Probability-2 Lecture-12

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Issue: X is an absolutely continuous random variable with density function f_X.

$Y = h(X)$ is a random variable.

Qs. When does Y have a density? How to find F_Y?

Theorem:

X - absolutely continuous with density $f_X > 0$.

If $h: I \to J$, I, J - open intervals is continuously differentiable with h never vanishing on I (i.e, $h(x) \neq 0 \ \forall \ x \in I$), then,

$$\boxed{f_Y(y) = f_X(h^{-1}(y)) \cdot \frac{1}{|h'(h^{-1}(y))|}} \quad \forall \ y \in J.$$

Remark: $h' \neq 0$ on I and continuous.

$\Rightarrow h' > 0$ or $h' < 0$ on I.

$\Rightarrow h$ is one-one on I.

$\Rightarrow h : I \xrightarrow[\substack{1-1, \\ \text{cont.}}]{} J$.

$\Rightarrow h$ has an inverse.

$$g := h^{-1} : J \to I,$$

which is differentiable and

$$g'(y) = \frac{1}{h'(g(y))}$$

Proof: fix $a \in J$.

$$F_Y(a) = P(h(X) \leq a) = P(X \leq h^{-1}(a)).$$

$$= P(X \leq g(a))$$

...

$$= P(X \leqslant g(a))$$

$$= \int_{-\infty}^{g(a)} f_X(x)\,dx$$

$$\Rightarrow F_Y(a) \underset{\text{(substitution)}}{=} \int_{-\infty}^{a} f_X(g(y))\cdot g'(y)\,dy \quad \Bigg| \quad \begin{aligned} x &= g(y) \\ dx &= g'(y)\,dy \end{aligned}$$

$$\therefore f_Y(y) = f_X(g(y))\cdot g'(y).$$

$$= f_X(h^{-1}(y))\cdot \frac{1}{|h'(g(y))|}$$

$$\underset{Y}{\text{pdf of}} \longrightarrow f_Y(y). = f_X(h^{-1}(y))\cdot \frac{1}{|h'(h^{-1}(y))|} \quad \blacksquare$$

Applications:

① X has density f_X.

Then, for any $c \neq 0$, $d \in \mathbb{R}$.

$Y = cX + d.$ has a density

$$f_Y(y) = f_X\left(\frac{y-d}{c}\right)\cdot \frac{1}{|c|}$$

Eg: $X \sim \text{Unif}(0,1)$

$Y = cX + d \sim \text{Unif}(d, c+d)$, $c > 0$

$\sim \text{Unif}(c+d, d)$, $c < 0$.

(ie, Uniform distⁿ is invariant under "shifting" or "scaling")

$cX + d$

② X has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\cdot e^{-x^2/2} \quad , \quad -\infty < x < \infty.$$

(i.e. $X \sim N(0,1)$)

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(i.e, $X \sim N(0,1)$).

$Y = cX + d$.

$$\therefore f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{y-d}{c}\right)^2} \cdot \frac{1}{|c|} = \frac{1}{|c| \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{y-d}{c}\right)^2}$$

write μ for d,

σ for $|c|$:

then, $$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}.$$

$\therefore Y \sim N(\mu, \sigma^2)$.

③ $X \sim \text{Unif}(0,1)$.

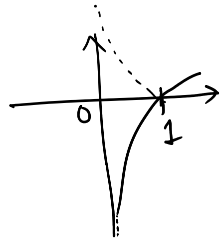
$Y = -\log(X)$.

$f_X(x) = 1 \quad \forall \; 0 < x < 1$

$\therefore h : (0,1) \to (0,\infty)$

$\therefore h(x) = -\log x$

$\therefore h'(x) = -\frac{1}{x}$.



$\therefore h \to$ continuously differentiable.

$h(x) = -\log x$

$\therefore g(y) = h^{-1}(y)$

$= e^{-y}$

$$f_Y(y) = \underbrace{f_X(h^{-1}(y))} \cdot \frac{1}{|h'(h^{-1}(y))|}$$

$$= 1 \cdot e^{-y} = e^{-y} = 1 \cdot e^{-1 \cdot y}.$$

$$\left|\frac{1}{h'(e^{-y})}\right| = \left|\frac{1}{-\frac{1}{e^{-y}}}\right| = e^{-y}$$

$\therefore Y \sim \text{Exp}(1)$.

Some examples "beyond" the theorem:

$\to$ ie, cases where the theorem cannot be applied mechanically.

① $X \sim N(0,1)$.

$Y = X^2$.

$a \geq 0$.

$$F_Y(a) = P(Y \leq a)$$
$$= P(X^2 \leq a)$$
$$= P(-\sqrt{a} \leq X \leq \sqrt{a})$$
$$= \int_{-\sqrt{a}}^{\sqrt{a}} \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \, dx.$$

$y = x^2$

$dy = 2x \, dx$

$\Rightarrow dx = \frac{dy}{2\sqrt{y}}$

$$= \frac{\cancel{2}}{\sqrt{2\pi}} \cdot \int_0^{\sqrt{a}} \frac{e^{-y/2}}{\cancel{2}\sqrt{y}} \cdot dy$$

$$\therefore f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-y/2}}{\sqrt{y}}, \quad y > 0.$$