Recall: Uniform Integrability.

Definition:

A seq. $\{X_n\}$ of r.v.s is called **Uniformly Integrable**

if $E\left(|X_n|\cdot 1_{|X_n|>\lambda}\right) \longrightarrow 0$ as $\lambda \longrightarrow \infty$

(uniformly integrable)

$$\Leftrightarrow \sup_n \left(E|X_n|\cdot 1_{|X_n|>\lambda}\right) \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty$$

$$\Leftrightarrow \forall \varepsilon>0, \exists \lambda=\lambda_\varepsilon \text{ s.t. } E\left(|X_n|\cdot 1_{|X_n|>\lambda}\right)<\varepsilon \quad \underline{\underline{\forall n}}$$

uniformity.

"X is integrate" $\equiv E|X|<\infty$

Recall:

X is integrable $\Longleftrightarrow E\left(|X|\cdot 1_{|X|>\lambda}\right) \longrightarrow 0$ as $\lambda \longrightarrow \infty$

(ie, tail expectation $\downarrow$ 0)

Observations:

① $\{X_n\}$ - uniformly integrable $\Rightarrow \{X_n\}$ is L-1 bounded.

(ie, $\sup_n \|X_n\|<\infty$).

for proof: take $\varepsilon = 1$.

Converse <u>not</u> true !!

② If $\exists$ integrable r.v. Y such that, $|X_n|\leq Y \ \forall n$, then $\{X_n\}$ is uniformly integrable.

Converse <u>not</u> true !! ie, DCT can be made stronger !!

③ $\{X_n\}$ is L_p bounded for some $p>1$, (say, even $p=1+\delta$, $\delta>0$.)

$\Rightarrow \{X_n\}$ is uniformly integrable

Converse <u>not</u> true !!

Ques: what is it then, that's equivalent ($\Leftrightarrow$) to uniform integrability?

Suppose, X - real-valued r.v. (*)

Proposition: X is "integrable" ~~iff~~. $\lim\limits_{P(A)\to 0} E(X\cdot 1_A)=0$

ie, $\forall \varepsilon>0, \exists \delta>0$ st, $\forall$ $A\in\mathcal{A}$, $P(A)<\delta$ $\Rightarrow E(|X|\cdot 1_A)<\varepsilon$.

Proof: Exercise. (All necessary tools taught in Sem-2.)

(*) Why "real-valued" is needed?
Counter eg: $\Omega=\mathbb{N}$. $\mathcal{A}=\mathbb{P}(\mathbb{N})$ (Power set of naturals)

$P(\{n\})=\frac{1}{2^n}$.

$X(n)=\begin{cases}+\infty, & \text{if } n=1\\ 0, & \text{otherwise.}\end{cases}$

take $\delta=\frac{1}{2}$. $\longrightarrow$ this works $\forall\,\varepsilon$.

BUT, this X is certainly NOT integrable.

Result 1:

A sequence X_n is uniformly integrable

$\Leftrightarrow$ ($\{X_n\}$ is L_1-bounded and

$\lim\limits_{P(A)\to 0}\sup\limits_n E(|X_n|\cdot 1_A)=0$.

ie, $\forall\,\varepsilon>0$, $\exists\,\delta=\delta_\varepsilon>0$ s.t. $\left.\begin{array}{l}A\in\mathcal{A},\\ P(A)<\delta\end{array}\right]\Rightarrow E(|X_n|\cdot 1_A)<\varepsilon$. $\forall n$)

this is referred to as "uniform absolute continuity" "absolute continuity" $P(A) < \delta \Big] \Rightarrow E(|X_n| \cdot 1_A) < \varepsilon$. $\forall n$.

Proof: ($\Rightarrow$) Assume, $\{X_n\}$ - uniformly integrable.

to prove, X_n is L_1 bounded $\to$ exercise !!

So, let $\varepsilon > 0$ be given.

$$E(|X_n| \cdot 1_A) = \underline{E(|X_n| \cdot 1_{|X_n| \le \lambda} \cdot 1_A)} + E(|X_n| \cdot 1_{|X_n| > \lambda} \cdot 1_A).$$

$$\le \underline{\lambda \cdot P(A)} + E(|X_n| \cdot 1_{|X_n| > \lambda} \cdot 1_A)$$

Now, choose $\lambda_0 = \lambda_0(\varepsilon)$ s.t.

(by uniform integrability) $E(|X_n| \cdot 1_{|X_n| > \lambda}) < \varepsilon/2$

Now, choose $\delta > 0$ s.t.

$\lambda_0 \delta < \varepsilon/2$ $\left(\delta < \frac{\varepsilon}{2\lambda_0} \right)$

$P(A) \le \delta \Rightarrow \therefore \lambda_0 P(A) \le \lambda_0 \delta < \lambda_0 \cdot \frac{\varepsilon}{2\lambda_0} < \varepsilon/2$

($\Leftarrow$) Let $\varepsilon > 0$ be given.

<u>To show:</u> We can λ_0 st $E(|X_n| \cdot 1_{|X_n| > \lambda_0}) < \varepsilon$. $\forall n$

ie, all that we have to do, is to find λ_0 s.t $P(|X_n| > \lambda_0) < \delta$.

By Chebyshev's Inequality:

$$P(|X_n| > \lambda_0) \le \frac{E|X_n|}{\lambda_0} = \frac{\|X_n\|_1}{\lambda_0}$$

$$\le \sup_n \frac{\|X_n\|_1}{}$$

this "removes"

this "removes" the dependence on n.

$$\leq \sup_n \frac{\|X_n\|_1}{\lambda_0}$$

So now, we can choose λ_0 such that

$$\sup_n \frac{\|X_n\|_1}{\lambda_0} < \delta.$$

(from given condition)

Result 2:

$X_n, n \geq 1$ and X - real r.vs on a Probability space.

(a) $X_n \xrightarrow{P} X$ and $\{X_n\}$ uniformly integrable

$\Rightarrow X \in L_1$, and $X_n \xrightarrow{L_1} X$

kind of an additional criteria over p-conv to imply L_1 conv.

(b) $X_n \xrightarrow{L_1} X \not\Rightarrow X_n \xrightarrow{P} X$ and $\{X_n\}$ is uniformly integrable.

Recall: DCT.

$X_n \xrightarrow{a.s} X$ & $|X_n| \leq Y$ for some Y-integrable is $E|Y| < \infty$

$\Rightarrow E(X_n) \longrightarrow E(X)$ $\quad E|X_n| < \infty$.

$X_n \xrightarrow{P} X$ $\quad$ ie, $X_n \xrightarrow{L_1} X$.

So, note that,

Result-2, part-a is a huge improvement over DCT. ie, we've weakened the conditions of DCT, yet getting the same result.

Proof: (a) $X_n \xrightarrow{P} X$

"same" result

Proof: (a) $X_n \xrightarrow{P} X$

$\Rightarrow X_{n_k} \xrightarrow{a.s.} X$ for a subsequence $\{n_k\}$.

Fatou's lemma $\Rightarrow E|X| \leq \underbrace{\liminf E|X_{n_k}|}_{\liminf \|X_{n_k}\|_1} \leq \sup_n \|X_n\|_1 < \infty$ (from u.i)

$E|X| = E|\liminf X_{n_k}|$, as $X = \liminf X_{n_k}$ as $X_{n_k} \xrightarrow{a.s} X$.

(Eg. real seq. $\{x_n\}$, $|x_n| \leq 10$. $x_n \to x$. then, $|x| \leq 10$)

$\Rightarrow X \in L_1$.

now, to show, $X_n \xrightarrow{L_1} X$

$$E|X_n - X| = E\left(|X_n - X| \cdot 1_{|X_n - X| \leq \varepsilon/3}\right) + E\left(|X_n - X| \cdot 1_{|X_n - X| > \varepsilon/3}\right)$$

$$\leq (\varepsilon/3) + \left(E\left(|X_n| \cdot 1_{|X_n - X| > \varepsilon/3}\right) + E\left(|X| \cdot 1_{|X_n - X| > \varepsilon/3}\right)\right)$$

Δ - inequality.

Now, $E|X| < \infty \Rightarrow$

$\exists\, \delta_1 > 0$ st, $A \in \mathcal{a}$, $P(A) < \delta_1$, ($\delta_1 = \delta_{1,\varepsilon/3}$)

$\Rightarrow E(|X| \cdot 1_A) < \varepsilon/3$

Also $\{X_n\}$ - u.i,

$\Rightarrow \exists\, \delta_2 > 0$, s.t $A \in \mathcal{a}$, $P(A) < \delta_2$ $\Rightarrow E(|X_n| \cdot 1_A) < \varepsilon/3$ $\forall n$

(using result-1).

$X_n \xrightarrow{P} X$

$\exists\, n_0$ s.t., $\forall n \geq n_0$, $P(|X_n - X| > \varepsilon/3)$

$X_n \longrightarrow X$

$\exists\, n_0$ s.t., $\forall\, n \geq n_0$, $P(|X_n - X| > \varepsilon/3) < \min\{\delta_1, \delta_2\}$ $\forall\, n \geq n_0$.

$\therefore E(|X| \cdot \mathbb{1}_{|X_n - X| > \varepsilon/3}) < \varepsilon/3$ $\forall\, n \geq n_0$,

as, $\forall\, n \geq n_0$, $P(|X_n - X| > \varepsilon/3) < \delta_2$.

&, $E(|X_n| \cdot \mathbb{1}_{|X_n - X| > \varepsilon/3}) < \varepsilon/3$ $\forall\, n \geq n_0$,

as, $\forall\, n \geq n_0$, $P(|X_n - X| > \varepsilon/3) < \delta_1$

$$\therefore E|X_n - X| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

$$X_n \xrightarrow{L_1} X.$$

(b) To prove: $X_n \xrightarrow{L_1} X \Rightarrow X_n \xrightarrow{P} X$ & $\{X_n\}$ is u.i

we know, $\forall\, p \geq 1$, $X_n \xrightarrow{L_p} X \Rightarrow X_n \xrightarrow{P} X$.

So, this part is trivial.

We're just left to show, $\{X_n\}$ is u.i (uniformly integrable)

Let $\lambda > 1$ [$\because$ we're supposed to choose "large" λ.]

$$E\left(|X_n| \cdot \mathbb{1}_{|X_n| > \lambda}\right) \leq E|X_n - X| \cdot \mathbb{1}_{|X_n| > \lambda} + E|X| \cdot \mathbb{1}_{|X_n| > \lambda}$$

(Δ-inequality)

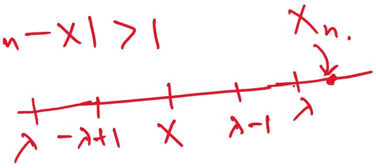
$$\leq \|X_n - X\|_1 + \underset{*}{E}\left(|X| \cdot \mathbb{1}_{|X_n| > \lambda,\ |X| \leq \lambda - 1}\right) + \underset{(*)}{E}\left(|X| \cdot \mathbb{1}_{|X_n| > \lambda,\ |X| > \lambda - 1}\right)$$

this just stays

$$\leq \|X_n - X\|_1 + \underset{(*)}{E} \cdot \left(|X| \cdot \mathbb{1}_{|X_n - X| > 1,\ \underline{|X| \leq \lambda - 1}}\right)$$

$(*) \rightarrow$ $\underbrace{|X_n| > \lambda}_{A}$, $\underbrace{|X| \leq \lambda - 1}_{B}$.

$(C) \Rightarrow |X_n - X| > 1$



$A \cap B \subseteq C$

$\therefore \mathbb{1}_{A \cap B} \leq \mathbb{1}_C$

i.e, 1.

i.e, $\mathbb{1}_{(X_n>\lambda,\ |X|\leq\lambda-1)} \leq \mathbb{1}_{(|X_n-X|>1)}$

$$\leq \|X_n-X\|_1 + \underset{(*)}{E}\left(|X|\cdot\mathbb{1}_{|X_n-X|>1,\ |X|\leq\lambda-1}\right)$$

$$\underset{(*)}{E}\left(|X|\,\mathbb{1}_{|X|>\lambda-1}\right)$$

$$\leq \underbrace{\|X_n-X\|_1 + (\lambda-1)\cdot P(|X_n-X|>1)}$$

$$+ E\left(|X|\cdot\mathbb{1}_{|X|>\lambda-1}\right)$$

$\|X_n-X\|_1 + (\lambda-1)\cdot\|X_n-X\|_1 = \lambda\cdot\|X_n-X\|_1.$

$$\leq (\lambda-1)\cdot\frac{E|X_n-X|}{1}$$

(Chebyshev)

$$= (\lambda-1)\cdot\|X_n-X\|_1$$

$\therefore$

$$\leq \lambda\cdot\|X_n-X\|_1 + E\left(|X|\cdot\mathbb{1}_{|X|>\lambda-1}\right).$$

Now, $X\in L_1$

tail expectation

$\therefore$ Choose $\lambda_0>1$ st

$$E\left(|X|\cdot\mathbb{1}_{|X|>\lambda_0-1}\right)<\varepsilon/2$$

$\therefore$ Choose n_0 s.t $\forall\, n\geq n_0$

$$\lambda_1\,\|X_n-X\|_1 < \varepsilon/2$$

[$\because X_n \xrightarrow{L_1} X$, $\therefore \|X_n-X\|$ can be made arbitrarily small.]

& choose $\lambda_1,\dots,\lambda_{n_0-1}$ s.t.

$$E\left(|X_n|\cdot\mathbb{1}_{|X_n|>\lambda_n}\right)<\varepsilon \quad \forall\ n=1,\dots,n_0-1.$$

$\|X_n-X\|_1<\varepsilon.$

Now, choose $\lambda=\mathrm{Max}.\{\lambda_0,\lambda_1,\dots,\lambda_{n_0-1}\}$.

$\therefore$ For this λ, $\forall\, n$,

$$E\left(|X_n|\cdot\mathbb{1}_{|X_n|>\lambda}\right)<\varepsilon.$$ ▨

— Midsem syllabus till. here —