STA 5107/4013: Home Assignment # 6

Spring 2021/Due Date: March 5th

1. Write a matlab program implementing a Gibbs sampler to sample from the Markov Random Field model for binary images introduced in the class:

$$f(s) \propto \exp\{-H \sum_i s_i - J \sum_{(i,j) \in \mathcal{N}} s_i s_j\} \ .$$

The full conditional density of s_i, given all other pixels, is:

$$f(s_i | s_{j \neq i}) = \frac{\exp\{-(H + J \sum_{j \in \mathcal{N}_i} s_j))(s_i + 1)\}}{1 + \exp\{-2(H + J \sum_{j \in \mathcal{N}_i} s_j)\}} \ .$$

Choose the image size to be 10×10. Use a random image of 1s and (-1)s to initialize the program. Study the four cases: (i) $H = 1$, $J = 1$, (ii) $H = -1$, $J = 1$, (iii) $H = 1$, $J = -1$, and (iv) $H = -1$, $J = -1$. Show a few states of the image along each Markov chain.

2. Let $X \in \mathbb{R}^{m \times n}$ be a matrix of random variables such that they form a Markov Random Field (MRF). The conditional density of an element is dependent only on the values of its vertical and horizontal neighbors (except for the boundaries where the neighbors are limited). Let the conditional density of a pixel be Gaussian with mean m and variance 0.1, where m is the mean of its neighbors. That is:

$$f(X_{i,j}|\text{all other pixels}) = f(X_{i,j}|X_{i,j-1}, X_{i,j+1}, X_{i-1,j}, X_{i+1,j}) = \mathcal{N}(m, 0.1) \ ,$$

where $m = (X_{i,j-1} + X_{i,j+1} + X_{i-1,j} + X_{i+1,j})/4$.

Write a matlab program to generate samples from this distribution using a Gibbs sampler. To set as initial condition download (use "save link as" to download) the four images from the class website. For each initial condition, run the Markov chain for at least 5 sweeps. Show the matrix after every sweep (you can use these commands: `imagesc`, `axis equal off`, `colormap(gray)`, etc) and comment on the effect of sampling on the images.

3. (Only for 5017) Use Gibb's sampler to generate samples from the posterior marginals of θ and σ^2 in the following problem. We are given that:

$$\begin{aligned} X_i|\theta, \sigma^2 &\sim \mathcal{N}(\theta, \sigma^2), \ i = 1, 2, \ldots, n \\ \theta &\sim \mathcal{N}(\theta_0, \tau^2), \ \ \sigma^2 \sim \mathcal{IG}(a, b) \ , \end{aligned}$$

where IG denotes inverted gamma distribution. Here θ_0, τ, a, and b are given and fixed. The full conditionals are given by:

$$\begin{aligned}
\theta|\mathbf{x},\sigma^2 &\sim \mathcal{N}\left(\frac{\sigma^2}{\sigma^2+n\tau^2}\theta_0+\frac{n\tau^2}{\sigma^2+n\tau^2}\bar{x},\frac{\sigma^2\tau^2}{\sigma^2+n\tau^2}\right)\\
\sigma^2|\mathbf{x},\theta &\sim \mathcal{IG}\left(\frac{n}{2}+a,\frac{1}{2}\sum_i(x_i-\theta)^2+b\right).
\end{aligned}$$

Set $\mathbf{x} = (91,\ 504,\ 557,\ 609,\ 693,\ 727,\ 764,\ 803,\ 857,\ 929,\ 970,\ 1043,\ 1089,\ 1195,\ 1384,\ 1713)$, and use $a = b = 3$, $\tau^2 = 100,000$, and $\theta_0 = 5$.

Use 5000 steps of the Markov chain and show resulting histograms of the values of θ and σ^2 in the chain.