Probability-3 Lecture-13

20 September 2024 11:23

Again, full respect to A.N. Kolmogorov

Classical SLLN:

$\{X_n\}$ - sequence of i.i.d r.vs with finite common mean μ.

Then, $\dfrac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s.} \mu$

$$\iff \frac{(X_1-\mu) + (X_2-\mu) + \cdots + (X_n-\mu)}{n} \xrightarrow{a.s.} 0$$

So, we'll prove, if $\{X_n\}$ - sequence of i.i.d r.vs with common mean, $EX_1 = 0$

then, $\dfrac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s.} 0$

Here, $\forall \varepsilon > 0$,

$$\sum_n P\left(\left| \frac{X_1 + \cdots + X_n}{n} \right| > \varepsilon \right) < \infty$$

ie, to show that, this series converges [which is stronger than WLLN!!]

Kronecker's Lemma:

$\rightarrow$ a result on real sequences.

Let $\{x_n\}$ be any real sequence.

Then, if the series $\sum_n \frac{x_n}{n}$ converges, then

$$\frac{1}{n}(x_1 + \cdots + x_n) \longrightarrow 0.$$

Proof:

Let $b_n = \sum_{k=1}^{n} x_k/k$, $n = 1, 2, 3, \dots$

partial sums

$b_0 = 0.$

Hypothesis : b_n converges.

say, $b_n \longrightarrow b.$

$\therefore \forall k \geq 1$, $b_k - b_{k-1} = k^{th}$ term $= \dfrac{x_k}{k}$

$$\Rightarrow x_k = k(b_k - b_{k-1})$$

$$x_1 + \dots + x_n = \sum_{k=1}^{n} k \cdot b_k - \sum_{k=1}^{n} k \cdot b_{k-1}$$

$$= \sum_{k=1}^{n} k b_k - \sum_{k=1}^{n} (k-1) \cdot b_{k-1} - \sum_{k=1}^{n} b_{k-1}$$

change of variable $k-1 \to j$

$$= \sum_{k=1}^{n} k \cdot b_k - \sum_{j=1}^{n-1} j b_j - \sum_{k=1}^{n-1} b_k$$

change of variable $k-1 \to k.$

$$= n b_n - \sum_{k=1}^{n-1} b_k$$

$$\therefore \frac{x_1 + \dots + x_n}{n} = \frac{\cancel{n} b_n}{\cancel{n}} - \left(\frac{1}{n} \sum_{k=1}^{n-1} b_k\right) \longrightarrow 0. \checkmark \quad \blacksquare$$

$b_n \to b$, $\quad \frac{1}{n}\sum_{k=1}^{n-1} b_k \to b.$

back to SLLN...

By Kronekar's lemma, its enough to show, (ie, sufficient) this "random series"

$$\sum_n \frac{X_n}{n} \text{ converges a.s.}$$

Q: So, what should I prove to get that

$$\sum_n \frac{X_n}{n} \text{ converges a.s. ??}$$

Let $\{y_n\}$ be a real sequence.

the series $\sum_n y_n$ converges iff the sequence of partial sums are cauchy.

ie, $\forall j \geq 1$, $\exists n \geq 1$ s.t $\forall$ $m' > m \geq n$,

$$\left| \sum_{k=1}^{m'} y_k - \sum_{k=1}^{m} y_k \right| \leq 1/j$$

$$\Leftrightarrow \left| \sum_{k=m+1}^{m'} y_k \right| \leq 1/j$$

So, back again...

$$Y_n := \frac{X_n}{n}.$$

$\{Y_n\}$ - sequence of real r.vs, then the series

$$\sum_n Y_n \text{ converges a.s.}$$

$$\Leftrightarrow P\left(\bigcap_{j\geq 1} \bigcup_{n \geq 1} \bigcap_{m' > m \geq n} \left\{ \omega : \left| \sum_{k=m+1}^{m'} Y_k(\omega) \right| \leq \frac{1}{j} \right\} \right) = 1$$

take complement

$$\Leftrightarrow P\left(\bigcup_{j\geq 1} \bigcap_{n \geq 1} \bigcup_{m' > m \geq n} \left\{ \omega : \left| \sum_{k=m+1}^{m'} Y_k(\omega) \right| > \frac{1}{j} \right\} \right) = 0$$

$$\Leftrightarrow P\left(\bigcup_{j\geq 1}\bigcap_{n\geq 1}\bigcup_{m'>m\geq n}\left\{\omega : \left|\sum_{k=m+1}^{m'} Y_k(\omega)\right| > \frac{1}{j}\right\}\right) = 0$$

$$\Leftrightarrow P\left(\bigcup_{j\geq 1}\bigcap_{n\geq 1}\left\{\omega : \sup_{m'>m\geq n}\left|\sum_{k=m+1}^{m'} Y_k(\omega)\right| > \frac{1}{j}\right\}\right) = 0.$$

this set is decreasing in n

ie, as $n\uparrow$, this set shrinks.

$\Leftrightarrow \forall j\geq 1,$

$$P\left(\bigcap_{n\geq 1}\left\{\omega : \sup_{m'>m\geq n}\left|\sum_{k=m+1}^{m'} Y_k(\omega)\right| > \frac{1}{j}\right\}\right) = 0$$

decreasing events. $\therefore$ By continuity of probability.

$$\Leftrightarrow \forall j\geq 1,\ P\left(\left\{\omega : \sup_{m'>m\geq n}\left|\sum_{k=m+1}^{m'} Y_k(\omega)\right| > \frac{1}{j}\right\}\right) \to 0 \text{ as } n\to\infty$$

take $m=n$, $m'=m$ $\Downarrow$ $\Uparrow$ Δ-inequality (* Exc: verify this)

$$\Leftrightarrow \forall j\geq 1,\ P\left(\left\{\omega : \sup_{m>n}\left|\sum_{k=n+1}^{m} Y_k(\omega)\right| > \frac{1}{j}\right\}\right) \to 0 \text{ as } n\to\infty.$$

So now, fix $\varepsilon>0$.

then,

$$\sup_{m>n}\left|\sum_{k=n+1}^{m} Y_k\right| = \lim_{m\to\infty}\uparrow \max_{n<l\leq m}\left|\sum_{k=n+1}^{l} Y_k\right|$$

Supremum $\equiv$ (increasing limit of partial maximums)

$$\therefore \sup_{m>n}\left|\sum_{k=n+1}^{m} Y_k\right| > \varepsilon$$

$m > n \quad \Big| \quad \overline{\quad}_{k=n+1} \quad \Big|$

$$\iff \max_{n < l \leq m} \left| \sum_{k=n+1}^{l} Y_k \right| > \varepsilon \quad \text{for some } m > n.$$

$\left[\text{Since } \max_{n<l\leq m} \left\{ \left| \sum_{k=n+1}^{l} Y_k \right| \right\} \text{ is an increasing sequence in } m, \therefore \forall \text{ for all } m' > m, \text{ it holds.} \right]$

Kolmogorov's Maximal Inequality

$\xi_1, \xi_2, \dots, \xi_n$ – independent r.vs with 0 means & finite variances.

$$P\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \xi_i \right| > \varepsilon \right) \leqslant \frac{\operatorname{Var}\left(\sum_{i=1}^{n} \xi_i \right)}{\varepsilon^2}$$

this set $\equiv A$ (say)

(stronger than Chebyshev !!)

$\left[\text{Compare: } P\left(\left| \sum_{i=1}^{n} \xi_i \right| > \varepsilon \right) \leqslant \frac{\operatorname{Var}\left(\sum_{i=1}^{n} \xi_i \right)}{\varepsilon^2} \right]$

But yes, Chebyshev doesn't need independence, but Kolmogorov's Maximal ineq. assumes independence.

Proof: "Think" of this as a random walk.

ξ_i – increment in the i^{th} step.

$$A_k := \left\{ \left|\sum_{i=1}^{k-1} \xi_i\right| \le \varepsilon \quad \forall\, i=1,2,\dots,k-1, \quad \left|\sum_{i=1}^{k} \xi_i\right| > \varepsilon \right\}$$

$\downarrow$ k is the first time the threshold $(-\varepsilon, \varepsilon)$ has been crossed

Clearly, $\bigcup_{i=1}^{n} A_k = A = P\left(\max_{1\le k\le n} \left|\sum_{i=1}^{k} \xi_i\right| > \varepsilon\right)$.

& , $A_1, A_2, \dots, A_n$ – disjoint (trivial)

$$\therefore P(A) = \sum_{k=1}^{n} \underbrace{P(A_k)}_{\overset{\shortparallel}{E(A_k)}} \le \frac{1}{\varepsilon^2}\cdot \sum_{k=1}^{n} E\left(\left|\sum_{i=1}^{k} \xi_i\right|^2 \cdot 1_{A_k} \right)$$

How ?

$\downarrow$ at k, $\left|\sum_{i=1}^{k} \xi_i\right| > \varepsilon$

$\therefore \left|\sum \xi_i\right|^2 > \varepsilon^2$ ✓

Hence, this inequality holds.

$$= \frac{1}{\varepsilon^2}\cdot\left[\sum_{k=1}^{n} E\left(\left|\sum_{i=1}^{n} \xi_i\right|^2 \cdot 1_{A_k} \right) - \sum_{k=1}^{n} E\left(\left|\sum_{i=k+1}^{n} \xi_i\right|^2 \cdot 1_{A_k} \right) - 2\sum_{k=1}^{n} E\left(\left(\sum_{i=1}^{n} \xi_i\right)\cdot\left(\sum_{i=1}^{k} \xi_i\right)\cdot 1_{A_k} \right) \right.$$

$\left[\because a^2 = b^2 - (b-a)^2 - 2a(b-a) \right.$

$\left[\because a^2 = b^2 - (b-a)^2 - 2a(b-a) \right.$

Here, $\left. \begin{aligned} a &= \sum_{i=1}^{k} \xi_i \\ b &= \sum_{i=1}^{n} \xi_i \end{aligned} \right]$

Now, last term: (ie, 3^{rd} term)

$$-2\sum_{k=1}^{n} E\left(\underbrace{\left(\sum_{i=k+1}^{n} \xi_i\right)}_{\text{depends on rest of the } \xi_i\text{'s}} \cdot \underbrace{\left(\sum_{i=1}^{k} \xi_i\right) \cdot 1_{A_k}}_{\text{depends on } \xi_1, \dots, \xi_k} \right)$$

Independent

$$= -2\sum \cdot \underbrace{E\left(\sum_{i=k+1}^{n} \xi_i\right)}_{\to 0 \text{ [as per hypothesis.]}} \cdot E\left(\quad \cdot \quad \right)$$

&, 2^{nd} term ≥ 0. So, removing (-2^{nd} term) preserves the inequality.

$$\therefore P(A) \leq \frac{1}{\varepsilon^2} \cdot \sum_{k=1}^{n} E\left(\left| \sum_{i=1}^{n} \xi_i \right|^2 \cdot 1_{A_k} \right)$$

$$\leq \frac{1}{\varepsilon^2} \cdot E\left(\left| \sum_{i=1}^{n} \xi_i \right|^2 \cdot 1_A \right) \quad \left[\because A = \bigcup_{k=1}^{n} A_k \right]$$

$$\leq \frac{1}{\varepsilon^2} \cdot Var\left(\sum_{i=1}^{n} \xi_i \right).$$

[$\because$ all cross terms $= 0$

"Fundamental law of Statistics"]

law of Statistics"

... Again, back to SLLN:

Have to show: $\forall \varepsilon > 0, \; P\left(\sup_{m > n} \left| \sum_{k=n+1}^{m} \frac{X_k}{k} \right| > \varepsilon \right) \longrightarrow 0.$

from Kolmogorov's maximal inequality,

$$P\left(\max_{n \le l \le m} \left| \sum_{k=n+1}^{l} \frac{X_k}{k} \right| > \varepsilon \right) \le \frac{1}{\varepsilon^2} \cdot \operatorname{Var}\left(\sum_{k=n+1}^{m} \frac{X_k}{k} \right)$$

$$= \frac{1}{\varepsilon^2} \cdot \sum_{k=n+1}^{m} \frac{1}{k^2} \cdot \operatorname{Var}(X_k)$$

bound by the entire series further

$$\le \frac{1}{\varepsilon^2} \cdot \sum_{k=n+1}^{\infty} \frac{1}{k^2} \cdot \operatorname{Var}(X_k)$$

to say, this $\downarrow 0$,

Goal: (next week) only thing that remains to show is that,

if $X_1, X_2, \ldots$ iid with 0 common mean,

then $\displaystyle\sum_{k=1}^{\infty} \frac{\operatorname{Var}(X_k)}{k^2} < \infty$.

(ie, only then, the tail

$$\sum_{k=n+1}^{\infty} \frac{\operatorname{Var}(X_k)}{k^2} \to 0,$$

and that's precisely
what we need.)