Homework 5

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0.1 Question 1:

1. Write a program to solve a driven pendulum problem that we discussed in the class. Suppose that a pendulum consists of a light string of length l and a point mass m attached to the end of the rod. The end of the string is tied to a heavy horizontal bar aligned along with the x axis and attached to a heavier stand. The pendulum swings in the yz plane. Three kinds of forces are acting on the pendulum: gravity, frictional force linearly proportional to the angular velocity $-kld\theta/dt$, and driving force $f_d^0 cos(\omega_0 t)$, where k is a proportional constant, fd0 is the amplitude, and ω_0 is the angular frequency of the driving force. Initially the pendulum is located vertically ($\theta = 0$), where θ is the angle between the rod and the vertical line. You may choose the range of the angle as $(-\Pi, \Pi)$ or $(0, 2\Pi)$. The initial angular velocity of the pendulum is $\omega = 11.431 sec^{-1}$. Use the following numbers for the parameters: $l = 30$ cm, (k/m) l/g = 0.5, $\omega l/g = 2/3$, $\tau = 3\Pi/100$ (time interval), and $t = 1000\tau$ (elapsed time).

Answer:

Driven Pendulum Problem

We have a pendulum consisting of a light string of length l and a point mass m attached to the end of the rod. The end of the string is tied to a heavy horizontal bar aligned along the x-axis and attached to a heavier stand. The pendulum swings in the yz plane.

Three types of forces act on the pendulum:

1. Gravity : mg
2. Frictional force linearly proportional to the angular velocity: $-k \cdot l \frac{d\theta}{dt}$
3. Driving force: $f_d^0 \cdot \cos(\omega_0 t)$

Where:

k is a proportional constant

f_d^0 is the amplitude of the driving force

ω_0 is the angular frequency of the driving force

Initially, the pendulum is located vertically, meaning $\theta = 0$, where θ is the angle between the rod and the vertical line. The initial angular velocity of the pendulum is $\omega = 11.431\,\sec^{-1}$.

We can use the following values for the parameters:

$$
\begin{aligned}
l &= 30\,\text{cm} \\
\frac{k}{m} \cdot \frac{l}{g} &= 0.5 \\
\omega_0 \cdot \frac{l}{g} &= \frac{2}{3} \\
\tau &= \frac{3\pi}{100} \;\; \text{(time interval)} \\
t &= 1000\tau \;\; \text{(elapsed time)}
\end{aligned}
$$

So, we have ordinary differential equation:

$$ml\frac{d^2\theta}{dx^2} = -mgsin(\theta) - k \cdot l\frac{d\theta}{dt} + f_d^0 \cdot \cos(\omega_0 t) \tag{1}$$

which finally becomes after some math,

$$\frac{d^2\theta}{dt\prime^2} = -sin(\theta) - \frac{k}{m} \cdot \sqrt{\frac{l}{g}}\frac{d\theta}{dt\prime} + \frac{f_d^0}{mg} \cdot \cos(\bar{\omega_0} t\prime) \tag{2}$$

$$t = \sqrt{\frac{l}{g}t\prime} \tag{3}$$

1. when, the $f_d^0 = 0$,the driven force is zero. So the angular acceleration is less. And that's why the pendulum will stop after some time as there is no driven force because of the friction and gravity pull.

2. when, the $f_d^0 = 0.89mg$, the driven force amplitude is less than 1. the effective force is there

Algorithm 1 Fourth-Order Runge-Kutta Solver

procedure RUNGEKUTTASOLVER($\theta_0, \omega_0, t_{\max}, \tau, \frac{k}{m}\sqrt{\frac{l}{g}}, \frac{\omega_0 l}{g}, \frac{f_{d0}}{mg}$)
  $\theta \leftarrow \theta_0$
  $\omega_{\text{bar}} \leftarrow \omega_0$
  $time \leftarrow [0.0]$
  $\theta_{\text{values}} \leftarrow [\theta]$
  $\omega_{\text{values}} \leftarrow [\omega_{\text{bar}}]$
  for t_{bar} in $0.0 : \tau : t_{\max}$ **do**
    $k1_\theta \leftarrow \tau \cdot \omega_{\text{bar}}$
    $k1_\omega \leftarrow \tau \cdot \text{angular\_acceleration}(\theta, \omega_{\text{bar}}, t_{\text{bar}}, \frac{f_{d0}}{mg}, \frac{k}{m}\sqrt{\frac{l}{g}}, \frac{\omega_0 l}{g})$
    $k2_\theta \leftarrow \tau \cdot (\omega_{\text{bar}} + 0.5 \cdot k1_\omega)$
    $k2_\omega \leftarrow \tau \cdot \text{angular\_acceleration}(\theta + 0.5 \cdot k1_\theta, \omega_{\text{bar}} + 0.5 \cdot k1_\omega, t_{\text{bar}} + 0.5 \cdot \tau, \frac{f_{d0}}{mg}, \frac{k}{m}\sqrt{\frac{l}{g}}, \frac{\omega_0 l}{g})$
    $k3_\theta \leftarrow \tau \cdot (\omega_{\text{bar}} + 0.5 \cdot k2_\omega)$
    $k3_\omega \leftarrow \tau \cdot \text{angular\_acceleration}(\theta + 0.5 \cdot k2_\theta, \omega_{\text{bar}} + 0.5 \cdot k2_\omega, t_{\text{bar}} + 0.5 \cdot \tau, \frac{f_{d0}}{mg}, \frac{k}{m}\sqrt{\frac{l}{g}}, \frac{\omega_0 l}{g})$
    $k4_\theta \leftarrow \tau \cdot (\omega_{\text{bar}} + k3_\omega)$
    $k4_\omega \leftarrow \tau \cdot \text{angular\_acceleration}(\theta + k3_\theta, \omega_{\text{bar}} + k3_\omega, t_{\text{bar}} + \tau, \frac{f_{d0}}{mg}, \frac{k}{m}\sqrt{\frac{l}{g}}, \frac{\omega_0 l}{g})$
    $\theta \leftarrow \theta + \frac{k1_\theta + 2 \cdot k2_\theta + 2 \cdot k3_\theta + k4_\theta}{6}$
    $\omega_{\text{bar}} \leftarrow \omega_{\text{bar}} + \frac{k1_\omega + 2 \cdot k2_\omega + 2 \cdot k3_\omega + k4_\omega}{6}$
    append t_{bar} to $time$
    append θ to θ_{values}
    append ω_{bar} to ω_{values}
  end for
  return $time, \theta_{\text{values}}, \omega_{\text{values}}$
end procedure

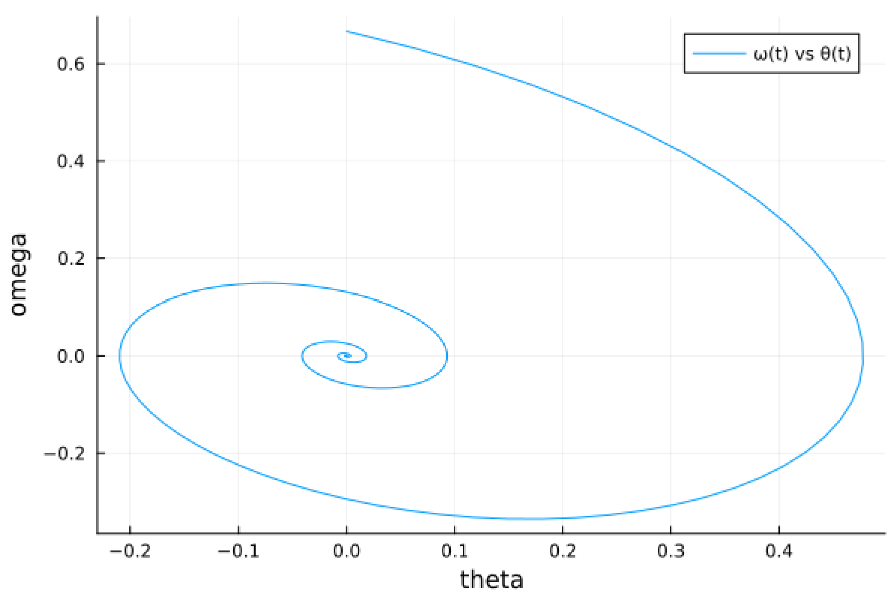

Figure 1: ω vs θ when $f_d^0 = 0mg$

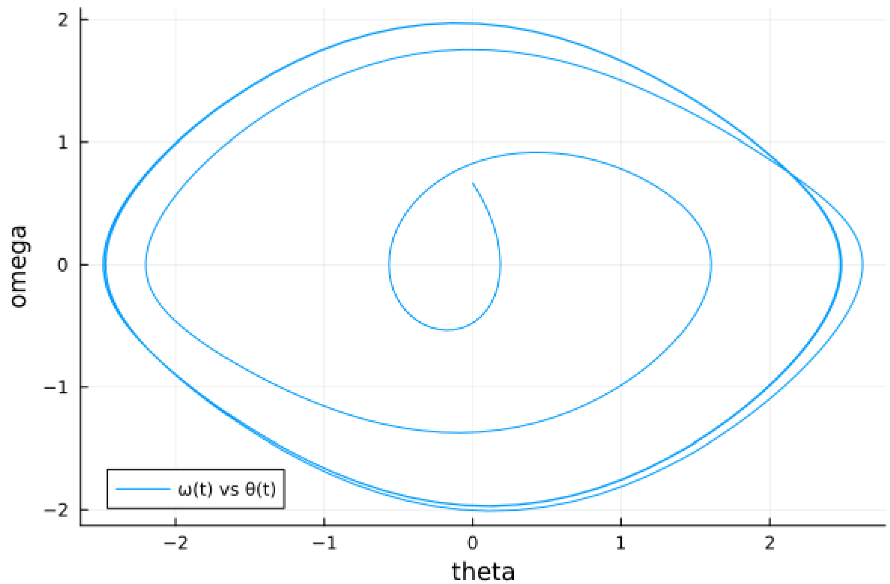

Figure 2: ω vs θ when $f_d^0 = 0.89mg$

but the driven force is not more than the dampening force. The the angular velocity is more than the initial one but after a certain time the angular velocity of pendulum will eventually die out.

3. when, the $f_d^0 = 0.89mg$, the driven force amplitude is greater than 1. The angular velocity of the pendulum is not in circular motion with the angle.

Time:

f_d^0	**Execution Time (s)**	**Allocations**	**Memory Usage (KiB)**
0mg	0.000296	16	43.969
0.89mg	0.000241	13	43.781
1.145mg	0.000156	13	43.781

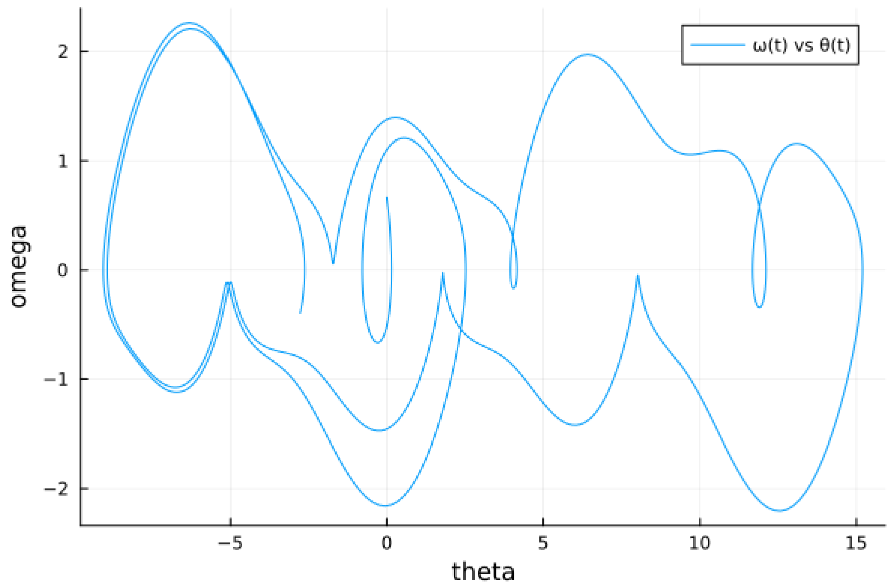

Figure 3: ω vs θ when $f_d^0 = 1.145mg$

0.2 question 2:

Consider the following ordinary differential equation (ODE) with two boundary conditions:

$$\frac{d^2y}{dx^2} = -4\pi^2 y(x)$$

subject to the boundary conditions:

$$y(x=0) = 1$$
$$\frac{dy}{dx}(x=1) = 2\pi$$

where x is defined in the interval $0 \leq x \leq 1$.

The objective is to solve this ODE using the shooting method, employing the 4th-order Runge-Kutta method in combination with a root-finding technique like the secant method.

Answer:

1. The given second-order linear homogeneous ordinary differential equation is:

$$\frac{d^2y}{dx^2} = -4\pi^2 y$$

with the initial conditions:

$$y(x = 0) = 1$$

and

$$\left.\frac{dy}{dx}\right|_{x=1} = 2\pi$$

The general solution to the differential equation is:

$$y(x) = A\sin(2\pi x) + B\cos(2\pi x)$$

Now, applying the initial conditions:

1. $y(x = 0) = 1$: Plugging in $x = 0$ and setting $y(0)$ equal to 1:

$$1 = B\cos(0) = B \cdot 1$$

Therefore, $B = 1$

2. $\frac{dy}{dx}(x = 1) = 2\pi$: Taking the derivative of $y(x)$ with respect to x:

$$\frac{dy}{dx} = 2\pi A\cos(2\pi x) - 2\pi B\sin(2\pi x)$$

Plugging in $x = 1$ and setting $\frac{dy}{dx}(1)$ equal to 2π:

$$2\pi = 2\pi A\cos(2\pi) - 2\pi \cdot 1 \cdot \sin(2\pi)$$

Since $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$, you have:

$$2\pi = 2\pi A$$

Therefore, $A = 1$

So, the specific solution for the given initial conditions is:

$$y(x) = \sin(2\pi x) + \cos(2\pi x)$$

This is the solution to the differential equation $\frac{d^2y}{dx^2} = -4\pi^2 y$ with the initial conditions $y(x = 0) = 1$

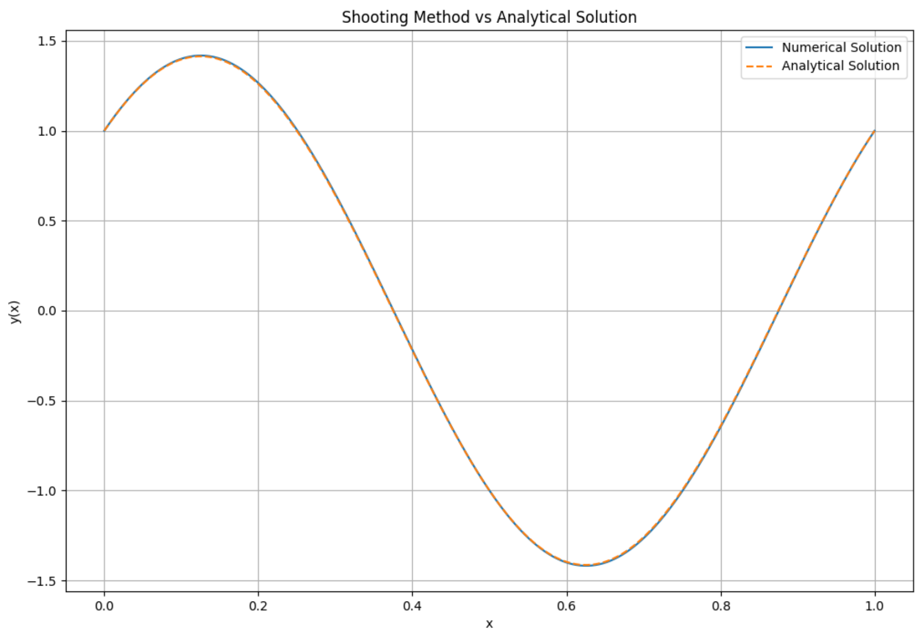

Figure 4: Shooting method solution vs analytical solution

and $\frac{dy}{dx}(x = 1) = 2\pi$ over the interval $0 \leq x \leq 1$.

2. I have used $\frac{dy}{dx}_{x=0} = 0$ as guess.

3. if we increase the number of points from 100 to 1000 , we can see the the convergence of numerical solution to analytical solution. Time: Execution Time: 1.8681316375732422 seconds