Probability-3 Lecture-1

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Office hours: Tuesday 4pm – 5:30 pm

Venue: Associate Dean's Office. (5th floor, S.N. Bose Bhavan)

Recall: (Q^n from midsem of Sem-2)

$X \in \{0,1,2,\dots\}$.

$p_n = P(X=n)$.

$$P(s) = \sum_{n=0}^{\infty} s^n p_n$$

$$P'(s) = \sum_{n=1}^{\infty} n s^{n-1} p_n.$$

$s \in (-1,1)$.

Q. Show that,

$\lim_{s \uparrow 1} P'(s)$ exists & is finite — ①

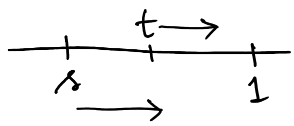
iff

$\lim_{s \uparrow 1} \dfrac{P(s)-1}{s-1}$ exists. — ②

and is finite & equal.

then, show that ① $\Leftrightarrow$ ② $\Leftrightarrow$ $E(X)$ is finite & equals to the limit in ① or ②.

Sol^n. for $0<s<1$..



$$\frac{P(s)-1}{s-1} = P'(t) \quad \text{for } s<t<1.$$

If ① is assumed, then $\lim_{t \uparrow 1} P'(t)$ exists & finite.

But, converse is NOT true!!

correction –

$P'(t)$ is increasing in t.

So, $\lim_{t \uparrow 1} P'(t)$ must exist.

$\therefore$ We have one such seq. $\{t_n\}$ given by

$\dfrac{P(s)-1}{s-1} = P'(t_n)$. $\therefore \lim_{s \uparrow 1} \dfrac{P(s)-1}{s-1}$ exists.

For each n, define $X_k = X \cdot \left(1-\frac{1}{k}\right)^{X-1}$

Now, what happens when $k \uparrow$?

X_k increases to X.

$\therefore X_k \geqslant 0$, $X_k \uparrow X$.

[MCT]

$\therefore X_k \geq 0, \; X_k \nearrow X.$

$\therefore E(X_k) \nearrow E(X).$ [By MCT]

$\therefore \lim_{s \uparrow 1} P'(s) = E(X).$

Revision of Sem-1 & Sem-2:

Random Variable:

$(\Omega, \mathcal{A}, P)$ ← Probability assignment over $\mathcal{A}$.

Ω: Sample space

$\mathcal{A}$: σ-field. "Class of events"

$X : \Omega \to \mathbb{R}$ (or $[-\infty, \infty]$) is a r.v.
if $\forall a \in \mathbb{R}$, $\{\omega : X(\omega) \leq a\} \in \mathcal{A}$.

This is equivalent to $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$ for all borel sets $B \subset \mathbb{R}$.

Given any real r.v. X,
probability distribution of X, $P_X(B) = P(X^{-1}(B))$ $\forall$ borel sets $B \subset \mathbb{R}$

i.e, X acts like the "carrier" of probability mass.

$F_X : \mathbb{R} \to \mathbb{R}$.

$F_X(a) = P_X((-\infty, a])$
$= P(X \leq a)$, $a \in \mathbb{R}$.

$\Delta F_X(a) = P_X(\{a\}) = P(X = a)$.

Properties:

- Right continuous
- Non-decreasing
- $F_X(a) \to \begin{cases} 0, & a \to -\infty \\ 1, & a \to \infty \end{cases}$

The distribution function completely defines the probability assignment.

$$F_X \xleftrightarrow[\text{correspondence}]{1-1} P_X$$

Special types:

1. X - discrete.

$\exists$ ... $D \subset \mathbb{R}$ such that $P_X(D) = 1$.

① X - discrete.

$\exists$ countable $D \subset \mathbb{R}$, such that $P_X(D) = 1$.

$$p_X(x) = P(X = x), \ x \in \mathbb{R}. \quad \text{pmf}.$$

$$p_X(x) = 0, \ x \notin D. \ \& \ \ \forall x \in D, \ p_X(x) > 0.$$

$$\&, \ P_X(B) = \sum_{x \in B \cap D} p_X(x).$$

$$F_X(a) = \sum_{x \leq a, x \in D} p_X(a)$$

② X (absolutely*) continuous.

* absolutely cts = has probability density.

if $\exists f \geq 0$ on $\mathbb{R}$.

s.t. $P_X(B) = \int_B f(x)\,dx$

$$F_X(a) = \int_{(-\infty, a]} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx.$$

[Note that, the "$\int$" need not always be a Riemann Integration.]

Say, X has density $f(x)$, $x \in I$ (open interval.)

take $Y := h(X)$.

Special case: $h : I \longrightarrow J$ is 1-1, onto.

$$g = h^{-1} : J \to I.$$

Assume: g is continuously differentiable.

Then, $Y = h(X)$ has density:

$$f_Y(y) = f(g(y)) \cdot |g'(y)|.$$

In general, for $Y = h(X)$, try to find

$$F_Y(b) = P(h(X) \leq b) = \int_{x : h(x) \leq b} f(x)\,dx.$$

$x : h(x) \leq b$

try & see if
$F_Y(y)$ is onto.

Expected Value:

Step-1: Non-ve, simple, real r.v X.

(can be written in the form:

$$X = \sum_{i=1}^{n} c_i \cdot 1_{A_i}, \quad A_1, \ldots, A_n \text{ is a partition of } \Omega.)$$

$$E(X) = \sum_{i=1}^{n} c_i \cdot P(A_i)$$

Step-2:
$X \geq 0$ extended real values.

$$E(X) = \sup_{Y \leq X} \{E(Y) : Y \text{ is a simple real non-ve r.v}\}$$

Result:
For any sequence $\{X_n\}$ of non-ve real simple r.vs with $X_n \uparrow X$,

$$E(X) = \lim_{n \to \infty} E(X_n).$$

Step-3:
For general X, $E(X) = E(X^+) - E(X^-)$ if RHS is defined.

Properties of $E(X)$:
Expectation is a linear, order preserving operator.

Inequalities: Chebyshev, Holder's, Minkowski's, Jensen.

M.G.F:
X - r.v.

X – r.v.

$m_X(t) = E(e^{tX})$ if RHS is finite.

Holder's inequality gives –

The set $I = \{t : E(e^{tX}) < \infty\}$ is an interval with $0 \in I$.

Eg: $I = \{0\}$ for Cauchy distn.

$$f_X(x) = \frac{c}{1+x^2}, \quad x \in \mathbb{R},$$

c – normalizing constant.

Incase 0 is an interior point of I, then MGF characterizes (or determines) the distribution of X. ie, $m_X(t)$ determines F_X.

$$m_X(t) = \sum_{n=0}^{\infty} \frac{E(X^n)}{\lfloor n} \cdot t^n \quad \text{for } t \in (-\varepsilon, \varepsilon).$$

radius of convergence