Probability-3 Lecture-23

05 November 2024 14:18

Set-3 (contd...)

4.(d) $P(\sup S_n = \infty, \ \inf S_n = -\infty) = 1.$

$\exists \varepsilon > 0, \ P(X_n > \varepsilon \ \text{i.o}, \ X_n < -\varepsilon \ \text{i.o}) = 1.$

Special Case:

$X_n = \pm$ with prob $= 1/2$ (S.S.R.W)

fix an integer $j \geq 1$ $\qquad S_0 = 0$

$\theta_j = P(\sup_n S_n \geq j) \qquad \theta_0 = 1$

$$\theta_j = \frac{1}{2}\theta_{j-1} + \frac{1}{2}\theta_{j+1}.$$

$$\theta_j - \theta_{j-1} = d$$

$$\therefore \theta_j = \theta_0 + j \cdot d$$

Here, $\theta_j \to$ a prob.

& $0 \leq \theta_j \leq 1 \ \forall j$

$\therefore \Rightarrow d = 0$ (forced)

$\Rightarrow \theta_j = 1 \ \forall j$

$\therefore \Rightarrow P\left(\left\{\bigcap_j \sup_n S_n \geq j\right\}\right) = 1$

$\Rightarrow P(\sup_n S_n = \infty) = 1.$

Similarly, $P(\inf_n S_n = -\infty) = 1$

Similarly, $P\left(\inf_n S_n = -\infty\right) = 1$

$\blacksquare$

General case:

$\exists\, \varepsilon > 0$ s.t. $P(X_n > \varepsilon) = \delta > 0$. $\longrightarrow$ ie, X_n <u>not</u> degenerate at 0.

take any $k \geq 1$.

$$P(X_1 > \varepsilon, X_2 > \varepsilon, \ldots, X_n > \varepsilon) = \delta^k > 0$$

$[\because X_i \text{ - iid}]$

Define: $A_1 = (X_1 > \varepsilon, \ldots, X_k > \varepsilon)$

$A_2 = (X_{k+1} > \varepsilon, \ldots, X_{2k} > \varepsilon)$

$A_3 = (X_{2k+1} > \varepsilon, \ldots, X_{3k} > \varepsilon)$

$\vdots$

$\rightarrow$ Breaking into blocks, each of k

$\therefore A_n$'s - independent.

$\therefore P(A_n) = \delta^k > 0 \quad \therefore \sum P(A_n) = \infty$

$\therefore$ By B.C-II:

$$P(\underbrace{A_n \text{ happens i.o}}_{C_k}) = 1.$$

ie, for every k, we can get a k-long run i.o s.t every term within that is $> \varepsilon$.

fix $M > 0$.

<u>Claim</u>: $\underbrace{\{|S_n| \leq M \ \forall\, n\} \cap C_k}_{} = \phi$ if $k > \frac{2M}{\varepsilon}$

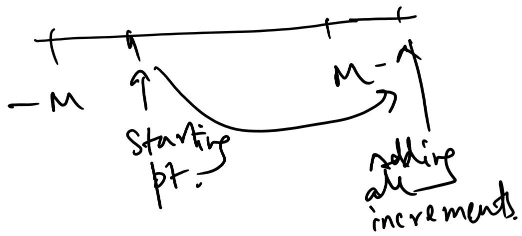
Why? take $\omega \in C$

Why? take $\omega \in$

i.e., $\exists$ a k-long run s.t. every $X_i > \varepsilon$ in that run. So, if $S_n < -M$, done.

if $S_n > -M$, & $k > 2M/\varepsilon$, $S_n > M$!! done.

$\varepsilon \times 2M/\varepsilon = 2M$



$$\therefore P(|S_n| \leq M) = 0.$$

$$\therefore P(|S_n| = \infty) = 1$$

$$\Rightarrow P(\sup S_n = \infty \text{ or } \inf S_n = -\infty) = 1$$

12. $X_n = 2$ or n^α each with $p = \theta_n$.

$X_n = \theta_n$ with prob. $= 1 - 2\theta_n$.

$P(\sum X_n \text{ conv}) = 1$ iff $\sum \theta_n < \infty$.

$$\Rightarrow \sum P(X_n \neq \theta_n) < \infty.$$

Set - 4

1.(b) Note: (ii) & (iii) are just complements.

$\therefore$ (ii) $\Leftrightarrow$ (iii).

to show: (i) $\Leftrightarrow$ (ii).

"$\Leftarrow$" Assume, $\liminf P(X_n \in V) \geq P(X \in V)$ $\forall$ V open.

take $V = (-\infty, a)$

take $V = (-\infty, a)$

$$F_n(a^-) = \lim P(X_n \in (-\infty, a)) \geq P(X \in (-\infty, a)) = F(a^-).$$

&, take $V = (a, \infty)$.

Using $\underrightarrow{\liminf} (1 - F_n(a)) \geq 1 - F(a)$

$$\Rightarrow \cancel{1} - \limsup F_n(a) \geq \cancel{1} - F(a)$$

$$\Rightarrow \limsup F_n(a) \leq F(a).$$

Now, refer to part 1.(a)

$$X_n \xrightarrow{d} X.$$

($\Rightarrow$) Assume, $X_n \xrightarrow{d} X$.

Use part 1. (a) to show that,

$$\lim P(X_n \in I) \geq P(X \in I) \quad \forall \text{ open interval } I.$$

Let V - open set.

Write $V = \bigcup_n I_n$

$\uparrow$ disjoint open intervals.

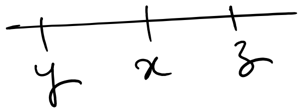
Given $\varepsilon > 0$, $\exists M > 0$ s.t.

$$P(X \in V) \leq \sum_{j=1}^{M} P(X \in I_j)$$

$$\liminf P(X_n \in V) \geq \liminf \sum_{j=1}^{M} P(X_n \in I_j)$$

$$\geq \sum_{j=1}^{M} P(X \in I_j) \geq P(X \in V) - \varepsilon$$

2. Idea: $x \in C(F)$. i.e. x is a fixed continuity pt.



Find $y < x < z$, $y, z \in C(F)$

s.t. $F(x) - F(y) < \varepsilon$.

$F(z) - F(x) < \varepsilon$.

(we can always do so, $\because C(F)$ is dense)

Now, $X_n \longrightarrow X$.

$\therefore$ after a large n, all x_n's are "trapped" within (y, z).

$\therefore \limsup_n F_n(x_n) \leq F_n(z)$

$\liminf_n F_n(x_n) \geq F_n(y)$

3. direct definition, & computation.

4. Let D be the countable set.

$P(X \in D) = 1$.

for any $\varepsilon > 1$.

$\exists$ a finite set $F \subseteq D$.

s.t. $P(X \notin F) < \varepsilon$.

$P(X \in V) > P(X_n \in V \cap F)$.

$$P(X \in V) > P(X_n \in V \cap F).$$

$$\downarrow$$

$$P(X \in V \cap F). > P(X \in V) - \varepsilon$$

$$\uparrow$$

By our choice of F.

$$\therefore \liminf P(X_n \in V \cap F) = P(X \in V \cap F).$$

5. (a) "local limit theorem"

6. (a) showing Δ - inequality:

F, G, H - 3 cdfs.

Suppose $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ are such that,

$$G(x-\varepsilon_1) - \varepsilon_1 < F(x) < G(x+\varepsilon_1) + \varepsilon_1 \quad \forall x \in \mathbb{R}.$$

and, $$H(y-\varepsilon_2) - \varepsilon_2 < G(y) < H(y+\varepsilon_2) + \varepsilon_2 \quad \forall y \in \mathbb{R}.$$

(should imply)

$$\Rightarrow \rho(F, H) \leq \varepsilon_1 + \varepsilon_2.$$

7. for any cdf F,

define $a_y = \sup\{x : F(x) < y\}$, $0 < y < 1$.

$\rightarrow$ left half

$$b_y = \inf\{x : \underbrace{F(x) > y}_{\text{!!}\ G(y)}\}.$$

Here, $\forall y$, $a_y \geqslant b_y$ [think & check]

& $y_1 < y_2 \Rightarrow b_{y_1} \leq a_{y_2}$

if $a_y = b_y$, $(a_y, b_y) = \phi$.

ie collect those $0 < y < 1$ s.t,

$(a_y, b_n) \neq \phi$,

ie, $a_y < b_y$

ie, $\{y : (a_y, b_y) \neq \phi\} =: D$ (say)

this is countable (think) $\therefore P(Y \in D) = 0$

Now, show: for $y \notin D$,

$G_n(y) \longrightarrow G(y)$

Step-1. $\lim G_n(y) \geqslant G(y) - \varepsilon$.

Step.-2. $\lim G_n(y) \leq G(y) + \varepsilon$.

for step 1: pick $x \in C(F)$

$G(y) > x$.

$G(y) > x.$

& then, show, $\lim G_n(y) \geq x.$

→ Not of use for exam.

9.(b) (converse)

$Y_n := e^{-X_n}, \quad Y = e^{-X}.$ → takes values in

All moments bounded.

Set-5

7. $1 - \frac{\lambda}{n} + \frac{\lambda}{n}\varphi(t)$ → convex. combination of C.F ∴ C.F ✓

↑ a C.F (a r.v degenerate at 0)

11. (b). $X_1, X_2, \ldots\ldots = 1$ w.p $\left(\frac{1}{2}\right)$

$\varphi_n(t) =$ Char. f^n $\sum_{k=1}^{n} X_k/2^n$

$\xi_k = \frac{X_k + 1}{2}$. $\quad \sum_{k=1}^{n} \frac{\xi_k}{2^k} = \frac{1}{2} \sum_{k=1}^{n} \frac{(X_k+1)}{2^k}.$

(first n erms of binary expansion)

$\sum_{k=1}^{n} \frac{\xi_k}{2^k} \xrightarrow{a.s.} \text{Unif}(0,1).$

$\frac{1}{2} \sum_{k=1}^{n} \frac{(X_k+1)}{2^k} \xrightarrow{a.s} \text{Unif}(-1,1)$

13-(c)

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} \cdot h(x)\, dx$$

Suppose, $\varphi(t)$ is a non-ve seq, real-valued integrable s.t, $\exists\, c > 0$