1. Write down the matrices of the following 3-ary quadratic forms:

   (a) $x_1^2 + x_2^2 - 3x_3^2 + 2x_1x_2 - 6x_1x_3$, (b) $x_1^2 + 2x_3^2 - x_1x_2$,

   (c) x_2x_3, (d) $(2x_1 - x_2 + 3x_3)^2$ and (e) $(\mathbf{u}^{\mathrm{T}}\mathbf{x})^2$.

2. If $\mathbf{A}$ is not symmetric, what is the matrix of $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ viewed as a quadratic form?
3. Show that $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = 0\ \forall\, \mathbf{x} \Rightarrow \mathbf{A} = \mathbf{0}$ and $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x}\ \forall\, \mathbf{x} \Rightarrow \mathbf{A} = \mathbf{B}$.
4. If $\overline{x} = \frac{1}{n}(x_1 + \cdots + x_n)$, find the matrices of the quadratic forms $n\overline{x}^2$ and $\sum_{i=1}^{n}(x_i - \overline{x})^2$. Verify that they are idempotent and add up to $\mathbf{I}$.
5. A map $\psi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is said to be a *bilinear form* if $\psi(\alpha\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x}, \alpha\mathbf{y}) = \alpha\psi(\mathbf{x}, \mathbf{y})$, $\psi(\mathbf{x}+\mathbf{z}, \mathbf{y}) = \psi(\mathbf{x}, \mathbf{y}) + \psi(\mathbf{z}, \mathbf{y})$ and $\psi(\mathbf{x}, \mathbf{y}+\mathbf{u}) = \psi(\mathbf{x}, \mathbf{y}) + \psi(\mathbf{x}, \mathbf{u})$.
   (a) Show that every bilinear form $\psi(\mathbf{x}, \mathbf{y})$ can be written as $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{y}$ for some $m \times n$ matrix $\mathbf{A}$.
   (b) When $m = n$ show that a bilinear form gives rise to a quadratic form if we put $\mathbf{x} = \mathbf{y}$.
   (c) If $m = n$ and $\mathbf{A}$ is symmetric, show that there is a unique bilinear form $\psi(\mathbf{x}, \mathbf{y})$ such that $\psi(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{y}, \mathbf{x})$ which gives rise to the quadratic form $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ as in (b).

1. Prepare a table showing the possible types of definiteness (p.d., p.s.d., n.d., n.s.d. and indefinite) of $\mathbf{A} + \mathbf{B}$ given those of $\mathbf{A}$ and $\mathbf{B}$.
2. Find the quadratic form to which $x_1^2 + 2x_2^2 - x_3^2 + 2x_1x_2 + x_2x_3$ transforms by the change of variables $y_1 = x_1 - x_3, y_2 = x_2 - x_3, y_3 = x_3$ by actual substitution. Verify that the matrix of the resulting quadratic form is congruent to the matrix of the original quadratic form.
3. Prove that congruence is an equivalence relation on the set of all $n \times n$ symmetric matrices.
4. If $\mathbf{A}$ and $\mathbf{B}$ are n.n.d., then show that $\mathrm{diag}(\mathbf{A}, \mathbf{B})$ is n.n.d. If $\mathbf{A}$ is p.d. and $\mathbf{B}$ is n.d., what can be said about $\mathrm{diag}(\mathbf{A}, \mathbf{B})$?
5. (a) If $\mathbf{A}$ is an n.n.d. matrix of order n and $\mathbf{P}$ is an $m \times n$ matrix, show that $\mathbf{P}\mathbf{A}\mathbf{P}^{\mathrm{T}}$ is n.n.d. Deduce that $\mathbf{P}\mathbf{P}^{\mathrm{T}}$ is n.n.d. for any matrix $\mathbf{P}$.
6. Let $\mathbf{A}$ and $\mathbf{B}$ be n.n.d. Show that $\mathbf{A} + \mathbf{B} = \mathbf{0}$ iff $\mathbf{A} = \mathbf{B} = \mathbf{0}$. Deduce that if $\mathbf{C}$ and $\mathbf{D}$ are symmetric and $\mathbf{C}^2 + \mathbf{D}^2 = \mathbf{0}$, then $\mathbf{C} = \mathbf{D} = \mathbf{0}$.

1. Prove that every orthogonal projector is an n.n.d. matrix.
2. If $\mathbf{B} = \mathbf{A}^{-1}$, show that $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ and $\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x}$ have the same signature.
3. Let $\mathbf{P}$ be of full column rank. Show that $\mathbf{A}$ and $\mathbf{P}\mathbf{A}\mathbf{P}^{\mathrm{T}}$ have the same rank and the same signature. What about the number of zero eigenvalues?

4. Prove that a quadratic form $\mathbf{x}^T\mathbf{A}\mathbf{x}$ can be written as the product of two linearly independent linear forms in $\mathbf{x}$ iff $\mathbf{A}$ has rank 2 and signature 0.

8. Find the rank and signature of each of the following quadratic forms:
   (a) $x_1^2-3x_2^2-8x_3^2-x_4^2+2x_1x_2-2x_1x_3+2x_1x_4-14x_2x_3+10x_2x_4+10x_3x_4$
   (b) $x_1x_2+x_3x_4+\cdots+x_{2k-1}x_{2k}$
   (c) $\sum_{i,j=1}^{n}(x_i-x_j)^2$

11. If $\mathbf{A}$ is any real symmetric matrix, show that there exists a real number α such that $\alpha\mathbf{I}+\mathbf{A}$ is positive definite.

12. Show that every real symmetric matrix can be written as the difference of two p.d. matrices

13. Show that the set $\{\mathbf{x} : \mathbf{x}^T\mathbf{A}\mathbf{x} \le 1\}$ is bounded iff $\mathbf{A}$ is p.d.

1. If $\mathbf{A}$ is n.n.d., show that $\mathbf{x}^T\mathbf{A}\mathbf{x}=0$ iff $\mathbf{A}\mathbf{x}=\mathbf{0}$. Show also that $\mathbf{x}^T\mathbf{A}\mathbf{x}=0$ iff $\mathbf{y}^T\mathbf{A}\mathbf{x}=0$ for all $\mathbf{y}$.

2. Let $\mathbf{A}$ be an $n\times n$ p.d. matrix and let $\mathbf{P}$ be an $n\times r$ matrix of rank r. Then show that $\mathbf{P}^T\mathbf{A}\mathbf{P}$ is p.d.

3. If $\mathbf{A}$ is an n.n.d. matrix of order n with rank r and if $k\ge r$, prove that there exists an $n\times k$ matrix $\mathbf{C}$ such that $\mathbf{A}=\mathbf{C}\mathbf{C}^T$. Note that if $k=r$, $(\mathbf{C},\mathbf{C}^T)$ is a rank-factorization of $\mathbf{A}$.

4. Let $(\mathbf{C},\mathbf{C}^T)$ be a rank-factorization of an n.n.d. matrix $\mathbf{A}$ of order n and let $\mathbf{C}_L^{-1}$ be a left inverse of $\mathbf{C}$.
   (a) Show that $([\mathbf{C}:\mathbf{u}],[\mathbf{C}:\mathbf{u}]^T)$ is a rank-factorization of $\mathbf{A}+\mathbf{u}\mathbf{u}^T$ if $\mathbf{u}\notin\mathcal{R}(\mathbf{A})$

6. Let $\mathbf{A}$ be an n.n.d. matrix and let p be a positive integer. Show that there is a unique n.n.d. matrix $\mathbf{B}$ such that $\mathbf{B}^p=\mathbf{A}$.

7. Show that $(1-\rho)\mathbf{I}+\rho\mathbf{1}\mathbf{1}^T$ is p.d. iff $-\frac{1}{n-1}<\rho<1$ where n is the order of the matrix and $\mathbf{1}^T=(1,1,\ldots,1)$.

9. If $\mathbf{A}$ is a p.s.d. matrix of order n, show that there exists an n.n.d. matrix $\mathbf{B}$ of order n such that $\rho(\mathbf{A}+\mathbf{B})=\rho(\mathbf{A})+\rho(\mathbf{B})=n$.

10. If $\mathbf{A},\mathbf{B}$ are symmetric matrices of the same order, write $\mathbf{A}\ge\mathbf{B}$ if $\mathbf{A}-\mathbf{B}$ is n.n.d. Then prove the following:
    (a) $\mathbf{A}\ge\mathbf{B}$ and $\mathbf{B}\ge\mathbf{A}\Rightarrow\mathbf{A}=\mathbf{B}$
    (b) $\mathbf{A}\ge\mathbf{B}$ and $\mathbf{B}\ge\mathbf{C}\Rightarrow\mathbf{A}\ge\mathbf{C}$
    (c) $\mathbf{B}$ n.n.d. and $\mathbf{A}\ge\mathbf{B}\Rightarrow|\mathbf{A}|\ge|\mathbf{B}|$
    (d) $\mathbf{B}$ p.d., $\mathbf{A}\ge\mathbf{B}$ and $|\mathbf{A}|=|\mathbf{B}|\Rightarrow\mathbf{A}=\mathbf{B}$

11. Let $\mathbf{A}$ be p.d. and $\mathbf{M}=\begin{bmatrix}\mathbf{A} & \mathbf{b}\\ \mathbf{b}^T & d\end{bmatrix}$. Show that $\mathbf{M}$ is p.d., p.s.d. or indefinite according as $d-\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}$ is positive, zero or negative.

13. Let $\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{bmatrix}$ be symmetric, where $\mathbf{A}$ is square.

(a) Prove that $\mathbf{M}$ is p.d. iff $\mathbf{A}$ and $\mathbf{D} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$ are p.d. Show also that $\mathbf{M}$ is p.d. iff $\mathbf{D}$ and $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^T$ are p.d.

(b) Prove that $\mathbf{M}$ is n.n.d. iff $\mathbf{A}$ and $\mathbf{D} - \mathbf{B}^T\mathbf{A}^-\mathbf{B}$ are n.n.d. and $\mathcal{C}(\mathbf{B}) \subseteq \mathcal{C}(\mathbf{A})$.

(c) If $\mathbf{M}$ is p.d. and $\mathbf{L}$ is the leading principal submatrix of $\mathbf{M}^{-1}$ with the same order as $\mathbf{A}$, prove that $\mathbf{L} - \mathbf{A}^{-1}$ is n.n.d.

(d) If $\mathbf{M}$ is n.n.d., prove that $|\mathbf{M}| \leq |\mathbf{A}| \cdot |\mathbf{D}|$. Suppose next $\mathbf{M}$ is p.d. Then prove that $|\mathbf{M}| = |\mathbf{A}| \cdot |\mathbf{D}|$ iff $\mathbf{B} = \mathbf{0}$.

14. If $\mathbf{A}$ is p.d., show that $\begin{bmatrix} \mathbf{A} & \mathbf{I} \\ \mathbf{I} & \mathbf{A}^{-1} \end{bmatrix}$ is p.s.d.

15. (a) Let $\mathbf{A}$ be p.d. Then show that $\mathbf{A} - \mathbf{b}\mathbf{b}^T$ is p.d. iff $\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b} < 1$.

(b) Let $\mathbf{A}$ be n.n.d. Then show that $\mathbf{A} - \mathbf{b}\mathbf{b}^T$ is n.n.d. iff $\mathbf{b} \in \mathcal{C}(\mathbf{A})$ and $\mathbf{b}^T\mathbf{A}^-\mathbf{b} \leq 1$.

*19. Let $\mathbf{A}$ and $\mathbf{B}$ be n.n.d. Show that the eigenvalues of $\mathbf{AB}$ are non-negative. If $\mathbf{AB} \neq \mathbf{0}$, show that $\mathbf{AB}$ has a positive eigenvalue.

*22. If $\mathbf{A}$ and $\mathbf{B}$ are n.n.d. and if $\rho(\mathbf{A} - \mathbf{B}) = \rho(\mathbf{A}) - \rho(\mathbf{B})$, show that $\mathbf{A} - \mathbf{B}$ is n.n.d.

3. (a) If α is an eigenvalue of an $n \times n$ real symmetric matrix $\mathbf{A}$, show that $|\alpha| \leq n(\max_{i,j} |a_{ij}|)$.

4. Show that for any $m \times n$ matrix $\mathbf{A}$, $\max\{\|\mathbf{Ax}\| : \|\mathbf{x}\| = 1\}$ is the square-root of the largest eigenvalue of $\mathbf{A}^T\mathbf{A}$. If $\mathbf{A}$ is a real $n \times n$ normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, show that $\max\{\|\mathbf{Ax}\| : \|\mathbf{x}\| = 1\} = \max_i |\lambda_i|$.

6. (a) Show that the largest singular value of a square matrix $\mathbf{A}$ is $\|\mathbf{A}\|$ where $\|\cdot\|$ is the matrix norm induced by the Euclidean norm for vectors.

(b) Deduce that the modulus of any eigenvalue of $\mathbf{A}$ cannot be greater than the largest singular value of $\mathbf{A}$.

1. Show that the quadratic forms $x_1^2 - x_2^2$ and $x_1^2 + x_2^2 + 2x_1x_2$ cannot be simultaneously diagonalized by a non-singular transformation.