Covariance Structures: Introduction

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1 Gauss Markov setup

We are aware that the Gauss Markov setup is frequently utilised to model our data. The setup is given below:

$$\vec{y} = X\vec{\beta} + \vec{\epsilon} \quad where \quad \vec{\epsilon} \sim N_n(\vec{0}, \sigma^2 I)$$

It conditions the error term of the model to have a distribution whose mean is $\vec{0}$ and variance equal to σ^2I, i.e. homoscedastic errors. Generally, we consider the distribution to be Gaussian even though it's not mentioned explicitly. However, in many instances it is not enough to take the variance of the Gaussian distribution simply as σ^2I because not all data have homoscedastic errors.

2 Covariance Structures

To fix this issue, we take $\Sigma(\vec{\theta})$ to be the general form for the covariance matrix where Σ is a known function and $\vec{\theta}$ is a vector of all the free parameters of the covariance matrix. In the previous cases, $\vec{\theta}$ was a single parameter equal to σ^2. However, one thing we need to make sure is that the $\Sigma(\vec{\theta})$ must be a positive definite matrix. This surely leads to many complications on $\vec{\theta}$ as the parameters can now only take a restricted set of values. The different structures that the covariance matrix $\Sigma(\vec{\theta})$ takes are called covariance structures.

3 Types of Covariance Structures

1. **Scalar Structure:** The first covariance structure is a common form in the Gauss-Markov setup which is a scalar matrix with all the diagonal entries equal to σ^2.

$$\sigma^2 \begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & 1 \end{bmatrix}$$

2. **Compound Symmetry Structure:** The next form is called the compound symmetry structure. The correlation matrix has all the diagonal entries equal to 1 and every other entry equal to ρ. The covariance matrix is obtained by multiplying the correlation matrix by σ^2. This is typically used when all the correlations are equal.

$$\sigma^2 \begin{bmatrix} 1 & & & & \\ & 1 & & \rho & \\ & & \ddots & & \\ & \rho & & 1 & \\ & & & & 1 \end{bmatrix}$$

We also need to keep in mind that ρ can only take a restricted set of values as the matrix must be positive definite for it to be a covariance structure. To calculate the values ρ can take, we can check if all the eigen values are positive as that is a necessary and sufficient condition for symmetric matrices to be positive definite.

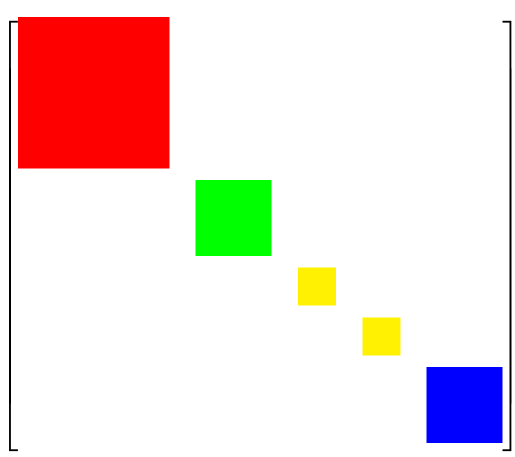
3. **Auto-regressive Structure:** The third covariance structure is called AR(1) or auto-regressive form of covariance matrix. All the diagonal elements of the correlation matrix are 1 and the super diagonal elements are ρ and the adjacent diagonal's elements are ρ^2 and so on.

$$\sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix}$$

   It is called AR(1) structure because it is used in time series analysis which we shall not further dwell upon. There is also a corresponding restriction on the set of values that ρ can take.

4. **Unstructured Matrix:** Another structure that is used is called the unstructured covariance matrix which is basically any positive definite matrix. All the entries are components of the theta parameters with just a condition that the matrix has to be positive definite.

5. **Block Diagonal Structure:** The final covariance structure is basically a combination of all the previous structures. It is a block diagonal matrix where the different blocks can take different covariance structures and the off-diagonal elements are 0. A simple example is shown below:



   In this case, we can take the red square to be a AR(1) structure, green to be a scalar structure, blue to be an unstructured matrix and the two yellow squares to be compound symmetry structures.