Probability-3 Lecture-3

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(X, Y) - random vector.

Conditional distribution of Y, given X.

Fix $x \in \mathbb{R}$. $B \subset \mathbb{R}$.

Suppose $Q(x, B)$ satisfies:

$(*)$
1) for each $x \in \mathbb{R}$, $Q(x, B)$ is a probability in B.
2) $\forall$ borel $B \subset \mathbb{R}$, $Q(x, B)$ is a measurable function.
3) $\forall$ borel $B, C \subset \mathbb{R}$,
$$P(Y \in B, X \in C) = E\left(Q(X, B) \cdot \mathbb{1}_C(X)\right)$$

$$Q(x, B) = P(Y \in B \mid X = x)$$

(interpretation)

Two "trivial" cases:

1) X is discrete.

Say, with $D \subset \mathbb{R}$ countable, satisfying $P(X = x) > 0 \;\; \forall\, x \in D \subset \mathbb{R}$.

Define
$$Q(x, B) = P(Y \in B \mid X = x) = \begin{cases} \dfrac{P(Y \in B, X = x)}{P(X = x)}, & x \in D. \\[2ex] \delta_{\{x\}}(B), & x \notin D. \end{cases}$$

Dirac mass.

(doesn't matter!!!)

2. (X, Y) jointly (absolutely) continuous with joint density f

has joint density.

Define $q(x, y) = \begin{cases} \dfrac{f(x, y)}{\quad} & , \text{ if } \int f(x, y)\, dy > 0 \end{cases}$

Define $g(x,y) = \begin{cases} \dfrac{f(x,y)}{\int f(x,y)\,dy} & , \text{ if } \int f(x,y)\,dy > 0 \\ \dfrac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}y^2} & , \text{ if } \int f(x,y)\,dy = 0. \end{cases}$

Marginal density of x.

Can be anything else as well!!

Define $Q(x, B) = \int_B g(x,y)\,dy$

A Special Case: X, Y – independent r.v.s.

$h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a measurable function.

$\therefore h(X,Y)$ is a random variable.

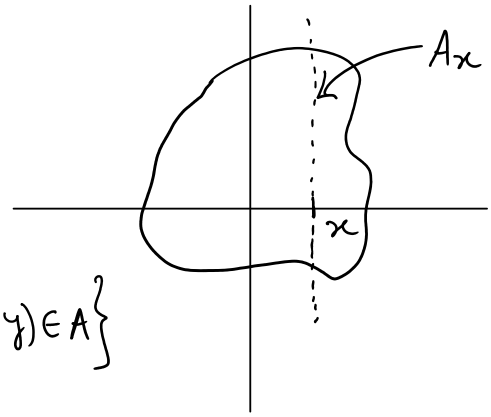
Q. What is the conditional distribution of $h(X,Y)$ given X.

- Proposed conditional distribution:

define $Q(x,B) = P(h(x,Y) \in B)$.

to show: this $Q(x,B)$ follows all the 3 results (*)

Consider the following setting:



Let $A \subset \mathbb{R}^2$ → borel subset of $\mathbb{R}^2$.

"x-section"

For $x \in \mathbb{R}$, denote $A_x = \{ y \in \mathbb{R} \mid (x,y) \in A \}$

* $\forall A \subset \mathbb{R}^2$ borel,

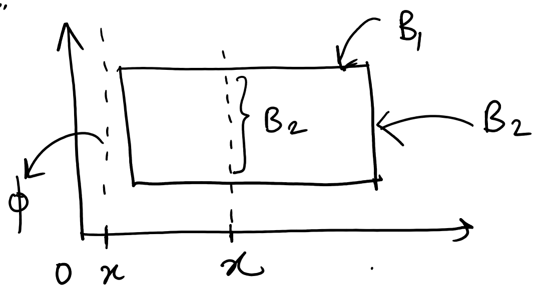
$A_x \subset \mathbb{R}$ is borel $\forall x \in \mathbb{R}$.

[ie, pick a borel in $\mathbb{R}^2$. take an x-section. that $A_x \subset \mathbb{R}$ is also borel.]

Take $A = B_1 \times B_2$ ← "rectangle" B_1

Take $A = B_1 \times B_2 \leftarrow$ "rectangle"

Clearly, $A_x = \begin{cases} B_2, & \text{if } x \in B_1 \\ \phi, & \text{if } x \notin B_1 \end{cases}$



So, we can talk about
$P(Y \in A_x)$, $\forall$ borel $A \subset \mathbb{R}^2$
& $x \in \mathbb{R}$.

$= \varphi(x, A)$

Now, fix a borel $A \subset \mathbb{R}^2$, & keep on varying x.
i.e., look at x-section.

look at
$x \longmapsto \varphi(x, A)$

$\therefore$ This is measurable $\forall$ $A \subset \mathbb{R}^2$ borel.

Clearly, this map $x \longmapsto \varphi(x, A)$ is a f^n from borel $\mathbb{R}$ to $[0,1]$.

$\therefore$ for $A = B_1 \times B_2$.
$P(Y \in A_x) = P(Y \in B_2) \cdot \mathbb{1}_{B_1}(x)$

Now, to show,
$\mathcal{Q} = \{ A \subset \mathbb{R}^2 : \varphi_x(A)$ is measurable in x
$\forall$ $B_1 \times B_2 \in \mathcal{Q} \}$

"Monotone class Theorem"

Theorem (Monotone Class Theorem):

Ω is a set. $\mathcal{F}$ is a field on Ω
→ NOT σ-field !!
→ closed under
- complementation
- finite intersection.

If M is a class of sets such that,
$\mathcal{F} \subset M$, and M is closed under monotone limits, then $M \supset \sigma(\mathcal{F})$

$\mathcal{F} \subset \mathcal{M}$, and $\mathcal{M}$ is closed under monotone limits, then $\mathcal{M} \supset \sigma(\mathcal{F})$

σ-field generated by field $\mathcal{F}$.

In our case, we have to first show - the set of rectangles in $\mathbb{R}^2$ form a field.

Where do we stand now? Let $A \subset \mathbb{R}^2$ any Borel set. $x \in \mathbb{R}$. $A_x = \{ y \in \mathbb{R} \mid (x,y) \in A \}$

Step-1: We showed:

$A_x \subset \mathbb{R}$ is Borel $\forall x$.

(first for rectangles, then ....)

then, $P(Y \in A_x)$ — an event; A_x — Borel set

$\therefore$ we can talk about this probability

Step-2: the f^n: $x \longmapsto \varphi(x, A)$ is measurable.

Step-3: $P((X,Y) \in A)$

Borel in $\mathbb{R}^2$

Hence, (X,Y) is an event.

So, we can talk about this probability.

Claim:

$$P((X,Y) \in A) = E(\varphi(X, A)).$$

a r.v. [$\because$ from Step-2: $x \longmapsto \varphi(x,A)$ is measurable.]

hence, we can talk about this expectation.

expectation.

Main thing: We can't do serious Probability without Measure Theory !!

Here, $A \subset \mathbb{R}^2$.

if $A = B_1 \times B_2$.

LHS $= P((X,Y) \in A) = P((X,Y) \in B_1 \times B_2)$

$= P(X \in B_1, Y \in B_2)$

$\left[\because X, Y \text{ independent}\right] = P(X \in B_1) \cdot P(Y \in B_2)$.

&, RHS $= E(\varphi(X, A))$ &, $\varphi(x, A) = P(Y \in A_x)$

$\therefore \varphi(X, A) = P(Y \in B_2) \cdot 1_{B_1}(X)$

the X-section.

$= E(P(Y \in B_2) \cdot 1_{B_1}(X))$

$= P(Y \in B_2) \cdot E(1_{B_1}(X))$

$= P(Y \in B_2) \cdot P(X \in B_1)$.

$\therefore$ LHS = RHS is immediate if $A = B_1 \times B_2 \subset \mathbb{R}^2$ borel.

Now, back to our last qn from last lecture (Lec-2)

proposed $Q(x, B) := P(h(x, Y) \in B)$

Is $Q(x, B)$ measurable in x?

$Q(x, B) = \varphi(x, A)$, where $A = h^{-1}(B)$.

$= \{(x, y) \mid h(x, y) \in B\}$

$\therefore$ the X-section,

$A_x = \{y \mid h(x, y) \in B\}$.

$\therefore \varphi(x, A) = P(Y \in A_x)$

$= P(h(x, Y) \in B)$, which is measurable.

$\therefore Q(x, B)$ is measurable.

Now, Is $Q(x, B)$ a probability in B?

$Q(x, B) = P(h(x, Y) \in B)$. • if $B = \mathbb{R}$, Is $\varphi(x,B)$ a probability ...:

$$\varphi(x,B) = P(h(x,Y)\in B).$$

- if $B = \mathbb{R}$, $P(h(x,Y)\in\mathbb{R}) = 1$ ✓
- take $B, B^c \subset \mathbb{R}$. $P(h(x,Y)\in B) = 1 - P(h(x,Y)\in B^c)$ ✓
- B_1, B_2 - disjoint borels in $\mathbb{R}$. $P(h(x,Y)\in B_1 \sqcup B_2) =$ ✓ $P(h(x,Y)\in B_1) + P(h(x,Y)\in B_2)$

& now, finally, is

$$P(h(X,Y)\in B, X\in C) = E(\varphi(X,B)\cdot 1_C(X)) \;??$$

take $A = \{(x,y) \mid h(x,y)\in B, x\in C\}$.

$$\therefore A_x = \begin{cases} \{y \mid h(x,y)\in B\}, & x\in C \\ \phi, & x\notin C \end{cases}$$

(empty set)

Now, the 3 steps above can be put to use.

Little relief: 🙂

All these proofs are <u>not</u> a part of the syllabus. Only the final result (i.e, the proposition) is!!

A consequence:

X, Y - independent r.v.s.

$h(x,y) = x+y$.

finding distⁿ of $Z = X+Y$

$$F_Z(a) = P(Z\le a) = P(X+Y\le a)$$

$$= E(\varphi(X) \mid \text{where } \varphi(x) = P(x+Y\le a))$$

$F_Z(a) = P(Z \leq a) = \ldots$

$$= E\left(\varphi(X)\right) \quad \text{where } \varphi(x) = P(X+Y \leq a)$$

$$= P(Y \leq a-x)$$

$$= F_Y(a-x)$$

$\therefore$

$$= E\left(F_Y(a-X)\right)$$

$$= E\left(F_X(a-Y)\right)$$

usually called

$F_X * F_Y$. This is called the <u>convolution</u>.

So, in general, X has dist^n f^n F

Y " " " G.

Then, $X+Y$ has dist^n $F * G$

<u>Fact</u>: take dist^n of X degenerate at 0.

It acts as identity !!

Also, in general, F does <u>not</u> have an inverse.

But, if F is degenerate, then inverse exists.

<u>Prove that</u>: No other dist^n will have an inverse.

Suppose F has density f $\quad X \sim F$, X, Y - ind.

G " " g. $\quad Y \sim G$.

$$F * G(a) = P(X+Y \leq a)$$

$$= E\left(G(a-X)\right)$$

$$= \int G(a-x) f(x)\, dx$$

$$= \int^{\infty} \int^{a-x} g(u)\, du\, f(x)\, dx.$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{a-x} g(y)\,dy\; f(x)\,dy\,dx$$

$u = y + x$
$\Rightarrow y = u - x$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{a} g(u-x) \cdot f(x)\, dy \cdot dx$$

$$= \int_{-\infty}^{a} \underbrace{\int_{-\infty}^{\infty} g(u-x) \cdot f(x)\, dx}_{} \cdot dy$$

→ Acts as density !!

$\therefore F * G$ has density given by —

$$f * g(u) = \int_{-\infty}^{\infty} g(u-x) \cdot f(x)\, dx .$$

More general cases:

Given general X, Y.

Try to find $P(Y \leq a \mid X = x)$

In general, the classical def^n does not work as the conditioning event $X = x$ might not have positive probability.

Take x s.t.

$$P(X \in (x-\varepsilon, x+\varepsilon)) > 0 \quad \forall\, \varepsilon > 0$$

Find $P(Y \leq a \mid X \in (x - \varepsilon, x + \varepsilon)) \quad \forall\, \varepsilon > 0$ using "classical definition"

Now, we'll do $\varepsilon \xrightarrow{lt} 0$ & check if a limit exists. → [Lebesgue Differentiation Theorem ⚠ (out of syllabus)]

(If it exists, then it works !! i.e, it gives a candidate for the conditional dist^n.)

$$P(Y \leq a \mid X = x).$$

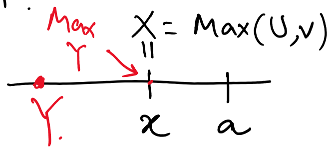
Eg: $U \sim Exp(1)$ } Independent.

Eg: $U \sim Exp(1)$ } Independent.
$V \sim Exp(2)$

$X = Max.\{U, V\}$

$Y = V$

$X > 0$
$Y > 0$, $X \geq Y$.



So, Y always ≤ a.

<u>to find:</u> $P(Y \leq a \mid X = x)$

Fix $x > 0$. $P(Y \leq a \mid X = x) = 1$ if $a \geq x$

for $a < x$, take $\varepsilon > 0$ st.

$P(Y \leq a \mid X \in (x-\varepsilon, x+\varepsilon))$

$= P(Y \leq a, X \in (x-\varepsilon, x+\varepsilon))$

$= V \leq a, U \in (x-\varepsilon, x+\varepsilon)$

$$\therefore f(a|x) = \begin{cases} \text{density}, & a < x \\ 1, & a \geq x. \end{cases}$$