3. (a) Show that the characteristic polynomial of the $n \times n$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

is $f(\lambda) := a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n$. (The matrix is called the *companion matrix* of $f(\lambda)$.)

(b) Given complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, not necessarily distinct, show that there is a matrix with characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove this in two ways: (i) directly and (ii) using (a).

4. Show that the characteristic roots of the $n \times n$ permutation matrix $\mathbf{P}$ with $p_{i,i+1} = 1$ for $i = 1, \ldots, n-1$ and $p_{n1} = 1$, are the n-th roots of unity.

7. Find a rank-factorization of the matrix

$$\mathbf{C} = \begin{bmatrix} 2 & 4 & 2 & 4 & 4 \\ 1 & 2 & 1 & 2 & 2 \\ 3 & 0 & 3 & 3 & 0 \\ 0 & -4 & 0 & -2 & -4 \\ 5 & 2 & 5 & 6 & 2 \end{bmatrix}$$

and hence the characteristic roots of $\mathbf{C}$.

8. Express the characteristic polynomial of $\alpha\mathbf{I} + \beta\mathbf{A}$ in terms of that of $\mathbf{A}$. Hence find the characteristic roots of $\alpha\mathbf{I} + \beta\mathbf{A}$. What are the characteristic roots of $-\mathbf{A}$?

9. Show that if β is a characteristic root of $\mathbf{A}$ and $\mathbf{A}$ is non-singular, $1/\beta$ is a characteristic root of $\mathbf{A}^{-1}$.

10. Show that the characteristic roots of a matrix do not determine rank (except when zero occurs as a characteristic root at most once;

12. Let $\mathbf{A}$ be a 2×2 matrix. Then show that $|\mathbf{I} + \mathbf{A}| = \mathbf{1} + |\mathbf{A}|$ iff $\text{tr}(\mathbf{A}) = 0$.

3. Let α be an eigenvalue of $\mathbf{A}$. Then show that $\text{ES}(\mathbf{A}^k, \alpha^k) \supseteq \text{ES}(\mathbf{A}, \alpha)$ if $k \geq 1$. Extend the result to $k = -1$ if $\mathbf{A}$ is non-singular. Show also that proper inclusion is possible.

4. If k, ℓ and n are integers such that $1 \leq k \leq \ell \leq n$, show that there exists an $n \times n$ matrix $\mathbf{A}$ and an eigenvalue α of $\mathbf{A}$ such that k and ℓ are the geometric and algebraic multiplicities of α with respect to $\mathbf{A}$.

5. If $\alpha_1, \ldots, \alpha_k$ are the distinct eigenvalues of an $n \times n$ matrix $\mathbf{A}$ with geometric multiplicities $n_1, \ldots, n_k$ respectively, then $n_1 + \cdots + n_k \leq n$.

6. (a) Let δ be an eigenvalue of $\mathbf{A}$ with algebraic multiplicity a and let $\beta \neq 0$. Then show that $\alpha + \beta\delta$ is an eigenvalue of $\alpha\mathbf{I} + \beta\mathbf{A}$ with algebraic multiplicity a and $\mathrm{ES}(\alpha\mathbf{I} + \beta\mathbf{A}, \alpha + \beta\delta) = \mathrm{ES}(\mathbf{A}, \delta)$.

   (b) Prove or disprove: if δ is an eigenvalue of $\mathbf{A}$, the algebraic and geometric multiplicities of $f(\delta)$ with respect to $f(\mathbf{A})$ are the same as those of δ with respect to $\mathbf{A}$ for any polynomial f.

8. If $\mathbf{A}$ is an $n \times n$ singular matrix with k distinct eigenvalues, show that $k - 1 \leq \rho(\mathbf{A}) \leq n - 1$. Also show by construction that $\rho(\mathbf{A})$ can take any value ℓ between $k - 1$ and $n - 1$

10. Let $\mathbf{A} = \mathbf{u}\mathbf{u}^*$ where $\mathbf{u}$ is a non-null vector.

    (a) Show that the eigenvalues of $\mathbf{A}$ are 0 and $\mathbf{u}^*\mathbf{u}$.

    (b) Show that $\mathbf{u}^*\mathbf{u}$ is a simple eigenvalue of $\mathbf{A}$.

    (c) Identify $\mathrm{ES}(\mathbf{A}, 0)$ and $\mathrm{ES}(\mathbf{A}, \mathbf{u}^*\mathbf{u})$ and deduce the result in (b).

    (d) Show that $\mathbf{A}$ is similar to a diagonal matrix.

11. Find the eigenvalues and their algebraic and geometric multiplicities for each of the real $n \times n$ matrices (i) $(\alpha - \beta)\mathbf{I} + \beta\mathbf{1}\mathbf{1}^{\mathrm{T}}$ and (ii) $\alpha\mathbf{I} + \mathbf{u}\mathbf{1}^{\mathrm{T}} + \mathbf{1}\mathbf{u}^{\mathrm{T}}$. Here $\mathbf{1}$ denotes a vector with all entries 1 and $\mathbf{u}$ is an arbitrary vector.

12. If α is an eigenvalue of $\mathbf{A}$, then it is an eigenvalue of $\mathbf{A}^{\mathrm{T}}$ also. An eigenvector of $\mathbf{A}^{\mathrm{T}}$ corresponding to α, i.e., a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x}^{\mathrm{T}}\mathbf{A} = \alpha\mathbf{x}^{\mathrm{T}}$, is called a *left eigenvector of* $\mathbf{A}$ *corresponding to* α. Viewed in the same spirit, eigenvectors of $\mathbf{A}$ as defined in *Definition* 8.3.1 are called *right eigenvectors*. Let λ_1 and λ_2 be distinct eigenvalues of $\mathbf{A}$. If $\mathbf{x}$ is a left eigenvector of $\mathbf{A}$ corresponding to λ_1 and $\mathbf{y}$ is a right eigenvector of $\mathbf{A}$ corresponding to λ_2, then show that $\mathbf{x}^{\mathrm{T}}\mathbf{y} = 0$.

13. Let λ be an eigenvalue of $\mathbf{A}$. Let r be the geometric multiplicity of λ. Show that the dimension of the space of the left eigenvectors of $\mathbf{A}$ corresponding to λ is also r

*15. Let $\mathbf{A}$ be an $n \times n$ matrix and let

$$\rho_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad (i = 1, \ldots, n)$$

    (a) If α is an eigenvalue of $\mathbf{A}$, show that $|\alpha - a_{ii}| \leq \rho_i$ for at least one i

16. Let α be an eigenvalue of $\mathbf{A}$. Then show that $|\alpha| \leq \|\mathbf{A}\|$ where $\|\cdot\|$ is the matrix norm induced by any vector norm (

17. (a) Let $\mathbf{A}$ be an $n \times n$ idempotent matrix. Then show that $\mathcal{C}(\mathbf{A}) = \mathrm{ES}(\mathbf{A}, 1)$ and $\mathcal{C}(\mathbf{I} - \mathbf{A}) = \mathrm{ES}(\mathbf{A}, 0)$ and that $\mathbf{A}$ has n linearly independent eigenvectors.

    (b) If each eigenvalue of $\mathbf{A}$ is 0 or 1, does it follow that $\mathbf{A}$ is idempotent?

*18. Le f be a linear operator on a complex vector space V.

(a) Show that there exists a subspace S of V with $d(S) = 1$ such that $f(S) \subseteq S$. An S satisfying the latter condition is said to be *invariant under* f.

19. Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{D}$ be the $n \times n$ matrix with (i,j)-th element $\mathrm{tr}(\mathbf{A}^{i+j-2})$. Show that the characteristic roots of $\mathbf{A}$ are distinct iff $\mathbf{D}$ is non-singular.

6. Prove Cayley-Hamilton theorem thus: let $\mathbf{H} := (\lambda\mathbf{I} - \mathbf{A})^{\circledast} = \mathbf{H}_0 + \lambda\mathbf{H}_1 + \cdots + \lambda^{n-1}\mathbf{H}_{n-1}$. Then $(\lambda\mathbf{I} - \mathbf{A})\mathbf{H} = \chi_{\mathbf{A}}(\lambda)\,\mathbf{I}$. Multiply by $\mathbf{A}^i$ the equation obtained by comparing the coefficients of λ^i on the two sides and sum up to get $\chi_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$.

8. $\mathbf{A}$ is said to be nilpotent if $\mathbf{A}^k = \mathbf{0}$ for some positive integer k. Show that $\mathbf{A}$ is nilpotent iff all the characteristic roots of $\mathbf{A}$ are 0.

9. Let $\mathbf{A}$ be nilpotent.

(a) If $\mathbf{A} \neq \mathbf{0}$, show that $\mathbf{A}$ cannot be similar to a diagonal matrix.

(b) What can you say about the minimal polynomial of $\mathbf{A}$?

10. Show that the constant term in the minimal polynomial of $\mathbf{A}$ is non-zero iff $\mathbf{A}$ is non-singular.

13. Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same order and let $\mathbf{C} = \mathbf{AB} - \mathbf{BA}$. Show that $\mathbf{I} - \mathbf{C}$ is not nilpotent.

14. What is the minimal polynomial of $\alpha\mathbf{A}$?

15. Find the minimal polynomial of the $n \times n$ matrix $\mathbf{J} = \mathbf{1}\mathbf{1}^{\mathrm{T}}$.

16. Prove that the minimal polynomial of $\mathrm{diag}(\mathbf{A}, \mathbf{B})$ is the L.C.M. of the minimal polynomials of $\mathbf{A}$ and $\mathbf{B}$.

1. If $\mathbf{A}$ is a 2×2 matrix such that $\mathbf{A}^2 = \mathbf{0}$, show that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{A}$ is similar to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

2. Show that $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is semi-simple iff either $\mathbf{A}$ is a scalar matrix or $(a-d)^2 + 4bc \neq 0$.

3. If $\mathbf{A}^k = \mathbf{I}$ for some positive integer k, show that $\mathbf{A}$ is semi-simple.

4. Show that $\mathbf{A}$ is idempotent iff each eigenvalue of $\mathbf{A}$ is 0 or 1 and $\mathbf{A}$ is semi-simple.

5. If $\mathbf{A}$ is a semi-simple matrix such that $\mathbf{A}^2 = \mathbf{A}^3$, show that $\mathbf{A}$ is idempotent. Show also that the condition that $\mathbf{A}$ is semi-simple cannot be dropped.

7. If $\mathbf{A}$ is semi-simple, show that any polynomial in $\mathbf{A}$ is also semi-simple.

8. Let $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Obtain a spectral decomposition of $\mathbf{A}$. Hence find $\mathbf{A}^{10}$ and write down a spectral decomposition of $\mathbf{A}^{-1}$.

9. A matrix $\mathbf{A}$ is said to be *stochastic* if $a_{ij} \geq 0$ for all i and j and $\sum_j a_{ij} = 1$ for all i. Let $\mathbf{A}$ be a 2×2 stochastic matrix $\neq \mathbf{I}$.
   (a) Find a spectral decomposition of $\mathbf{A}$.
   (b) Obtain an expression for $\mathbf{A}^k$ where k is an arbitrary positive integer.
   (c) Show that there exists a 3×3 stochastic (upper triangular) matrix which is not semi-simple.

13. Let $\mathbf{A}$ be semi-simple and let $\mathbf{A} = \sum_{i=1}^{k} \alpha_i \mathbf{E}_i$ be the spectral form of $\mathbf{A}$. Then prove that $\mathbf{B}$ commutes with $\mathbf{A}$ iff $\mathbf{B}$ commutes with $\mathbf{E}_i$ for $i = 1, \ldots, k$.

14. Let $\mathbf{A}$ be a square matrix with real eigenvalues such that $\rho(\mathbf{A}) = \rho(\mathbf{A}^2)$ and $\operatorname{tr}(\mathbf{A}^2) \neq 0$. Then show that

$$\rho(\mathbf{A}) \geq \frac{(\operatorname{tr}(\mathbf{A}))^2}{\operatorname{tr}(\mathbf{A}^2)}$$

16. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ semi-simple matrices. Show that the following statements are equivalent:
   (a) $\mathbf{AB} = \mathbf{BA}$,
   (b) $\mathbf{A}$ and $\mathbf{B}$ are simultaneously diagonable (i.e., there exists a non-singular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{P}^{-1}\mathbf{BP}$ are diagonal),
   (c) $\mathbf{A}$ and $\mathbf{B}$ are polynomials in a common semi-simple matrix.

4. Let $\mathbf{A}$ be a real skew-symmetric matrix of order n.
   (a) If n is odd, show that $|\mathbf{A}| = 0$.
   (b) If n is even, show that $|\mathbf{A}| \geq 0$.
   (c) For any n, show that $|\mathbf{I} + \mathbf{A}| \geq 1$.
   (b) Find the spectral form of the $(k+1) \times (k+1)$ matrix $\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1}^{\mathrm{T}} & 0 \end{bmatrix}$.
   (Hint: use rank-factorization.)

8. Find a normal matrix which is none of: hermitian, skew-hermitian, unitary and diagonal.

9. Show that a normal matrix is unitary iff every eigenvalue has unit modulus.

11. Prove or disprove: every complex symmetric matrix is normal.

12. Show that the $n \times n$ matrix $\mathbf{1}\mathbf{1}^{\mathrm{T}}$ is similar to the $n \times n$ matrix $\begin{bmatrix} n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

13. Show that $\mathbf{A}$ is an orthogonal projector iff $\mathbf{A}$ is hermitian and each eigenvalue of $\mathbf{A}$ belongs to $\{0, 1\}$.
14. Prove that $\mathbf{A}$ is normal iff $\|\mathbf{A}^*\mathbf{x}\| = \|\mathbf{A}\mathbf{x}\|$ for all $\mathbf{x}$.
15. Let $\mathbf{A}$ be a real symmetric matrix.
    (a) If $\mathbf{A}^k = \mathbf{I}$ for some positive integer k, show that $\mathbf{A}^2 = \mathbf{I}$.
    (b) If the eigenvalues of $\mathbf{A}$ are all positive and if $\mathbf{A}^k = \mathbf{I}$ for some positive integer k then show that $\mathbf{A} = \mathbf{I}$.
    (c) If $\mathbf{A}^k = \mathbf{0}$ for some positive integer k, then show that $\mathbf{A} = \mathbf{0}$.