# STA 5107: Mid-Term Project

Spring 2021/Due Date: March 25

**Bayesian Analysis of Noisy Images**

1. **Goal**: Given observed noisy images, our goal is to perform a Bayesian analysis of this data. We will assume a prior probability model and an observation model to obtain a posterior density, and will generate samples from the posterior.

2. **Models**:

   (a) **Prior Model**: Let $I \in I\!R^{m \times n}$ be a matrix of random variables such that they form a Markov Random Field (MRF). The conditional density of an element is dependent only on the values of its vertical and horizontal neighbors (except for the boundaries where the neighbors are limited). Let the conditional density of a pixel be Gaussian with mean $m$ and variance $\sigma_1^2$, where $m$ is the mean of its neighbors. That is:

   $$f(I_{i,j}|\text{all other pixels}) = f(I_{i,j}|I_{i,j-1}, I_{i-1,j}, I_{i+1,j}, I_{i,j+1}) = \mathcal{N}(m, \sigma_1^2) \ .$$

   This model specifies the prior probability density $f(I)$ on the image space.

   (b) **Observation Model**: Let $D$ be a noisy observation of $I$ given by the model:

   $$D = I + W \ ,$$

   where each element of $W$ is an independent normal random variable with mean zero and variance $\sigma_2^2$. This equation specifies the likelihood function $f(D|I)$ for a given observation $D$.

   (c) **Posterior Density**: The posterior density on the image space can be written as:

   $$f(I|D) \propto f(D|I)f(I) \ .$$

   Given the structure of $f(I)$ and the independence of the elements of $W$, the full conditional of this posterior density can be written as:

   $$\begin{aligned} f(I_{i,j}|D, \text{all other pixels}) &= f(I_{i,j}|D_{i,j}, I_{i-1,j}, I_{i,j+1}, I_{i,j-1}, I_{i+1,j}) \\ &= \mathcal{N}(m, \sigma_1^2)\mathcal{N}(D_{i,j}, \sigma_2^2) \end{aligned}$$

3. **Sampling from the Posterior**: We will use an MCMC technique for sampling from the posterior. To sample from the prior density we have already used a Gibbs sampler that sequentially samples from each full conditional (of the prior). Using the same idea for $f(I|D)$ we can sequentially sample from the full conditional (of the posterior). I suggest that you investigate the use of Metropolis-Hastings for sampling from the conditionals. Write a matlab program to sample from the posterior density on the image space.

4. **Experiment**: Download five data images from the class website and run use your program to generate posterior samples. For each image document the evolution of the Gibbs sampler by showing intermediate results. Use $\sigma_2 = 30$, and try different values of $\sigma_1 = 10$, 20, and 100.

5. **Report**: Prepare a report for this project describing completely all parts of the project. Ten percent of the grade is allocated to the quality of presentation.