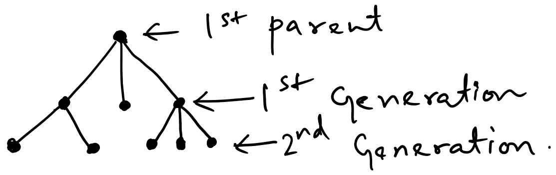
Probability-2 Lecture-4

18 January 2024 11:25

Recall: (Galton- Watson Model)



X_n = size of n^{th} generation

$X_0 \equiv 1$

$p_0, p_1, p_2, \ldots$ pmf on $\{0, 1, 2, \ldots\}$

$p_j = P$ (a node branches to j nodes).

$\therefore$ $q = P$(tree becomes extinct). $\quad 0 \leq q \leq 1$.

Let φ denote the pgf of $\{p_0, p_1, \ldots\}$

$$\varphi(t) = \sum_{j \geq 0} p_j t^j, \quad |t| \leq 1.$$

$$\therefore \varphi_n(t) = \underbrace{\varphi(\varphi(\varphi(\ldots\ldots(\varphi}_{n\text{-fold}}(t))\ldots))$$

$$\varphi_n(t) = \varphi(\varphi_{n-1}(t))$$

$$q = \lim_{n\to\infty} P(X_n = 0) = \lim_{n\to\infty} \varphi_n(0)$$

$$= \lim_{n\to\infty} \varphi(\varphi_{n-1}(0))$$

$$\boxed{q = \varphi(q)}$$

Case-1: $p_0 = 0 \cdot \longrightarrow$ every node will branch out to atleast one node.

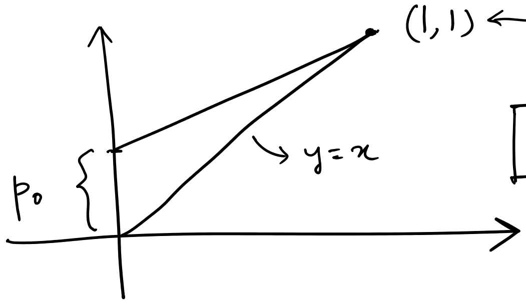
$\therefore X_1 \geq 1$. ; $X_2 \geq X_1 \geq 1 \ldots$

$\therefore X_n \geq X_{n-1} \geq \ldots \geq X_1 \geq 1$.

$$\boxed{\therefore q = 0}.$$

Case - 2: $p_0 > 0.$ $p_0 + p_1 = 1$

$\therefore \varphi(t) = p_0 + p_1 t$



$\boxed{q=1}$

[pts. where $f(x)=x$]

$p_1 = 1 - p_0.$

$\therefore \varphi(t) = p_0 + (1-p_0)\cdot t$

Also, $\varphi(t) = t$

$\therefore p_0 + (1-p_0)\cdot t = t$

$\Rightarrow p_0 = t(1-(1-p_0))$

$\Rightarrow p_0 = t \cdot p_0$

$\Rightarrow \boxed{t=1} \rightarrow (t, \varphi(t)) = (1,1)$ (fixed pt.)

Assume: $p_0 > 0.$

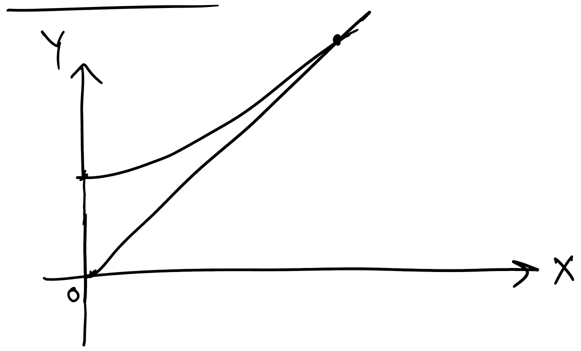
$p_0 + p_1 < 1.$

$$\varphi(t) = p_0 + p_1 t + p_2 t^2 + \cdots$$

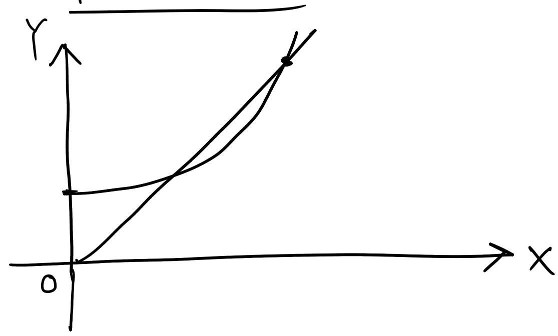
Claims:

1. φ is strictly increasing.
2. $\varphi'(t) = p_1 + 2p_2 t + 3p_3 t^2 + \cdots$ is strictly $\uparrow$ on $[0,1)$.

Picture - 1:



Picture - 2:



Analytical description of the pictures:

Picture-1: $\underset{t \nearrow 1}{\text{Lt}}\ \varphi'(t) \leq 1$

"t increases to 1."

$\because \varphi'$ is strictly increasing, $\varphi(t) < 1 \ \forall \ t < 1.$

For any $s \in [0,1)$ $\therefore \dfrac{\varphi(1) - \varphi(s)}{1-s} = \varphi'(t)$ for some constant $t \in (s,1)$ [LMVT]

$1 - \varphi(s) = (1-s)\cdot\varphi'(t)$ [$\because \varphi(1) = 1$]

$< 1 - s$ [$\because \varphi'(t) < 1$]

$$1-\varphi(s) = (1-s)\cdot\varphi'(t) \qquad [\because \varphi(1)=1]$$
$$< 1-s \qquad [\because \varphi'(t)<1]$$

$$\therefore \cancel{1}-\varphi(s) < \cancel{1}-s$$
$$\Rightarrow \varphi(s) > s.$$

$\therefore$ If the "expected no. of nodes produced by each node" ≤ 1, then the tree is bound to go extinct with probability $=1$

<u>Picture-2:</u> $\lim_{t\nearrow 1} \varphi'(t) > 1$.

$\therefore \exists$ a $t_0<1$ such that, $\varphi'(t)>1 \;\forall\; t\in(t_0,1)$.

$$\therefore \frac{\varphi(1)-\varphi(t_0)}{1-t_0} = \varphi'(t) \text{ for some } t\in(t_0,1) \quad [\text{LMVT}]$$
$$\Rightarrow 1-\varphi(t_0) = (1-t_0)\cdot\varphi'(t).$$
$$\Rightarrow \cancel{1}-\varphi(t_0) > \cancel{1}-t_0$$
$$\Rightarrow \varphi(t_0) < t_0. \Rightarrow \varphi(t_0)-t_0<0$$

$\therefore g(t) = \varphi(t)-t$. $\quad g(0) = p_0 > 0.$
$\quad g(t_0) = \varphi(t_0)-t_0 < 0.$

| $\varphi(t) = p_0+p_1t+p_2t^2+\dots$
| $\therefore \varphi(0) = p_0$
| $\varphi(1) = p_0+p_1+\dots = 1$

$\therefore$ By Rolle's theorem,
$g(t)=0$ for some $t\in(0,t_0)$.

Suppose $0<t_1<t_2<1$
s.t. $g(t_1)=g(t_2)=g(1)=0$.

$g(t)=\varphi(t)-t$
$\therefore g'(t)=\varphi'(t)-1$

$\Rightarrow \exists\, s_1\in(t_1,t_2),\; s_2\in(t_2,1)$
s.t. $g'(s_1)=g'(s_2)=0 \Rightarrow \varphi'(s_1)-1 = \varphi'(s_2)-1$
$\Rightarrow \varphi'(s_1)=\varphi'(s_2).$

This is a contradiction!!
$[\because \varphi'$ is $\nearrow]$.

ie, there cannot exist more than 1 fixed point.

Claim: Incase $\lim \varphi'(t)>1$., we have $q=\alpha$

Claim: Incase $\lim_{t \nearrow 1} \varphi'(t) > 1$, we have $q = \alpha$

Recall: Let θ be any fixed point of φ on $[0,1]$.

Then, $q \leq \theta$.

Proof: $0 \leq \theta \Rightarrow \varphi(0) \leq \varphi(\theta) = \theta$. $\quad [\because P(X_n = t) = \varphi(t)]$

$$\Rightarrow P(X_1 = 0) \leq \theta.$$

$$\therefore P(X_2 = 0) = \varphi_2(0) = \varphi(\varphi(0))$$
$$= \varphi(P(X_1 = 0))$$
$$\leq \varphi(\theta) = \theta$$

Use induction to show that,

$$P(X_n = 0) \leq \theta.$$

$$\Rightarrow q = \lim_{n \to \infty} P(X_n = 0) \leq \theta. \quad \blacksquare$$

X, Y - discrete r.vs.

$$p(x,y) = P(X = x, Y = y).$$

Conditional distribution of X, given $Y = y$:

$$p(x|y) = P(X = x, Y = y) = \frac{p(x,y)}{\sum_x p(x,y)} \rightarrow \text{Marginal} = P(Y=y)$$

$P(\cdot|y)$ is called <u>conditional pmf</u> of X given $Y = y$.

$$E(X|Y=y) = \sum_x x \cdot p(x|y) \rightarrow \text{Conditional Expectation of } X, \text{given } Y=y$$

Claim: If $E(|X|) < \infty$,
(equivalently, X having finite expectation),
then $E(X|Y=y)$ is also finite for every value of y.

$$\therefore \sum |x| \cdot p(x|y) = \frac{\sum_x |x| \cdot p(x,y)}{\sum_x p(x,y)} \leq \frac{E(|X|)}{\sum_x p(x,y)} \leq E(|X|) < \infty$$

$$\sum_x |x| \cdot p(x,y) \leq \sum_x |x| \cdot p(x) = E(|X|)$$

<u>Result</u>: Assume X has finite expectation.

$$\therefore E(X) = \sum_y E(X|Y=y) \cdot P(Y=y)$$

"Law of Total Expectations"

X, Y — r.vs.

$g : D_X \longrightarrow \mathbb{R}$.

$$\therefore E(g(X)|Y=y) = \sum_x g(x) \cdot p(x|y)$$

$$E(g(X)) = \sum_y E(g(X)|Y=y) \cdot P(Y=y).$$

$\psi(y) = E(g(X)|Y=y)$ defines a real-valued function on D_Y.

$$Y : \Omega \longrightarrow D_Y \xrightarrow{\psi} \mathbb{R}.$$

$\therefore \psi(Y)$ is a random variable.

$\therefore E(g(X)|Y)$ is a random variable.

$\therefore E(g(X)|Y)(\omega) = E(g(X)|Y=y)$ if $Y(\omega) = y$

<u>Fact</u>:

$$E(E(g(X)|Y)) = E(g(X)).$$

Fact:

$$E(E(g(X)|Y)) = E(g(X)).$$

"Expectation of conditional expectation is the original expectation."

Proof: $E(E(g(X)|Y))$

$$= E\left(\sum_x g(x)\cdot P(X=x|Y)\right)$$

$$= \sum_x E(g(x)\cdot P(X=x|Y))$$

$$= \sum_x \left(\sum_y g(x)\cdot P(X=x|Y=y)\cdot P(Y=y)\right) *$$

$$= \sum_x \left(\sum_y g(x)\cdot \frac{p(x,y)}{\cancel{p(y)}}\cdot \cancel{p(y)}\right)$$

$$= \sum_x g(x)\cdot \underbrace{\sum_y p(x,y)}_{\to \text{Marginal}} = \sum_x g(x)\cdot p(x)$$

$$= \sum_x g(x)\cdot P(X=x) = E(g(X)). \quad \blacksquare$$

* Here, $E(X) = \sum x\cdot p(X=x)$.
Here, we replace x by $g(x)\cdot P(X=x|Y=y)$ & $p(X=x)$ by $P(Y=y)$ [Not the whole function.]

$E(g(X)|Y) = \psi(Y)$ $\quad [B \subseteq \mathbb{R}]$

Claim: $E(\psi(Y)\cdot \mathbb{1}_{Y\in B}) = E(g(X)\cdot \mathbb{1}_{Y\in B})$.

i.e., if any ψ satisfies the above equation, then ψ has to be $\psi(Y) = E(g(X)|Y)$.

Tutorial-2

X, Y - function on Ω. $\quad p(x,y)$.

$g:\mathbb{R}\to\mathbb{R}$.

$g(X):\Omega\to\mathbb{R}$.

Assume $E(g(X))$ to be finite.

Define $Z : \Omega \to \mathbb{R}$

$$Z(w) = E(g(X) \mid Y = y) \text{ if } Y(\omega) = y.$$

Z is a random variable.

Result:

① $E(Z)$ is finite.

② $E(Z \, \mathbb{1}_{Y \in B}) = E(g(X) \cdot \mathbb{1}_{Y \in B})$ for all $B \subset \mathbb{R}$.

Proof: ①
$$E(|Z|) = \sum_y \left| E(g(X) \mid Y = y) \right| \cdot P(Y = y)$$
$$\leq \sum_y E\big(|g(X)| \;\big|\; Y = y\big) \cdot P(Y = y).$$
$$= E(|g(X)|) < \infty \quad \left[\because \text{We've assumed, } g(X) \text{ has finite expectation.} \right]$$

$\blacksquare$

②
$$LHS = E(Z \cdot \mathbb{1}_{Y \in B})$$
$$= \sum_{y \in B} E(g(X) \mid Y = y) \cdot P(Y = y)$$

(for $y \notin B$, those terms $= 0$).

$$= \sum_{y \in B} \left(\sum_x g(x) \cdot p(x \mid y) \cdot p(y) \right)$$
$$= \sum_{y \in B} \left(\sum_x g(x) \cdot \frac{p(x,y)}{\cancel{p(y)}} \cdot \cancel{p(y)} \right)$$
$$= \sum_y \sum_x \underbrace{g(x) \cdot \mathbb{1}_B(y)}_{h(x,y)} \cdot p(x,y)$$
$$= E(g(X) \, \mathbb{1}_B(Y))$$

$\blacksquare$

$$= E\Big(\underbrace{g(X)\cdot \mathbb{1}_B(Y)}_{h(X,Y)}\Big) \quad \blacksquare$$

<u>Result</u>: Let W be a random variable on Ω, s.t.:

① W is a function on Y.

② W satisfies-

$$E(W\cdot \mathbb{1}_{Y\in B}) = E(g(X)\cdot \mathbb{1}_{Y\in B}) \quad \forall\ B\subset \mathbb{R}. \quad (*)$$

Then, $W = E(g(X)\,|\,Y)$. [ie, W is identical to Z] (uniqueness)

<u>Proof</u>: Have to show, $\forall\ \omega\in\Omega$, if $Y(\omega)=y$, then

$$W(\omega) = E(g(X)\,|\,Y=y).$$

Fix $y\in D_Y$. Take $B=\{y\}\subset\mathbb{R}$.

$\therefore$ Apply $(*)$:

$$E(W\cdot \mathbb{1}_B(y)) = E(g(X)\cdot \mathbb{1}_{Y\in\{y\}})$$

Suppose, for ω s.t. $Y(\omega)=y$,

$$W(\omega)=c.$$

$$\therefore\ c\cdot P(Y=y) = E(g(X)\cdot \mathbb{1}_{Y\in\{y\}})$$

$$= \sum_x g(x)\cdot \frac{p(x,y)}{p(y)}\cdot p(y).$$

$\underline{\underline{Q}}$: Given X, Y & function $g:\Omega\to\mathbb{R}$

s.t. $E(g(X))$ finite,

does there exist a random variable Z s.t.

① Z is a function of Y ?

② for any $B\subset\mathbb{R}$,

$$E(Z\cdot \mathbb{1}_{Y\in B}) = E(g(Y)\cdot \mathbb{1}_{Y\in B}).$$

② for any $B \subset \mathbb{R}$,

$$E(Z \cdot \mathbb{1}_{Y \in B}) = E(g(Y) \cdot \mathbb{1}_{Y \in B}) .$$

- for every $B \subset \mathbb{R}$.

$$\nu(B) = E(g(X) \cdot \mathbb{1}_{Y \in B})$$

set function ("Measures")
i.e, we attach a number
to every subset B of $\mathbb{R}$.

} will be taught later !!

<u>Theorem</u>: (Radon Nikodym theorem): - this guarantees existence of such a Z.

(We "<u>might</u>" learn this in M.Stat-II).