Probability-3 Lecture-7

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$(\Omega, \mathcal{A}, P)$ - probability space.

$X_n, n \geq 1.$ - seq. of real r.vs on $(\Omega, \mathcal{A}, P)$.

Def^n: $X_n \xrightarrow{P} X$ if $\forall \varepsilon > 0$, $P(|X_n - X| > \varepsilon) \longrightarrow 0$ as $n \longrightarrow \infty$.

ie, $\forall \varepsilon > 0$, $\forall \delta > 0$,

$\exists n_0$ st $\forall n > n_0$,

$P(|X_n - X| > \varepsilon) < \delta$

(Enough to this for $\varepsilon = \delta$)

(*) $\left.\begin{array}{l} X_n \xrightarrow{P} X \\ Y_n \xrightarrow{P} Y \end{array}\right\} X = Y.$

(*) $\left.\begin{array}{l} X_n \longrightarrow X \\ Y_n \longrightarrow Y \end{array}\right\} \Rightarrow$
- $cX_n \xrightarrow{P} cX \quad \forall c \in \mathbb{R}$
- $X_n + Y_n \longrightarrow X + Y.$
- $X_n \cdot Y_n \longrightarrow X \cdot Y.$

$\underbrace{|a+b| > \varepsilon}_{D} \Rightarrow$ either $\underbrace{|a| > \varepsilon/2}_{A}$, or $\underbrace{|b| > \varepsilon/2}_{= B}$.

$D \subset A \cup B \Rightarrow$
$P(D) \leq P(A) + P(B)$

(*) $X_n \xrightarrow{P} X$

$\Rightarrow f(X_n) \xrightarrow{P} f(X)$ for any continuous function f.

if this is proved,

then, $X_n Y_n = \frac{1}{4}\left((X_n+Y_n)^2 - (X_n - Y_n)^2\right)$

then, $\left.\begin{array}{l} X_n \xrightarrow{P} X \\ Y_n \xrightarrow{P} Y \end{array}\right\} X_n Y_n \xrightarrow{P} XY$

becomes obvious by simple algebra.

quick review of the proof:

"Tightness":

Suppose, Y is a real r.v.

$P(|Y| > M) \longrightarrow 0$ as $M \rightarrow \infty$.

$\therefore \forall \varepsilon > 0$, $\exists M_\varepsilon$ s.t. $P(|Y| > M_\varepsilon) < \varepsilon$.

$Y_1, Y_2, \ldots, Y_k$ are real r.v's

$\therefore \forall i = 1, \ldots, k$,

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$\exists M_i = M_i(\varepsilon)$ s.t. $P(|Y_i| > M_i) < \varepsilon$.

$\therefore M = \text{Max}\{M_1, \ldots, M_k\}$.

$\Rightarrow P(|Y_i| > M) < \varepsilon$. $\forall i = 1, \ldots, k$

We did this for a finite no. of r.vs.

Q: Given a sequence of $\{Y_n\}$ of real random r.vs,

(exercise given in Lec-6) Can we get, $\forall \varepsilon > 0$, an M_ε s.t. $P(|Y_i| > M_\varepsilon) < \varepsilon$?

In general, NO!!!

→ Eg: $Y_n = \begin{cases} n & \text{with prob.} = 1/2 \\ -n & \text{with prob} = 1/2 \end{cases}$

Here, for every M,

$$P(|Y_n| > M) = 1 \quad \forall n \geq M.$$

So, here's a counter example.

But, what if, $\{Y_n\}$ is such a sequence s.t.

$Y_n \xrightarrow{P} Y$, then? (this is what we did in last class)

Definition: (Tightness)

A sequence of $\{X_n\}$ - real r.v is said to be tight if $\forall \varepsilon > 0$, $\exists M = M_\varepsilon$, such that:

Independent of n

$$P(|X_n| > M) < \varepsilon \quad \forall n.$$

→ ie, the tail probability goes to 0 uniformly.

What we proved last time (lec-6) is:

$$X_n \xrightarrow{P} X \Rightarrow \{X_n\} \text{ is tightness.}$$

Proof:

Let $\varepsilon > 0$. be given.

X is a real r.v., get M_0 st $P(|X| > M_0) < \varepsilon$.

look at: $P(|X_n| > M_0 + 1) \leq P(|X| > M_0) + P(|X_n - X| > 1)$

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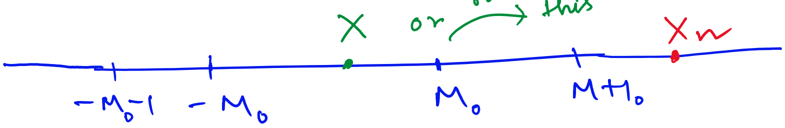
(just a convenient choice. i.e. could have been something else as well.)

<u>Claim</u>: $|X_n| > M_0 + 1 \Rightarrow$ Either $|X| > M_0$ or $|X_n - X| > 1$

↳ when $|X| \leq M_0$



So, $P(|X_n| > M_0+1) \leq \underbrace{P(|X| > M_0)}_{< \varepsilon} + \underbrace{P(|X_n - X| > 1)}_{\text{we need this small}}$

So, get n_0 large enough, st, RHS $< \varepsilon$.

$\therefore P(|X| > M_0 + 1) < \varepsilon$

& $P(|X_n| > M_0 + 1) < \varepsilon \quad \forall n \geq n_0$.

& $X_1, \ldots, X_{n_0 - 1}$ are finitely many. So, we can take care of that.

<u>Result</u>: $X_n \xrightarrow{a.s} X \Leftrightarrow \sup_{k \geq n} |X_k - X| \xrightarrow{p} 0$

$\Rightarrow |X_k - X| \xrightarrow{p} 0 \Leftrightarrow X_n \xrightarrow{p} X$

So, almost sure converges

$\Downarrow$

converges in probability.

The converse is <u>NOT</u> true!!! (Refer L-6 for counter example).

(*) One special case, where $X_n \xrightarrow{a.s} X \Leftarrow\Rightarrow X_n \xrightarrow{a.s} X$

(*) One special case, where $X_n \xrightarrow{a.s} X \Longleftrightarrow X_n \xrightarrow{a.s} X$ is when we have a Discrete Probability Space.
(Proof: Exercise)

Theorem:

$$X_n \xrightarrow{P} X \Rightarrow \exists \text{ a subsequence } X_{n_k} \xrightarrow{a.s} X$$

ie, convergence in $P \not\Rightarrow$ a.s convergence, but we can get a subsequence that exhibits a.s convergence.

Proof: $X_n \xrightarrow{P} X$

$\therefore$ By definition, $\forall k \geq 1, \exists n_k \geq 1$ s.t.

$$P(|X_m - X| > 2^{-k}) < 2^{-k} \quad \forall m \geq n_k$$

We may, & do assume $1 \leq n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$

Denote $A_k = \{|X_{n_k} - X| > 2^{-k}\}$,
event
then, $P(A_k) < 2^{-k}$ $[\because n_k > k]$

$$\therefore \sum_k P(A_k) < \infty$$

(Recall)
* Borel-Cantelli Lemma:

If $A_k, k \geq 1$ - events on the same probability space.

s.t, $\sum_k P(A_k) < \infty$.

Then, $P(\limsup_k A_k) = 0$.

ie, $P\left(\bigcap_{k \geq 1} \bigcup_{j \geq k} A_j\right) = 0$.

(just a consequence of continuity of Probability)

$$\Rightarrow \lim_{k \to \infty} P\left(\bigcup_{j \geq k} A_j\right)$$

$\therefore$ By Borel-Cantelli lemma—

$\therefore$ By Borel-Cantelli lemma

$$P\left(\bigcap_{k} \bigcup_{k' \geq k} \left\{ |X_{n_{k'}} - X| > 2^{-k'} \right\}\right) = 0.$$

$$\Rightarrow P\Big(\underbrace{\bigcup_{k \geq 1} \bigcap_{k' \geq k} \left\{ |X_{n_{k'}} - X| \leq 2^{-k'} \right\}}_{A \text{ (say)}}\Big) = 1.$$

then, $\forall\ \omega \in A$, $X_{n_k}(\omega) \longrightarrow X(\omega)$

$\Downarrow$

$\exists\ k_0 = k_0(\omega)$ s.t. $\left| X_{n_{k'}}(\omega) - X(\omega) \right| \leq 2^{-k'}$ $\forall\ k' \geq k_0(\omega)$

comes from $\bigcup_{k \geq 1}$

comes from $\bigcap_{k' \geq k_0}$

$\therefore$ This has reduced to the definition of a.s. convergence.

<u>Theorem</u>:

$X_n \xrightarrow{P} X \iff$ For every subsequence $\{X_{n_k}\} \subseteq \{X_n\}$ $\exists$ a further subsequence $\{X_{n_{k''}}\}$ s.t. $X_{n_{k''}} \xrightarrow{a.s} X$.

$(\Rightarrow)$

* clearly, $X_n \xrightarrow{P} X \Rightarrow X_{n_k} \xrightarrow{P} X$. $\therefore$ By prev. theorem, $\exists\ \{X_{n_{k''}}\} \subseteq \{X_{n_k}\}$ s.t. $X_{n_{k''}} \xrightarrow{a.s.} X$

(proof: exercise)

$(\Leftarrow)$ Fix $\varepsilon > 0$.

Have to show:

$$a_n = P(|X_n - X| > \varepsilon) \longrightarrow 0$$

Now, let $\{a_{n_k}\}$ a subsequence of $\{a_n\}$.

$\therefore a_{n_k} = P(|X_{n_k} - X| > \varepsilon)$

$\therefore \exists\ n_{k''}$ s.t. $X_{n_{k''}} \xrightarrow{a.s.} X \Rightarrow X_{n_{k''}} \xrightarrow{P} X$

$\therefore \exists\, n_k''$ s.t. $X_{n_k''} \xrightarrow{a.s.} X. \Rightarrow X_{n_k''} \xrightarrow{p} X$

$$\Rightarrow a_{n_k''} \longrightarrow 0 .$$

Corollary: (Exercise) $X_n \xrightarrow{p} X \Rightarrow g(X_n) \xrightarrow{p} g(X)$ $\forall$ continuous fn g.

Definition: (Convergence in the p^{th} moment).

$\{X_n\}$ is said to converge to X in L_p (or, in p^{th} moment)

if $\|X_n - X\|_p \longrightarrow 0$.

$$\left(E|X_n - X|^p \longrightarrow 0\right).$$

We write, $X_n \xrightarrow{L_p} X$

$\left\{\begin{array}{l} X_n \xrightarrow{L_p} X \\ X_n \xrightarrow{L_p} Y \end{array}\right.$

$\Rightarrow X = Y$ a.s.

why? By Minkowski's inequality,

$\|X - Y\|_p \le \|X_n - X\|_p + \|X_n - Y\|_p$, where $\|X_n - X\|_p \downarrow 0$ and $\|X_n - Y\|_p \downarrow 0$.

$\Rightarrow \|X - Y\|_p \longrightarrow 0$

$\Rightarrow X = Y$ a.s.

(*) $X_n \xrightarrow{L_p} X \Rightarrow cX_n \longrightarrow cX$

(*) $X_n \xrightarrow{L_p} X$, $Y_n \xrightarrow{L_p} Y$

$$\Rightarrow X_n + Y_n \xrightarrow{L_p} X + Y.$$

(*) $X_n \xrightarrow{L_p} X$.

$$\therefore P(|X_n - X| > \varepsilon) \le \frac{\|X_n - X\|_p^p}{\varepsilon^p} \longrightarrow 0.$$

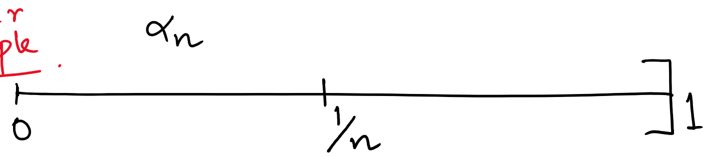
$$\| \quad P(|X_n - X|^p > \varepsilon^p)$$

[By Chebyshev's inequality]

$\Rightarrow X_n \xrightarrow{p} X$. ie, convergence in L_p $\Rightarrow$ convergence in p.

However, converge is not true!!

Counter example.



$$X_n = \alpha_n \cdot \mathbb{1}_{(0, 1/n]}, \quad \alpha_n > 0$$

$X_n = \alpha_n \cdot \mathbb{1}_{(0, 1/n]}$, $\alpha_n > 0$

$\therefore X_n \xrightarrow{p} 0$

$$E(|X_n|^p) = \alpha_n^p \cdot \frac{1}{n}.$$

Clearly, we can choose α_n such that this does not converge.

infact, we can choose α_n s.t. $E(|X_n|^p)$ diverges to ∞.

So, convergence in $p \not\Rightarrow$ convergence in L_p

$\therefore$ Convergence in probability is the "weakest" in a sense.