Probability-3 Lecture-19

18 October 2024 11:14

(Recall):

Definition:

for a real r.v, its characteristic f^n φ_X is defined as

$$\varphi_X(t) = E(e^{itX}), \quad t \in \mathbb{R}$$
$$= E(\cos tX + i \sin tX)$$
$$= E(\cos tX) + i\, E(\sin tX)$$

$$\varphi_X : \mathbb{R} \to \mathbb{C}$$

We showed, $X \overset{d}{=} Y \iff \varphi_X \equiv \varphi_Y$.

Properties:

1. $\varphi_X(0) = 1$
2. $|\varphi_X(t)| \leq 1$ $\left[\because |EZ| \leq E|Z|\right]$
3. $\varphi_{a+bX}(t) = e^{iat} \cdot \varphi_X(bt)$.
4. $\varphi_{-X}(t) = \varphi_X(-t) = \overline{\varphi_X(t)}$
5. If X has a symmetric distribution,
   $X \overset{d}{=} -X$.
   then, $\varphi_{-X}(t) = \varphi_X(t)$
   $\Rightarrow \overline{\varphi_X(t)} = \varphi_X(t)$
   $\Rightarrow \varphi_X(t)$ is real-valued.
6. $\varphi_X(t)$ is uniformly continuous.

Proof: $|\varphi_X(t+y) - \varphi_X(t)| \leq E|e^{itX} - 1| \xrightarrow[DCT]{} 0$

(6) $\varphi_X(t)$ is uniformly continuous.

Proof: (detailed proof in last lecture) $\quad |\varphi_X(t+y) - \varphi_X(t)| \leq E|e^{itX} - 1| \xrightarrow[\text{DCT}]{} 0$

Some examples:

Eg 1. $X \sim \text{Bin}(n, p)$

$$\varphi_X(t) = (pe^{it} + q)^n, \quad q = 1-p$$

Eg. 2. $X \sim \text{Poi}(\lambda)$

$$\varphi_X(t) = e^{-\lambda(1-e^{it})}$$

Eg. 3. $X \sim \text{Exp}(\lambda)$.

$$\varphi_X(t) = \frac{\lambda}{\lambda - it}$$

Eg. 4. $Z \sim N(0,1)$

$$\varphi_Z(t) = e^{-t^2/2}$$

if $X \sim N(\mu, \sigma^2)$, $\quad \varphi_X(t) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$

In particular, if $Y \sim N(0, \sigma^2)$

$$\varphi_Y(t) = e^{-\frac{1}{2}\sigma^2 t^2}$$

Recall (a result from last lecture)

for independent random variables X, Y, we use the notation $X * Y$ for the random variable $(X+Y)$.

If Y has density function f, then $X * Y$ has density $E(f(v - X))$.

Suppose,

X has characteristic f^n $\varphi_X(t)$.

... with the integral

X has characteristic f^n $\varphi_X(t)$.

We start with the integral

$$\frac{1}{\sqrt{2\pi n}} \cdot \int_{\mathbb{R}} \varphi_X(v) \cdot e^{-itv} \cdot e^{-v^2/2n} \cdot dv$$

$$= \frac{1}{\sqrt{2\pi n}} \int_{\mathbb{R}} E\left(e^{ivx}\right) \cdot e^{-itv} \cdot e^{-v^2/2n} \, dv$$

$$= E\left(\frac{1}{\sqrt{2\pi n}} \cdot \int_{\mathbb{R}} e^{-iv(t-x)} \cdot e^{-v^2/2n} \, dv\right)$$

Note that: $\frac{1}{\sqrt{2\pi n}} \cdot e^{-v^2/2n} \rightarrow$ Density of $N(0,n)$

$\therefore$ this is nothing but, the characteristic f^n of $N(0,n)$ evaluated at $-(t-x)$.

$$= E\left(e^{-\frac{1}{2}(t-x)^2 \cdot n}\right)$$

$\therefore$ Multiply both sides by $\frac{\sqrt{n}}{\sqrt{2\pi}}$,

$$\frac{1}{2\pi} \cdot \int_{\mathbb{R}} \varphi_X(v) \cdot e^{-itv} \cdot e^{-v^2/2n} \, du = \frac{\sqrt{n}}{\sqrt{2\pi}} \cdot E\left(e^{-\frac{1}{2} \cdot (t-x)^2 \cdot n}\right)$$

$$= E\Bigg(\underbrace{\frac{\sqrt{n}}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} \cdot (t-x)^2 / (1/\sqrt{n})^2}}_{f(t-x)}\Bigg)$$

$f(t-x)$, where

$f(t-x)$, where f is the density of $N(0, 1/n)$

$$= E(f(t-X))$$

$\hookrightarrow$ density of $X_n = X * N(0,\frac{1}{n})$

All of these imply,

if $\varphi_X = \varphi_Y$, then

$$\begin{array}{ccc} X * N(0,\frac{1}{n}) & = & Y * N(0,\frac{1}{n}) \\ \downarrow d & \text{(by Slutsky)} & \downarrow d \\ X & & Y \end{array}$$

$\therefore X \overset{d}{=} Y$. ✓

ie, that's Feller's way of showing, that characteristic function uniquely determines a distribution.

Eg. C.F of Binomial (n,p)

$X \sim Bin(n,p)$. $\qquad q = 1-p$

$$\varphi_X(t) = (pe^{it}+q)^n$$

$$\varphi_X(0) = 1.$$

$$\varphi_X(2\pi) = (pe^{2i\pi}+q)^n$$
$$= (p+q)^n = 1.$$

infact, $\varphi_X(t) = 1$ for any $t = 2n\pi$, $n \in \mathbb{Z}$.

Eg: $X \sim \mathrm{Poi}(\lambda)$

$$\varphi_X(t) = e^{-\lambda(1-e^{it})}$$

Again, $t = 2n\pi$, $n \in \mathbb{Z}$,

$$\varphi_X(t) = \varphi_X(2n\pi) = e^{-\lambda(1-e^{i2n\pi})}$$
$$= e^{-\lambda(1-1)} = e^0 = 1.$$

(Again!!)

Now,

take $X \sim \mathrm{Exp}(\lambda)$.

$$\varphi_X(t) = \frac{\lambda}{\lambda - it}$$

$\varphi_X(t) = 1$ only at $t = 0$,

& $\forall\, t \neq 0$, $|\varphi_X(t)| < 1$

– same conclusion in the case that $X \sim N(\mu, \sigma^2)$

Eg. take this discrete case:

$X = 0, \pm\sqrt{2}, \pm 1$; each with $p = 1/5$.

$$\therefore\ \varphi_X(t) = \frac{1}{5}\left(1 + 2\cos\sqrt{2}\,t + 2\cos t\right).$$

check that, Here too,

$\varphi_X(t) = 1$ only if $t = 0$,

& $\forall\, t \neq 0$, $|\varphi_X(t)| < 1$.

* Such distributions, like $\mathrm{Bin}(n,p)$, or $\mathrm{Exp}(\lambda)$ are called Lattice distributions, or Arithmetic distributions (as used by Feller)

Def^n: A real r.v X is said to be **Lattice distribution**, or **Arithmetic distribution** if:

X is discrete, and its countable support is contained in

$$S = \{ a + jd : j \in \mathbb{Z} \},$$

for some real a, d.

* **Result:**

Let X be a real r.v. with characteristic f^n - φ_X, then $\exists\, t_0 \neq 0$, s.t. $|\varphi_X(t_0)| = 1$ **iff**.

X has a Lattice distribution with its support contained in

$$S = \left\{ a_0 + j \cdot \frac{2\pi}{t_0} \;\middle|\; j \in \mathbb{Z} \right\}$$

further, in this case -

$|\varphi_X(t)|$ is periodic with period $= t_0$.

Proof: "if" part:

Let support of X be contained in S.

denote $p_j = P\left(X = a_0 + j \cdot \frac{2\pi}{t_0}\right)$

$$\varphi_X(t_0) = E\left(e^{it_0 X}\right)$$
$$= \sum_j e^{it_0\left(a_0 + j \cdot 2\pi / t_0\right)} \cdot p_j$$

corresponding probability

[we can have $p_j = 0$ for many j's. Eg. for Bin (n, p), $p_j = 0 \;\forall j > n$]

$$= e^{ia_0 t_0} \sum_j \underbrace{e^{ij \cdot 2\pi}}_{\substack{\shortparallel \\ 1}} \cdot p_j$$

$$= e^{ia_0 t_0} \qquad \left[\because \sum p_j = 1\right]$$

$$= e^{ia_0 t_0} \cdot 1. \qquad \left[\because \sum p_j = 1\right]$$

$$\therefore \left|\varphi_X(t_0)\right| = 1 \qquad \blacksquare$$

"only if" part:

WLOG, take $t_0 > 0$. $\left[\because \text{ if } t_0 < 0, \; -t_0 > 0, \; \& \; |\varphi_X(t_0)| = 1 \iff \left|\overline{\varphi_X(t_0)}\right| = 1\right]$

why can we take ?

$$\left|\varphi_X(t_0)\right| = 1$$

$$\Rightarrow \varphi_X(t_0) = e^{i\theta_0}$$

$$\Rightarrow E\left(e^{it_0X}\right) = e^{i\theta_0}$$

$$\Rightarrow E\left(e^{it_0X - i\theta_0}\right) = 1$$

$$\Rightarrow E\left(\underbrace{e^{it_0(X - \theta_0/t_0)}}_{\text{complex r.v}}\right) = \underset{\substack{\downarrow \\ \text{Real expected value}}}{1.}$$

$$\therefore E(\text{Real part}) = 1, \qquad E\left(\text{imaginary part}\right) = 0$$

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$$\Rightarrow E\left(\cos t_0\left(X - \frac{\theta_0}{t_0}\right)\right) = 1.$$

ie, X must take values - multiples of 2π with prob $= 1$.

$$\Rightarrow P\left(X \in \left\{\frac{\theta}{t_0} + \frac{2\pi j}{t_0} : j \in \mathbb{Z}\right\}\right) = 1.$$

$$\blacksquare$$

Now, take $\varphi_X(t+t_0)$

$$= E\left(e^{i(t+t_0)X}\right)$$

$$= \sum_j e^{i(t+t_0)\cdot\left(\frac{\theta_0}{t_0}+\frac{2\pi j}{t_0}\right)} \cdot p_j$$

$$= \sum_j e^{it\left(\frac{\theta_0}{t_0}+\frac{2\pi j}{t_0}\right)} \cdot \underbrace{e^{it_0\left(\frac{\theta_0}{t_0}+\frac{2\pi j}{t_0}\right)}}_{e^{i\theta_0}\cdot \underbrace{e^{ij2\pi}}_{=1}} \cdot p_j$$

$$= e^{i\theta_0}\cdot \sum_j e^{it\left(\theta_0/t_0 + 2\pi j/t_0\right)} \cdot p_j$$

$$= \underbrace{e^{i\theta_0}}_{1\cdot 1=1} \cdot \varphi_X(t). \qquad \therefore \left|\varphi_X(t+t_0)\right| = \left|\varphi_X(t)\right|.$$

i.e., for every t_0, $|\varphi_X(t)|$ is periodic with period $= t_0$. $\boxtimes$

Theorem:

Let φ be a characteristic f^n.

Then,

either: (i) $|\varphi(t)| < 1 \quad \forall\, t \neq 0$.

or (ii) $|\varphi(t)| \equiv 1$.

or (iii) $|\varphi(t)| = 1$ for a countable set of isolated points.

Proof: We'll simply prove, if there is a

Proof: We'll simply prove, if there is a sequence $\{t_n\}$ of distinct reals, with $t_n \longrightarrow s$ real..

s.t, $|\varphi(t_n)| = 1 \ \forall n$,

then $|\varphi(t)| \equiv 1$.

[Proving this works. Think why !!]

if such t's are uncountable, then $\exists$ a compact set in which there are infinitely many such t's, which imply $|\varphi(t)| \equiv 1$

[$\because$ By Bolzano Weierstrass, $\exists$ a seq that converges.]

else, if t's are countable. there are no limit pts. Hence, the t's are isolated.

Observe: for any integer k, and any $m \neq n$,

$$\left| \varphi\left(k(t_n - t_m)\right) \right| = \left| \varphi(k t_m) \right| = 1.$$

Take any integer in (a, b).

$\exists\, n \neq m$ s.t., $0 < (t_n - t_m) < b - a$

$\Rightarrow$ k - integer s.t. $k(t_n - t_m) \in (a, b)$

$\therefore$ Every interval contains a t s.t $|\varphi(t)| = 1$

So, the set of such t's is dense.

But, φ is continuous.

But, φ is continuous,
So, that set is closed.
$\therefore$ that set of t's, where $|\varphi(t)|=1$ has to be the entire real line $\mathbb{R}$.

$\blacksquare$

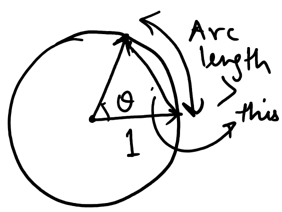
[Exercise: if $t_1, t_2 \in \mathbb{R}$ s.t $|\varphi(t_1)| = 1 = |\varphi(t_2)|$, (WLOG, $t_1 > t_2$) &, $t_1/t_2 \in \mathbb{Q}^c$ (irrational), then show that: $|\varphi(t)| \equiv 1$.]

Inversion formula.

If φ_X is a characteristic function of X, then for any two pts. $a < b$ in the "continuity of X",

$$F_X(b) - F_X(a) = \frac{1}{2\pi} \cdot \lim_{T\to\infty} \underbrace{\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \cdot \varphi_X(t)\, dt}_{\text{is it integrable? YES.}}$$

$\because |e^{i\theta} - 1| \le |\theta|$



$\therefore |e^{-ita} - e^{-itb}|$

$$\therefore \left| \frac{e^{-ita} - e^{-itb}}{it} \right| \leq |b-a|.$$

$$\left| \frac{e^{-it(a-b)} - 1}{it} \right|$$

$\therefore$ it is integrable over any compact set.

Suppose, the characteristic function φ_X of a r.v. X is integrable over $\mathbb{R}$, ie,

$$\int_{\mathbb{R}} |\varphi_X(t)|\, dt < \infty.$$

In this case, we have, for continuity pts $a < b$.

$$F_X(b) - F_X(a) = \frac{1}{2\pi} \cdot \int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \cdot \varphi_X(t)\, dt$$

* verify: $\displaystyle\int_a^b e^{-itu}\, du = \frac{-\left(e^{-itb} - e^{-ita}\right)}{it}.$

[$\because$ "pretend" that, this is real.]

$$F_X(b) - F_X(a) = \frac{1}{2\pi} \cdot \int_{\mathbb{R}} \underbrace{\frac{e^{-ita} - e^{-itb}}{it}}_{\shortparallel} \cdot \varphi_X(t)\, dt$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_a^b e^{-itu}\, du \right) \cdot \varphi_X(t)\, dt$$

$$= \int_a^b \left(\frac{1}{2\pi} \int e^{-itu} \cdot \varphi_X(t)\, dt \right) du$$

→ "Inverse Fourier

$$F_X(b) - F_X(a) = \int_a^b \underbrace{\left(\frac{1}{2\pi} \cdot \int_{\mathbb{R}} e^{-itv} \cdot \varphi_X(t)\, dt\right)}_{} dv$$

→ "Inverse Fourier Transform".

→ it "looks like", this is some density!!

Denote $f(v) = \frac{1}{2\pi} \cdot \int_{\mathbb{R}} e^{-itv} \cdot \varphi_X(t)\, dt \quad , v \in \mathbb{R}.$

↓ this is integrable because,

$$\left| e^{-itv} \cdot \varphi_X(t) \right| \leq \left| \varphi_X(t) \right| < \infty$$

$[\because \varphi_X$ is integrable.$]$

* firstly, f is continuous. (Proof: take seq. $v_n \to v$.
$\Rightarrow e^{-itv_n} \cdot \varphi_X(t) \to e^{-itv} \cdot \varphi_X(t)$
$\therefore$ By DCT, integral converges)

* for every pair $a < b$ of continuity points, ↘ which is dense
& $F_X(b) - F_X(a) = \int_a^b f(v)\, dv$.

↘ Right cont. at every (a, b)

↘ cont. for every (a, b)

& LHS, RHS agree on a dense set,

then, by Lebesgue criterion, this holds.

In particular, F is continuous.
[the word "continuous" has "nb" written below it]

In particular, F is continuous.

Also, $F_X(b) - F_X(a) = \int_a^b f(v)\,dv$ $\forall (a,b)$.

$\therefore$ By Fundamental Theorem of Calculus, F is differentiable, & $F'(v) = f(v)$,

& Hence, we're done. $\boxed{/\!/}$

Exercise: $f_X(x) = \frac{1}{2} e^{-|x|}$. $\leftarrow$ symmetric 2-sided Exponential.

- ✓ compute C.F,
- ✓ observe that C.F is integrable
- Write the inverse fourier transform formula,

[Original density can be written as inverse transform formula of the C.F.]

Something should click!!.

$$\varphi_X(t) = E(e^{itX}) = \int_{\mathbb{R}} e^{itx} \cdot \frac{1}{2} e^{-|x|}\,dx$$

$$= \int_{-\infty}^{0} e^{itx} \cdot \frac{1}{2} \cdot e^{x}\,dx + \int_0^\infty e^{itx} \cdot \frac{1}{2} e^{-x}\,dx$$

$$= \int_{-\infty}^{0} \frac{1}{2} e^{(it+1)x}\,dx + \int_0^\infty \frac{1}{2} e^{(it-1)x}\,dx$$

$$= \frac{1}{2} \cdot \frac{e^{(it+1)x}}{it+1} \Bigg|_{-\infty}^{0} + \frac{1}{2} \cdot \frac{e^{(it-1)x}}{it-1} \Bigg|_{0}^{\infty}$$

$$= \frac{1}{2} \left[\frac{1}{it-1} + \frac{1}{it+1} \right] = \frac{1}{1+t^2}$$ ✓ $\boxed{\therefore \varphi_X(t) = \frac{1}{1+t^2}}$

$$\int_{\mathbb{R}} |\varphi_X(t)| \, dt = \int_{\mathbb{R}} \frac{1}{1+t^2} \cdot dt = \tan^{-1} t \Big|_{-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\pi/2\right) = \pi.$$

$\therefore \varphi_X(t)$ is integrable. ✓.

Now, By Inverse Fourier Transform formula -

$$f(u) = \int_{\mathbb{R}} e^{-itu} \cdot \varphi_X(t) \, dt$$