Probability-2 Lecture-21

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Definition:

X, Y are said to be independent if

$$P(X\in B_1, Y\in B_2) = P(X\in B_1)\cdot P(Y\in B_2) \quad \forall \text{ borel sets } B_1, B_2 \subset \mathbb{R}.$$

This is equivalent to

$$P(X\le a,\ Y\le b) = P(X\le a)\cdot P(X\le b) \quad \forall\ a,b\in\mathbb{R}.$$

Corollary: If X, Y are independent, then $h(X), g(Y)$ independent for any two measureable functions $h, g : \mathbb{R}\longrightarrow \mathbb{R}$.

Result: If X, Y are independent, and if $E(X)$ & $E(Y)$ are both finite, then $E(XY)$ is also finite, and

$$\boxed{E(XY) = E(X)\cdot E(Y).}$$

Proof: Case-I:

X, Y both non-negative.

Define $X_n = h_n(X)$, $Y_n = h_n(Y)$,

where $h_n : [0,\infty)\longrightarrow\mathbb{R}$ is

$$h_n(x) = \sum_{k=0}^{n\cdot 2^n-1}\frac{k}{2^n}\cdot \mathbb{1}_{\left(\frac{k}{2^n},\frac{k+1}{2^n}\right]}(x) + n\cdot\mathbb{1}_{(n,\infty)}(x).$$

X_n, Y_n – independent, and X_n, Y_n both non-ve, real simple r.vs with finite expectations.

$X_n \nearrow X$ converges pointwise.

$Y_n \nearrow Y$ converges pointwise

pdt. of 2 simple f^ns is also simple. $\Rightarrow$ $X_nY_n \nearrow XY$ converges pointwise.

$$E(X_nY_n) = E(X_n)\cdot E(Y_n)$$

f's also is simple.

from Sem-1: $E(X_n Y_n) = E(X_n) \cdot E(Y_n)$

$\downarrow \quad \downarrow \quad \downarrow$ (taking $\lim_{n\to\infty}$).

$E(XY) = E(X) \cdot E(Y)$.

<u>Case-II</u>: $X = X^+ - X^-$ $\quad\therefore\quad$ $\left.\begin{matrix} X^+, Y^+ \\ X^+, Y^- \\ X^-, Y^+ \\ X^-, Y^- \end{matrix}\right\} \to$ independent.

$Y = Y^+ - Y^-$

$\therefore \to E(X^+Y^+) = E(X^+) \cdot E(Y^+)$

$\to E(X^+Y^-) = E(X^+) \cdot E(Y^-)$

$\to E(X^-Y^+) = E(X^-) \cdot E(Y^+)$

$\to E(X^-Y^-) = E(X^-) \cdot E(Y^-)$

$\therefore (XY)^+ = X^+Y^+ + X^-Y^-$

$(XY)^- = X^+Y^- + X^-Y^+$

$\therefore XY = (XY)^+ - (XY)^-$

$\therefore E(XY) = \left(E(XY)^+ - E(XY)^-\right)$

$= E(X^+Y^+ + X^-Y^- - X^+Y^- - X^-Y^+)$

$= E(X^+Y^+) + E(X^-Y^-) - E(X^+Y^-) - E(X^-Y^+)$

$= E(X^+) \cdot E(Y^+) + E(X^-) \cdot E(Y^-)$
$\quad - E(X^+) \cdot E(Y^-) - E(X^-) \cdot E(Y^+)$

$= E(X^+)\left(E(Y^+) - E(Y^-)\right) - E(X^-)\left(E(Y^+) - E(Y^-)\right)$

$= \left(E(X^+) - E(X^-)\right) \cdot \left(E(Y^+) - E(Y^-)\right)$

$$\boxed{E(XY) = E(X) \cdot E(Y).}$$ ▨

Result:

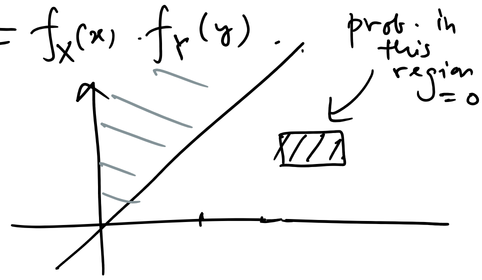
2 absolutely continuous r.vs X, Y are independent if and only if (X,Y) has a joint density which equals :

$$\boxed{f(x,y) = f_X(x) \cdot f_Y(y).}$$

Example: $f_{X,Y}(x,y) = 2x y^{-2} \cdot e^{-y}$, $0 < x < y < \infty$.

Q. firstly, show that, $X \,\&\, Y$ are independent.

first way: calculate the marginals:
Show that $f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$.



Second & clever way:

take (x,y) s.t. $x > y$.
then, $f_X(x) \neq 0$, $f_Y(y) \neq 0$.
But, $f_{X,Y}(x,y) = 0 \neq f_X(x) \cdot f_Y(y)$

Q. find density of $\frac{X}{Y}$.

Solution 1:

$$F(a) = P\left(\frac{X}{Y} \leq a\right) \quad , \quad a \in (0,1).$$

$$= P(X \leq aY)$$

$$= \iint_{x \leq ay} f(x,y)\,dx\,dy.$$

$$= \int_{y=0}^{\infty} \int_{x=0}^{ay} 2x \cdot \frac{e^{-y}}{y^2}\, dx\, dy$$

$$= \int_{y=0}^{\infty} \left(\frac{e^{-y}}{y^2} \cdot (x^2) \Big|_0^{ay} \right) dy$$

$$\int_{y=0} \left(\frac{\quad}{y^2} \cdots \Big|_0 \right)$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{\cancel{y^2}} \times a^2 \cancel{y^2}\, dy = a^2 \cdot -e^{-y} \Big|_0^{\infty}$$

$$\therefore \Rightarrow F(a) = a^2 \,, \quad a \in (0,1) . \quad \underline{\text{Ans.}}$$

<u>Solution 2</u>: take the transformation:

$$(X,Y) \longmapsto \left(\frac{X}{Y}, Y \right) \equiv (U, V) .$$

$$(x,y) \longmapsto (u,v).$$

$$u = \frac{x}{y} \in (0,1) . \;\; ; \quad v = y .$$

$$\therefore \quad x = uy = uv \quad v = y$$

$\therefore$ Inverse of the transformation, $g : (u,v) \longmapsto (x,y) \equiv (uv, v)$

$$M = \{ (x,y) : 0 < x < y < \infty \}$$

$$N = \{ (u,v) : 0 < u < 1 \,, \; 0 < v < \infty \} .$$

$$\therefore \text{Jacobian, } J(u,v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \det \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}$$

$$= v$$

$\therefore$ (U,V) has joint density:

$$f_{U,V}(u,v) = f_{X,Y}(g(u,v)) \cdot J \;//$$ [Refer Lecture-20]

$$f_{U,V}(u,v) = f_{X,Y}(g(u,v)) \cdot J \quad [\text{[illegible]}]$$

$$= f_{X,Y}(uv, v) \cdot v$$

$$= 2(uv) \cdot v^{-2} \cdot e^{-v} \cdot v$$

$$f_{U,V}(u,v) = 2u e^{-v}, \quad 0 < u < 1, \quad 0 < v < \infty.$$

Here, we needed $F_U(a) = P(U \leq a)$

$\therefore$ Marginal density of U:

$$f_U(u) = \int_0^\infty 2u e^{-v} \, dv = 2u$$

$$\therefore F_U(a) = \int_0^a f_U(u)\, du = \int_0^a 2u\, du = a^2, \quad a \in (0,1)$$

Ans.

(X,Y) has joint density $f(x,y)$.

for any $h: \mathbb{R}^2 \to \mathbb{R}$ measureable,

$$E(h(X,Y)) = \int\int h(x,y) f(x,y)\, dx\, dy,$$

provided the integral exists.

$$\therefore E(h(X)) = \int\int h(x) \cdot f(x,y)\, dx\, dy$$

$$= \int h(x) \cdot f_X(x)\, dx$$

<u>Definition</u>:

For any two random variables X, Y with

finite second moment

For any two random variables X, Y with finite second moment,

Covariance : $Cov(X,Y) = E(XY - E(X)E(Y))$

$= E((X-E(X))(Y-E(Y)))$

Variance : $V(X) = Cov(X,X) = E((X-E(X))^2)$

$= E(X^2) - (E(X))^2$

CONVOLUTION:

Example: (X, Y) has a joint density $f(x,y)$

$Z = X+Y$. Does f has a density?

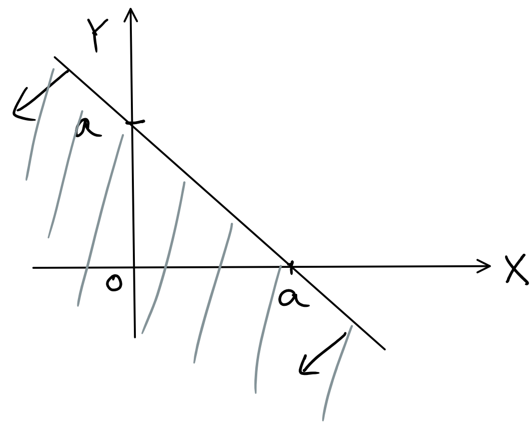
If yes, find it.

Solution 1:

for real a,

$P(Z \le a) = P((X,Y) \in B)$, where $B = \{(x,y) : x+y \le a\}$.

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{a-y} f(x,y)\,dx\right)dy$$



we want to remove y from limit of the inner integral.

So, put $u = x+y$

$\therefore x = u - y$

$dx = du$

x	u
$-\infty$	$-\infty$
$a-y$	a

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{a} f(u-y,y)\,du\right)dy \quad \text{——— } (1)$$

$$= \int_{-\infty}^{a} \underbrace{\left(\int_{-\infty}^{\infty} f(u-y, y)\, dy \right)}_{\substack{\text{a function} \\ \text{of } u. \\ \text{Call it } g(u)}} \cdot du$$

$$\therefore \quad P(Z \leq a) = F_Z(a) = \int_{-\infty}^{a} g(u)\, du.$$

$$\therefore f_Z(a) = g(a) = \int_{-\infty}^{\infty} f_{X,Y}(a-y, y)\, dy \quad \text{—} (2)$$

Ans.

Corollary: Suppose X, Y - independent.

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

$\Rightarrow Z = X + Y$ has density

$$(2) \rightarrow \quad f_Z(a) = \int_{-\infty}^{\infty} f_X(a-y) \cdot f_Y(y)\, dy$$

$$= E\left(f_X(a-Y)\right)$$

from (1):

$$\therefore \quad F_Z(a) = \int_{-\infty}^{\infty} \int_{-\infty}^{a} f(u-y, y)\, du\, dy.$$

$$= \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{a} f_X(u-y)\, du \right)}_{\substack{F_X(a-y) \\ = P(X \leq a-y)}} \cdot f_Y(y)\, dy \qquad [\because X, Y \text{ - independent}]$$

$$F_X(a-y)$$
$$= P(X \leq a-y)$$

$$= \int_{-\infty}^{\infty} F_X(a-y) \cdot f_Y(y)\, dy$$

$$F_Z(a) = E\left(F_X(a-Y)\right)$$

. . . .

f_1, f_2 - any two probability density f^{ns}.

$\Rightarrow f_1 * f_2(z) = \int_0^z f_1(z-y) \cdot f_2(y) \cdot dy$ is a probability density.

Note that, $f_1 * f_2 \equiv f_2 * f_1$.

This is called "Convolution" of density f^{ns}.

<u>Convolution of Distribution Functions:</u>

F_1, F_2 distribution functions.

Define $F(a) = E\left(F_1(a-X)\right)$, where $X \sim F_2$.

This F is a distribution function. (<u>Exercise</u>: verify this)

$$F = F_1 * F_2$$

<u>Exercise</u>: Show that:

$$F_1 * F_2 = F_2 * F_1.$$