Probability-2 Lecture-17

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- X – non-ve with density f, then $E(X) = \int_0^x x f(x)\,dx$

- X - real valued random variable with density. f

$$E(X) \text{ exists} \iff \int_0^\infty x f(x)\,dx < \infty \quad \text{or} \quad \underbrace{\int_0^\infty x f(-x)\,dx < \infty}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow \ \left(\text{change of variable } x \to -x\right)$$

$$\left(E(X^+) \ \& \ E(X^-) \text{ exist.}\right) \qquad\qquad \int_{-\infty}^0 x f(x)\,dx > -\infty$$

& in that case,

$$-\infty < E(X) = \int_{-\infty}^\infty x f(x)\,dx < \infty .$$

We show, $E(X^+) = \int_0^\infty x \cdot f(x)\,dx$ —— $(*)$

Suppose done:

Note that, $X^- = (-X)^+$

Also, $Y = -X$ has density $f(-y)$

$$\Rightarrow E(X^-) = \int_0^\infty x f(-x)\,dx .$$

Fix $M > 0$.

define $Y_M := \begin{cases} X, & X \leq M \quad \text{or} \quad X \leq 0 . \\ 0, & X > M \end{cases}$

Clearly, $E(Y_M) \longrightarrow E(X^+)$ as $M \to \infty$.

$$= \lim_{M \to \infty} \int_0^M x \cdot f(x)\,dx .$$

$$= \int_0^\infty x f(x)\,dx .$$

ie, enough to show:

$$\boxed{E(Y_M) = \int_0^M x f(x)\,dx}$$

$\therefore$ Fix $M > 0$.

Define $\{Y_{M,n}\}_{n \geq 1}$ as sequence of real-valued simple random variables converging to Y_M.

$$Y_{M,n} := \sum_{0 \leq k \leq M \cdot 2^n - 1} \frac{k}{2^n} \cdot \mathbb{1}_{\frac{k}{2^n} < x < \frac{k+1}{2^n}}$$

Clearly, $Y_{M,n} \nearrow Y_M$.

$$\therefore E(Y_{M,n}) = \sum_{0 \leq k \leq M \cdot 2^n - 1} \frac{k}{2^n} \cdot \int_{k/2^n}^{(k+1)/2^n} f(x)\, dx .$$

$\therefore$ By MCT,

$$\lim_{n \to \infty} E(Y_{M,n}) = E(Y_M)$$

$$= \lim_{n \to \infty} \sum_{0 \leq k \leq M \cdot 2^n - 1} \frac{k}{2^n} \cdot \int_{k/2^n}^{(k+1)/2^n} f(x)\, dx .$$

$$= \int_0^M x \cdot f(x)\, dx .$$

X is a r.v. with density f.

$h : \mathbb{R} \to \mathbb{R}$ measureable function.

$\therefore h(X)$ is a real random variable.

Theorem: (Law Of The Unconcious Statistician).

$$\boxed{E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f(x)\, dx}$$

Case - I: h is non-negative.

$$E(h(X)) = E\left(\int_0^{h(X)} dy\right)$$

$\because a = \int_0^a da$

$$= E\left(\int_0^{\infty} \mathbb{1}_{(y \le h(X))} \cdot dy\right)$$

hence, we make the limits of integration free of $h(X)$.

$\because \mathbb{1}_{y \le h(X)} = \begin{cases} 1, & y \le h(X) \\ 0, & y > h(X) \end{cases}$

"Fubini's Theorem" (Expectation & "$\int$" can be swapped iff the integrand is non-ve.)

$$= \int_0^{\infty} E\left(\mathbb{1}_{y < h(X)}\right) \cdot dy$$

$[\because E(\mathbb{1}_A) = P(A)]$

$$\boxed{E(h(X)) = \int_0^{\infty} P(h(X) \geqslant y)\, dy}$$

an intermediate result which is itself important.

Now, X has density f.

$$\therefore P(X \in B) = \int_B f(x)\, dx$$

$$= \int_0^{\infty} \left\{ \int_{x : h(x) \geqslant y} f(x)\, dx \right\} \cdot dy$$

$$= \int_0^{\infty} \int_{-\infty}^{\infty} \mathbb{1}_{\{h(x) \geqslant y\}} \cdot f(x) \cdot dx \cdot dy$$

$$= \int_{-\infty}^{\infty} \left(\int_0^{\infty} \mathbb{1}_{\{y \le h(x)\}} \cdot dy \right) f(x) \cdot dx$$

$\left[\because \int^{\infty} \mathbb{1}_{y \le h(x)}\, dy\right]$

$$\int_{-\infty}\left(\int_0 \mathbb{1}_{\{y \le h(x)\}}\right)$$

$$\boxed{E(h(X)) = \int_{-\infty}^{\infty} h(x)\cdot f(x)\,dx}$$

$$\left[\because \int_0 \mathbb{1}_{y \le h(x)}\, dy = h(x)\right]$$

$$\therefore \quad E(h^+(X)) = \int_{-\infty}^{\infty} h^+(x)\cdot f(x)\,dx \qquad * \; h^+(X) = (h(X))^+$$

$$E(h^-(X)) = \int_{-\infty}^{\infty} h^-(x)\cdot f(x)\,dx$$

Definition:

For a real random variable X, and $p \geq 1$, we say that X has finite p^{th} moment if $E|X|^p < \infty$, and, in that case, $E(X^p)$ is called the p^{th} moment of X.

We know: If X has finite p^{th} moment for some $p > 1$, then X has finite p'^{th} moment for $p' < p$.

If X has finite second moment, then

$$\boxed{\begin{array}{c}\text{Variance of } X \text{ is defined as} \\ V(X) := E(X - E(X))^2\end{array}}$$

$$\Rightarrow V(X) = E(X^2) - (E(X))^2$$

Fact: X has finite p^{th} moment $\Leftrightarrow$ $X + c$ has finite p^{th} moment for $c \in \mathbb{R}$.

why? - $\|X+c\|_p \le \|X\|_p + |c|$

why? $-\ \|X+c\|_p \le \|X\|_p + |c|$

$\downarrow$ bounded.

ie, for every bounded random variable, all of its moments are bounded.

Example:

① $X \sim U(a,b)$.

$$E(X) = \frac{a+b}{2}$$

$$\&,\ V(X) = E(X^2) - (E(X))^2$$

$$= \int_a^b x^2 \cdot f(x)\,dx - \left(\frac{a+b}{2}\right)^2$$

$$= \int_a^b \frac{x^2\,dx}{b-a} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{(a-b)^2}{12}$$

[* Fact: Suppose X has density $f(x)$ which is an even function.

Then, Either $\int_0^\infty x f(x)\,dx < \infty$ in which case $E(X) = 0$.

Or, $\int_0^x x f(x)\,dx = \infty$, in which case, $E(X)$ does not exist.]

② $X \sim N(0,1)$. $\quad f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$

$$\therefore E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = 0.$$

$\therefore E(X) = \int_{-\infty} x f(x)\,dx = 0.$

$V(X) = E(X^2) - (E(X))^2$

$\searrow E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$

$= 2 \times \int_0^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}\,dx.$

$\frac{x^2}{2} = t$

$x\,dx = dt$

$dx = \frac{dt}{\sqrt{2t}}$

$= \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} 2t \cdot e^{-t} \frac{dt}{\sqrt{2t}}$

$= \frac{2}{\pi} \times \int_0^{\infty} t^{1/2} \cdot e^{-t}\,dt \rightarrow \alpha - 1 = 1/2,\ \alpha = 3/2$

$= \frac{2}{\pi} \times \Gamma(3/2) = \frac{2}{\pi} \times \frac{\pi}{2}$

$= 1$

$\therefore V(X) = E(X^2) - (E(X))^2$

$= 1 - 0 = 1.$

Note that, if $X \sim N(0,1)$

odd moments $\rightarrow E(X^{2p-1}) = 0.$ $\quad \forall p \geq 1$

even moments $\rightarrow E(X^{2p}) = (2p-1)(2p-3)\cdots 5\cdot 3\cdot 1.$

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Defn: $X \sim \text{Gamma}(\lambda, \alpha).$

$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot e^{-\lambda x} \cdot x^{\alpha-1}, \quad x \in (0, \infty).$

∞

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot e^{-\lambda x} \cdot x^{\alpha-1},$$

$$\therefore E(e^{tX}) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \int_0^\infty e^{tx} \cdot e^{-\lambda x} \cdot x^{\alpha-1}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \int_0^\infty e^{-(\lambda - t)\cdot x} \cdot x^{\alpha-1}\, dx \quad \left(< \infty \Leftrightarrow \begin{array}{l} \lambda - t > 0 \\ t < \lambda \end{array} \right)$$

put $(\lambda - t)x = u$

$$\therefore dx = \frac{du}{\lambda - t}.$$

$$\therefore \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \int_0^\infty e^{-u} \cdot \frac{u^{\alpha-1}}{(\lambda-t)^{\alpha-1}} \cdot \frac{du}{(\lambda - t)}$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{1}{(\lambda - t)^\alpha} \cdot \underbrace{\int_0^\infty e^{-u} \cdot u^{\alpha-1} \cdot du}_{\Gamma(\alpha)}$$

$$= \frac{\lambda^\alpha}{\cancel{\Gamma(\alpha)}} \cdot \frac{1}{(\lambda - t)^\alpha} \cdot \cancel{\Gamma(\alpha)}$$

$$= \frac{\lambda^\alpha}{(\lambda - t)^\alpha}$$

$\therefore X \sim \text{Gamma}(\lambda, \alpha)$. $\quad f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot e^{-\lambda x} \cdot x^{\alpha-1}, \quad x > 0.$

$$\therefore m(t) = E(e^{tX}) = \begin{cases} \left(\frac{\lambda}{\lambda - t}\right)^\alpha, & t < \lambda \\ +\infty, & t \geq \lambda. \end{cases}$$

Is this continuous $\forall\, t \in (-\infty, \lambda)$?

Choose a sequence $\{t_n\}_{n \geq 1}$, s.t. $t_n \in (-\infty, \lambda)\ \forall n$

& $t_n \longrightarrow t$

Choose a sequence $\{t_n\}_{n\geq 1}$, s.t. $t_n < \ldots$,

& $t_n \longrightarrow t$

$\Rightarrow \exists\, \omega$ s.t. $e^{t_n X(\omega)} \longrightarrow e^{tX(\omega)}$

$E\left(e^{t_n X}\right) \xrightarrow{?} E\left(e^{tX}\right)$ Yes! (By DCT)

ie, is there a Z with $E(Z) < \infty$.

such that, $\left|e^{t_n X}\right| \leq Z \quad \forall n$?

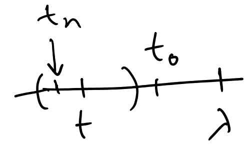
$t < \lambda$. & $t_n \rightarrow t$.

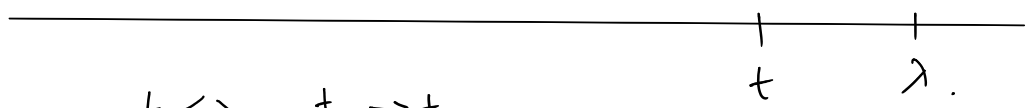
$\Rightarrow \exists\, t_0 < \lambda$ s.t. $t_n < t_0$

$\Rightarrow \left|e^{t_n X}\right| \leq \left|e^{t_0 X}\right| \quad \forall\, n \geq N$

$t + \varepsilon < t_0$

$\therefore$ choose: $\varepsilon < t_0 - t$.





$t < \lambda$, $t_n \rightarrow t$

$$\frac{m(t_n) - m(t)}{t_n - t} = \frac{E\left(e^{t_n X}\right) - E\left(e^{tX}\right)}{t_n - t}$$

$$= E\left(\frac{e^{t_n X} - e^{tX}}{t_n - t}\right) \xrightarrow{?} E\left(Xe^{tX}\right)$$

$\downarrow$

Xe^{tX}

$\left[\frac{d}{dt}\left(e^{tX}\right) = Xe^{tX}\right]$

$\therefore$ By MVT, $\dfrac{e^{t_n X} - e^{tX}}{t_n - t} = Xe^{s_n X}$ for some $s_n \in (t_n, t)$.

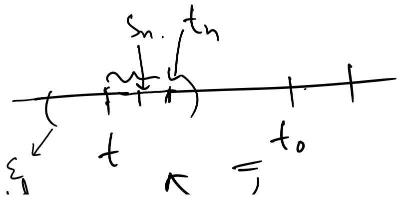
Now, this s_n is also bounded by t_0.

then, by DCT,

$\because \left|\dfrac{e^{t_n X} - e^{tX}}{t_n - t}\right| = \left|Xe^{s_n X}\right| \leq \left|Xe^{t_0 X}\right|$

ie, bounded.

why t_0 works?



ϵ nbd. t t_0 $|t_n - t|$ ie, bounded.

$s_n < t_0$

$$\therefore E\left(\frac{e^{t_n X} - e^{tX}}{t_n - t}\right) \longrightarrow E(X e^{tX}).$$

$$\left[\text{ie, } \frac{d}{dt} E(e^{tX}) = E\left(\frac{d}{dt}(e^{tX})\right) = E(Xe^{tX})\right.$$ $\longrightarrow$ ie, differentiation & expectation swapped.

$$\frac{d}{dt}\left(\left(\frac{\lambda}{\lambda - t}\right)^{\alpha}\right) = \lambda^{\alpha} \cdot \frac{d}{dt}\left((\lambda - t)^{-\alpha}\right)$$
$$= \lambda^{\alpha} \cdot (-\alpha) \cdot (\lambda - t)^{-\alpha - 1} \cdot (-1)$$
$$= \alpha \lambda^{\alpha} \cdot (\lambda - t)^{-\alpha - 1}$$

$$\therefore E(Xe^{tX}) = \alpha \cdot \lambda^{\alpha} \cdot (\lambda - t)^{-\alpha - 1}.$$

putting $t = 0$: $E(X) = \frac{\alpha}{\lambda}$.

$\therefore$ X - real r.v.

for $t \in \mathbb{R}$, $m_X(t) = E(e^{tX})$

$$I = \{ t \in \mathbb{R} : m_X(t) < \infty \}.$$

Is $I \neq \phi$?

Yes! as $0 \in I$. $[E(e^{0 \cdot X}) = E(1) = 1]$

Ex: $I = \{0\}$. X - r.v. that takes only +ve integer values (n) with prob $\propto \frac{1}{n^2}$

$$\therefore P(X = n) = c/n^2, \quad n \in \mathbb{Z}^+$$

$$\therefore E(e^{tX}) = \sum_{n \in \mathbb{Z}^+} e^{tn} \cdot (c/n^2) \longrightarrow \infty \quad \forall\, t \neq 0.$$

<u>Result</u>:

convex combination.

Now, $\left.\begin{aligned} &E(e^{t_1 X}) < \infty \\ &E(e^{t_2 X}) < \infty. \end{aligned}\right\} \Rightarrow E\left(e^{(\alpha t_1 + (1-\alpha) t_2) X}\right) < \infty$

<u>Proof</u>: $E\left(e^{(\alpha t_1 + (1-\alpha) t_2) X}\right)$

Proof: $E\left(e^{(\alpha t_1 + (1-\alpha) t_2)X}\right)$

$$= E\left(\left(e^{t_1 X}\right)^{\alpha} \cdot \left(e^{t_2 X}\right)^{1-\alpha}\right)$$

$$\leq \Big(\underbrace{E\left(e^{t_1 X}\right)^{\alpha p}}_{<\infty}\Big)^{1/p} \cdot \Big(\underbrace{E\left(e^{t_2 X}\right)^{(1-\alpha)\cdot q}}_{<\infty}\Big)^{1/q}$$

$$< \infty .$$

$\blacksquare$