Probability-3 Lecture-21

29 October 2024 11:09

Recall: qs. from midsem.

$(\Omega, \mathcal{A}, P)$ – a discrete probability space.

i.e, Ω – countable.

$\Omega = \{\omega_1, \omega_2, \ldots\}$

& $P(A) = \sum_{i : \omega_i \in A} P(\{\omega_i\})$

$X_n \xrightarrow{P} X$.

to show: $X_n \xrightarrow{a.s} X$

Proof: Suppose, $X_n \overset{a.s}{\not\to} X$

$\Rightarrow P(\underbrace{\{\omega : X_n(\omega) \not\to X(\omega)\}}_{A \text{ (say)}}) > 0$ (ie, $\neq 0$)

$\because$ Ω is discrete.

$\Rightarrow \exists\, \omega_0 \in A$ s.t, $P(\{\omega_0\}) = \underset{(\text{say})}{\delta} > 0$.

$X_n(\omega_0) \not\to X(\omega_0)$

$\Rightarrow \exists\, \varepsilon > 0$ and a subsequence $\{n_k\}$ such that:

$$|X_{n_k}(\omega) - X(\omega)| > \varepsilon$$

$P(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) \xrightarrow{?} 0$

But, $\exists\, n_k \geq N$, s.t. $P(\{\omega : |X_{n_k}(\omega) - X(\omega)| > \varepsilon\}) \geq \delta$ $\forall\, n \geq N$.

$\therefore X_n \overset{P}{\not\to} X$ [which is a contradiction!!].

$\therefore X_n \xrightarrow{a.s} X$.

$\therefore X_n \xrightarrow{a.s} X.$ [contradiction !!]

$\blacksquare$

* whenever you write density functions, write their supports (always !!)

$\underset{\sim}{X} = (X_1, X_2, \dots, X_m)$ is a random vector which has a joint density.

$f_{\underset{\sim}{X}}(x_1, \dots, x_m)$ for $(x_1, \dots, x_m) \in I$ (a connected open subset $\subseteq \mathbb{R}^m$)

Suppose, we have

$$\varphi : \underset{\mathbb{R}^m}{\overset{\cap}{I}} \longrightarrow \underset{\mathbb{R}^m}{\overset{\cap}{J}}$$

[Jacobian rule applies only on same dim. spaces $\mathbb{R}^m \longrightarrow \mathbb{R}^m$.]

$\therefore \underset{\sim}{Y} = \varphi(\underset{\sim}{X}) \in J$

$= (Y_1, Y_2, \dots\dots, Y_m)$ (say)

<u>Sufficient</u> conditions for Y to have joint density:

① $\varphi : I \longrightarrow J$ is 1-1, onto.

: let $\varphi^{-1} : J \longrightarrow I$.

then, φ^{-1} is 1-1, onto.

(2) Let us write:

$$\psi(y_1, y_2, \ldots, y_m) = (x_1, x_2, \ldots, x_m)$$

[ie, each x_i is a real-valued function of $y_1, y_2, \ldots, y_m$.]

So, we can talk about $\left(\frac{\partial x_i}{\partial y_j}\right)$

<u>Condition</u>: $\left(\frac{\partial x_i}{\partial y_j}\right)$ exist & is 1-1, onto in $\mathcal{J}$ for all (i, j)

Define $J(\underset{\sim}{y}) = \det \left(\left(\frac{\partial x_i}{\partial y_j}\right)\right)_{m \times m}$

"Jacobian"

(Has to be a square matrix..)

J is <u>continuous</u>.

(3) $J(\underset{\sim}{y}) \neq 0 \quad \forall \underset{\sim}{y} \in \mathcal{J}$.

ie, $J(\underset{\sim}{y}) \neq 0$, & J continuous.

$\Rightarrow J$ non-zero

Under all these conditions, $\underset{\sim}{Y}$ has a density

$$f_{\underset{\sim}{Y}}(y_1, \ldots, y_m) = f_{\underset{\sim}{X}}(\underbrace{\psi}_{\varphi^{-1}}(y_1, \ldots, y_n)) \cdot \left| J(y) \right|, \quad \underset{\sim}{y} \in \mathcal{J}.$$

Eg. $X, Y \overset{iid}{\sim} \mathcal{E}xp(\lambda)$.

$U = X+Y$
$V = X-Y$. find the joint density of (U,V).

Here, $\underset{\sim}{X} = (X,Y)$ has a density—

$$f(x,y) = \lambda^2 \cdot e^{-\lambda(x+y)}, \quad x>0, y>0.$$

or, $I = (0,\infty)\times(0,\infty)$

$\varphi : I \longrightarrow J$

$(x,y) \longmapsto (U,V)$.
$= (x+y, x-y)$

ie, $\begin{bmatrix} U \\ V \end{bmatrix} = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$,

A - 2×2.

$A = ?$

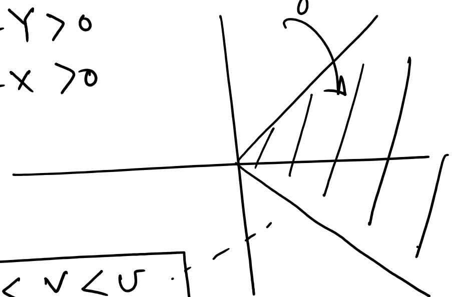
$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Now, $U = X+Y$
$V = X-Y$

$U - V = 2Y > 0$
$U + V = 2X > 0$
$\Rightarrow V < U$
$\Rightarrow V > -U$

$\boxed{\therefore -U < V < U \ \&, U > 0}$

region.



$\therefore \psi : (U,V) \longmapsto (x,y)$
$= \left(\frac{1}{2}(U+V), \frac{1}{2}(U-V)\right)$

$\therefore D\psi(U,V) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$\therefore |J| = \left|\det\left(D\psi(U,V)\right)\right|$
$= 1/2$.

$\therefore$ The density of (U,V) is $g(u,v) = \lambda^2 \cdot e^{-\lambda u} \cdot \frac{1}{2}$; $\forall (u,v) \in J$.

$$J = \left\{ (u,v) : \begin{array}{l} u > 0, \\ -u < v < u \end{array} \right\}.$$

Eg. Y_1, Y_2, Y_3, Y_4 i.i.d Gamma with parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

ie, $Y_i \sim \text{Gamma}(1, \alpha_i) \equiv \text{Gamma}(\alpha_i)$

scale parameter

$$X_1 = \frac{Y_1}{Y_1 + Y_2 + Y_3 + Y_4}$$

$$X_2 = \frac{Y_2}{Y_1 + Y_2 + Y_3 + Y_4}$$

$$X_3 = \frac{Y_3}{Y_1 + Y_2 + Y_3 + Y_4}$$

find joint density of X_1, X_2, X_3.

take: $Z = Y_1 + Y_2 + Y_3 + Y_4$
(to make a $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$ transformation, hence to get a square matrix for det $|\cdot|$.

$$\therefore f(y_1, y_2, y_3, y_4) = \frac{e^{-(y_1+y_2+y_3+y_4)}}{\Gamma(\alpha_1) \cdot \Gamma(\alpha_2) \cdot \Gamma(\alpha_3) \, \Gamma(\alpha_4)} \cdot y_1^{\alpha_1 - 1} y_2^{\alpha_2 - 1} y_3^{\alpha_3 - 1} y_4^{\alpha_4 - 1}$$

$\therefore$ Consider the transformation

$$\varphi : I \longrightarrow J$$

$$\varphi : (y_1, y_2, y_3, y_4) \longmapsto (x_1, x_2, x_3, z) \; ;$$

$$= \left(\frac{y_1}{\sum y_i}, \frac{y_2}{\sum y_i}, \frac{y_3}{\sum y_i}, \sum y_i \right)$$

$I = (0, \infty)^4$

$J = ?$

$$J = \left\{ (x_1, x_2, x_3, z) \;\middle|\; \begin{array}{l} x_1 > 0 \\ x_2 > 0 \\ x_3 > 0 \\ x_1 + x_2 + x_3 \leq 1 \end{array} \right\}$$

$\because 0 < x_1 < 1$
$0 < x_2 < 1$
$0 < x_3 < 1$

$x_1 + x_2 + x_3 \leq 1$

$\therefore$ the inverse transformation:

$$\therefore \psi : J \longrightarrow I.$$

$$(x_1, x_2, x_3, z) \longmapsto (y_1, y_2, y_3, y_4)$$

$$= \left(x_1 z, \; x_2 z, \; x_3 z, \; (1 - x_1 - x_2 - x_3) \cdot z \right)$$

Notice $\nearrow$
all coordinates > 0 ✓

[$\therefore$ J, as we evaluated is correct.]

$$J = \det \begin{pmatrix} z & 0 & 0 & x_1 \\ 0 & z & 0 & x_2 \\ 0 & 0 & z & x_3 \\ -z & -z & -z & 1 - x_1 - x_2 - x_3 \end{pmatrix}$$

$$= \quad {}^{3}$$

$$J = z^3$$

$$\left[\because z = y_1 + y_2 + y_3 + y_4\right]$$

$$\therefore g(x_1, x_2, x_3, z) = \frac{1}{\Gamma(\alpha_1)\cdot\Gamma(\alpha)\cdot\Gamma(\alpha_3)\cdot\Gamma(\alpha_4)} \cdot e^{-z} \cdot$$

$$(x_1 z)^{\alpha_1 - 1} (x_2 z)^{\alpha_2 - 1} \cdot (x_3 z)^{\alpha_3 - 1} \cdot z^{\alpha_4 - 1} \cdot (1 - x_1 - x_2 - x_3)^{\alpha_4 - 1} \cdot \underset{\substack{\uparrow \\ |J|.}}{z^3}$$

$$(x_1, x_2, x_3, x_4) \in J.$$

$$= \frac{1}{\Gamma(\alpha_1 + \cdots + \alpha_4)} \cdot \frac{\Gamma(\alpha_1 + \cdots + \alpha_4)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_4)} \cdot e^{-z} \cdot z^{\alpha_1 + \cdots + \alpha_4 - 1} \cdot x_1^{\alpha_1 - 1} \cdot x_2^{\alpha_2 - 1} \cdot x_3^{\alpha_3 - 1} \cdot (1 - x_1 - x_2 - x_3)^{\alpha_4 - 1}$$

itself a density. $\left(\int (\cdot)\, dz = 1\right)$.
integrating out this whole expression w.r.t z, this becomes the density of Dirichlet $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

Concept : Tail event.

$\{X_1, X_2, \ldots\}$.
An event A is a tail event of $A \in \sigma(X_n, X_{n+1}, \ldots)$ $\forall\, n \geq 1$.

Eg. Event $A := \{ \omega : X_1(\omega) + X_2(\omega) + X_3(\omega) \le 3 \}$.

then, $A \in \sigma(X_1, X_2, X_3)$.

ie, if A is a tail event, with just any given tail of the seq. $X_1, X_2, \ldots$ we can say whether A occurs or <u>not</u>.

Eg. $A = \{ \omega : \lim_n X_n(\omega) \text{ exists} \}$.

$A = \{ \omega : \sum_n X_n(\omega) \text{ converges} \}$.

these are tail events.

$A = \{ \omega : \underline{\sum_n X_n > 0} \text{ converges} \}$. $\longrightarrow$ <u>NOT</u> a tail event.

for $\sum X_n > 0$ we require to know each of the X_i's.

Eg. $A = \{ \omega : \sup_n X_n > 23 \}$.

$\rightarrow$ Not tail event.

why?

we can have

$$X_i = \begin{cases} 25, & i = 1 \\ 0, & i \ge 2 \end{cases}$$

So, information on any tail isn't enough.

What's a tail r.v ?

Say, Y is a r.v.

if the event $A_B = Y \in B$ $\forall B$ is

Borel set.

a tail event, then

Y is a tail r.v.

In practice, we don't need to see for all such Borel set 'B'

We have to check for things whether,

- the event $Y \geq a$ for some a, or $\forall a$ is a tail event or not

- the event $Y \leq a$ for some a, or $\forall a$ is a tail event or not.

Eg: $Y = \sum_n X_n$ is NOT a tail r.v.,

$\because A := \{\omega : \sum_n X_n > 0\}$ is NOT a tail event.