1. If $\mathbf{A}$ is of order 7, find the signs of the terms: $a_{13}a_{26}a_{32}a_{45}a_{54}a_{67}a_{71}$, $a_{41}a_{52}a_{73}a_{34}a_{15}a_{66}a_{27}$ and $a_{17}a_{24}a_{35}a_{41}a_{53}a_{66}a_{72}$ in the expansion of $|\mathbf{A}|$.
2. What is $|\alpha\mathbf{A}|$ if $\mathbf{A}$ is of order n?
3. Show that the determinant of a real skew-symmetric matrix of odd order is 0.
6. Prove that

$$\begin{vmatrix} 1+a_1 & a_2 & \cdots & a_n \\ a_1 & 1+a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & 1+a_n \end{vmatrix} = 1 + a_1 + \cdots + a_n$$

11. Let $\mathbf{A}$ be an $n \times n$ matrix whose elements are (differentiable) functions of a real variable x. Then show that

$$\frac{d}{dx}|\mathbf{A}| = \sum_{k=1}^{n} |\mathbf{A}_k|$$

where $\mathbf{A}_k$ is the matrix obtained from $\mathbf{A}$ by differentiating the elements of the k-th row with respect to x.

12. Let f be a map from $F^n \times \cdots \times F^n$($n$ copies) to F. Consider the following conditions:
    (i) $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = 0$ whenever two of the $\mathbf{x}_i$'s are equal. (Such an f is said to be *alternating*.)
    (ii) $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is linear in each $\mathbf{x}_i$ when all the other $\mathbf{x}_j$'s are kept fixed. (Such an f is said to be *multilinear*.)
    (iii) $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is multiplied by α if any $\mathbf{x}_i$ is multiplied by α.
    (iv) $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is not altered if a scalar multiple of $\mathbf{x}_j$ is added to $\mathbf{x}_i$, $i \neq j$.
    (v) $f(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) = 1$.

    (a) If f satisfies (i) and (ii), show that $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = c|\mathbf{A}|$ for some constant $c \in F$, where $\mathbf{A} = [\mathbf{x}_1 : \cdots : \mathbf{x}_n]^{\mathrm{T}}$. (Hint: Use *Exercise* 6.3.10.) If, moreover, f satisfies (v), show that $f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = |\mathbf{A}|$.
    (b) Do (a) when f satisfies (iii) and (iv) instead of (i) and (ii).

Thus determinant can be defined as a function of the rows (or columns) satisfying either '(i), (ii) and (v)' or '(iii), (iv) and (v)'.

prove that the area of the triangle formed by the prove that the area of the triangle formed by the three points $A = (x_1, y_1), P = (x_2, y_2)$ and $Q = (x_3, y_3)$ in $\mathbb{R}^2$ is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Deduce that A, P and Q are collinear iff the determinant is 0.

3. (a) Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two distinct points in $\mathbb{R}^2$. Show that the equation of the line passing through P and Q is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

by showing that this equation represents a line, i.e., the coefficients of x and y are not both zero, and that it contains P and Q.

(b) State and prove an analogous result for the plane containing three non-collinear points in $\mathbb{R}^3$.

(c) Let $P_i = (u_i, v_i, w_i)$, $i = 1, 2, 3$, be points in $\mathbb{R}^3$ such that O, P_1, P_2 are not collinear. Show that the equation of the plane containing P_3 and parallel to the plane OP_1P_2 is

$$\begin{vmatrix} x - u_3 & y - v_3 & z - w_3 \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = 0$$

4. If $\mathbf{A}$ is the $n \times n$ matrix with $a_{ij} = i + j - 2$ for all i and j, show that $|\mathbf{A}| = 0$ whenever $n \geq 4$.

9. If $A_{ij} \neq 0$, show that by changing the single element a_{ij}, we can change $|\mathbf{A}|$ to any specified scalar.

11. Prove that if $x_1, x_2, \ldots, x_n$ are distinct real numbers and $y_1, y_2, \ldots, y_n$ are arbitrary real numbers then there exists a unique polynomial $p(t) \in \mathcal{P}_n$ such that $p(x_i) = y_i$ for $i = 1, \ldots, n$.

8. Show that the linear operator $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ on $\mathbb{R}^2$ takes a parallelogram in $\mathbb{R}^2$ to another in $\mathbb{R}^2$ whose area is $|\mathbf{A}|$ times the area of the former.

7. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ be $m \times m$ matrices.

   (a) If $\mathbf{A}$ is non-singular and $\mathbf{A}$ commutes with $\mathbf{C}$, show that

   $$\begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = |\mathbf{AD} - \mathbf{CB}|$$

   (b) Find similar expressions for the determinant on the left hand side in each of the cases: (i) $\mathbf{D}$ is non-singular and $\mathbf{D}$ commutes with $\mathbf{B}$ and (ii) $\mathbf{A}$ is non-singular and $\mathbf{A}$ commutes with $\mathbf{B}$.

   (c) Show that the result in (a) is false even if $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are non-singular and $m = 2$ if $\mathbf{A}$ and $\mathbf{C}$ do not commute.