Gauss-Markov Model: Estimability

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The concept of identifiability is central to all statistical models. But in the special case of linear models with which we are dealing, the condition of identifiability is reduced to a linear algebraic equivalence given as:

$$\vec{c}^{\,'}\vec{\beta} \text{ is identifiable} \Leftrightarrow \vec{c}^{\,'} \in \Re(X) \tag{1}$$

If $\vec{c}^{\,'} \notin \Re(X)$, then it is a meaningless problem to try to estimate $\vec{c}^{\,'}\vec{\beta}$, whichever technique we use. As we saw earlier, trying to estimate β_1 when it is always present as $\beta_1 + \beta_2$ will amount to nothing because the vector $\begin{bmatrix}1 & 0 & 0\end{bmatrix}$ was not present in the row space of the design matrix.

On the other hand, if $\vec{c}^{\,'} \in \Re(X)$, then we can obtain $\vec{c}^{\,'}$ by some linear combination of the rows of X. If we perform the same operations on our data, then we can find a linear function of our data which can be used to estimate $\vec{c}^{\,'}\vec{\beta}$. To demonstrate this, let us consider an example. Suppose we have the equations

$$\begin{aligned} y_1 &= \beta_1 \qquad\;\;\, + \epsilon_1 \\ y_2 &= \beta_1 + \beta_2 + \epsilon_2 \end{aligned}$$

and we have to estimate β_2. So, we have to estimate $\vec{c}^{\,'}\vec{\beta}$ with $\vec{c}^{\,'} = \begin{bmatrix}0 & 1\end{bmatrix}$. The design matrix in this case is $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and it is easily seen that we can obtain $\begin{bmatrix} 0 & 1 \end{bmatrix}$ by subtracting the first row from the second. So, we perform the same operations on our data. That is, we subtract the first entry from the second entry to obtain $y_2 - y_1$. We claim this to be an unbiased estimator of β_2. This can be verified as subtracting the first equation from the second equation gives us

$$y_2 - y_1 = \beta_2 + \epsilon_2 - \epsilon_1$$

and $E(y_2 - y_1) = \beta_2 + E(\epsilon_2 - \epsilon_1) = \beta_2$.

Mathematically, let $\vec{c}^{\,'} = \vec{v}^{\,'}X$ for some $\vec{v}$. Then, we claim that $\vec{v}^{\,'}\vec{y}$, which is linear in $\vec{y}$, is as an unbiased estimator for $\vec{c}^{\,'}\vec{\beta}$.

Conversely, if we can find a linear combination of our data which is an unbiased estimator for $\vec{c}^{\,'}\vec{\beta}$, then by applying the same linear combination to the rows of X, we can obtain $\vec{c}^{\,'}$. In our example, we see that $y_2 - y_1$ is a linear combination of the data which is also an unbiased estimator for β_2. So, we apply the same linear combination to the rows of X. We subtract the first row from the second to obtain $\begin{bmatrix} 0 & 1 \end{bmatrix}$, which is precisely $\vec{c}^{\,'}$.

Mathematically, if there exists a $\vec{v}$ such that $\vec{v}^{\,'}\vec{y}$ is an unbiased estimator for $\vec{c}^{\,'}\vec{\beta}$, then $\vec{c}^{\,'} = \vec{v}^{\,'}X$ i.e., $\vec{c}^{\,'} \in \Re(X)$.

The above discussion leads us to the following conclusion:

$$\vec{c}^{\,'}\vec{\beta} \text{ is identifiable} \Leftrightarrow \vec{c}^{\,'} \in \Re(X) \Leftrightarrow \exists \ \vec{v} \ \ni \ E(\vec{v}^{\,'}\vec{y}) = \vec{c}^{\,'}\vec{\beta} \ \forall \ \vec{\beta} \tag{2}$$

So, whenever we have a linear parametric function which is identifiable, we know that it is a meaningful thing to try to estimate it. In addition, we also get one way by which we can estimate it using a linear function of the data. This estimator will also be unbiased though it may not necessarily be the best. Hence, identifiable turns out to be synonymous with linearly, unbiasedly estimable. This linearly, unbiasedly estimable concept occurs so very frequently in the world of linear models that we tend to drop these two adjectives and refer to it as simply estimable. Thus, identifiable is equivalent to estimable.