Probability-2 Lecture-22

02 April 2024 11:25

(X,Y) has a joint density $f(x,y)$.

$Z = X+Y$ has density $f_Z(z) = \int f(z-y, y)\,dy$

$$= \int f(x, z-x)\,dx$$

<u>Special Case:</u>

X has density f_1, Y has density f_2,

X, Y - independent

$(\Longleftrightarrow)$ joint density $f(x,y) = f_1(x) \cdot f_2(y)$.

In this case, density of $Z = X+Y$ is

$$f_Z(z) = \int f_1(x) \cdot f_2(z-x)\,dx$$

$$= \int f_1(z-y) \cdot f_2(y)\,dy$$

$f_Z = f_1 * f_2 = f_2 * f_1$ is called the Convolution of densities f_1 & f_2.

Eg. $X \sim \text{Gamma}(\lambda, \alpha_1)$ $\qquad X, Y$ - independent

$Y \sim \text{Gamma}(\lambda, \alpha_2)$

$Z = X+Y$.

$$f_1 * f_2(z) = \int_0^z f_1(x) \cdot f_2(z-x)\,dx \qquad z \in (0,\infty)$$

$$= \frac{\lambda^{\alpha_1} \cdot \lambda^{\alpha_2}}{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2)} \cdot \int_0^z e^{-\lambda x} \cdot e^{-\lambda(z-x)} \cdot x^{\alpha_1 - 1} \cdot (z-x)^{\alpha_2 - 1}\,dx.$$

$$= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\cdot\Gamma(\alpha_2)} \cdot e^{-\lambda z} \int_0^z x^{(\alpha_1-1)} \cdot (z-x)^{(\alpha_2-1)}\, dx$$

$u = x/z$ i.e., $x = zu$

$du = \frac{1}{z} dx \Rightarrow dx = z\, du$

u	x
0	0
1	z

$$= \frac{\lambda^{\alpha_1+\alpha_2} \cdot e^{-\lambda z}}{\Gamma(\alpha_1)\cdot\Gamma(\alpha_2)} \cdot z^{\alpha_1-1} \cdot z^{\alpha_2-1} \cdot z \underbrace{\int_0^1 u^{\alpha_1-1} \cdot (1-u)^{\alpha_2-1}\, du}_{\beta(\alpha_1,\alpha_2)}$$

$$= \frac{\lambda^{\alpha_1+\alpha_2} \cdot e^{-\lambda z} \cdot z^{(\alpha_1+\alpha_2)-1}}{\Gamma(\alpha_1)\cdot\Gamma(\alpha_2)} \cdot \underbrace{\beta(\alpha_1,\alpha_2)}_{= \frac{\Gamma(\alpha_1)\cdot\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}}$$

$$= \frac{\lambda^{\alpha_1+\alpha_2} \cdot e^{-\lambda z} \cdot z^{(\alpha_1+\alpha_2)-1}}{\cancel{\Gamma(\alpha_1)}\cdot\cancel{\Gamma(\alpha_2)}} \cdot \frac{\cancel{\Gamma(\alpha_1)}\cdot\cancel{\Gamma(\alpha_2)}}{\Gamma(\alpha_1+\alpha_2)}$$

$$f_1 * f_2(z) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} \cdot z^{(\alpha_1+\alpha_2)-1} \cdot e^{-\lambda z},$$ which is the density of Gamma$(\lambda, \alpha_1+\alpha_2)$.

Eg. $X \sim N(0, \sigma_1^2)$
$Y \sim N(0, \sigma_2^2)$ X, Y - independent.

$Z = X + Y \sim (\,?\,)$

$$\int^{\infty} e^{-\frac{1}{2}\left(\frac{x^2}{\cdots} + \frac{(z-x)^2}{\cdots}\right)}$$

$Z = X + Y \sim (\,!\,)$

$$f_Z(z) = \frac{1}{2\pi\sigma_1 \cdot \sigma_2} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_1^2} + \frac{(z-x)^2}{\sigma_2^2}\right)} dx.$$

complete the square & proceed.

$$= \cdots$$

$$\vdots$$

$$= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \cdot e^{-\frac{1}{2} \cdot \frac{z^2}{\sigma_1^2+\sigma_2^2}} \longrightarrow \text{density of } N(0, \sigma_1^2+\sigma_2^2).$$

Q: What to do for $\left.\begin{array}{l} X \sim N(\mu_1, \sigma_1^2) \\ Y \sim N(\mu_2, \sigma_2^2) \end{array}\right\}$ for any μ_1, μ_2?

X, Y – independent

- $\tilde{X} = X - \mu_1 \sim N(0, \sigma_1^2)$

  $\tilde{Y} = Y - \mu_2 \sim N(0, \sigma_2^2)$

$\therefore \tilde{Z} = \tilde{X} + \tilde{Y} = N(0, \sigma_1^2 + \sigma_2^2).$

$\therefore Z = X + Y = \tilde{Z} + (\mu_1 + \mu_2) \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$

<u>Alternative way</u>: (Using MGFs).

<u>Result</u>: X, Y – independent

$\Rightarrow m_{X+Y}(t) = m_X(t) \cdot m_Y(t)$. $\forall\, t$ s.t. $m_X(t) < \infty$, $m_Y(t) < \infty$.

[* <u>Note</u>: whenever $0 \in I := \{t : m_X(t) < \infty\}$ (convex set), $m_X(t)$ uniquely determines the distribution.

ie, there should exist an open interval containing 0, for mgf to uniquely determine the distⁿ.]

uniquely determine the distⁿ.

$X \sim \text{Gamma}(\lambda, \alpha)$. $\qquad m_X(t) = \left(\dfrac{\lambda}{\lambda - t}\right)^{\alpha}$, $t < \lambda$.

$X \sim N(\mu, \sigma^2)$. $\qquad m_X(t) = e^{\mu t + \frac{1}{2}\mu^2 t^2}$, $t \in \mathbb{R}$.

Eg: $X \sim \text{Unif}(0,1)$ $\qquad X, Y$ – independent.

$Y \sim \text{Unif}(0,1)$.

$Z = X + Y \sim (\,?\,)$. ← Case where mgf is NOT helpful. We resort back to Convolution again.

Z takes values in $(0,2)$.

Fix $z \in (0,2)$.

$0 < x < 1$; $0 < z < 2$

$$f_Z(z) = \int_0^z f_1(x) \cdot f_2(z-x)\, dz, \quad z \le 1$$

$0 < z - x < 1 \to x > z-1$

$z > 1$

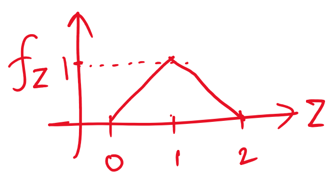
$z - x > 1 - x$

$\boxed{z - 1 < x < 1}$

$$= \int_{z-1}^{1} f_1(x) \cdot f_2(z-x)\, dx = \begin{cases} z, & 0 < z < 1 \\ 2 - z, & 1 < z < 2 \end{cases}$$

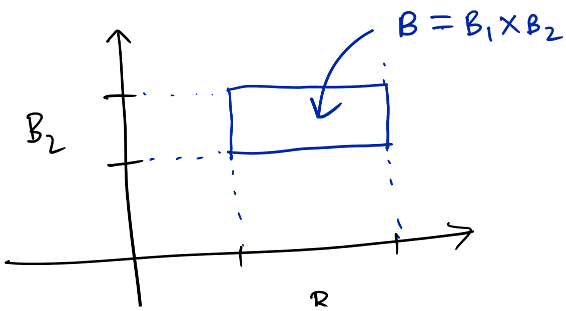
→ known as "tent" density.

Note that: this no longer remains Uniform.



X, Y independent $\iff P(X \in B_1, Y \in B_2) = P(X \in B_1) \cdot P(Y \in B_2)$.

$$P((X,Y)) \in B_1 \times B_2$$
$$= P(X \in B_1) \cdot P(Y \in B_2).$$



$B_x := \{y : (x,y) \in B\}$

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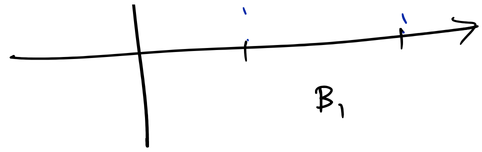


$$\therefore B_x = \begin{cases} B_2 , & x \in B_1 \\ \phi , & x \notin B_1 \end{cases}$$

$$\therefore P(Y \in B_x) = \begin{cases} P(Y \in B_2), & \text{if } x \in B_1 \\ 0 , & \text{if } x \notin B_1 \end{cases}$$

$$= P(Y \in B_2) \cdot \mathbb{1}_{B_1}(x)$$

$\therefore P(Y \in B_X) = \underset{\text{(Say)}}{\Psi(X)} = P(Y \in B_2) \cdot \mathbb{1}_{B_1}(X)$, where $\Psi : [0,1] \to \mathbb{R}$.

$$\therefore \boxed{\begin{aligned} &P((X,Y)) \in B_1 \times B_2 \\ &\qquad = P(X \in B_1) \cdot P(Y \in B_2). \end{aligned}}$$

$\Updownarrow$ If B is a "rectangle",

Cartesian pdt of 2 borel sets, NOT the geometric rectangle.

then $P((X,Y) \in B)$

$$= E(\Psi(X)),$$

where, $\Psi(x) = P(Y \in B_x)$

- -

X has distribution function F_1,

Y has distribution function F_2.

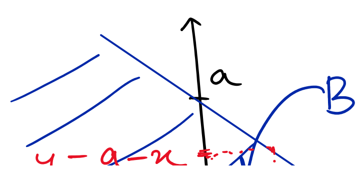
X, Y - independent.

$Z = X + Y$. What is the distribution function of F_Z?

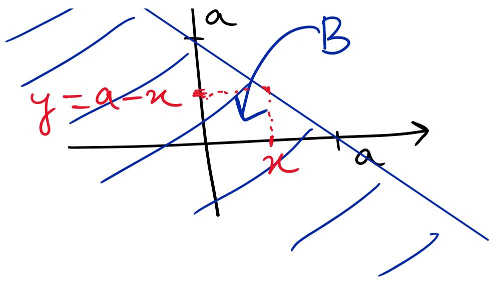
$P(Z \leq a) = P((X,Y) \in B)$, where $B = \{ (x,y) \in \mathbb{R}^2 : x + y \leq a \}$.

$$= E(\Psi(X))$$

$$= E(F_2(a - X))$$



$$= E\left(F_2(a-X)\right)$$



$$\therefore \Psi(x) = P(Y \in B_x).$$
$$= P(Y \leq a - x).$$
$$= F_2(a-x)$$

Conditional Distribution and Conditional Expectation:

(X,Y) - pair of real r.v.s.

Conditional distribution of Y, given $X = x$.

$$P(Y \in B \mid X = x) = \Psi(x, B).$$

[We define it for only those x's that belong to the support of X.]

$$\Psi(x, B) = E\left(1_B(Y) \mid X = x\right).$$

$\Psi(\cdot, B)$ must satisfy:

$$P(X \in A, Y \in B) = E\left(\Psi(X,B) \cdot 1_A(X)\right)$$

for all $A \subset \mathbb{R}$.

[Recall: Problem Sheet - 1 (Sem - 2)

$$\Psi(X) = E(Y \mid X). \Leftrightarrow$$

$$E\left(Y \cdot 1_A(X)\right) = E\left(\Psi(X) \cdot 1_A(X)\right)$$

$\forall A$.]

(X,Y) - pair of real r.v.s.

Let S = support of X (ie, set of all values X can take.).

We want a f^n $\Psi : S \times \{\text{Borel sets}\} \longrightarrow [0,1]$,

such that,

(i) $\forall x \in S$, $B \longmapsto \Psi(x, B)$ is a probability.

(ii) $\forall B$, $P(X \in A, Y \in B) = E(\psi(X,B) \cdot \mathbb{1}_A(X))$

for all borel $A \subset \mathbb{R}$.

So, if such a ψ exists, then $\psi(x,B)$ is called $P(Y \in B \mid X = x)$.

All of this is subjected to a big "IF".
ie, does such a ψ exist in general?
Ans: Yes !!!

Proof: Aukaat ke bahar ka

(X,Y) has density $f(x,y)$.

Let f_X - marginal density of X.

define $$g(y|x) = \frac{f(x,y)}{f_X(x)} \quad \forall x \text{ s.t. } f_X(x) > 0.$$

Claim: $g(y)$ is density in y.

Proof: (trivial). $$g(y) = \int_y g(y|x)\, dy = \int \frac{f(x,y)}{\underbrace{f_X(x)}_{\text{const.}}} dy$$

$$= \frac{f_X(x)}{f_X(x)} = 1 \checkmark$$ valid density.

Define $$\psi(x,B) := \int_B g(y|x) \cdot dy$$

Fix $x \in S$.

$B \longmapsto \psi(x,B)$ is a probability.

$(\mathbb{R}, \mathcal{B}, \psi(x,\cdot))$ $\quad h: \mathbb{R} \to \mathbb{R}$ measureable.

$$E_{\psi(x,\cdot)}(h) = E(h(Y) \mid X = x).$$