Probability-2 Lecture-6

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Recall: Basic setting:

Ω : non-empty set.

$\mathcal{A}$: a non-empty class of subsets of Ω satisfying:

(1) $\Omega \in \mathcal{A}$

(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(3) $A_n \in \mathcal{A},\ n \geq 1 \Rightarrow \bigcup_n A_n \in \mathcal{A}$.

Essentially, this class $\mathcal{A}$ will be closed under all countable set operations.

Axioms of Probability:

P: a "set function" on $\mathcal{A}$

$P : \mathcal{A} \longrightarrow [0,1]$ satisfying

(i) $P(\Omega) = 1$

(ii) $A_n \in \mathcal{A},\ n \geq 1$, disjoint $\Rightarrow P\left(\bigcup_n A_n\right) = \sum_n P(A_n)$

This set function P is called Probability or Probability Measure.

Definition:

For any non-empty set Ω, a class $\mathcal{A}$ of subsets of Ω satisfying (1), (2) & (3) is called a σ-field on Ω.

Definition:

For any non-empty set Ω, a class $\mathcal{S}$ of subsets of Ω is called a semifield if:

(i) $\Omega \in \mathcal{S}$

(ii) $A_1, A_2 \in \mathcal{S} \Rightarrow A_1 \cap A_2 \in \mathcal{S}$.

(iii) $A \in \mathcal{S} \Rightarrow A^c = A_1 \cup A_2 \cup \dots \cup A_n$ where $A_1, \dots, A_n \in \mathcal{S}$, disjoint.

ie the complement of any element of $\mathcal{S}$ must be expressible as a finite union of disjoint sets is the complement of any element of $\mathcal{S}$ must be expressible as a finite union of disjoint sets belonging to $\mathcal{S}$.

disjoint.

Eg. Consider $\mathcal{S}$, a class of all intervals on $\mathbb{R}$.

Recall: Random Walks

Ω = set of all segments of ± 1.

$\omega \in \Omega \iff \omega = (\omega_1, \omega_2, \ldots)$, where each $\omega_i \in \{-1, 1\}$

$P(\{\omega\}) = 0 \quad \forall\ \omega \in \Omega$

For every finite-dimensional subset $A \subset \Omega$, we have $P(A)$.

Defn: A set $A \subset \Omega$ is called a finite dimensional (f.d.) set if, for some $1 \leq n_1 < n_2 < \ldots \leq n_k$ and some choice of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in \{-1, 1\}$,

$$A = \{\omega : \omega_{n_1} = \varepsilon_1, \ldots, \omega_{n_k} = \varepsilon_k\}$$

ie, finite no. of positions fixed in ω.

Axiomatic (ϕ is also a f.d. subset).

Carotheodory's Extension Theorem:

Let Ω be a non-empty set and $\mathcal{S}$ a semifield on Ω. If P is a set function on $\mathcal{S}$ satisfying:

(1) $P(\Omega) = 1$

(2) $A_n \in \mathcal{S}$, $n \geq 1$, disjoint, $\bigcup_n A_n \in \mathcal{S} \Rightarrow P(\bigcup_n A_n) = \sum_n P(A_n)$.

then,

P has a unique extension to a probability P on the smallest σ-field containing $\mathcal{S}$.

$$\text{smallest } \sigma\text{-field containing } \mathcal{S} = \sigma(\mathcal{S}).$$

(However, its difficult to see how the sets in this "smallest σ-field" look like.)

Eg.

Eg.

$\Omega = (0,1]$.

$\mathcal{S} = \{(a,b] : 0 \leq a \leq b \leq 1\}$

Here, $\mathcal{S}$ is a semifield.

Define P on $\mathcal{S}$ by $P((a,b]) = b-a$.

$\therefore$ $(a_n, b_n] \in \mathcal{S}$, $n \geq 1$.

$$\bigcup_n (a_n, b_n] = (a,b] \Rightarrow \sum_n (b_n - a_n) = (b-a)$$

$(n, n+1]$.

$\mathcal{B}((n,n+1]) =$ Borel σ-field on $(n, n+1]$

= the smallest σ-field on $(n, n+1]$ containing

$\mathcal{S} = \{(a,b] : n \leq a < b \leq n+1\}$,

and by Carotheodory,

$P (\equiv \text{length})$ can be defined for all sets in $\mathcal{B}((n,n+1])$.

Defn :

A set $B \subset \mathbb{R}$ is called a Borel Set if $B = \bigcup_n B_n$ where $B_n \in \mathcal{B}((n,n+1])$, $n \in \mathbb{N}$.

The collection of all borel sets in $\mathbb{R}$ is called

Borel σ-field on $\mathbb{R}$.

Define "length"$(B) = \sum_n$ "length"(B_n), for a borel set B.

Defn: (another definition of Borel σ-field).

On $\mathbb{R}$, consider the following class of subsets

$\mathcal{C} = \{(a,b] : a, b \in \mathbb{R}\}$

The smallest σ-field on $\mathbb{R}$ containing $\mathcal{C}$ is called the

The smallest σ-field on $\mathbb{R}$ containing $\mathcal{C}$ is called the Borel σ-field on $\mathbb{R}$.

<u>Facts:</u> (a) $\mathcal{C}_1 =$ all bounded open intervals.

(b) $\mathcal{C}_2 =$ all bounded closed intervals.

(c) $\mathcal{C}_3 =$ all intervals of the form $[a, b)$

(d) $\mathcal{C}_4 =$ all intervals of the form $(-\infty, a]$, $a \in \mathbb{R}$.

The smallest σ-field containing $\mathcal{C}_i \,(i = 1, 2, \ldots, 5)$ are all same and equals $\mathcal{B}(\mathbb{R})$.

<u>To show:</u> $\sigma(\mathcal{C}) = \sigma(\mathcal{C}_1)$.

ie, $\mathcal{C} \subset \sigma(C_1) \Rightarrow \sigma(\mathcal{C}) \subset \sigma(C_1)$.

(If a class $\mathcal{C}$ is contained in $\sigma(C_1)$, the smallest σ-field containing $\mathcal{C}$ must also be contained in $\sigma(C_1)$.)

ie, $(a, b] = \bigcap_n (a, b + 1/n)$.

& $C_1 \subset \sigma(\mathcal{C}) \Rightarrow \sigma(C_1) \subset \sigma(\mathcal{C})$.

ie, $(a, b) = \bigcap_n (a, b - 1/n]$

ie, for a Borel σ-field, there are many classes that generate that same σ-field.

<u>Discrete Setting</u>

Ω is countable.

$\mathcal{A} = \mathcal{P}(\Omega)$.

$X : \Omega \to \mathbb{R}$.

Suppose, $D_X =$ set of values of X.

D_X ← countable set.. $x \in D_X$.

For any $A \subset \mathbb{R}$, $\{X = x\}$,
$$\{X = x\} = \{\omega \in \Omega : X(\omega) = x\}.$$

$$\{X \in A\} = \{\omega : X(\omega) \in A\}.$$

$(\Omega, \mathcal{A}, P) \to$ Probability space.

Definition:
A real random variable is a function $X : \Omega \to \mathbb{R}$.
such that, $\forall\, a \in \mathbb{R}$,
$$\{\omega : X(\omega) \leq a\} \in \mathcal{A}.$$

Eg. $\{\omega : X(\omega) \in (-2, 3)\} \to$ this set expressed as a countable $\bigcap$. (here, finite)

$$= \{\omega : X(\omega) < 3\} \bigcap \{\omega : X(\omega) > -2\}$$

$\{\omega : X(\omega) < 3\} = \{\omega : X(\omega) \leq 3 - 1/n\}$, $\{\omega : X(\omega) \leq 3 - 1/n\} \in \mathcal{A}$

$\{\omega : X(\omega) > -2\} \in \mathcal{A}$ $\because$ $\{\omega : X(\omega) \leq -2\}^c \in \mathcal{A}$, & $\mathcal{A}$ is closed under complementation.

Corollary: If X is a random variable, then for every interval I,
$$\{\omega : X(\omega) \in I\} \in \mathcal{A}.$$

Corollary: For every Borel Set $B \subset \mathbb{R}$,
$$\{\omega : X(\omega) \in B) \in \mathcal{A}.$$