Gauss Markov Model: UMVUE of σ^2

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Now, in case of estimating β, we got something which was both unbiased and best in some sense (BLUE). Can we make such a statement about σ^2 which is best in some sense ? Does this have the smallest possible variance ? Well we can not really talk about that because it is already a quadratic expression in y. So, if we want to talk about its variance , we will run into fourth order moments of y and our Gauss-Markov Model has assumptions only up to the second order moments, from which we can not make any assumption to the fourth order moment , they may not even exist. As a result , just based on the Gauss-Markov assumptions we can not claim any optimal result in this case for $\hat{\sigma}^2$, other than the fact that it is very natural we are looking at the norm square of the error and σ^2 basically controls the amount of error , so we have taken norm square of that and divide it by a suitable factor and this should be a natural thing . However, if we put some further assumptions, it is possible to show optimality and this is not very difficult. If, we assume further that the distribution of ϵ is also Gaussian, we already have mean 0 and variance σ_i^2. Now we also put the assumption that it is gaussian, which completely specifies the distribution you have moments of all orders and etc. In that case we can have a much stronger result , which says same $\hat{\sigma}^2$ will also be UMVUE(Uniformly Minimum Variance Unbiased Estimator) . The fact that it is an unbiased estimator is something which is already proved and we don't need the gaussianity assumption for that. But, under gaussianity assumption , we might show , we don't want to go in to that . It is much straight forward from a result that we will learn that says for an exponential distribution under certain assumptions - interior of parameter space no being empty the natural sufficient statistics also happens to be complete sufficient. So, from that we can see that the $\hat{\sigma}^2$ expression that we have given is a function of complete sufficient statistics and then we know it must be the UMVUE. Interestingly under the same gaussian assumption the BLUE we showed for $\hat{\beta}$ if X^TX is non-singular or those $c^T\hat{\beta}$ things if X^TX is possibly singular. So, all the estimable parametric functions , the least square estimator that we saw $c^T\hat{\beta}$, they are also UMVUE. So, we are making a stronger statement. Earlier we have said look at the class of all linear unbiased estimators they have the minimum variance among that clusters . Now, we are extending the class , which is not necessarily linearly unbiased but just unbiased. We will now allow even non-linear estimators . Even then this will have the least possible variance. So, we are not removing the unbiasedness condition, we are just removing the linearity condition. Again the same proof. we will see that x^Ty will be a complete sufficient statistics for the $c^T\beta$ vector and from that we will use the Rao-Blackwell theorem and etc. thing from parametric inference course. Often the error distribution is not gaussian and even if it is gaussian it nearly says the the least square estimators is BLUE is not already a good property , it has yet another good property .