Probability-2 Lecture-10

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X - real r.v. on a probability space $(\Omega, \mathcal{A}, P)$:

Distribution function:

$$P(X^{-1}(B)) = P(X \in B) \quad , \; B \in \mathcal{B}.$$

$\hookrightarrow$ this is a probability on $\mathcal{B}$, denoted by P_X.

Distribution function:

$$F_X : \mathbb{R} \to \mathbb{R}$$

$$F_X(a) = P(X \le a) = P((-\infty, a])$$

fact: $F_X(\cdot)$ determines P_X uniquely.

Properties:

① F_X is non-decreasing.

② F_X is right continuous

③ $\lim_{a \to \pm\infty} F_X(a) = \begin{cases} 1 \\ 0 \end{cases}$.

$F_X(a^-) = \lim_{y \nearrow a} F_X(y) = P_X((-\infty, a))$
$= P(X < a)$.

F_X - continuous at a
$\iff P_X(\{a\}) = 0$.

Definition:

A function $F : \mathbb{R} \to \mathbb{R}$ with properties ①, ②, ③ (above) is called a (Probability) Distribution Function.

Result: Given any distribution function F on $\mathbb{R}$, then exists a unique probability Q on $\mathcal{B}$, such that,

$$F(a) = Q((-\infty, a])$$

Corollary: Given any dist^n f^n F on $\mathbb{R}$, $\exists$ a prob. space $(\Omega, \mathcal{A}, P)$ & a real r.v. X on it s.t. $F = F_X$.

take $\Omega = \mathbb{R}$.

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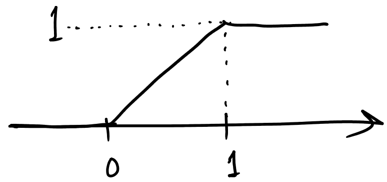
$\mathcal{A} = \mathcal{B}$.

$X : \mathbb{R} \longrightarrow \mathbb{R}$ is the identity function.

Take P to be the Q in the prev. result.

Consider the function $F : \mathbb{R} \rightarrow \mathbb{R}$ defined as:

$$F(a) = \begin{cases} 0, & a < 0 \\ a, & 0 \leq a \leq 1 \\ 1, & a > 1 \end{cases}$$



$\therefore \exists$ a unique prob. Q_1 on $\mathcal{B}$ s.t.

$$Q_1((-\infty, a]) = F(a) \quad \forall a \in \mathbb{R}$$

observe, $Q_1((0,1)) = 1$

$$\Rightarrow Q_1(B) = 0 \quad \forall B \subset (0,1)^c$$

Let $I = (0,1)$

$$\mathcal{A} = \{B \in \mathcal{B} : B \subset (0,1)\}.$$

$\rightarrow$ σ-field on I.

$P(A) = Q_1(A)$ for $A \in \mathcal{A}$ is a prob. on $\mathcal{A}$.

$(I, \mathcal{A}, P)$: probability space

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be any distribution function.

Define $X : I \longrightarrow \mathbb{R}$ as

$$X(u) := \sup\{x \in \mathbb{R} ; F(x) < u\}, \quad u \in I = (0,1).$$

We show, $\{u : X(u) \leq a\} = \{u : u \leq F(a)\}$

i.e, We show, $X(u) \le a \iff u \le F(a)$

X is non-decreasing. Hence, a real random variable.

<u>To show:</u>

for any $u \in (0,1)$ & $a \in \mathbb{R}$,

$X(u) \le a \iff u \le F(a)$.

"$\Rightarrow$" i.e, we'll show, $X(u) \le a \Leftarrow u \le F(a)$.

equivalently, $X(u) > a \Leftarrow u > F(a)$

i.e, $F(a) < u$,

then, $\exists\, b > a$ s.t. $F(b) < u$ (right continuity of F)

$\Rightarrow X(w) \geq b > a$

Thus, $u > F(a) \Rightarrow X(u) > a$.

"$\Leftarrow$" Suppose $X(u) > a$

Then, $u > F(a)$. ▨

For any distribution function F on $\mathbb{R}$:

define: $G(\alpha) := \sup\{x : F(x) < \alpha\}$.

α^{th} quantile of F.

$G \equiv X$

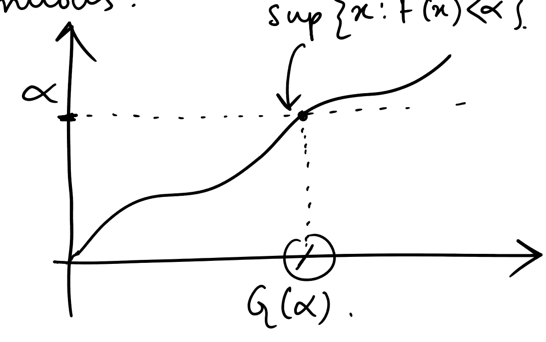
$\alpha \equiv u$

(from prev. section)

<u>Case-I</u>: F is strictly increasing, continuous.

$\Rightarrow$ F - 1-1 f^n on $(0,1)$

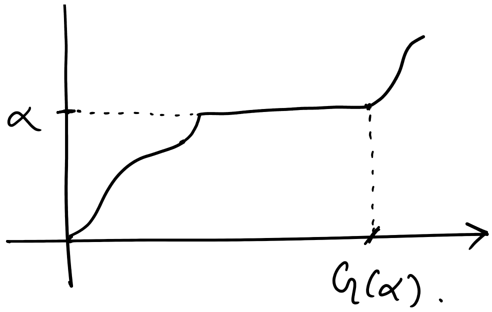
Here, $G(\alpha) = F^{-1}(\alpha)$



<u>Case-II</u>: F is non-decreasing, continuous.

Here, $G(x) = \inf\{x : F(x) > \alpha\}$

Here, $G(\alpha) := \inf\{x : F(x) \geq \alpha\}$



Definition:

A real random variable X is said to be discrete if $\exists$ a countable set $D (= D_X) \subset \mathbb{R}$ s.t.,

$$P(X \in D) = 1.$$

In this case, $p(x) = P(X = x)$, $x \in D$ is called the pmf of X.

$$P_X(B) = \sum_{x \in B} p(x), \quad B \in \mathcal{B}.$$

$$F_X(a) = \sum_{x : x \leq a} p(x), \quad a \in \mathbb{R}.$$

Def^n: A random variable is said to be Continuous if F_X is continuous. $(\Leftrightarrow P(X = x) = 0 \ \forall\, x \in \mathbb{R})$.

A random variable X is said to be absolutely continuous if $\exists$ a non-ve f^n f_X on $\mathbb{R}$ s.t.

$$F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx \quad \forall\, a \in \mathbb{R} \text{ ——— } (*)$$

f_X is called the density f^n of X.

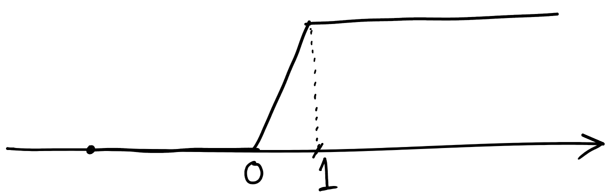
Theorem: (Lebesgue differentiation theorem)
$(*)$ holds iff F_X is absolutely continuous in every closed, bounded interval. (Proof is out of scope of this course.)

Definition:

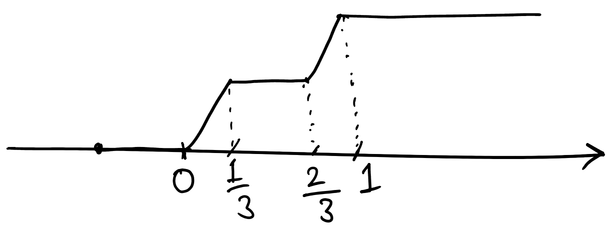
A function $g : \mathbb{R} \to \mathbb{R}$ is said to be absolutely continuous on $[a, b]$ if $\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ s.t. $\forall$ choice of $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ and

$$\sum_{i=1}^{n} |y_i - x_i| < \delta \implies \sum_{i=1}^{n} |g(y_i) - g(x_i)| < \varepsilon$$

$$F_0(x) := \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \geq 1. \end{cases}$$



$$F_1(x) := \begin{cases} 0, & x \leq 0, \\ \ , & 0 < x \leq 1/3 \\ \ , & 1/3 < x \leq 2/3 \\ \ , & 2/3 < x < 1 \\ 1, & x \geq 1 \end{cases}$$



$$F_{n+1}(x) := \begin{cases} 0, & x < 0 \\ \frac{1}{2} \cdot F_n(3x), & 0 \leq x \leq \frac{1}{3} \\ \frac{1}{2}, & \frac{1}{3} < x < \frac{2}{3} \\ \frac{1}{2} + F_n(3x-2), & \frac{2}{3} \leq x \leq 1 \\ 1, & x > 1. \end{cases}$$

Exercise: prove that

$$|F_{n+1}(x) - F_n(x)| \leq \frac{1}{2} \cdot |F_n(x) - F_{n-1}(x)|$$

$$\Downarrow$$

$$\sup_x | \quad\cdot\quad | \leq \frac{1}{2} \cdot \sup | \quad\cdot\quad |$$

$$\therefore \sup_x |F_{n+1}(x) - F_n(x)| \leq \left(\frac{1}{2}\right)^n \sup_x \cdot |F_1(x) - F_0(x)|$$

$$\therefore \sup_x \left| F_m(x) - F_n(x) \right| \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty$$

(Cauchy).

Here, F_1 does <u>not</u> give any "mass" to $\left(\frac{1}{3}, \frac{2}{3}\right)$

F_2 does <u>not</u> give any mass to $\left(\frac{1}{3}, \frac{2}{3}\right)$,

$\left(\frac{1}{9}, \frac{2}{9}\right)$, $\left(\frac{7}{9}, \frac{8}{9}\right)$

$\vdots$

In general,

F_n does <u>not</u> give any mass to the complement of the Cantor's Middle Third Set.

[think!]