Probability-2 Lecture-23

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(X, Y) - a pair of real r.v.s.

support of X.

<u>Definition</u>: A function Ψ defined on $S_X \times \mathcal{B}$ satisfying:

(a) $\forall\, x \in S_X$, $\Psi(x, \cdot)$ is a probability on $\mathcal{B}$.

(b) $\forall\, B \in \mathcal{B}$, $P(X \in A, Y \in B) = E\left(\Psi(X, B) \cdot 1_A(X)\right)$ $\forall$ borel sets A.

is called the conditional distribution of Y, given X.

More specifically, $\Psi(x, B) = P(Y \in B \mid X = x)$.

<u>Result</u>: Such a Ψ always exists. (Can't prove this now. Out of scope).

<u>Two special cases</u> where we have explicit formula for such a Ψ:

<u>Case-I</u>: X is a discrete r.v. with values in the countable set D_X.

For every $x \in D_X$ & $B \in \mathcal{B}$,

define $\Psi(x, B) = P(Y \in B \mid X = x)$

$$= \frac{P(Y \in B, X = x)}{P(X = x)}.$$

[<u>Exercise</u>: Check that Ψ satisfies (a) & (b).]

<u>Case-II</u>: (X, Y) has a joint density.

Let f represent the joint density, and f_X be the (marginal) density of X. $\left[f_X(x) = \int f(x,y)\,dy\right]$

Let $g(y \mid x) = \frac{f(x,y)}{}$ for x s.t. $f(x) > 0$, Let $g(y|x) = \frac{f(x,y)}{f_X(x)}$ for x s.t. $f_X(x) > 0$,

$g(\cdot|x)$ is a joint density.

Define $\Psi(x, B) = \int_B g(y|x)\, dy$

Here, property (a) is trivially satisfied, as $\Psi(x, B)$ is just the integral of a density over B.

Exc: check that it follows (b).

Example: (X,Y) has joint density.

$$f(x,y) = 2\lambda x y^{-2} e^{-\lambda y}, \quad 0 < x < y < \infty.$$

first, let's verify that this is a joint density.

$$\int\int f(x,y)\, dx\, dy = \int_{y=0}^{\infty} \int_{x=0}^{y} 2\lambda x y^{-2} e^{-\lambda y}\, dx\, dy = \int_{y=0}^{\infty} \lambda y^{-2} \cdot x^2 \Big|_0^y dy$$

$$= \int_{y=0}^{\infty} \lambda e^{-\lambda y}\, dy. = 1 \checkmark$$

Now, • conditional distribution of $X|Y$:

$$F_Y(y) = \int_{x=0}^{y} f(x,y)\, dx = \lambda e^{-\lambda y}$$

$$\therefore g(x|y) = \frac{f(x,y)}{f_X(x)} = \frac{2\lambda x y^{-2} e^{-\lambda y}}{\lambda e^{-\lambda y}} = \frac{2x}{y^2}$$

- Conditional Expectation of $E(X^2 | Y=y) = \int_0^y x^2 \cdot g(x|y)\, dx$

$$= \int_0^y x^2 \cdot \frac{2x}{y^2}\, dx$$

$$= \frac{1}{y^2} \cdot \frac{x^4}{2} \Big|_0^y$$

$$= \frac{y^2}{2}.$$

- Conditional distribution of $Y|X$:

Example:

U, V are independent $Exp(\lambda)$.

Find conditional distribution of $U-V$, given $U+V$.

$X = U+V$

$Y = U-V$.

(U,V) takes values on

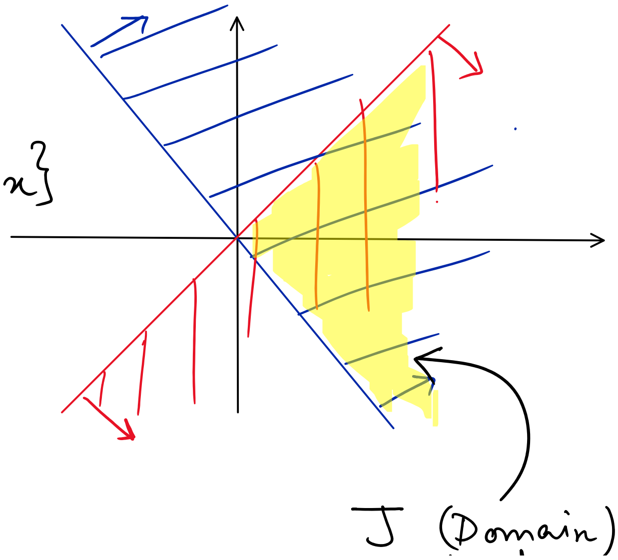
$I = (0,\infty) \times (0,\infty)$.

$\therefore U = \frac{X+Y}{2}, \quad V = \frac{X-Y}{2}.$

$\therefore X+Y > 0, \quad X-Y > 0.$

$(u,v) \longmapsto (x,y)$

$I \longrightarrow \boxed{J} = \{(x,y) : -x < y < x\}$



$\therefore \quad u = \frac{x+y}{2}, \quad v = \frac{x-y}{2}$

J (Domain) obtained, after applying all constraints.

$\therefore$ The Jacobian,

$$J = \left| \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \right| = \left| \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right|$$

$$= \left(\frac{1}{2}\right)$$

$$f_{U,V}(u,v) = f_U(u) \times f_V(v)$$
$$= \lambda e^{-\lambda u} \times \lambda e^{-\lambda v}$$
$$= \lambda^2 e^{-\lambda(u+v)}$$

$$\therefore \quad f_{X,Y}(x,y) = \frac{1}{2}\lambda^2 e^{-\lambda x}$$

- Marginal density of X: $f_X(x) = \int_{y=-x}^{x} f_{X,Y}(x,y)\,dy = \lambda^2 x e^{-\lambda x}$

$$\boxed{\therefore X \sim \text{Gamma}(\lambda, 2)}$$

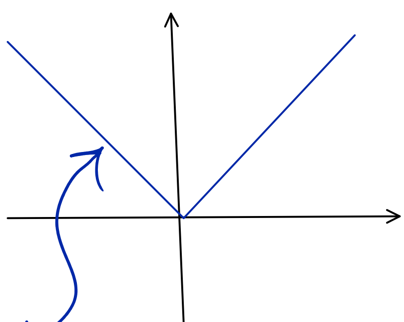
- Conditional density of $Y|X$: $g(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{1}{2x}, \quad -x < y < x$

<u>Example:</u>

$X \sim N(0,1)$.

$Y = |X|$.

Here, the dist^n $Y=|X|$ lies only on this $y=|x|$



Find the

this $y=|x|$ line (1D).

Find the conditional distribution of X, given Y.

$\because$ This is a continuous dist^n,

$\therefore$ mass over lower dimension $=0$.

$\therefore$ X & Y have no joint density.

$$\Psi(y,B) = \frac{1}{2}\cdot\delta_{\{y\}}(B) + \frac{1}{2}\delta_{\{-y\}}(B)$$

Dirac mass at y — Dirac mass at $-y$.

(if $Y=|X|=5$, $X=5$ or -5, with equal probability $=\frac{1}{2}$

why? Since, $N(0,1)$ is symmetric about y-axis.)

$*\ \mathbb{1}_{(y\in B)} = \delta_y(B)$

ie, $P(Y=y)=1$.

So, if $y\in B$, $\delta_y(B)=1$

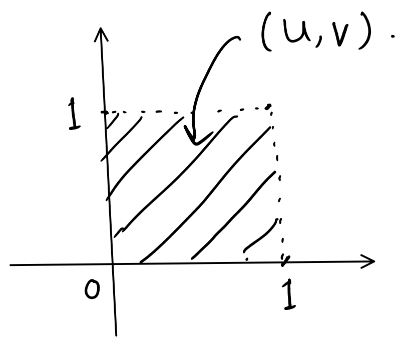
$y\notin B$, $\delta_y(B)=0$.

Example:

U, V - independent Uniform $(0,1)$

$X=\max\{U,V\}$.

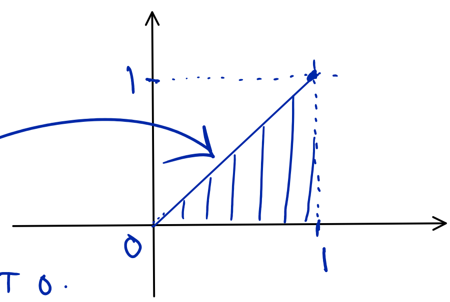
$Y=U$



clearly, $U\leq \max\{U,V\}$.

ie, $Y\leq X$.

$P(X=Y)=\frac{1}{2}$



Here, this line (with 0 area) has positive prob. mass, NOT 0.

So, (X,Y) cannot have a joint density.

a joint density.

Still, we want the conditional distribution of Y, given X.

For $x \in (0,1)$.

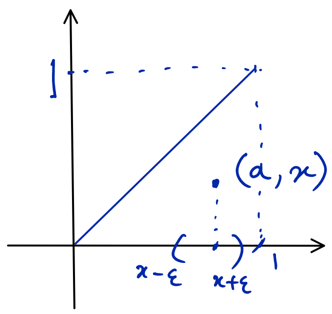
$$\Psi(x,(0,a)) = P(Y \leq a, X = x)$$

$$(\text{intuitively}) = \begin{cases} \dfrac{a}{2x}, & a < x \quad (*) \\ 1, & a \geq x. \end{cases}$$

we think of the distribution of Unif $(0,1)$.

Let's check this result formally.

Fix $0 < x < 1$, & $0 < a < x$.



$$P(Y \leq a \mid X = x) = P(Y \leq a \mid X \in (x-\varepsilon, x+\varepsilon))$$

Can't calculate wrt a point. Hence, we take a small interval around $X = x$, $\because$ that small interval has +ve mass.

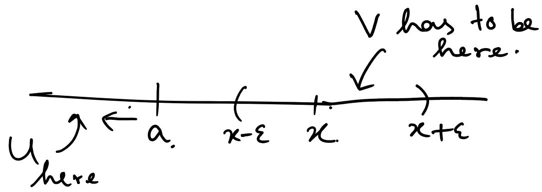
$$= \frac{P(Y \leq a, X \in (x-\varepsilon, x+\varepsilon))}{P(X \in (x-\varepsilon, x+\varepsilon))}$$

$\Big[P(X \leq x) = P(\text{Max.}\{U,V\} \leq x)$
$= P(U \leq x, V \leq x)$
$= P(U \leq x) \cdot P(V \leq x)$
$= x \times x = x^2$

$(\because U,V \text{ ind.}) \rightarrow$
$$= \frac{P(U \leq a) \cdot P(V \in (x-\varepsilon, x+\varepsilon))}{(x+\varepsilon)^2 - (x-\varepsilon)^2}$$

$\because P(X \leq x) = x^2$
$\therefore P(X \in (x-\varepsilon, x+\varepsilon)) = P(X < x+\varepsilon) - P(X \leq x-\varepsilon)$

$$= \frac{a \cdot (x+\varepsilon-(x-\varepsilon))}{4x\varepsilon}$$



$$= \frac{2a\varepsilon}{4x\varepsilon} = \frac{a}{2x}.$$

$\therefore$ Our initial $(*)$ intuition was wrong. We have $\frac{1}{2}$ mass

$\therefore$ Our initial (*) intuition was wrong. We have $\frac{1}{2}$ mass over $y=x$ line, & remaining half mass is distributed over the region.

So, it's $\frac{a}{2x}$, <u>not</u> $\frac{a}{x}$.