Probability-2 Lecture-20

26 March 2024 11:19

A random vector (X,Y) is said to be absolutely continuous if $\exists$ a non-ve f^n f on an "open region" I s.t.

$$P((X,Y)\in B) = \iint_B f(x,y)\,dx\,dy \quad \forall \text{ Borel set } B\subset\mathbb{R}^2$$

Clearly, $P((X,Y)\in I)=1$, &

$$F_{X,Y}(a,b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy$$

$$((-\infty,a]\times(-\infty,b]\cap I).$$

f is called the joint density of (X,Y)

(X,Y) has joint density f

$$\left.\begin{array}{l}\Rightarrow X \text{ has "density" } f_X(x) = \int f(x,y)\,dy \\ \& \ Y \text{ has "density" } f_Y(y) = \int f(x,y)\,dx\end{array}\right\}\text{Marginals.}$$

Result: (from 1st Semester)

Suppose X has density f_X on an open interval I. If h is a continuously differentiable function on I onto J, and if h never vanishes on I, then $y=h(X)$ has a density given by

$$f_Y(y) = f_X(g(y))\cdot|g'(y)|, \quad y\in J.$$

where $g\equiv h^{-1}: J\to I$.

Its analogue in 2D:

Let $h: I \longrightarrow J$ be a function on open region I onto open region J

Define random vector –

$$(U, V) = h(X, Y)$$

Denote

$$h(x,y) = \left(h_1(x,y), \, h_2(x,y) \right)$$

where,

$$h_1, \, h_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

<u>Suppose</u>:

① Both h_1 and h_2 continuously differentiable in both variables.

②

$$\det \begin{pmatrix} \dfrac{\partial h_1}{\partial x} & \dfrac{\partial h_1}{\partial y} \\[2ex] \dfrac{\partial h_2}{\partial x} & \dfrac{\partial h_2}{\partial y} \end{pmatrix} \neq 0 \quad \text{everywhere on } I.$$

Jacobian of the map $h: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

ie, if h is differentiable, then taking derivative to be a linear operator, this matrix is the matrix of the linear transformation.

Then, (U, V) has a joint density given by –

$$\boxed{f_{U,V}(u,v) = f_{X,Y}(g(u,v)) \cdot \left| \det \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \right|}\,,$$

$$(u,v) \in J.$$

Here,

$$g = h^{-1} \qquad (x,y) \xmapsto{h} (u,v)$$

$$(u,v) \xmapsto{g} (x,y)$$

$\therefore$ Hence $(u,v) \xmapsto{g_1} x$

$(u,v) \mapsto (x,y)$

$\therefore$ Here, $(u,v) \xmapsto{g_1} x$

& $(u,v) \xmapsto{g_2} y$

Example 1

(X,Y) has density $f_{X,Y}(x,y) = C(x+2y)$, $0<x<1$, $0<y<1$.

Q: find 'c'.

$$\therefore \int_{y=0}^{1}\int_{x=0}^{1} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\Rightarrow c\int_0^1\left(\int_0^1 (x+2y)\,dx\right)dy = 1$$

$$\Rightarrow c.\int_0^1\left(\frac{1}{2}+2y\right)dy = 1$$

$$\Rightarrow c\left(\frac{1}{2}+1\right) = 1 \qquad \Rightarrow c \times \frac{3}{2} = 1$$

$$\Rightarrow c = 2/3.$$

Now, let $U = X+Y$

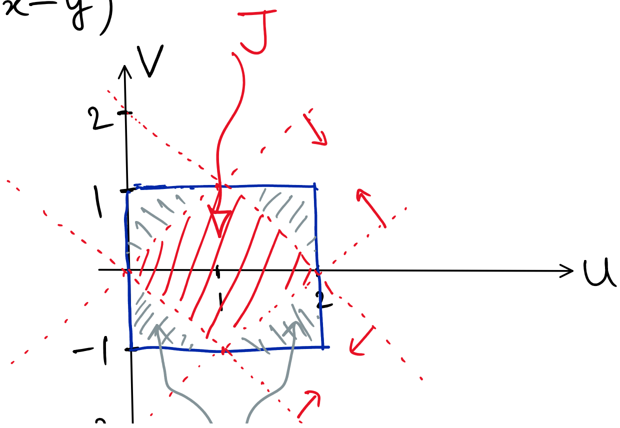
$V = X-Y$

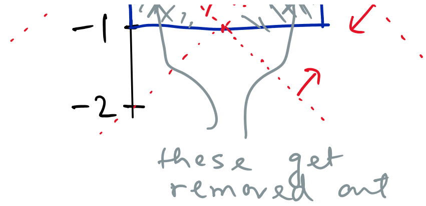
$I = (0,1)\times(0,1).$

$h:(x,y) \longmapsto (u,v) = (x+y,\, x-y)$

$J = \{(u,v):\ 0<u<2,\ -1<v<1,$

$\left(\begin{array}{l}\because u = x+y \\ v = x-y\end{array}\right) \Rightarrow \left.\begin{array}{l} x = \frac{u+v}{2} \in (0,1) \\ y = \frac{u-v}{2} \in (0,1)\end{array}\right\}$





$$x = \frac{u}{2} + \frac{v}{2}, \quad y = \frac{u}{2} - \frac{v}{2}$$

Here,

$$\text{Jacobian} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{\partial}{\partial u}\left(\frac{u}{2}+\frac{v}{2}\right) & \frac{\partial}{\partial v}\left(\frac{u}{2}+\frac{v}{2}\right) \\ \frac{\partial}{\partial u}\left(\frac{u}{2}-\frac{v}{2}\right) & \frac{\partial}{\partial v}\left(\frac{u}{2}-\frac{v}{2}\right) \end{pmatrix}$$

$$= \det\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = 1/2$$

$$\therefore f_{U,V}(u,v) = \frac{2}{3}\left(x + 2y\right) \cdot \frac{1}{2}$$

$$= \frac{2}{3}\left(\frac{u+v}{2} + 2\left(\frac{u-v}{2}\right)\right) \cdot \frac{1}{2}$$

$$= \frac{\cancel{2}}{3}\left(\frac{3u}{2} - \frac{v}{2}\right) \cdot \frac{1}{\cancel{2}}$$

$$f_{U,V}(u,v) = \frac{1}{2}\left(u - \frac{v}{3}\right)$$

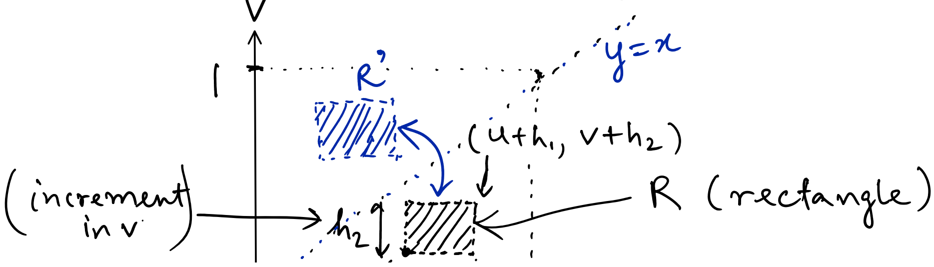
Exercise: take $U = \max\{X, Y\}$
$V = \min\{X, Y\}$

find $f_{U,V}$.

1st way: find joint distⁿ $F_{U,V}$

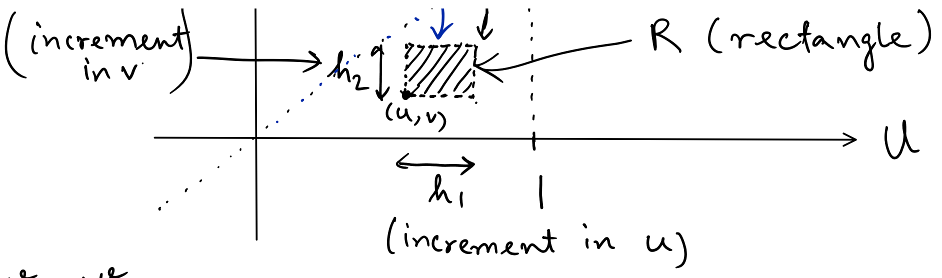
then, $f_{U,V}(u,v) = \dfrac{\partial^2 F_{U,V}}{\partial u \, \partial v}$

2nd Way: Here, $J = \{(u,v) : 0 < u < v < 1\}$.



$h_1, h_2 > 0$

[analogue to 'h' in $x+h$ in 1D]



[analogue to 'h' in $x+h$ in 1D. (increment in x.)]

Analogue of derivative in 2D.

$$\therefore \frac{P((U,V)\in R)}{h_1 h_2} = \frac{1}{h_1 h_2}\Big[P((X,Y)\in R) + \underbrace{P((Y,X)\in R}_{\parallel} \Big]$$

$$= \frac{1}{h_1 h_2}\Big[P((X,Y)\in R) + P((X,Y)\in R') \Big]$$

$$\therefore = \frac{1}{h_1 h_2}\left[\int_{x=u}^{u+h_1}\int_{y=v}^{v+h_2} f(x,y)\,dy\,dx + \int_{x=v}^{v+h_2}\int_{y=u}^{u+h_1} f(x,y)\,dy\,dx \right]$$

$$\therefore \lim_{h_1\to 0}\ \lim_{h_2\to 0} \frac{P((U,V)\in R)}{h_1 h_2} = f_{U,V}(u,v)$$

Independence:

Definition: Random Variables X and Y are said to be independent if

$$P(X\in B_1,\ Y\in B_2) = P(X\in B_1)\cdot P(Y\in B_2)$$ for any pair of borel sets B_1, B_2.

Result:

X and Y are independent $\iff F_{X,Y}(a,b) = F_X(a)\cdot F_Y(b)$

How?

fix $a\in\mathbb{R}$.

$$P(X\le a,\ Y\in B) = P(X\le a)\cdot P(Y\in B) \quad \forall \text{ borel sets } B\subset\mathbb{R}$$

$$P(X \leq a, Y \in B) = P(X \leq a) \cdot P(Y \in B) \quad \forall \text{ borel sets } B \subset \mathbb{R}$$

Assume $P(X \leq a) > 0$.

$$\therefore \frac{P(X \leq a, Y \in B)}{P(X \leq a)} = P(Y \in B)$$

$$\therefore \frac{P(X \leq a, Y \leq b)}{P(X \leq a)} = P(Y \leq b).$$

So, by fixing a, we can show,
this is a probability distn in b.
(by Carotheodory Extension thm).
proof: Aukaat ke bahar ka hain!!
(M. Stat 1st Yr.)

Result:
For (X, Y) absolutely continuous, X, Y are independent iff

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y).$$

Result: (X, Y) are independent iff

$$f_{X,Y}(x, y) = g(x) \cdot h(y) \quad \forall (x, y).$$

(ie, joint density can be factored into two single variable functions.)

(Why?) $f_X(x) = \int_y f(x, y)\, dy$

$$= g(x) \cdot \int_y h(y)\, dy.$$

$$= g(x) \underbrace{\int_y h(y)\,dy}_{c \text{ (some constant value)}}$$

$$\therefore \frac{f_X(x)}{c} = g(x)$$ ← scalar multiple of the marginal density of x.

Similarly, for Y.

⚠ <u>Example</u>: $f(x,y) = e^{-x}\cdot\left(\frac{2y}{x^2}\right), \quad 0 < y < x < \infty.$

Can this be factored into 2 single variable $f^n s$?

its tempting to take $g(x) = \frac{e^{-x}}{x^2}$ & $h(y) = 2y$.

BUT! this doesn't hold for all (x,y).
(it holds only for $0<y<x<\infty$)

So, we can rewrite $f(x,y)$ as -

$$f(x,y) = e^{-x}\cdot\frac{2y}{x^2}\cdot \mathbb{1}_{(0,x)}y.$$

Now, clearly, this is <u>no longer</u> a product of 2 single variable functions.