The Banach-Tarski Paradox

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Indian Statistical Institute

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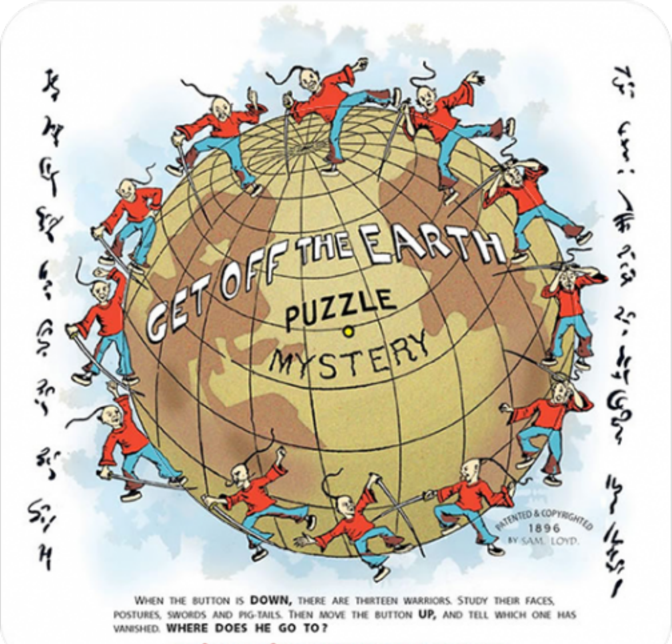
0+2=2

Get off the earth!



$0+2=2$

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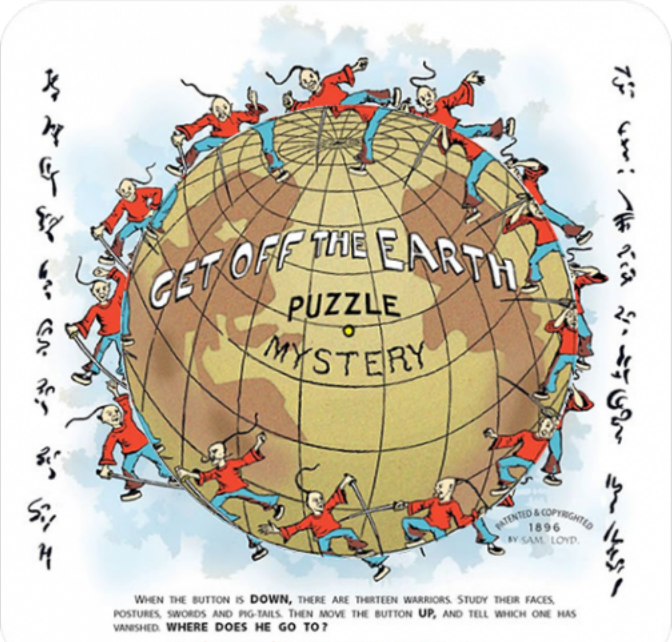

0+2=2

Get off the earth!

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Everyday experience...

Certain transformations are not expected to change certain things.

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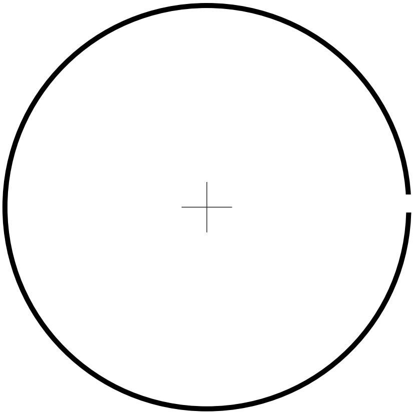
Rotation and translation should not change length, area or volume.

4+2=6

The Banach-Tarski theorem

It is possible to split a solid ball into finitely many non-overlapping pieces, and then rotate and refit them together to form two solid balls identical to the original ball!

Circle with a single point missing



Can you break it into two non-overlapping parts, rotate and refit to get a complete circle?

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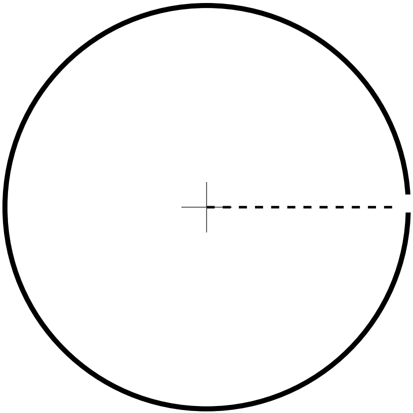

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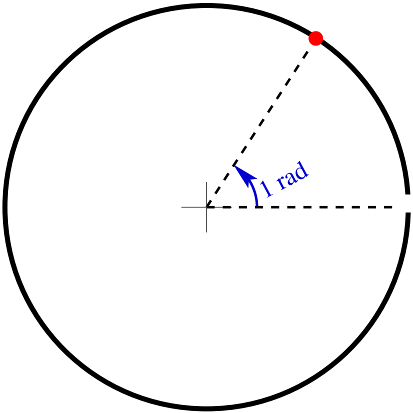

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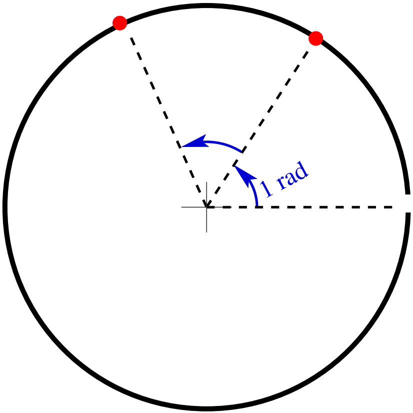

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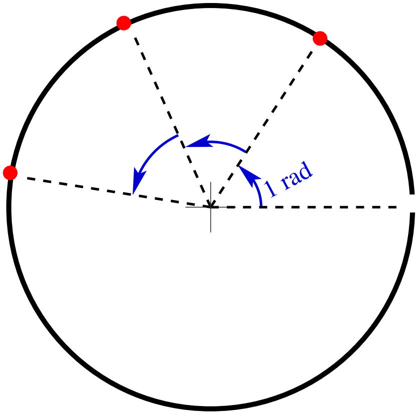
6+2=8

Circle with a single point missing



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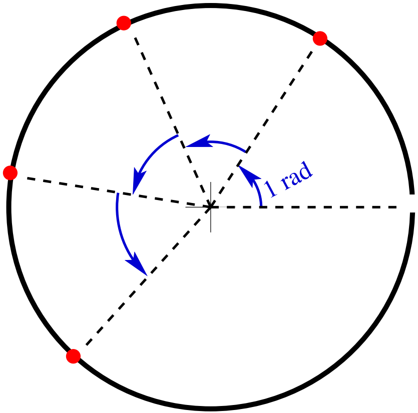

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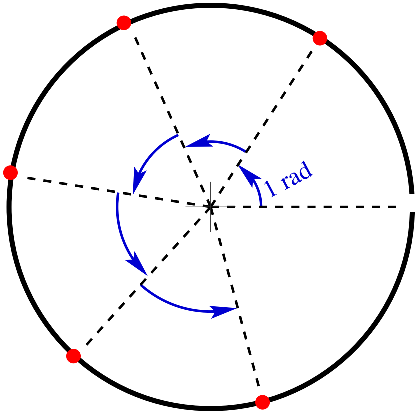

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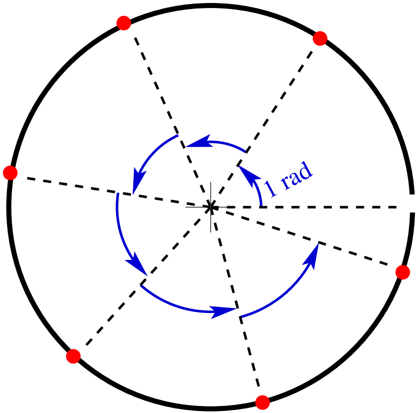

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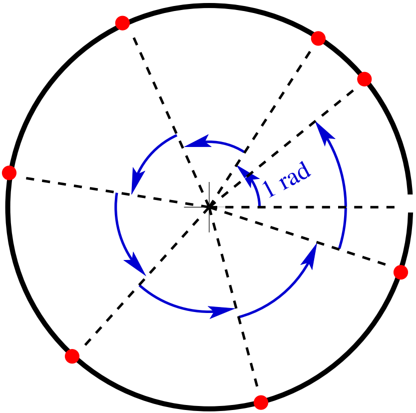

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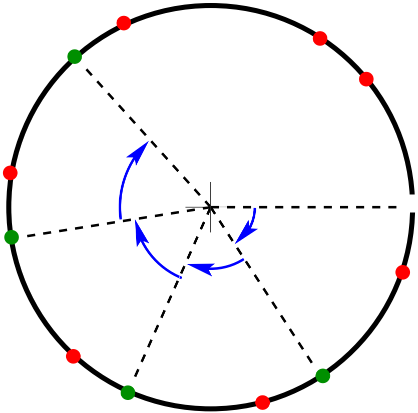

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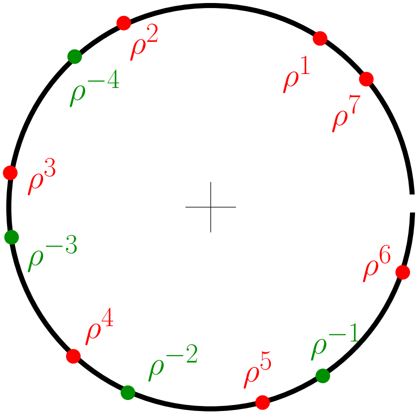
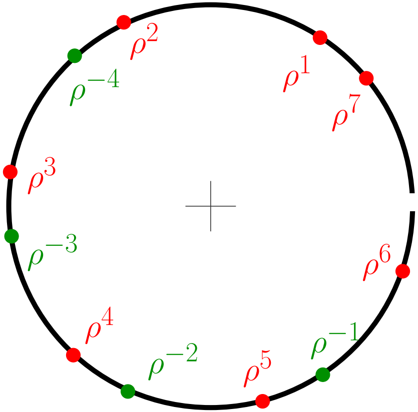


Can you break it into two non-overlapping parts, rotate and refit to get a complete circle?

8+2=10

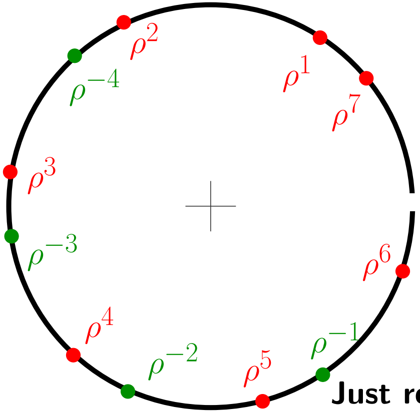
The solution



Three parts:

- $\{\rho^{-n}(\textbf{gap}) \ : \ n \in \mathbb{N}\}$
- $\{\rho^{n}(\textbf{gap}) \ : \ n \in \mathbb{N}\}$
- **rest**

The solution



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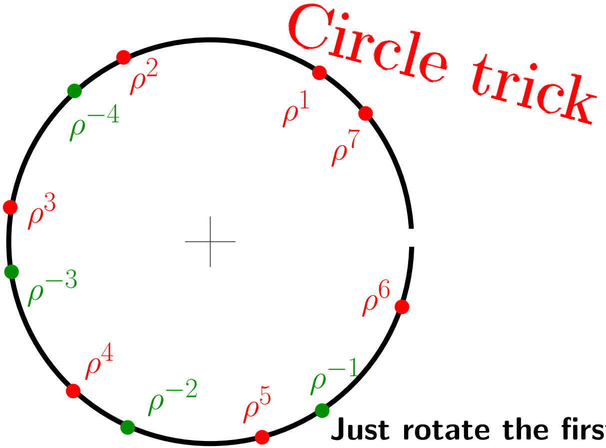
Just rotate the first part by 1 radian counter-clockwise.

8+2=10

The solution



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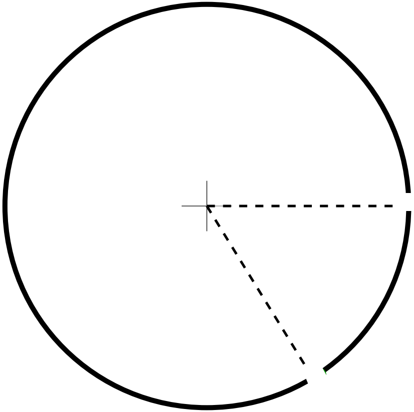
More gaps

What if there is also a gap at ρ^{-1}?



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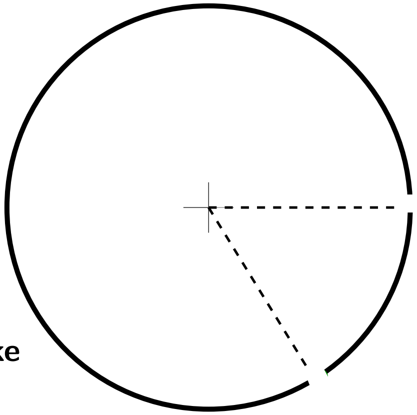

Just choose an angle that

- does not take one gap to another
- is an irrational multiple of π.

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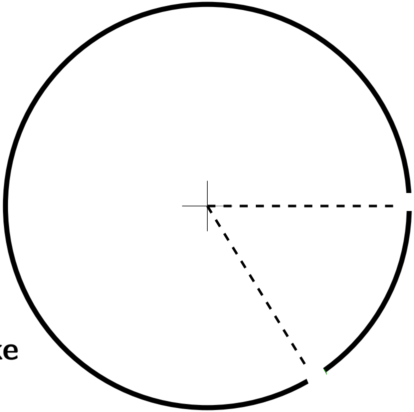

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Works for any finite number of gaps.

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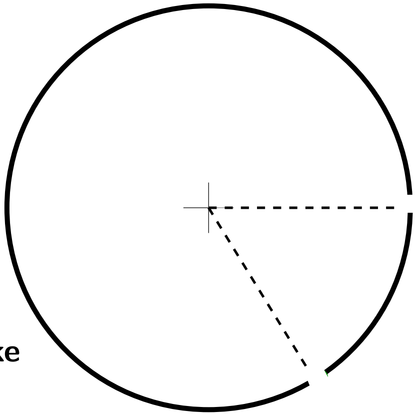

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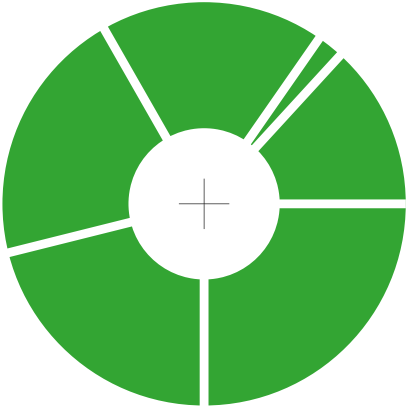
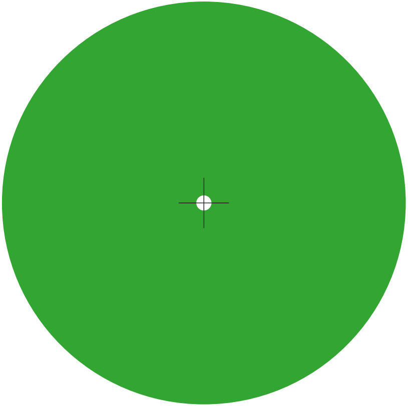
12+2=14

The plot thickens...



14+2=16

Puzzle 1



Split the punctured disk into two parts. Rotate one to get a complete disk.

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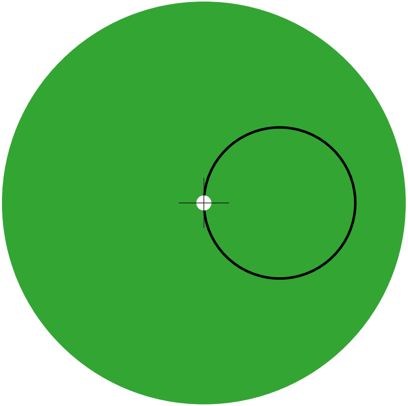
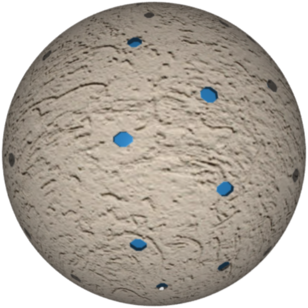


Split the punctured disk into two parts. Rotate one to get a complete disk.

16+2=18

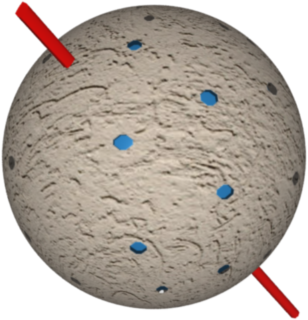
Puzzle 2



Consider a sphere with countably many holes. Split into two parts, rotate and refit to remove all the holes.

16+2=18

Puzzle 2



Just pick an axis not passing through any of the points.

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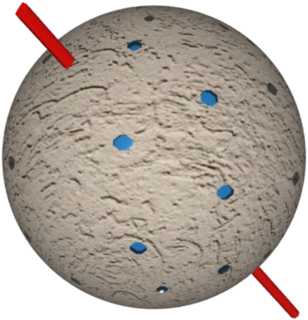

Just pick an axis not passing through any of the points.Now pick an angle such that no hole goes to another.

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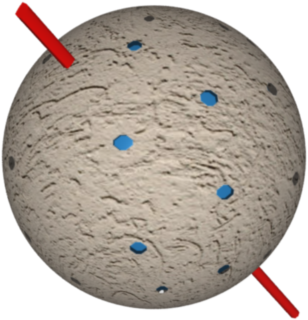
Puzzle 2

Ctbl holes trick



Just pick an axis not passing through any of the points.Now pick an angle such that no hole goes to another.

Hausdorff's paradox

It is possible to split a sphere into finitely many non-overlapping parts, rotate and refit to create two spheres identical to the original!

Independent rotation pair

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We shall need two rotations a and b such that all possible combinations like

- ab
- ba
- aa
- $ab^{-1}aaba^{-1}$
- etc

are distinct.

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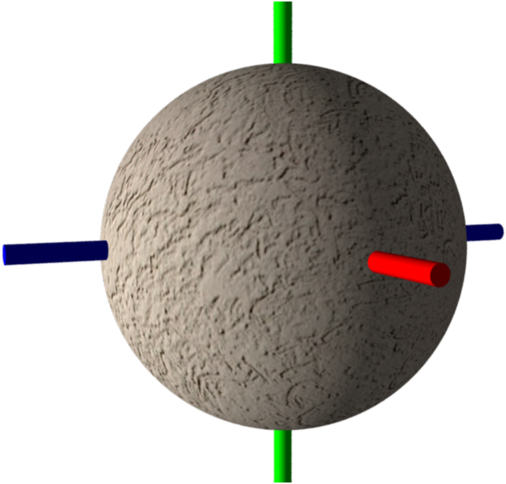

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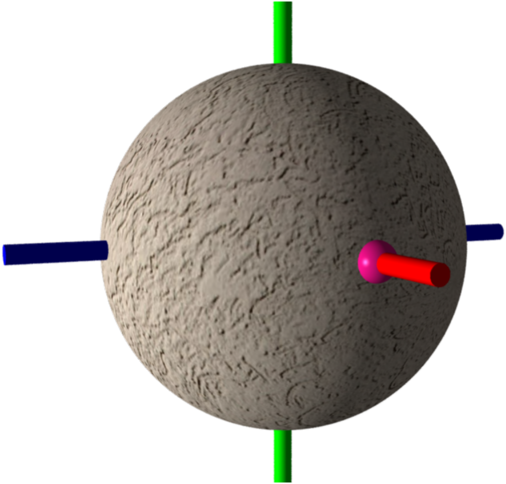

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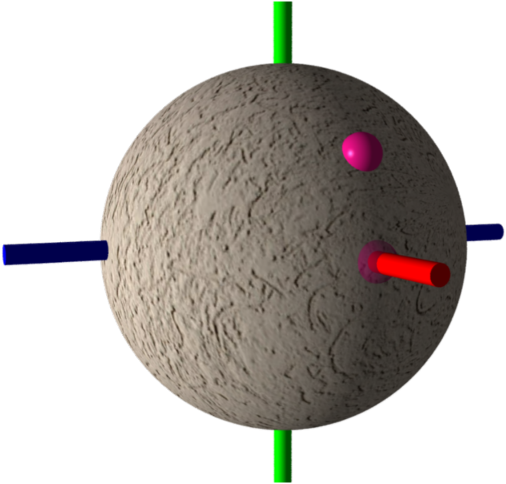

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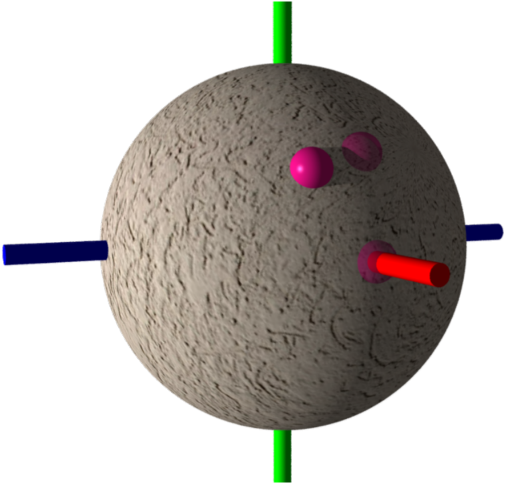

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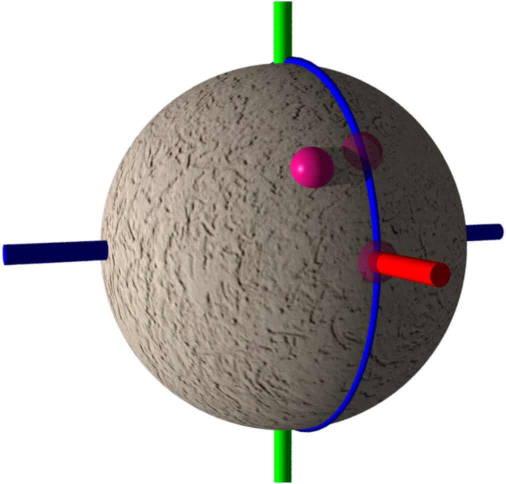

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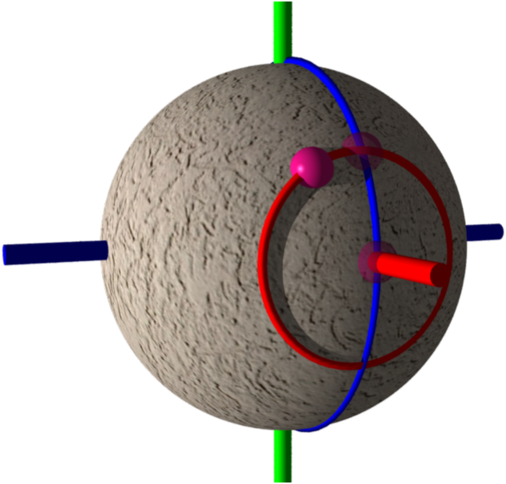
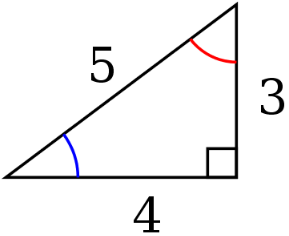
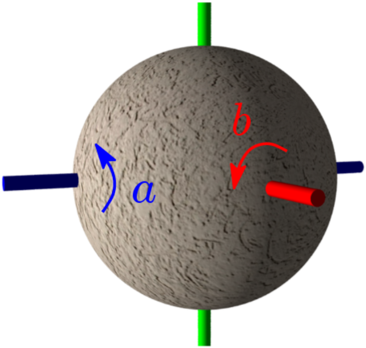


Independent rotation pair

Many such pairs exist.





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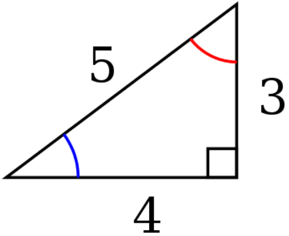

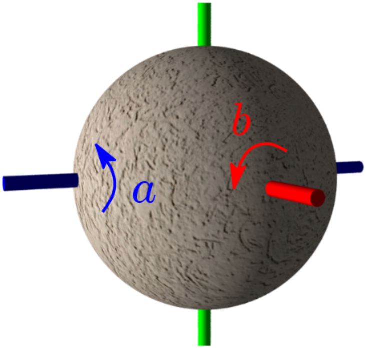

The set G of all possible rotations is countable: All possible finite sequences made of a, b, a^{-1} and b^{-1} :

$$a, b, a^{-1}, b^{-1}, ab, ba, aa, bb, a^{-1}b, ab^{-1}, \ldots$$

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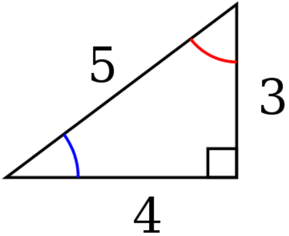
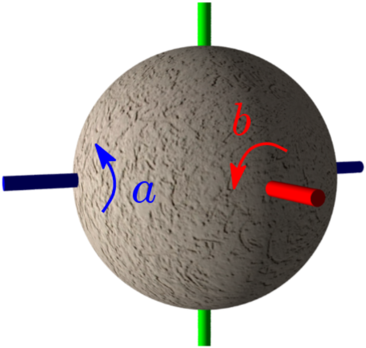
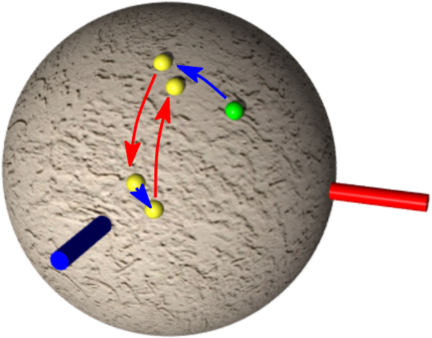




Convention: Cancel out $aa^{-1}, b^{-1}b,$ **etc:**

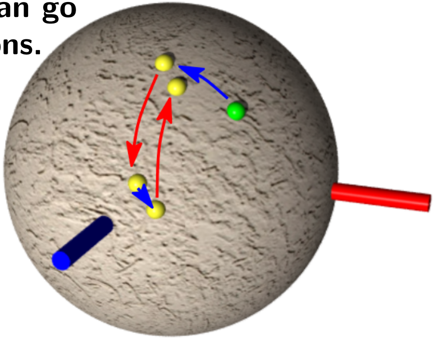
$$abb^{-1}a = aa.$$

Orbits and addresses



Orbits and addresses

Orbit of a point: The set of all positions it can go to after the rotations.



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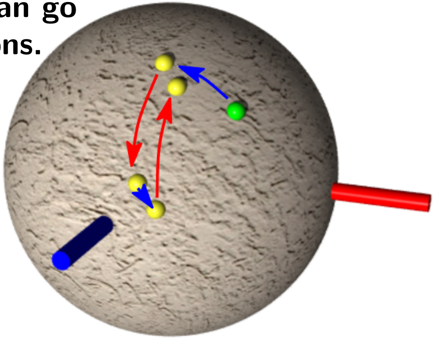

The different orbits are disjoint.

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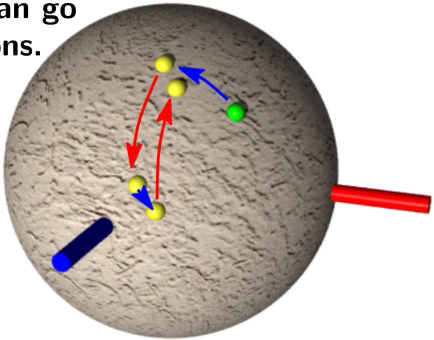

The different orbits are disjoint.

Choice set: A set M formed by picking exactly one point (**monitor**) from each orbit.

Orbits and addresses

Address of a point: (Monitor, rotation).

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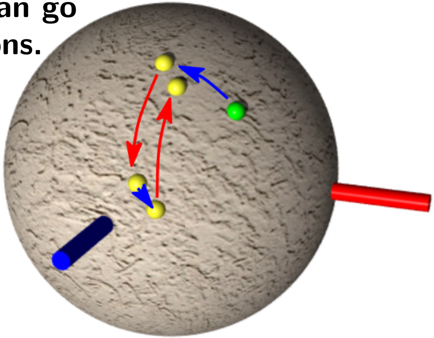

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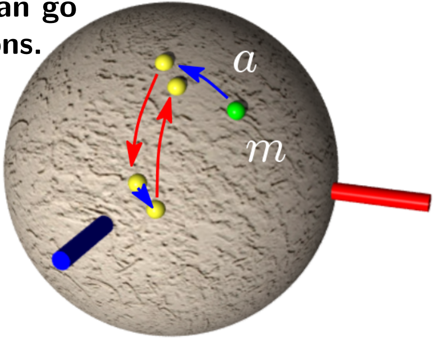

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Address $= (m, a)$.

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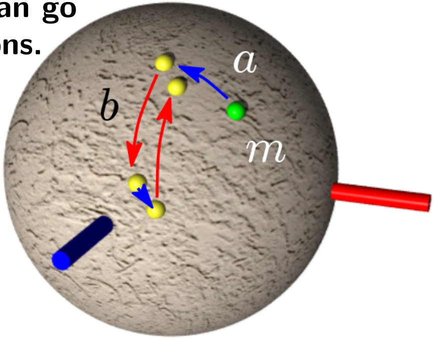

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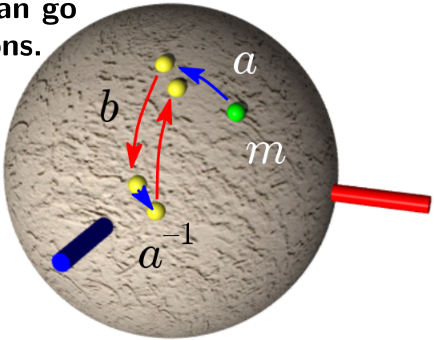

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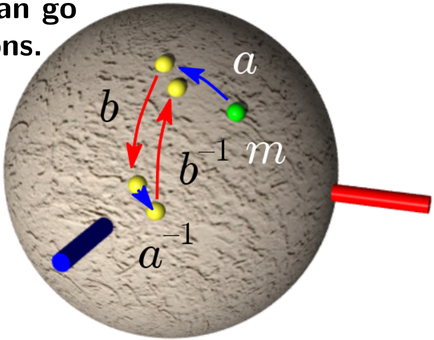

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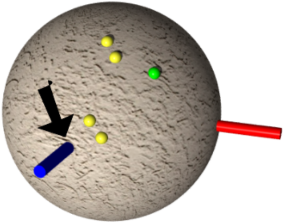
Are addresses unique?

Yes, except for the orbits of axis endpoints.



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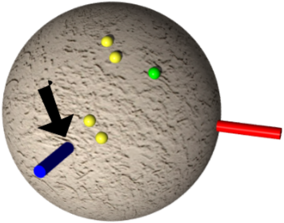
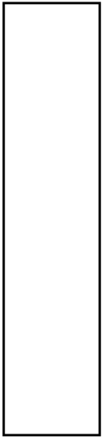


Only countable number of points with non-unique addresses.

$28+2=30$

5 parts



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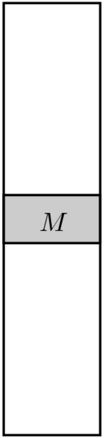

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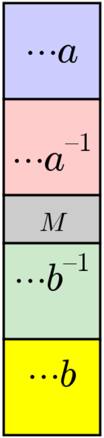

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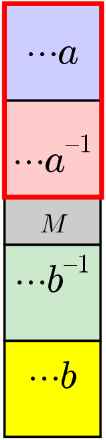

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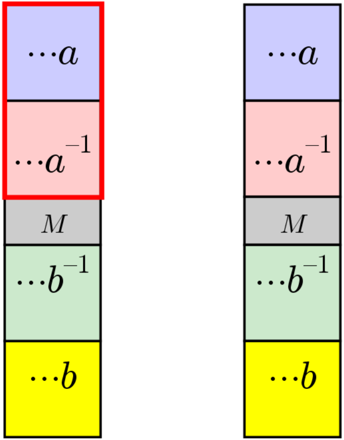
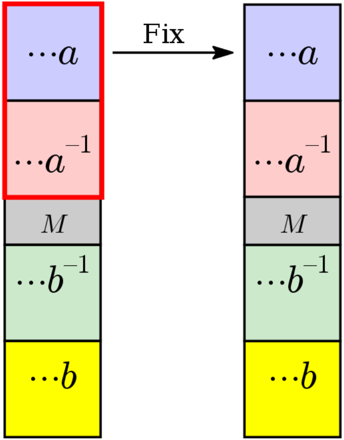


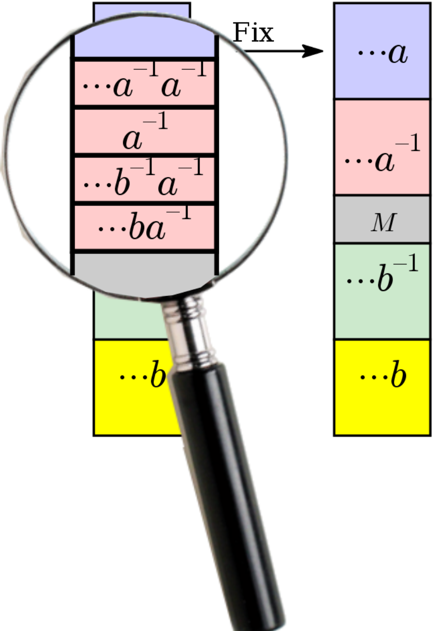
28+2=30

5 parts



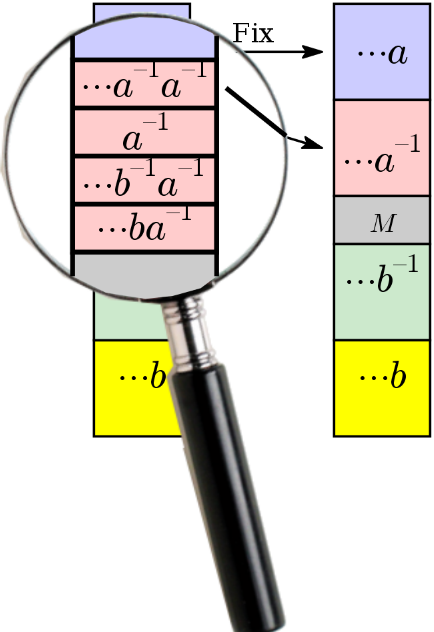
28+2=30

5 parts



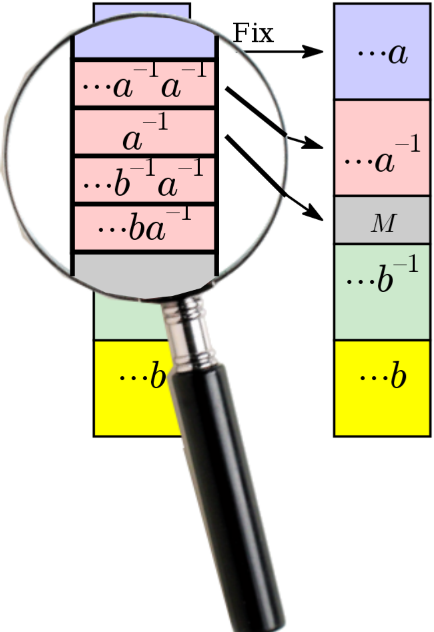
28+2=30

5 parts



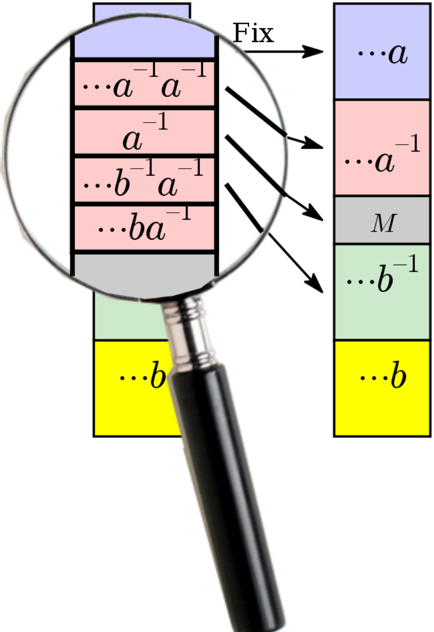
28+2=30

5 parts



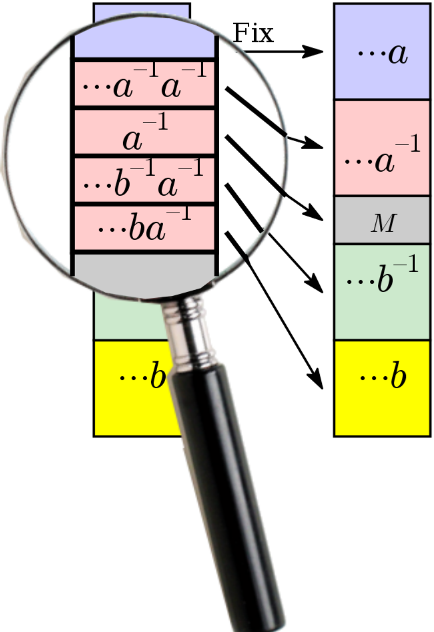
28+2=30

5 parts



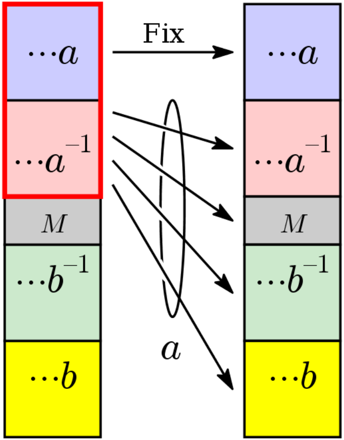
28+2=30

5 parts



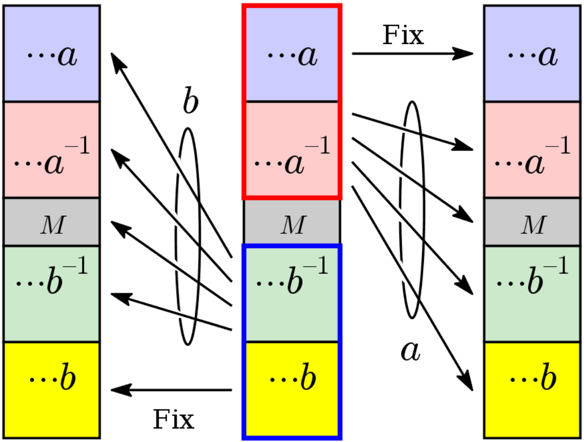
28+2=30

5 parts

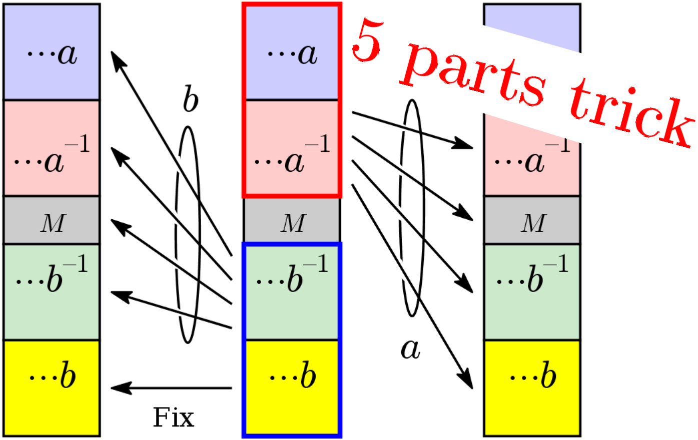


28+2=30

5 parts

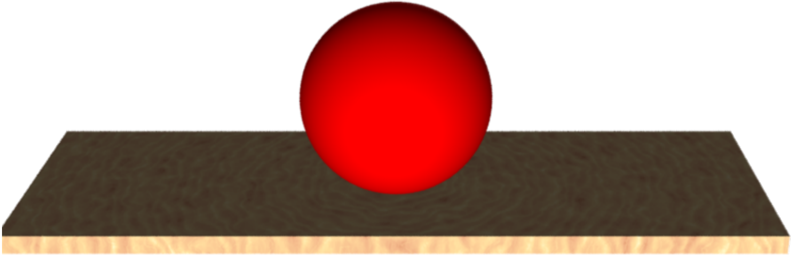


5 parts



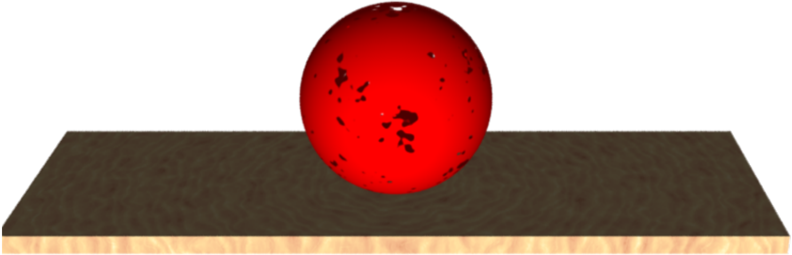
Full Hausdorff

Start with a sphere.



Full Hausdorff

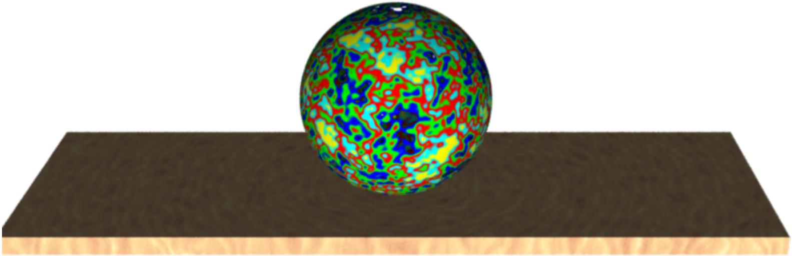
Remove the points with non-unique addresses.



30+2=32

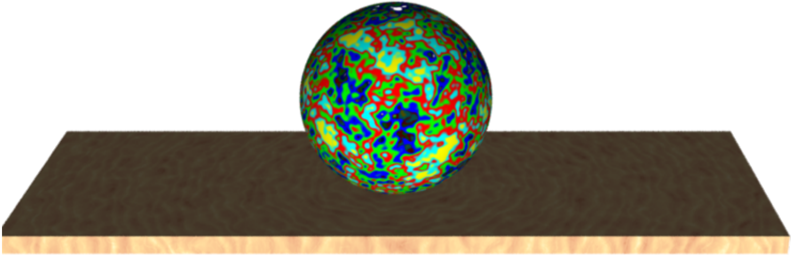
Full Hausdorff

GREEN	RED	BLUE	YELLOW	CYAN
$...a$	$...a^{-1}$	M	$...b^{-1}$	$...b$



Full Hausdorff

GREEN	RED	BLUE	YELLOW	CYAN
$\ldots a$	$\ldots a^{-1}$	M	$\ldots b^{-1}$	$\ldots b$



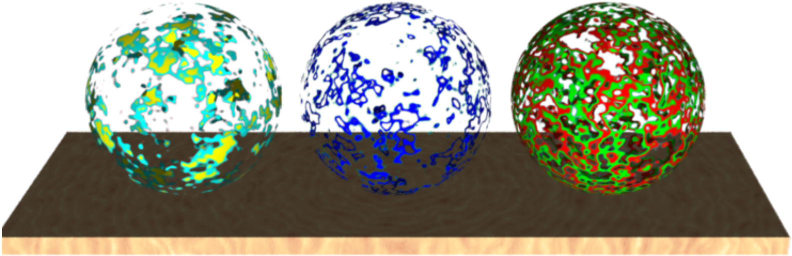
30+2=32

Full Hausdorff

GREEN	RED
$\ldots a$	$\ldots a^{-1}$

BLUE
M

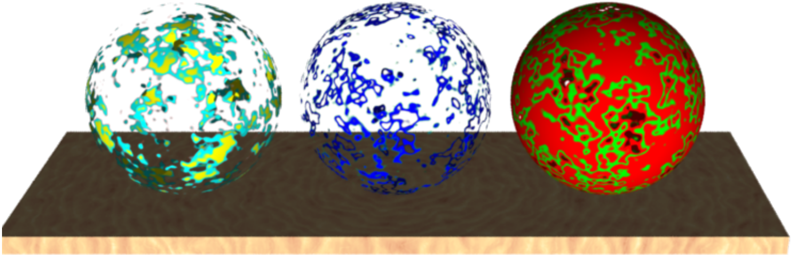
YELLOW	CYAN
$\ldots b^{-1}$	$\ldots b$



30+2=32

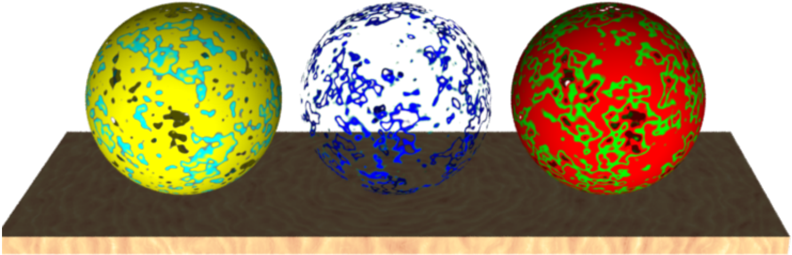
Full Hausdorff

GREEN	RED	BLUE	YELLOW	CYAN
$\ldots a$	$\ldots a^{-1}$	M	$\ldots b^{-1}$	$\ldots b$



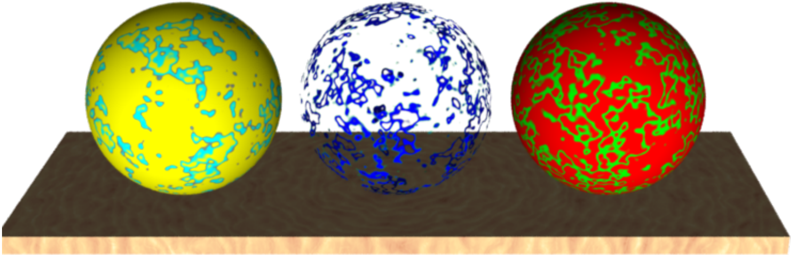
Full Hausdorff

GREEN	RED	BLUE	YELLOW	CYAN
$\ldots a$	$\ldots a^{-1}$	M	$\ldots b^{-1}$	$\ldots b$

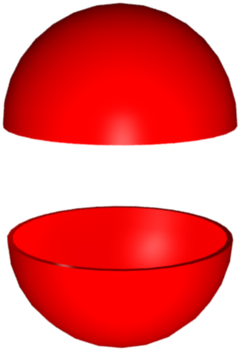


Full Hausdorff

Restore the points with non-unique addresses.

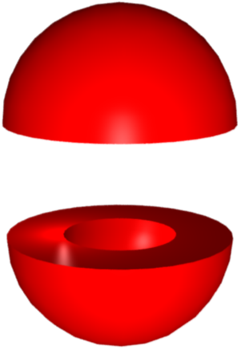


The plot thickens...(again)

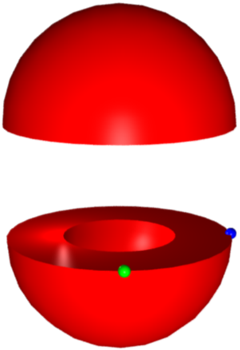


32+2=34

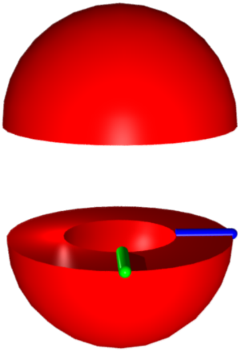
The plot thickens...(again)



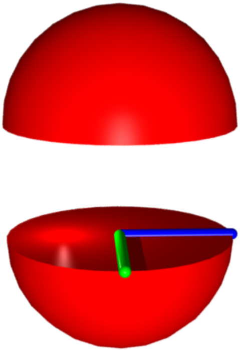
The plot thickens...(again)



The plot thickens...(again)

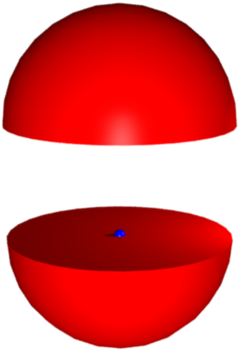


The plot thickens...(again)



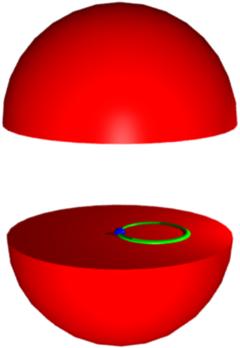
If we extend to full ball, then parts overlap at centre

The plot thickens...(again)



If we extend to full ball, then parts overlap at centre

The plot thickens...(again)



Apply circle trick at start to remove the centre. Apply circle trick again at end to bring it back.

34+2=36
$\therefore 1 \geq 2?$

$$volume(parts\ 1,2) = volume(parts\ 4,5) = volume(ball)$$

$\Longrightarrow$ 

A "similar(?)" paradox

$$\left(2 - \frac{5}{2}\right)^2 = \left(3 - \frac{5}{2}\right)^2$$

A "similar(?)" paradox

$$\left(2 - \frac{5}{2}\right)^2 = \left(3 - \frac{5}{2}\right)^2$$

$$\therefore 2 - \frac{5}{2} = 3 - \frac{5}{2}$$

A "similar(?)" paradox

$$
\begin{aligned}
\left(2-\frac{5}{2}\right)^2 &= \left(3-\frac{5}{2}\right)^2 \\
\therefore 2-\frac{5}{2} &= 3-\frac{5}{2} \\
\therefore 2 &= 3
\end{aligned}
$$

A "similar(?)" paradox

$$\begin{aligned} \left(2-\frac{5}{2}\right)^2 &= \left(3-\frac{5}{2}\right)^2 \\ \therefore 2-\frac{5}{2} &= 3-\frac{5}{2} \\ \therefore 2 &= 3 \end{aligned}$$

Holmes to Watson

How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?

Ways out

Is M really a set?

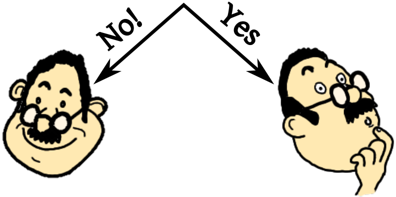
Ways out

Is M really a set?

No! **Yes**

Ways out

Is M really a set?



Ways out

Is M really a set?

No!

Yes

Does M have volume?

Ways out

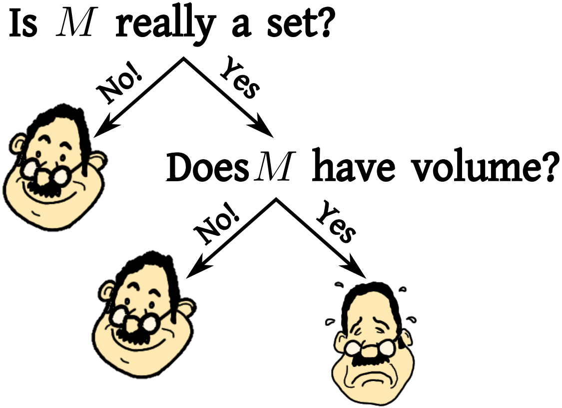
Is M really a set?

No! — Yes

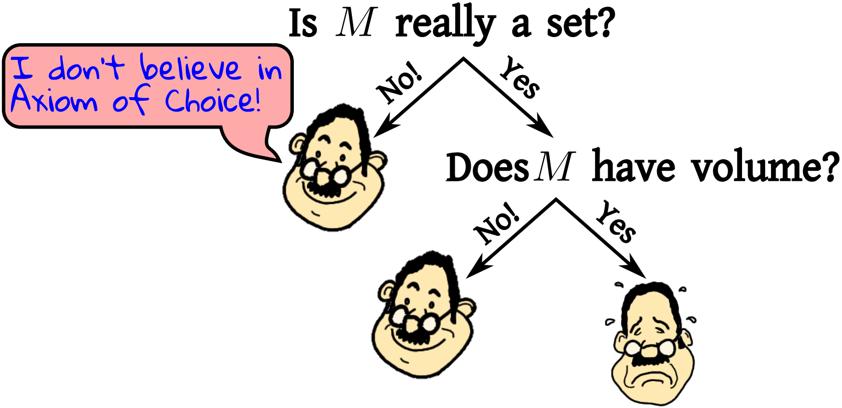
Does M have volume?

No! — Yes

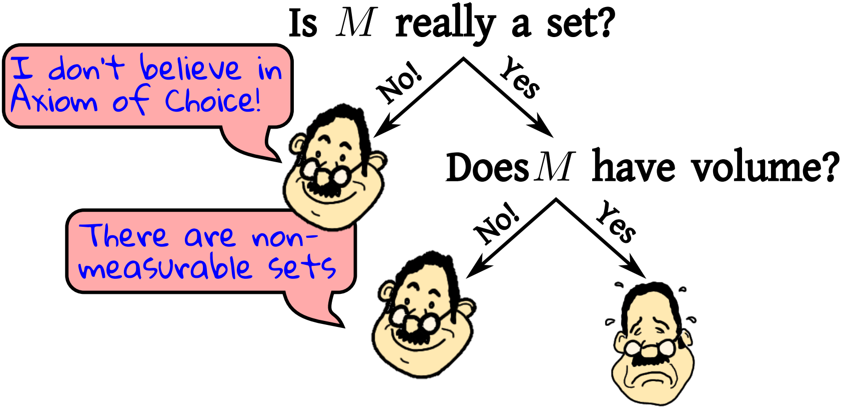
Ways out



Ways out


Is M really a set?
No!
Yes
I don't believe in Axiom of Choice!
Does M have volume?
No!
Yes

Ways out


Is M really a set?
No!
Yes
I don't believe in Axiom of Choice!
Does M have volume?
No!
Yes
There are non-measurable sets

THANK YOU!