Probability-2 Lecture-15 (First Lecture after midsem)

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$(\Omega, \mathcal{A}, P)$: probability space

a non-empty set "sample space"

a σ-algebra on Ω "class of events."

a probability defined on sets in $\mathcal{A}$.

Definition: Real Random Variable.

A real random variable X on $(\Omega, \mathcal{A}, P)$ is a function $X : \Omega \to \mathbb{R}$ such that -

$$\{\omega \in \Omega : X(\omega) < a\} \in \mathcal{A} \quad \forall \; a \in \mathbb{R}.$$

Fact: If X is a real r.v. on $(\Omega, \mathcal{A}, P)$, then

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{A}$$

$\forall$ borel set $B \subset \mathbb{R}$.

Defn:

An extended real r.v. X on $(\Omega, \mathcal{A}, P)$ is a function $X : \Omega \longrightarrow [-\infty, \infty]$ such that

$$\{\omega \in \Omega : X(\omega) < a\} \in \mathcal{A} \quad \forall \; a \in \mathbb{R}$$

Fact: If X is a real r.v. on $(\Omega, \mathcal{A}, P)$, then $X^{-1}(B) \in \mathcal{A}$ $\forall$ borel set $B \subset \mathbb{R}$, and, $X^{-1}(\{-\infty\}) \in \mathcal{A}$,

$X^{-1}(\{\infty\}) \in \mathcal{A}$.

$$\left[\text{Note: } \{\omega : X(\omega) = \infty\} = \bigcap_n \{\omega : X(\omega) < n\} \right]$$

Aim: to define Expectation $E(X)$.

Step 1: X is a real simple r.v.

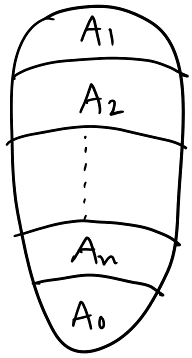
Recall, X is a real simple r.v.

$\iff$ X has a representation:

$$X = \sum_{i}^{n} c_i \cdot 1_{A_i}, \quad \text{where } A_1, A_2, \ldots, A_n \in \mathcal{A}$$

$$X = \sum_{i=1}^{n} c_i \cdot 1_{A_i}, \quad \text{where } A_1, A_2, \ldots, A_n \in \mathcal{a} \text{ are disjoint.}$$

Clearly X can take only valued $c_1, c_2, \ldots, c_n$ and 0.
So, we can make sure that $A_1, A_2, \ldots, A_n$ form of partition of Ω.



$$\therefore X = \sum_{i=1}^{n} c_i \cdot 1_{A_i} + 0 \cdot 1_{A_0}$$

Define: $\boxed{E(X) = \sum_{i=1}^{n} c_i \cdot P(A_i)}$.

Qs. Is this well defined?

Ans: Suppose, $X = \sum_{i=1}^{n} c_i \cdot A_i = \sum_{j=1}^{n} d_j \cdot B_j$

are two canonical representations.

W.L.O.G: assume $\{A_1, \ldots, A_n\}$ is a partition of Ω,
& $\{B_1, \ldots, B_n\}$ is also a partition of Ω.

$$X = \sum_{i=1}^{n} a_i 1_{A_i}$$

$$= \sum_{i=1}^{n} a_i \cdot 1_{\bigcup_{j=1}^{m} (A_i \cap B_j)}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i 1_{A_i \cap B_j}.$$

$\because A_i \cap B_j$ are disjoint $\forall (i,j)$

$\therefore 1_{\bigcup_{j=1}^{m} (A_i \cap B_j)} = \sum_{j=1}^{m} 1_{A_i \cap B_j}$

[1 (Indicator) "breaks" linearly over disjoint sets.]

Similarly, $X = \sum_{i=1}^{n} \sum_{j=1}^{m} b_j \cdot 1_{A_i \cap B_j}$

Claim:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_i \, 1_{A_i \cap B_j} = \sum_{i=1}^{n} \sum_{j=1}^{m} b_j \cdot 1_{A_i \cap B_j}$$

Proof of claim:

If $A_i \cap B_j = \phi$, then $P(A_i \cap B_j) = 0$.

hence, nothing to prove.

So, let $A_i \cap B_j \neq \phi$.

take $w \in A_i \cap B_j$.

$$\therefore a_i = X(w) = b_j$$

$$\therefore \sum_i a_i P(A_i) = \sum_j b_j \cdot P(B_j). \quad \blacksquare$$

This proves well-definedness.

Now, we prove Linearity of Expectation:

If $X = \sum_{i=1}^{n} a_i \cdot 1_{A_i}$ is canonical,

then, $\alpha X = \sum_{i=1}^{n} \alpha a_i \cdot 1_{A_i}$ is canonical.

$$\therefore E(\alpha X) = \sum_i \alpha a_i \cdot 1_{A_i} = \alpha \cdot \sum_i a_i \cdot 1_{A_i} = \alpha \cdot E(X)$$

&, $X = \sum_i c_i \cdot 1_{A_i}$

$Y = \sum_j d_j \cdot 1_{B_j}$ $\Big\} \rightarrow$ Canonical.

$$\Rightarrow X + Y = \sum_i \sum_j (c_i + d_j) \cdot 1_{A_i \cap B_j}$$

$$= \sum (c_i + d_j) \, P(A_i \cap B_j) = E(X) + E(Y).$$

$$\Rightarrow E(X+Y) = \sum_i \sum_j (c_i + d_j) \cdot P(A_i \cap B_j) = E(X) + E(Y)$$

$$\boxed{\begin{aligned} &\therefore\ E(\alpha X) = \alpha \cdot E(X) \\ &\&\ E(X+Y) = E(X) + E(Y). \end{aligned}}$$

Now, we prove <u>Monotonicity of Expectation:</u>

X, Y – real simple r.v.

$$X \leq Y$$

$$X = \sum_i c_i \cdot 1_{A_i} = \sum_i \sum_j c_i \cdot 1_{A_i \cap B_j}$$

$$Y = \sum_j d_j \cdot 1_{B_j} = \sum_i \sum_j d_j \cdot 1_{A_i \cap B_j}$$

$$X \leq Y \Rightarrow c_i \leq d_j \quad \forall (i,j). \text{ with } A_i \cap B_j \neq \phi.$$

$$\Rightarrow E(X) \leq E(Y). \quad \blacksquare$$

Observe that,

If X is a non-ve, real, simple, then

$$0 \leq E(X) < \infty.$$

Step 2:

X – any non-ve random variable.

Define $E(X) = \sup\{E(Y) : Y \text{ is a simple r.v., } 0 \leq Y \leq X\}$.

(Here, the construction itself guarantees well-definedness.)

$$0 \leq E(X) \leq +\infty.$$

Easy to see that, if X - non -ve, real, simple,
then $E(X)$ is same as defined in Step 1.

If $X_1 \leqslant X_2$ are two non -ve r.v.s,
it is clear from the definition that:

$$E(X_1) \leq E(X_2).$$

[Monotonicity of Expectation.]

Result:

Let X be any non -ve random variable.
If $X_n, n \geqslant 1$ is any sequence of non -ve r.v.s
with $X_n \nearrow X$, then $E(X_n) \nearrow E(X)$.

Assume this:

X, Y are non -ve r.v.s.

$$E(X+Y) = E(X) + E(Y).$$
$$E(\alpha X) = \alpha \cdot E(X). \quad \text{for } \alpha \geqslant 0.$$

Proof: follows from the above result.

X is non -ve r.v.

(X_n) - sequence of non -ve real r.v.s, $X_n \nearrow X$.

to prove, $\lim_{n \to \infty} E(X_n) = E(X)$.

- $\lim_{n \to \infty} E(X_n)$ exists (trivial).
- We clearly have $X_n \nearrow X$.
  ie, $X_n \leq X$.

$$\therefore \lim_{n \to \infty} E(X_n) \leq E(X). \quad \text{—— (1)}$$

only left to show,
$$\lim_{n \to \infty} E(X_n) \geqslant E(X).$$

Enough to show, that for <u>ANY</u> real simple Y with $0 \leq Y \leq X$,

$$\lim_{n\to\infty} E(X_n) \geq E(Y).$$

Fix $0 < \alpha < 1$

We'll prove,

$$\lim_{n\to\infty} E(X_n) \geq \alpha \cdot E(Y).$$

for each n, let $\Omega_n = \{\omega \in \Omega : X_n(\omega) \geq \alpha \cdot Y(\omega)\}$

Clearly, $\Omega_{n+1} = \{\omega \in \Omega : X_{n+1}(\omega) \geq \alpha \cdot Y(\omega)\}$

$$\downarrow$$

$$X_{n+1}(\omega) \geq X_n(\omega) \geq \alpha \cdot Y(\omega)$$

$$\therefore \Omega_{n+1} \supseteq \Omega_n$$

$\therefore \Omega_n$ - increasing sets :

$$\therefore \Omega_n \nearrow \Omega.$$

If $Y(\omega) = 0$, then $\omega \in \Omega_n \ \forall n$

If not,

ie, if $Y(\omega) > 0$, then

$$\alpha \cdot Y(\omega) < Y(\omega) \leq X(\omega).$$

$\Rightarrow$ for <u>some</u> n :

$$X_n(\omega) \geq \alpha \cdot Y(\omega)$$

$$\Rightarrow \omega \in \Omega_n.$$

$\forall n$, $X_n \geq X_n \cdot 1_{\Omega_n} \geq \alpha \cdot Y \cdot 1_{\Omega_n}$

(trivial)

$$= \sum_{i=1}^{n} \alpha \cdot c_i \cdot 1_{A_i \cap \Omega_n}$$

$$E(X_n) \geq \alpha \cdot \sum_{i=1}^{n} c_i \cdot P(A_i \cap \Omega_n)$$

taking $\lim_{n\to\infty}$,

$$\lim_{n\to\infty} E(X_n) \geqslant \lim_{n\to\infty} \alpha \cdot \sum_{i=1}^{n} c_i \cdot P(A_i \cap \Omega_n)$$

$$= \alpha \cdot \sum_{i=1}^{n} c_i \cdot P(A_i) \qquad \left[\begin{array}{l} \text{as } n \to \infty, \\ \Omega_n \to \Omega \end{array}\right]$$

$$= \alpha \cdot E(Y)$$

$$\therefore \lim_{n\to\infty} E(X_n) \geqslant \alpha \cdot E(Y).$$

Now, choice of $\alpha \in (0,1)$ was arbitrary.

$\therefore$ Now, let $\alpha \nearrow 1$.

$$\therefore \lim_{n\to\infty} E(X_n) \geqslant E(Y).$$

$$\therefore \lim_{n\to\infty} E(X_n) \geqslant E(X). \quad \text{—— (2)}$$

$\therefore$ Using (1) & (2): $\boxed{\lim_{n\to\infty} E(X_n) = E(X).}$

This completes the proof. ▨

Step-3:

X is any r.v.

Define $X^+ := \text{Max.}\{X, 0\}$

$X^- := \text{Max}\{-X, 0\}$.

Note: $X = X^+ - X^-$

(1) Both X^+, X^- are both non-ve random variables

So, $E(X^+)$, $E(X^-)$ defined as in Step 2.

(2) Define $\boxed{E(X) := E(X^+) - E(X^-)}$, if

atleast one of $E(X^+), E(X^-)$ infinite atleast one of $E(X^+), E(X^-)$ infinite

$(-\infty \leq E(X) \leq \infty)$.

otherwise,
say that $E(X)$ does not exist.

Now, to prove:

① $X \leq Y$, and $E(X), E(Y)$ both exist

$$\Rightarrow E(X) \leq E(Y).$$

Proof of ①: $X^+ \leq Y^+$, $X^- \geq Y^-$

$\Rightarrow E(X^+) \leq E(Y^+)$ $\Rightarrow E(X^-) \geq E(Y^-)$

If, either $E(X) = -\infty$ or $E(Y) = +\infty$,
then nothing to prove.

So, assume, $E(X) > -\infty$ and $E(Y) < \infty$

$$\Downarrow \qquad\qquad \Downarrow$$

$$E(X^-) < \infty \qquad E(Y^+) < \infty$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$E(Y^-) < \infty \qquad E(X^+) < \infty$$

$\therefore$ $E(X^+), E(X^-), E(Y^+), E(Y^-)$ are all real.

$$E(X^+) - E(X^-) \leq E(Y^+) - E(Y^-).$$

$$\Rightarrow E(X) \leq E(Y).$$ ▨ (Monotonicity of Expectation)

② $\left.\begin{array}{l} E(X) \text{ exists} \\ \alpha \in \mathbb{R} \end{array}\right\} \Rightarrow E(\alpha X) = \alpha \cdot E(X).$

Proof: Exercise:
(Hints: $\alpha > 0 \Rightarrow (\alpha X)^+ = \alpha X^+$
$(\alpha X)^- = \alpha X^-$

(HINTS: $\alpha > 0 \Rightarrow (\alpha X)^+ = \dots$

$$(\alpha X)^- = \alpha X^-$$

$$\alpha < 0 \Rightarrow (\alpha X)^+ = (-\alpha)\cdot X^-$$

$$(\alpha X)^- = (-\alpha)\cdot X^+ .)$$

(3) X, Y — r.v.s, • $E(X)$, $E(Y)$ both exist.
• $X+Y$ is well defined pointwise.
• $E(X)+E(Y)$ is well defined.

$$\Rightarrow E(X+Y) \text{ exists } \& \quad E(X+Y) = E(X) + E(Y).$$

Proof: Under the hypotheses above,
Either $E(X^+), E(Y^+)$ both $< \infty$

OR $E(X^-), E(Y^-)$ both $< \infty$.

Suppose, $E(X^+) = +\infty$

$$\Downarrow \quad \text{(because } E(X) \text{ exists.)}$$

$$\boxed{E(X^-) < \infty}$$

$$\Downarrow$$

$$E(X) = \infty \Rightarrow E(Y) > -\infty \Rightarrow \boxed{E(Y^-) < \infty}$$

$\because E(X) + E(Y)$ is well-defined.

Similarly, the other statement can also be proved.

$$\underbrace{(X+Y)^+ - (X+Y)^-}_{\text{LHS}} = (X+Y) = (X^+ + Y^+) - \underbrace{(X^- + Y^-)}_{\text{RHS}} \quad \text{—— } (*)$$

$$\therefore (X+Y)^+ + X^- + Y^- = (X+Y)^- + X^+ + Y^+$$

Suppose, LHS $= +\infty$.

$\Rightarrow$ RHS $= +\infty$

why? $X^- := \max\{0, -X\}$
$\therefore (X+Y)^- := \max\{0, -(X+Y)\}$

$\Rightarrow$ RHS $= +\infty$

$\Rightarrow X^- < \infty,\ Y^- < \infty$

$\Rightarrow (X^- + Y^-) < \infty$, & $(X+Y)^- \leq (X^- + Y^-)$.

$\Rightarrow (X+Y)^- < \infty$.

why? $X^- := \operatorname{Max}\{0, -X\}$

$\therefore (X+Y)^- := \operatorname{Max}\{0, -(X+Y)\}$

$\leq \operatorname{Max}\{0, -X\} + \operatorname{Max}\{0, -Y\}$.

Similarly,

$(X+Y)^+ \leq X^+ + Y^+$

$$\therefore E(X+Y)^+ + E(X^-) + E(Y^-) = E(X+Y)^- + E(X^+) + E(Y^+)$$

Fact: $E(X)$ is finite

$\Leftrightarrow E(X^+), E(X^-)$ both finite.

$\Leftrightarrow E|X| < \infty$

If $E(X)$ exists,

$|E(X)| \leq E|X|$.

We know, $-|X| \leq X < |X|$ is always true.

$E(X)$ exists $\Rightarrow E(|X|)$ exists.

$-E|X| \leq E(X) \leq E|X|$

Fact:

$E(X)$ is finite

$\Rightarrow P(|X| < \infty) = 1$.

Proof: (Exercise)

Q. $(X_n) \longrightarrow X$ pointwise.

$E(X), (E(X_n))$ all exist.

Can we say: $\lim_{n \to \infty} E(X_n) = E(X)$. ?

Ans: In general, No!!

Q. then, what are the additional conditions needed to

Q. then, what are the additional conditions needed to be imposed?

Monotone Convergence Theorem (MCT):

$X_n, n \geq 1$ is a sequence of non-negative random variables s.t. $X_n \nearrow X$ (increases monotonically) $\Rightarrow \lim_{n \to \infty} E(X_n) = E(X)$.

Proof:

X_{ij}'s are simple r.v.s.

$$\begin{array}{l}
X_{11} \leq X_{12} \leq X_{13} \leq \cdots \leq \boxed{X_{1n}} \leq \cdots \nearrow X_1 \\
X_{21} \leq X_{22} \leq X_{23} \leq \cdots \leq \boxed{X_{2n}} \leq \cdots \nearrow X_2 \\
\vdots \\
X_{n1} \leq X_{n2} \leq X_{n3} \leq \cdots \leq \boxed{X_{nn}} \leq \cdots \nearrow X_n \\
\hfill \downarrow \\
\hfill X
\end{array}$$

choose this column of simple r.v.s.

Put $Y_n = \max_{1 \leq k \leq n} X_{kn}$

Clearly, $Y_{n+1} \geq Y_n$.

$\therefore Y_n$ is a sequence of non-ve, simple, increasing random variables

For all $n \geq k$,

$X_{kn} \leq Y_n \leq X_n$.

Let $n \to \infty$.

$\therefore \lim_{n \to \infty} X_{kn} \leq \lim_{n \to \infty} Y_n \leq \lim_{n \to \infty} X_n$

$\Rightarrow X_k \leq \lim_{n \to \infty} Y_n \leq X$.

Now, let $k \to \infty$

$\Rightarrow \lim \ldots \leq \lim . Y_n \leq X$

Now, let $k \to \infty$

$$\Rightarrow \lim_{k\to\infty} X_k \leq \lim_{n\to\infty} Y_n \leq X$$

$$\Rightarrow X \leq \lim_{n\to\infty} Y_n \leq X.$$

$\therefore$ By Sandwich theorem,

$$\lim_{n\to\infty} Y_n = X.$$

i.e, $Y_n \nearrow X$ pointwise,

& $\lim_{n\to\infty} E(Y_n) = E(X)$

Back to fix k:

for $n \geq k$, $E(X_{kn}) \leq E(Y_n) \leq E(X_n)$

Let $n \to \infty$: $\downarrow$ $\qquad\qquad\qquad\qquad \downarrow$

$$\Rightarrow E(X_k) \leq E(X) \leq \lim_{n\to\infty} E(X_n)$$

Now, let $k \to \infty$: $\downarrow$

$$\Rightarrow \lim_{k\to\infty} E(X_k) \leq E(X) \leq \lim_{n\to\infty} E(X_n)$$

$$\therefore \lim_{n\to\infty} E(X_n) = E(X). \quad ▨$$

Dominated Convergence Theorem (DCT):

Let $X_n \to X$.

If $\exists Y$ - r.v. s.t $E(|Y|) < \infty$

s.t. $|X_n| \leq Y \quad \forall n$

then, $E(X_n) \to E(X)$.

Proof: $|X_n - X| \leq 2Y$

Put $Z_n = 2Y - |X_n - X|$. ——— $(*)$

$\therefore Z_n \geq 0$.

$$\inf_{k \geq n} Z_k \nearrow \liminf Z_n = 2Y \quad \text{(by defn of inf.)}$$

$\underbrace{\inf_{k\geq n}}$ tail infimum.

$$E\left(\inf_{k \geq n} Z_k\right) \longrightarrow 2 \cdot E(Y).$$

$$\Rightarrow \liminf_n E(Z_n) \geq 2 \cdot E(Y).$$

from (*): $Z_n = 2Y - |X_n - X|$

Apply Expectation on both sides.

$$E(Z_n) = 2E(Y) - E(|X_n - X|)$$

Now, Apply liminf on both sides:

$$2E(Y) - \limsup_n E|X_n - X| \geq 2E(Y)$$

$$\therefore \limsup_n (E|X_n - X|) \leq 0$$

But, this is non -ve.

$$\therefore \limsup_n (E|X_n - X|) = 0.$$

$$\therefore \lim_{n \to \infty} E(|X_n - X|) = 0.$$