Gauss-Markov Model:BLUE, Gauss-Markov theorem

Srikar Babu Pilli

August 2022

We have seen that the Least squares estimator,assuming that it exists uniquely, has the following interesting properties under the Gauss-Markov setup:

$$E(\hat{\vec{\beta}}) = \vec{\beta}$$
$$var(\hat{\vec{\beta}}) = \sigma^2(X^T X)^{-1}$$

$$\boxed{\begin{array}{l} E(\vec{\beta}) = \vec{\beta} \\ V(\hat{\vec{\beta}}) = \sigma^2(X'X)^{-1} \end{array}}$$

Now,we want to prove that the Least square estimator is the best estimator(in some sense).In general,it is not true to say that the least square estimator is the best estimator,hence we shall give some restricted sense to this "best". In particular,we shall work with those estimators which are linear in data(i.e. which are of the form a matrix times $\vec{y}$).We have already seen that $\hat{\vec{\beta}}$ satisfies this property and is unbiased,so we have to work among all the unbiased estimators which are linear(i.e. linear unbiased estimators)and at the same time, as we have to talk about best,we want to minimize the variance.
Now $\hat{\vec{\beta}}$ is a vector valued estimator,and hence its variance is actually a matrix,so we cannot talk about one matrix being lesser or more than another matrix.So,instead we shall work with linear combinations of $\vec{\beta}$,we shall not try to estimate data,but linear combinations of those β_i's,which we shall call $\vec{c}'\vec{\beta}$,like if $\vec{c}$ is $\vec{e_1} = (1, 0, 0, ..., 0)^T$ then $\vec{c}'\vec{\beta} = \beta_1$.Similarly,if $\vec{c}$ is $\vec{e_2} = (0, 1, 0, ..., 0)^T$ then $\vec{c}'\vec{\beta} = \beta_2$ and so on.
If we work analogously with $\vec{c}'\hat{\vec{\beta}}$, then obviously following the same previous result,we can see that whatever $\vec{c}$ we choose,we have:

$$E(\vec{c}'\hat{\vec{\beta}}) = \vec{c}'\vec{\beta}, \forall \vec{c}.$$

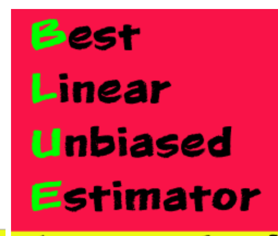
$$var(\vec{c}\prime\hat{\vec{\beta}}) = \sigma^2\vec{c}\prime(X^TX)^{-1}\vec{c}, \forall\vec{c}.$$

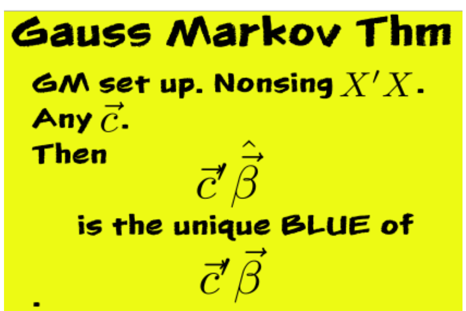
$$E(\hat{\vec{\beta}}) = \vec{\beta}$$
$$V(\hat{\vec{\beta}}) = \sigma^2(X'X)^{-1}$$

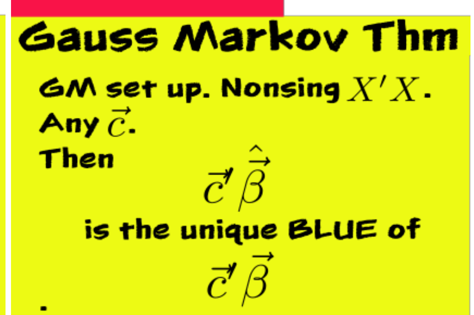
$$\forall\vec{c} \quad E(\vec{c}'\hat{\vec{\beta}}) = \vec{c}'\vec{\beta}$$
$$V(\vec{c}'\hat{\vec{\beta}}) = \sigma^2\vec{c}'(X'X)^{-1}\vec{c}$$

Now,as the variance is a number,it will be very nice if we can say that this is the smallest possible variance that we can achieve,subject to certain conditions.Also,as it is a number,we can talk about larger or smaller. There is a very important theorem of Linear Models to do this,and that is called the Gauss-Markov theorem.
If we start with the Gauss-Markov setup,assume that our X^TX is non-singular(in fact,this condition may be dropped,so ideally we can call it Gauss-Markov simple version if we drop this condition) then the Gauss-Markov theorem says that:

Take any vector $\vec{c} then \vec{c}\prime\hat{\vec{\beta}}$(this linear combination of all the $\hat{\beta}_i$'s is the unique BLUE of the corresponding linear parametric function $\vec{c}\prime\vec{\beta}$.







BLUE means Best Linear Unbiased Estimator,i.e. among all the estimators that are linear in $\vec{y}$(i.e. a matrix times $\vec{y}$)and that are unbiased,this is the best(i.e. it has the minimum possible variance,the variance expression that we saw above is the minimum possible among all the linear unbiased estimators).It also happens to be the unique estimator(i.e. if we can find any other linear unbiased estimator with variance equal to the variance of this Least Squares Estimator,then we can claim with probability 1 that our estimate equals this estimate)