STA 5107/4013: Home Assignment # 4

Spring 2021/Due Date: February 16th

1. Let X_t be a Markov chain generated using some initial probability $P[1]$ and the transition matrix Π,

$$\Pi = \begin{bmatrix} 0.1 & 0.3 & 0.4 & 0.2 \\ 0.2 & 0.1 & 0.3 & 0.4 \\ 0.4 & 0.2 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.2 & 0.1 \end{bmatrix} .$$

First verify if X_t is (i) irreducible, and (ii) aperiodic, and then find the stationary probability vector for X_t.
Then, using your program from the last homework verify that the averages along a sample path converge to the stationary probability. Define a function $f : \{a_1, a_2, a_3, a_4\} \to \mathbb{R}$ as follows:

$$f(a_1) = 2.0,\ f(a_2) = 1.0,\ f(a_3) = 2.5,\ f(a_4) = -1.0 .$$

Show through simulation that

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i) \overset{n\to\infty}{\to} \sum_{j=1}^{4} f(a_j)P(a_j) ,$$

where P is the stationary probability.

2. Repeat Problem 2 for the transition matrix

$$\Pi = \begin{bmatrix} 0.1 & 0.3 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.0 & 0.4 \\ 0.0 & 0.3 & 0.5 & 0.2 \\ 0.5 & 0.3 & 0.2 & 0.0 \end{bmatrix} .$$

3. Let Π be an $m \times m$ transition matrix of a irreducible, homogeneous Markov chain on a finite state space. Suppose the Π is idempotent, i.e. $\Pi^2 = \Pi$. Prove that the Markov chain is aperiodic and that all rows of Π are identical.

4. Write a matlab program implementing the Metropolis-Hastings algorithm to sample a random variable X with the density

$$f(x) = \frac{x^2|sin(\pi x)|e^{-|x|^3}}{\int_{\mathbb{R}_+} x^2|sin(\pi x)|e^{-|x|^3}dx} , \quad x > 0 .$$

You have to decide what q (proposal density) you want to use. Choose positive numbers to start the Markov chain.

(a) Plot the density function $f(x)$.

(b) Histogram the values attained by Markov chain and compare it to the plot of $f(x)$.

(c) Estimate the value of $E[X]$ and $var(X)$ using values of Markov chain.