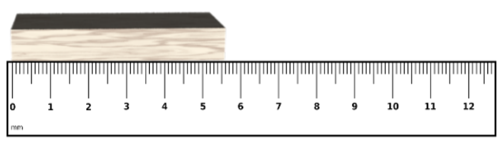
Chapter 9

Gauss-Markov Model

9.2 Example: Measuring a Bar

Let us try to understand the set up of the Gauss-Markov model through an example. Suppose we want to measure the length of a bar as shown below.



This is a simple laboratory experiment that all of us have performed in high school. We could do this by simply placing a ruler along the length of the bar such that the 0 marking on the ruler aligns with the edge of the bar. Say the measured length turns out to be y_1.

Our linear model for this measurement looks something like:

$$y_1 = \beta + \epsilon_1$$

where y_1 is the measured length, β is the true length of the bar, ϵ_1 is the random error which accounts for deviations in our measured length from the true length

We now repeat this trial n times and record the measured length for each trial. Therefore, we will have a system of n linear equations.

$$y_1 = \beta + \epsilon_1$$
$$\vdots$$
$$y_n = \beta + \epsilon_n$$

We can model this situation by the equation:

$$y_i = \beta + \epsilon_i$$

where y_i and ϵ_i are the measured length and random error observed in the i-th trial respectively. Note that only y_i's and ϵ_i's are different across the trials, which is what we expected anyway as the true length of the bar β is fixed.
It is important to note that all measurements must be taken in the same unit. For example, if y_1 is recorded in centimetres, then all subsequent measurements must be made in *cms* as well.

Writing our linear model in $\vec{Y} = X\vec{\beta} + \vec{\epsilon}$ form, we get:

$$\begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} \beta + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ . \\ . \\ \epsilon_n \end{bmatrix}$$

Here, the design matrix X is the nx1 column matrix $[1, 1, ..., 1]^T$ and the vector $\vec{\beta}$ has just one entry and hence is just the scalar β.

The normal equation $X'X\hat{\beta} = X'\vec{y}$ is given by:

$$\begin{bmatrix} 1 & 1 & . & . & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} \hat{\beta} = \begin{bmatrix} 1 & 1 & . & . & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix}$$

It is easy to see that $X'X = 1$ and hence the least squares estimate for β is:

$$\hat{\beta} = \frac{1}{n} \sum_{i=1}^{n} y_i$$

which is quite intuitive as the true length of the bar β can be best estimated by the average of all the measured lengths $\bar{y}$.

In the next section, we shall explore why $\bar{y}$ is considered a good approximation of the true value.