Probability-2 Lecture-11

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X is said to be <u>absolutely continuous</u> if

$\exists$ a non-negative (measureable) function

f on $\mathbb{R}$ such that—

$$F_X(a) = \int_{-\infty}^{a} f(x)\,dx.$$

f is called the <u>density</u> of X.

It follows, $P_X(B) = \int_B f(x)\,dx.$

<u>Special Case:</u>

I is an open-interval.

f is Riemann integrable (infact, continuous) on I,

& $\int_I f(x)\,dx = 1$

this may also be an improper integral.

$\therefore f$ is a valid candidate for density f^{n}.

If $I \subset \mathbb{R}$ an open interval, and $f: \mathbb{R} \to \mathbb{R}$ is such that—

1. $f(x) > 0$ and continuous on I.
2. $f(x) = 0$ outside I.
3. $c = \int_I f(x)\,dx < \infty$

Then, $\frac{1}{c} \cdot f$ is a density.

Eg: $f(x) = \begin{cases} x^{-\alpha}, & x > 1. \\ 0, & x \leq 1 \end{cases}$ $\quad I = (a, \infty)$

$\int^{\infty} \cdots \quad \int^{1} \cdots + \int^{\infty} x^{-\alpha} \cdot dx < \infty$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{1} 0 \cdot dx + \int_{1}^{\infty} x^{-\alpha} \cdot dx < \infty$$

$$\&,\ \begin{cases} (\alpha-1)\cdot x^{\alpha}, & x>1 \quad \text{if } \alpha>1 \\ 0, & x\le 1 \end{cases}$$ is a density.

Eg. $f(x) = \begin{cases} x^{-\alpha}, & 0<x<1. \\ 0, & \text{otherwise.} \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} x^{-\alpha}\,dx = \lim_{\varepsilon\to 0} \left.\frac{x^{1-\alpha}}{1-\alpha}\right|_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon\to 0} \frac{1}{1-\alpha} - \frac{\varepsilon^{1-\alpha}}{1-\alpha} = \lim_{\varepsilon\to 0} \frac{1-\varepsilon^{1-\alpha}}{(1-\alpha)} < \infty$$

$$\Rightarrow 1-\alpha > 0$$

$$\Rightarrow \alpha < 1$$

$\therefore \int_0^1 x^{-\alpha}\,dx < 0$ iff $\alpha < 1$.

$\&\ \int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 k\cdot x^{-\alpha}\,dx = 1 \Rightarrow \frac{k}{1-\alpha} = 1 \Rightarrow k = 1-\alpha$

$$\therefore \begin{cases} (1-\alpha)\cdot x^{-\alpha}, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$ is a density.

Ex. Let $\alpha<1$, $\beta>1$.
find possible c_1 & c_2 s.t this is a density

$$f(x) = \begin{cases} c_1 \cdot x^{-\alpha}, & 0<x<1 \\ c_2 \cdot x^{-\beta}, & 1<x<\alpha \\ \text{otherwise.} \end{cases}$$

Some standard distributions:

① Uniform on (a, b) $(a < b$ - reals$)$. : Denoted by $\boxed{U(a,b)}$.

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise.} \end{cases}$$

Often, continuous dist^{n}s can be seen as a limit of sequence of discrete dist^{n}s. We take such an example here.

Special case:

$U(0,1)$

Let X_n be a discrete r.v.

$X_n = k/n$ with probability $\frac{1}{n}$.

$$F_{X_n}(x) = P(X_n \leq x)$$
$$= \sum_{k=1}^{[nx]} \frac{1}{n} = \frac{[nx]}{n} \longrightarrow x = F(x).$$

The sequence of random variables $\{X_n\}$ converge in distribution to a random variable X,

$\Longleftrightarrow$ for every bounded, continuous f^{n}s f,
$$E(f(X_n)) \longrightarrow E(f(X))$$

Proof: next semester.

② $$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$ → Exponential distribution.

Notation: $\boxed{Exp(\lambda)}$

why is this constant λ?

$$\int_0^\infty k e^{-\lambda x} = 1 \Rightarrow \frac{-k}{\lambda} \cdot e^{-\lambda x}\Big|_0^\infty = 1 \Rightarrow k/\lambda = 1 \Rightarrow \boxed{k = \lambda}$$

Eg. We look at a system which emits signals. (Poisson process).
$X =$ waiting time till first signal

(No. of signals emitted in disjoint time intervals is considered to be independent.).

(No. of signals emitted in time t) $\sim Poi(\lambda t)$; $\lambda =$ emission rate.

$$\therefore P(X \leq t) = 1 - P(X > t)$$
$$= 1 - e^{-\lambda t} \stackrel{!}{=} \int_0^t \underbrace{\lambda e^{-\lambda x}}_{\exp(\lambda)} dx$$

③ $f(x) = \begin{cases} (?) \cdot x^{\theta} \cdot e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$ $\quad \underline{\theta > -1, \ \lambda > 0}$.

first, let's consider $\theta < 0$.

$$\alpha \cdot \int_{\varepsilon}^{1} x^{\theta} dx \leq \int_{\varepsilon}^{1} x^{\theta} \cdot e^{-\lambda x} dx \leq \beta \cdot \int_{\varepsilon}^{1} x^{\theta} dx$$

③ $f(x) = \begin{cases} \frac{1}{\Gamma(\alpha, \lambda)} \cdot x^{\alpha - 1} \cdot e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise,} \end{cases}$ $\leftarrow$ Gamma Distribution

Notation: $\boxed{Gamma(n, \lambda)}$

where, $\Gamma(\alpha, \lambda) = \int_0^{\infty} x^{\alpha - 1} e^{-\lambda x} \cdot dx$.

special case, $\lambda = 1$.

$$\therefore \Gamma(\alpha) = \int_0^1 x^{\alpha - 1} e^{-x} dx$$

Gamma f^n.

↳ Gamma f^n.

Result:

$$\Gamma(\alpha) = (\alpha-1)\cdot\Gamma(\alpha-1). \quad \forall \ \alpha-1>0$$
ie, $\alpha > 1$.

[How to get this?
- Integration by Parts.]
(Exc.)

for any natural no. $n \in \mathbb{N}$,

$$\Gamma(n) = (n-1)\cdot\Gamma(n-1) = (n-1)(n-2)\cdot\Gamma(n-2)$$
$$= (n-1)(n-2)\cdots 2\cdot 1\cdot \underbrace{\Gamma(0)}_{\substack{\shortparallel \\ 1}}$$

$$\boxed{\Gamma(n) = \lfloor n-1 .}\ , n\in\mathbb{N}.$$

Now, in Poisson process,

$Y =$ waiting time till n^{th} emissions.

$$P(Y\le t) = 1 - P(Y>t)$$
$$= 1 - P(Pois(\lambda t) \le n-1)$$
$$= 1 - \sum_{k=0}^{n-1} e^{-\lambda t}\cdot\frac{(\lambda t)^k}{\lfloor k} \underset{(ex.)}{\overset{?}{=}} \frac{1}{\lfloor n-1}\cdot\int_0^t x^{n-1}\cdot e^{-\lambda x}\,dx$$

Gamma(n,λ)

Here, $\Gamma(1,\lambda) = \int_0^\infty e^{1-1}\, e^{-\lambda x}\,dx$

$$= \int_0^\infty e^{-\lambda x}\,dx = \frac{-1}{\lambda}e^{-\lambda x}\Big|_0^\infty = 1/\lambda$$

when $\alpha = 1$

$$\therefore f(x) = \frac{x^{1-1}\cdot e^{-\lambda x}}{1} = \lambda e^{-\lambda x}$$

$$\therefore f(x) = \frac{x^{1-1} \cdot e^{-\lambda x}}{1/\lambda} = \lambda e^{-\lambda x}$$

$$\boxed{\therefore \text{Gamma}(1, \lambda) \equiv \text{Exp}(\lambda).}$$

④ $f(x) = \begin{cases} \frac{1}{\pi \cdot \sqrt{x(1-x)}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$ → Arcsine distribution

?

$$\int_0^1 \frac{k}{\sqrt{x(1-x)}} \, dx$$

$x = \sin^2\theta$

$\therefore dx = 2\sin\theta\cos\theta \, d\theta$

$$= \int_0^{\pi/2} \frac{k \cdot 2\sin\theta \cdot \cos\theta}{\sin\theta \cdot \cos\theta} \, d\theta = \pi.$$

⑤ $f(x) = \begin{cases} \frac{1}{\beta(\alpha_1, \alpha_2)} \cdot x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$ $\alpha_1, \alpha_2 > 0.$

Beta distribution

where, $\beta(\alpha_1, \alpha_2) = \int_0^1 x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1} \, dx.$

(Beta function)

Notation: $\text{Beta}(\alpha_1, \alpha_2)$

Fact: $\beta(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$ $\forall \alpha_1, \alpha_2 > 0.$

(5) $f(x) = \dfrac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \quad \forall\ x \in (-\infty, \infty).$

<u>Standard Normal distribution:</u>

$\boxed{\begin{array}{l}\underline{\text{Notation:}} \\ N(0,1)\end{array}}$

<u>Proof (using Walli's Integrals):</u>

$$I_n = \int_0^{\pi/2} \sin^n dx = \int_0^{\pi/2} \cos^n dx \quad , \quad n \in \mathbb{N} \cup \{0\}.$$

<u>fact</u> $\rightarrow 0 \leqslant I_0.$

$\rightarrow I_{n+1} \leqslant I_n \quad \forall\ n$

<u>fact</u>: $I_0 = \dfrac{\pi}{2}$, $I_1 = 1.$

<u>fact</u>: $I_n = \left(\dfrac{n-1}{n}\right) \cdot I_{n-2}$, $n \geqslant 2$.

$\left(\underline{\text{Hint}}: I_n = \int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} (1-\cos^2 x) \cdot \sin^{n-2} x\, dx.\right.$

Now, use Integration by Parts.$\Big)$.

<u>fact</u>: $I_{2n} = C_n \cdot \dfrac{\pi}{2}$. ; $I_{2n+1} = D_n$

Here, $I_n = \left(\dfrac{n-1}{n}\right) \cdot I_{n-2} \longrightarrow$ put $n = n+1$

$(n+1) I_{n+1} = n \cdot I_{n-1}.$

$\Rightarrow (n+1) I_{n+1} \cdot I_n = n I_n \cdot I_{n-1}$

$= (n-1) I_{n-1} \cdot I_{n-2}.$

$\vdots$

$$\vdots$$

$$= 1 \cdot I_1 \cdot I_0 = 1 \cdot \frac{\pi}{2} \cdot 1 = \frac{\pi}{2}$$

Now,

$$\frac{n I_n^2}{(n+1) \cdot I_{n+1} \cdot I_n} \longrightarrow 1.$$

$$\Rightarrow n I_n^2 \longrightarrow \frac{\pi}{2}$$

$$\Rightarrow \sqrt{n} \cdot I_n \longrightarrow \sqrt{\frac{\pi}{2}}$$

$$\therefore \int_0^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n dx \le \underbrace{\int_0^{\sqrt{n}} e^{-x^2} dx}_{} \le \int_0^{\infty} e^{-x^2} dx \le \int_0^{\infty} \left(1 + \frac{x^2}{n}\right)^{-n} dx$$

(show) $\|$ $\quad$ $\|$ (show)

$\sqrt{n} \cdot I_{2n} \to \sqrt{\frac{\pi}{2}}$ $\qquad$ $\sqrt{n} \cdot I_{2n-2} \to \sqrt{\frac{\pi}{2}}$

We know, $e^x \geqslant 1 + x$

put $x \to -x^2$

$\therefore \quad e^{-x^2} \geqslant 1 - x^2$

$$\left(1 - \frac{x^2}{n}\right)^n = \left(1 - \frac{\cancel{n} x^2}{\cancel{n}} + - \; - \; -\right)$$

$$1 - x^2 + \frac{\binom{n}{2} \cdot x^4}{\lfloor 2} > 1 - x^2$$

(will complete the minor details later).