Probability-3 Lecture-15

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Lemma:

Let $F_n, n \geq 1$ and F be cdf on $\mathbb{R}$

Assume, on a dense set $D \subset \mathbb{R}$

$$F_n(x) \longrightarrow F(x) \quad \forall x \in D, \ \&,$$

$\forall \ x \in J(F) =$ set of all discontinuities of F.

$$F_n(x) - F_n(x^-) \longrightarrow F(x) - F(x^-)$$

Then, $F_n \longrightarrow F$ on $\mathbb{R}$ uniformly, i.e,

$$\sup_{x \in \mathbb{R}} \left| F_n(x) - F(x) \right| \longrightarrow 0.$$

Definitions: $(\Omega, \mathcal{A}, P)$ — probability space (fixed).

$\mathcal{G}_1$ & $\mathcal{G}_2$ be sub-σ fields of $\mathcal{A}$ are said to be independent if

$$P(G_1 \cap G_2) = P(G_1) \cdot P(G_2)$$

$$\forall \ G_1 \in \mathcal{G}_1, \ G_2 \in \mathcal{G}_2$$

* In practice, proving independence through this might be difficult.

Result:

$\mathcal{G}_2$ - smallest σ- field gen. by $\mathcal{S}_2$

If $\mathcal{S}_1$ & $\mathcal{S}_2$ are semifields s.t,

$$\sigma(\mathcal{S}_1) = \mathcal{G}_1, \quad \sigma(\mathcal{S}_2) = \mathcal{G}_2,$$

and if $P(S_1 \cap S_2) = P(S_1) \cdot P(S_2) \quad \forall \ S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2$

then, $\mathcal{G}_1$ & $\mathcal{G}_2$ are independent.

Definition:

Let Λ- indexing set.

Let $\{\mathcal{G}_\alpha, \alpha \in \Lambda\}$ be a family of sub-σ fields of $\mathcal{A}$.

sub-σ fields of $\mathcal{A}$.

Then, $\{\mathcal{G}_\alpha, \alpha \in \Lambda\}$ are said to be mutually independent if for any choice of $\alpha_1, \ldots, \alpha_n \in \Lambda$,

$$P(G_1 \cap G_2 \cap \ldots \cap G_n) = \prod_{i=1}^{n} P(G_i) \quad \forall$$

$$G_1 \in \mathcal{G}_{\alpha_1}, \ldots, G_n \in \mathcal{G}_{\alpha_n}.$$

Result:

If for each $\alpha \in A$, $\mathcal{S}_\alpha$ is a semi-field s.t. $\sigma(\mathcal{S}_\alpha) = \mathcal{G}_\alpha$.

then

$$P\left(\bigcap_{i=1}^{n} S_i\right) = \prod_{i=1}^{n} P(S_i) \quad \text{for all choices of } \alpha_1, \ldots, \alpha_n \in \Lambda.$$

is sufficient for $\{\mathcal{G}_\alpha, \alpha \in \Lambda\}$, and all choices of $S_i \in \mathcal{S}_{\alpha_i}$ to be independent. $\forall i = 1, \ldots, n$.

$(\Omega, \mathcal{A}, P)$.

Given a family $\{X_\alpha, \alpha \in \Lambda\}$ of r.vs, the smallest σ-field on Ω w.r.t. which all $X_\alpha, \alpha \in \Lambda$ are measurable, is called the σ-field generated by $\{X_\alpha, \alpha \in A\}$, denoted by

$$\sigma\left(\{X_\alpha, \alpha \in A\}\right).$$

$\mathcal{C} = \{S \subset \Omega : S = \bigcap^{n} X^{-1}(B \ldots) \ldots \alpha \ldots \in \Lambda\}$

$$\mathcal{S} = \left\{ S \subset \Omega : S = \bigcap_{i=1}^{n} X_{\alpha_i}^{-1}(B_i), \; \begin{matrix} \alpha_1, \dots, \alpha_n \in \Lambda \\ B_1, \dots, B_n \in \mathcal{B} \end{matrix} \right\}$$

$$\sigma(\mathcal{S}) = \sigma(\{ X_\alpha, \alpha \in A \}).$$

KOLMOGOROV'S 0-1 LAW

Setup: Let $\{X_n, n \geq 1\}$ - sequence of independent r.vs

ie, $\{\sigma(X_n), n \geq 1\}$ - is an independent sequence of σ- fields.

For each $n \geq 1$, define

$$\mathcal{A}_n = \sigma(X_1, \dots, X_n)$$

these are $\uparrow$ in n.

Any event determined by the first n random variables.

check: $\bigcup_n \mathcal{A}_n$ is a field.

($\because$ increasing union of σ- fields is a field.)

$\therefore$ the σ- field generated by this,

$$\sigma\left(\bigcup_n \mathcal{A}_n\right) = \mathcal{A}_\infty.$$

Check: this is the smallest σ- field wrt which all the X_n's are measurable.

Now, take $\mathcal{T}_n := \sigma(X_{n+1}, X_{n+2}, \dots)$

$\hookrightarrow$ any event that depends only on the tail.

Note: $\mathcal{T}_n$ decreases $\downarrow$ with n.

Note: $\mathcal{T}_n$ decreases $\downarrow$ with n.

[Eg. of an event that belong to $\mathcal{T}_n$: $\{\omega : X_n(\omega) = 0$ for infinitely many $n\}$]

ie, $X_n = 0$ a.s.

Also, <u>note</u>: $\sum X_n \geq 0$ is <u>not</u> a tail event.

$\mathcal{T}$ = the "tail" σ-field.

Any set $A \in \mathcal{T}$ is called a <u>"tail event"</u>.

Any r.v X measurable w.r.t $\mathcal{T}$ is a tail r.v.

K's 0-1 law:

If $\{X_n\}$ - independent seq. of r.vs, then for every tail event A, $P(A)$ is either 0 or 1.

Proof: <u>Step 1</u>: for every $n \geq 1$, $\mathcal{A}_n$ - independent of $\mathcal{T}_n$. [Exercise]

$\Downarrow$

<u>Step 2</u>: $\mathcal{A}_n$ is independent of $\mathcal{T}$ $\forall n$.

$\Downarrow$

<u>Step 3</u>: $\sigma\left(\bigcup_n \mathcal{A}_n\right)$ independent of $\mathcal{T}$.

$\Downarrow$

<u>Step 4</u>: $\mathcal{T}$ is independent of $\mathcal{T}$.

$\therefore$ any $A \in \mathcal{T}$.

$\therefore P(A \cap A) = P(A) \cdot P(A)$

$\mathcal{T} = \bigcap \mathcal{T}_n$

$\mathcal{J} := \bigcap_n \mathcal{J}_n$

&, any $\mathcal{J}_n$ is a sub-σ field of $\sigma(\bigcup_n a_n)$

$\therefore \bigcap_n \mathcal{J}_n = \mathcal{J} \subseteq \sigma(\bigcup_n a_n)$.

$\therefore P(A \cap A) = P(A) \cdot P(A)$

$\Rightarrow P(A) = (P(A))^2$

$\Rightarrow P(A) = 0$ or 1 ▨

* X - tail r.v. then $P(X \leq c) =$ either 0 or 1.

ie, X - degenerate r.v.

Jessen - Wintner.

Suppose $\{X_n\}$ is an independent seq. of r.vs, each of which is discrete. s.t.

$\sum_n X_n$ converges a.s.

Then, the limit r.v. X is of "pure" type.

ie, either • X is discrete,

or • X is continuous (ie, distⁿ fⁿ continuous) ie, no point mass, but supported by a set of measure 0.

or • X is absolutely continuous (ie, has a density fⁿ.)

Proof:

Let $D_n, n \geq 1$ be the countable set of possible values (ie, support) of X_n, & let $D = \bigcup D_n$

& let $D = \bigcup_n D_n$

Let G be a Subgroup (!!!) of $\mathbb{R}$, generated by D.

$$G = \left\{ g : g = \sum_{i=1}^{n} k_i x_i \ : \ \begin{array}{l} x_1, \dots, x_n \in D. \\ k_1, \dots, k_n \in \mathbb{Z}. \end{array} \right\}$$

G - countable

Observe: for any Borel Set B, the set $\{X \in B + G\}$ is a tail set. (How !?)

$\sum X_n \in B + G.$

$\iff \sum X_n - b \in G$ for some $b \in B.$

$\iff \sum_{i=1}^{n} X_i + \sum_{i=n+1}^{\infty} X_i - b \in G.$

$\left[\begin{array}{l} \text{take } k_i = 1 \\ g = \sum x_i \in G \end{array} \right]$

$\iff \sum_{i=n+1}^{\infty} X_i - b \in G.$ ✓

$\Rightarrow \sum_{i=n+1}^{\infty} X_i \in B + G.$

$\therefore \sum_{i=n+1}^{\infty} X_i$ - tail r.v. ✓.

$\therefore \forall$ borel set B,

$P(X \in B + G) = 0$ or 1

[By K's 0-1 law]

Case 1: $\exists$ a countable set B,

<u>Case 1</u>: $\exists$ a countable set B,
s.t $P(X \in B + G) = 1$.
$\therefore X$ - discrete. ✓

<u>Case 2</u>: Otherwise,
$P(X \in B + G) = 0$ for every countable B.

$\therefore$ take $B = \{x\}$.

$\therefore P(X = x) \le P(X \in \{x\} + G)$

$= P(X \in \{x\})$ $\because 0 \in G$.

$= P(B)$

$= 0$ (assumed)

$\Rightarrow X$ - continuous.

<u>Case 2a</u>: $\exists$ a Borel set B of $\text{leb}(B) = 0$

leb $\rightarrow$ lebesgue measure.

s.t, $P(X \in B + G) = 1$.

$$\text{leb}(B + G) = \text{leb}\left(\bigcup_{g \in G}(B + g)\right)$$

$$\le \sum_{g \in G} \text{leb}(B + g)$$

each $= 0$.

$= 0$. $\left[\because G \text{ - ctbl}, \therefore \sum_{g \in G} \text{ is a countable sum.}\right]$

<u>Case 2b</u>: for every borel set B with $\text{leb}(B) = 0$.

$P(X \in B + G) = 0$.

$$P(X \in B + G) = 0.$$

? $\Rightarrow P(X \in B) = 0,$

? $\Rightarrow X$ - absolutely continuous.