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An anti-Hausdorff Fréchet space in which convergent sequences have unique limits

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Abstract

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We shall discuss the possibility of slicing MAD families. As an application we present an example of an anti-Hausdorff Fréchet space in which convergent sequences have unique limits.

Keywords: MAD family, Fréchet space, anti-Hausdorff space.

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1. Introduction

We answer an old question of Novák by constructing an anti-Hausdorff Fréchet space in which convergent sequences have unique limits. An essential ingredient in our construction is that there exists an infinite maximal almost disjoint family $\mathscr{A}$ on ω and a function $L:\mathscr{A}\to\omega$ such that for all $X\subseteq\omega$, if $\mathscr{F}=\{A\in\mathscr{A}: |A\cap X|=\omega\}$ is infinite then $L^{\to}\mathscr{F}=\omega$. We also discuss possible strengthenings and generalizations of this result.

2. Set theoretic conventions

We use ξ or η or ζ to denote an ordinal (=set of smaller ordinals), α or γ to denote a cardinal (=initial ordinal), and κ to denote an infinite cardinal. ω denotes ω_0, $\mathfrak{c}$ denotes 2^{ω} and $\mathrm{cf}(\kappa)$ denotes the cofinality of κ. Also,

$$[X]^{\kappa}=\{A\in\mathscr{P}(X): |A|=\kappa\} \quad\text{and}\quad [X]^{<\kappa}=\{A\in\mathscr{P}(X): |A|<\kappa\}.$$

$A\subseteq_{\kappa}B$ abbreviates $|A-B|<\kappa$.

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A collection $\mathscr{A} \subseteq [\kappa]^\kappa$ is called AD (almost disjoint) if $(\forall A \neq B \in \mathscr{A})[|A \cap B| < \kappa]$, and $\mathscr{A}$ is called κ-MAD is $\mathscr{A}$ is a maximal AD subcollection of $[\kappa]^\kappa$ with $|\mathscr{A}| \geq \mathrm{cf}(\kappa)$. We put

$$\mathfrak{a}_\kappa = \min\{|\mathscr{A}| : \mathscr{A} \text{ is } \kappa\text{-MAD}\}, \quad \mathfrak{a} = \mathfrak{a}_\omega.$$

Remark 2.1. It is easy to see that $\mathfrak{a}_\kappa > \mathrm{cf}(\kappa)$, and also that $\mathfrak{a}_\kappa \leq \mathfrak{a}_{\mathrm{cf}(\kappa)}$, see Proposition 8.1, hence $\mathrm{cf}(\kappa) < \mathfrak{a}_\kappa \leq 2^{\mathrm{cf}(\kappa)}$. So if κ is regular and $2^\kappa = \kappa^+$ then $\mathfrak{a}_\kappa = 2^\kappa$. For regular κ it is consistent that $\mathfrak{a}_\kappa = 2^\kappa > \kappa^+$ (use a generalized Martin's axiom) and also that $\mathfrak{a}_\kappa < 2^\kappa$ (see [7, Chapter VIII, Theorem 2.3] or (for $\kappa > \omega$) recall it is consistent $[\kappa]^\kappa$ has no AD subfamily of size 2^κ [1]). We have no information about $\mathfrak{a}_\kappa$ for singular κ other than $\mathfrak{a}_\kappa \leq \mathfrak{a}_{\mathrm{cf}(\kappa)}$. We expect $\mathfrak{a}_\kappa < \mathfrak{a}_{\mathrm{cf}(\kappa)}$ to be consistent, for each singular κ.

Remark 2.2. We have "$|\mathscr{A}| \geq \mathrm{cf}(\kappa)$" in the definition of "$\kappa$-MAD" because of practical reasons. A justification is that if $\mathscr{A}$ is a maximal AD subfamily of $[\kappa]^\kappa$, then $\mathscr{A}$ is trivial in the sense that $(\exists R \in [\kappa]^{\kappa})(\forall A \neq B \in \mathscr{A})[A \cap B \subseteq R]$ iff $|\mathscr{A}| < \mathrm{cf}(\kappa)$.

3. Terminology

All spaces considered are T_1. We say $C \subseteq X$ *converges* to $x \in X$ if $C = \{x\}$ or if C is infinite and if ($\forall$ neighborhood U of x)$[C \subseteq_\omega U]$. The *sequential closure* of $A \subseteq X$ is

$$\bar{A}^s = \{x \in X : (\exists C \subseteq A)[C \text{ converges to } x]\}.$$

X is called *Fréchet* if $(\forall A \subseteq X)[\bar{A} = \bar{A}^s]$, *sequential* if $(\forall A \subseteq X)[A = \bar{A} \Leftrightarrow A = \bar{A}^s]$.

We call $U \subseteq X$ an s-neighborhood of $x \in X$ if $x \notin \overline{X - U}^s$. Clearly ordinary neighborhoods are s-neighborhoods, and the converse holds iff X is Fréchet.

We call X *s-Hausdorff* if every two points of X have disjoint s-neighborhoods, *s-anti-Hausdorff* if no two points have disjoint s-neighborhoods; *anti-Hausdorff* is self-explanatory.

Finally, we say that X is *USL* (unique sequential limits) if no subset of X converges to more than one point.

4. Anti-Hausdorff USL Fréchet spaces: history

Clearly each s-Hausdorff space is USL. In 1926 Urysohn has given an example of a USL, [6, p. 212]. In his 1937 topology seminar Čech has asked if there is an s-anti-Hausdorff USL space. This question was answered affirmatively by Novák in 1939, [9, pp. 16–17]. At this point we mention the examples of Urysohn and Novák are sequential, for if they were not they could be made sequential by an obvious change of topology which does not affect which sets converge.

Since Novák's example is not Fréchet, he has asked if there is an (s-)anti-Hausdorff USL Fréchet space, [loc. cit.]. There are examples of non-Hausdorff Fréchet spaces, one by Katětov, [2, 6.4.11, 6.4.12].

(I am indebted to V. Koutnik for supplying me with the above information, and with providing me with a reprint of [9].) (I have translated this history from the language of convergence structures into topology.)

An additional example of an s-anti-Hausdorff USL sequential non-Fréchet space was given in 1971 by Franklin and Rajagapolan, [5], who were unaware of Novák's example. Unlike Novák's example theirs is homogeneous.

In Section 5 we will answer Novák's question affirmatively by constructing a compact countable anti-Hausdorff USL Fréchet space Ω in ZFC; this requires information about MAD families given in Sections 6 and 7. Under the additional axiom $\mathfrak{a}=\mathfrak{c}$, Ω can be made homogeneous; I do not know how to do this in ZFC. For details see Section 9.

5. An anti-Hausdorff space

We here construct the example Ω promised in Section 4. Rather than just presenting the construction we motivate why we do it the way we do it. The underlying set of Ω is ω.

Let $\mathscr{C}$ denote the collection of infinite convergent subsets of Ω. We plan to construct Ω from $\mathscr{C}$. But $\mathscr{C}$ may be unmanageably big. So we work with an $\mathscr{A}\subseteq\mathscr{C}$ such that

$$(\forall C\in\mathscr{C})(\exists A\in\mathscr{A})[C\subseteq_\omega A]. \tag{1}$$

We also need to know for each member of $\mathscr{C}$ what it converges to. Of course it suffices to know this for members of $\mathscr{A}$ only. So we need

a suitable collection $\mathscr{A}$ of infinite subsets of ω and a suitable function $L:\mathscr{A}\to\omega$ with (just for convenience) $(\forall A\in\mathscr{A})[L(A)\notin A]$. (*)

We explain what suitable means as we investigate Ω. All (set theoretic) properties of $\mathscr{A}$ and L we find desirable will be labelled with Roman capitals, and all (topological) properties of Ω will be labelled with numerals.

Our intention behind (*) is, of course, to have

$$(\forall A\in\mathscr{A})[A \text{ converges to } L(A), \text{ and to } L(A) \text{ only}], \tag{2}$$

so then each member of

$$\mathscr{S}=\bigcup_{A\in\mathscr{A}}\{B\cup\{L(A)\}: B\subseteq A\}$$

must be closed in Ω. In order to have good control over Ω we give it the smallest topology for which this is true, i.e., we topologize ω by declaring $\mathscr{S}$ to be a subbase for the closed sets. In other words

$$(\forall G \subseteq \Omega)[G \text{ is closed in } \Omega \Leftrightarrow (\forall x \in \Omega - G)(\exists \text{ finite } \mathscr{F} \subseteq \mathscr{S})$$
$$[G \subseteq \bigcup \mathscr{F} \text{ and } x \notin \bigcup \mathscr{F}]]. \tag{3}$$

For later use we observe the obvious

$$(\forall G \subseteq \Omega)[G \text{ not dense} \Rightarrow (\exists \text{ finite } \mathscr{F} \subseteq \mathscr{A})\Big[G \subseteq \bigcup_{F \in \mathscr{F}} (F \cup \{L(F)\})\Big]. \tag{4}$$

It is easy to see that a necessary condition on $\mathscr{A}$ and L for (2) to hold is

$$(\forall A, B \in \mathscr{A})[L(A) \neq L(B) \Rightarrow A \cap B \text{ finite}]. \tag{A}$$

It turns out that this condition is also sufficient. First observe that no $A \in \mathscr{A}$ converges to a point different from $L(A)$ since $(\forall a \in A)[(A - \{a\}) \cup \{L(A)\}]$ is closed. Next, consider any $A \in \mathscr{A}$ and let U be a neighborhood of $L(A)$. Because of (3) we may assume there are finite $\mathscr{F} \subseteq \mathscr{A}$ and $B(F) \subseteq F$ for $F \in \mathscr{F}$ such that $\Omega - U = \bigcup_{F \in \mathscr{F}} (B(F) \cup \{L(F)\})$. For each $F \in \mathscr{F}$ we have $L(F) \neq L(A)$ since $L(A) \in U$ but $L(F) \notin U$. So $A \cap (\bigcup \mathscr{F})$ is finite, hence $A \subseteq_\omega U$ since $\Omega - U \subseteq \bigcup_{F \in \mathscr{F}} (F \cup \{L(F)\}) \subseteq_\omega \bigcup \mathscr{F}$. We simplify (A) to

$$\mathscr{A} \text{ is an AD subfamily of } [\Omega]^\omega = [\omega]^\omega. \tag{B}$$

We now quickly take care of three more properties of Ω. From (2) and (4) we see that each proper closed subset of Ω is compact, hence

$$\Omega \text{ is compact.} \tag{5}$$

Next, since $(\forall A \in \mathscr{A})[\{L(A)\} \text{ is closed}]$, a simple way to have

$$\Omega \text{ is } T_1 \tag{6}$$

is to have

$$L^{\to}\mathscr{A} = \Omega. \tag{C}$$

(There are other ways, but we will have (C) anyway.) Finally, since $\mathscr{A}$ is infinite by (C) we see from (B) that $(\forall \text{ finite } \mathscr{F} \subseteq \mathscr{A})[\bigcup_{F \in \mathscr{F}} (F \cup \{L(F)\}) \neq \Omega]$. Hence Ω is not the union of two proper closed subsets. The dual of this statement is

$$\Omega \text{ is anti-Hausdorff.} \tag{7}$$

Using (6), (4), (B) and (2) one can prove that (1) is true, and hence

$$\Omega \text{ is USL} \tag{8}$$

because of (2). However, we omit the proof since we do not need (1). (We will give a different proof of (8).)

Because of (2) the natural way to prove

$$\Omega \text{ is Fréchet} \tag{9}$$

is to have

$$(\forall S \subseteq \Omega)(\forall x \in \Omega)[x \in \overline{S - \{x\}}] \Rightarrow [S \cap A \text{ infinite and } L(A) = x]]. \tag{10}$$

(In fact, because of (1) this is the only way.)

It is a pleasant surprise that (10) implies (8): Let C be infinite and let it converge to p and to q. There is $A \in \mathscr{A}$ with $C \cap A$ infinite. $C \cap A$ also converges to p and to q, but it converges to $L(A)$ only because of (2).

Clearly $(\forall \text{ infinite } S \subseteq \Omega)(\exists x \in \Omega)[x \in \overline{S-\{x\}}]$ since Ω is compact. Hence a necessary condition for (10) to hold is the following strengthening of (8):

$$\mathscr{A} \text{ is } \omega\text{-MAD.} \tag{D}$$

To see what we need for (10) consider any $S \subseteq \Omega$ and $x \in \Omega$ with $x \in \overline{S-\{x\}}$. Define

$$\mathscr{F} = \{A \in \mathscr{A} : S \cap A \text{ is infinite}\}.$$

Case 1: *$\mathscr{F}$ is finite.* Clearly $S \subseteq_\omega \bigcup \mathscr{F}$ because of (D), hence $x \in \overline{S \cap (\bigcup \mathscr{F}) - \{x\}}$ since Ω is T_1, hence there is $A \in \mathscr{F}$ with $x \in \overline{S \cap A - \{x\}}$; now $S \cap A$ is infinite, since Ω is T_1, and $x = L(A)$ because of (2).

Case 2: *$\mathscr{F}$ is infinite.* Since $\mathscr{A}$ is infinite and AD, we see from (4) that S is dense. So we must have $(\exists A \in \mathscr{A})[S \cap A$ infinite and $L(A) = y]$ for all $y \in \Omega$ (not just for $y = x$), or, equivalently, $L^{\to}\mathscr{F} = \Omega$. This and (10) lead to the following idea to make (*) precise.

$\mathscr{A}$ is ω-MAD and $L : \mathscr{A} \to \Omega$ are such that for each $S \subseteq \Omega$, if $\mathscr{F} = \{A \in \mathscr{A} : S \cap A \text{ infinite}\}$ is infinite then $L^{\to}\mathscr{F} = \Omega$. (E)

As promised above, this implies (C) since if $S = \Omega$ then $\mathscr{F} = \mathscr{A}$ in (E), and ω-MADs are infinite by convention. We will construct $\mathscr{A}$ and L as in (E) in Section 6. That construction will not necessarily yield $\mathscr{A}$ and L satisfying $(\forall A \in \mathscr{A})[L(A) \notin A]$, but this is easy to fix. (Also, the condition is not really needed anyway.)

6. Slicing MAD families: Easy results

Recall that α, γ, κ are cardinals, with $\kappa \geq \omega$. For Section 5 we only need the special case $\alpha = \kappa = \omega$ of the discussion below. We limit our discussion of slicings to slicings of MAD families since we have no applications of generalizations.

Throughout this section $\mathscr{A} \subseteq [\kappa]^\kappa$. Define

$$\mathscr{A}\#X = \{A \in \mathscr{A} : |A \cap X| = \kappa\} \quad \text{for } X \subseteq \kappa;$$

$$\mathscr{A}^+ = \{X \subseteq \kappa : |\mathscr{A}\#X| \geq \mathrm{cf}(\kappa)\};$$

$$\mathrm{tk}(\mathscr{A}) = \min\{|\mathscr{A}\#X| : X \in \mathscr{A}^+\} \ (\min(\emptyset) = 0), \text{ the true cardinality of } \mathscr{A}.$$

Fact 6.1. *If $\mathscr{A}$ is κ-MAD then* $\mathrm{tk}(\mathscr{A}) \geq \mathfrak{a}_\kappa$.

Proof. Consider any $X \in \mathscr{A}^+$. The obvious function from $\mathscr{A}\#X$ to $\mathscr{A} \restriction X = \{A \cap X : A \in \mathscr{A}\#X\}$ is a bijection, and of course $\mathscr{A} \restriction X$ is a maximal AD subfamily of $[X]^\kappa$. $\square$

Call L a *γ-slicing* of $\mathscr{A}$ if L is a function $\mathscr{A} \to \gamma$ satisfying

$$(\forall X \subseteq \kappa)[X \in \mathscr{A}' \Rightarrow L^{\to}(\mathscr{A} \# X) = \gamma].$$

(Note "$\Rightarrow$" becomes "$\Leftrightarrow$" if $\gamma \geq \mathrm{cf}(\kappa)$.) Call $\mathscr{A}$ *γ-sliceable* if it admits a γ-slicing. Clearly $\mathscr{A}$ is γ'-sliceable if $\mathscr{A}$ is γ-sliceable and $\gamma' \leq \gamma$.

Fact 6.2. *If* $\mathrm{tk}(\mathscr{A}) = 2^\kappa$ *then* $\mathscr{A}$ *is* 2^κ*-sliceable.*

Proof. We omit the proof: something quite similar will be proved in Section 7. □

Corollary 6.3. *If* $\mathfrak{a} = 2^\kappa$ *then every* κ*-MAD is* 2^κ*-sliceable.*

This suggests the following questions. Note $\mathscr{A}$ never is $\mathrm{tk}(\mathscr{A})^+$-sliceable.

Question 6.4. Let $\mathscr{A}$ be κ-MAD. Is $\mathscr{A}$ γ-sliceable for $\gamma = \mathrm{tk}(\mathscr{A})$? $\gamma = \mathfrak{a}_\kappa$? $\gamma = \kappa^+$? $\gamma = \kappa$? $\gamma = \omega$? $\gamma = 2$?

Question 6.5. Suppose there is an AD subfamily of $[\kappa]^\kappa$ of size α. Is there a γ-sliceable κ-MAD for $\gamma = \alpha$? $\gamma = \mathfrak{a}_\kappa$? $\gamma = \kappa^+$? $\gamma = \kappa$? $\gamma = \omega$? $\gamma = 2$?

For regular κ the only honest answer we have is that there is an ω-sliceable ω-MAD, and fortunately that is all we need for Section 5; see Section 8 for singular κ. Note that the question of whether there is a $\mathfrak{c}$-sliceable ω-MAD is equivalent to the known question of whether there is an ω-MAD of true cardinality $\mathfrak{c}$. Recall that for suitable $\kappa > \omega$, e.g. $\kappa = \omega_1$, it is consistent that $[\kappa]^\kappa$ has no AD subfamily of size 2^κ, [1]; this explains the "α" in Question 6.5. (We do not know if this is consistent for all $\kappa > \omega$, nor if it is consistent for suitable $\kappa > \omega$ that $\{|\mathscr{A}|: \mathscr{A} \in [\kappa]^\kappa$ is AD$\}$ has no maximum.)

Remark 6.6. One can generalize the concept of a γ-slicing and consider $L: \mathscr{A} \to \gamma$ such that

$$(\forall X \subseteq \kappa)[|\mathscr{A} \# X| \geq \alpha \Rightarrow L^{\to}(\mathscr{A} \# X) = \gamma]. \qquad (*)$$

We have no applications if $\alpha > \mathrm{cf}(\kappa)$. We would have applications if $\alpha = \omega$, but it is easy to see that if $\mathscr{A}$ is κ-MAD and $\alpha < \mathrm{cf}(\kappa)$ then no $L: \mathscr{A} \to \gamma$ satisfies $(*)$.

7. Slicing MAD families: An ω-sliceable ω-MAD

For $\mathscr{A} \subseteq [\omega]^\omega$ we defined

$$\mathscr{A} \# X = \{A \in \mathscr{A}: |A \cap X| = \omega\} \quad \text{for } X \subseteq \omega;$$

$$\mathscr{A}' = \{X \subseteq \omega: |\mathscr{A} \# X| \geq \omega\}.$$

We now also define

$$\mathscr{A} \upharpoonright X = \{A \cap X : A \in \mathscr{A} \# X\}, \quad X \subseteq \omega.$$

Clearly, if $\mathscr{A}$ is an ω-MAD then for each $Y \in \mathscr{A}^+$ the collection $\mathscr{A} \upharpoonright Y$ is an infinite maximal AD subfamily of $[Y]^\omega$ (and $\mathscr{A} \upharpoonright Y \cup \{\omega - Y\}$ is ω-MAD), we prove the existence of an ω-sliceable ω-MAD by proving

Theorem 7.1. $(\forall \omega\text{-}MAD\ \mathscr{A})(\exists Y \in \mathscr{A}^+)[\mathscr{A} \upharpoonright Y$ *is* ω*-sliceable*$]$.

Call $\mathscr{Q} \subseteq [\omega]^\omega$ a *P-collection* if for every sequence $\langle Q_n : n \in \omega \rangle$ in $\mathscr{Q}$ with $(\forall n \in \omega)$ $[Q_n \supseteq Q_{n+1}]$ there is $Q \in \mathscr{Q}$ with $(\forall n \in \omega)[Q \subseteq_\omega Q_n]$. It is well known that $[\omega]^\omega$ is a P-collection and that in fact the following holds.

Lemma 7.2. *Let $\mathscr{K}$, $\mathscr{L}$ be at most countable subcollections of $[\omega]^\omega$ such that*

$$(\forall K \in \mathscr{K})(\forall \mathscr{F} \subseteq \mathscr{L})[1 \leqslant |\mathscr{F}| < \omega \Rightarrow |K \cap \bigcap \mathscr{F}| = \omega], \quad \text{and} \quad \mathscr{K} \neq \emptyset.$$

Then there is $Q \subseteq \omega$ with $(\forall K \in \mathscr{K})[|K \cap Q| = \omega]$ and $(\forall L \in \mathscr{L})[Q \subseteq_\omega L]$.

For the proof of Theorem 7.1 we need the following result of Dočkalkova [3], which is a consequence of the fact that $\mathscr{A}^+$ is a happy family in the sense of Matthias [8]. We supply a proof as a service to the reader.

Lemma 7.3. *If $\mathscr{A}$ is ω-MAD then $\mathscr{A}^+$ is a P-collection.*

Proof. Let $\langle Q_n : n \in \omega \rangle$ be a sequence in $\mathscr{A}^+$ with $(\forall n \in \omega)[Q_n \supseteq Q_{n+1}]$. We claim

$$\mathscr{F} = \bigcap_{n \in \omega} (\mathscr{A} \# Q_n)$$

is infinite. If not then for each $n \in \omega$ the set $Q_n - \bigcup \mathscr{F}$ is infinite since $Q_n \in \mathscr{A}^+$ and since $\mathscr{A}$ is AD, and is a superset of $Q_{n+1} - \bigcup \mathscr{F}$. Hence by Lemma 7.2 there is $P \in [\omega]^\omega$ with $(\forall n \in \omega)[P \subseteq_\omega Q_n - \bigcup \mathscr{F}]$. Since $\mathscr{A}$ is ω-MAD there is $A \in \mathscr{A}$ with $|P \cap A| = \omega$. Then $A \notin \mathscr{F}$ since $\mathscr{A}$ is AD and $|A \cap \bigcup \mathscr{F}| < \omega$, but $(\forall n \in \omega)$ $[|A \cap Q_n| = \omega]$, so $A \in \bigcap_{n \in \omega}(\mathscr{A} \# Q_n)$, which is absurd.

Now use Lemma 7.2 with $\mathscr{K}$ a countably infinite subfamily of $\mathscr{F}$ and with $\mathscr{L} = \{Q_n : n \in \omega\}$ to find $Q \subseteq \omega$ with $(\forall n \in \omega)[Q \subseteq_\omega Q_n]$ and $\mathscr{L} \subseteq \mathscr{A} \# Q$ (so that $Q \in \mathscr{A}^+$). $\square$

Note that this implies the known result that $\mathfrak{a} \geqslant \omega_1$. The following is the key to the proof of Theorem 7.1.

Lemma 7.4. *Let $\mathscr{A} \subseteq [\omega]^\omega$ be AD, and assume $\mathscr{A}^+$ is a P-collection. Also, let $\mathscr{B} \subseteq \mathscr{A}$. If $\mathscr{A}^+ \subseteq \mathscr{B}^+$ then*

$$(\forall S \in \mathscr{A}^+)(\exists T \in \mathscr{A}^+)(\exists \mathscr{U} \subseteq \mathscr{B})$$
$$[T \subseteq S \text{ and } (\mathscr{A} \upharpoonright T)^+ \subseteq (\mathscr{U} \upharpoonright T)^+ \cap ((\mathscr{B} - \mathscr{U}) \upharpoonright T)^+].$$

(Note that for $\mathscr{C}, \mathscr{D} \subseteq [\omega]^\omega$, if $\mathscr{D} \subseteq \mathscr{C}$ then $\mathscr{C}^+ \subseteq \mathscr{D}^+$ iff $\mathscr{C}^+ = \mathscr{D}^+$.)

Proof. We first point out that

$$(\forall Y \in \mathscr{A}^+)(\forall \mathscr{U} \subseteq \mathscr{B})[Y \notin \mathscr{U}^+ \Rightarrow (\exists T \in \mathscr{A}^+)[T \subseteq Y \text{ and } \mathscr{U} \# T = \emptyset]]. \tag{1}$$

Indeed, $Y \notin \mathscr{U}^+$ means $|\mathscr{U} \# Y| < \omega$, hence if $Y \in \mathscr{A}^+$ then $T = Y - \bigcup(\mathscr{U} \# Y)$ belongs to $\mathscr{A}^+$ since $\mathscr{A}$ is an AD and since $\mathscr{U} \# Y \subseteq \mathscr{A}$, and clearly $T \subseteq Y$ and $\mathscr{U} \# T = \emptyset$.

We also make the elementary observation that

$$(\forall X, Y \subseteq \omega)(\forall \mathscr{V} \subseteq \mathscr{A})[X \in (\mathscr{V} \restriction Y)^+ \text{ iff } X \cap Y \in \mathscr{V}^+]. \tag{2}$$

Finally, note that since $|\mathscr{B}| \leq \mathfrak{c} = |\mathbb{R}|$, $\mathscr{B}$ carries some second countable T_1-topology, hence there is an indexed collection $\langle \mathscr{B}_{n,i} : n \in \omega, i \in 2 \rangle$ of subsets of $\mathscr{B}$ such that

$$(\forall n \in \omega)[\mathscr{B}_{n,1} = \mathscr{B} - \mathscr{B}_{n,0}], \tag{3}$$

$$(\forall B \neq C \in \mathscr{B})(\exists n \in \omega)[B \in \mathscr{B}_{n,0} \text{ and } C \in \mathscr{B}_{n,1}]. \tag{4}$$

Now consider any $S \in \mathscr{A}^+$. We attempt to construct a sequence $\langle T_n : n \in \omega \rangle$ and a function $s : \omega \to 2$ such that

$$(\forall n \in \omega)[S \supseteq T_n \supseteq T_{n+1} \text{ and } T_n \in \mathscr{A}^+], \tag{5}$$

$$(\forall n \in \omega)[\mathscr{B}_{n,s(n)} \# T_{n+1} = \emptyset]. \tag{6}$$

Let $T_0 = S$. Next, let $n \in \omega$, and assume T_n known. If $(\mathscr{A} \restriction T_n)^+ \subseteq \bigcap_{i \in 2} (\mathscr{B}_{n,i} \restriction T_n)^+$ then our construction terminates but we establish the lemma with $T = T_n$ and $\mathscr{U} = \mathscr{B}_{n,0}$, because of (3). Otherwise there are $s(n) \in 2$ and $Y \in (\mathscr{A} \restriction T_n)^+$ with $Y \notin (\mathscr{B}_{n,s(n)} \restriction T_n)^+$, i.e., $Y \cap T_n \in \mathscr{A}^+$ but $Y \cap T_n \notin \mathscr{B}^+_{n,s(n)}$ by (2), and now (1) tells us how to find T_{n+1}.

We claim the construction terminates. Indeed, if not there is by (5) a $Q \in \mathscr{A}^+$ with $(\forall n \in \omega)[Q \subseteq_\omega T_n]$. From (6) we see that

$$(\forall n \in \omega)[\mathscr{B}_{n,s(n)} \# Q = \emptyset]. \tag{7}$$

However, as $Q \in \mathscr{A}^+ \subseteq \mathscr{B}^+$ there are $B \neq C \in \mathscr{B} \# Q$. By (4) there is $n \in \omega$ with $B \in \mathscr{B}_{n,0}$ and $C \in \mathscr{B}_{n,1}$, which contradicts (7). $\square$

Because of Lemma 7.3 the following implies Theorem 7.1.

Lemma 7.5. *If $\mathscr{A}$ is an infinite AD subfamily of $[\omega]^\omega$ such that $\mathscr{A}^+$ is a P-collection then there is $Y \in \mathscr{A}^+$ such that $\mathscr{A} \restriction Y$ is ω-sliceable.*

Proof. For all $Y \in \mathscr{A}^+$ the collection $(\mathscr{A} \restriction Y)^+$ is a P-collection, since, as observed above,

$$(\forall X \subseteq \omega)[X \in (\mathscr{A} \restriction Y)^+ \Rightarrow X \cap Y \in \mathscr{A}^+].$$

It follows from Lemma 7.4 that there are a sequence $\langle \mathscr{A}_n : n \in \omega \rangle$ of subfamilies of $\mathscr{A}$ and a sequence $\langle Y_n : n \in \omega \rangle$ such that $\mathscr{A}_0 = \mathscr{A}$ and

$$(\forall n \in \omega)[Y_n \supseteq Y_{n+1} \text{ and } Y_n \in \mathscr{A}^+],$$

$$(\forall n \in \omega)[\mathscr{A}_n \supseteq \mathscr{A}_{n+1} \text{ and}$$

$$(\mathscr{A}_n \restriction Y_n)^+ \subseteq (\mathscr{A}_{n+1} \restriction Y_n)^+ \cap ((\mathscr{A}_n - \mathscr{A}_{n-1}) \restriction Y_n)^+].$$

Hence there is $Y \in \mathscr{A}^+$ such that $(\forall n \in \omega)[Y \subseteq_\omega Y_n]$. Then

$$(\forall n \in \omega)[(\mathscr{A} \restriction Y)^+ \subseteq ((\mathscr{A}_n - \mathscr{A}_{n+1}) \restriction Y)^+].$$

So we can define an ω-slicing L of $\mathscr{A} \restriction Y$ by

$$L(A \cap Y) = n \quad \text{if } A \in \mathscr{A}_n - \mathscr{A}_{n+1} \text{ or if } n = 0 \text{ and } A \in \bigcap_{k \in \omega} \mathscr{A}_k.$$

(L is well defined since $(\forall A \neq B \in \mathscr{A})[A \cap Y = B \cap Y]$ because $\mathscr{A}$ is AD and $Y \in \mathscr{A}^+$.) $\square$

Remark 7.6. [Call $\mathscr{A} \subseteq [\omega]^\omega$ *sane* if $(\forall X \in \mathscr{A}^+)[\mathscr{A} \restriction X$ is not ω-MAD.]

(a) The proof of Lemma 7.4 is essentially Simon's proof that there are an ω-MAD $\mathscr{A}$ and $\mathscr{B} \subseteq \mathscr{A}$ such that $\mathscr{B}$ and $\mathscr{A} - \mathscr{B}$ are sane, [10]. This is only natural since

$$(\forall \omega\text{-MAD } \mathscr{A})(\forall \mathscr{B} \subseteq \mathscr{A})[\mathscr{A}^+ \subseteq \mathscr{B}^+ \cap (\mathscr{A} - \mathscr{B})^+ \Leftrightarrow \mathscr{B} \text{ and } \mathscr{A} - \mathscr{B} \text{ are sane}].$$

(b) It is easy to see that if $\mathscr{A}^+$ is a P-collection and $\mathscr{B} \subseteq \mathscr{A}$ then there is $X \in \mathscr{A}^+$ such that (at least) one of $\mathscr{B} \restriction X$ and $(\mathscr{A} - \mathscr{B}) \restriction X$ is a P-collection. It follows from Simon's result quoted in Remark 7.6(a) that there is an AD $\mathscr{A} \subseteq [\omega]^\omega$ such that $\mathscr{A}^+$ is a P-collection yet $\mathscr{A}$ is sane, cf. Lemma 7.3.

Remark 7.7. Now let $\mathscr{A}$ be a κ-MAD. The proofs in this section can easily be modified to prove

$$(\exists Y \in \mathscr{A}^+)[\mathscr{A} \restriction \text{ is } \omega\text{-sliceable}] \tag{1}$$

provided Lemma 7.3 would generalize to

> if $\lambda = \log \kappa$ $(= \min\{\alpha : 2^\alpha \geqslant \kappa\})$, then for each $\gamma \leqslant \lambda$ and each γ-sequence $\langle Q_\xi : \xi \in \gamma \rangle$ in $\mathscr{A}^+$ satisfying $(\forall \xi, \eta \in \gamma)[\xi < \eta \Rightarrow Q_\eta \subseteq_\kappa Q_\xi]$ there is $Q \in \mathscr{A}^+$ with $(\forall \xi \in \gamma)[Q \subseteq_\kappa Q_\xi]$. (2)

Now (2) does hold if $\mathrm{cf}(\kappa) = \omega$ and $\lambda = \omega$; however, it never holds if $\mathrm{cf}(\kappa) > \omega$. To see this let $\mathscr{A}$ be any κ-MAD, with $\mathrm{cf}(\kappa) > \omega$. We shall find $\langle Q_n : n \in \omega \rangle$ in $\mathscr{A}^+$ with $(\forall n \in \omega)[Q_n \supseteq Q_{n+1}]$ but $\bigcap_{n \in \omega} Q_n = \emptyset$; then for all $Q \subseteq \kappa$, if $(\forall n \in \omega)[Q \subseteq_\kappa Q_n]$ then $|Q| < \kappa$ hence $Q \notin \mathscr{A}^+$. To find $\langle Q_n : n \in \omega \rangle$ find a sequence $\langle \mathscr{B}_n : n \in \omega \rangle$ in $[\mathscr{A}]^{\mathrm{cf}(\kappa)}$ such that $(\forall n \in \omega)[\mathscr{B}_n \supseteq \mathscr{B}_{n+1}]$ and $\bigcap_{n \in \omega} \mathscr{B}_n = \emptyset$. For $B \in \mathscr{B}_0$ find $B' \in [B]^\kappa$ such that $(\forall B \neq C \in \mathscr{B}_0)[B' \cap C' = \emptyset]$. Define $Q_n = \bigcup \{B' : B \in \mathscr{B}_n\}$ $(n \in \omega)$.

A similar argument shows (2) fails if $\mathrm{cf}(\kappa) = \omega$ but $\lambda > \omega$.

8. Singular cardinals

In the previous sections we used $\mathrm{cf}(\kappa)$ in some definitions involving κ where $\mathrm{cf}(\kappa)$ may have been more natural. Remark 2.2 gives one reason. The following result is another reason.

Proposition 8.1. *Let α abbreviate* $\mathrm{cf}(\kappa)$.

(a) $\mathfrak{a}_\kappa \leqslant \mathfrak{a}_\alpha$;

(b) *if there is a γ-sliceable α-MAD then there is a γ-sliceable κ-MAD.*

Proof. Let $\langle \kappa_\xi : \xi \in \alpha \rangle$ be an α-sequence of cardinals such that

(1) $\sup_{\xi \in \alpha} \kappa_\xi = \kappa$ and $(\forall \xi, \eta \in \alpha)[\xi < \eta \Rightarrow \kappa_\xi < \kappa_\eta]$ (so $(\forall \xi \in \alpha)[|\kappa_\xi| < \kappa]$).

Let $\langle K_\xi : \xi \in \alpha \rangle$ be an α-sequence of subsets of κ such that

(2) $\bigcup_{\xi \in \alpha} K_\xi = \kappa$ and $(\forall \xi \neq \eta \in \alpha)[K_\xi \cap K_\eta = \emptyset]$ and $(\forall \xi \in \alpha)[|K_\xi| = \kappa_\xi]$.

For any $A \subseteq \alpha$ let Q_A denote $\bigcup_{\xi \in A} K_\xi$. Note

(3) $(\forall A, B \subseteq \alpha)[Q_A \cap Q_B = Q_{A \cap B}]$;

(4) $(\forall A \subseteq \alpha)[|Q_A| = \lambda \Leftrightarrow |A| = \kappa]$;

(5) $(\forall A, B \subseteq \alpha)[Q_A = Q_B \Leftrightarrow A = B]$.

Proof of (a). Let $\mathcal{A}$ be an α-MAD. Define $\mathcal{Q} = \{Q_A : A \in \mathcal{A}\}$. Then $\mathcal{Q}$ is an AD subfamily of $[\kappa]^\kappa$ with $|\mathcal{Q}| = |\mathcal{A}|$ by (3), (4) and (5). To see $\mathcal{Q}$ is a κ-MAD consider any $X \in [\kappa]^\kappa$. From (1) and (2) and we see there is an injection $s : \alpha \to \alpha$ such that $(\forall \xi \in \alpha)[|X \cap K_{s(\xi)}| \geqslant \kappa_\xi]$. As $|\mathrm{ran}(s)| = \alpha$ and $\mathcal{A}$ is α-MAD there is $A \in \mathcal{A}$ with $|A \cap \mathrm{ran}(s)| = \alpha$. Clearly $|Q_A \cap X| = \kappa$ because of (1). (This is due to Erdös and Hajnal, [4].)

Proof of (b). The argument is similar. Let $\mathcal{A}$ be an α-MAD and let L be a γ-slicing of $\mathcal{A}$. Define $\mathcal{Q}$ as in (a), the $\mathcal{Q}$ is κ-MAD. Define $\mathcal{L} : \mathcal{Q} \to \gamma$ in the obvious way:

$$\mathcal{L}(Q_A) = L(A) \quad \text{for } A \in \mathcal{A}.$$

($\mathcal{L}$ is well defined by (5).) To prove $\mathcal{L}$ is a γ-slicing of $\mathcal{Q}$ consider any $X \subseteq \kappa$ such that $\mathcal{Q} \# X \geqslant \mathrm{cf}(\kappa) = \alpha$. Find $\mathcal{B} \in [\mathcal{A}]^\alpha$ with $(\forall B \in \mathcal{B})[Q_B \in \mathcal{A} \# X]$, i.e., $(\forall B \in \mathcal{B})$ $[|Q_B \cap X| = \kappa]$. Let $b : \alpha \to \mathcal{B}$ satisfy

(6) $(\forall B \in \mathcal{B})[|b^{\leftarrow}\{B\}| = \quad]$,

and find an injection $s : \alpha \to \alpha$ such that

(7)(*a*) $(\forall \xi \in \alpha)[s(\xi) \in b(\xi)]$, and (b) $(\forall \xi \in \alpha)[\,|X \cap K_{s(\xi)}| \geqslant \kappa_\xi]$.

From (6) and (7)(a) we see that $(\forall B \in \mathcal{B})[|B \cap \mathrm{ran}(s)| = \alpha]$, hence $|\mathcal{A} \# \mathrm{ran}(s)| \geqslant |\mathcal{B}| = \alpha$. Also, from (7)(b) we see that $(\forall A \in (\mathcal{A} \# \mathrm{ran}(s)))[Q_A \in \mathcal{Q} \# X]$. Since L is a γ-slicing of $\mathcal{A}$ it follows that $\mathcal{L}^{\rightarrow}(\mathcal{Q} \# X) \supseteq L^{\rightarrow}(\mathcal{A} \# \mathrm{ran}(s)) = \gamma$. $\square$

Corollary 8.2. *If* $\mathrm{cf}(\kappa) = \omega$ *then there is an ω-sliceable κ-MAD.*

Corollary 8.3. *If* $\mathfrak{a}_{\mathrm{cf}(\kappa)} = 2^{\mathrm{cf}(\kappa)}$ *then there is an* $2^{\mathrm{cf}(\kappa)}$*-sliceable κ-MAD.*

9. A consistent homogeneous example

We now show the example of Section 5 can be made homogeneous if $\mathfrak{a} = \mathfrak{c}$.

Identify ω with a countable subgroup of some compact metrizable group G, whose operation we denote by juxtaposition. Define

$$\mathscr{B} = \{B \in [\omega]^\omega : (\forall x \neq y \in \omega)[\,|xB \cap yB| < \omega]\} \quad (xB = \{xb : b \in B\}).$$

Lemma 9.1. $(\forall X \in [\omega]^\omega)(\exists B \in \mathscr{B})[B \subseteq X]$.

Proof. If $X \in [\omega]^\omega$ then, since G is compact metrizable, there are $B \in [X]^\omega$ and $g \in G$ such that B converges to g. Clearly $(\forall x \in \omega)[xB$ converges to $xg]$. Hence $B \in \mathscr{B}$. (The lemma can be proved with any group operation on ω, but we won't bother.) $\square$

Call $\mathscr{A} \subseteq [\omega]^\omega$ *invariant* if $(\forall x \in \omega)(\forall A \in \mathscr{A})[xA \in \mathscr{A}]$.

Lemma 9.2. *There is an invariant MAD $\mathscr{A}$ with $\mathscr{A} \subseteq \mathscr{B}$.*

Proof. Let $\mathscr{A}$ be a maximal invariant AD subfamily of $\mathscr{B}$. It is not hard to see from Lemma 9.1 that $\mathscr{A}$ is MAD. $\square$

Lemma 9.3 ($\mathfrak{a} = \mathfrak{c}$). *Let $\mathscr{A}$ be as in Lemma 9.2. There is $L : \mathscr{A} \to \omega$ such that*

(1) $(\forall X \in \mathscr{A}^+)(\forall n \in \omega)(\exists A \in \mathscr{A})[|A \cap X| = \omega$ *and* $L(A) = n]$,

(2) $(\forall x \in \omega)(\forall A \in \mathscr{A})[L(xA) = xL(A)]$.

Proof. Define an equivalence relation $\sim$ on $\mathscr{A}$ by $A \sim B \Leftrightarrow (\exists x \in \omega)[xA = B]$. Note that each $\sim$-equivalence class is countable. Since $|\mathscr{A}^+| = \mathfrak{c} = \mathfrak{a}$, we can with an easy transfinite construction find $\Lambda \subseteq \mathscr{A} \times \omega$ such that

(3) $(\forall X \in \mathscr{A}^+)(\forall k \in \omega)(\exists A \in \mathscr{A} \# X)[\langle A, k\rangle \in \Lambda]$,

(4) $(\forall \langle A, k\rangle \neq \langle A', k'\rangle \in \Lambda)[A \not\sim A']$.

Upon enlarging Λ if necessary we find such Λ satisfying additionally

(5) $(\forall B \in \mathscr{A})(\exists \langle A, k\rangle \in \Lambda)[B \sim A]$.

Now define L as follows: For $B \in \mathscr{A}$ there is a unique $\langle A, k\rangle \in \mathscr{A}$ with $B \sim A$, by (5) and (4), and there is a unique $x \in \omega$ with $B = xA$ since $\mathscr{A} \subseteq \mathscr{B}$, let $L(B) = xk$. $\square$

From these $\mathscr{A}$ and L construct Ω as in Section 5. Then for each $x \in \Omega$ the function $y \mapsto xy$ $(y \in \Omega)$ is a homeomorphism, because of (2) of Lemma 9.3. Hence Ω is homogeneous.

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