Probability-3 Lecture-10

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Set-2, Q-9. $\{X_n\}$ → u.i, $\{Y_n\}$ → tight. $\Rightarrow \{X_n \cdot Y_n\}$ - u.i

This is false!!!

i.e, $\forall \varepsilon > 0, \exists K > 0$ s.t. $P(|Y_n| > K) < \varepsilon \ \forall n$

take $Y_n \sim U(0,1)$ with prob. $= \frac{n-1}{n}$

& $Y_n = e^n$ with prob $= 1/n$.

this is NOT u.i.

$\because \|Y_n\|_{\ell_1} \geq \frac{E(e^n)}{n} \to \infty$

Correct question:

$\{X_n\}$ → u.i, $\{Y_n\}$ → bounded. $\Rightarrow \{X_n \cdot Y_n\}$ u.i (prove)

$\S$ $\{X_n\}$ - u.i, but X_n is not bounded in any L_p for $p>1$

I construct a r.v $X \geq 1$.

s.t. $E(X \log X) < \infty$.

But, $E(X^p) = \infty \ \forall p > 1$.

Now, seq. of r.vs,

$$X_n = X \wedge n$$

$\hookrightarrow$ Min. $\{X, n\}$

<u>claim</u>: $\{X_n\}$ is u.i,

but $\sup_n E(X_n^p) = \infty$.

clearly, X_n's are increasing $\uparrow$

$\therefore \sup_n E(X_n^p) = \lim_{n\to\infty} E(X_n^p)$

$= E(X^p) = \infty$.

$$E(X_n \cdot 1_{X_n > \lambda}) = E\left(\frac{X_n}{X_n \log X_n}(X_n \log X_n) 1_{X_n > \lambda_0}\right)$$

$\frac{x}{x \log x} \searrow 0$ as $x \nearrow \infty$

$\therefore$ Given $\varepsilon > 0$,

<u>get λ_0</u> s.t. $x > \lambda_0 \Rightarrow \frac{x}{x \log x} < \varepsilon / E(X \log X)$.

$$\ldots < \frac{\varepsilon}{E(X\log X)} \cdot E\left(X_n \log X_n \cdot 1_{X_n > \lambda_0}\right).$$

$$< \frac{\varepsilon}{E(X\log X)} \cdot E(X\log X).$$

$$< \varepsilon.$$

* Z has density.

$$F(z) = e^{1-z}, \quad z > 1.$$

$$X = \frac{e^Z}{Z^{2+\varepsilon}} \leftarrow \text{anything larger than 2 would do.}$$

Borel - Cantelli Lemma.

If $\{A_n\}_{n\geq 1}$ is a sequence of events s.t.

$$\sum_n P(A_n) < \infty \left(\Rightarrow \sum_{k\geq n} P(A_n) \longrightarrow 0 \text{ as } n\to\infty \; (\text{tail sum}) \right)$$

then, $P(\limsup_n A_n) = P\left(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k\right) = 0.$

Proof: Clearly, $\bigcup_{k\geq n} A_k$ - decreasing unions.

$$\therefore P\left(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k\right) \underset{\substack{\text{Continuity} \\ \text{of} \\ \text{probability.}}}{=} \lim_{n\to\infty} P\left(\bigcup_{k\geq n} A_k\right).$$

$$\leq \lim_{n\to\infty} \sum_{k\geq n} P(A_k) = 0$$

* What are the ω's that belong to $\underbrace{\limsup A_n}_{\text{set.}}$?

$\omega \in \limsup_n A_n \iff$ for every n,
$\omega \in \bigcup A_k$

$\omega \in \limsup_n A_n \iff$ for every n, $\omega \in \bigcup_{k \geq n} A_k$.

$\iff$ for every n, there exists $k \geq n$ s.t $\omega \in A_k$.

ie, ω should belong to infinitely many of these A_k's.

Borel-Cantelli Second Lemma

If $\{A_n\}$ - sequence of independent events s.t,

$\sum_n P_n = \infty$ (ie, sequence of partial sum diverges)

then $P(\limsup_n A_n) = 1$

(kind of, partial converse of Borel-Cantelli lemma)

* Kolmogorov's 0-1 law

(ie, if A_n - seq. of ind. events then $P(\limsup_n A_n) = 0$ or 1

ie, no intermediate value)

Proof: $P((\limsup A_n)^c) = P\left(\left(\bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k\right)^c\right)$

$$= P\left(\bigcup_{n=1}^{\infty} \bigcap_{k \geq n} A_k^c\right)$$

countable union.

So, enough to show that

$P\left(\bigcap_{k \geq n} A_k^c\right) = 0$ for every n.

then $P((\limsup A_n)^c) = 0$.

$$k \geq n \qquad\qquad \cup$$

then, $P\left((\limsup A_n)^c \right) = 0.$

$\because \bigcap_{k \geq n} A_k^c$ is decreasing limit of partial intersections.

$\therefore$ fix n.

$$\therefore P\left(\bigcap_{k \geq n} A_k^c\right) = \lim_{m \to \infty} P\left(\bigcap_{k=n}^{n+m} A_k^c\right)$$

$$= \lim_{m \to \infty} \prod_{k=n}^{n+m} P(A_k^c)$$

$$= \lim_{m \to \infty} \prod_{k=n}^{n+m} (1 - P(A_k))$$

$^{*}1 - x < e^{-x} \longrightarrow$

$$= \lim_{m \to \infty} \prod_{k=n}^{n+m} e^{-P(A_k)}$$

$$= \lim_{m \to \infty} e^{-\sum_{k=n}^{n+m} P(A_k)} \longrightarrow 0$$

tail diverges

$$= 0. \quad \blacksquare$$

Set-1, Q.12:

$$d(X,Y) := \inf\{\varepsilon > 0 : P(|X-Y| > \varepsilon) \leq \varepsilon\}.$$

$(\Omega, \mathcal{A}, P)$

$L_0 \left(= L_0(\Omega, \mathcal{A}, P)\right)$

$= \{$all real r.vs on $(\Omega, \mathcal{A}, P)\}$.

Another metric, $\rho(X,Y) := E\left(\dfrac{|X-Y|}{1+|X-Y|}\right)$

So, we have 2 diff. metrics which are complete in L_0.

Recall:

$L_p(\Omega, \mathcal{A}, P) = \{X$ - real r.v. s.t. $E(|X|^p) < \infty\}$

For $X, Y \in L_p$. $d_p(X,Y) = E(|X-Y|^p)^{1/p} =: \|X-Y\|_p$

(we showed that this is a valid metric)

for L_p, $p \geq 1$,

to show: L_p is complete.

Let $\{X_n\}$ - Cauchy in metric d_p

ie, $d_p(X_m, X_n) = \|X_m - X_n\|_p \longrightarrow 0$ as $m,n \longrightarrow \infty$

x_n - Cauchy $\Rightarrow x_n$ bounded

Proof:

$\forall \varepsilon > 0, \exists N$ st $\forall m,n \geq N$ $|x_n - x_m| < \varepsilon$.

$\big||x_n| - |x_N|\big| < \varepsilon$

$\Rightarrow |x_n| < \varepsilon + |x_N|$

$\forall n > N$.

Take max

$x_0 = \max\{|x_1|, |x_2|, \dots, |x_N|\}$

$|x_n| \leq \varepsilon + x_0 < \infty$

Claim, $\{X_n\}$ is bounded in L_p.

$\forall \varepsilon > 0, \exists N$ st $\forall m,n > N$.

$\|X_m - X_n\|_p < \varepsilon$.

$\therefore \|X_n - X_N\|_p < \varepsilon$.

$\Rightarrow \|X_n\|_p \leq \varepsilon + \|X_N\|_p$. $\forall n > N$.

(Minkowski's ineq.)

Let $X_0 = \max\{\|X_1\|_p, \dots, \|X_N\|_p\}$.

$\therefore \|X_n\|_p \leq \varepsilon + X_0$.

$\boxed{\therefore X_n \text{- bounded in } L_p.}$

Using Cauchy property, we get a subsequence

Using Cauchy property, we get a subsequence

$1 \leq n_1 < n_2 < \cdots < n_k < \cdots$

s.t. $\|X_m - X_n\|_p < 3^{-k}$ $\forall$ $m, n \geq n_k$.

ie, put $k=1$. get n_1

put $k=2$, get n_2.

if $n_2 > n_1$ ✓

if $n_2 \leq n_1$

take $n_2 = n_1 + n_2$

& so on.

$\therefore$ get these indices in this manner.

$\therefore$ By Chebyshev's inequality.

$$P\left(|X_{n_{k+1}} - X_{n_k}| > 2^{-k}\right) \leq \frac{E\,|X_{n_{k+1}} - X_{n_k}|^p}{2^{-kp}}$$

$$< \left(\left(\frac{2}{3}\right)^p\right)^k \longrightarrow 0$$

infinite g.p, series - finite.

$$\therefore \sum_k P\left(|X_{n_{k+1}} - X_{n_k}| > 2^{-k}\right) < \infty$$

$\therefore$ By Borel-Cantelli Lemma,

$$P\left(|X_{n_{k+1}} - X_{n_k}| > 2^{-k} \text{ for infinitely many } k\right) = 0.$$

$$\therefore P\left(\left(|X_{n_{k+1}} - X_{n_k}| > 2^{-k} \text{ for infinitely many } k\right)^c\right) = 1$$

$$\Rightarrow P\left(|X_{n_{k+1}} - X_{n_k}| \leq 2^{-k} \text{ for all } k \text{ after } \underbrace{\text{some stage}}_{\text{depending on } \omega}\right) = 1$$

[$\{a_n\}$ - real seq.

on ω.)

[$\{a_n\}$ - real seq.

$|a_{k+1} - a_k| < 2^{-k} \quad \forall\, k \geq k_0$

then, a_n must converge to some real limit: $a_n \to a$.]

$$\Rightarrow P\left(X_{n_k}(\omega) \longrightarrow X(\omega) \text{ real}\right) = 1$$

ie, $X_{n_k} \xrightarrow{a.s.} X$ define $X(\omega) = \lim_{n\to\infty} X_n(\omega)$, where lim exists
(verify that this works) $= 0$, otherwise.

We have shown,

if $\{X_n\}$ is Cauchy in L_p, then $\exists$ a subsequence

Step 1: $\{X_{n_k}\}$ s.t. $X_{n_k} \xrightarrow{a.s.} X$ - real r.v.

Step 2: next, to show: $X_n \xrightarrow{a.s.} X$

Step 3: then, to show: $X \in L_p$.

$$\therefore E(|X|^p) = E\left(\liminf_k |X_k|\right) \leq \liminf_k E(|X_k|)$$

$$\leq \sup_n \|X_n\|_p^p < \infty$$

($\because X_n$ is Cauchy)

Fix k.

$$E\left(|X_m - X_{n_k}|^p\right) \leq 3^{-kp} \quad \forall\, m \geq n_k.$$

$$E\left(|X - X_{n_k}|^p\right) \leq \lim_j E\left(|X_{n_j} - X_{n_k}|^p\right) \leq 3^{-kp} \quad \forall\, j > k.$$

$\longrightarrow 0$ as $k \to \infty$.

$\therefore X_{n_k} \xrightarrow{L_p} X$, & $\{X_n\}$- Cauchy

$\Downarrow$

$X_n \xrightarrow{L_p} X$.

Set-0: X, Y- independent.

Y- cont $\longrightarrow P(Y=y) = 0 \ \forall\, y \in \mathbb{R}$.

?

Y - cont $\longrightarrow P(Y=y)=0 \ \forall y \in \mathbb{R}$.

(a) $P(X=Y) \overset{?}{=} 0$

$= P((X,Y) \in D)$

$= E(Q(X,D))$,

where

$Q(x,D) = P(Y \in D_x)$

$= P(Y=x)$

$= 0$.

where,

$D = \{(x,y) : x = y\}$.

$\therefore D_x = \{y : (x,y) \in D\}$

"x - section".

(b) $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\forall\ a, x \in \mathbb{R}$.

$h(x,y) = a$ has countably many sol^n.

$A_a = \{(x,y) : h(x,y) = a\}$.

$P(h(X,Y) = a) = P$

$\vdots$

$\sup_{k \geq n} |X_k - X_n|$

$\leq \sup_{k \geq n} |X_k - X| + |X_n - X|$

$\sup_{k \geq n} |X_k - X| \to 0$, $|X_n - X| \to 0$

$X_n \xrightarrow{a.s.} X$

$X_n \xrightarrow{P} X$

$P\left(\bigcup_{(a,b) \in \mathbb{Q} \times \mathbb{Q}} N_{(a,b)} \right) = 0$.

$\downarrow$

X_n does $) = 0$.

P X.