LRT for linear models

Now, we shall use the likelihood ratio for hypothesis testing in a linear model.

The linear model we are working with, is

$$\vec{y} = X\vec{\beta} + \vec{\epsilon}$$

$$\vec{\epsilon} \sim N(\vec{0}, \sigma^2 I)$$
$$\vec{\beta} \in \mathbb{R}^p, \sigma^2 > 0$$

Under normality assumption, the likelihood as a function of the parameters is given by the multivariate normal density,

$$L(\vec{\beta}, \sigma^2) \propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\|\vec{y} - X\vec{\beta}\|^2}.$$

This function will be maximum when the parameters are substituted by the maximum likelihood estimators (MLEs). The MLE of $\vec{\beta}$ will be its *least square estimator* (if X is not full column rank, it will simply be any least square estimator). The MLE of σ^2 will be $\frac{1}{n}||\vec{y} - X\vec{\beta}||^2$ (may not be unbiased).

Upon substitution, the expression (excluding the constant part, which is independent of the parameters or data) simplifies to $supL(\vec{\beta}, \sigma^2) \propto RSS^{-n/2}$.

Under H_0, RSS will be the residual sum of squares in the restricted model, RSS_0.

Hence, the likelihood ratio becomes

$$\frac{sup_H L}{sup_{H_0} L} \propto \left(\frac{RSS_0}{RSS}\right)^{n/2}$$

Hence, the test rejects H_0 if RSS_0 far exceeds RSS, i.e, the ratio $\left(\frac{RSS_0}{RSS}\right)$ is large, or equivalently, if $\frac{RSS_0 - RSS}{RSS}$ is large.

This also shows that the theoretical likelihood ratio test is the same as our intuitive test, which rejects if the additional RSS due to the restricted null model is much larger than the acceptable RSS in the full model.

$$\frac{RSS_0 - RSS}{RSS}$$