Probability-2 Lecture-7

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Probability space : $(\Omega, \mathcal{A}, P)$

a non-empty set

a σ-field on Ω

a probability defined on $\mathcal{A}$.

A Special Case:

Ω - countable, $\mathcal{A} = P(\Omega)$.

$$P(A) = \sum_{w \in A} P(\{w\}).$$

A discrete probability space.

Definition:

A real random variable on a probability space $(\Omega, \mathcal{A}, P)$ is a function $X : \Omega \to \mathbb{R}$ satisfying :

$$(*) \quad \underbrace{\{w : X(w) \leq a\}}_{\text{we will abbreviate it as } \{X \leq a\}} \in \mathcal{A} \quad \forall\, a \in \mathbb{R}.$$

Fact: $(*)$ can be replaced by any of the following three : (equivalently).

1. $\{w : X(w) < a\} \in \mathcal{A} \quad \forall\, a \in \mathbb{R}$.
2. $\{w : X(w) > a\} \in \mathcal{A} \quad \forall\, a \in \mathbb{R}$.
3. $\{w : X(w) \geq a\} \in \mathcal{A} \quad \forall\, a \in \mathbb{R}$

Consequence:

If X is a real random variable on a probability space, $(\Omega, \mathcal{A}, P)$ then,

$$\{w : X(w) \in B\} \in \mathcal{A} \quad \forall \text{ Borel subset } B \subset \mathbb{R}.$$

Q. How to show?

Notation: $h : E \longrightarrow F$. for any $A \subset F$,

$$h^{-1}(A) = \{x \in E : h(x) \in A\}$$

then :

- $h^{-1}\left(\bigcup_{\alpha} A_\alpha\right) = \bigcup_{\alpha} h^{-1}(A_\alpha)$.

- $h^{-1}\left(\bigcap_{\alpha} A_\alpha\right) = \bigcap_{\alpha} h^{-1}(A_\alpha)$

- $h^{-1}(A^c) = \left(h^{-1}(A)\right)^c$

complementation in domain.

complementation in codomain

$\mathcal{G} := \{ B \subset \mathbb{R} : X^{-1}(B) \in \mathcal{A} \}$

- Claim: $\mathcal{G}$ contains $\mathbb{R}$.

Why? $\because X : \Omega \to \mathbb{R}$.

$\Rightarrow X^{-1}(\mathbb{R}) = \Omega \in \mathcal{A}$. [$\because \mathcal{A}$ is a σ-field].

$\Rightarrow \mathbb{R} \in \mathcal{G}$. ▨

- Claim: $\mathcal{G}$ is closed under complementation.

Why? $B \in \mathcal{G} \Rightarrow X^{-1}(B) \in \mathcal{A}$.

$\Rightarrow (X^{-1}(B))^c \in \mathcal{A}$ [$\because \mathcal{A}$ is a σ-field]

$\Rightarrow X^{-1}(B^c) \in \mathcal{A}$.

$\Rightarrow B^c \in \mathcal{G}$. ▨

- Claim: $\mathcal{G}$ is closed under countable union.

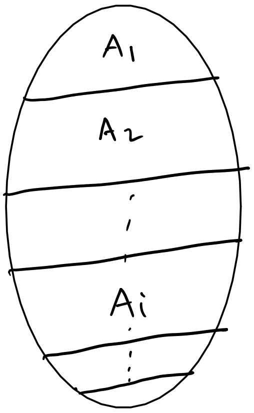
Why? (Exc.)

Definition:

A real random variable X on a probability space $(\Omega, \mathcal{A}, P)$ is called discrete if X takes only countably many values.

Let $\{c_1, c_2, \ldots\}$ be the countable set of values of X.

$\therefore$ For each $i \geq 1$ $\underbrace{\{w : X(w) = c_i\}}_{\text{call this } A_i \text{ (an event)}} \in \mathcal{A}$



Each $A_i \in \mathcal{a}$.
$\{A_1, A_2, \ldots\}$ is a partition on Ω.
$X = c_i$ on A_i.

- constant functions are trivially r.vs.
  Eg. $X(w) = 10 \quad \forall\ w \in \Omega$.
  then $\{w : X(w) \geqslant 10\} = \Omega \in \mathcal{a}$. $\blacksquare$)

- X is a real r.v. on $(\Omega, \mathcal{a}, P)$:

$\overset{?}{\Longrightarrow} \alpha X$ is a real r.v $\forall\ \alpha \in \mathbb{R}$.

$\alpha = 0$: $\alpha X \equiv 0$ is a r.v. (trivial)

WLOG, Suppose, $\alpha > 0$. $\{w : \alpha X \cdot (w) \leqslant c\} = \{w : X(w) \leq c/\alpha\}$, $c/\alpha \in \mathbb{R}$.
$\therefore \alpha X$ is a real r.v. $\blacksquare$

- X, Y — r.vs on $(\Omega, \mathcal{a}, P)$:

$\overset{?}{\Longrightarrow} (X+Y)$ is a random variable.

Fix $c \in \mathbb{R}$.

$\{w : X(w) + Y(w) \leq c\} \overset{?}{\in} \mathcal{a}$.

$\{w : X(w) + Y(w) \leq c\} = \{w : X(w) < c - Y(w)\}$

$= \bigcup_r \{w : X(w) < r,\ Y(w) < c - r,\ r \in \mathbb{Q}\}$.

$= \bigcup_r \{w : X(w) < r < c - Y(w)\}$.

between any 2 reals $\exists$ a rational.
here: $X(w)$, $c - Y(w) \in \mathbb{R}$.
$X(w) < c - Y(w)$.
then, $\underline{\exists\ r \in \mathbb{Q}}$, s.t.

then, $\exists\, r \in \mathbb{Q}$, s.t.
$X(\omega) < r < c - Y(\omega)$. ▨

- X, Y — r.vs on $(\Omega, \mathcal{A}, P)$:

$\overset{?}{\Longrightarrow}$ XY is a random variable.

Proof: We'll show that, X is a r.v $\Rightarrow X^2$ is a r.v.

fix $c \in \mathbb{R}$. for $c < 0$,

$$\{\omega : X^2(\omega) < c\} = \phi \in \mathcal{A}. \quad \text{(trivial)}$$

$\therefore$ for $c \geq 0$,

$$\{\omega : X^2(\omega) \leq c\} = \{\omega : X(\omega) \leq \sqrt{c}\} \bigcap \{\omega : X(\omega) \geq -\sqrt{c}\} \in \mathcal{A}.$$ ▨

$\therefore$, $\because$ $(X+Y)$ is r.v $\Rightarrow (X+Y)^2$ is r.v.

$(X-Y)$ is r.v $\Rightarrow (X-Y)^2$ is r.v.

$\Rightarrow \dfrac{(X+Y)^2 - (X-Y)^2}{4}$ is r.v.

$\Rightarrow XY$ is a r.v. ▨

Q. $h : \mathbb{R} \to \mathbb{R}$.
X - real r.v. on $(\Omega, \mathcal{A}, P)$
is $h(X)$ a r.v.?

Soln: fix $c \in \mathbb{R}$.

$$\{\omega : h(X(\omega)) < c\} \overset{?}{\in} \mathcal{A}$$

$$= \{\omega : X(\omega) \in \underline{h^{-1}((-\infty, c])}\}$$

$\rightarrow$ Now, all we require is,
$h^{-1}((-\infty, c])$ is a Borel Set $\forall\, c \in \mathbb{R}$.

ie, the inverse image of the left "half line" is a Borel set.

left half line is a
Borel Set.

<u>Result</u>: If X is a real r.v, then $h(X)$ is also a real r.v.

$(\Omega, \mathcal{A}, P)$ - Probability Space.

<u>Definition</u>: A function $X : \Omega \longrightarrow [-\infty, \infty]$ closed at ∞ !
is called an extended real valued r.v. if

$$\{w : X(w) \leq a\} \in \mathcal{A} \quad \forall\, a \in \mathbb{R}.$$

This forces, $\{w : X(w) = -\infty\} \in \mathcal{A}$

$$\{w : X(w) = +\infty\} \in \mathcal{A}.$$

X, Y - r.vs.

$$\left.\begin{aligned} X \vee Y &:= \max\{X, Y\} \\ X \wedge Y &:= \min\{X, Y\} \end{aligned}\right\} \longrightarrow \text{they are also r.vs.}$$

$$a \in \mathbb{R} \Rightarrow \{X \vee Y \leq a\} = \{X \leq a\} \cap \{Y \leq a\} \in \mathcal{A}. \checkmark$$

$$\{X \wedge Y \geq a\} = \{X \geq a\} \cap \{Y \geq a\} \in \mathcal{A}. \checkmark$$

<u>Consequence</u>:

$X_1, X_2, \ldots, X_n$ r.v.s

$\bigvee_{i=1}^{n} X_i$, $\bigwedge_{i=1}^{n} X_i$ are r.vs. (proof: similar to above case involving 2 r.vs.)

(<u>finite</u> max.) (<u>finite</u> min.)

Now, for <u>countable</u>:

$\{X_n\}_{n \geq 1}$ sequence of r.v.s

$\bigvee_n X_n$, $\bigwedge_n X_n$ are r.vs. (Maybe extended real values.)

(countable max.) (countable min.)

" "

(countable max.) (... min.)

$$\sup_n (X_n) \qquad \inf_n (X_n).$$

$\limsup X_n$, $\liminf X_n$ are all r.v.s.
(sup. of tail) (inf. of tail).

[limit of a seq. of r.v. is a r.v as: if it exists, it must be equal to the limsup (= liminf).].

<u>Food for thought:</u>

Consider a sequence of real valued r.v.s $\{X_n\}_{n \geq 1}$.
(Consider all ω s.t. $\lim_n X_n(\omega)$ exists.)
$\rightarrow$ Call this set A.
Show that, $A \in \mathcal{a}$.