Deterministic Finite Automata

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Indian Statistical Institute

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Don't worry...

...this is not a G-pay QR code:

...just a link to a Youtube playlist containing this lecture.

Deterministic Finite Automaton (DFA)

= Finite Automaton

= Finite state machine

= State machine

= Internal state machine

Application areas

Computer science:

- **Robotics**
- **Digital electronics**
- **Compilers for languages like C, Java etc**
- **Embedded systems (*e.g.*, IoT)**

Mathematics:

1. **Algebraic structures**
2. **Combinatorics**

Humptydumpty

7+3=10

Definition

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Three sets:

- A = **set of inputs** = $\{$**Smile, Punch**$\}$
- B = **set of outputs** = $\{$**Normal, Smile back, Raise eyebrow, Scowl, Growl, Grimace**$\}$
- S = **set of states** = $\{$**Ordinary, Happy, Surprised, Angry, Furious**$\}$

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Two functions:

- $f : A \times S \to B$, **the output function.**
- $g : A \times S \to S$, **the next state function.**

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One state:

- $s_0 \in S$: **the initial state = Ordinary.**

10+3=13

Transition diagram

Why care?

Why care?

Any machine with a memory must be a state machine!

C comments


```
void main() {
  printf("Hello world!"); /* Correct this.*/
}
```

16+1=17

Look for /* ... */

16+1=17

Look for /* ... */

Look for /* ... */

Look for /* ... */

x=4

Look for /* ... */

x=4/

Look for /* ... */

x=4/5

Look for /* ... */

x=4/5+

Look for /* ... */

x=4/5+3

16+1=17

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```
x=4/5+3/*
```

Look for /* ... */

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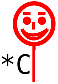

Look for /* ... */

```
x=4/5+3/*Ch
```

Look for /* ... */


```
x=4/5+3/*Cha
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```
x=4/5+3/*Change / to *.*/;
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17+3=20

Inside our brain

Looking for start of comment.

17+3=20

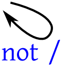
Inside our brain

Looking for start of comment.

not /

Inside our brain

Looking for start of comment.



17+3=20

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Inside our brain

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Looking for start of comment.

/

not /

Possible start.

Inside our brain

x=4/5

Looking for start of comment.

not *

/

not /

Possible start.

Inside our brain

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x=4/5+3

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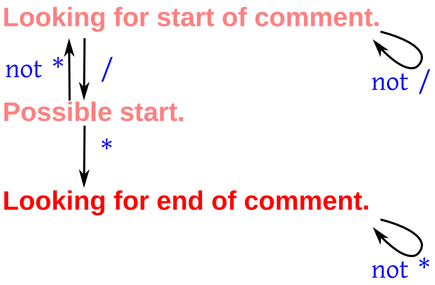

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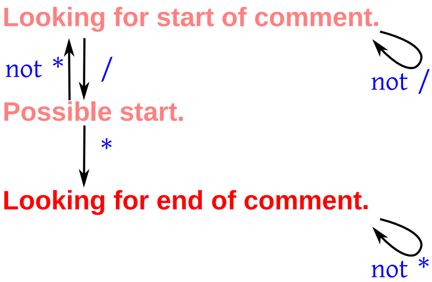

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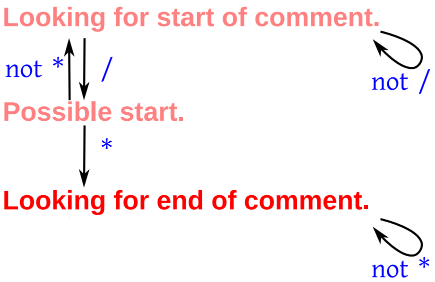

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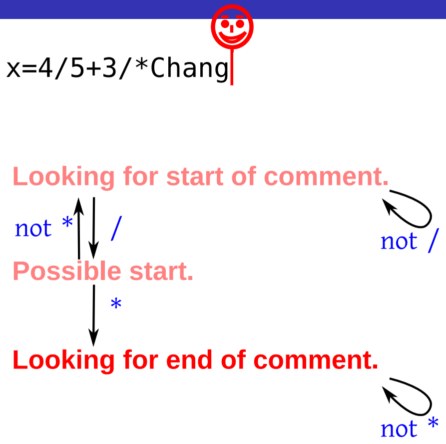

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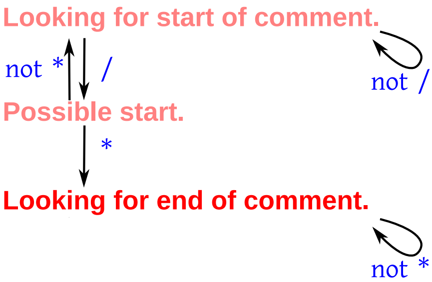

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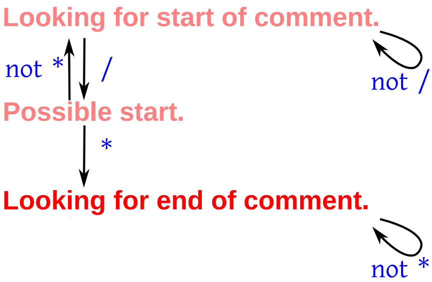

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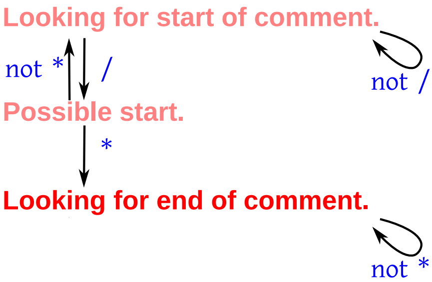

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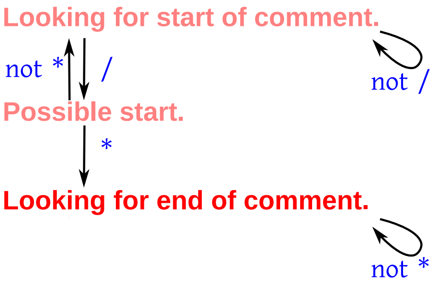

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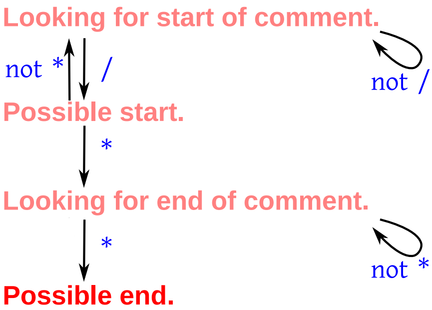

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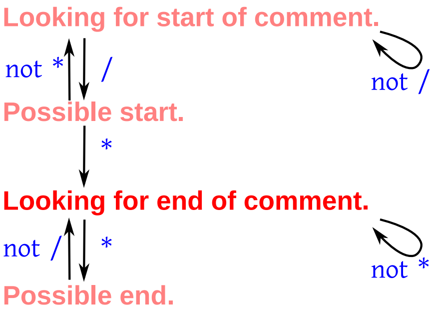

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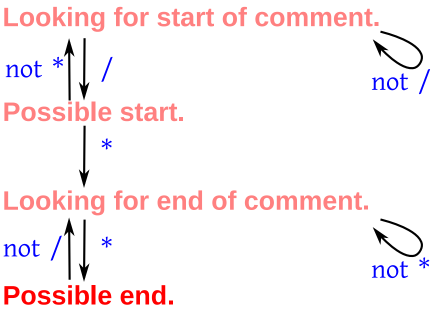

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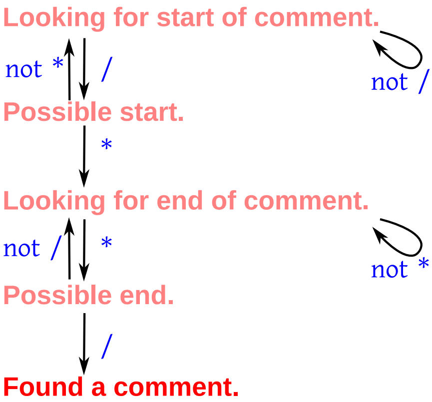

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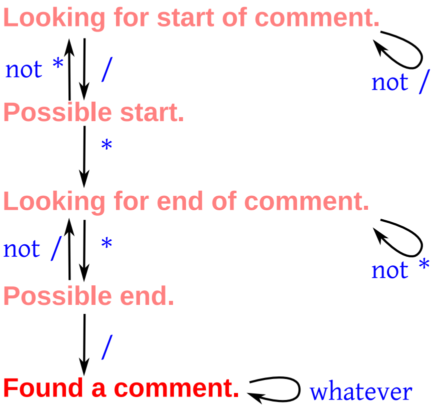

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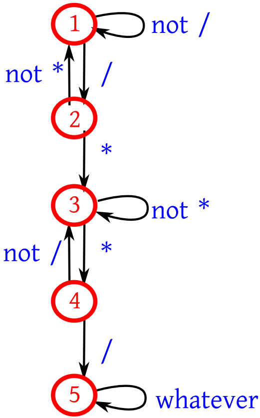

$20+2=22$

Demo

English to Bengali transliteration

Application in hardware

Application in hardware

Arduino

24+2=26

Application in hardware

Arduino

Raspberry Pi

Coin dropper example

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Mathematically...

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$(\{0, 1, 2, ...\}, +)$.

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Example:

$(\{0, 1, 2, ...\}, +)$. **Identity = 0.**

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All finite length words using the letters a, b, c under the concatenation operator: $"abc" \bullet "ab" = "abcab"$.

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The monoid is always finite.

From monoid to finite state machine

Start with a finite monoid, $(M, +)$. Then we can define a FSM with state set and input set both equal to M. For any state $s \in M$ and any input $i \in M$ we define the next state as $m + i$.

Some interesting questions

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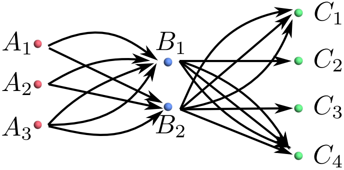
- **FSM to monoid to FSM.**
- **Size of the monoid for a given FSM.**
- **Same monoid $\implies$ same transition diagram?**

Transfer matrix method



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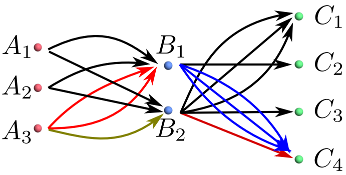
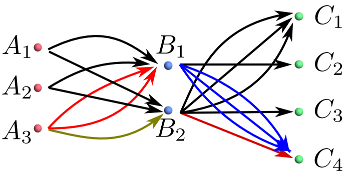


$$2 \times 3 + 1 \times 1 = 7.$$

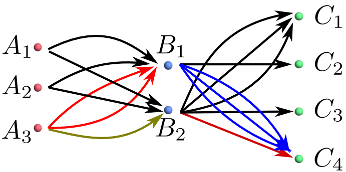
Transfer matrix method



$$\color{red}{2} \times \color{blue}{3} + \color{green}{1} \times \color{purple}{1} = 7.$$

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \mathbf{2} & \mathbf{1} \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ B_1 \\ B_2 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & \mathbf{3} \\ 3 & 0 & 1 & \mathbf{1} \end{bmatrix} \end{array}$$

Transfer matrix method



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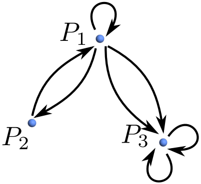
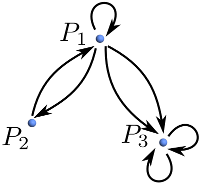
Same set

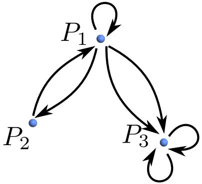


Same set



$$\begin{array}{c c} & \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array} & \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{array}$$

Same set



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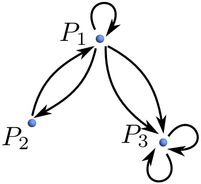

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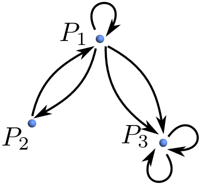
$$A^2 = \begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

Same set



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$$A^2 = \begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

The (i, j)-th entry of A^k gives the number of paths of length k from P_i to P_j.

Example: a counting problem

Want to count the number (b_n) of sequences with , and . Total number of squares is n. Must have two consecutive 's somewhere before three consecutive 's.

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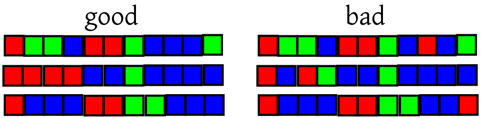
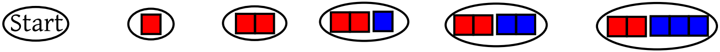
Want to count the number (b_n) of sequences with 🟥, 🟩 and 🟦. Total number of squares is n. Must have two consecutive 🟥's somewhere before three consecutive 🟦's.



Solution



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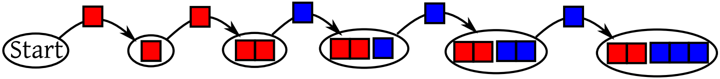

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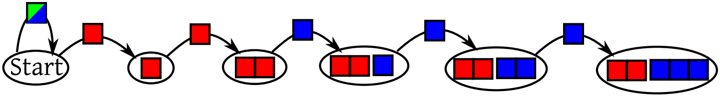

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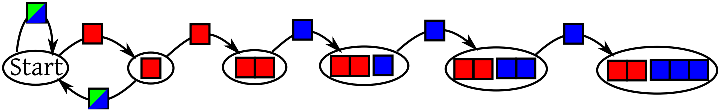

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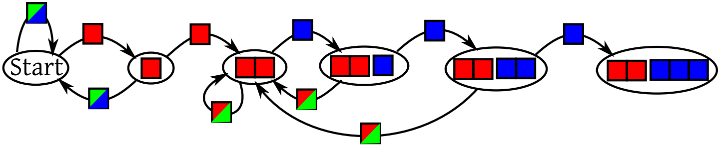

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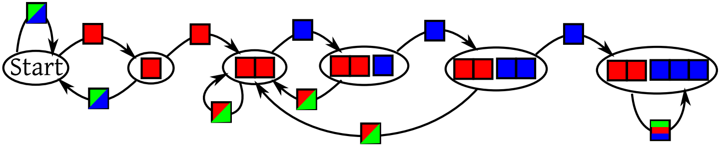

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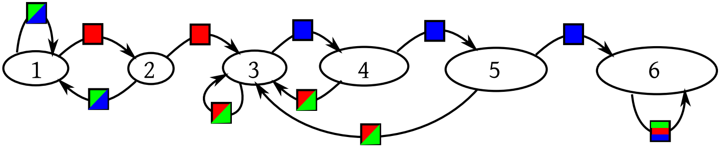

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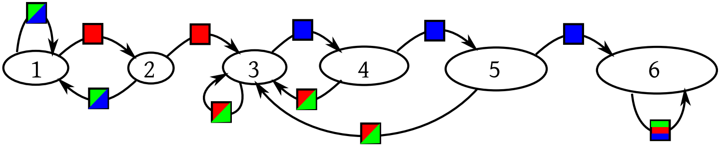
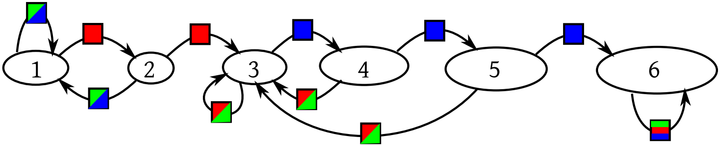


$$A = \begin{array}{c} \text{From}\backslash\text{To} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \end{array}$$

Solution



$$A = \begin{array}{c|c} {}_{\text{From}}\backslash^{\text{To}} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \end{array}$$

The $(1,6)$-th entry of A^n gives the number of such patterns of length n, *i.e.* b_n.

Example

`http://maxima.cesga.es`

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```
A: matrix([2, 1, 0, 0, 0, 0],
          [2, 0, 1, 0, 0, 0],
          [0, 0, 2, 1, 0, 0],
          [0, 0, 2, 0, 1, 0],
          [0, 0, 2, 0, 0, 1],
          [0, 0, 0, 0, 0, 3]);
```

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A: matrix([2, 1, 0, 0, 0, 0],
          [2, 0, 1, 0, 0, 0],
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          [0, 0, 2, 0, 1, 0],
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          [0, 0, 0, 0, 0, 3]);

A^^45;
```

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          [2, 0, 1, 0, 0, 0],
          [0, 0, 2, 1, 0, 0],
          [0, 0, 2, 0, 1, 0],
          [0, 0, 2, 0, 0, 1],
          [0, 0, 0, 0, 0, 3]);

A^^45;
```

$$\begin{bmatrix} 3458540538388755 2512 & 12659136970685939712 & 86357776463132858 7776 & 29467125259992591 5648 & 10052059169947344 8960 & 16482985552655322 54035 \\ 25318273941371879424 & 9267131442515673088 & 80304282556948466 8928 & 27423525943147675 6480 & 9363006920097901 7728 & 17488191469650057 02995 \\ 0 & 0 & 62392791058373938 3808 & 21370032036963283 7632 & 7319407603252985 8560 & 20434903995649316 18643 \\ 0 & 0 & 57378879280432539 2384 & 19652726984447370 8544 & 6731216830457312 0512 & 21166844755974614 77203 \\ 0 & 0 & 42740064073926567 5264 & 14638815206505971 7120 & 5013911777941399 1424 & 23303847959670943 14835 \\ 0 & 0 & 0 & 0 & 0 & 29543127065508336 98643 \end{bmatrix}$$

16482985552655322 54035

Analytically...

The generating function of $\{b_n\}_n$ is the formal power series

$$b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots .$$

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Just like

$$1 + x + x^2 + \cdots = \frac{1}{1-x} \textbf{ if } |x| < 1.$$

Example


```
(1-t*A);
```

$$\begin{bmatrix} 1-2\,t & 1-t & 1 & 1 & 1 & 1 \\ 1-2\,t & 1 & 1-t & 1 & 1 & 1 \\ 1 & 1 & 1-2\,t & 1-t & 1 & 1 \\ 1 & 1 & 1-2\,t & 1 & 1-t & 1 \\ 1 & 1 & 1-2\,t & 1 & 1 & 1-t \\ 1 & 1 & 1 & 1 & 1 & 1-3\,t \end{bmatrix}$$

Inverse

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((I-t*A)^^-1)[1,6];
```

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$$\frac{t^5}{-12\,t^6 - 20\,t^5 - 10\,t^4 + 6\,t^3 + 12\,t^2 - 7\,t + 1}$$

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From this we get

$$b_{n+5} = 7b_{n+4} - 12b_{n+3} - 6b_{n+2} + 10b_{n+1} + 20b_n + 12b_{n-1}$$

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How?

Reference