Consistency of normal equations

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1 Introduction

We know the normal equations is given by:
$X\vec{\beta} \approx \vec{y}$, where this system could be inconsistent.
We want to solve this in least square sense.
This can also be written as:

$$\vec{y} = X\vec{\beta} + \vec{\epsilon}, where\ \epsilon \sim (\vec{0}, \sigma^2) \tag{1}$$

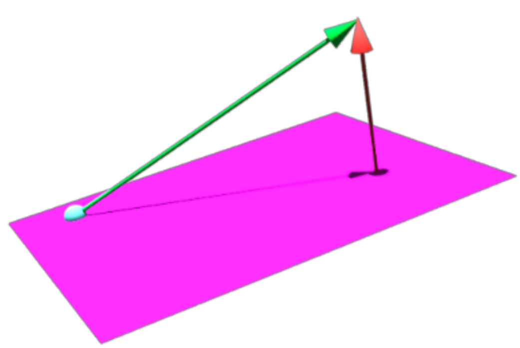
The Normal equations states that:

$$X'X\vec{\beta} = X'\vec{y} \tag{2}$$

The question remains:
Why should $Eqn1 = Eqn2$?
Proof.
We have,

$$X = \begin{pmatrix} X_1 & X_2 & ... & X_P, \end{pmatrix}$$

where X_i is a nx1 vector $\forall\ i = 1, 2, ..., p$.



$X_i \in \mathcal{C}(X)$ and $(\vec{y} - X\vec{\beta}) \perp \mathcal{C}(X)$
$\implies X_i'(\vec{y} - X\vec{\beta}) = \vec{0} \;\forall\ i = 1, 2, ..., p,\ \forall\ \vec{\beta}$
$\implies X_i'\vec{y} = X_i'X\vec{\beta} \;\forall\ i = 1, 2, ..., p$
$\implies [X_1'\ X_2'\ ...\ X_p']\ \vec{y} = [X_1'\ X_2'\ ...\ X_p']\ X\vec{\beta}$

$\Longrightarrow\ X'\vec{y} = X'X\vec{\beta}$

Can we always solve the *normal* Eq^n.1?
i.e., is the *normal* Eq^n.1 *consistent* ?
To show this, we know,
$X'\vec{y} = X'X\vec{\beta}\ \ \forall\ \vec{\beta}$
$<=> X'\vec{y} \in \mathcal{C}(X'X)\ \ \forall\ \vec{y}$
$<=> \mathcal{C}(X') \subseteq \mathcal{C}(X'X)$
However, we know that,
$\mathcal{C}(X'X) \subseteq \mathcal{C}(X')\ \ \mathcal{R}(X') = \mathcal{R}(X'X)$
Therefore, $\mathcal{C}(X'X) = \mathcal{C}(X')$
In particular, $\mathcal{C}(X') \subseteq \mathcal{C}(X'X)$

Hence,the *normal* Eq^n.1 is always *consistent*.