2.2 Show that $M(u,v) = \min(u,v)$, $W(u,v) = \max(u+v-1,0)$, and $\Pi(u,v) = uv$ are indeed copulas.

2.3 (a) Let C_0 and C_1 be copulas, and let θ be any number in $\mathbf{I}$. Show that the weighted arithmetic mean $(1-\theta)C_0 + \theta C_1$ is also a copula. Hence conclude that any convex linear combination of copulas is a copula.
(b) Show that the geometric mean of two copulas may fail to be a copula. [Hint: Let C be the geometric mean of Π and W, and show that the C-volume of the rectangle $[1/2,3/4]\times[1/2,3/4]$ is negative.]

2.4 *The Fréchet and Mardia families of copulas.*
(a) Let α, β be in $\mathbf{I}$ with $\alpha + \beta \le 1$. Set

$$C_{\alpha,\beta}(u,v) = \alpha M(u,v) + (1-\alpha-\beta)\Pi(u,v) + \beta W(u,v).$$

Show that $C_{\alpha,\beta}$ is a copula. A family of copulas that includes M, Π, and W is called *comprehensive*. This two-parameter comprehensive family is due to Fréchet (1958).
(b) Let θ be in $[-1,1]$, and set

$$C_\theta(u,v) = \frac{\theta^2(1+\theta)}{2} M(u,v) + (1-\theta^2)\Pi(u,v) + \frac{\theta^2(1-\theta)}{2} W(u,v). \quad (2.2.9)$$

Show that C_θ is a copula. This one-parameter comprehensive family is due to Mardia (1970).

2.9 The *secondary diagonal section* of C is given by $C(t,1-t)$. Show that $C(t,1-t) = 0$ for all t in $\mathbf{I}$ implies $C = W$.

2.10 Let t be in $[0,1)$, and let C_t be the function from $\mathbf{I}^2$ into $\mathbf{I}$ given by

$$C_t(u,v) = \begin{cases} \max(u+v-1,t), & (u,v) \in [t,1]^2, \\ \min(u,v), & \text{otherwise.} \end{cases}$$

(a) Show that C_t is a copula.

(b) Show that the level set $\left\{(u,v) \in \mathbf{I}^2 \middle| C_t(u,v) = t\right\}$ is the set of points in the triangle with vertices $(t,1)$, $(1,t)$, and (t,t), that is, the shaded region in Fig. 2.3. The copula in this exercise illustrates why the term "level set" is preferable to "level curve" for some copulas.