Probability-3 Lecture-6

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Definition: $X_n \xrightarrow{a.s} X$ if

$P(\{w : X_n(w) \to X(w)\}) = 1.$

ie, $\exists$ a P- null set N s.t.

for $w \notin N$, $X_n(w) \longrightarrow X(w)$.

* $X_n \xrightarrow{a.s} X$, $Y_n \xrightarrow{a.s} Y \Rightarrow$
  - $X_n + Y_n \xrightarrow{a.s} X + Y.$
  - $X_n \cdot Y_n \xrightarrow{a.s} X \cdot Y.$

* $\left.\begin{array}{l} X_n \xrightarrow{a.s} X \\ X_n \xrightarrow{a.s} Y \end{array}\right\} \Rightarrow X = Y$ a.s .

* $X_n \xrightarrow{a.s} X \Rightarrow f(X_n) \xrightarrow{a.s.} f(X)$ for any continuous function f.

* $X_n \xrightarrow{a.s} X$, $X \neq 0$ a.s .

  $\Rightarrow \frac{1}{X_n} \longrightarrow \frac{1}{X}$.

Proof: $P(X \neq 0) = 1$. & , $X_n \xrightarrow{a.s} X$

$$\Rightarrow P\left(\bigcup_k \{ |X| > 1/k \} \cap \{ X_n \to X \} \right) = 1$$

intersection of 2 sets having probability 1.

$$\Rightarrow P\Big(\underbrace{\bigcup_k \{|X| > 1/k\} \cap \{X_n \to X\}}_{A} \Big) = 1$$

& ,

$$\left\{ \frac{1}{X_n} \longrightarrow \frac{1}{X} \right\} = B \text{ (say)}.$$

$A \subseteq B$. [think why ?]

$$\Rightarrow P\left(\left\{ \frac{1}{X_n} \longrightarrow \frac{1}{X} \right\} \right) = 1 \qquad \boxtimes$$

Definition: (Converges in Probability)

$\{X_n\}$ - real r.v.s.

X - real r.v.

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X - real r.v.

Say that, $\{X_n\}$ converges in probability to X,
if $\forall \varepsilon > 0, \forall \delta > 0, \exists n_0 \in \mathbb{N}$ s.t $\forall n > n_0$,

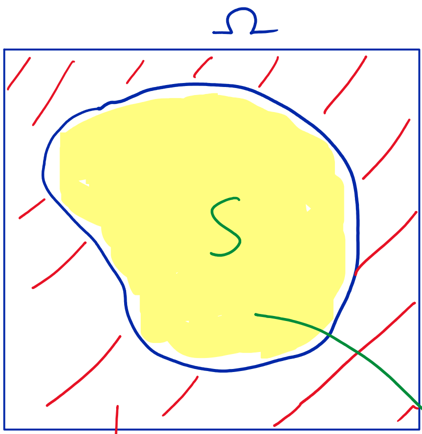
$$P(|X_n - X| > \varepsilon) < \delta.$$

ie, if $\forall \varepsilon > 0$, $P(|X_n - X| > \varepsilon) \longrightarrow 0$ as $n \longrightarrow \infty$

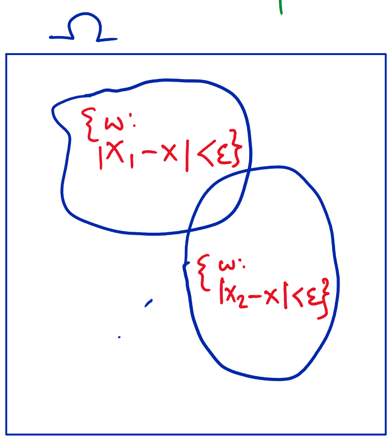
Notice that:
for $\varepsilon' = \min\{\varepsilon, \delta\}$, this holds.

<u>Notation:</u> $X_n \xrightarrow{P} X$

Bit of explanation (a.s convergence v/s convergence in probability)



this shaded area is that one P-Null set.
If we "throw this away"
in this
$\forall w \in$ remaining set = S (say)
$X_n(w) \longrightarrow X(w)$. $P(S^c) = 0$.



fix $\varepsilon > 0$.
We look at the sets
$\{w : |X_i - X| < \varepsilon\}$.
& hence, the sequence
$P(\{w : |X_i - X| < \varepsilon\})$
converges.

$*$ $\left.\begin{array}{l} X_n \xrightarrow{P} X \\ X_n \xrightarrow{P} Y \end{array}\right\} \Rightarrow X = Y$

ie, to show, $P(X \neq Y) = 0 \Leftrightarrow P\left(\bigcup_k \{|X - Y| > 1/k\}\right) = 0$

replacement for ε.

countable

Proof: fix $\varepsilon > 0$.

ie, to show, $P(X \neq Y) = 0 \iff P\Big(\underbrace{\bigcup_k \ldots}_{\text{countable union.}}\Big)$

Proof: fix $\varepsilon > 0$.

$$P(|X-Y| > \varepsilon) = 0.$$

So, we just need to show,

$$P(\{|X-Y| > 1/k\}) = 0 \quad \forall k$$

[then, by countable additivity, the result would follow.]

$$P(\underbrace{|X-Y| > \varepsilon}_{A}) \leq P(\underbrace{|X_n - X| > \varepsilon/2}_{B}) + P(\underbrace{|X_n - Y| > \varepsilon/2}_{C.})$$

to show this, its enough to show:

$A \subset B \cup C$,

or $A^c \supset B^c \cap C^c \longrightarrow$ we use Δ-inequality to show this.

Here,

$B^c =$ event that $|X_n - X| \leq \varepsilon/2 = \{\omega : |X_n(\omega) - X(\omega)| \leq \varepsilon/2\}$

$C^c =$ event that $|X_n - Y| \leq \varepsilon/2 = \{\omega : |X_n(\omega) - Y(\omega)| \leq \varepsilon/2\}$.

$\therefore$ By Δ-inequality,

$$|X-Y| \leq \underbrace{|X_n - X|}_{<\varepsilon/2} + \underbrace{|X_n - Y|}_{<\varepsilon/2} < \varepsilon.$$

$\therefore A \subset B \cup C$.

$$\therefore P(A) \leq P(B \cup C) \leq \underbrace{P(B)}_{\substack{<\delta/2 \\ \forall n > N_1}} + \underbrace{P(C)}_{\substack{<\delta/2 \\ \forall n > N_2}} < \delta. \quad [\forall n > N = \max\{N_1, N_2\}]$$

$\boxed{///}$

(*) $X_n \xrightarrow{p} X \Rightarrow c \cdot X_n \xrightarrow{p} cX$.

Case-I: $c = 0$. (trivial: nothing to prove)

Case-II. $c \neq 0$. then, for $\varepsilon > 0$.

$$P(|cX_n - cX| > \varepsilon)$$
$$= P(|X_n - X| > \varepsilon/|c|) \longrightarrow 0$$

as $X_n \xrightarrow{p} X$

(*) $\left.\begin{array}{l} X_n \xrightarrow{p} X \\ Y_n \xrightarrow{p} Y \end{array}\right\} X_n + Y_n \xrightarrow{p} X + Y$.

$$P(\overbrace{|X_n + Y_n - X - Y|}^{A} > \varepsilon)$$

$\ldots P(|\ldots| > \ldots) + P(|X_n - X| > \varepsilon/\ldots)$

$P(|X_n + Y_n - X - Y| > \varepsilon)$

$$\leq P(\underbrace{|X_n - X| > \varepsilon/2}_{B}) + P(\underbrace{|Y_n - Y| > \varepsilon/2}_{C})$$

Again, enough to show, $A \subset B \cup C$

$\Rightarrow A^c \supset B^c \cap C^c$

B^c = event that $|X_n - X| \leq \varepsilon/2$

C^c = event that $|Y_n - Y| \leq \varepsilon/2$.

Clearly, $|X_n - X| \leq \varepsilon/2$ & $|Y_n - Y| \leq \varepsilon/2$

Again, by Δ-inequality, $|X_n + Y_n - X - Y| < \underbrace{|X_n - X|}_{\leq \varepsilon/2} + \underbrace{|Y_n - Y|}_{\leq \varepsilon/2} < \varepsilon$.

$\therefore A \subset B \cup C$.

$\therefore P(A) \leq P(B \cup C) \leq \underbrace{P(B)}_{<\delta/2 \ \forall n > N_1} + \underbrace{P(C)}_{<\delta/2 \ \forall n > N_2} < \delta$. $[\forall n > N = \text{Max}.\{N_1, N_2\}]$ ▨

(*) $X_n \xrightarrow{P} X \Rightarrow f(X_n) \xrightarrow{P} f(X)$ for any continuous function f.

Proof: We're going to show, that

$X_n \xrightarrow{P} X$ implies: why compact set? $\because$ on a compact set, every cont. f^n is bounded

for every $\varepsilon > 0$, we can find atleast one compact set $K_\varepsilon \subset \mathbb{R}$ s.t $\forall n \geq 1$,

Called "Tightness Property". (we'll prove this later)

$$P(X_n \in K_\varepsilon, X \in K_\varepsilon) > 1 - \varepsilon/2$$

$$\Leftrightarrow P(X_n \notin K_\varepsilon \text{ or } X \notin K) < \varepsilon/2$$

for the time being, we'll assume: proving this (above) $\Rightarrow$ we're done.

So, for $\varepsilon > 0$.

$P(|f(X_n) - f(X)| > \varepsilon)$.

Get K_ε as stated:

$P(|f(X_n) - f(X)|)$

$$P(|f(X_n) - f(X)|)$$

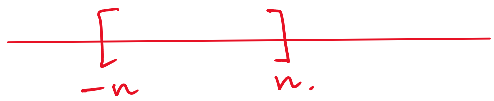
$$= P\left(X_n \notin K_\varepsilon \text{ or } X \notin K_\varepsilon,\ |f(X_n) - f(X)| > \varepsilon\right)$$

$$+ P\left(X_n \in K_\varepsilon, X \in K_\varepsilon,\ |f(X_n) - f(X)| > \varepsilon\right)$$

$$< \varepsilon/2 + P\left(|X_n - X| > \delta\right),$$ where $\delta > 0$ is s.t, $\forall\ x, y \in K_\varepsilon$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

[$\because$ On a compact set, f continuous $\Rightarrow$ f-uniformly continuous]

Notion:



* Given a real r.v X, we can find a compact set K_ε s.t $P(X \in K_\varepsilon) < 1 - \varepsilon$.

ie, $P(X \in \mathbb{R}) = 1$,

& $\bigcup_n [-n, n] = \mathbb{R}$. ie, $[-n, n] \nearrow$ to $\mathbb{R}$

$\therefore\ P\left(X \in [-n, n]\right) \nearrow 1$

s.t, $\forall\ 0 < \varepsilon < 1$, $\exists\ N$ s.t., $\forall\ n > N$,

$$P\left(X \in [-n, n]\right) > 1 - \varepsilon.$$

Exercise:

Seq. $\{X_n\}$.

$\exists$ compact set K_n. s.t. $P(X \in K_n) > 1 - \varepsilon$.

prove that:

You cannot find a single $K \subset \mathbb{R}$ (a compact set) s.t. $\forall\ n$, $P(X_n \in K) > 1 - \varepsilon$.

Now, we prove our assumption: $\hookrightarrow$ that tightness $\Rightarrow$ "we're done."

Get $M \in \mathbb{N}$ s.t.

$$\alpha = P\left(X \in [-M, M]\right) > 1 - \varepsilon/2$$

$$X_n \xrightarrow{P} X \Rightarrow P\left(|X_n - X| > 1\right) \longrightarrow 0.$$

get n_0 s.t,

$$P\left(|X_n - X| > 1\right) < \alpha - (1 - \varepsilon/2) \quad \forall\ n \geq n_0$$

get n_0 s.t.,

$$P(|X_n - X| > 1) < \alpha - (1 - \varepsilon/2) \quad \forall n \geq n_0$$

$\therefore$ for $n \geq n_0$,

$$* \quad \underbrace{P(|X_n| > M+1)}_{A} \leq \underbrace{P(|X_n - X| > 1, |X| \leq M)}_{B} + \underbrace{P(|X| > M)}_{C}$$

$\alpha - (1 - \varepsilon/2)$ $\quad 1 - \alpha$ $\quad < \varepsilon/2$

Again, we'll try using the same argument.

Explanation.

$$* \quad P(|X_n| > M+1) = P(|X_n| > M+1, |X| \leq M) + P(|X_n| > M+1, |X| > M)$$

$\Longleftrightarrow P(|X_n - X| > 1)$.

$|X_n| > M+1 \Rightarrow |X| > M$.

$\therefore P(|X_n| > M+1) < \varepsilon/2$

$\Rightarrow P(|X_n| \leq M+1) > 1 - \varepsilon/2$

$\therefore$ We found an $M_0 \in \mathbb{R}$ s.t. for $\varepsilon > 0$,

$$P(X \in [-M_0, M_0]) > 1 - \varepsilon/2,$$

$M_0 = \text{Max.}\{M, M+1\}$.

(*). $X_n \xrightarrow{P} X \Rightarrow X_n^2 \xrightarrow{P} X$

$$\left.\begin{array}{l} X_n \xrightarrow{P} X \\ Y_n \xrightarrow{P} Y \end{array}\right\} X_n \cdot Y_n \xrightarrow{P} X \cdot Y$$

Proof: $X_n \cdot Y_n = \frac{1}{4} \cdot \left[(X_n + Y_n)^2 - (X_n - Y_n)^2 \right]$.

$$\downarrow P \qquad \downarrow P$$

$$(X+Y)^2 - (X-Y)^2 = XY. \quad \blacksquare$$

Exercise: $X_n \xrightarrow{P} X$ $\quad X \neq 0$ a.s.

Prove that: $\frac{1}{X_n} \xrightarrow{P} \frac{1}{X}$.

(*) $X_n \xrightarrow{a.s} X$

$$\iff P\left(\bigcap_{j=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{k\geq n}\left\{\omega : |X_k(\omega)-X(\omega)| < \frac{1}{j}\right\}\right) = 1$$

take complement

$$\iff P\left(\bigcup_{j=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{k\geq n}\left\{\omega : |X_k(\omega)-X(\omega)| > \frac{1}{j}\right\}\right) = 0$$

$$\iff P\left(\bigcap_{n=1}^{\infty}\bigcup_{k\geq n}\left\{\omega : |X_k(\omega)-X(\omega)| > \frac{1}{j}\right\}\right) = 0 \quad \forall j \in \mathbb{N}$$

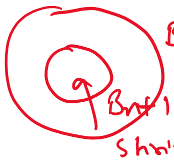
$$\iff P\left(\bigcap_{n=1}^{\infty}\left\{\sup_{k\geq n}|X_k(\omega)-X(\omega)| > \frac{1}{j}\right\}\right) = 0 \quad \forall j \in \mathbb{N}$$

(*)

$$\iff P\left(\sup_{k\geq n}|X_k - X| > 1/j\right) \longrightarrow 0 \quad \text{as } n\to\infty \quad \forall j \in \mathbb{N}$$

(* if $B_n = \left\{\sup_{k\geq n}|X_k(\omega)-X(\omega)| > 1/j\right\}$,
& $B_{n+1} = \left\{\sup_{k\geq n+1}|X_k(\omega)-X(\omega)| > 1/j\right\}$,

then, $B_{n+1} \subseteq B_n$. ∴



shrinking, so, $n\to\infty$ makes sense.)

$$\iff P\left(\sup_{k\geq n}|X_k - X| > \varepsilon\right) \longrightarrow 0 \quad \forall \varepsilon > 0.$$

$$\iff \forall \varepsilon>0, \quad P\left(\sup_{k\geq n}|X_n - X| > \varepsilon\right) \longrightarrow 0$$

$$\equiv \sup_{k\geq n}|X_n - X| \xrightarrow{P} 0.$$

∴ <u>Theorem:</u>

$$X_n \xrightarrow{a.s.} X \quad \text{iff} \quad \sup_{k\geq n}|X_k - X| \xrightarrow{P} 0$$

$\Downarrow$

P [∵ sup → 0]

$$\Downarrow$$

$$|X_k - X| \xrightarrow{P} 0 \quad [\because \sup \to 0]$$

ie, a.s. convergence $\Rightarrow$ convergence in p.

However, converse is <u>NOT</u> true !!

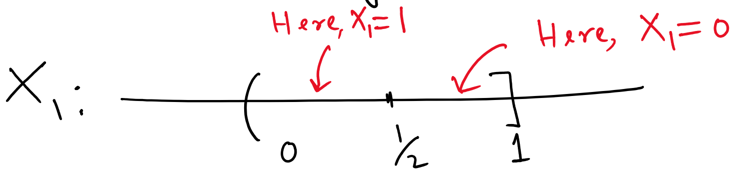
Counter eg:

to show: $X_n \xrightarrow{P} X \not\Rightarrow X_n \xrightarrow{a.s} X$

Take $\Omega = (0,1]$. $\mathcal{A}$ = Borel sets on $\mathbb{R}$.

P = Lebesgue measure.

fix we define X_1 & X_2



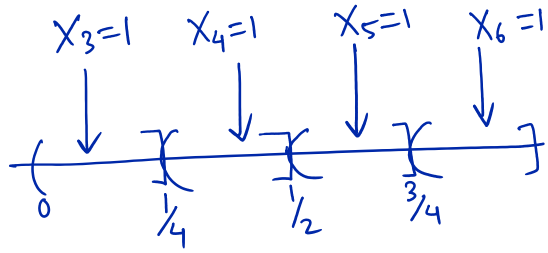
ie, $X_1 := \mathbb{1}_{(0,\frac{1}{2}]}$, &

$X_2 := \mathbb{1}_{(\frac{1}{2},1]}$.

Now, we define X_3, X_4, X_5, X_6.

$X_3 := \mathbb{1}_{(0,\frac{1}{4}]}$

$X_4 := \mathbb{1}_{(\frac{1}{4},\frac{1}{2}]}$



$X_5 := \mathbb{1}_{(\frac{1}{2},\frac{3}{4}]}$

$X_6 := \mathbb{1}_{(\frac{3}{4},1]}$.

Now, it's clear how next 8 X_i's would be defined.

$\therefore$ from the def^n, the seq^n: $X_n \xrightarrow{P} 0$.

Q. Does $X_n \xrightarrow{a.s} 0$?

– NO !!! a.s

Q. Does $X_n \xrightarrow{\because} 0$?

– NO!!! $X_n \overset{a.s}{\not\longrightarrow} 0$.

for any $\omega \in (0,1]$,

Exercise:

to show: in a discrete probability space (ie, Ω – countable).

$$X_n \xrightarrow{a.s} X \iff X_n \xrightarrow{P} X$$

(ie, the reverse implication becomes true in a discrete probability space.)

"Strict non-atomicity" property:

A – a set.

if $P(A) = \alpha > 0$,

$\forall \beta \in (0, \alpha)$ $\exists$ $B \subseteq A$ s.t. $P(B) = \beta$.

("kind of" I.V.T in probability).