Probability-3 Lecture-12

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$X_1, X_2, \ldots$ i.i.d. sequence of r.vs, all on same probability space $(\Omega, \mathcal{A}, P)$ with finite common mean μ.

$E|X_1| < \infty$.

$$S_n = \frac{X_1 + \cdots + X_n}{n}$$

Weak Law of Large numbers (WLLN):

$$\frac{S_n}{n} \xrightarrow{P} \mu$$

Strong law of large numbers (SLLN):

$$\frac{S_n}{n} \xrightarrow{a.s} \mu.$$

for WLLN,

we need to prove: $P\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) \longrightarrow 0 \quad \forall \varepsilon > 0$.

$$P\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) = P\left(|S_n - n\mu| > n\varepsilon\right)$$

tail probability of the r.v: $(S_n - n\mu)$.

Assume: 2^{nd} moment finite.

ie, $E|X_1|^2 < \infty$

$$P\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) = P\left(|S_n - n\mu| > n\varepsilon\right) \leq \frac{E|S_n - n\mu|^2}{n^2\varepsilon^2}$$

(Chebyshev)

$$= \frac{Var(S_n)}{n^2\varepsilon^2}$$

pairwise independence

$$= \frac{\cancel{n} \cdot Var(X_1)}{n^{\cancel{2}}\varepsilon^2}$$

$\longrightarrow 0$ as $n \to \infty$

Q: How to prove if only $E|X_1| < \infty$ is given?

i.e, only first moment finite is given.

New idea: "Truncation".

↳ Replace X_n's by Y_n's, where

Y_n's are X_n's "truncated" at an "appropriate threshold".

Here, define $Y_n := X_n \cdot \mathbb{1}_{|X_n| \le n}$.

$Y_1, Y_2, \ldots$ while are still independent

they are no longer identically distributed, as each Y_i has different truncation levels.

Define $T_n := Y_1 + Y_2 + \cdots + Y_n$.

We will show: $\dfrac{T_n - E T_n}{n} \xrightarrow{P} 0$ ——— $(*)$

& this will then imply, ? Why? *

$\dfrac{S_n - n\mu}{n} \xrightarrow{P} 0$

* firstly, $\sum P(X_n \neq Y_n) = \sum P(|X_n| > n)$

$= \sum P(|X_1| > n)$ $\quad (\because X_i$'s - iid$)$

$< \infty$.

because: $E|X_1| < \infty$

$\therefore$ By Borel-Cantelli lemma:

$P(X_n \neq Y_n$ for infinitely$) = 0$

$\therefore$ By Borel-Cantelli lemma:

take complement

$$P\left(X_n \neq Y_n \text{ for infinitely many } n\right) = 0$$

$$\Rightarrow P\left(\{\omega : X_n(\omega) = Y_n(\omega) \text{ for all } n \geq n_0(\omega)\}\right) = 1$$

$$\Rightarrow P\left(\{\omega : \left(\frac{1}{n} S_n - \frac{1}{n} T_n\right) \longrightarrow 0\}\right) = 1.$$

$$\therefore \frac{S_n}{n} - \frac{T_n}{n} \xrightarrow{a.s} 0 \quad \text{———} \quad 1$$

$$\Downarrow$$

Now, $\dfrac{T_n - ET_n}{n} \xrightarrow{P} 0$ & $\dfrac{S_n}{n} - \dfrac{T_n}{n} \xrightarrow{P} 0$

sum up these two

$$\frac{T_n}{n} - \frac{ET_n}{n} + \frac{S_n}{n} - \frac{T_n}{n} \xrightarrow{P} 0 + 0$$

$$\Rightarrow \frac{S_n}{n} - \frac{ET_n}{n} \xrightarrow{P} 0 \quad \text{———} \quad 2$$

$$\therefore E(Y_n) = E\left(X_1 \cdot 1_{|X_1| \leq n}\right) \xrightarrow[\text{DCT}]{} \mu$$

$$\frac{E(T_n)}{n} = \frac{E(Y_1) + \cdots + E(Y_n)}{n} \longrightarrow \mu.$$

(real seq. $\rightarrow \mu$ $\Rightarrow$ their cesaro mean $\rightarrow \mu$) ——— 3

$*$ & 3 $\Rightarrow$

$$\frac{S_n}{n} - \mu \xrightarrow{P} 0.$$

$\therefore$ Now, we have to show: $\dfrac{T_n - ET_n}{n} \xrightarrow{P} 0$.

Fix $\varepsilon > 0$.

$$P\left(\left|\frac{T_n - ET_n}{n}\right| > \varepsilon\right) = P\left(|T_n - ET_n| > n\varepsilon\right)$$

$$\leq \frac{Var(T_n)}{n^2\varepsilon^2}$$

pairwise independent

$$= \frac{1}{n^2\varepsilon^2} \cdot \sum_{k=1}^{n} V(Y_k)$$

$$\leq \frac{1}{n^2\varepsilon^2} \cdot \sum_{k=1}^{n} E|X_1|^2 \cdot \mathbb{1}_{|X_1| \leq k}$$

we're stuck -.

(New trick:
fix any non-ve seq. $\{a_n\} \nearrow \infty$, but $\dfrac{a_n}{n} \to 0$,
e.g. $a_n = \sqrt{n}$.)

$$\leq \frac{1}{n^2\varepsilon^2} \cdot \left(\sum_{k=1}^{n} E\left(|X_1|^2 \cdot \mathbb{1}_{|X_1| \leq a_k}\right) + \sum_{k=1}^{n} E\left(|X_1|^2 \cdot \mathbb{1}_{a_k \leq |X_1| \leq n}\right)\right)$$

2nd term

1st term:

$$\leq \sum_{k=1}^{n} a_k \cdot E\left(|X_1| \cdot \mathbb{1}_{|X_1| \leq a_k}\right)$$

$$\leq \sum_{k=1}^{n} a_k \cdot E(|X_1|) =$$

$$\therefore \frac{1}{n^2\varepsilon^2} \cdot (1^{st} \text{ term}) = \frac{1}{n^2\varepsilon^2} \cdot \sum_{k=1}^{n} a_n \cdot E(|X_1|)$$

$$\therefore \frac{1}{n^2\varepsilon^2}\cdot\left(1^{st}\text{ term}\right) = \frac{1}{n^2\varepsilon^2}\cdot\sum_{k=1} a_n \cdot E\left(|X_1|\right)$$

$$= \frac{\cancel{n} a_n \cdot E|X_1|}{n^{\cancel{2}}\varepsilon^2} \longrightarrow 0$$

&, 2^{nd} term,

$$\sum_{k=1}^{n} E\left(X_1^2 \cdot \mathbb{1}_{a_n < |X_1| \leq n}\right)$$

$$\leq \sum_{k=1}^{n} n\, E\left(|X_1| \cdot \mathbb{1}_{a_n < |X_1| \leq n}\right)$$

$$\leq \sum_{k=1}^{n} n \cdot E\left(|X_1| \cdot \mathbb{1}_{|X_1| > a_n}\right)$$

$$\therefore \frac{1}{n^2\varepsilon^2}\left(2^{nd}\text{ term}\right) = \frac{1}{n^{\cancel{2}}\varepsilon^2}\cdot \cancel{n}\cdot\sum E\left(|X_1|\cdot\mathbb{1}_{|X_1|>a_n}\right)$$

$$= \frac{1}{\cancel{n}\varepsilon^2}\times\cancel{n}\cdot E\left(|X_1|\cdot\mathbb{1}_{|X_1|>a_n}\right)$$

$$= \frac{E\left(|X_1|\cdot\mathbb{1}_{|X_1|>a_n}\right)}{\varepsilon^2} \xrightarrow{(D.C.T.)^*} 0$$

$\because X_1$ — finite 1^{st} moment $\therefore E|X_1| \leq M < \infty$

Hence, we are done. ▨

Claim: for WLLN, we only need to show,

$$P\left(\left|\frac{S_n - ES_n}{n}\right| > \varepsilon\right) \longrightarrow 0$$

for SLLN, we need to show

$$\sum P\left(\left|\frac{S_n - ES_n}{n}\right| > \varepsilon\right) < \infty.$$

$$\sum_n P\left(\left|\frac{S_n - ES_n}{n}\right| > \varepsilon\right) < \infty .$$

Borel's Strong Law of Large Numbers:

(supposedly) Proved SLLN under stronger assumption that,

$$E|X_1|^4 < \infty \quad \left(4^{th} \text{ moments finite}\right)$$

essentially centering the X_is.

Denote $Y_n := X_n - EX_n = X_n - \mu$

$T_n = Y_1 + \cdots + Y_n$.

We have to now show: $\frac{T_n}{n} \xrightarrow{a.s} 0$.

ie, $\sum_n P\left(\left|\frac{T_n}{n}\right| > \varepsilon\right) < \infty$.

$$P\left(\left|\frac{T_n}{n}\right| > \varepsilon\right) \leq \frac{E(T_n^4)}{n^4 \varepsilon^4} \quad [\text{Chebyshev.}]$$

$$E(T_n^4) = E\left(\left(\sum_{i=1}^n Y_i\right)^4\right)$$

Y_i's . independent

$4 . \sum_{i \neq j} E(Y_i^3 . Y_j) = 4 . \sum E(Y_i^3) . E(Y_j) \to 0$

Now, $E\left(\sum_{i=1}^4 Y_i\right)^4 = E\left(\sum_{i=1}^n Y_i^4\right) + 4 . E\left(\sum_{i \neq j}^n Y_i^3 . Y_j\right) \to 0 +$

$4\left(\quad\right) \to 0 + \left(\quad\right) \to 0 +$ (similarly)

$6 . E\left(\sum_{i \neq j} Y_i^2 . Y_j^2\right)$

$$= n\, E(Y_1^4) + 6 . n(n-1) . E(Y_1^2 . Y_2^2)$$

$$= n\,E(Y_1^4) + 6 \cdot n(n-1) \cdot E(Y_1^2 \cdot Y_2^2)$$

$$\therefore P\left(\left|\frac{T_n}{n}\right| > \varepsilon\right) \le \frac{E(T_n^4)}{n^4 \varepsilon^4} = \frac{n E(Y_1^4) + 6 n(n-1) E(Y_1^2 Y_2^2)}{n^4 \varepsilon^2}$$

of the order $\frac{1}{n^2}$

$$\therefore \Sigma(\cdot) < \infty \checkmark$$

$$\therefore \frac{S_n}{n} \xrightarrow{a.s} E|X_1|.$$

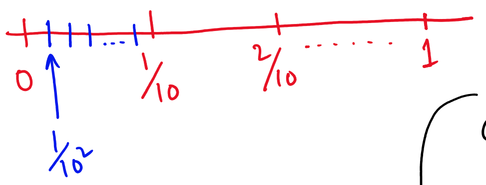
$\Omega = (0,1]$, $\mathcal{A}$ = Borel σ-field

P = leb. measure

$$X(\omega) = \omega$$

$$X_1(\omega) = \begin{cases} 0, & \text{if } \omega \in (0, 1/10] \\ 1, & \text{if } \omega \in (1/10, 2/10] \\ \vdots & \\ 9, & \text{if } \omega \in (9/10, 1] \end{cases}$$

$$|\omega - X_1(\omega)| \le \frac{1}{10}$$



$$X_2(\omega) = \begin{cases} 0, & \text{if } \omega - \frac{X_1(\omega)}{10} \in \left(0, \frac{1}{10^2}\right] \\ 1, & \text{if } \omega - \frac{X(\omega)}{10} \in \left(\frac{1}{10^2}, \frac{2}{10^2}\right] \\ \vdots & \\ 9, & \text{if } \omega - \frac{X(\omega)}{10} \in \left(\frac{9}{10^2}, 1/10\right] \end{cases}$$

This is basically the non-terminating

This is basically the non-terminating decimal expansion of ω

ie, $\omega = 0 . X_1(\omega) X_2(\omega) \cdots$

decimal

1st place after decimal

2nd place after decimal.

Note that, X_i's are identically distributed.

Q. are they independent?

$P(X_1 = 2, X_2 = 5, X_3 = 1)$

$= P\left(\omega : \omega \in \left(\frac{2}{10} + \frac{5}{10^2} + \frac{1}{10^3}, \frac{2}{10} + \frac{5}{10^2} + \frac{2}{10^3}\right)\right)$

$= \frac{1}{10^3} = P(X_1 = 2) \cdot P(X_2 = 5) \cdot P(X_3 = 1)$

Yes ✓.

$\therefore$ We have i.i.d seq of r.vs, bounded

ie, all moments finite.

Common mean $= \frac{0+1+2+\cdots+9}{10}$

$= 9/2$

$\therefore$ By Borel SLLN,

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s} 9/2$$

Now, say, $Z_n := 1_{X_n = 3}$.

Q. are Z_n's iid = ? Yes!!

Yes!!

$\because P(Z_n = 1)$
$= P(X_n = 3)$,
& X_n's are i.i.d.

$$E(Z_n) = 1 \cdot \underbrace{\frac{1}{10}}_{P(X_n=3)} + 0 + 0 + \cdots + 0$$

this is actually the proportion of 3's in the first n decimal places

$$\left(\frac{Z_1 + Z_2 + \cdots + Z_n}{n}\right) \xrightarrow{a.s} \frac{1}{10}.$$

$$P\left(w : \text{For any } i \in \{0, 1, \ldots, 10\}. \text{ proportion of } \underline{i} \text{ in decimal expansion of } w \longrightarrow \frac{1}{10}\right) = 1$$

* Borel's Normal Number Theorem:

↳ "almost every no. is normal".

↓ a no. for which, for every k, the k-adic expansion of the that no., proportion of every no. in that expansion $\rightarrow \frac{1}{k}$