Probability-3 Lecture-8

20 August 2024 14:19

"Learn it, give exams, but when it comes to real life, Probability doesn't matter." — Prof. AG, 20th Aug. '24

$(\Omega, \mathcal{A}, P)$

$p > 0$.

Defn: $L_p := \{X \text{ - r.v} : E|X|^p < \infty\}$ → set of all r.vs with finite p^{th} moment.

Firstly, L_p is a Vector Space.

- Scalar multiplication & pt-wise addition are to be checked

$X \in L_p \Rightarrow cX \in L_p$ ✓ (trivial)

Now if $X, Y \in L_p$, does $X+Y \in L_p$

$$\|X+Y\|_p = \left(E|X+Y|^p\right)^{1/p} \leq \|X\|_p + \|Y\|_p < \infty$$

[By Minkowski's inequality]

$\therefore X+Y \in L_p$ ✓.

$\therefore L_p$ is a vector space. ▨

<u>Inequality</u>: for any two real nos. a, b.

$$|a+b|^p \leq |a|^p + |b|^p \quad \text{———} \quad (*)$$

Then: for $\omega \in \Omega$, $X(\omega), Y(\omega) \in \mathbb{R}$. $a = X(\omega)$, $b = Y(\omega)$

$$\therefore |X(\omega) + Y(\omega)|^p \leq |X(\omega)|^p + |Y(\omega)|^p.$$

$\therefore$ Taking Expectation on both sides:

$$E|X(\omega)+Y(\omega)|^p \leq \underbrace{E|X(\omega)|^p}_{<\infty} + \underbrace{E|Y(\omega)|^p}_{<\infty} \quad \forall \omega \in \Omega.$$

<u>Proof</u>: <u>Case-I</u> if any of $a = 0$ or $b = 0$, then nothing to prove!! (trivial)

Case-II $a \neq 0,\ b \neq 0$.

for (*), enough to prove $\underbrace{(|a|+|b|)^p \leq |a|^p + |b|^p}_{(**)}$.

then (*) follows by Δ-inequality.

WLOG, assume

$$0 < |a| \leq |b|$$

$$\therefore\ r = \frac{|b|}{|a|} \geq 1.$$

$f(r) = (1+r)^p - (1+r^p)$

Exercise: Show that $f'(r) < 0$, i.e., $f(r) \downarrow$ $r \geq 1$.

$\Rightarrow f(r) \leq f(1) = 2^p - 2 \leq 0$ $[\because 0 < p \leq 1]$

$\therefore$ LHS of (**)

$$= |a|^p (1+r)^p$$

RHS of (**)

$$= |a|^p (1+r^p)$$

$\therefore$ L_p is a vector space $\forall\, p > 0$.

But specifically: L_p is a Normed Linear Space $\forall\, p \geq 1$.

Definition: (Convergence in L_p)

For $p > 0$, we say that $\{X_n\}$ converges to X in L_p norm

OR,

$\{X_n\}$ converges in p^{th} moment

[for $p = 1$, $\{X_n\}$ converges in mean]

if $X_n \in L_p$, $n \geq 1$.

& $E|X_n - X|^p \longrightarrow 0$.

Observe:

* $X_n, n \geq 1$ in L_p and $X_n \xrightarrow{L_p} X \Rightarrow X \in L_p$. (Proof: Exercise)

* $X_n \xrightarrow{L_p} X \Rightarrow cX_n \xrightarrow{L_p} cX$.

* $X_n \xrightarrow{L_p} X,\ Y_n \xrightarrow{L_p} Y \Rightarrow X_n + Y_n \xrightarrow{L_p} X + Y$

* $X_n \xrightarrow{L_p} X \Rightarrow X_n \xrightarrow{p} X$

[Proof by chebyshev's inequality : $\forall Z$ - r.v. $P(|Z|>\lambda) \leq \frac{E(|Z|)}{\lambda}$

$P(|X_n - X| > \varepsilon) \leq \frac{E|X_n - X|^p}{\varepsilon^p} \to 0$]

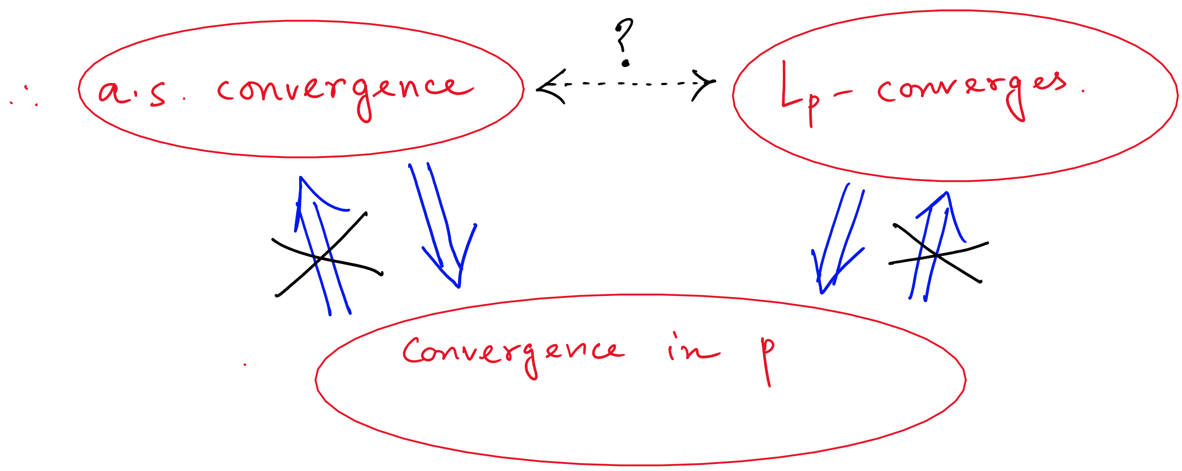
$\therefore X_n \xrightarrow{p} X$.

* $X_n \xrightarrow{L_p} X \Rightarrow X_n \xrightarrow{L_r} X \quad \forall\ r \leq p$.

[Consequence of Holder's Inequality.

We had proved: rth moment $\leq$ pth moment $\forall\ r \leq p$.]

What we did so far.....



We already know,

L_p convergence $\not\Rightarrow$ a.s. convergence

Recall that example of convergence in p $\not\Rightarrow$ a.s convergence.

Also, Eg...

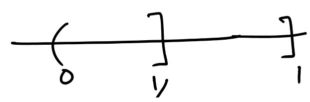
$\left((0,1], \mathcal{B}(0,1], P = leb\right)$

$$X_n = n^{\alpha} \cdot 1_{(0, 1/n]} \xrightarrow{a.s.} X \equiv 0.$$

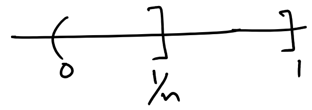
take $\omega \in (0,1]$.

$\Rightarrow \exists\ n_0$ s.t $1/n < \omega,\ \forall\ n \geq n_0$

$X_n = 0$.



$$X_n = 0 .$$



$\therefore \forall \omega, \exists$ such n_0 exists.

$$\therefore X_n \xrightarrow{a.s} 0$$

BUT: $E\,|X_n|^p = n^{\alpha p} \cdot P(X \in (0, 1/n])$

$= n^{\alpha p} \cdot \frac{1}{n} . \not\longrightarrow 0$ for certain values of α.

Precisely, $\alpha p > 1$.

$\Rightarrow \boxed{\alpha > 1/p}$

$\therefore$ a.s convergence $\not\Longrightarrow$ L.p convergence

L.p convergence $\not\Longrightarrow$ a.s convergence.

Now, we only consider $\boxed{p \geq 1}$.

We define $\|X\|_p = (E|X|^p)^{1/p}$

Minkowski $\Rightarrow \|\cdot\|_p$ is a norm

(Modulo identification)

(i.e., 2 r.vs are "same" if they are equal a.s., i.e., equal except over a measure zero set.)

Result: $\|\cdot\|_p$ is complete.

Proof: let $X_n, n \geq 1$ be a Cauchy sequence in L_p.

We can get a subsequence: $1 \leq n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$

such that, for each $k \geq 1$,

$$\|X_m - X_n\|_p < 3^{-k} \quad \forall m, n \geq n_k .$$

Now, we're going to show, this subsequence converges a.s.

$\therefore \forall k \geq 1,$

$$P(|X_{n_{k+1}} - X_{n_k}| > 2^{-k}) \underset{\text{(Chebyshev)}}{\leq} \frac{E\,|X_{n_{k+1}} - X_{n_k}|^p}{2^{-k \cdot p}}$$

$$= \frac{(\|X_{n_{k+1}} - X_{n_k}\|_p)^p}{2^{-kp}}$$

$$< \frac{3^{-kp}}{\cdots} \quad (\cdots^{p})^{k}$$

$$\leq \frac{3^{-kp}}{2^{-kp}} = \left(\frac{2^p}{3^p}\right)^k < \infty$$

2^{-kp}

$\therefore$ This is a convergent series.

$$\Rightarrow \sum_k P(|X_{n_{k+1}} - X_{n_k}| > 2^{-k}) < \infty.$$

$\therefore$ By Borel-Cantelli Lemma,

$$P\left(|X_{n_{k+1}} - X_{n_k}| > 2^{-k} \text{ for infinitely many } k\right) = 0$$

converse

$$\Longrightarrow P\left(|X_{n_{k+1}} - X_{n_k}| \leq 2^{-k} \text{ for all large } k\right) = 1$$

(will hold for finitely many k's. So, won't hold for large k's.)

$$\Rightarrow X = \lim X_{n_k} \text{ exists } P\text{- a.s.}$$

Now, take $X = \limsup X_{n_k}$

$$X_{n_k} \xrightarrow{a.s.} X.$$

[Every Cauchy seq. in L_p has a subsequence which convergences a.s.]

Now, we're left to show, this "X" is the limit of the entire sequence.

ie, we'll show, $X_{n_k} \xrightarrow{L_p} X$. Then,

$\left\{\begin{array}{l}\text{Sequence - Cauchy} \\ \text{Subsequence - Converges to } X\end{array}\right.$

$\Rightarrow$ Entire Sequence converges to X.

$$E(|X|^p) = E\left(\liminf |X_{n_k}|^p\right)$$

$\because$ a.s convergence holds

$\therefore$ $\limsup\limits_{X_{n_k}} = \liminf\limits_{X_{n_k}} = \text{limit } X.$

[Fatou's lemma] $\leq \liminf \left(E|X_{n_k}|^p\right) < \infty$

$\left[\begin{array}{l}\text{Fatou's}\\ \text{lemma}\end{array}\right] \leq \liminf \left(E\, |X_{n_k}| \right) < \infty$

$\Rightarrow \sup_{n} \|X_n\|^p < \infty$ $\left[\because X_n \text{ is Cauchy in } L_p, \therefore X_n \text{ is bounded in } L_p.\right]$

$\therefore X \in L_p$. ✓

Now, fix k.

$\forall\, j > k$, $E\,|X_{n_j} - X_{n_k}|^p \leq 3^{-kp}$

$n_j > n_k$

$\because X_{n_j} \xrightarrow{a.s} X$.

Let $j \nearrow \infty$. (remember, k is fixed)

$\therefore$ Again, by Fatou's lemma,

$$E\,|X - X_{n_k}|^p \leq 3^{-kp}.$$

Now, let $k \nearrow \infty$.

$$\therefore E\,|X - X_{n_k}|^p \longrightarrow 0.$$

Ques: trying to build something "extra" to build the bridge betn a.s convergence & L_p convergence.

Also, conv. in p + "Extra" $\equiv$ a.s convergence.

Some rephrasing: X - r.v

"X is integrable" $\equiv E|X| < \infty$. (finite expectation)

↳ just a name!!!

Exercise:

$\Longleftrightarrow E\left(|X| \cdot \mathbb{1}_{|X|>\lambda}\right) \longrightarrow 0$ as $\lambda \longrightarrow \infty$. (Sort of "tail integral" $\to 0$)

$\Rightarrow$ DCT

$\Leftarrow$ (trivial).

Definition:

A sequence of r.vs $\{X_n\}$ is said to be uniformly integrable (u.i.) if

$$E\left(|X_n| \cdot \mathbb{1}_{|X_n|>\lambda}\right) \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty$$

$$E\left(|X_n|\cdot \mathbb{1}_{|X_n|>\lambda}\right) \longrightarrow 0 \quad \text{as } \lambda \longrightarrow \infty$$

UNIFORMLY in n.

(ie, for a given ε, one single λ works $\forall n$) $\to \sup_n$.

Equivalently,

$$\forall \varepsilon>0, \exists \lambda>0 \text{ s.t } \sup_n E\left(|X_n|\cdot \mathbb{1}_{|X_n|>\lambda}\right)<\varepsilon$$

Simple observations:

1. if X_i - "integrable" $\forall i=1,\dots,k$ finite collection of r.vs, then $\{X_i\}$ is uniformly integrable.

   [for X_i, given ε, get $\lambda_i>0$ then choose $\lambda=\max_i\{\lambda_i\}$]

2. If $\{X_n\}$ is u.i, then $\{X_n\}$ is bounded in L_1.

   ie, $\exists c>0$ s.t $E(|X_n|)<c \quad \forall n\in\mathbb{N}$.

   Exercise: show that, the converse is NOT true!!! (try to get a counter example. Hint: a simple r.v. might as well work!!!)

3. If $\exists Y$ with $E(|Y|)<\infty$, s.t. $|X_n|\leq|Y| \quad \forall n$.

   then $\{X_i\}$ is uniformly integrable.

   Converse is again NOT true!!!

   ie, for $\{X_i\}$ u.i, there might not exist such a Y.

4. If $\sup_n E|X_n|^p<\infty$ for any $p>1$.

   then $\{X_n\}$ is uniformly integrable.

   AGAIN!! Converse is NOT true.

Note: $E|X| < \infty \Longleftrightarrow E(|X| \cdot \mathbb{1}_{|X| > \lambda}) \longrightarrow 0$ as $\lambda \longrightarrow \infty$. proved last sem

$\Updownarrow$

$E(|X| \cdot \mathbb{1}_A) \longrightarrow 0$ as $P(A) \longrightarrow 0$

(Proof needed)

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall A$,
$P(A) < \delta \Rightarrow E(|X| \cdot \mathbb{1}_A) < \varepsilon$.

5 Theorem:

$\{X_n\}$ - uniformly integrable.

$\Longleftrightarrow \sup_n E(|X_n| \cdot \mathbb{1}_A) \longrightarrow 0$ as $P(A) \longrightarrow 0$.

ie, given $\varepsilon > 0$, $\exists \delta > 0$ s.t.
$E(|X_n| \cdot \mathbb{1}_A) < \varepsilon \quad \forall n$ whenever $P(A) < \delta$.