Probability-3 Lecture-20

22 October 2024 14:22

Result: φ is a characteristic function which is integrable on $\mathbb{R}$.

Then, the underlying distribution function is continuously differentiable on $\mathbb{R}$, & has a continuous density given by–

$$f(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} e^{-itx} \cdot \varphi(t)\, dt$$

"Inverse-Fourier Transform".

⚠ Converse NOT true!!!

ie, a distribution f^n with continuously differentiable density need not have a integrable characteristic f^n.

Result: Suppose X is a r.v. taking only integer values,

then, the characteristic f^n φ_X would be periodic with period $= 2\pi$, & for $k \in \mathbb{Z}$,

$$P(X=k) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} e^{-itk} \varphi_X(t)\, dt.$$

[Proof: Exercise.]

Here, $\varphi_X(t) = E(e^{itX})$

$$= \sum_{m \in \mathbb{Z}} e^{itm} \cdot P(X=m).$$

Moments & Characteristic Functions:

Let X – a real r.v and φ – its characteristic f^n.

Result 1: If $E|X| < \infty$, then φ is continuously differentiable, and $\varphi'(t) = E(iXe^{itX})$, $\forall t \in \mathbb{R}$.

Proof:

Now, $\left|\frac{e^{i\theta}-1}{i\theta}\right| \leq 1$, and $\frac{e^{i\theta}-1}{i\theta} \longrightarrow 1$ as $\theta \longrightarrow 0$.

$$\frac{e^{i\theta}-1}{i\theta} = \underbrace{\frac{\cos\theta - 1}{i\theta}}_{\to 0} + \underbrace{\cancel{i}\frac{\sin\theta}{\cancel{i}\theta}}_{\to 1}$$

as $\theta \longrightarrow 0$.

$$\therefore \frac{\varphi(t+h) - \varphi(t)}{h}$$

$$= E\left(\frac{e^{itX}(e^{ihX}-1)}{h}\right)$$

$$= E\left(\underbrace{e^{itX}}_{\leq 1} \underbrace{\frac{(e^{ihX}-1)}{ihX}}_{\leq 1} \cdot \underbrace{iX}_{\leq |X|}\right), \ \& \ E|X| < \infty$$

$\therefore$ By DCT $\xrightarrow{\text{this}} E\left(e^{itX} \cdot iX\right)$ $\left[\because \frac{e^{ihx}-1}{ihx} \to 1 \text{ as } h \to 0\right]$

$\therefore \varphi'(t) = E\left(e^{itX} \cdot iX\right)$. ▨

Result - 2:

Suppose $E(X^2) < \infty$.

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Then, $\varphi'(t)$ is continuously differentiable

& , $\varphi''(t) = -E(X^2 \cdot e^{itX})$.

Proof: $$\frac{\varphi'(t+h) - \varphi'(t)}{h} = E\left(iX \cdot e^{itX} \frac{(e^{ihX} - 1)}{ihX} \cdot iX \right)$$

$$= E\left(-X^2 \cdot e^{itX} \cdot \underbrace{\left(\frac{e^{ihX} - 1}{ihX} \right)}_{\downarrow \; 1} \right)$$

Again, DCT : $\downarrow$

$$E\left(-X^2 e^{itX}\right)$$

$$\therefore \varphi''(t) = -E(X^2 e^{itX})$$

$\therefore$ Result 'k'th :

If $E(|X|^k) < \infty$,

then φ is k-times continuously differentiable, and

$$\varphi^{(j)}(t) = E\left(i^j \cdot X^j \cdot e^{itX}\right)$$

$$\forall \; j = 1, 2, \ldots, k$$

In particular, if $E(X^k) < \infty$ for some $k \geq 1$.

then,

$$\varphi(t) = 1 + \sum^{k} \frac{i^j}{j!} \cdot E(X^j) + o(t^k)$$

$$\varphi(t) = 1 + \sum_{j=1}^{n} \frac{i^j}{\lfloor j} \cdot E(X^j) + o(t^k)$$

$$\frac{o(t^k)}{t^k} \to 0 \text{ as } t \downarrow 0$$

(finite Taylor Expansion)

"The ultimate aim is to do well in exams. Don't deny that!" – Prof. AG, 22nd Oct.

$X_{n}, n \geq 1$, X – r.vs.

$\varphi_{n}, n \geq 1$ φ – c.Fs.

Levy's Continuity Theorem:

$$X_n \xrightarrow{d} X \quad \text{iff}$$

$$\varphi_n \longrightarrow \varphi \text{ pointwise.}$$

Proof:

"only if": $X_n \xrightarrow{d} X$.

$$\Rightarrow E(\cos tX_n) \longrightarrow E(\cos tX)$$

$$\&, \ E(\sin tX_n) \longrightarrow E(\sin tX). \quad \forall t \in \mathbb{R}.$$

$$\therefore E(\cos tX_n + i \sin tX_n) \longrightarrow E(\cos tX + i \sin tX).$$

$$\therefore \text{Clearly, } \varphi_n(t) \longrightarrow \varphi(t) \quad \forall t \in \mathbb{R}.$$

Levy's Theorem.

Suppose, $\varphi_n, n \geq 1$ – sequence of c.Fs.

Suppose, $\varphi_n \longrightarrow$ some function g pointwise uppose, $\varphi_n \longrightarrow$ some function g pointwise

If g is continuous at 0, then g is a characteristic f^n and $F_n \xrightarrow{d} F$.

["if" part will follow from here.].

Helly's Selection Theorem:

Given any sequence $\{F_n\}$ of probability distribution f^ns, there is a subsequence $\{F_{n_k}\}$ and a non-decreasing right continuous f^n $G: \mathbb{R} \to [0,1]$ s.t.

$$F_{n_k}(a) \longrightarrow G(a) \quad \forall a \in C(G)$$

set of all cont. pts of G.

* A subset $A \subseteq \mathbb{R}$ is called "Relatively Compact" if A has a compact closure. [characterized by: every seq. has a subseq. which converges, (not necessarily within A.)]

Eg. $F_n =$ pdf of $X_n = \pm n$, each with prob $= 1/2$.

verify that: we can never get a "G" s.t. G is a cdf.

Eg. $F_n =$ pdf of $\text{Unif}(-n, n)$.

Again, verify this

[Intuition: as $n \nearrow \infty$, the probability mass is

[Intuition: as $n \nearrow \infty$, the probability mass is "dissipated" over ∞.]

* In addition, if $\{F_n\}$ is "tight", i.e,

$\forall \varepsilon > 0, \exists K_\varepsilon > 0$ s.t.

$$F_n(K_\varepsilon) - F_n(K_\varepsilon^-) > 1 - \varepsilon \quad \forall n.$$

then, the limit f^n G is indeed a probability dist^n f^n.

[i.e, the compact set $[-K_\varepsilon, K_\varepsilon]$ covers more than $1-\varepsilon$ mass.
i.e, this prevents the "escaping" of probability mass.]

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Let $r_1, r_2, \ldots$ be an enumeration of $\mathbb{Q}$.

$\{F_n(r_1)\}$ – seq. of cdfs.

$\therefore$ By <u>Bolzano-Weierstrass</u> thm–

$\exists$ subsequence $n(1,j)$ s.t.

$$F_{n(1,j)}(r_1) \longrightarrow l_1 .$$

Now, take $\{F_{n(1,j)}(r_2)\}$ - new seq.

Again, by Bolzano-Weierstrass thm,

$\Rightarrow \exists$ subseq. $n(2,j)$

s.t.

$$F_{n(2,j)}(r_i) \longrightarrow l_i, \quad i = 1, 2$$

[$\longrightarrow l_1$ as, any further subsequence of a convergent subsequence is also convergent,

& $\longrightarrow l_2$, by Bolzano-Weierstrass (applied freshly)]

Proceeding this way, and then looking at the diagonal subsequence will give us $\{F_{n_k}\}$

$$F_{n_k}(r_i) \longrightarrow l_i \quad \forall i$$

as $k \longrightarrow \infty$

Define $H : \mathbb{Q} \longrightarrow [0,1]$.

$$H(r_i) = l_i$$

H - non decreasing.

why? say, $r, s \in \mathbb{Q}$, $r < s$.

$$F_{n_k}(r) < F_{n_k}(s)$$

$$\downarrow \qquad\qquad \downarrow$$

$$H(r) \qquad H(s).$$

Define $G : \mathbb{R} \longrightarrow [0,1]$ by $G(x) = \inf.\left\{H(r) : \begin{array}{l} r > x \\ r \in \mathbb{Q} \end{array}\right\}$

Now, <u>show that</u>. if 'a' is a continuity point of G, then $F_{n_k}(a) \longrightarrow G(a)$

- -

∴ Levy's real thm:

φ_n's - characteristic f^{ns}.

$\varphi_n \longrightarrow \varphi$ pointwise, & $\boxed{\varphi \text{ is continuous at } 0}$

g is continuous at 0

Then, g is a C.F. & $F_n \xrightarrow{d.} F$

Sketch of proof:

$\{F_n\}$ - seq. of cdfs corresponding to $\{\varphi_n\}_{n\geq 1}$.

$\therefore$ By Helly, $\exists$ a subsequence n_k & a sub p.d.f $G : \mathbb{R} \to [0,1]$

s.t. $F_{n_k}(a) \longrightarrow G(a) \ \forall a \in C(G)$.

Now, $\varphi_n \longrightarrow g$ pointwise, & g is cont. at 0 $\Big\} \Rightarrow \{F_n\}$ is tight

[show: by "delicate" integral estimates]

Now, as $\{F_n\}$ - tight

$\Rightarrow G$ is a cdf.

(Helly, additional part.)

$\therefore$ subseq. $F_{n_k}(a) \longrightarrow G(a)$

$\therefore \quad \varphi_{n_k} \longrightarrow \varphi_G$ ptwise.

But, $\varphi_n \longrightarrow g$ ptwise

$\Bigg\} \Rightarrow \boxed{\varphi_G = g}$

We can apply Helly's further on sub-subsequences.

CENTRAL LIMIT THEOREM (C.L.T)

$\{X_n\}$ - iid with mean μ, variance σ^2.

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$$\Rightarrow \frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1), \quad S_n = X_1 + \cdots + X_n$$

Proof: Because of Levy's continuity theorem, its enough to show:

$$\varphi_{\frac{S_n - n\mu}{\sigma\sqrt{n}}}(t) \longrightarrow e^{-t^2/2}$$

Can't compute this!!

So, Denote $Y_k := \frac{X_k - \mu}{\sigma}$ $\left(\equiv \text{"}Y\text{"} \right)$

$\therefore Y_k$ - iid with mean 0, variance 1.

$$\therefore \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{T_n}{\sqrt{n}}, \quad \text{where } T_n = Y_1 + \cdots + Y_n.$$

$\therefore$ to show: $\varphi_{T_n/\sqrt{n}}(t) \longrightarrow e^{-t^2/2}$

Now, $\varphi_{T_n/\sqrt{n}}(t) = \varphi_{T_n}(t/\sqrt{n})$

$$= \left(\varphi_Y(t/\sqrt{n})\right)^n$$

Now, $\varphi_Y = 1 - \frac{h^2}{2} + o(h^2)$ $\left[\because \begin{array}{l} E(Y) = 0 \\ E(Y^2) = 1 \end{array} \right]$

[Finite Taylor Expansion]

$$\therefore \left(\varphi_Y(t/\sqrt{n})\right)^n = \left(1 - t^2/_{\ldots} + R_n\right)^n,$$

$$\therefore \Big(\underbrace{\varphi_Y(t/\sqrt{n})}_{\overset{\shortparallel}{a_n\,(\text{say})}}\Big)^n = \Big(1 - t^2/2n + R_n\Big)^n,$$

where $nR_n(t) \longrightarrow 0$ for $t \neq 0$ [fixed]

$$\therefore \lim_{n\to\infty} a_n \overset{?}{=} \lim_{n\to\infty}\left(1 - t^2/2n\right)^n$$

$$= e^{-t^2/2}. \quad \text{(done!!)}$$

Left to show: this

Lemma: for complex numbers $z_1, z_2, \ldots, z_n$ & $w_1, w_2, \ldots, w_n$, all with modulus ≤ 1,

$$\left|z_1 \cdot z_2 \cdots z_n - w_1 \cdot w_2 \cdots w_n\right| \leq \sum_{k=1}^{n} |z_k - w_k|$$

$\therefore$ By Induction,

$$|z_1 \cdots z_n - w_1 \cdots w_n|$$

$$\leq |z_1 \cdots z_{n-1} \cdot z_n - z_1 \cdots z_{n-1} \cdot w_n| + |z_1 \cdots z_{n-1} \cdot w_n - w_1 \cdots w_{n-1} \cdot w_n|$$

$$\leq |z_n - w_n| + |z_1 \cdots z_{n-1} - w_1 \cdots w_{n-1}|$$

$[\because |w_n| < 1]$

$$\therefore \left|\left(1 - t^2/2n + R_n\right)^n - \left(1 - t^2/2n\right)^n\right|$$

$$\leq n \cdot |R_n| \longrightarrow 0. \quad \left[\because R_n = o\left(\tfrac{1}{n}\right)\right]$$

Hence, done!! $\boxtimes$

Hence, done !!

Course ends here!!

Respect to Prof. AG!