Probability-3 Lecture-22

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<u>Set-3</u>.

2. (i) $A = \{\omega : \{X_n(\omega)\} \text{ remains bounded}\}$.

take <u>any $k \in \mathbb{N}$.</u>

$$A^{(k)} = \{\omega : \{X_{n+k}(\omega)\}_{n\geq 1} \text{ is bounded.}\}.$$

Clearly, $A = A^{(k)}$ for every k.

$$\because A^{(k)} \in \underbrace{\sigma(X_{k+1}, X_{k+2}, \dots)}_{\substack{\| \\ \mathcal{T}_k}}$$

[a subset $A \subseteq \mathbb{R}$ is bounded iff $\forall$ finite subsets $F \subseteq A$, $A \setminus F$ is bounded.]

$\therefore A \in \mathcal{T}_k \quad \forall k$

$\therefore A \in \bigcap_k \mathcal{T}_k = \mathcal{T}$. ▨

(ii) $\{S_n > 0 \text{ i.o}\} \equiv \{X_1 + \dots + X_n > 0 \text{ i.o}\}$

$\equiv \{X_1 > -(X_2 + \dots + X_n)$ for infinitely many $n.\}$

$\notin \sigma(X_{k+1}, \dots) \quad \forall k \geq 1$

$\therefore$ this is <u>NOT</u> a tail event.

3.(b) $\{X_n\}$-independent. seq.

Fix $k \geq 1$.

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$$\limsup_n \left(\frac{S_n - S_k}{a_n} \right)$$ this r.v is measurable w.r.t $\sigma(X_{k+1}, \ldots, X_{k+n}) = \mathcal{T}_k$.

$$\overset{?}{=} \limsup_n \frac{S_n}{a_n}$$

Yes.

2 seq: $(a_n - b_n)$ s.t $b_n \to 0$

$\therefore \limsup_n (a_n - b_n) = \limsup_n a_n$.

$\because \lim_{n \to \infty} \frac{S_k}{a_n} = 0$, as S_k - fixed, & $a_n \nearrow \infty$.

4.(b) $\{X_n\}$ - i.i.d.

(by contradiction)

Suppose, $\{X_n\}$ - not degenerate at 0.

$\Rightarrow \exists\ a > 0$ s.t. either

$P(X_n > a) = P(X_1 > a) = \delta > 0$

or $P(X_n < -a) = P(X_1 < -a) = \delta > 0$

$\therefore \sum_n P(X_n > a) = \sum_n P(X_1 > a) = \infty$

$\therefore$ By B.C - II,

$P(X_n > a \ \text{ i.o}) = 1.$

$\Rightarrow P(\sum_n X_n \text{ conv.}) = 0.$ ▨

(d) Again $\{X_n\}$ - i.i.d.

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Here, X_n's are "symmetric".

i.e, X_n & $-X_n$ have same r.v.

So, such an r.v, if it is degenerate, it has to be " at only & only $X_n = 0$

So, by contradiction: if X_n - not degenerate,

$\Rightarrow$ X_n - not degenerate at 0.

$\exists$ $a > 0$, s.t $P(X_n > a) = P(X_n < -a) = \delta > 0$.

$\therefore P(\limsup S_n = \infty) = 1$

$P(\liminf S_n = -\infty) = 1$.

Qs. 7. think, but don't waste much time on that.

Qs. 8 $\rightarrow$ out of syllabus for the time being.
(requires knowledge of "infinite products")

* Unless specified, r.v $\equiv$ real r.v
(ie, NOT extended real)

9. (a) $\{X_n\}$ - i.i.d.

$M_n = \text{Max.}\{|X_1|, |X_2|, \ldots, |X_n|\}$.

(a) $\frac{M_n}{n} \xrightarrow{P} 0 \iff n \cdot P(|X_1| > n) \to 0$.

" $\Leftarrow$ " $\frac{M_n}{n} \xrightarrow{P} 0$

$$\therefore P\left(\frac{M_n}{n} > \varepsilon\right) = P\left(M_n > n\varepsilon\right)$$

$$= 1 - P\left(M_n \leq n\varepsilon\right)$$

$$= 1 - P\left(|X_1| \leq n\varepsilon,\ |X_2| \leq n\varepsilon,\ \cdots |X_n| \leq n\varepsilon\right)$$

$\{X_i\}$ - iid

$$= 1 - \left[P\left(|X_1| \leq n\varepsilon\right)\right]^n$$

$$= 1 - \left(1 - P\left(|X_1| > n\varepsilon\right)\right)^n$$

$$\overset{?}{\leq} n P\left(|X_1| > n\varepsilon\right)$$

[Expand using Taylor (upto 2nd term ?!)]

$$\leq n P\left(|X_1| > [n\varepsilon]\right)$$

Greatest integer f^n.

$$= \frac{1}{\varepsilon} \cdot \frac{n\varepsilon}{[n\varepsilon]} \cdot [n\varepsilon] \cdot P\left(|X_1| > [n\varepsilon]\right)$$

$\downarrow$

0 from hypothesis.

"$\Rightarrow$" $\frac{M_n}{n} \xrightarrow{P} 0$ ..

$\therefore \frac{X_n}{n} \xrightarrow{P} 0.$

to prove:

[$(1-a_n)^n \longrightarrow 1 \Leftrightarrow n a_n \to 0$]

"$\Leftarrow$"

$(1-a_n)^n \geq 1 - n a_n.$

$\Rightarrow n a_n \geq 1 - (1-a_n)^n$

$\Rightarrow na_n \geq 1-(1-a_n)^n$

we used this previously.

9. (b) (discussion)

$\{x_n\}$ - real seq.

$$m_n := \text{Max}\{|x_1|, \dots, |x_n|\}.$$

$$\frac{x_n}{n} \to 0 \iff \frac{m_n}{n} \to 0$$

"$\Leftarrow$" trivial.

"$\Rightarrow$" $\frac{x_n}{n} \to 0$.

take $\varepsilon > 0$. $\therefore \exists N$ s.t, $\frac{|x_n|}{n} < \varepsilon \quad \forall n \geq N$.

$$\therefore \frac{m_n}{n} < \frac{(|x_1| \vee |x_2| \vee \dots \vee |x_{n-1}|)}{n} \vee \varepsilon.$$

$$\therefore \frac{X_n}{n} \xrightarrow{a.s} 0$$

$$\iff \forall j \geq 1, \ P\left(\bigcap_n \bigcup_{k \geq n} \left\{ \left|\frac{X_k}{k}\right| > \frac{1}{j} \right\}\right) = 0.$$

$$E|X_1| < \infty \iff \forall j \geq 1, \ E|j \cdot X_1| < \infty$$

[ie, any r.v has finite mean $\Rightarrow$ any multiple of it has finite mean.]

$$\Rightarrow \sum_n P(|jX_1| > n) < \infty$$

B.C.-I: $\Leftrightarrow P(|jX_1| > n \text{ i.o.}) = 0$

for independent case

(as X_i's iid), this becomes $\Leftarrow\!\!\!\!\Rightarrow$.

10. Hint: Truncate the seq. at λ. Apply SLLN.

$\lambda \to \infty$ (MCT or something)

$$\frac{S_n}{n} \xrightarrow{a.s.} E(X^\lambda).$$

$$\liminf \frac{S_n}{n} \geqslant \lim \frac{S_n^*}{n} \overset{a.s.}{=\!=} E(X_1^\lambda) \quad \forall \lambda.$$

$\lambda \uparrow \infty$. MCT.

$$E(X) = +\infty.$$

[Instead of λ,

use M – some large integer.

Why? Some "Null-sets" or smth.

(dealing with uncountable null sets)]

11. (b) $\{X_n\}$ – i.i.d.

$$\frac{X_n - c_n}{n} \xrightarrow{a.s.} 0 \quad \text{for } \underline{\text{some}} \text{ real } \{c_n\}.$$

$$\Leftrightarrow E(|X_1|) < \infty.$$

& in that case, $\frac{c_n}{n} \to 0$.

($\Leftarrow$) $E(|X_1|) < \infty$.

$\therefore$ we exhibit one real seq,

$c_n = 0$. $\quad \frac{X_n - c_n}{n} = \frac{X_n}{n} \xrightarrow{a.s} 0$

So, done. ✓

✓.

($\Longrightarrow$) we now have, for some real seq.

$\{c_n\}$,

$\frac{X_n - c_n}{n} \xrightarrow{a.s} 0$ for some $\{c_n\}$.

11\.
part (a)

$\frac{X_n}{n} \xrightarrow{P} 0$ (always)

$\therefore \Rightarrow \frac{X_n - c_n}{n} \xrightarrow{P} 0$

$\therefore \frac{c_n}{n} \longrightarrow 0$.