Probability-3 Lecture-18

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Recall – result:

$$X_n \xrightarrow{d} X \quad \text{iff} \quad E(f(X_n)) \longrightarrow E(f(X)) \quad \forall f \in C_b(\mathbb{R})$$

bounded real f^ns.

This is a characterization of convergence of distributions.

Some standard notations:

- $X_n \xrightarrow{d} F$ — a cdf

  this means, X_n converges to some r.v whose CDF is F.

- $F_n \xrightarrow{d} F$

  F_n - cdf of X_n. $\xrightarrow{d}$ cdf F (of some r.v).

Complex Random Variables.

Z is a complex valued r.v on $(\Omega, \mathcal{A}, P)$ if $Z = X + iY$, where X, Y are real valued r.v.s

Define expectation,

$$E(Z) := E(X) + i \cdot E(Y), \text{ if}$$

$E(X)$ & $E(Y)$ are both finite.

Properties:

(a) Z_1, Z_2 - complex r.vs.

α_1, α_2 - complex nos.

$E(Z_1)$ & $E(Z_2)$ both exist.

then, $\Rightarrow E(\alpha_1 Z_1 + \alpha_2 Z_2) = \alpha_1 \cdot E(Z_1) + \alpha_2 \cdot E(Z_2)$

[Easy to prove.
take $\alpha_1 = u_1 + i \cdot v_1$, $\alpha_2 = u_2 + i v_2$
$Z_1 = X_1 + i \cdot Y_1$; $Z_2 = X_2 + i \cdot Y_2$
& do some algebra.]

(b) For any complex r.v Z,

$$|E(Z)| \leq E(|Z|).$$

$\underbrace{E(Z)}_{\text{a complex no.}}$

⚠ looks analogous to $|EX| \leq E(|X|)$, but it isn't !!

Proof of (b):

For every complex w, $\exists$ a complex no. α with $|\alpha| = 1$

s.t. $|w| = \alpha . w$.

if $w = 0$, take any α.
[Easy to check]

$\therefore$ Applying with $w = E(Z)$,

$\exists \ \alpha \in \mathbb{C}$ with $|\alpha| = 1$.

s.t, $\underbrace{|E(Z)|}_{\text{Real}} = \underbrace{\alpha . E(Z)}_{\text{complex}}$.

$$\Rightarrow |E(Z)| = |Re(\alpha E(Z))|$$

Easy to check !!

$$= |Re \cdot E(\alpha Z)|$$

$$= |E(Re(\alpha Z))|$$

$$\leq E\,\underbrace{|Re(\alpha Z)|}_{\leq} .$$

$$\leq E|\alpha Z|$$

$$= E|Z| \quad [\because |\alpha| = 1]$$

Suppose, $Z = X + iY$.

then, $E(Z) := E(X) + iE(Y)$.

$$\therefore |E(Z)| = \sqrt{(E(X))^2 + (E(Y))^2} \;,$$

whereas,

$$E|Z| = E\left(\sqrt{X^2+Y^2}\right)$$

$\therefore$ for real r.vs X, Y,

we're saying:

$$\sqrt{(E(X))^2 + (E(Y))^2} \leq E\left(\sqrt{X^2+Y^2}\right)$$

This is <u>not</u> a L_1 ineq.

This is stronger.

Consequence of (b):

DCT holds for complex r.vs as well !!

$Z_n, n \geq 1$ is a sequence of complex r.vs &

Z is a complex r.v,

Z is a complex r.v,

and if $Z_n \xrightarrow{a.s} Z$, & if

$\exists\, Y$ with $E(Y) < \infty$ s.t,

$|Z_n| \le |Y| \;\; \forall n,$

then $E\,|Z_n - Z| \longrightarrow 0$. [the "real-DCT"]

in particular,

$E\,Z_n \longrightarrow E\,Z$.

[using part (b)].

Definition:

Complex r.vs

$$Z_1 = X_1 + i\cdot Y_1$$

$$\vdots$$

$$Z_n = X_n + i\cdot Y_n\,.$$

are said to be independent if

$(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent random vectors.

Consequence:

Z_1, Z_2 - complex independent r.vs,

then $E(Z_1 \cdot Z_2) = E(Z_1) \cdot E(Z_2)$.

Proof: Write $Z_1 = X_1 + i\cdot Y_1$

$Z_2 = X_2 + i\cdot Y_2$

, just do it !!

just do it !!

Definition: Characteristic Function. (C.F.)

For a real r.v X, its characteristic f^n φ_X is defined by

$$\varphi_X(t) = E\left(e^{itX}\right), \quad \varphi_X : \mathbb{R} \longrightarrow \mathbb{C}.$$

$$\varphi_X(t) = E(\cos tX) + i\,E(\sin tX).$$

* C.F is a function $\varphi_X : \mathbb{R} \longrightarrow \mathbb{C}$,

$\varphi_X(t)$ exists $\forall\, t \in \mathbb{R}$. [$\because$ $E(\cos tX)$ & $E(\sin tX)$ are always bounded within $[-1,1]$]

[Unlike MGF.]

Properties:

(a) $\varphi_X(0) = 1$

(b) $|\varphi_X(t)| \leq 1.$

Proof:

$$|\varphi_X(t)| = |E(e^{itX})|$$
$$\leq E\left|e^{itx}\right| = 1.$$

(c) φ_X is a uniformly continuous f^n on $\mathbb{R} \longrightarrow \mathbb{C}$.

Proof:

$$\left|\varphi_X(t+h)-\varphi_X(t)\right| = \left|E\left(e^{i(t+h)X}\right)-E\left(e^{itX}\right)\right|$$

$$= \left|E\left(e^{itX}\left(e^{ihX}-1\right)\right)\right|$$

$$\leq E\left|e^{itX}\cdot\left(e^{ihX}-1\right)\right|$$

$[\because E|e^{itX}|=1]$

$$= E\left|e^{ihX}-1\right| \xrightarrow[(DCT)]{} 0$$

$\because |e^{ihX}-1| \longrightarrow 0,$

& $|e^{ihX}-1| \leq 2 < \infty.$

constant r.v.

$\boxtimes$

(d) $\varphi_{a+bX}(t) = e^{iat}\cdot\varphi_X(bt)$

Proof: easy to check.

$$\varphi_{a+bX}(t) = E\left(e^{it(a+bX)}\right)$$

$$= E\left(e^{iat}\cdot e^{i(bt)\cdot X}\right) = e^{iat}\cdot E\left(e^{i(bt)X}\right)$$

$$= e^{iat}\cdot\varphi_X(bt).$$

(e) $\varphi_{-X}(t) = \varphi_X(-t) = \overline{\varphi_X(t)}$

* if 2 r.vs have same CDF, they have same C.F.

(f) If $X \stackrel{d}{=} -X$ [ie, X has symmetric dist^n eg. $X = N(0,1)$]

then $\varphi_X(t) = \varphi_{-X}(t)$

$$\varphi_X(t) = \overline{\varphi_X(t)}$$

$\therefore \varphi_X(t)$ is real-valued.

Is the converse true?

ie if $\varphi_X(t)$ is real-valued, can we say: X has symmetric dist^n?

Not until we prove that C.F. uniquely determines the distribution.

(g) X, Y - independent.

$$\varphi_{X+Y}(t) = \varphi_X(t) + \varphi_Y(t).$$

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* For a discrete distribution with values $x_1, x_2, \ldots$ & pmf $p(x_1), p(x_2), \ldots$

$$\varphi_X(t) = \sum_i e^{itx_i} \cdot p(x_i)$$

For a random variables with density f,

$$\varphi_X(t) = \int e^{itx} \cdot f(x)\, dx$$

Some examples.

(a) $X \sim \mathrm{Bin}(n, p)$. Write $q = 1 - p$.

$$\varphi_X(t) = \sum_{k=0}^{n} \binom{n}{k} e^{itk} \cdot p^k \cdot q^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cdot (pe^{it})^k \cdot q^{n-k}$$

$$= (pe^{it} + q)^n$$

(b) $X \sim \mathrm{Poi}(\lambda)$.

$$\varphi_X(t) = e^{-\lambda} \cdot \sum_{k=0}^{\infty} e^{itk} \cdot \lambda^k / \underline{|k}$$

$$= e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{\underline{|k}} = e^{-\lambda} \cdot e^{\lambda e^{it}}$$

$$= e^{-\lambda(1 - e^{it})}$$

(c) $X \sim \exp(\lambda)$.

$$\varphi_X(t) = \lambda \cdot \int_0^{\infty} e^{itX} \cdot e^{-\lambda x} \cdot dx$$

$$= \lambda \cdot \int_0^{\infty} e^{-(\lambda - it)} dx \overset{?}{\underset{\text{check}}{=}} \left(\frac{\lambda}{\lambda - it}\right)$$

quick thinking: assume $\lambda - it = \theta$.

$$\therefore \int_0^{\infty} e^{-\theta} = \frac{1}{\theta}.$$

that shortcut works!!

0 $\qquad\qquad\qquad\qquad$ $)$

exc: work out: $= \left(\lambda \int \cos tx \cdot e^{-\lambda x} dx + i\lambda \int \sin tx \cdot e^{-\lambda x} \cdot dx \right)$

(d) $X \sim N(0,1)$.
$\searrow$ symmetric.

$\therefore \varphi_X(t)$ should be real-valued.

$$\varphi_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \cos tx \cdot e^{-x^2/2} dx$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \int_0^{\infty} \cos tx \cdot e^{-x^2/2} dx.$$

$$\varphi_X'(t) = \frac{-2}{\sqrt{2\pi}} \cdot \int_0^{\infty} \sin tx \cdot x e^{-x^2/2} \cdot dx \qquad \left[* \int x e^{-x^2/2} dx = = e^{-x^2/2} \right]$$

Int. by parts. $= \frac{-2}{\sqrt{2\pi}} \cdot \left[\left(\sin tx \left(-e^{-x^2/2} \right) \right) \Big|_{x=0}^{x=\infty} + t \int_0^{\infty} \cos tx \cdot e^{-x^2/2} dx \right]$

$\therefore \varphi_X'(t) = -t \cdot \varphi_X(t)$, $\varphi_X(0) = 1$.

$\searrow$ a simple differential eq^n.

Sol^n: $\boxed{\varphi_X(t) = e^{-t^2/2}.}$

f density of $N(0,1)$

$$\hat{f}(t) = \sqrt{2\pi}\cdot f(t).$$

$\therefore$ for $X \sim N(\mu, \sigma^2)$

$$\varphi_X(t) = e^{i\mu t}\cdot e^{-\sigma^2 t^2/2}.$$

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Characteristic f^n $\xleftrightarrow[\text{correspondance}]{1-1}$ distribution.

Inversion formula:

Let φ_X be the characteristic function of X & F_X be the CDF.

then, for any $a<b$ in $\mathbb{R}$,

$$P(a<X<b) + \frac{1}{2}\Big(P(X=a) + P(X=b)\Big) =$$

$$\underbrace{\frac{1}{2\pi}\cdot \lim_{T\to\infty}\int_{-\infty}^{\infty} \frac{e^{ita} - e^{itb}}{it}\cdot \varphi_X(t)\,dt.}$$

In particular, for $a, b \in C(F_X)$, $F(b) - F(a) =$

$C(F_X)$: set of all continuity pts. of cdf of X.

* fact to recall from Sem - 2:

Suppose, let X, Y - independent r.vs.

Y has a density f.

Y has a density f.

then, $P(X+Y \leq a) = E(g(X))$, where $g(x) = P(Y \leq a-X)$

$$= \int_{-\infty}^{a-X} f(y)\,dy.$$

$$= \int_{-\infty}^{a-x} f(y)\,dy$$

change of variable:

call $v = y + X$.

v	y
$-\infty$	$-\infty$
a	$a-X$

$$= E\left(\int_{-\infty}^{a} f(v-X) \cdot dv \right)$$

$$= \int_{-\infty}^{a} E\big(f(v-X)\big)\,dv$$

$\therefore$ X, Y - ind. rvs, Y has density f. $\Rightarrow$

$\boxed{X+Y \text{ has a density } g \text{ given by } g(v) = E\big(f(v-X)\big).}$

Now, Feller's way of proving:

C.f $\longleftrightarrow$ CDF
1-1 correspondance.

Let X have a c.f φ_X.

$$\frac{1}{\sqrt{2\pi n}} \int_{\mathbb{R}} e^{-itv} \varphi_X(v)\, e^{-v^2/2n}.\,dv \qquad , t \in \mathbb{R}.$$

some real.

$$= \frac{1}{\sqrt{2\pi n}} \cdot \int_{\mathbb{R}} e^{-itv} \cdot E\left(e^{itv}\right) \cdot e^{-v^2/2n} \, dv$$

$$= \frac{1}{\sqrt{2\pi n}} \cdot \int_{\mathbb{R}} E\left(e^{-iv(t-X)}\right) \cdot e^{-v^2/2n} \cdot dv$$

$$= E\left(\frac{1}{\sqrt{2\pi n}} \cdot \int_{\mathbb{R}} e^{-iv(t-X)} \cdot e^{-v^2/2n} \, dv \right)$$

$\searrow$ c.f of $N(0,1)$ evaluated at $(t-X)$

$\sigma^2 = n$.

$$= E\left(e^{-\frac{n}{2}(t-X)^2} \right)$$

$$= E\left(\frac{\sqrt{n}}{\sqrt{2} \cdot \sqrt{n}} \cdot e^{-n/2 \, (t-X)^2} \right)$$

$|||$

density of $X + \frac{1}{\sqrt{n}} \cdot Z$,

$Z \sim N(0,1)$

$\therefore$ LHS $\to$ a crazy integral using c.f. of X

RHS $\to$ density of $X + \frac{1}{\sqrt{n}} \cdot Z$.

So now, if

$$\varphi_X = \varphi_Y ,$$

then,

$$\Rightarrow X + \frac{1}{\sqrt{n}} Z \overset{d}{=} Y + \frac{1}{\sqrt{n}} Z .$$

Now, as $n \to \infty$, $\frac{1}{\sqrt{n}} \to 0$.

$\therefore$ $\therefore$ By slutsky, $\frac{1}{\sqrt{n}} \cdot Z \xrightarrow{d} 0$

$$\therefore \boxed{X \overset{d}{=} Y .} \quad \blacksquare$$