Course : Linear Statistical Models Oct 05, 2022

Video 90 : LRT IN GENERAL

Professor: *Arnab Chakraborty* *Semanti Dutta*

Likelihood Ratio tests are a routine way of constructing Hypothesis tests. The basic idea is that we would start with a given hypothesis H_0 and an alternating hypothesis H_1. What we usually do is that we consider the likelihood of the given data under both H_0 and H_1 and reject H_0 if L_{H_1} far exceeds from L_{H_0}. Here, L_{H_1} refers to the chance of observing our data if H_1 is true. Similarly, L_{H_0} refers to the chance of observing the data if H_0 is true. But since our data may be continuous, we cannot take probability, so instead, we take likelihood. Thus, the intuitive idea is :

$$Reject \ \ H_0 \ \ if \ \ \frac{L_{H_1}}{L_{H_0}} \ \ is \ \ large.$$

Now, if both H_0 and H_1 are simple, in that case, Neyman Pearson Lemma guarantees that this is the unique most powerful test. However, if one of these hypothesis is composite, i.e, if H_0 and/or H_1 is composite, then the *likelihood under* H_0 *or* H_1 does not make sense because if H_0 is composite then it can take a lot of different possible values and thus the parameter is not completely specified under H_0. In this case, we cannot use $\frac{L_{H_1}}{L_{H_0}}$, so we work with the following :

$$Reject \ \ H_0 \ \ if \ \ \frac{\sup_{H_1} \ L}{\sup_{H_0} \ L} \ \ is \ \ large.$$

To understand why we are working with supremum, let us take the following example. Suppose we want to know who runs faster among two people. We would ask them to run in a race and whoever wins is our desired candidate. Now if we want to know whether the state Bengal has faster runners or the state Bihar, we could do the following. We could observe who runs the fastest in Bengal and similarly who runs fastest in Bihar. Now among these two champions, if the faster runner among these two runners is from Bengal, we could claim that Bengal runs faster than Bihar, and vice versa.

Following this idea, in our model, we consider the maximum likelihood we can attain under the parameter space specified by H_0 and the maximum likelihood we can attain under the parameter space specified by H_1. If the latter far exceeds the former, we reject the null hypothesis. Unfortunately, we cannot guarantee that this test is optimal but in most cases it turns out to be optimal and it is a very reasonable test. One advantage of this test is that it is practical and the test *can* be performed to do the maximizations and comparisons.

Now generally this H_0 is specified in a nice way so the parameter space under H_0 is well-specified but the alternating hypothesis H_1 is often not well specified.

For example, let us consider our model:

$$
\begin{aligned}
&\vec{y} = X\vec{\beta} + \vec{\epsilon} \\
&\vec{\epsilon} \sim N(\vec{0}, \sigma^2 I) \\
&\vec{\beta} \in \mathbb{R}^p,\ \sigma^2 > 0
\end{aligned}
$$

And the sub-model (under H_0) :

$$
\begin{aligned}
&\vec{y} = X_0\vec{\beta} + \vec{\epsilon} \\
&\vec{\epsilon} \sim N(\vec{0}, \sigma^2 I) \\
&\vec{\beta} \in \mathbb{R}^q,\ \sigma^2 > 0
\end{aligned}
$$

$$
\text{where } \mathscr{C}(X_0) \subseteq \mathscr{C}(X)
$$

The parameter space under H_0 is the column space of X_0 but H_1 says that our expected value of y will lie in column space of X but not in column space of X_0. So, this is not very well-defined and thus is very difficult to work with. Thus, we take a slightly different but equivalent formulation of likelihood ratio test. We compare the supremum of the likelihood under H_0 with the supremum of the likelihood in general, i.e, we compare the supremum of the likelihood under H_0 with the supremum of the likelihood under $\{H_0 \cup H_1\}$. The latter case is the unrestricted model where we consider the design matrix as X and under null hypothesis we consider the model with design matrix X_0. Thus, our final test is :

$$
Reject\ \ H_0\ \ if\ \ \frac{\sup\ L}{\sup_{H_0}\ L}\ \ is\ \ large.
$$