Probability-2 Lecture-16

12 March 2024 11:25

Recall: DCT (Dominated Convergence Theorem):

$\{X_n\}_{n\geq 1}$ - sequence of Random Variables.

$$X_n \to X$$

Suppose $\exists$ a r.v Y s.t. $E(Y) < \infty$ and $|X_n| \leq Y \quad \forall\, n \in \mathbb{N}$.

Then, $E(X_n) \longrightarrow E(X)$.

Indeed, $E|X_n - X| = 0$

Lemma: (Fatou's Lemma):

For any sequence $\{X_n\}$ of non-ve r.v.s,

$$\boxed{E(\liminf X_n) \leq \liminf (E(X_n)).}$$

Proof: Define $Y_n = \inf\limits_{k\geq n} X_k$

$Y_n \geq 0$. $Y_n \nearrow \liminf X_n$

$\overset{(MCT)}{\Longrightarrow}$ $E(Y_n) \to E(\liminf X_n)$

$E(X_n) \geq E(Y_n) \;\forall n$

$\liminf E(X_n) \geq E(\liminf X_n)$ ▨

Proof (DCT):

Since $|X_n| \leq Y \quad \forall\, n$.

& $X_n \longrightarrow X$.

we have $|X| \leq Y$.

$$|X_n - X| \leq \underbrace{|X_n|}_{\leq Y} + \underbrace{|X|}_{\leq Y} \leq 2Y$$

Set $Z_n := 2Y - |X_n - X|$. $\therefore E(Z_n) := 2E(Y) - E|X_n - X|$.

non -ve r.v

$\therefore \liminf Z_n = 2Y - \underbrace{\limsup |X_n - X|}_{\overset{(*)}{=} 0}$

$\therefore E(\liminf Z_n) \leq \liminf (E(Z_n))$ [By Fatou's Lemma]

$\Rightarrow 2E(Y) \leq 2E(Y) - \limsup E(|X_n - X|)$

[$\because 0 \leq \liminf |X_n - X| \leq \limsup |X_n - X| \leq 0$]

$$\left[\begin{array}{l} \because\ 0 \leq \liminf |X_n - X| \leq \limsup |X_n - X| \leq 0 \\ (*) \qquad\quad \downarrow \qquad\quad \text{as } X_n \to X. \qquad \downarrow \\ \qquad\qquad 0 \qquad\qquad\qquad\qquad\qquad\quad 0 \\ \therefore \lim_{n\to\infty} |X_n - X| = 0 \end{array}\right]$$

$$\left.\begin{array}{l} \Rightarrow \limsup E|X_n - X| \leq 0 \\ \&,\ |X_n - X| \geq 0 \Rightarrow \\ \qquad E|X_n - X| \geq 0. \end{array}\right\} \lim_{n\to\infty} E|X_n - X| = 0.$$

$\blacksquare$

$\therefore$

Notation:

for any r.v. X on a probability space, & any $p > 0$, denote $\|X\|_p = \left(E|X|^p\right)^{1/p}$.

$$0 \leq \|X\|_p \leq \infty$$

Holder's Inequality:

For any two random variables X, Y on the same probability space

$$\boxed{E|XY| \leq \|X\|_p \cdot \|X\|_q.}\ ,$$

where $p, q > 1$ are conjugate (ie, $\frac{1}{p} + \frac{1}{q} = 1$)

Proof:

Case-I: RHS $= 0$

Say, $\|X\|_p = 0$.

$\Rightarrow E|X|^p = 0.$

$\Rightarrow P(X = 0) = 1$

$\Rightarrow P(XY = 0) = 1$

$\Rightarrow P(X=0) = 1$

$\Rightarrow P(XY=0) = 1$

$\Rightarrow LHS = 0.$

Case - 2: $\|X\|_p > 0 \;, \; \|X\|_q > 0.$

In case atleast one of $\|X\|_p$ and $\|Y\|_q$ ∞, the inequality holds trivially.

Case - 3: (Non-trivial case)

$0 < \|X\|_p < \infty \;, \quad 0 < \|X\|_q < \infty.$

Define $X' := \dfrac{X}{\|X\|_p}$, $Y' := \dfrac{Y}{\|Y\|_q}$ ("Normalizing a r.v")

Clearly, $\|X'\|_p = \|Y'\|_q = 1.$

Use inequality:

$\forall$ reals $a, b \geq 0.$

$ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}.$

$\therefore \; |X'| \cdot |Y'| \leq \frac{1}{p} \cdot |X|^p + \frac{1}{q} \cdot |Y|^q$

$\therefore \; \underbrace{E\left(|X'| \cdot |Y'|\right)}_{\left(\frac{|X| \cdot |Y|}{\|X\|_p \cdot \|Y\|_q}\right)} \leq \frac{1}{p} E|X|^p + \frac{1}{q} E|Y|^q = 1$

$\therefore E|XY| \leq \|X\|_p \cdot \|X\|_q.$

Cauchy - Schwarz Inequality:

$\boxed{E|XY| \leq \|X\| \cdot \|Y\|}$

Cauchy - Schwarz Inequality:

$$\boxed{E|XY| \le \|X\|_2 \cdot \|Y\|_2}$$

Fact: for any r.v. X,

$$\|X\|_{p_1} \le \|X\|_{p_2} \quad \forall \quad 0 < p_1 < p_2.$$

Proof:

$E|X|^{p_1}$ $\qquad W = |X|^{p_1}$

$= E|WZ|$ $\qquad Z = 1$

$\le (E(W^p))^{1/p} \cdot (E(Z^q))^{1/q}$ $\longrightarrow$ [By Holder's inequality]

$p = p_2/p_1 > 1$, q - conjugate of p.

$= \left(E|X|^{p_2}\right)^{p_1/p_2}$

$\therefore \left(E|X|^{p_1}\right)^{1/p_1} \le E\left(|X|^{p_2}\right)^{1/p_2}$

$\Rightarrow \quad \|X\|_{p_1} \le \|X\|_{p_2}$

Minkowski's Inequality:

Let $p \geq 1$.

for any two X, Y

$$\|X+Y\|_p \le \|X\|_p + \|Y\|_p$$

Proof: $p=1$: (trivial).

$\because |X+Y| \le |X| + |Y|$.

$p>1$: Let q be the conjugate of p. ie, $\frac{1}{p} + \frac{1}{q} = 1$

$\therefore E|X+Y|^p \le E\left(|X| \cdot |X+Y|^{p-1}\right) + E\left(|Y| \cdot |X+Y|^{p-1}\right)$ (triangle ineq.)

$\le \|X\|_p \cdot \| |X+Y|^{p-1} \|_q + \|Y\|_p \cdot \| |X+Y|^{p-1} \|_q$. (Holder's inequality)

$= \left(\|X\|_p + \|Y\|_p\right) \left(\| |X+Y|^{p-1} \|_q \right)$. $\qquad \frac{1}{p} + \frac{1}{q} = 1$

$= \left(\|X\|_p + \|Y\|_p\right) \cdot \left(E|X+Y|^{(p-1)\cdot q}\right)^{1/q}$. $\qquad \therefore (p-1)q = p$

$= \left(\|X\|_p + \|Y\|_p\right) \cdot \left(E|X+Y|^p\right)^{1/q}$ $\qquad$ if $\left(E|X+Y|^p\right)^{1/q} = 0$

$$= \left(\|X\|_p + \|Y\|_p\right) \cdot \left(E|X+Y|^p\right)^{1/q}$$

if $\left(E|X+Y|^p\right)^{1/q} = 0$ Minkowski holds trivially.

$\leftarrow$ if not,

$$\left(E|X+Y|^p\right)^{1-1/q} \le \left(\|X\|_p + \|Y\|_p\right) \qquad 1-1/q \leftarrow 1/p$$

$$\Rightarrow \left(E|X+Y|^p\right)^{1/p} \le \left(\|X\|_p + \|Y\|_p\right). \qquad \blacksquare$$

Jensen's Inequality:

Let X be a real random variable with $E(X) < \infty$ (i.e, finite)

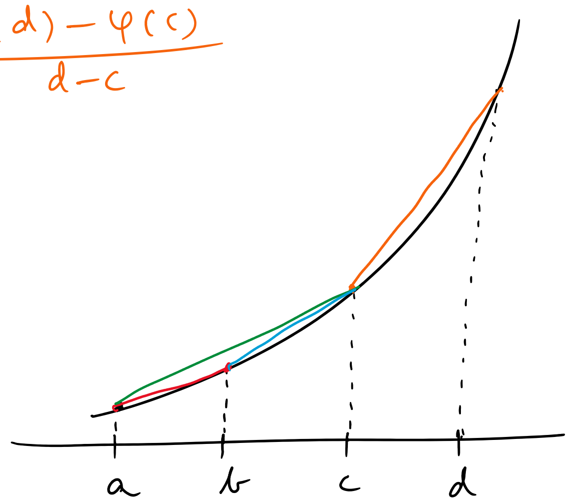
Then, for convex function $\varphi : \mathbb{R} \to \mathbb{R}$. with $E(\varphi(X))$ finite,

$$\boxed{\varphi(E(X)) \le E(\varphi(X)).}$$

Some facts about Convex functions:

① $\forall\ a, b, c, d$.

$$\frac{\varphi(b) - \varphi(a)}{b-a} \le \frac{\varphi(d) - \varphi(c)}{d-c}$$



② $\forall\ a, b, c$

$$\frac{\varphi(c) - \varphi(a)}{c-a} < \frac{\varphi(c) - \varphi(b)}{c-b}$$

③ For $a < b < c$.

$$\frac{\varphi(c) - \varphi(a)}{c-a} \geqslant \frac{\varphi(b) - \varphi(a)}{b-a}$$

Consequence:

$\forall\ x \in \mathbb{R}$, $\varphi'(x^-)$ & $\varphi'(x^-)$ exist &

$$\varphi'(x^-) \le \varphi'(x^+)$$

In particular, φ is continuous.

In particular, φ is continuous.

Fact:

For every $x \in \mathbb{R}$.

$\exists$ real nos. $a (= a_x)$ and $b (= b_x)$.

depending on x.

s.t. $a + by \leq \varphi(y)$

with equality holding only at $y = x$, φ - convex.

Proof: take any $x \in \mathbb{R}$.

take any real b such that $\varphi'(x^-) \leq b \leq \varphi'(x^+)$

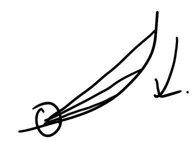
take $a = \varphi(x) - bx$.

Clearly, $\varphi(x) = a + bx$.

take $y \neq x$.

W.L.O.G, take $y > x$.

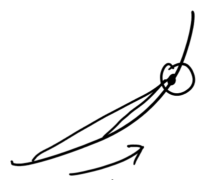
$$\frac{\varphi(y) - \varphi(x)}{y - x} \geq \varphi'(x^+) \geq b.$$



(RHDs - seq. of values decreasing)

$$\Rightarrow \quad \varphi(y) - (a + bx) \geq by - bx$$

$$\Rightarrow \quad \varphi(y) \geq a + by.$$



LHDs - seq. of increasing values.

proof of Jensens (...contd):

$\because$ E(X) is finite.

$E(X) \in \mathbb{R}$.

$\therefore \exists$ reals a, b s.t.

$a + by \leq \varphi(y) \;\; \forall y.$ (using the result above).

& $a + b\,E(X) \leq \varphi(E(X))$

$a + bX \leq \varphi(X)$.

$\therefore a + bE(X) \leq E(\varphi(x)$

$$\underbrace{\qquad\qquad}_{\varphi(E(X))}$$

<u>Aim</u>: X is a real random variable with density f - non -ve.

This means, f is Riemann integrable on $\mathbb{R}$ with $\int_{-\infty}^{\infty} f(x)\,dx = 1$, and, for any interval I,

$$P(X\in I) = \int_I f(x)\,dx.$$

If $h:\mathbb{R}\to\mathbb{R}$ is a function such that, $h(X)$ is a random variable, then $h(X)$ has finite expectation iff $\int_{-\infty}^{\infty} |h(x)|\cdot f(x)\,dx < \infty$.

& in that case, $E(h(X)) = \int_{-\infty}^{\infty} h(x)\cdot f(x)\,dx.$

<u>Proof</u>: Start with $0 \leq X \leq M$.

X has density function f : $\int_0^M f(x)\,dx = 1$.

Define $X_n := \sum_{0\leq k\leq M\cdot 2^n - 1} \frac{k}{2^n}\cdot \mathbb{1}_{k/2^n \leq X \leq \frac{k+1}{2^n}}$

Each X_n is non-negative simple r.v; $X_n \nearrow X$

$$\Rightarrow E(X) = \lim_{n\to\infty} E(X_n)$$

$$= \sum_{0\leq k\leq M\cdot 2^n - 1} \frac{k}{2^n}\cdot \int_{k/2^n}^{(k+1)/2^n} f(x)\,dx.$$

proof: (exercise)

$$= \int_0^M x f(x)\,dx.$$

Now, X has density f : $\int_0^\infty f(x)\,dx = 1$.

<u>Claim</u>: $E(X) = \int_0^\infty x \cdot f(x)\,dx$.

<u>Fix M</u>:

define $X_M = \begin{cases} X, & \text{if } X \leq M \\ 0, & \text{if } X > M. \end{cases}$

$X_M \geq 0$.

$X_M \nearrow X$. $\xRightarrow[\text{MCT})]{\text{(by}}$ $E(X) = \lim_{M \to \infty} E(X_M)$.

$$X_{n,M} = \sum_{0 \leq k \leq M \cdot 2^n - 1} \frac{k}{2^n} \cdot \mathbb{1}_{\frac{k}{2^n} < X < \frac{k+1}{2^n}}$$

$$\therefore E(X_M) = \lim_{n \to \infty} E(X_{n,M}) = \int_0^M x f(x)\,dx$$

$$\therefore E(X) = \lim_{M \to \infty} E(X_M) = \lim_{M \to \infty} \int_0^M x \cdot f(x)\,dx = \int_0^\infty x f(x)\,dx$$

$\blacksquare$