Probability-3 Lecture-4

02 August 2024 11:20

$X \sim F_X$, $Y \sim F_Y$ } Independent.

$Z = X+Y \sim F_X * F_Y$.

(Convolution).

Special Case:

X - density f_X.

Y - " f_Y.

$Z = X+Y$ has density:

$$f_Z(z) = \int f_Y(z-x) \cdot f_X(x)\, dx$$

$$= \int f_X(z-y) \cdot f_Y(y)\, dy$$

$$f_Z = f_X * f_Y.$$

Suppose $X \sim \text{Gamma}(\lambda, \alpha_1)$, $Y \sim \text{Gamma}(\lambda, \alpha_2)$ } independent.

$$Z = X+Y \sim \text{Gamma}(\lambda, \alpha_1+\alpha_2).$$

$$f_Z(z) = \int_0^z f_Y(z-x) \cdot f_X(x)\, dx \quad , z>0.$$

$x > 0$ → lower limit,

$z - x > 0$

$\Rightarrow x < z$ → upper limit

Eg.: $X, Y \sim \text{Unif}(0,1)$ are independent.

$Z = X+Y$. $\quad 0 < Z < 2$

$$f_Z(z) = \int f_Y(z-x) \cdot f_X(x)\, dx.$$

$$= \begin{cases} \displaystyle\int_0^z 1 \cdot 1 \cdot dx, & 0 < z < 1 \\ \displaystyle\int_{z-1}^1 1 \cdot 1 \cdot dx, & 1 < z < 2 \end{cases}$$

$0 < x < 1$

$0 < z - x < 1$

$x < z < 1$

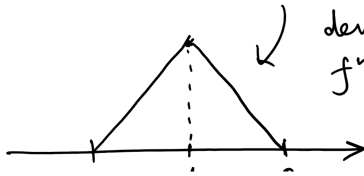
$0 < x < 1$

$0 < z - x < 1$

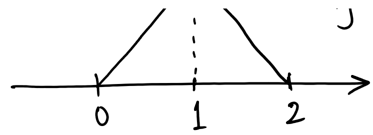
$x > z - 1 > 0$

$$= \begin{cases} z, & 0 < z < 1 \\ 2-z, & 0 \leq z < 2 \end{cases}$$

that's how density f^n looks like.



$\left\lfloor 2-z, \quad 0 \leq z < 2 \right.$



looks like.

<u>Review of Multivariate:</u>

$\underset{\sim}{X} = (X_1, \ldots, X_k)^T$

$E(\underset{\sim}{X}) = \left(E(X_1), \ldots, E(X_k)\right)^T$ - mean vector.

$D(\underset{\sim}{X}) = ((Cov(X_i, X_j)))_{k \times k}$ - "dispersion matrix", or "Variance - Covariance Matrix".

$E(A\underset{\sim}{X} + \underset{\sim}{\beta}) = A \cdot E(\underset{\sim}{X}) + \underset{\sim}{\beta}$ [Linearity of Expectation].

$D(A\underset{\sim}{X} + \beta) = A\, D(\underset{\sim}{X}) \cdot A^T$.

Clearly, $D(\underset{\sim}{X}) = E\left((X - E(X))(X - E(X))^T\right)$

* $D(\underset{\sim}{X})$ is a real, symmetric, non-negative definite (nnd)

$\underset{\sim}{a}^T D(X)\, \underset{\sim}{a} = Var(\underset{\sim}{a}^T X) \geq 0$

* $D(\underset{\sim}{X})$ is singular.

$\Rightarrow P(\underset{\sim}{X} - E(\underset{\sim}{X}) \in \text{Column space}(D(\underset{\sim}{X}))) = 1$

X has density.

$\Rightarrow D(X)$ is non-singular, positive definite.

Conversely, every p.d is matrix is the dispersion matrix of some random vector, namely the normal random vector.

[take the $\underbrace{\text{quadratic form}}$ of the pd matrix.

$e^{-\frac{1}{2}(\quad)}$ is integrable.

(normalise by some constant to get the density of the normal random vector)]

<u>Some special Multivariate Distributions:</u>

① Discrete Multinomial $(n; p_1, \ldots, p_k)$: $\quad 0 < p_i < 1$

$\sum^{k}$

① Discrete Multinomial $(n;\ p_1, \dots, p_k)$: $\quad 0 < p_i < 1$, $\sum_{i=1}^{k} p_i = 1$

$$P(X_1 = x_1, \dots, X_k = x_n) =$$

$$\frac{\lfloor n}{\lfloor n_1 \cdot \lfloor n_2 \cdot \dots \cdot \lfloor n_k} \cdot p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_k^{n_k}.$$

(Classical example: rolling a die.)

Eg.:

$X \sim Bin(n, p)$.

$\therefore X, n-X \sim Multi(n, p, 1-p)$.

<u>Marginals:</u>

$j < k$.

$(X_1, \dots, X_j,\ X_{j+1} + \dots + X_k) \sim$

$$Multi(n;\ p_1, \dots, p_j,\ p_{j+1} + \dots + p_k).$$

(Exercise: conditional distributions.)

② <u>Dirichlet:</u>

$D_k(\alpha_1, \dots, \alpha_k;\ \alpha_{k+1})$, $\quad \alpha_i > 0 \quad \forall i$

Density is given by:

$$f(x_1, x_2, \dots, x_k) = \frac{\Gamma(\alpha_1 + \alpha_2 + \dots + \alpha_{k+1})}{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2) \cdot \dots \cdot \Gamma(\alpha_{k+1})} \cdot x_1^{\alpha_1 - 1} \cdots x_k^{\alpha_k - 1} (1 - x_1 - \dots - x_k)^{\alpha_{k+1} - 1}$$

for $\underset{\sim}{x} \in S = \{(x_1, \dots, x_k) :\ x_i > 0,\ \forall i,\ x_1 + \dots + x_k < 1\}$.

"Simplex"

$X \sim Beta(\alpha_1, \alpha_2) \iff X \sim D_1(\alpha_1; \alpha_2)$.

$$\therefore \int\int_S \cdots \int x_1^{\alpha_1 - 1} \cdot \dots \cdot x_k^{\alpha_k - 1} \cdot (1 - x_1 - \dots - x_k)^{\alpha_{k+1} - 1}\, dx_k \cdots dx_1$$

the inner-most integral.

$\int^{1 - x_1 - \dots - x_{k-1}}$

$$\int_0^{1-x_1-\cdots-x_{k-1}} x_k^{(\alpha_k-1)} \cdot (1-x_1-\cdots-x_{k-1}-x_k)^{\alpha_{k+1}-1}\, dx_k .$$

change of variable:

$$u = \frac{x_k}{(1-x_1-\cdots-x_{k-1})}$$

$$\Rightarrow x_k = u(1-x_1-\cdots-x_{k-1})$$

$$= (1-x_1-\cdots-x_{k-1})^{\alpha_k+\alpha_{k+1}-1} \cdot \underbrace{\int_0^1 u^{\alpha_k-1} \cdot (1-u)^{\alpha_{k+1}-1} \cdot du}$$

$$= (1-x_1\cdots-x_{k-1})^{\alpha_k+\alpha_{k+1}-1} \cdot \frac{\Gamma(\alpha_k)\cdot\Gamma(\alpha_{k+1})}{\Gamma(\alpha_k+\alpha_{k+1})} \qquad \beta(\alpha_k,\alpha_{k+1}).$$

$\vdots$

(proceed further... Exc.)

$$(X_1,\ldots,X_j)^T \sim D_j(\alpha_1,\ldots,\alpha_j;\alpha_{j+1}+\cdots+\alpha_{k+1}).$$

$1 \leq j < k$.

* $\pi : \{1,\ldots,k\} \longrightarrow \{1,\ldots,k\}$ is a permutation.

$$(X_{\pi(1)},\ldots,X_{\pi(k)})^T \sim D_k(\alpha_{\pi(1)},\ldots,\alpha_{\pi(k)};\alpha_{k+1})$$

* $Y_1,\ldots,Y_{k+1}$ – independent.

$$Y_i \sim \text{Gamma}(\alpha_i), \quad 1 \leq i \leq k+1.$$

Show that (Exercise): $(X_1,\ldots,X_k)^T \sim D_k(\alpha_1,\ldots,\alpha_k;\alpha_{k+1})$,

where $X_j = \dfrac{Y_j}{Y_1+\cdots+Y_{k+1}}$.

[Best book for Dirichlet (or just multivariate dist^ns in general)
• Samuel Wilks : Mathematical Statistics

L • Samuel Wilks : Mathematical Statistics.

dist^ns in general)

"Ordered" Dirichlet :

$$\underset{\sim}{Y} \sim D_k^*(\alpha_1, \ldots, \alpha_k, \alpha_{k+1}) \text{, where } Y_1 = X_1,$$
$$Y_2 = X_1 + X_2$$
$$\vdots$$
$$Y_k = X_1 + \cdots + X_k.$$

$$f(\underset{\sim}{y}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_{k+1})}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{k+1})} \cdot y_1^{\alpha_1 - 1} \cdot (y_2 - y_1)^{\alpha_2 - 1} \cdots (y_k - y_{k-1})^{\alpha_k - 1} (1 - y_k)^{\alpha_{k+1} - 1}$$

for $0 < y_1 < y_2 < \cdots < y_k < 1$

Order Statistics:

$X_1, X_2, \ldots, X_k$ i.i.d.

The order Statistic obtained from $(X_1, \ldots, X_k)$

$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(k)}$.

$X_{(j)}$ is called the j^{th} order statistic.

Assume, each X_j has density f.

Q. How to find joint density of $(X_{(1)}, \ldots, X_{(n)})^T$?

Strategy - Take a $y_1 < y_2 < \cdots < y_k$

take a "rectangle"

$$R = (y_1, y_1 + \varepsilon] \times \cdots \times (y_k, y_k + \varepsilon].$$

$$P\left((X_{(1)}, \ldots, X_{(k)})^T \in R\right) = \underline{|k} \cdot \prod_{j=1}^{k} \int_{y_j}^{y_j + \varepsilon} f(x) \cdot dx$$

$\because X_{(1)}, \ldots, X_{(k)}$ is some permutation of $X_1, \ldots, X_k$, & # total permutations $= \underline{|n}$

Also, they can be factored because the events

$X_{(i)} \in (y_i, y_i + \varepsilon]$

because the events
$X_{(j)} \in (y_i ; y_i+\varepsilon]$
are <u>disjoint</u>.

$$\therefore \lim_{\varepsilon \downarrow 0} \frac{P((X_{(1)}, \ldots\ldots, X_{(k)}) \in R)}{\varepsilon^k} = \underline{\lfloor k} \cdot \prod_{j=1}^{k} f(y_j) \quad \text{for } y_1 < y_2 < \cdots < y_k.$$

(By FTC)

<u>Note:</u>

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\iint_{y_1<y_2} f(y_1) \cdot f(y_2) = \frac{1}{\underline{\lfloor 2}}$$

$$\iiint_{y_1<y_2<y_3} f(y_1) \cdot f(y_2) \cdot f(y_3) = \frac{1}{\underline{\lfloor 3}}$$

⋮

and so on.

[in case of iid rv, order statistic ≡ sufficient statistic.]

$X_1, \ldots, X_k$ – independent with a continuous distⁿ. fⁿ F.
then, the order statistic $(F(X_{(1)}), F(X_{(2)}), \ldots, F(X_{(k)}))$
$\sim D_k^*(1, 1, \ldots, 1 ; 1)$.

(<u>Exercise:</u> Verify)

[Recall X – r.v. has cont. distⁿ fⁿ F. then, define: $Y := F(X)$ then, $Y \sim \text{Unif}(0,1)$]

<u>Multivariate Normal:</u>

$\Sigma_{k \times k}$ symmetric, p.d. matrix.

$\Sigma_{k\times k}$ symmetric, p.d. matrix.

$\underset{\sim}{\mu}$ is a vector.

X has density:

$$f_X(\underset{\sim}{x}) = (2\pi)^{-k/2} \cdot \Sigma^{-1/2} \cdot e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}, \quad \underset{\sim}{x} \in \mathbb{R}^k.$$

* $\underset{\sim}{\mu} = E(\underset{\sim}{X})$; $\Sigma = D(\underset{\sim}{X})$.

* $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, \Sigma) \Rightarrow$ for non-singular A & vector $\underset{\sim}{\beta}$,
  $A\underset{\sim}{X} + \underset{\sim}{\beta} \sim N(A\underset{\sim}{\mu} + \underset{\sim}{\beta}, A\Sigma A^T)$

$$(X_{\pi(1)}, \ldots, X_{\pi(k)}) \sim N_k(\mu_\pi, \Sigma_\pi).$$

& , for $m < k$,

$$(X_1, \ldots, X_m) \sim N_m(\cdot\, , \,\cdot)$$

* for non-null $\underset{\sim}{a}$,

$$a^T \underset{\sim}{X} = N(a^T\mu, \underset{\sim}{a}^T \Sigma \underset{\sim}{a})$$

* $\underset{\sim}{X} \sim N_k(\underset{\sim}{\mu}, \Sigma) \Rightarrow \exists$ non-singular A s.t. $\longrightarrow$ $\left[\begin{array}{l}\text{Here,} \\ A = \Sigma^{-1}\end{array}\right]$
  $A(\underset{\sim}{X} - \underset{\sim}{\mu}) \sim N_k(\underset{\sim}{0}, I_k)$

* $\sigma_{ij} = 0 \Rightarrow X_i$ & X_j are independent.

* $(\underset{\sim}{X} - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{X} - \underset{\sim}{\mu}) \sim \chi^2_{(k)} \equiv \text{Gamma}\left(\frac{1}{2}, \frac{k}{2}\right)$.

- -

$X_1, \ldots, X_k \overset{iid}{\sim} N(0,1)$.

$Q_1(\underset{\sim}{x}), \ldots, Q_n(\underset{\sim}{x})$ are quadratic forms. | say, $Q_j(\underset{\sim}{x}) = \underset{\sim}{x}^T A_j \underset{\sim}{x}$.

$$Q_1(\underset{\sim}{x}) + \cdots + Q_n(\underset{\sim}{x}) = \underset{\sim}{x}^T x.$$

Let $r = \text{rank}(A_j)$, $1 \le j \le n$

$Q_1(\underset{\sim}{X}), \ldots\ldots, Q_n(\underset{\sim}{X})$ are independent with

$Q_j(\underset{\sim}{X}) \sim \chi^2_{(r_j)} \iff r_1 + r_2 + \cdots + r_n = k.$

(Version of Fisher-Cochran theorem)

Exercise: prove it!!! (think)

$\Rightarrow$ is trivial

$\Leftarrow$ Linear algebra.

One "easy" consequence:

$\underset{\sim}{X}^T A \underset{\sim}{X}$ is χ^2

$\iff A^2 = A$. (ie, A-idempotent).

In that case,

degrees of freedom,

$d.f = r(A) = tr(A)$.

Q. $\underset{\sim}{X}^T A \underset{\sim}{X}$ - quadratic form.

$\underset{\sim}{X}^T \sim N(\underset{\sim}{0}, \Sigma)$ not necessarily I.

What is a necessary & sufficient condition for the above to hold ??

$\Sigma^{-1/2} \cdot \underset{\sim}{X}^T \sim N(0, I)$. (Now think !!!).