Definition: $X_1, \ldots, X_n$ are said to be independent if –

$$P(X_1 \in B_1, \ldots, X \in B_n) = \prod_{i=1}^{n} P(X_i \in B_i) \quad \forall \text{ Borel Sets } B_1, \ldots, B_n \subset \mathbb{R}$$

$$\Leftrightarrow F_X(\underset{\sim}{a}) = \prod_{i=1}^{n} F_X(a_i) \qquad \forall\, a \in \mathbb{R}^n.$$

Consequences:

a finite subgroup from the set of n ind. r.vs.

① $X_1, \ldots, X_n$ independent $\Leftrightarrow$ $X_{i_1}, \ldots, X_{i_k}$ independent for any $1 \le i_1 < \cdots < i_k \le n$

② for any $1 \leq m < n$,
$h(X_1, \ldots, X_m)$ and $g(X_{m+1}, \ldots, X_n)$ are independent. (dividing the set of n ind. r.vs into 2 finite subfamilies.
↳ also holds for > 2 subfamilies.)

*

$X_1, \ldots, X_n$ absolutely continuous and $X_1, \ldots, X_n$ independent.

$\Leftrightarrow$ $\underset{\sim}{X} = (X_1, \ldots, X_n)^T$ has joint density

$$f_X(\underset{\sim}{x}) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

Fact: X, Y – independent and hence finite means

$$\Rightarrow Cov(X,Y) = 0.$$

[Converse is NOT true!].

Remark: (X,Y) – bivariate normal. (just X, Y separately being univariate normal DOES NOT suffice!!!)

$Cov(X,Y) = 0.$

$\Rightarrow X \& Y$ are independent.

$\Rightarrow X \& Y$ are independent. suffice!!!

Recall: our discussion on Bivariate Normal started with –
$q(x,y)$ is a quadratic form in (x,y).

ie, $q(x,y) = \alpha x^2 - 2\beta xy + \gamma y^2$

$$\iint e^{-\frac{1}{2} q(x,y)}\, dx\, dy < \infty \quad \text{(When?)}$$

$$\Longleftrightarrow \quad \alpha > 0, \ \gamma > 0, \ \beta^2 < \alpha\gamma$$

$\therefore$ Under this condition,

$$f_{X,Y}(x,y) = C e^{-\frac{1}{2} q(x,y)}$$

The standard form of expressing a bivariate normal density:

$$f_{X,Y}(x,y) = C \cdot e^{\frac{-1}{2(1-\rho^2)} \cdot \left(\left(\frac{x}{\sigma_1}\right)^2 - \frac{2\rho xy}{\sigma_1 \sigma_2} + \left(\frac{y}{\sigma_2}\right)^2 \right)} ; \quad \sigma_1 > 0, \ \sigma_2 > 0, \ -1 < \rho < 1.$$

taking $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$,

we get $C = \dfrac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}}$

$\therefore (X,Y) \sim N(0, 0, \sigma_1, \sigma_2, \rho)$,

$$\boxed{f_{X,Y}(x,y) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \cdot e^{\frac{-1}{2(1-\rho^2)} \cdot \left(\left(\frac{x}{\sigma_1}\right)^2 - \frac{2\rho xy}{\sigma_1 \sigma_2} + \left(\frac{y}{\sigma_2}\right)^2 \right)}}$$

[Ex. from Set-6: $E(X) = E(Y) = 0$.
$V(X) = E(X^2) = \sigma_1^2$
$V(Y) = E(Y^2) = \sigma_2^2$
$Cov(X,Y) = \rho\sigma_1\sigma_2$]

$V(Y) = E(Y^2) - \sigma_2^2$
$\mathrm{Cov}(X,Y) = \rho\sigma_1\sigma_2$]

Note, if $\mathrm{Cov}(X,Y) = \rho\sigma_1\sigma_2 = 0$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2} \cdot e^{\frac{-1}{2}\cdot\left(\left(\frac{x}{\sigma_1}\right)^2 + \left(\frac{y}{\sigma_2}\right)^2\right)}$$

$$= \left(\frac{1}{\sigma_1\sqrt{2\pi}} \cdot e^{-\frac{x^2}{\sigma_1^2}}\right) \cdot \left(\frac{1}{\sigma_2\sqrt{2\pi}} \cdot e^{-\frac{y^2}{\sigma_2^2}}\right)$$

$$= f_X(x) \cdot f_Y(y). \quad \Rightarrow X, Y \text{ - independent.}$$

(i.e, the converse is true for Normal dist^ns.)

Slightly more general form:

$(x,y) \longmapsto (x-\mu_1,\ y-\mu_2)$; $\mu_1, \mu_2 \in \mathbb{R}$.

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot e^{\frac{-1}{2(1-\rho^2)}\cdot\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho xy}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)}$$

[can be shown: $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, $\mathrm{Cov}(X,Y) = \rho\sigma_1\sigma_2$]

Idea: The quadratic form is p.d.
So, do its spectral decomposition for it & see!!

Now, in $\mathbb{R}^k$:

$q(x) = \vec{x}^T A \vec{x}$ is a Quadratic form
$A \to$ real symmetric matrix $\to BAB^T$ - diagonal $= \mathrm{diag}(\alpha_1, \ldots, \alpha_k)$.

When is $\displaystyle\int_{\mathbb{R}^k} e^{-\frac{1}{2}\cdot q(\vec{x})} \cdot d\vec{x} < \infty$?

(*)

$\exists$ an orthogonal transformation $\vec{y} = B\vec{x}$ s.t.

$$q(\vec{x}) \longmapsto \alpha_1 y_1^2 + \cdots + \alpha_k y_k^2$$

The Jacobian for the transformation $x \mapsto y$ is $J = \det(B) = 1$ [$\because$ B-orthonormal matrix.].

$\therefore$ (*) $\to$ Same as asking:

when is $\displaystyle\int_{\mathbb{R}^k} e^{-\frac{1}{2} q(y)}\, dy < \infty$?, where

$q(y) = \alpha_1 y_1^2 + \cdots + \alpha_k y_k^2 > 0$,

i.e., $\alpha_1, \ldots, \alpha_k > 0$,

i.e., A is pd.

$$\therefore \int e^{-\frac{1}{2} q(\vec{x})}\, d\vec{x} < \infty$$

$$\Longleftrightarrow \int e^{-\frac{1}{2} q(\vec{y})}\, dy < \infty$$

$$\Longleftrightarrow A \text{ is p.d.}$$

$$\int e^{-\frac{1}{2}(\alpha_1 y_1^2 + \cdots + \alpha_k y_k^2)}\, d\vec{y} < \infty$$

$$\Rightarrow \int \prod_{i=1}^{k} e^{-\frac{1}{2}\cdot \alpha_i y_i^2}\, d\vec{y} < \infty$$

$$* \int_{\mathbb{R}} e^{-\frac{1}{2}\alpha y^2}\, dy = \sqrt{\frac{2\pi}{\alpha}} = (2\pi)^{1/2} \cdot \alpha^{-1/2}$$

$$\Rightarrow \prod_{i=1}^{k} \int e^{-\frac{1}{2}\alpha_i y_i^2}\, d\vec{y} < \infty$$

$$\Rightarrow (2\pi)^{k/2} \cdot \left(\prod_{i=1}^{k} \alpha_i\right)^{-k/2} < \infty$$

$$A = \begin{bmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_k \end{bmatrix}$$

$$\Rightarrow (2\pi)^{k/2} \cdot (\det A)^{-k/2} < \infty$$

$\therefore \Leftrightarrow A$ is p.d., & then,

$$\left[(2\pi)^{-k/2} \cdot (\det(A))^{1/2} \cdot e^{-\frac{1}{2}\cdot x^T A x}\right] \text{ is}$$

a density. $\rightarrow$ normalizing constant

Standard formulation:

Write $A = \Sigma^{-1}$

$$f(\vec{x}) = (2\pi)^{-k/2} \cdot (\det(\Sigma))^{-1/2} \cdot e^{-1/2 \cdot \vec{x}^T \Sigma^{-1} \vec{x}}.$$

Σ is a p.d matrix.

Now, make the transformation $\vec{x} \mapsto \vec{x} - \vec{\mu}$.

then,

$$f(\vec{x}) = (2\pi)^{-k/2} \cdot (\det(\Sigma))^{-1/2} \cdot e^{-1/2 \cdot (\vec{x}-\vec{\mu})^T \Sigma (\vec{x}-\vec{\mu})}$$

$$\sim N_k(\vec{\mu}, \Sigma).$$

Definition: X is said to have a k-variate normal density, ie, $X \sim N_k(\vec{\mu}, \Sigma)$.

$$\boxed{f(\vec{x}) = (2\pi)^{-k/2} \cdot (\det(\Sigma))^{-1/2} \cdot e^{-1/2 \cdot (\vec{x}-\vec{\mu})^T \Sigma (\vec{x}-\vec{\mu})}}$$

Consequences:

1. $\vec{X} \sim N_k(\vec{\mu}, \Sigma)$. $\Rightarrow$ for any non-singular A, any $\vec{\beta} \in \mathbb{R}^k$, $\vec{Y} = (A\vec{X} + \vec{\beta}) \sim N_k(A\vec{\mu} + \vec{\beta}, A\Sigma A^T)$.

2. $\vec{X} \sim N_k(\vec{\mu}, \Sigma)$
   $\Rightarrow$ for any permutation $\Pi : \{1, 2, \ldots, k\} \longrightarrow \{1, 2, \ldots, k\}$.

$\Rightarrow$ for any permutation $\pi : \{1, 2, \ldots, k\} \longrightarrow \{1, 2, \ldots, k\}$.

$$(X_{\pi(1)}, \ldots, X_{\pi(k)}) \sim N_k(\mu_\pi, \Sigma_\pi).$$

<u>Interpretation of μ & Σ:</u>

Let $X \sim N_k(\vec{\mu}, \Sigma)$

Get a non-singular matrix such that, $A \Sigma A^T = I_k$.

Put $Y = A(X - \vec{\mu})$

then, $\vec{Y} \sim N_k(\vec{0}, I)$ [By consequence 1].

$\Rightarrow Y_1, \ldots, Y_k$ - independent, & $Y_i \sim N(0,1)$.

$\therefore E(\vec{Y}) = 0$, Dispersion matrix of $\vec{Y} = I$.

is $D(\vec{Y}) = I$

Now, apply the transformation

$$\vec{X} = A^{-1}\vec{Y} + \vec{\mu}$$

(*) $E(\vec{X}) = A^{-1} \cancel{E(\vec{Y})}^{\,0} + \vec{\mu} = \vec{\mu}$

& , (*) $D(\vec{X}) = A^{-1} D(Y) (A^{-1})^T$

$$= A^{-1} \cdot I \cdot (A^{-1})^T$$

$$= (A^T A)^{-1} = \Sigma.$$

<u>Result:</u>

$\vec{X} \sim N_k(\vec{\mu}, \Sigma)$.

$\Rightarrow (X_1, \ldots, X_m)^T \sim N_m \quad \forall \; 1 \leq m < k$.

<u>Proof:</u> Assume $\mu = 0$.

We'll prove it for $m = k-1$.

Let $\Sigma^{-1} = ((a_{ij}))$. Observe that, $a_{kk} \neq 0$

[$\because \Sigma$ is pd,

$\therefore \Sigma^{-1}$ is pd

We do a non-singular linear transformation:

We do a non-singular linear transformation:

[$\because \Sigma$ is pd, $\therefore \Sigma^{-1}$ is pd $\Rightarrow$ all diagonal entries > 0]

$\vec{x} \mapsto \vec{y}$,

$$y_1 = x_1$$
$$y_2 = x_2$$
$$\vdots$$
$$y_{k-1} = x_{k-1}$$
$$y_k = \alpha_{1k} x_1 + \alpha_{2k} x_2 + \cdots + \alpha_{kk} x_k .$$

writing the quadratic form in terms of y_k^2, we have

$$q(\vec{x}) = \frac{1}{\alpha_{kk}} \cdot y_k^2 + q(y_1, \ldots, y_{k-1}) .$$

Quadratic form in $y_1, \ldots, y_{k-1}$.

Now, we know, any such non-singular transformation converts one k-variate normal dist^n to another k-variate normal dist^n. [check : exercise].

$\therefore \vec{Y} \sim N_k$.

But, from the quadratic form $q(\vec{y})$, it is separable into 2 normals, Y_k, & $(Y_1, \ldots, Y_{k-1})^T$.

i.e, the density of $\vec{Y}$ factors into the density of $\vec{Y_k}$ & $(Y_1, \ldots, Y_{k-1})^T$.

$\therefore (Y_1, \ldots, Y_{k-1})^T \sim N_{k-1}$.

Now, by induction, we can get down to any lesser k & prove the same result.

ie, any permutation of m Y_i's (picked from $\vec{Y}$) $\sim N_m$, $1 \leq m < k$.

Hence, any permutation of m X_i's (picked from $\vec{X}$) $\sim N_m$. $\square$

Hence, any permutation of m X_i's (picked from $\vec{X}$) $\sim N_m$. $\blacksquare$

ie, Consequences:

① $X_1, \ldots, X_k \sim N_k(\vec{\mu}, \Sigma)$ for any $1 \le i_1 < \cdots < i_m \le k$.

$(X_{i_1}, \ldots, X_{i_m}) \sim N_m$. (Proved above).

② $\vec{X} \sim N_k(\vec{\mu}, \Sigma)$,

then $\forall\ \vec{a} \neq 0$, $\vec{a}^T \vec{X} \sim N(\vec{a}^T \vec{\mu}, \vec{a}^T \Sigma \vec{a})$

univariate normal.

Proof:

Take a matrix A with first row to be $\vec{a}^T$. Then, $A\vec{X}$ is k-dimensional normal, & the first component

take $\vec{a}$. Extend to a basis of $\mathbb{R}^k$. So, $A = \begin{bmatrix} \vec{a}^T \\ \vdots \end{bmatrix}$ thus obtained is a non-singular matrix.

$A\vec{x} = \begin{bmatrix} \circ \\ \\ \\ \end{bmatrix}$ follows univariate normal.

Using just the consequence above.

Exercise: assume $\mu = 0$. $\vec{X} \sim N_k(0, \Sigma)$

Make the transformation $\vec{x} \mapsto \vec{y}$

$$y_1 = x_1$$
$$y_2 = x_2$$
$$\vdots$$
$$y_{k-1} = x_{k-1}$$
$$y_k = \alpha_{1k} x_k + \cdots\cdots + \alpha_{kk} \cdot x_k$$

$$y_k = \alpha_{kk}(x_k - \beta_1 x_1 - \cdots - \beta_{k-1} x_{k-1}),$$

where $\beta_i = -\dfrac{\alpha_{ik}}{\alpha_{kk}}$

$$x_k = \frac{y_k}{\alpha_{kk}} + \beta_1 x_1 + \cdots + \beta_{k-1} \cdot x_{k-1}$$

Example: Z, X - independent.

$Z \sim N(0, \sigma^2)$.

... conditional distribution of

$Z \sim N(0, \sigma^2)$.

Q. What is the conditional distribution of Z given X?

Ans. $N(0, \sigma^2)$ $[\because x, Z$ - ind.$]$

Q. What is the conditional distribution of $Z + X$ given $X = x$?

Ans: $N(x, \sigma^2)$ [$\because$ Here, x acts as a constant!!!]

. . . .

$\vec{X} \sim N_k(\vec{0}, I_k)$

$\bar{X} = \frac{1}{k} \cdot \sum_{i=1}^{k} X_i$, $\quad S^2 = \frac{1}{k-1} \cdot \sum_{i=1}^{k} (X_i - \bar{X})^2$

to prove: $\bar{X}$, S^2 - independent.

$\bar{X} \sim N\left(0, \frac{1}{k}\right)$ & $S^2 \sim \chi^2_{(k-1)}$

Result (we'll use for this):

$\vec{X} \sim N_k(\vec{0}, \Sigma)$.

$Z = \frac{1}{2} \vec{X}^T \cdot \Sigma^{-1} \cdot \vec{X} \sim \chi^2_{(k)}$ (sum of squares of normal).

(Using mgf approach):

For $t < 1$,

$$E\left(e^{tZ}\right) = (2\pi)^{-k/2} \cdot (\det(\Sigma))^{-1/2} \cdot \int e^{\frac{1}{2} \cdot t \cdot \vec{x}^T \Sigma^{-1} \vec{x}} \cdot e^{-\frac{1}{2} \vec{x}^T \Sigma^{-1} \vec{x}} \, dx$$

$\shortparallel$

$e^{tz} \cdot f_Z(z)$.

e

$$= \left(\quad ? \quad \right) \cdot \int e^{-\frac{1}{2} \cdot \vec{x}^T \underbrace{(1-t) \cdot \Sigma^{-1}}_{A} \cdot \vec{x}} \, dx$$

$$= \left(\quad ? \quad \right) (2\pi)^{k/2} \cdot \left(\det\left((1-t) \cdot \Sigma^{-1}\right)\right)^{1/2}$$

$$\vdots$$

$$= (1-t)^{-k/2}$$

$$= \left(\frac{1}{1-t}\right)^{k/2} \longrightarrow \text{looks like } \left(\frac{\lambda}{\lambda-t}\right)^{\alpha}.$$

$\therefore$ mgf of Gamma $\left(k/_2, 1\right)$.

[Check : $\chi^2_{(n)} \equiv \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$]

But, we got Gamma$\left(k/_2, \underline{1}\right)$??

Now, back to the proof:

we make a transformation

$$\vec{x} \longmapsto \vec{y}.$$

$$y_1 = x_1 - \overline{x}$$

$$y_2 = x_2 - \overline{x}$$

$$\vdots$$

$$y_{k-1} = x_{k-1} - \overline{x}$$

$$y_k = \overline{x}.$$

$$\therefore \vec{x}^T x = k\, y_k^2 + \underbrace{(k-1)\cdot S^2}_{\sum_{i=1}^{k-1} y_i^2 + \left(\sum_{i=1}^{k-1} y_i\right)^2}$$