Probability-3 Lecture-14

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Recall:

<u>Result:</u>

$\{\xi_i\}$ - an independent, mean zero seq. of r.vs.

then, $\sum_n V(\xi_n) < \infty$ is a sufficient condition for random series $\sum_n \xi_n$ to converge a.s.

<u>Corollary:</u>

Say, $\eta_n = \pm 1$, with prob $= 1/2$ each; independent

$\therefore$ By the result above: $\sum_n \eta_n \cdot \frac{1}{n}$ converges a.s.

$\left[\because \{\eta_n\} \text{ - mean } 0 \text{ seq.} \quad E(\eta_n) = 1\cdot\frac{1}{2} + (-1)\cdot\frac{1}{2} = 0 \right]$

[check that: $\sum_n V(\eta_n) < \infty$]

"The Random Signs Problem"

$$\sum_n V(\eta_n) = \sum \left(\pm 1/n\right)^2 = \sum \frac{1}{n^2} < \infty. \checkmark$$

<u>Corollary:</u>

$\{\xi_n\}$ - independent seq.

Then, $\underbrace{\sum E(\xi_n) \text{ converges } \& \sum V(\xi_n) < \infty}$

(Exercise) $\Rightarrow \sum_n \xi_n$ converges a.s.

[Converse <u>not</u> true]

But, partial converse: if $|\xi_n| \leq K$, then the converse holds.

* $\sum_n (\xi - E(\xi))$ converges a.s.

complete this.

Back to SLLN:

$\{X_n\}$ - iid sequence with finite mean μ.

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} \mu.$$

Step 1: Truncation.

for each $n \geq 1$, let $Y_n = X_n \cdot \mathbb{1}_{|X_n| \leq n}$

clearly, Y_n's - independent, but not identically distributed.

$$Y_n \stackrel{d}{=} X_1 \cdot \mathbb{1}_{|X_1| \leq n}.$$

Let $T_n = \sum_{k=1}^{n} (Y_k - E(Y_k))$ ← is, centering the Y_n's & taking their partial sums.

clearly, $\left\{ \underbrace{\frac{Y_n - E(Y_n)}{n}} \right\}$ is an independent, 0-mean sequence.

think of this as ξ_n.

We want to show,

$$\sum_n \frac{Y_n - E(Y_n)}{n} \text{ converges a.s.}$$

If we can do that, then by Kronekar's lemma,

$$\Rightarrow \frac{1}{n} \cdot \sum_{k=1}^{n} (Y_k - E(Y_k)) \xrightarrow{a.s.} 0, \quad \text{—— (1)}$$

& that suffices. ??

& that suffices.
??

&, $\frac{1}{n}\sum_{k=1}^{n}\left(X_k - E(Y_k)\right) \xrightarrow{a.s} 0$ $\overset{??}{\Longrightarrow}$ classical SLLN

Observation 1:

$$\sum_n P(X_n \neq Y_n) = \sum_n P(|X_n| > n)$$

$$= \sum_n P(|X_1| > n) < \infty$$

$[\because E|X_1| < \infty]$

$\Rightarrow$ By Borel-Cantelli Lemma.

$P(X_n \neq Y_n$ for infinitely many n's$) = 0$

$\Rightarrow P(X_n = Y_n$ for all "large" n's$) = 1$.

$\Rightarrow \frac{1}{n}\sum_{k=1}^{n}(Y_k - X_k) \xrightarrow{a.s} 0$ —— 2

[think why? only finitely many terms in numerator are significant, while denominator $\nearrow \infty$]

$\therefore$ 1 & 2 $\Rightarrow$

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - EY_k) \xrightarrow{a.s.} 0 \text{ —— } 3$$

Observation 2:

$$E(Y_n) = E\left(X_1 \cdot 1_{|X_1| \le n}\right)$$

$\downarrow$

$E(X_1)$ as $n \to \infty$ [By DCT]

$= \mu$.

Also, $\frac{1}{n}\sum_{k=1}^{n} E(Y_k) \longrightarrow \mu = E(X_k)$. —— 4

Also, $\frac{1}{n}\sum_{k=1}^{n} E(Y_k) \longrightarrow \mu = E(X_k)$. —— (4)

(3) + (4) $\Rightarrow$

$$\frac{1}{n}\sum_{k=1}^{n}\left(X_k - E(X_k)\right) \xrightarrow{a.s} 0$$

$$\Longleftrightarrow \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s} \mu$$

SLLN ✓.

So, left to show:

$$\frac{1}{n}\sum_{k=1}^{n}\left(Y_k - E(Y_k)\right) \xrightarrow{a.s} 0$$

$$\sum_n V\left(\frac{Y_n - EY_n}{n}\right) \leq \sum_n \frac{1}{n^2}\cdot E(Y_n^2)$$

$$= \sum_n \frac{1}{n^2}\cdot E\left(|X_1|^2\cdot 1_{|X_1|\leq n}\right)$$

$$= \left(\sum_n \frac{1}{n^2}\right)\sum_{j=1}^{n} E\left(|X_1|^2\cdot 1_{j-1<|X_1|\leq j}\right)$$

take inside.

$$= \sum_{j=1}^{\infty}\left(E\left(|X_1|^2\cdot 1_{j-1<|X_1|\leq j}\right)\cdot \underbrace{\sum_{n=j}^{n}\frac{1}{n^2}}_{\text{(check!!)} \ \leq \frac{2}{j}}\right)$$

[$|X_1| = 0$ does not contribute to the expection.]

$$\leq \sum_{j=1}^{\infty}\frac{2}{j}\cdot E\left(|X_1|^2\cdot 1_{j-1<|X_1|\leq j}\right)$$

pull $|X_1|$ out

$$\leq \sum_{j=1}^{\infty}\frac{2}{\cancel{j}}\cdot\cancel{j}\cdot E\left(|X_1|\cdot 1_{j-1<|X_1|\leq j}\right)$$

$$j=1$$

$$= 2 \cdot \sum_{j=1}^{\infty} \underbrace{E\left(|X_1| \cdot 1_{j-1<|X_1|\le j}\right)}_{E|X_1|}$$

$$= 2 \cdot E|X_1| \cdot < \infty \cdot \quad \boxed{}$$

SLLN proved !!!

A Trivial Application: (of SLLN).

F - unknown distribution. (ie, kind of "non-parametric": ie, not just the parameter, but also the distribution is not known.)

$X_1, X_2, \ldots \overset{iid}{\sim} F.$

Fix $x \in \mathbb{R}$.

Goal: to estimate $F(x)$.

Approach: take a sample of size n.

$$X_1, \ldots, X_n.$$

$$\therefore F_n(x) := \frac{1_{X_1\le x} + \cdots + 1_{X_n \le x}}{n}$$

$\nwarrow$ (proportion of X_i's $\le x$)

each of these r.vs,

$1_{X_i \le x}$ - iid, with common mean

$$E\left(1_{X_i\le x}\right) = P(X_i \le x) = F(x).$$

ie, by SLLN,

$= F(x)$.

ie, by SLLN,

$$F_n(x) \xrightarrow{a.s.} F(x)$$ & hence, as well as: $$F_n(x) \xrightarrow{P} F(x)$$

F_n : Empirical distribution f^n based on n observations.

→ discrete distⁿ.

Glivenko - Cantelli Lemma:

$$\sup_x \left| F_n(x) - F(x) \right| \xrightarrow{a.s.} 0 \quad \text{as } n \to \infty.$$

"the Kolmogorov Metric"

ie, ∃ a single P-null set, s.t, ∀ ω ∉ that P-null set, the function $F_n(x) \longrightarrow F(x)$ uniformly.

Recall: Ana-2

Proof:

Trivial consequence of SLLN:

for each $x \in \mathbb{R}$, $F_n(x) \xrightarrow{a.s} F(x)$

Let N_1 be a P-null set s.t. ∀ ω outside N_1.

$$F_n(r)(\omega) \longrightarrow F(r)(\omega) \quad \forall \; r \in \mathbb{Q}.$$

[for each r, we get a P-null set N_r that countable union & define

$$N_1 = \bigcup_{r \in \mathbb{Q}} N_r \qquad \text{——— } (1)$$]

Another consequence of SLLN:

If x is a discontinuity point of F,

then, $$F_n(x) - F_n(x^-) \xrightarrow{a.s} F(x) - F(x^-).$$

$, F_n(x) - F_n(x^-) \longrightarrow F(x) - F(x^-)$

our required iid seq: $1_{X_1=x}, 1_{X_2=x}, \ldots$

$$E\left(1_{X_1=x}\right) = F(x) - F(x^-).$$

$$\| \qquad\qquad //$$

$$P(X \leq x) - P(X < x)$$

& , $F_n(x) - F_n(x^-) = \dfrac{\#\{k : X_k \leq x\} - \#\{k : X_k < x\}}{n}.$

$\therefore$ Let N_2 be a null set s.t. outside N_2,

$$F_n(x) - F_n(x^-) \longrightarrow F(x) - F(x^-).$$

$\bigcup_x N_x$, where x - discont. pts [countably many at most]

——②

$\therefore$ By taking the required p-null set to be $N = N_1 \cup N_2$,

we get, outside of N, both ① & ② hold.

<u>Claim:</u> Outside of N,

$$\sup_x |F_n(x) - F(x)| \xrightarrow{a.s} 0 \quad \text{as} \quad n \longrightarrow \infty.$$

<u>Proof by contradiction:</u>

Suppose <u>not</u>.

ie, $\exists\, \varepsilon > 0$, & a subsequence

$$1 < n_1 < n_2 < \cdots < n_k < \cdots \uparrow \infty$$

& a real sequence $\{x_k\}$

s.t. $|F_{n_k}(x_k) - F(x_k)| \geq \varepsilon.$

ie, n_1 is the first index where,

first index where, the sup$|\cdot| > \varepsilon$.

n_2 is the next index where this happens.

s.t. $|F_{n_k}(x_k) - F(x_k)| \geq \varepsilon$.

Now, for $\sup_x |F_{n_1}(x) - F(x)| > \varepsilon$

$\exists\, x = x_1$ for which this holds.

Similarly, $\sup_x |F_{n_2}(x) - F(x)| > \varepsilon$

for some $x = x_2$.

that's how we obtain $\{x_k\}$.

<u>Claim</u>: the sequence $\{x_k\}$ - cannot be unbounded.

i.e, It has to be both bounded above & bounded below.

$\because F(x) \to 0$ as $x \to -\infty$.

$\therefore \exists\, r \in \mathbb{Q}$ (largely -ve)

s.t

$F(r) < \varepsilon/2$

$\Rightarrow F_{n_k}(r) < \varepsilon/2$

just a subsequence of $F_n(r)$

$\left[\because F_{n_k}(r) \to F(r)\right]$

If $\{x_k\}$ - not bounded below, $\therefore$ (for this r, there must exist x_{n_k} s.t. $x_{n_k} < r$, as we've assumed x_{n_k} to be unbounded below.)

then $x_k < r$ for infinitely many k

$\Rightarrow F(x_k) \leq F(r) < \varepsilon/2$

$F_{n_k}(x_k) \leq F(r) < \varepsilon/2$

for infinitely

$F_{n_k}(x_k) \le F(r) < \varepsilon/2$) for infinitely many k. below .

$\therefore \left| F_{n_k}(x_k) - F(x_k) \right| < \varepsilon/2$, which is a contradiction!!!

$\therefore \{x_k\}$ must be bounded below.

Similarly, $\because F(x) \to 1$ as $x \to \infty$

$\therefore \exists\, r \in \mathbb{Q}$ st,

$F(r) > 1 - \varepsilon/2$

$\Rightarrow 1 - F(r) < \varepsilon/2$

By similar argument $\Rightarrow 1 - F_{n_k}(r) < \varepsilon/2 \ \forall$ large k.

then, we can conclude that $\{x_k\}$ is bounded above.

Now, by passing to a subsequence, if necessary, (& still calling it $\{x_k\}$), we may assume $x_k \longrightarrow a \in \mathbb{R}$.

Exercise: (Ana-1) By again passing through subsequences, if necessary, we assume atleast one of the following holds:

(1) $x_k < a$, $x_k \uparrow a$, $F_{n_k}(x_k) - F(x_k) > \varepsilon$.

(2) $x_k < a$, $x_k \uparrow a$, $F(x_k) - F_{n_k}(x_k) \geq \varepsilon$

(3) $x_k \geq a$, $x_k \downarrow a$, $F_{n_k}(x_k) - F(x_k) \geq \varepsilon$

(4) $x_k \geq a$, $x_k \downarrow a$, $F(x_k) - F_{n_k}(x_k) \geq \varepsilon$

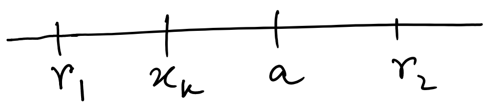
$\therefore$ No. of 'k's for which all 4 above holds is all $k \in \mathbb{N}$. $\leftarrow$ infinite

$\therefore$ Atleast one of the 4 cases above must happen for infinitely many k's.

W.L.O.G, say, (1) holds for infinitely many k's.

So, Suppose, we are in case-(1) this $a \in \mathbb{R}$.

<u>first</u>, let's assume, 'a' is a continuity point of F



$$\varepsilon \leq \underbrace{F_{n_k}(x_k)}_{\leq} - \underbrace{F(x_k)}_{\geq}$$

$$\leq F_{n_k}(r_2) - F(r_1) + F(r_2) - F(r_2)$$

$$= \underbrace{F_{n_k}(r_2) - F(r_2)}_{\downarrow \atop 0} - \big(F(r_1) - F(r_2)\big)$$

$$= \underbrace{F(r_2) - F(r_1)}_{\downarrow \atop 0 \text{ as } r_1 \uparrow a}$$

[$\because$ 'a' is a continuity point.]

as $r_1 \downarrow a$

$r_2 \nearrow a$

"You'd see it if you want to see it,

you'd not see if you don't want to see it."

— Prof. AG,
24^{th} Sept, '24

Now, assume a to be a point of discontinuity of F

$$\varepsilon \leq F_{n_k}(x_k) - F(x_k)$$

$$\leq F_{n_k}(a^-) - F_{n_k}(a) + F_{n_k}(a^-) - F(x_k)$$