Problem: $x'' = x \sin t + x' \cos t - \exp t$ for $t \in (0,1)$

$x(0) = 0, \quad x(1) = 1.$

Solution: Replacing

$$x'' = \frac{1}{h^2}\left[x_{i+1} - 2x_i + x_{i-1}\right]$$

$h \to$ step size

$n \to$ number of steps.

$$x' = \frac{1}{2h}\left[x_{i+1} - x_{i-1}\right]$$

We are at t_i:

$$\frac{1}{h^2}\left[x_{i+1} - 2x_i + x_{i-1}\right] = x_i \sin t_i + \frac{1}{2h}\left[x_{i+1} - x_{i-1}\right]\cos t_i - \exp(t_i)$$

multiplying the above equation with h^2.

$$\Rightarrow x_{i+1} - 2x_i + x_{i-1} = h^2 x_i \sin t_i + \left(\frac{h}{2} x_{i+1} - \frac{h}{2} x_{i-1}\right)\cos t_i - h^2 \exp(t_i)$$

multiplying with 2.

$$\Rightarrow 2x_{i+1} - 4x_i + 2x_{i-1} = 2h^2 x_i \sin t_i + \left(h x_{i+1} - h x_{i-1}\right)\cos t_i - 2h^2 \exp(t_i)$$

$$\Rightarrow (2 - h\cos t_i)\, x_{i+1} + (-4 - 2h^2 \sin t_i)\, x_i + (2 + h\cos t_i)\, x_{i-1} = -2h^2 \exp(t_i)$$

where $i = 1 \longrightarrow (n-1)$

Let

$c_i = 2 - h\cos t_i$

$d_i = -4 - 2h^2 \sin t_i$

$a_i = 2 + h\cos t_i$

$b_i = -2h^2 \exp(t_i)$

$i=1$: $c_1 x_2 + d_1 x_1 + a_1 x_0 = b_1$

$\Rightarrow d_1 x_1 + c_1 x_2 = b_1 - a_1 x_0$

But $x_0 = 0$, $x_n = 1$ as given

$i=2$: $c_2 x_3 + d_2 x_2 + a_2 x_1 = b_2$

$i=3$: $c_3 x_4 + d_3 x_3 + a_3 x_2 = b_3$

$\vdots$

$i=n-1$: $c_{n-1} x_n + d_{n-1} x_{n-1} + a_{n-1} x_{n-2} = b_{n-1}$

$a_{n-1} x_{n-2} + d_{n-1} x_{n-1} = b_{n-1} - c_{n-1} x_n$

$\therefore$ Writing the above equations in matrix form.

$$\underbrace{\begin{bmatrix} d_1 & c_1 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & d_2 & c_2 & 0 & \cdots & 0 & 0 \\ 0 & a_3 & d_3 & c_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1} & d_{n-1} \end{bmatrix}}_{a[i][j]} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{Bmatrix}}_{y[i]} = \underbrace{\begin{Bmatrix} b_1 - a_1 x_0 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} - c_{n-1} x_n \end{Bmatrix}}_{b[i]}$$

The above set of Linear equations is solved in coding using the gauss - elimination of tri-diagonal matrix.