Homework 6

Arnab Bachhar

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2. Solving the Laplace Equation Using Relaxation Method

Part I: Energy Functional

The "energy" functional E for the Laplace equation is defined as:

$$E = \int (\nabla\phi)^2 dV,$$

where $\nabla\phi$ is the gradient of ϕ and dV represents the volume element. The variation of E with respect to ϕ gives the Laplace equation.

Part II: Discretized Forms

The Laplace equation can be discretized on a grid with lattice spacing h as:

$$\nabla^2\phi \approx \frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}}{h^2}$$

The discretized energy functional can be written as:

$$E = \sum(\nabla\phi)^2 = \sum \left(\frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}}{h^2}\right)^2$$

Part III: Omega dependency

Omegas = [1.0, 1.2, 1.4, 1.6] has been used to show that converged "energy" functional does not depend on the values of the relaxation parameter for a fixed lattice spacing, h. We can see that from the figure 1.

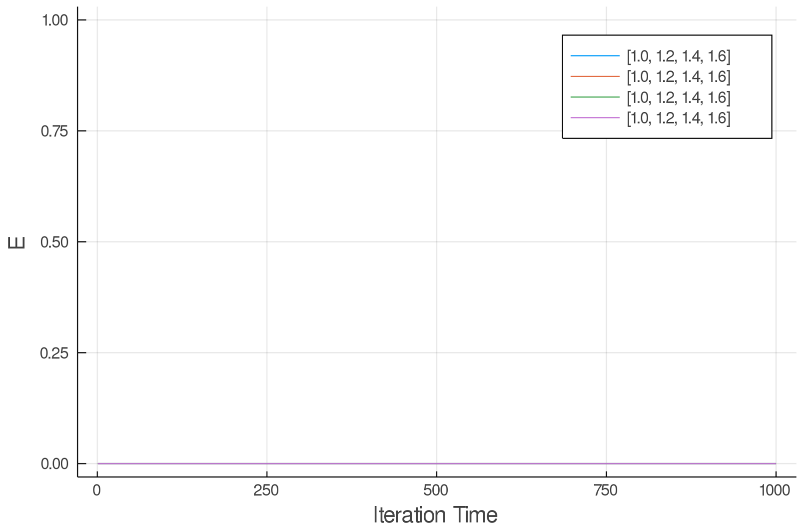

Figure 1: Energy functional vs iteration time to show dependency of relaxation parameter, $\bar{\omega}$ for "energy" functional

Part IV: Analytical Solution for E

The convergence of the energy functional E doesn't depend on the relaxation parameter $\bar{\omega}$. Typically, $\bar{\omega}$= 1 value is used, corresponding to the Gauss-Seidel method.

The Laplace equation is given as:

$$\nabla^2\phi = 0,$$

with the following boundary conditions:

$$\phi(x, y = 0) = x(1 - x)$$

$$\phi(x = 0, y) = 0$$

$$\phi(x, y = \infty) = 0, \quad 0 \le x \le 1$$

$$\phi(x = 1, y) = 0, \quad 0 \le y \le \infty$$

Using the separation of variables, the equation can be written as:

$$\Phi(x, y) = X(x)Y(y)$$

$$\frac{\partial^2\Phi(x, y)}{\partial x^2} + \frac{\partial^2\Phi(x, y)}{\partial y^2} = 0$$

Dividing by $\Phi(x, y)$ in both sides,

As $x\&y$ are independent, each term can be equal to a Constant,

$$\frac{\partial^2 x}{\partial x^2} = \lambda^2 X, \frac{\partial^2 Y}{\partial y^2} = -\lambda^2 Y.$$

The general solution can be written as,

$$Y = \alpha \cos \lambda y + \beta \sin \lambda y, \tag{1}$$

$$X = \gamma \cosh \lambda x + \delta \sinh \lambda x \tag{2}$$

The analytical solution of E is 0 after applying the boundary conditions.

Part V: Lattice spacing dependency

To check the convergence with finer meshes, we can change the lattice spacing h. Smaller h values lead to finer meshes, and the solution should converge closer to zero (analytical value of E).

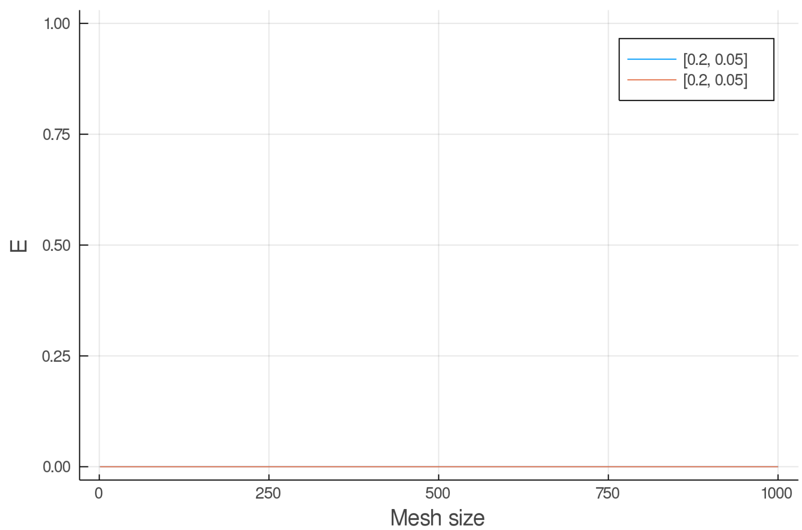

Figure 2: Energy functional vs iteration time to show dependency of relaxation parameter, $\bar{\omega}$ for "energy" functional

Part VI: Fitting using a polynomial function of h

E is being calculated for different meshes, and I have fitted the data points using a quadratic polynomial function of h.

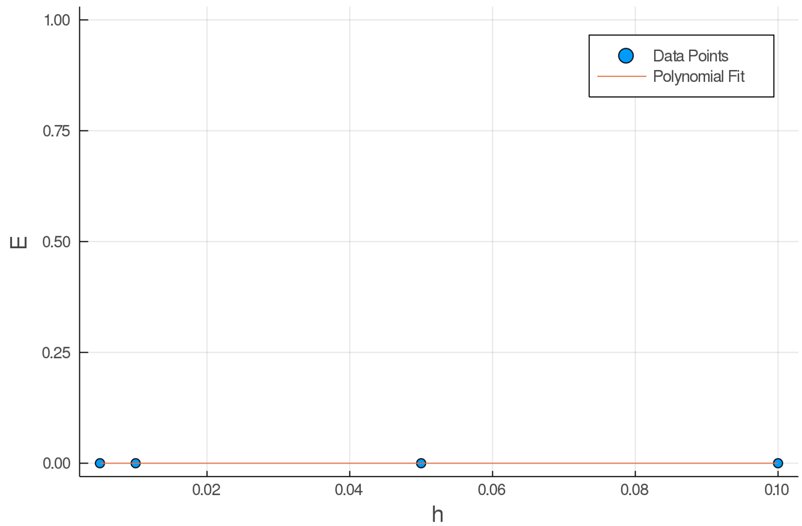

Figure 3: Fitting using a polynomial function of h

Part VII: Comparison with the analytical value

The result of $(E(h \to 0))$ is 0, same as the analytical value of E.

Part VIII: Numerical and analytical solution comparison

I have plotted both the analytical and numerical solutions to the problem. Both are giving the same result. As the numerical one is plotted step by step, it looks slightly different, like a layered one. But both plots are actually giving the same results.

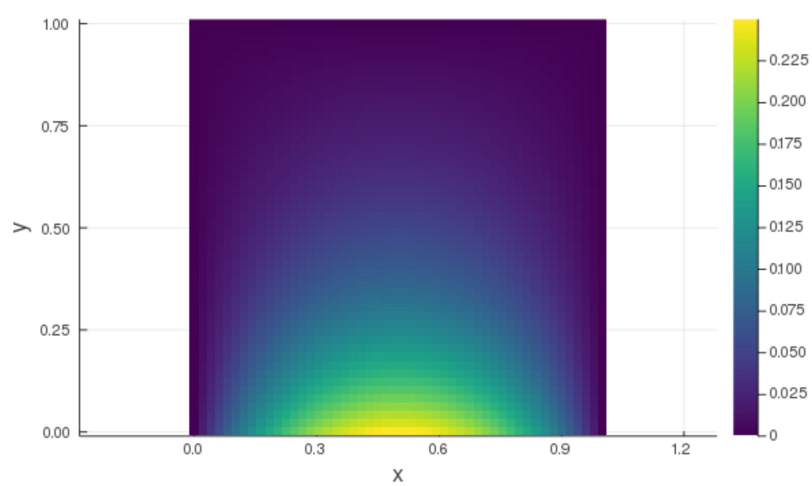

Figure 4: Analytical solution of differential equation

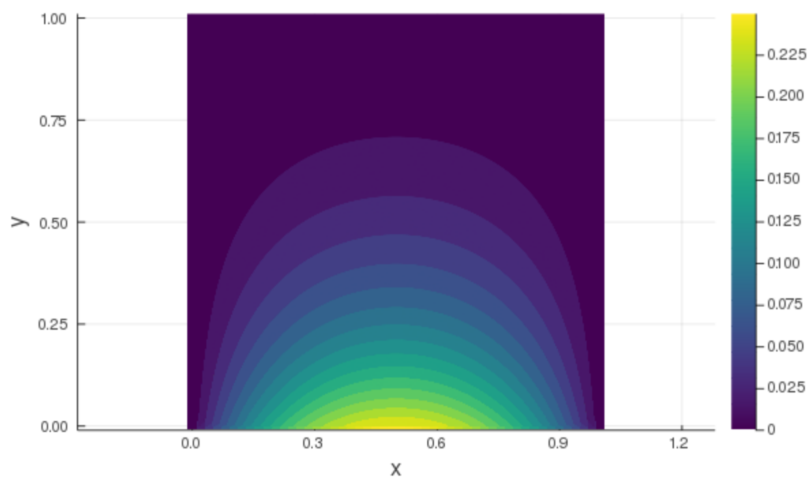

Figure 5: Numerical solution of differential equation