Probability-3 Lecture-16

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Convergence in Distributions.

Def^n: Let $\{X_n\}_{n\geq 1}$ and X be real r.vs.

Let $F_n, n\geq 1$ and F be cdfs of $X_n, n\geq 1$ & X respectively

We say, $\{X_n\}$ converges in distribution to X, or

$$X_n \xrightarrow{d} X \quad \text{if} \quad F_n(x) \longrightarrow F(x) \ \forall\ x \in C_F$$

$C_F \equiv$ set of all continuity pts. of F.

If F is a cdf, then C_F denotes the set of continuity points of F.

Since $\mathbb{R}\setminus C_F$ is countable, $\therefore C_F$ is dense in $\mathbb{R}$.

<u>Example</u> to show why pointwise convergence is relaxed to only convergence at pts. in C_F.

$X_n \sim \text{Unif}\left(-\frac{1}{n}, \frac{1}{n}\right)$.

X_n has density $f_n(x) = \frac{n}{2}$, $-\frac{1}{n} < x < \frac{1}{n}$.

$$X_n \xrightarrow{d} X \equiv 0$$

$$F_n = \begin{cases} 0, & x \leq -\frac{1}{n} \\ \dfrac{x + \frac{1}{n}}{2n}, & -\frac{1}{n} < x < \frac{1}{n} \\ 1, & x \geq \frac{1}{n} \end{cases}$$

$$\begin{cases} 1, & x \geq 1/n \end{cases}$$

Let $n \to \infty$,

$x < 0$. $F_n(x) = 0$ for all n large enough

$F_n(x) \to 0$

& $x > 0$. $F_n(x) = 1$ for all n large enough.

$F_n(x) \to 1$

at $x = 0$, $F_n(0) = \frac{1}{2}$ $\forall n$.

So, $F = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$

This is the limit required for pointwise convergence.

$\therefore$ If we had "wanted" ptwise conv.,

then $X_n \not\to X \equiv 0$

$\downarrow$

r.v degenerate at 0.

$\therefore$ Hence, we relax pt-wise convergence.

<u>Remark:</u>

① Conv. in dist^n does <u>not</u> require X_n's, X all to lie in the same probability space.

② $\left.\begin{array}{l} X_n \xrightarrow{d} X \\ X_n \xrightarrow{d} Y \end{array}\right\} \Longleftrightarrow F_X = F_Y$, or $X \overset{d}{=} Y$ (ie, X, Y must have the same cdf.)

③ $\left.\begin{array}{l} X_n \xrightarrow{d} X \\ Y_n \xrightarrow{d} Y \end{array}\right\} \not\Longrightarrow X_n \pm Y_n \xrightarrow{d} X \pm Y$

[counter eg: let $Z \sim N(0,1)$.
take $X_n = Z$ $\forall n \geq 1$, $Y_n = -Z$ $\forall n \geq 1$.

$X_n \to Z$
$Y_n \to -Z$

$X_n + Y_n = 0$.

But, $(X_n + Y_n) \longrightarrow 2Z \neq 0$.
$\because Z = -Z$.]

Result:

$$X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$$

Proof: take any $x \in \mathbb{R}$.

$$\liminf_n F_n(x) \geq F(x^-)$$
$$\limsup_n F_n(x) \leq F(x^+).$$

Proving these two are enough, as, for $x \in C_F$, $F(x^+) = F(x^-)$.

Remark: this is necessary & sufficient (ie, equivalent) condition for convergence in

condition for convergence in distribution.

$$F_n(x) = P(X_n \le x)$$
$$= \underset{\shortparallel}{P(X_n \le x, X \le x+\varepsilon)} + \underset{\shortparallel}{P(X_n \le x, X \ge x+\varepsilon)}$$
$$\le F(x+\varepsilon) + \underbrace{P(|X_n - X| > \varepsilon)}_{\downarrow\, 0}$$

(ie, limit exists, $\because X_n \xrightarrow{P} X$) $\therefore \limsup P(|X_n - X| > \varepsilon) = 0$

taking limsup,

$$\limsup_n F_n(x) \le F(x+\varepsilon)$$

letting $\varepsilon \searrow 0$,

$$\limsup_n F_n(x) \le F(x^+)$$

the other side:

$$P(X \le x-\varepsilon)$$
$$= P(X \le x-\varepsilon, X_n \le x) + P(X \le x-\varepsilon, X_n > x).$$
$$\le F_n(x) + P(|X_n - X| > \varepsilon)$$

$$F_n(x) \ge F(x-\varepsilon) - \underbrace{P(|X_n - X| > \varepsilon)}_{\to \liminf P(|X_n - X| > \varepsilon) = 0}.$$

$\therefore$ taking liminf.

$$\liminf_n F_n(x) \ge F(x-\varepsilon)$$

letting $\varepsilon \searrow 0$

$$\therefore \liminf_n F_n(x) \ge F(x^-).$$

Conv. in dist^n $\equiv$ "Weak convergence".

$$X_n \xrightarrow{a.s} X \underset{\not\Leftarrow}{\Rightarrow} X_n \xrightarrow{d} X$$
$$X_n \xrightarrow{P} X \underset{\not\Leftarrow}{\Rightarrow} X_n \xrightarrow{d} X$$

Exercise: $X_n \xrightarrow{d} X \not\Rightarrow X_n \xrightarrow{P} X$

$Z \overset{d}{=} N(0,1)$.

Take $X_{2n} = Z$, $X_{2n-1} = -Z$. $\quad n = 1, 2, \ldots$

$\therefore \quad X_{2n} \overset{d}{=} X_{2n-1} \overset{d}{=} Z \qquad Z = -Z$ (symm. around 0)

$\therefore X_n \xrightarrow{d} Z$.

But, $X_{2n} - X_{2n-1} = Z - (-Z)$
$= 2Z$.

$\therefore X_{2n} - X_{2n-1}$ is <u>not</u> even Cauchy !!

Result: $X_n \xrightarrow{d} X \equiv c$ (X - degenerate at c. i.e., $X = c$ with prob $= 1$)

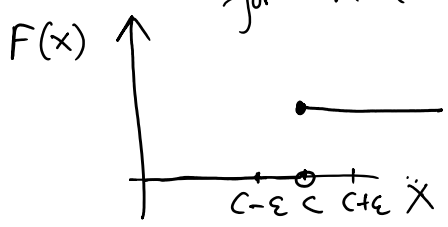
$\Rightarrow X_n \xrightarrow{P} X$

Proof: Fix $\varepsilon > 0$.

$$P(|X_n - X| > \varepsilon) = P(|X_n - c| > \varepsilon)$$
$$= P(X_n < c - \varepsilon) + 1 - P(X_n \le c + \varepsilon)$$
$$\le F_n(c-\varepsilon) + 1 - F_n(c+\varepsilon)$$
$$\to 0 + 1 - 1 = 0$$

$\left[\because F_n(c-\varepsilon) \to F(c-\varepsilon) = 0 \right.$
$\left. F_n(c+\varepsilon) \to F(c+\varepsilon) = 1 \right]$ for n large.

$\therefore \boxed{X_n \xrightarrow{P} X}$.



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Slutsky "type" result:

(we saw just now, if limit r.v X is degenerate at some c, then, $X_n \xrightarrow{d} X \iff X_n \xrightarrow{p} X$)

$$X_n \xrightarrow{d} X, \quad Y_n \xrightarrow{p} 0.$$

$$\Rightarrow X_n + Y_n \xrightarrow{d} X.$$

<u>Proof</u>: Let $X_n \sim F_n$. $n \geq 1$.

$$X_n + Y_n \sim G_n.$$

$$X \sim F.$$

Fix pts. $x, y \in C_F$

WLOG, $x < y$.

We'll prove:

$$(*) \begin{cases} \liminf G_n(y) \geq F(x) \\ \limsup G_n(x) \leq F(y). \end{cases}$$

We'll first accept this $(*)$ & complete the proof.

We'll prove this $(*)$ later then.

Take any $z \in C_F$.

We have to show,

$$\lim_n G_n(z) = F(z).$$

We'll consider 2 sequences,

one strictly increasing to z,

another strictly decreasing to z.

another strictly decreasing to z.

So, let $\{x_k\}_{k\geq 1}$ be a sequence in C_F.

$$x_k < z \quad \forall k$$

$$x_k \nearrow z$$

&, let $\{y_k\}_{k\geq 1}$ be a sequence in C_F,

$$y_k < z \quad \forall k$$

$$y_k \searrow z$$

By (*): $\liminf G_n(z) \geq F(x_k) \quad \forall k$

[for pair (x_k, y_k), $x_k \in C_F$, $y_k \in C_F$]

&, $\limsup G_n(z) \leq F(y_k) \quad \forall k$

$\therefore$ By letting $k \nearrow \infty$, & $z \in C_F$,

we get,

$$F(z) \leq \liminf G_n(z) \leq \limsup G_n(z) \leq F(z)$$

$$\therefore \liminf_n G_n(z) = \limsup_n G_n(z) = \lim_n G_n(z) = F(z)$$

So, we're done !!!

(modulo this (*)).

Now, proof of (*):

$$G_n(x) = P(X_n + Y_n \leq x)$$

$$= P(X_n + Y_n \leq x, X_n \leq y) + P(X_n + Y_n \leq x, X_n > y)$$

$$\leq F_n(y) + P(|Y_n| > y - x).$$

$\downarrow$ 0.

$y > x$,

$Y_n \xrightarrow{P} 0$.

taking limsup,

$$\limsup G_n(x) \leq F(y)$$

Similarly, $\liminf G_n(y) \geq F(x)$. [Prove this: exercise].

F_n, $n \geq 1$ - cdfs of X_n, $n \geq 1$

F - cdf of X.

Let $S \subset \mathbb{R}$ be a dense set.

Suppose, ① $F_n(x) \longrightarrow F(x)$.

② $\forall x \in J$ $(= C_F)$.

$$F_n(x) - F_n(x^-) \longrightarrow F(x) - F(x^-)$$

Then $\sup_n |F_n(x) - F(x)| \longrightarrow 0$

Corollary: Pólya's Theorem.

If $X_n \xrightarrow{d} X$, and if X has a continuous cdf F,

then, $F_n \longrightarrow F$ uniformly on $\mathbb{R}$.

(From previous result, set $S = \mathbb{R}$. & check that, this thm. is proved.)

*. Some "Characterization" of convergence in distribution:

Step 1:

Suppose, X_n, $n \geq 1$ is a seq. of real r.vs and X - a real r.v.

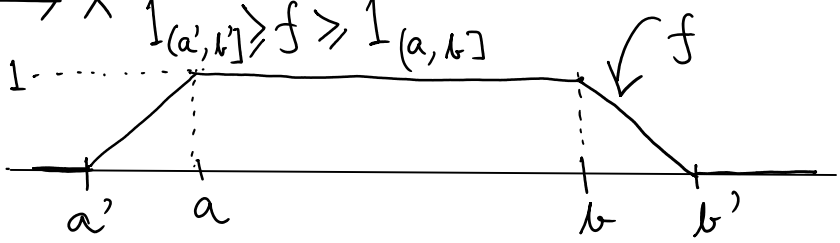
Given,

$$E f(X_n) \longrightarrow E f(X). \quad \forall f \in C_b(\mathbb{R})$$

set of all bounded, continuous functions over $\mathbb{R}$.

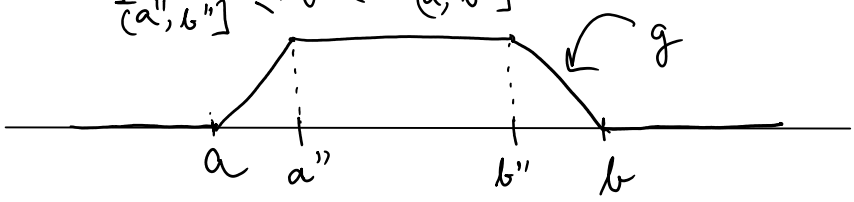
We'll show: $X_n \xrightarrow{d} X$

$1_{(a',b']} \geq f \geq 1_{(a,b]}$



$f, g \in C_b(\mathbb{R})$

$1_{(a'',b'']} \leq g \leq 1_{(a,b]}$



$$\therefore E(f(X)) = \lim_n E(f(X_n))$$

Using f, $\Rightarrow F(b) - F(a) \leq \lim_n E(f(X_n))$

$$\leq \liminf_n \left(F_n(b') - F_n(a')\right)$$

Using g,

$$\limsup_n \left(F_n(b'') - F_n(a'')\right) \leq F(b) - F(a).$$

Fix $a, b \in C_F$.

Using this, show that,

$$\boxed{\lim_n F_n(b) - F_n(a) = F(b) - F(a)}$$

[Hint / Sketch: C_F is dense. So, we can take $a', a'' \in C_F$ as close to 'a' we wish to, & same for $b', b'' \in C_F$.]

Now, take $a \searrow -\infty$.

Now, take $a \searrow -\infty$.

$$\therefore \lim_n F_n(b) = F(b).$$

$$\therefore X_n \xrightarrow{d} X.$$

Step-2: (the other side)

Given $X_n \xrightarrow{d} X$

We'll show, $Ef(X_n) \longrightarrow Ef(X)$. for all $f \in C_b(\mathbb{R})$

Proof: WLOG, we assume, $|f(x)| \leq 1 \ \forall x \in \mathbb{R}$.

Let $\varepsilon > 0$ be given.

Pick $M > 0$ with $M, -M \in C_F$ s.t.

$$F(-M) + (1 - F(M)) < \varepsilon.$$

ie, we want $-M$ very large $-$ve, & M very large +ve s.t. the probability mass of X beyond M & $-M$ adds up to $< \varepsilon$.

Realize this: it is enough to show,

$$E\left(f(X_n) \cdot 1_{X_n \in (-M, M]}\right) - E\left(f(X) \cdot 1_{X \in (-M, M]}\right) \to 0$$

call $J = (-M, M]$.

$$\therefore \left| E(f(X)) \cdot 1_{X \notin J} \right| \leq E\Big(\underbrace{|f(X)|}_{\substack{<1 \\ \text{(assumed)}}} \cdot 1_{X \notin J}\Big)$$

$$\leq E\left(1_{X \notin J}\right)$$

$$= P(X \notin J)$$

&,

$$\left| E(f(X_n) \cdot 1_{X \notin J} \right| \leq E\Big(\underbrace{|f(X_n)|}_{\substack{<1 \\ \text{(assumed)}}} \cdot 1_{X \notin J} \Big)$$

$$\leq E\left(1_{X_n \notin J}\right)$$

$$= P(X_n \notin J)$$

$$= P(X_n \leq -M) + 1 - P(X_n < M)$$

$$= \underset{\downarrow \atop F(-M)}{F_n(-M)} + 1 - \underset{\downarrow \atop F(M)}{F_n(M)} \quad \left[\because M \in C_F \,\&\, -M \in C_F\right]$$

$$\longrightarrow F(-M) + 1 - F(M)$$

$$= P(X \notin J) < \varepsilon' \quad \forall\, n \geq N_1$$

Choose $\delta > 0$ s.t, $x, y \in [-M, M]$.
& $|x-y| < \delta$
$\Rightarrow |f(x) - f(y)| < \varepsilon'$.
[using uniform continuity of f on compact set $[-M, M]$.]

Divide $[-M, M]$ into finitely many intervals, each of length $\delta/2$.

Let the partition points be called:

$$-M = y_0 < y_1 < y_2 < \cdots < y_k < y_{k+1} = M.$$

Now, between y_0 & y_1,

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i.e., in $[y_0, y_1]$,

pick $x_1 \in [y_0, y_1] = [-M, y_1]$

s.t. $x_1 \in C_F$. [We can do that, $\because C_F$ is dense.]

Similarly, pick

$x_2 \in (y_1, y_2)$,

$x_3 \in (y_2, y_3)$

$\vdots$

$x_{k+1} \in (y_k, y_{k+1}) = (y, M)$, . each $x_i \in C_F$.

$\therefore$ Due to this choices of x_i's,

$$|x_{i+1} - x_i| < \delta/2$$

Now,

define $g(x) := \sum_{i=1}^{k+1} f(x_i) \cdot 1_{(x_i, x_{i+1}]}(x)$

function on $[-M, M]$, <u>not</u> cont, agrees with f on every "left" point.

Convince yourself that — $|f(x) - g(x)| < \varepsilon \quad \forall x \in J = [-M, M]$

$$\left| E\left(f(X_n) \cdot 1_{X_n \in J}\right) - E\left(g(X_n) \cdot 1_{X_n \in J}\right) \right| < \varepsilon',$$

&,

$$\left| E\left(f(X)\cdot 1_{X\in J}\right) - E\left(g(X)\cdot 1_{X\in J}\right)\right| < \varepsilon',$$

$$\leq \left| E\left(g(X_n)\cdot 1_{X_n\in J} - E\left(\left(g(X)\cdot 1_{X\in J}\right)\right)\right| + 2\varepsilon'$$

[adding & subtracting terms. Δ-ineq.]

$$= \left| \sum_{i=1}^{k+1} f(x_i)\cdot\left(F_n(x_{i+1}) - F_n(x_i)\right) - \sum_{i=1}^{k+1} f(x_i)\cdot\left(F(x_{i+1}) - F(x_i)\right)\right| + 2\varepsilon'$$

$$= \left| \sum_{i=1}^{k+1} f(x_i)\left[\left(F_n(x_{i+1}) - F_n(x_i)\right) - \left(F(x_{i+1}) - F(x_i)\right)\right]\right| + 2\varepsilon'$$

(How? think!!) $< \varepsilon' + 2\varepsilon'$ $\forall$ $n \geq N_2$ large enough.

$= 3\varepsilon'$

$\blacksquare$