Probability-2 Lecture-3

16 January 2024 11:24

Convolution:

X, Y - independent

$Z = X + Y$.

Distribution of Z is called the "Convolution" of the distributions of X & Y.

pmf of X : $p_X(x)$, $x \in D_X$

pmf of Y : $p_Y(y)$, $y \in D_Y$.

(X, Y) — set of values from D_X and D_Y.

& $p(x,y) = p_X(x) \cdot p_Y(y)$. $[\because X, Y - \text{ind.}]$

$$D_Z = \{x + y : x \in D_X, \ y \in D_Y\}.$$

pmf of Z:

$$p_Z(z) = P(Z = z)$$

$$= P(X + Y = z)$$

$$= \sum_{x \in D_X} P(X = x, Y = z - x) = \sum_{y \in D_Y} P(X = z - y, Y = y)$$

$$= \sum_{x \in D_X} P(X = x) \cdot P(Y = z - x) = \sum_{y \in D_Y} P(X = z - y) \cdot P(Y = y)$$

$$= \underbrace{\sum_{x \in D_X} p_X(x) \cdot p_Y(z - x)}_{p_X * p_Y(z)} = \underbrace{\sum_{y \in D_Y} p_X(z - y) \cdot p_Y(y)}_{p_Y * p_X(z)}$$

D_1, D_2 - countable sets $\subset \mathbb{R}$

p_1 p_2 are pmfs supported on D_1 & D_2 respectively.

D_1, D_2 : countable

p_1, p_2 are pmfs supported on D_1 & D_2 respectively.

$$D = \{x+y \mid x \in D_1,\ y \in D_2\}$$

Define: $$p_1 * p_2(z) := \sum_{x \in D_1} p_1(x) \cdot p_2(z-x), \qquad z \in D.$$

Show that: ① $p_1 * p_2$ is a pmf.

② $p_1 * p_2 = p_2 * p_1$

③ $p_1 * e = p_1$, where $e(0) = 1$

$e(x) = 0 \ \forall x \neq 0$.

ie, $e \to$ degenerate r.v.

(degenerate at $x = 0$).

Eg. $p_1 = Bin(n, p)$

$p_2 = Bin(m, p)$.

Intuitively,

$p_1 \to n$ independent Bernoulli trials with prob$(H) = p$.

$p_2 \to m$ independent Bernoulli trials with prob$(H) = p$.

$\therefore p_1 + p_2 \to (m+n)$ independent Bernoulli trials with prob$(H) = p$.

$\therefore p_1 + p_2 = Bin(m+n, p)$.

Eg. $p_1 = Geo(p)$.

$p_2 = Geo(p)$.

$\Rightarrow p_1 * p_2 = N.B(2, p)$

$\downarrow$

negative binomial.

(no. of trials for 2 H, prob$(H) = p$)

Eg. $p_1 = \text{poi}(\lambda_1)$
$p_2 = \text{poi}(\lambda_2)$

$$\Rightarrow p_1 * p_2(z) = \sum p_1(x) \cdot p_2(z-x).$$

$$= \sum_{x=0}^{z} \frac{e^{-\lambda_1} \cdot \lambda_1^x}{\lfloor x} \cdot \frac{e^{-\lambda_2} \cdot \lambda_2^{z-x}}{\lfloor z-x}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{\lfloor z} \cdot \underbrace{\sum_{x=0}^{z} \binom{z}{x} \lambda_1^x \cdot \lambda_2^{z-x}}_{\text{Binomial expansion of } (\lambda_1+\lambda_2)^z}.$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{\lfloor z} \cdot (\lambda_1+\lambda_2)^z.$$

$$\therefore p_1 * p_2 = \text{Poi}(\lambda_1+\lambda_2).$$

Digression:

Power series around 0

Let $\{a_n\}_{n \geq 0}$ be any real sequence.

For $t \in \mathbb{R}$, $\varphi(t) = \sum_{n=0}^{\infty} a_n t^n$.

Suppose, radius of convergence $r > 0$

Facts: (1) $\varphi(t)$ converges for $|t| < r$, i.e, for $t \in (-r, r)$

(2) For any $0 < s < r$,
$\varphi(t)$ converges uniformly on $[-s, s]$

[Given $\varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ s.t, $\forall t_1, t_2 \in [-s, s]$,
$|\varphi(t_1) - \varphi(t_2)| < \varepsilon \quad \forall n > N_\varepsilon$]

[Given $\varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ . . . . . .
$$\left|\varphi(t_1) - \varphi(t_2)\right| < \varepsilon \quad \forall\ n > N_\varepsilon$$]

3. $\varphi : (-r, r) \longrightarrow \mathbb{R}$ is a continuous function.

φ is differentiable on $(-r, r)$, and

$$\varphi'(t) = \sum_{n=1}^{\infty} n \cdot a_n \cdot t^{n-1}$$

φ' is continuously differentiable.

ie, φ' — cont. on $(-r, r)$.

$\therefore \varphi''$ is continuously diff.

Carrying on: φ is infinitely differentiable.

Note: if $\tilde{\varphi}(t) = \varphi(t) \quad \forall \quad t\,(-r_0, r_0)$ $\left(r_0 = \text{min. of the radius of convergence of } \varphi \,\&\, \tilde{\varphi}. \right)$

$$\sum_{n=0}^{\infty} b_n \cdot t^n \qquad \sum_{n=0}^{\infty} a_n t^n$$

then, $b_n = a_n$.

ie, 2 functions "agreeing" can't give rise to 2 different power series.
The 2 power series must be the same.

Proof: If $\varphi(t) = \sum_{n=0}^{\infty} a_n t^n$ has a +ve radius of convergence, then.

$$a_n = \frac{\varphi^n(0)}{\lfloor n}$$

[$\because$ φ & $\tilde{\varphi}$ "agree" on a non-degenerate open interval $(-r_0, r_0)$ around 0, their n^{th} derivatives must "agree" as well.

Hence, $\varphi = \tilde{\varphi}$ on $(-r_0, r_0)$

$$\Rightarrow \varphi^n(0) = \tilde{\varphi}^n(0)$$

$$\Rightarrow \frac{\varphi^n(0)}{\lfloor n} = \frac{\tilde{\varphi}^n(0)}{\lfloor n}$$

$$\Rightarrow \frac{\varphi^{(n)}(0)}{\lfloor n} \qquad \frac{\ \ }{\lfloor n}$$

$\Rightarrow a_n = b_n.$ ∎

X is a random variable taking values in $\{0, 1, 2, \ldots\}$

<u>Def</u>^{n}: The probability generating function X is defined to be
(P<u>G</u>F)

$$\varphi_X(t) = \sum_{n=0}^{\infty} p_n \cdot t^n, \quad \text{where} \quad p_n = P(X=n)$$

$$|\varphi_X(t)| \leq \sum_{n=0}^{\infty} p_n |t|^n \leq 1 \qquad \text{for } |t| \leq 1$$

- $\varphi_X(\cdot)$ determines the pmf $\{p_n\}$ of X.

Indeed, $p_n = \dfrac{\varphi_X^{n}(0)}{\lfloor n}$.

<u>Note</u>: $\varphi_X(t) = \sum_{n=0}^{\infty} P(X=n) \cdot t^n = E(t^X)$.

<u>Eg</u>. $X \sim Poi(\lambda)$.

$$\varphi_X(t) = \sum_{n=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^n}{\lfloor n} \cdot t^n = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{\lfloor n} = e^{-\lambda} \cdot e^{\lambda t}$$

(Taylor series)

$$\Rightarrow \varphi_X(t) = e^{-\lambda(1-t)} \longrightarrow$$ PGF determines distribution.

(i.e, 2 different distributions cannot have the same PGF.
i.e, PGF is a "characteristic" of any dist^n.)

<u>Eg</u>.

$X \sim Poi(\lambda_1)$

$Y \sim Poi(\lambda_2)$

X, Y – ind.

$$\therefore \varphi_{X+Y}(t) = E(t^{X+Y}) = E(t^X \cdot t^Y)$$

$$= E(t^X)\cdot E(t^Y) \qquad \because X, Y \text{ - ind.}$$

$$= e^{-(\lambda_1+\lambda_2)\cdot(1-t)} \qquad E(t^X\cdot t^Y) = E(t^X)\cdot E(t^Y)$$

$\to$ PGF of $Poi(\lambda_1+\lambda_2)$.

$$\therefore \left.\begin{matrix} X \sim Poi(\lambda_1) \\ Y \sim Poi(\lambda_2) \end{matrix}\right\} \Rightarrow X+Y \sim Poi(\lambda_1+\lambda_2).$$

$\varphi_X(t)$ has radius of convergence ≥ 1

$\varphi_X(t)$ is differentiable of any order on $(-1,1)$.

$$\therefore \varphi_X(t) = \sum_{n=0}^{\infty} p_n\cdot t^n.$$

$$\varphi_X'(t) = \sum_{n=1}^{\infty} n\cdot p_n\cdot t^n, \qquad t\in(-1,1).$$

$\lim_{t\to 1} \varphi_X'(t)$ exists, & $\lim_{t\to 1} \varphi_X'(t) = \sum n p_n = E(X)$.

$$\therefore \varphi_X''(t) = \sum_{n=2}^{\infty} n(n-1)\cdot p_n\cdot t^{n-2}$$

$$\lim_{t\to 1} \varphi_X''(t) = \sum n(n-1)\, p_n = E(X(X-1)).$$

Offspring distribution: $\{p_0, p_1, \dots\} = \pi$

$X_0 \equiv 1 \leftarrow 1^{st}$ parent.

$X_1 = \#$ offspring produced by the ancestor $\sim \pi$

If $X_1 \equiv 0$, then $X_2 \equiv 0$. (extinct).

If $X_1 = k > 0$, then, $X_2 \sim Y_1 + \dots + Y_k$, If $X_1 = k > 0$, then, $X_2 \sim Y_1 + \cdots + Y_k$,

where $Y_1, \ldots, Y_k \sim \pi$.

⋮

If $X_n \equiv 0$, then $X_{n+1} \equiv 0$ (extinct).

If $X_n = k > 0$, then $X_{n+1} = Y_1 + \cdots + Y_k$,

$Y_1, \ldots, Y_k \sim \pi$.

<u>probability of extinction:</u>

$$q = P\left(\bigcup_n \{X_n = 0\} \right)$$

$$= \lim_{n\to\infty} P(X_n = 0).$$

(working with just the distributions seems difficult, let's work with the PGFs.)

$\therefore \varphi_n =$ PGF of X_n.

$$\varphi_1(t) = \varphi(t)$$

$$\varphi_2(t) = E(t^{X_2})$$

$$= \sum_k E(t^{X_2} \mid X_1 = k) \cdot P(X_1 = k).$$

$$= \sum_k (\varphi(t))^k \cdot P(X_1 = k) \leftarrow \text{expression for expectation}$$

$$= \varphi(\varphi(t))$$

$X_1 = k \Rightarrow$

$X_2 = Y_1 + \cdots + Y_k$

$\therefore E(t^{X_2} \mid X_1 = k)$

$= E(t^{Y_1 + \cdots + Y_k} \mid X_1 = k)$

$= \prod_{i=1}^{k} E(t^{Y_i} \mid X_1 = k)$

⋮

$$\varphi_n(t) = E(t^{X_n}) = \underbrace{\varphi(\varphi \ldots (\varphi(t))\ldots)}_{\text{n-fold composition.}}$$

$\therefore$ extinction probability, $q = P\left(\bigcup_n \{X_n = 0\} \right) = \lim_{n\to\infty} P(X_n = 0) = \lim_{n\to\infty} \varphi_n(0)$

$$\varphi_n(t) = \varphi(\varphi_{n-1}(t))$$

$$\varphi_n(t) = \varphi(\varphi_{n-1}(t))$$

$t=0$: $\varphi_n(0) = \varphi(\varphi_{n-1}(0))$

$\therefore \lim_{n\to\infty} \varphi_n(0) = \lim_{n\to\infty} \varphi(\varphi_{n-1}(0))$

$$\Rightarrow \boxed{q = \varphi(q)}$$

Food for thought: can there be other solutions?