3.3. (Sec. 3.2) Compute $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\Sigma}}$, S, and $\hat{\rho}$ for the following pairs of observations: (34, 55), (12, 29), (33, 75), (44, 89), (89, 62), (59, 69), (50, 41), (88, 67). Plot the observations.

3.5. (Sec. 3.2) Let x_1 be the body weight (in kilograms) of a cat and x_2 the heart weight (in grams). [Data from Fisher (1947b).]

(a) In a sample of 47 female cats the relevant data are

$$\Sigma \boldsymbol{x}_\alpha = \begin{pmatrix} 110.9 \\ 432.5 \end{pmatrix}, \qquad \Sigma \boldsymbol{x}_\alpha \boldsymbol{x}_\alpha' = \begin{pmatrix} 265.13 & 1029.62 \\ 1029.62 & 4064.71 \end{pmatrix}.$$

Find $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\Sigma}}$, S, and $\hat{\rho}$.

(b) In a sample of 97 male cats the relevant data are

$$\Sigma \boldsymbol{x}_\alpha = \begin{pmatrix} 281.3 \\ 1098.3 \end{pmatrix}, \qquad \Sigma \boldsymbol{x}_\alpha \boldsymbol{x}_\alpha' = \begin{pmatrix} 836.75 & 3275.55 \\ 3275.55 & 13056.17 \end{pmatrix}.$$

Find $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\Sigma}}$, S, and $\hat{\rho}$.

3.7. (Sec. 3.2) *Invariance of the sample correlation coefficient.* Prove that r_{12} is an invariant characteristic of the sufficient statistics $\bar{\boldsymbol{x}}$ and S of a bivariate sample under location and scale transformations ($x_{i\alpha}^* = b_i x_{i\alpha} + c_i$, $b_i > 0$, $i = 1, 2$, $\alpha = 1, \ldots, N$) and that every function of $\bar{\boldsymbol{x}}$ and S that is invariant is a function of r_{12}. [*Hint:* See Theorem 2.3.2.]

Theorem 2.3.2. *The correlation coefficient ρ of any bivariate distribution is invariant with respect to transformations $X_i^* = b_i X_i + c_i$, $b_i > 0$, $i = 1, 2$. Every function of the parameters of a bivariate normal distribution that is invariant with respect to such transformations is a function of ρ.*

3.10. (Sec. 3.2) *Estimation of $\boldsymbol{\Sigma}$ when $\boldsymbol{\mu}$ is known.* Show that if $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ constitute a sample from $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{\mu}$ is known, then $(1/N)\Sigma_{\alpha=1}^N (\boldsymbol{x}_\alpha - \boldsymbol{\mu})(\boldsymbol{x}_\alpha - \boldsymbol{\mu})'$ is the maximum likelihood estimator of $\boldsymbol{\Sigma}$.

3.19. (Sec. 3.4) Prove $(1/N)\Sigma_{\alpha=1}^N (\boldsymbol{x}_\alpha - \boldsymbol{\mu})(\boldsymbol{x}_\alpha - \boldsymbol{\mu})'$ is an unbiased estimator of $\boldsymbol{\Sigma}$ when $\boldsymbol{\mu}$ is known.

CHAPTER 4