Probability-2 Lecture-19

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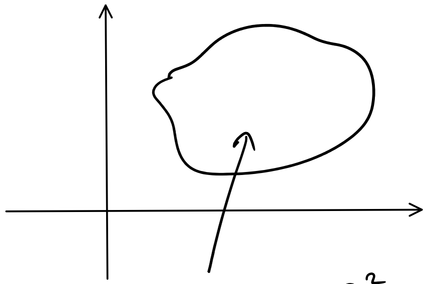
$(\Omega, \mathcal{A}, P)$ – probability space.

X, Y – real r.v.s on $(\Omega, \mathcal{A}, P)$.

For any Borel set $B \subset \mathbb{R}$,

$X^{-1}(B), \ Y^{-1}(B) \in \mathcal{A}$.



Q. for which $A \subset \mathbb{R}^2$ is

$P((X,Y) \in A)$ is defined?

$$P((X,Y)\in A) = P(\omega : (X(\omega), Y(\omega)) \in A)$$

Certain $A \subset \mathbb{R}$ are obvious,

like:– $A = B_1 \times B_2 \subset \mathbb{R}^2$ where, $B_1, B_2 \subset \mathbb{R}$ are 1-D Borel sets.

(a rectangle in $\mathbb{R}^2$)

$$\therefore \{\omega : (X(\omega), Y(\omega)) \in A\} = \{\omega : (X(\omega), Y(\omega)) \in B_1 \times B_2\}$$

$$= \{\omega : X(\omega) \in B_1, \ Y(\omega) \in B_2\}.$$

$$= X^{-1}(B_1) \cap Y^{-1}(B_2) \in \mathcal{A}.$$

$\mathcal{B}(\mathbb{R}^2) :=$ smallest σ-algebra on $\mathbb{R}^2$,

that contains all sets $\{B_1 \times B_2 : B_1, B_2 \in \mathcal{B}(\mathbb{R})\}$

that contains all sets $\{(a,b) \times (c,d)\}$

" " ... all open subsets of $\mathbb{R}^2$.

* Essence:

Consider X as a "carrier" of Probability Mass.

It takes a $B \subset \mathbb{R}$, checks its probability mass in Ω (by $X^{-1}(B)$), & then assigns that mass to B.

That's how X "distributes" the mass among all Borel sets in $\mathbb{R}$.

Recall: for a real r.v. X, the distribution of X is defined as:

$$P_X(B) = P(X^{-1}(B)) = P(\omega : X(\omega) \in B)$$

Joint Distribution of (X,Y):

$P_{X,Y}$ – a probability on $\mathcal{B}(\mathbb{R}^2)$.

defined as $\boxed{\begin{aligned} P_{X,Y}(B) &= P\left((X,Y)^{-1}(B)\right) \\ &= P\left(\omega : (X(\omega), Y(\omega)) \in B\right) \end{aligned}}$

Joint Distribution Function of (X,Y):

$F_{X,Y} : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\boxed{F_{X,Y}(a,b) = P(X \leq a,\ Y \leq b).}$$

Fact:

$F_{X,Y}$ uniquely determines (X,Y).

Properties of $F_{X,Y}$:

① $0 \leq F_{X,Y} \leq 1$

* ② $F_{X,Y}$ – non-decreasing in each coordinate.

ie, $a_1 < a_2$ (first coordinate). Fix b. (second coordinate)

$\Rightarrow F_{X,Y}(a_1, b) \leq F_{X,Y}(a_2, b)$.

$\Rightarrow P(X \leq a_1, Y \leq b) \leq P(X \leq a_2, Y \leq b)$

* ③ $F_{X,Y}$ is right continuous in each variable.

ie, $a_n \searrow a \Rightarrow F_{X,Y}(a_n, b) \searrow F_{X,Y}(a,b)$

& , $b_n \searrow b \Rightarrow F_{X,Y}(a, b_n) \searrow F_{X,Y}(a,b)$

④ $F_{X,Y}(a,b) = \begin{cases} 1, & \text{if } a \wedge b \longrightarrow \infty \\ 0, & \text{if } a \wedge b \longrightarrow -\infty \end{cases}$

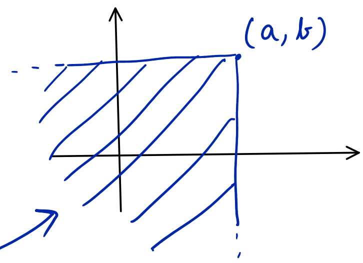
$\left[a \wedge b = \text{Max.}\{a, b\} \right]$

$\uparrow$ (a,b)

$0 \, , \quad \text{if} \quad a \wedge b \longrightarrow -\infty$

Now, note that, for (a, b),

$$F_{X,Y}(a,b) = P(X \le a, Y \le b) \;\equiv$$

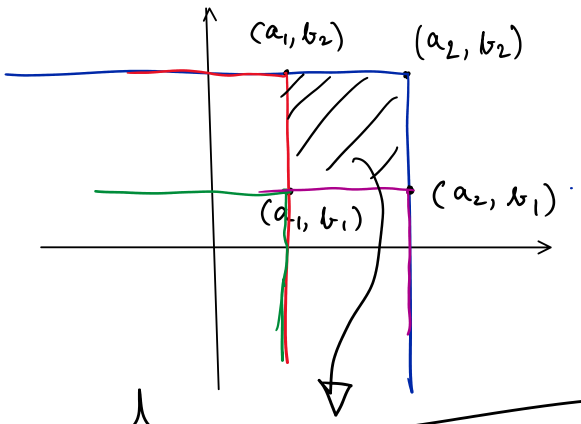


Probability "mass" of this box.

Now,

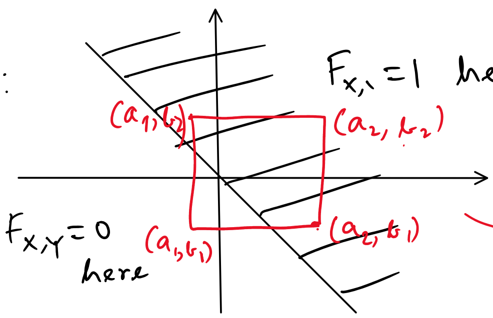
$a_2 \geqslant a_1$

$b_2 \geqslant b_1$



$$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1) \geqslant 0.$$

Eg:



In this case, the above expression:

$$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$

$= 1 - 1 - 1 + 0 = -1 \not\geqslant 0$, but in each coordinate it's increasing!

$\therefore$ This is a stronger condition than 2'!

$\therefore$ * 2' becomes 2:

$F_{X,Y}$ satisfies $\Delta F_{X,Y}((a_1, b_1), (a_2, b_2))$

2D difference.

$$= F(a_2, b_2) - F(a_2, b_1) - F(a_1, b_2) + F(a_1, b_1)$$

$$\geqslant 0 \quad \forall \quad a_2 \geqslant a_1, \; b_2 \geqslant b_1$$

$\geq 0 \quad \forall \quad a_2 \geq a_1, \; b_2 \geq b_1$

ie, $(a_2, b_2) \geq (a_1, b_1)$

Also, * ③' becomes ③ :

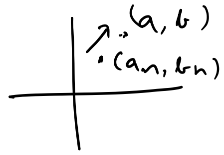
if $(a_n, b_n) \searrow (a, b)$ [ie, the sequence (a_n, b_n) approaches (a, b) from the North - East.]

$F_{X,Y}(a_n, b_n) \longrightarrow F_{X,Y}(a, b)$

Discontinuities of $F_{X,Y}$:

Let $(a_n, b_n) \nearrow (a, b)$, [ie, (a_n, b_n) approaches (a, b) from South-West]

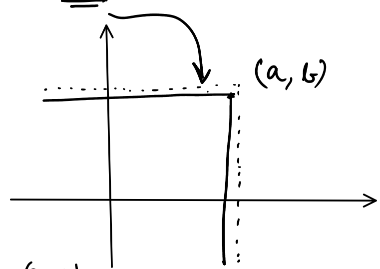
where $a_n < a$, $b_n < b$. $\forall n$.



then, $F_{X,Y}(a_n, b_n) \longrightarrow F_{X,Y}(a, b)$

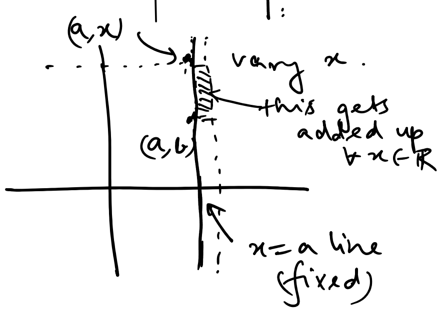
ie, $P(X \leq a_n, Y \leq b_n) \longrightarrow P(X < a, Y < b)$.

$F_{X,Y}$ is discontinuous at $(a, b) \Leftrightarrow P((X, Y) \in L_{a,b}) > 0$,



Here, $F_{X,Y}$ can have uncountably many discontinuities, as,

if, WLOG, we fix a, & keep varying x in (a, x), then the line will keep "distributing" the positive mass for all such x (the second argument.)



$\therefore F_{X,Y}$ is discontinuous at (a, b)

$\Leftrightarrow P((X, Y) \in L_{a,b}) > 0$.

$\therefore$ in a 2D distribution, how many discontinuities are possible?

— Either None or Uncountably many !!

Definition:

(X, Y) is said to be jointly continuous if $F_{X,Y}$ is continuous everywhere.

(X,Y) is said to be jointly continuous

if $F_{X,Y}$ is continuous everywhere.

(X,Y) is said to be jointly absolutely continuous

if $\exists$ a non-ve f^n f on $\mathbb{R}^2$ s.t.

$$\boxed{F_{X,Y}(a,b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy}$$

this "f" is called the density function.

The condition of jointly absolutely continuity is equivalent to:

$B \in \mathcal{B}(\mathbb{R}^2)$

"Area"$(B) = 0 \Rightarrow P_{X,Y}(B) = 0$.

"lebesgue measure"

i.e., the distribution assigns 0 probability mass to those 2D Borel sets whose "area" is 0.

["Radon Nikodym Theorem"]

If (X,Y) is jointly absolutely continuous with f as joint density, then $P((X,Y)\in B) = \iint_B f(x,y)\,dx\,dy$

Examples:

① Let (X,Y) have joint density.

$$f(x,y) = c\cdot\left(xy + \tfrac{1}{2}x^2\right), \quad 2<x<4, \quad 0<y<1$$

first question? $c = ?$

$$\int_{x=2}^{4}\int_{y=0}^{1} c\left(xy+\tfrac{1}{2}x^2\right)\cdot dy\,dx = 1$$

$$\int^{4} \quad\quad {}^{1}$$

$$\Rightarrow c \cdot \int_2^4 \left(\frac{x y^2}{2} + \frac{x^2 y}{2} \right) \Bigg|_0^1 dx = 1$$

$$\Rightarrow c \int_2^4 \left(\frac{x}{2} + \frac{x^2}{2} \right) dx = 1$$

$$\Rightarrow c \cdot \left(\frac{x^2}{2} + \frac{x^3}{3} \right) \Bigg|_2^4 = 2$$

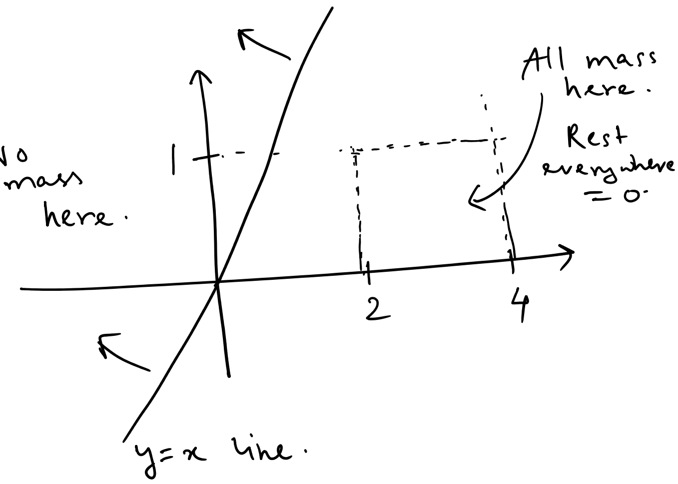
$$\Rightarrow c\left(8 + \frac{64}{3} - 2 - \frac{8}{3} \right) = 2$$

$$\Rightarrow c\left(6 + 56/3 \right) = 2$$

$$\Rightarrow c \times \frac{\cancel{74}^{\,37}}{3} = \cancel{2} \quad \Rightarrow \boxed{c = \frac{3}{37}}$$

Now, $P(X<Y) = ?$

Ans: $\underline{0}$



$P(XY<1) = ?$

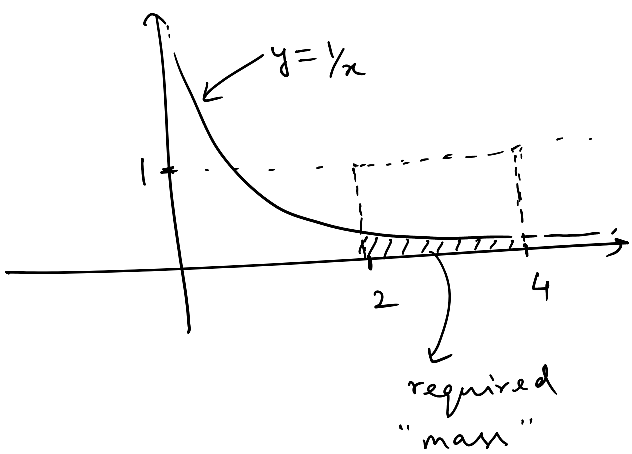
$xy < 1$

$y < 1/x$

$\downarrow$

at $x = 2$,

$y = 1/2$



$$\therefore P(XY<1) = \int_{x=2}^{4} \int_{y=0}^{1/2} f(x,y)\, dx\, dy.$$

(2) $f(x,y) = Cx(y-x)\cdot e^{-2y}, \quad 0<x<y<\infty$.

$C = ?$

$$\therefore \int_{y=0}^{\infty}\int_{x=0}^{y} Cx(y-x)\cdot e^{-2y}\, dx\, dy = 1.$$

$$\Rightarrow c\int_{y=0}^{\infty}\left(\int_{x=0}^{y} x(y-x)\, dx\right)\cdot e^{-2y}\, dy = 1$$

$$\Rightarrow c\cdot\int_{y=0}^{\infty}\left(y\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^y \cdot e^{-2y}\, dy = 1$$

$$\Rightarrow c\cdot\int_0^{\infty}\left(\frac{y^3}{2} - \frac{y^3}{3}\right)\cdot e^{-2y}\, dy = 1.$$

$$\Rightarrow \frac{c}{6}\underbrace{\int_0^{\infty} y^3\cdot e^{-2y}\, dy}\ = 1$$

$2y = t \Rightarrow dy = \frac{dt}{2}$

$y = t/2$

$$\frac{c}{6}\cdot\frac{1}{2^4}\underbrace{\int_0^{\infty} t^3\cdot e^{-t}\, dt}_{\Gamma(4) = \lfloor\underline{4-1} = \lfloor\underline{3} = 6}$$

$$\Rightarrow \frac{c}{6}\times\frac{1}{24}\times\Gamma(4) = 1$$

$$\Rightarrow \frac{c}{\cancel{6}}\times\frac{1}{2^4}\times\cancel{6} = 1$$

$$\Rightarrow \boxed{C = 16}.$$

$F_{X,Y}$ - joint distribution of (X,Y).

$\Rightarrow$ For each $a \in \mathbb{R}$,

$F(a,\infty) = \underset{b\nearrow\infty}{Lt}\ F_{X,Y}(a,b)$ is a distⁿ fⁿ, & distⁿ fⁿ of X.

$F(\infty, b) = \underset{a \nearrow \infty}{\mathrm{Lt}} \; F_{X,Y}(a,b)$ is a distⁿ fⁿ, & distⁿ fⁿ of Y.

(X,Y) has joint density f.

$\Rightarrow$ X has density $f_X(x) = \int f(x,y)\,dy$

Y " " $f_Y(y) = \int f(x,y)\,dx$

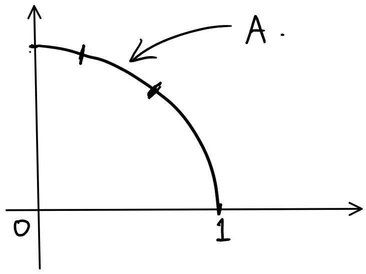
$\Big\}\longrightarrow$ integrating out x and y.

However, the CONVERSE IS NOT TRUE!

<u>Counter Example:</u>

define.

$P((X,Y) \in \text{arc } A) = \frac{2}{\pi} \cdot \text{arc length}(A)$



only boundary of 1st quadrant of unit circle. $x^2+y^2=1$.

$S \equiv$ complete arc.

Suppose, if possible,

let (X,Y) have a joint density.

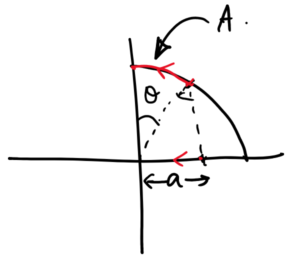
then, $\int_S \int f(x,y)\,dx\,dy = 1$ (should be!)

But, $\int_{x=0}^{1} \int_{y=\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx = 0$ (why? as, for a specific x, only one value of y possible, i.e., $y=\sqrt{1-x^2}$)

So, (X,Y) does <u>NOT</u> have a joint density.

But, X & Y have individual densities!

$P(X \leq a) = \frac{2}{\pi} \cdot (\text{Arc length } A)$.



$= \frac{2}{\pi} (\sin^{-1}(a))$

$= \frac{2}{\pi} \cdot \int_0^a \frac{dx}{\sqrt{1-x^2}}$.

Similarly for <u>Y</u>.

ie, joint density exists $\Longrightarrow$ individual densities exist

But individual densities exist $\not\Longrightarrow$ joint density exists.

Q. $g(x,y) = e^{-(ax^2+2bxy+cy^2)}$

What are the conditions on a, b, c for g(x,y) to be integrable?

$$\int_y \int_x e^{-(ax^2+2bxy+cy^2)}\, dx\, dy$$

$$= \int_y \int_x e^{-a\left(x^2+2\frac{b}{a}\cdot y\cdot x+\frac{b^2}{a^2}\cdot y^2\right)} \cdot e^{-\left(c-\frac{b^2}{a}\right)\cdot y^2} \cdot dx \cdot dy$$

$$= \int_y \left(\int_x e^{-a\left(x+\frac{b}{a}y\right)^2} \cdot dx \right) \cdot e^{-\left(c-\frac{b^2}{a}\right)y^2} \cdot dy.$$

Integrable when $a>0$. (this "looks like" shifted Normal distn).

& $\int_y e^{-\left(c-\frac{b^2}{a}\right)\cdot y^2} dy$ is integrable

when $c-\frac{b^2}{a}>0$

$\Rightarrow c > \frac{b^2}{a}$.

$\Rightarrow ac > b^2$ (& $a>0$, $ac>b^2\geq 0$ $\therefore c>0$)

$\therefore$ g(x,y) is integrable when $a>0$, $c>0$ & $ac>b^2$

<u>Note that:</u>

$$ax^2+2bxy+cx^2 = (x \;\; y)\underbrace{\begin{pmatrix} a & b \\ b & c \end{pmatrix}}_{= A}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\ddot{A}$

ie, $g(x,y) = e^{-(x\ y)\cdot A\cdot \binom{x}{y}}$

This is a quadratic form.

$\therefore$ $g(x,y)$ to be integrable $\Leftrightarrow$ $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is a positive definite matrix.

Usual: $a = \dfrac{1}{2\sigma^2}$

Usual redressing for 2D :

$$a = \frac{1}{2\sigma_1^2(1-\rho^2)}$$

$$b = \frac{1}{2\sigma_1\sigma_2\cdot(1-\rho^2)}.$$, where $\rho^2 \leq 1$.

$$c = \frac{1}{2\sigma_2^2(1-\rho^2)}$$