STA 5107/4013: Home Assignment # 9

Spring 2021/Due Date: April 13th

1. Let $\{x_k \in \mathcal{A}, k = 1, 2, \ldots, N\}$ represent a discrete-time and discrete-state process in the state space $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$. Our goal is to

---

Algorithm 1 *(***Dynamic Programming Algorithm***)*

(a) Initialize $C_{i,N} = 0$ for all $i = 1, 2, \ldots, n$

(b) For each $k = N - 1 : -1 : 1$ and all i, perform the following:

$$C_{i,k} = \min_{j \in 1,2,\ldots,n} (A_{ij} + C_{j,k+1}) \ .$$

Also, store the minimizing j^* as $p(i,k) = j^*$.

(c) Find the smallest starting cost at time $k = 1$: $i^* = \text{argmin}_i C_{:1}$.

(d) Reconstruct the optimal path: for $k = 1, \ldots, N - 1$,

$$\alpha(k+1) = p(\alpha(k), k), \quad \alpha(1) = i^* \ .$$

Write a matlab program to implement this algorithm. To demonstrate your code, use the transition costs given by the matrix:

$$A = \begin{bmatrix} Inf & 60 & 7 & 91 & 71 & 70 & 17 & 57 & 100 & 64 \\ 57 & Inf & 30 & 38 & 24 & 2 & 40 & 56 & 72 & 13 \\ 21 & 26 & Inf & 53 & 18 & 77 & 3 & 34 & 22 & 87 \\ 10 & 66 & 75 & Inf & 24 & 34 & 43 & 99 & 65 & 49 \\ 7 & 33 & 32 & 96 & Inf & 33 & 1 & 15 & 13 & 28 \\ 3 & 53 & 50 & 82 & 5 & Inf & 5 & 86 & 56 & 32 \\ 34 & 13 & 63 & 43 & 56 & 39 & Inf & 79 & 79 & 67 \\ 56 & 46 & 76 & 82 & 95 & 22 & 21 & Inf & 42 & 51 \\ 85 & 100 & 70 & 45 & 79 & 57 & 61 & 2 & Inf & 45 \\ 63 & 40 & 14 & 39 & 25 & 23 & 24 & 27 & 59 & Inf \end{bmatrix} .$$

You can also download this matrix from the course website. Present your results in form of plot as shown in Figure 1.

2. **Spatial Dynamic Programming**: Here we are given a set of n locations (cities) $\{X_i \in \mathbb{R}^2, n = 1, 2, \ldots, n\}$. We have the pairwise distances between then D_{ij} –

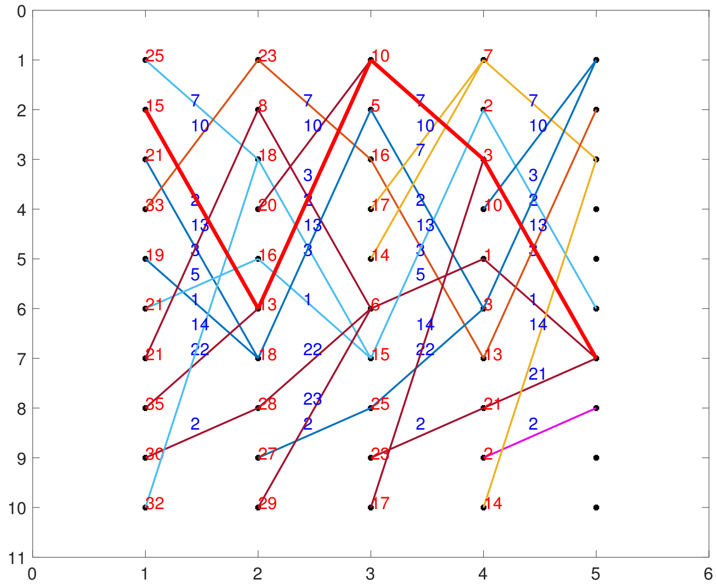

Figure 1: Optimal path obtained using dynamic programming algorithm.

$D_{ij} = \|X_i - X_j\|$ if X_i and X_j are connected, otherwise $D_{ij} = \infty$. Our goal is to find shortest path connecting any two arbitrarily chosen cities. We can apply Algorithm 1 to solve this problem except we need to restrict to the two cities that are of interest. We create an extra variable I that keeps track of cities that are relevant at each time.

Algorithm 2 (**Spatial Dynamic Programming Algorithm**) *Given a pairwise distance matrix D, and two cities X_{i_0} and X_{j_0}, find the shortest path connecting them.*

(a) *Choose $N = n$. Initialize $C_{i,N} = 0$ for all $i = 1, 2, \ldots, n$. Set $I_{i,N} = 0$ for all i, except set $I_{j_0,N} = 1$. Set $k = N - 1$.*

(b) *For all i, perform the following:*

$$C_{i,k} = \min_{j \in 1,2,\ldots,n} \left(D_{ij} + C_{j,k+1} + M * (1 - I(j, k+1)) \right),$$

where M is a large positive number (say $M = 1000$). Store the minimizing j^* as $p(i,k) = j^*$. If $C(i,k) < M$, set $I(i,k) = 1$, else it is 0.

(c) *Check if the starting city is reached: If $I(i_0, k) = 1$, then we have found a path and go to the next step. Otherwise, set $k = k - 1$ and return to Step (b).*

(d) *Draw the path city sequence: for $kk = k, k+1, \ldots, N-1$,*

$$\alpha(kk+1) = p(\alpha(kk), kk), \quad \alpha(k) = i_0.$$

Implement this algorithm in matlab. Use the data given on the class website (it contains $X \in \mathbb{R}^{n \times 2}$ and $D \in \mathbb{R}^{n \times n}$) to test your code and demonstrate it using

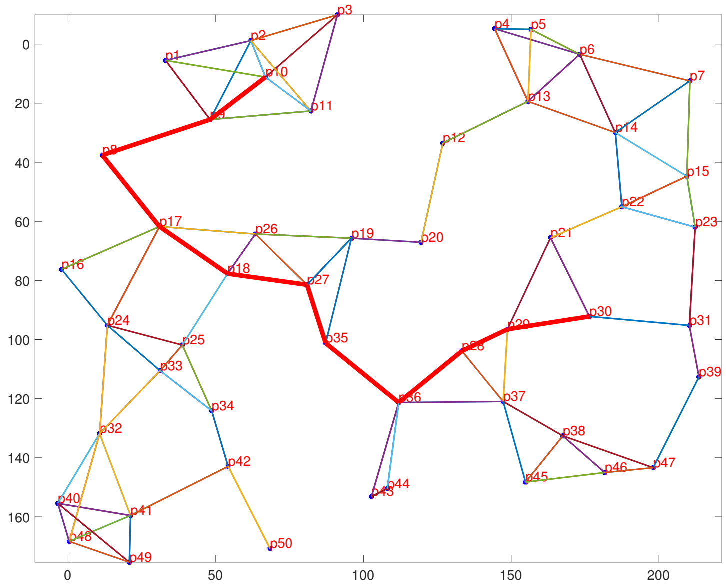

Figure 2: Optimal path from city 10 to city 30.

pictures of the kind shown in Fig. 2.