$\langle X, X \rangle \geq 0$

& $\langle X, X \rangle = 0 \Leftrightarrow P(X=0)=1$.

11/01/23

- V non-null vectorspace with a norm $\|\cdot\|$

Define $d: V \times V \longrightarrow [0, \infty)$

by $d(x,y) = \begin{cases} \|x\| + \|y\| & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

Ex. show d is a metric on V. s.t. $\nexists$ a norm on V that induces d.

- Recall X, Y are independent if

$$P(X=x, Y=y) = P(X=x)\,P(Y=y)$$

for all ~~$x \in X$ & $y \in Y$~~ $x \in D_X$ & $y \in D_Y$.

(Equivalently,

$$P(X \in B_1, Y \in B_2) = P(X \in B_1)\,P(Y \in B_2)$$

for any $B_1, B_2 \in \mathbb{R}$)

- Defn. $X_1, \ldots, X_n$ are mutually independent

if $P(X_1 = x_1, \ldots, X_n = x_n)$

$= P(X_1 = x_1) \ldots P(X_n = x_n)$

$\forall\, x_1, x_2, \ldots, x_n$.

This is equiv. to $P(X_1 \in B_1, \ldots, X_n \in B_n)$

$= P(X_1 \in B_1) \ldots P(X_n \in B_n)$

for any choice of y_j

$$1 \leq t_1 < \cdots < t_K \leq n.$$

$$P(X_{t_1} = y_1, \ldots, X_{t_K} = y_K)$$

$$= P(X_{t_1} = y_1) \cdots P(X_{t_K} = y_K).$$

[Note; If we take $B_1 = \{x_1\}$, $B_2 = \{x_2\}$ & all other $B_i = \mathbb{R}$ then we are done!]

Defn. $X_1, \ldots, X_n$ are pairwise indep. if X_i, X_j are indep. for all $i \neq j$

[This is a much weaker condn.

Ex. show 3 r.v.s which are pairwise indep. but not indep.]

Defn. A family $\{X_\alpha : \alpha \in A\}$ of discrete r.v.s are said to be independent if every finite subcollection is indep.

Examples X_1, X_2, X_3 indep.

Claim: For any $B \in \mathbb{R}^2$, $C \in \mathbb{R}$

$$P((X_1, X_2) \in B, X_3 \in C)$$

$$= P((X_1, X_2) \in B)\, P(X_3 \in C)$$

$$\text{LHS} = \sum_{\substack{x_1, x_2, x_3 \\ (x_1, x_2) \in B \\ x_3 \in C}} P(X_1 = x_1, X_2 = x_2, X_3 = x_3)$$

$$= \sum P(X_1 = x_1)\, P(X_2 = x_2)\, P(X_3 = x_3)$$

$$= \sum P(X_1 = x_1, X_2 = x_2)\, P(X_3 = x_3)$$

$$= \sum P(X_1 = x_1, X_2 = x_2) \sum P(X_3 = x_3)$$

$$= P((X_1, X_2) \in B)\, P(X_3 \in C)$$

- $V(X_1 + \cdots + X_n)$

$$= \sum_{1}^{n} V(X_i) + 2 \sum_{i \neq j} cov(X_i, X_j)$$

<u>Pf</u> $\quad LHS = cov(\sum X_i, \sum X_i)$

$$= \cdots$$

<u>*</u> $V(X_1 + \cdots + X_n) = V(X_1) + \cdots + V(X_n)$

if $X_1, \ldots, X_n$ are pairwise independent.

<u>Ex.</u> Each purchase of a product gives you one of r different type of toys

X = # diff. types of toys after n choices

Y = # purchases needed to get all r types.

① $X = I_1 + \cdots + I_r$.

where $I_j = \begin{cases} 1 & \text{if } j\text{-th type is obtained} \\ 0 & \text{if not.} \end{cases}$

$$E(I_j) = P(j\text{-th type is there})$$

$$= 1 - \left(\frac{r-1}{r}\right)^n.$$

$$\therefore E(X) = r\left(1 - \left(\frac{r-1}{r}\right)^n\right)$$

$$V(X) = \sum_{j=1}^{r} V(I_j) + \sum_{j<k} \operatorname{cov}(I_j, I_k)$$

$$V(I_j) = E(I_j) - [E(I_j)]^2$$

$$= E(I_j)(1 - E(I_j))$$

$$= \left(\frac{r-1}{r}\right)^n \left[1 - \left(\frac{r-1}{r}\right)^n\right]$$

$$\operatorname{cov}(I_j, I_k) = E(I_j I_k) - E(I_j)E(I_k)$$

$$= \underbrace{1 - 2\left(\frac{r-1}{r}\right)^n + \left(\frac{r-2}{r}\right)^n}_{\text{from inclusion-exclusion}}$$

$$- \left(1 - \left(\frac{r-1}{r}\right)^n\right)^2$$

$$= \underline{\left(\frac{r-2}{r}\right)^n - \left(\frac{r-1}{r}\right)^n}.$$

So, finally $V(X) = \cdots$

$Y = Y_1 + Y_2 + Y_3 + \cdots + Y_r$. Clearly $Y_1 \equiv 1$

$Y_2 \sim Geo\left(\frac{r-1}{r}\right)$

$Y_3 \sim Geo\left(\frac{r-2}{r}\right)$

$\vdots$

$Y_j \sim Geo\left(\frac{r-j+1}{r}\right)$

$\vdots$

$Y_r \sim Geo\left(\frac{1}{r}\right)$

$$E(Y) = \sum E(Y_i)$$

$$= 1 + \frac{r}{r-1} + \frac{r}{r-2} + \cdots + \frac{r}{1} \qquad \left[\because E(Y_j) = \frac{r}{r-j+1}\right]$$

$$= r\left(\frac{1}{r} + \frac{1}{r-1} + \cdots + 1\right)$$

$$V(Y) = \sum_{1}^{r} V(Y_j) + 2\sum_{j<k} cov(Y_j, Y_k)$$

$X \sim Geo(p)$

$$E(X(X-1)) = \sum_{k=1}^{\infty} k(k-1)pq^{k-1}$$

$$= pq \sum_{k=2}^{\infty} \underbrace{k(k-1)\, q^{k-2}}_{\frac{d^2}{dq^2}(q^k)}$$

$$= pq \sum \frac{d^2}{dq^2}(q^k)$$

$$= pq \frac{d^2}{dq^2} \sum q^k$$

Easy calculation! $\left\{ = pq\left(\frac{d^2}{dq^2}\left(\frac{1}{1-q}\right)\right) = \frac{2q}{p^2}\right.$

$$Var(X) = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \cdots\cdots$$

$$= \frac{q}{p^2}$$

Now, $q = \frac{j-1}{r}$

$p = \frac{r-j+1}{r}.$

$$\therefore \sum V(Y_j) = \sum_{j=2}^{r} \frac{j-1}{r} \frac{r^2}{(r-j+1)^2}$$

Ex. Now, show that the r.v.'s Y_j's are pairwise indep. so, $cov(Y_j, Y_K) = 0$.

$$\text{so, } V(Y) = \sum_{j=2}^{r} \frac{j-1}{r} \frac{r^2}{(r-j+1)^2}$$

X, Y discrete r.v.s.

$$p(x, y) = P(X = x, Y = y)$$

given $Y = y$,

the conditional dist. of X

$$P(x|y) = P(X = x | Y = y)$$

$$= \frac{P(x, y)}{\sum_x P(x, y)}$$

Ex. $X = \#$ signal transmitted from a transmission ~~on~~ centre.

$X \sim Poi(\lambda)$

The signal transmitted is received by the reception centre with prob. p & not received by prob $q = 1-p$; independently of other signals.

$Y = \#$ signals received.

$D = \{ (i,j) \mid 0 \leq j \leq i \}$

i, j integers.

$$p(i,j) = P(X=i, Y=j)$$

$$= P(Y=j \mid X=i) \underbrace{P(X=i)}_{e^{-\lambda}\frac{\lambda^i}{i!}}$$

$$= \binom{i}{j} p^j q^{i-j} \left(e^{-\lambda}\frac{\lambda^i}{i!}\right)$$

$$P(X=i \mid Y=j)$$

$$= \frac{e^{-\lambda}\frac{\lambda^i}{i!}\binom{i}{j} p^j q^{i-j}}{\sum_{K=1}^{\infty} e^{-\lambda}\frac{\lambda^K}{K!}\binom{K}{j} p^j q^{K-j}}$$

Let, $Z = X - Y = \#$ missed signals.

$$P(Z=K \mid Y=j)$$

$$\cancel{= P(Z=K, Y=j) / P(Y=j)}$$

$$= P(X=K+j, Y=j)$$

$$= e^{-\lambda}\frac{\lambda^{K+j}}{(K+j)!}\binom{K+j}{j} p^j q^K$$

$$= e^{-\lambda} \frac{\lambda^{k+j}}{k!\,j!} p^j q^k$$

$$\cancel{e^{-\lambda p} \frac{(\lambda)}{k!}}$$

$$= \left[e^{-\lambda p} \frac{(\lambda p)^j}{j!} \right] \left[e^{-\lambda q} \frac{(\lambda q)^k}{k!} \right]$$

So, ~~they~~ are independent.

(*) Poisson & Normal distr. can be written as some K numbers of same distribution; i.e. ~~K numbers~~ we can divide it in K poisson or K normal distribution.

Ex Coin with $P(H) = p$.

$X = \#$ tosses position with 7th H

$Y = \#$ " " 10th H.

$$D = \{ (i,j) \mid 7 \le i < i+3 \le j \}$$

$$P(X=i, Y=j) = \binom{i-1}{6} \binom{j-i-1}{2} p^{10} q^{j-10}$$

$$P(X=i \mid Y=j) = \frac{\binom{i-1}{6}\binom{j-i-1}{2} \cancel{p^{10} q^{j-10}}}{\binom{j-1}{9} \cancel{p^{10} q^{j-10}}}$$

$\Downarrow$

(Semi) Hyper. geo. distribution.

~~Tak~~ Toss a coin

$$P(H) = p$$

Times of 1st H, 2nd H & 3rd H.

as a vector in $\mathbb{R}^3$.

Let, $X_1 \rightarrow$ time till 1st H

$X_2 \rightarrow$ " " 2nd H

$X_3 \rightarrow$ " " 3rd H.

Find cond. prob. of (X_1, X_2) given X_3.

More generally $(X_1, X_2, \ldots, X_n)$ give X_{n+1}.

Analysis Weierstrass Approx. Thm.

Let, $f : [0,1] \longrightarrow \mathbb{R}$ be a cont. fn.

Fact 1 : $\exists M > 0$ s.t.

$|f(t)| \leq M \quad \forall t \in [0,1]$.

Fact 2. Given any $\varepsilon > 0$, $\exists \delta > 0$.

Uniform cont.

Fix an $\varepsilon > 0$ for any $n \geq 1$

$$P_n(t) = \sum_{K=0}^{n} \binom{n}{K} f\left(\frac{K}{n}\right) t^K (1-t)^{n-K}$$

$t \in [0,1]$

P_n is a polynomial of degree n.

$$|f(t)-P_n(t)|$$

$$= \left|f(t)-\sum_{K=0}^{n}\binom{n}{K}f\left(\frac{K}{n}\right)t^K(1-t)^{n-K}\right|$$

$$= \left|\sum_{K=0}^{n}\left(f(t)-f\left(\frac{K}{n}\right)\right)\binom{n}{K}t^K(1-t)^{n-K}\right|$$

$$\leqslant \sum_{K=0}^{n}\left|f(t)-f\left(\frac{K}{n}\right)\right|\binom{n}{K}t^K(1-t)^{n-K}$$

$$= \sum_{\substack{0\leqslant K\leqslant n \\ |\frac{K}{n}-t|<\delta}} \cdots + \sum_{\substack{0\leqslant K\leqslant n \\ |\frac{K}{n}-t|\geqslant\delta}} \cdots$$

Now the first sum

$$< \varepsilon \left(\sum_{0\leqslant K\leqslant n}\binom{n}{K}t^K(1-t)^{n-K}\right)$$

this term is bdd. by 1.

$$\text{2nd sum} \leqslant 2M \sum_{K:\, |\frac{K}{n}-t|\geqslant\delta}\binom{n}{K}t^K(1-t)^{n-K}$$

If $X \sim Bin(n,t)$ then,

$$\text{2nd sum} \leqslant 2M \cdot P\left(\left|\frac{X}{n}-t\right|\geqslant\delta\right)$$

$$= 2M \cdot P(|X-nt|\geqslant n\delta)$$

[$\because$ for Bin. r.v. $E(X)=nt$]

$$= 2M \cdot P(|X-EX|\geqslant n\delta$$

$$\leq 2M \cdot \frac{V(X)}{n^2\delta^2}$$ → By Chebyshev's inequality

$$\leq 2M \; \frac{nt(1-t)}{n^2\delta^2}.$$

Now $t(1-t) \leq \frac{1}{4}$

$$\leq 2M \cdot \frac{n \cdot \frac{1}{4}}{n^2\delta^2}$$

$$= \frac{M}{2n\delta^2} < \varepsilon$$

~~[Now take n as you wish!]~~

[Now we can fix this δ accordingly]