Probability-2 Lecture-8

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$(\Omega, \mathcal{A}, P)$ – probability space.

$$X(\omega) = 1_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A. \end{cases}$$

↑ Indicator function

$X = 1_A$ is a random variable iff $A \in \mathcal{A}$.

Definition:

A real-valued, simple random variable is a finite linear combination of indicator r.v.s.

Then, a simple real random variable:

$$X = \sum_{j=1}^{n} c_j \cdot 1_{A_j} \quad \text{when } n \geq 1$$

$A_j \in \mathcal{A} \ \forall \ 1 \leq j \leq n.$

$c_j \in \mathbb{R} \ \forall \ 1 \leq j \leq n.$

Fact:

Any simple random variable takes only finitely many real values.

Conversely, any real r.v. taking finitely many values is a simple real r.v.

ie, Suppose X is a r.v. which takes finitely many real values, say, $\{c_1, \ldots, c_n\}$.

Let $A_j = \{\omega : X(\omega) = c_j\} \in \mathcal{A}$.

then, $X = \sum_{j=1}^{n} c_j \cdot 1_{A_j}$

Infact,

$A_1, \ldots, A_n$ is a partition of Ω by sets in $\mathcal{A}$.

Fact: Any real simple r.v. X can be written as

$$X = \sum_{k=1}^{m} a_k \cdot 1_{B_k}, \text{ where } B_1, \ldots, B_m \text{ is}$$

a partition of Ω by sets in $\mathcal{A}$.

$\overline{}_{k=1} \quad n$

a partition of Ω by sets in $\mathcal{A}$.

<u>Fact</u>: X, Y – real simple r.v.

$\Rightarrow \alpha X,\ X \pm Y,\ XY,\ X \vee Y,\ X \wedge Y$ are all simple r.v.s.

<u>Theorem</u>:

Let X be any extended real valued r.v. on a probability space $(\Omega, \mathcal{A}, P)$.

Then, there <u>exists</u> a sequence of $\{X_n\}_{n \geq 1}$ of simple real r.v.s such that

$$X_n(\omega) \longrightarrow X(\omega) \quad \forall\ \omega \in \Omega.$$

(converges pointwise)

further,

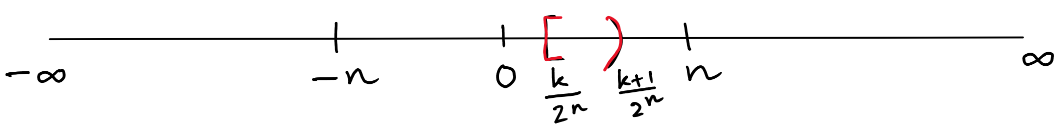
(1) If X is bounded, then the convergence is uniform.

(2) If X is a non-negative, then $X_n \nearrow X$ pointwise.

(increases monotonically.)

Fix $n \geq 1$:

<u>here is how X_n is constructed</u>:

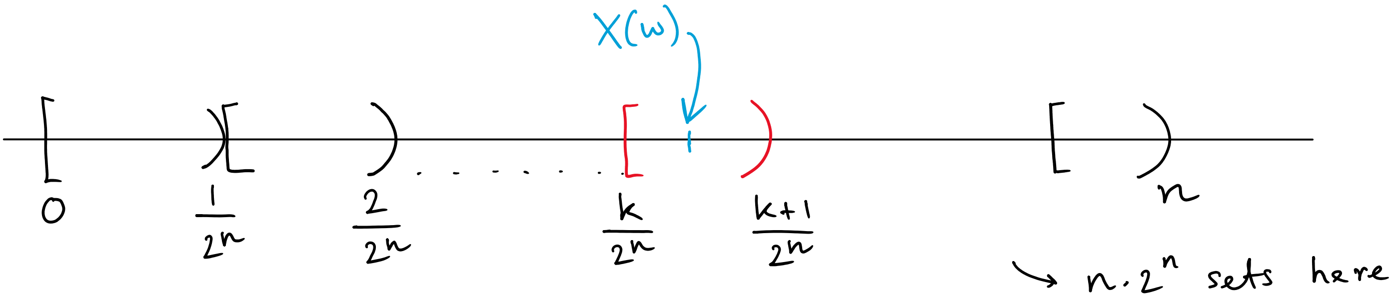


$$A_n = \{\omega : X(\omega) \geq n\} \in \mathcal{A}$$

$$A_{-n} = \{\omega : X(\omega) < -n\} \in \mathcal{A}$$

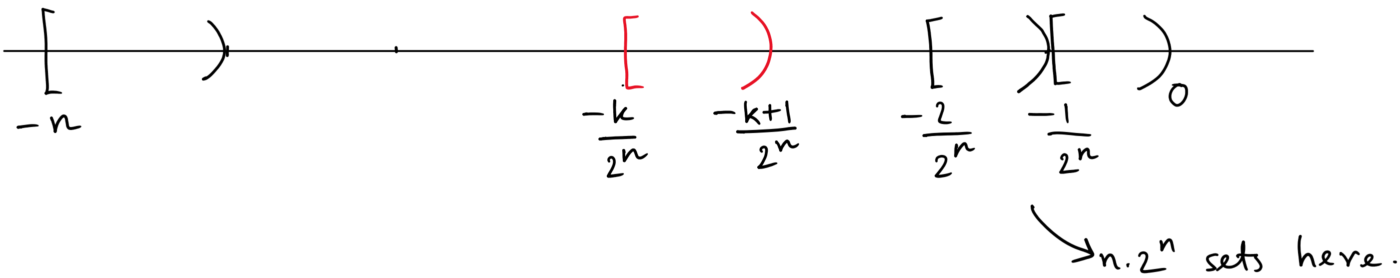
$$A_{k,n} = \left\{\omega : \frac{k}{2^n} \leq X(\omega) < \frac{k+1}{2^n}\right\}, \quad 0 \leq k \leq n \cdot 2^n - 1$$

ie, if some $X(\omega) \in [0, n)$, it must lie within ie, if some $X(\omega) \in [0, n)$, it must lie within one of such $A_{k,n}$.



Similarly,

$$A_{-k,n} = \left\{ \omega : \frac{-k}{2^n} \leq X(\omega) < \frac{-k+1}{2^n} \right\}, \quad 0 \leqslant k \leqslant n \cdot 2^n$$



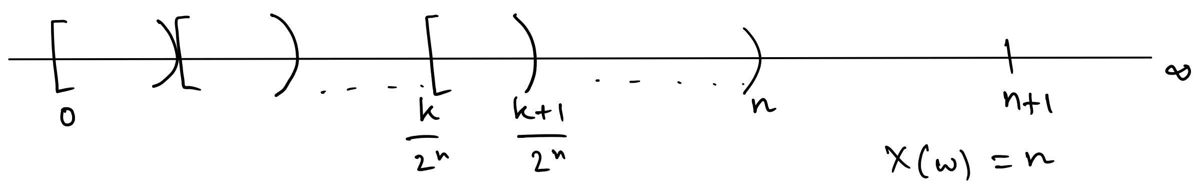
$\therefore$ total : $2n \cdot 2^n$ sets here

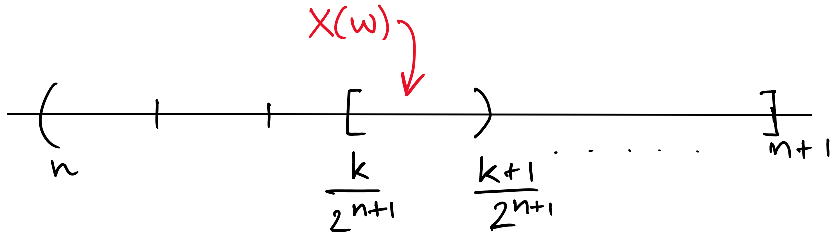
Define X_n on Ω by

$$\begin{aligned} X_n(\omega) &= n \quad \text{if } \omega \in A_n \\ &= -n \quad \text{if } \omega \in A_{-n} \\ &= \frac{k}{2^n} \quad \text{if } \omega \in A_{k,n} \\ &= \frac{-k}{2^n} \quad \text{if } \omega \in A_{-k,n} \end{aligned}$$

$$\boxed{\therefore \quad X_n = n \cdot \mathbb{1}_{A_n} + (-n) \cdot \mathbb{1}_{A_{-n}} + \sum_{k=0}^{n \cdot 2^n - 1} \frac{k}{2^n} \cdot \mathbb{1}_{A_{k,n}} + \sum_{k=0}^{n \cdot 2^n} \frac{-k}{2^n} \cdot \mathbb{1}_{A_{-k,n}}}$$

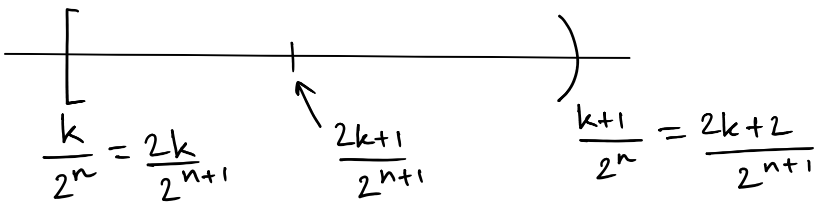
Now, $n < X(\omega) \leqslant n+1$.





$$X_{n+1}(\omega) = \frac{k}{2^{n+1}}$$

$$X_n(\omega) = \frac{k}{2^{n+1}} \cdot X(\omega)$$



Here, we can clearly see why we chose $\frac{1}{2^n}$ instead of $\frac{1}{n}$.

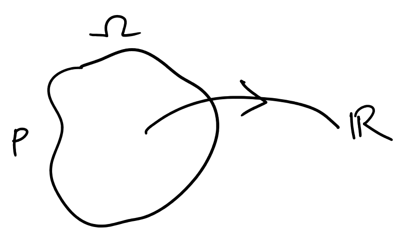
(ie, here, each interval can be divided further into equal subintervals without disturbing the other intervals.

ie, instead of $\frac{1}{2^n}$, we could have chosen $\frac{1}{3^n}$, $\frac{1}{4^n}$ or any $\frac{1}{N^n}$, $N \in \mathbb{N}$, $N > 1$.)

Probability Distribution of real random variables:

$(\Omega, \mathcal{A}, P)$ – probability space

X – a real r.v. on $(\Omega, \mathcal{A}, P)$.



$\forall\ B \in \mathcal{B}$, $\quad P_X(B) = P(X^{-1}(B)) = P(\{\omega \in \Omega : X(\omega) \in B\})$

$\therefore$ P, which is a probability measure on $\mathcal{A}$,

$\therefore P$, which is a probability measure on $\mathcal{A}$, gets converted to a "mass" assignment to every Borel set.

then, $P_X(\mathbb{R}) = 1$.

Now, is this P_X countably additive?

<u>Yes</u>! $\because B_1, B_2, \ldots\ldots$ are disjoint Borel sets

$$\Rightarrow P_X\Big(\bigcup_n B_n\Big) = \sum_n P_X(B_n)$$

$$\begin{aligned} &\| \\ &P\Big(X^{-1}\Big(\bigcup_n B_n\Big)\Big) \\ &= P\Big(\bigcup_n X^{-1}(B_n)\Big) \\ &= \sum_n P(X^{-1}(B_n)) \end{aligned}$$

<u>Definition</u>:

The probability P_X defined as:

$$P_X(B) = P(X^{-1}(B)) \quad \forall\, B \in \mathcal{B}$$

is a probability on $\mathcal{B}$ (borel σ-field) and is called the <u>probability distribution of X</u>.

<u>Definition: Cumulative Distribution Function: (CDF)</u>

The function $F_X : \mathbb{R} \longrightarrow [0,1]$ defined by

$$F_X(a) = P(X^{-1}(-\infty, a])$$

$$= P_X((-\infty, a])$$ is called the distribution function of X.

Cumulative Distribution Function (CDF)

distribution function of X.

or, Cumulative Distribution Function (CDF)

Properties of F_X (CDF):

1. $a \leq b \Rightarrow F_X(a) \leq F_X(b)$

2. $\left.\begin{array}{l} a \in \mathbb{R} \\ a_n \downarrow a \end{array}\right| \Rightarrow F_X(a_n) \to F_X(a)$.

3. $\left.\begin{array}{l} a \in \mathbb{R}. \\ a_n \nearrow a \\ a_n < a. \end{array}\right| \Rightarrow F_X(a_n) \to P_X((-\infty, a))$ open!

4. F_X is continuous at $a \Leftrightarrow P_X(\{a\}) = 0$.

5. $\lim_{a \nearrow \infty} F_X(a) = 1$

   $\lim_{a \downarrow \infty} F_X(a) = 0$.