (\overline{P}_g^\star - P_g^\star)C_g''(P_g^\star). \tag{14}$$

Under assumptions [A1]–[A2], prove that if the AGC participation factors are picked per (13), then the steady-state generator electrical power outputs following the net-load change correspond to KKT point, $\overline{P}_g^\star$, i.e.,

$$\overline{P}_g^{\rm e} = \overline{P}_g^\star,\ \forall\, g \in \mathcal{G}. \tag{15}$$

Comment on the implication/significance of the above result.

***Part (v).*** Can you provide specific cases (types of cost functions, types of networks) under which assumptions [A1]–[A2] are automatically satisfied?

**Problem 2.** *Notation.* The matrix transpose will be denoted by $(\cdot)^{\mathrm{T}}$, complex conjugate by $(\cdot)^*$, real and imaginary parts of a complex number by $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$, respectively, magnitude of a complex scalar by $|\cdot|$, and $\mathrm{j} := \sqrt{-1}$. A diagonal matrix formed with entries of the vector $x$ is denoted by $\mathrm{diag}(x)$; $\mathrm{diag}(x/y)$ forms a diagonal matrix with the $\ell$th entry given by $x_\ell/y_\ell$, where $x_\ell$ and $y_\ell$ are the $\ell$th entries of vectors $x$ and $y$, respectively; and $\mathrm{diag}(1/x)$ forms a diagonal matrix with the $\ell$th entry given by $x_\ell^{-1}$. For a vector $x = [x_1, \ldots, x_N]^{\mathrm{T}}$, $\cos(x) := [\cos(x_1), \ldots, \cos(x_N)]^{\mathrm{T}}$ and $\sin(x) := [\sin(x_1), \ldots, \sin(x_N)]^{\mathrm{T}}$. We will routinely decompose the complex-valued vector $x \in \mathbf{C}^N$ (complex-valued matrix $X \in \mathbf{C}^{N\times N}$) into its real and imaginary parts as follows: $x = x_{\mathrm{re}} + \mathrm{j}x_{\mathrm{im}}$ ($X = X_{\mathrm{re}} + \mathrm{j}X_{\mathrm{im}}$, respectively). The spaces of $N \times 1$ real-valued and complex-valued vectors are denoted by $\mathbf{R}^N$ and $\mathbf{C}^N$, respectively; $\mathbf{T}^N$ denotes the $N$-dimensional torus. With $0_N$ and $1_N$, we denote $N$-dimensional column vectors with all entries equal to 0 and 1, respectively. The $N \times N$ identity matrix is denoted by $I_{N\times N}$ and the $N \times N$ matrix with all 0 entries is denoted by $0_{N\times N}$.
*Power System Model.* Consider a power system with $N+1$ buses collected in the set $\mathcal{N}$. We model loads as the parallel interconnection of constant impedance and a constant power component. Without loss of generality, the slack bus is fixed to be the $N+1$ bus, and its voltage is denoted by $V_\circ \mathrm{e}^{\mathrm{j}\theta_\circ}$. Let $V = [V_1, \ldots, V_N]^{\mathrm{T}} \in \mathbf{C}^N$, where $V_\ell = |V|_\ell \angle \theta_\ell \in \mathbf{C}$ represents the voltage phasor at bus $\ell$. In subsequent developments, we will find it useful to define the vectors $|V| = [|V_1|, \ldots, |V_N|]^{\mathrm{T}} \in \mathbf{R}^N_{>0}$ and $\theta = [\theta_1, \ldots, \theta_N]^{\mathrm{T}} \in \mathbf{T}^N$. Given our focus on rectangular coordinates, we will also routinely express $V = V_{\mathrm{re}} + \mathrm{j}V_{\mathrm{im}}$, where $V_{\mathrm{re}}, V_{\mathrm{im}} \in \mathbf{R}^N$ denote the real and imaginary components of $V$. Let $I = [I_1, \ldots, I_N]^{\mathrm{T}}$, where $I_\ell \in \mathbf{C}$ denotes the current injected into bus $\ell$. Kirchhoff's current law for the buses in the power system can be compactly represented in matrix-vector form as follows:

$$
\begin{bmatrix} I \\ I_{N+1} \end{bmatrix} = \begin{bmatrix} Y & \overline{Y} \\ \overline{Y}^{\mathrm{T}} & y \end{bmatrix} \begin{bmatrix} V \\ V_\circ \mathrm{e}^{\mathrm{j}\theta_\circ} \end{bmatrix}, \tag{16}
$$

where $V_\circ \mathrm{e}^{\mathrm{j}\theta_\circ}$ is the slack-bus voltage, $I_{N+1}$ denotes the current injected into the slack bus, and the entries of the admittance matrix have the following dimensions: $Y \in \mathbf{C}^{N\times N}$, $\overline{Y} \in \mathbf{C}^N$, and $y \in \mathbf{C}\setminus\{0\}$. We will decompose the matrix $Y$ and its inverse $Y^{-1}$ as follows:

$$
Y = G + \mathrm{j}B, \quad Y^{-1} = R + \mathrm{j}X. \tag{17}
$$

Denote the vector of complex-power bus injections by $S = [S_1, \ldots, S_N]^{\mathrm{T}}$, where $S_\ell = P_\ell + \mathrm{j}Q_\ell$. By convention, $P_\ell$ and $Q_\ell$ are positive for generators and negative for loads.
***Part (i).*** Using (16), show that the complex-power bus injections can be compactly written as

$$
S = \mathrm{diag}\,(V) \left( Y^* V^* + \overline{Y}^* V_\circ \mathrm{e}^{-\mathrm{j}\theta_\circ} \right). \tag{18}
$$

***Part (ii).*** Express $V = V^{\mathrm{nom}} + \Delta V$, where $V^{\mathrm{nom}}$ is some a priori determined nominal voltage vector, and entries of $\Delta V$ capture perturbations around $V^{\mathrm{nom}}$. With the choice

$$
V^{\mathrm{nom}} = -Y^{-1}\overline{Y} V_\circ \mathrm{e}^{\mathrm{j}\theta_\circ}, \tag{19}
$$

show that $\Delta V$ can be solved from the following linear equation (once second-order terms are neglected)

$$
\mathrm{diag}\,((V^{\mathrm{nom}})^*)\, Y \Delta V = S^*. \tag{20}
$$

***Part (iii).*** Define the following matrices

$$K = \mathrm{diag}(V^{\mathrm{nom}})Y^*, \quad J = \begin{bmatrix} \mathrm{Re}(K) & \mathrm{Im}(K) \\ \mathrm{Im}(K) & -\mathrm{Re}(K) \end{bmatrix}. \tag{21}$$

Show that the real and imaginary parts of $\Delta V$, $\Delta V_{\mathrm{re}}$ and $\Delta V_{\mathrm{im}}$ are given by the solution of the following linear equations

$$\begin{bmatrix} \Delta V_{\mathrm{re}} \\ \Delta V_{\mathrm{im}} \end{bmatrix} = J^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}. \tag{22}$$

***Part (iv).*** Consider a complex scalar $\eta$ such that $|\eta| << 1$. Show that

$$|1+\eta| \approx 1 + \mathrm{Re}(\eta), \tag{23}$$

$$\angle(1+\eta) \approx \mathrm{Im}(\eta). \tag{24}$$

***Part (v).*** Leveraging the approximations in (23)–(24) and the expressions in (21)–(22), show that

$$|V| \approx |V^{\mathrm{nom}}| + [\mathrm{diag}(\cos(\theta^{\mathrm{nom}})) \quad \mathrm{diag}(\sin(\theta^{\mathrm{nom}}))] J^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}. \tag{25}$$

***Part (vi).*** Leveraging the approximations in (23)–(24) and the expressions in (21)–(22), show that

$$\theta \approx \theta^{\mathrm{nom}} + \mathrm{diag}(|V^{\mathrm{nom}}|)^{-1} [-\mathrm{diag}(\sin(\theta^{\mathrm{nom}})) \quad \mathrm{diag}(\cos(\theta^{\mathrm{nom}}))] J^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}. \tag{26}$$

***Part (vii).*** Consider a two-bus power system composed of a single $PQ$ load connected to bus 1 and bus 2 representing the slack bus. At buses 1 and 2, there are shunt admittances of 0.02j; and the impedance of the line connecting bus 1 and 2 is $0.01 + 0.05$j. The real power drawn by the load at bus 1 is 2 p.u., and the reactive power drawn is 0.75 p.u. The voltage of the slack bus, $V_{\mathrm{o}}\angle\theta_{\mathrm{o}} = 1\angle 30^{\mathrm{o}}$. Compute the voltage magnitude and phase angle at bus 1 using the Newton-Raphson approach run out for 5 iterations. (Choose the bus-2 voltage as the initial guess for the voltage magnitude and phase angle.) Verify the accuracy of the approximations in (25)-(26) with the result you obtain from the Newton-Raphson method. Include your code in the solution.