Probability-2 Lecture-5

23 January 2024 11:17

2nd Semester content starts today!

Basic settings:

Ω : a non-empty set

$\mathcal{A}$: a class of subsets of Ω.

Sets in "$\mathcal{A}$" are called "events".

For $A \in \mathcal{A}$, then an assignment $A \mapsto P(A)$ to be called "probability of A"

$$P(A) \geq 0 \quad \forall \; A \in \mathcal{A}.$$

Axioms:

① $P(\Omega) = 1$

② For every sequence $\{A_n\}_{n \geq 1}$ of disjoint sets belonging to $\mathcal{A}$,

$$P\left(\bigcup_n A_n\right) = \sum P(A_n)$$

Special Case:

Ω is a countable set.

Algorithm:

For each $\omega \in \Omega$, assign a number $P(\{\omega\}) \geq 0$ such that, $\sum_{\omega} P(\{\omega\}) = 1$

Take $\mathcal{A}$ = all subsets of Ω.

Define $P(A) = \sum_{\omega \in A} P(\{\omega\})$

$$(\Omega, \mathcal{A}, P)$$

Recall: Random Walks (Sem-1).

Back then, we only dealt with countable sample spaces.

Uncountable Sample Spaces.

Back then, we only dealt with countable sample spaces.
Now, we'll learn to deal with <u>Uncountable Sample Spaces</u>.

Ω = the set of all possible paths

= set of all infinite sequences of ± 1

[infinite sequences of +1 & −1, hence uncountable. (Recall: Cantor's diagonalization argument)]

$= \{\omega, \omega = (\omega_1, \omega_2, \ldots\ldots) : \omega_i = \pm 1\}$.

. <u>Ω is uncountable</u>.

Fix a sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots.)$, $\varepsilon_i = \pm 1$.

What is $P(\{\varepsilon\}) = ?$

$\{\varepsilon\} \subset \{\omega, \omega = (\omega_1, \omega_2, \ldots\ldots) : \omega_1 = \varepsilon_1\}$.

fix first coordinate.

$\therefore P(\quad) = \frac{1}{2}$

then, fixing second element, we again get $P(\cdot) = 1/2$, which can't be possible !!

$\therefore P(\{\varepsilon\}) \leq 1/2$ (must be).

By "similar argument" (ie, fixing values in the sequences & taking the probabilities)

$0 \leq P(\{\varepsilon\}) \leq \frac{1}{2^n}$ $(\longrightarrow 0$ as $n \to \infty)$

ie, $\boxed{P(\omega) = 0 \ \forall \ \omega \in \Omega \text{ (uncountable)}}$.

A "finite-dimensional" subset of A is
We have indices $1 \leq i_1 < i_2 < \cdots < i_k$,
and a fixed k-tuple $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ of ± 1.

$A = \{\omega \in \Omega : \omega_{i_j} = \varepsilon_j, j = 1, \ldots, k\}$.

Here, $P(A) = 2^{-k}$ $(\neq 0)$ Counter-intuitive !!

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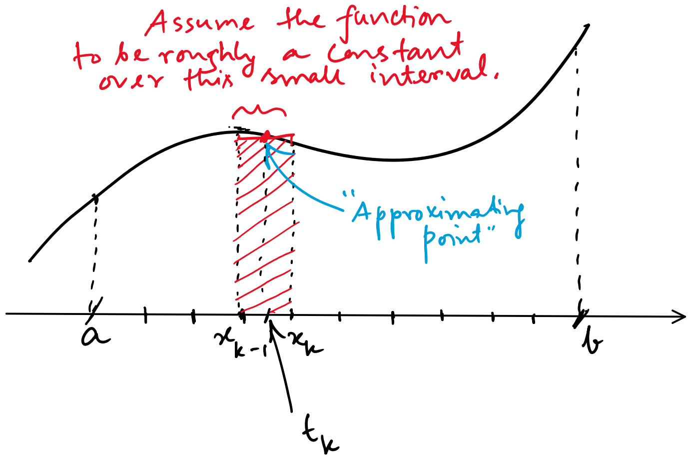
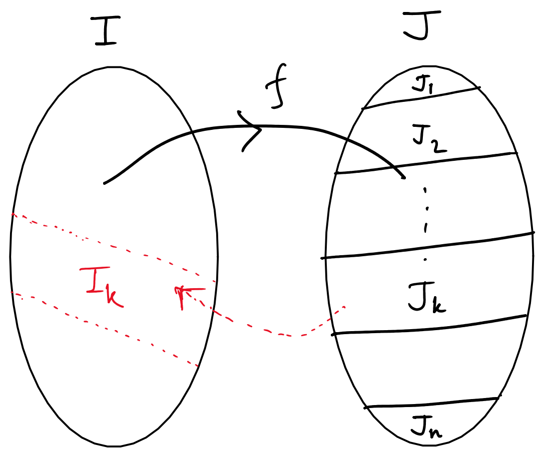
Digression:

Quick recap of Riemann Integration theory:



If the value of the function at t_k, i.e., $f(t_k)$ is "close" to the values that the function takes over other points in $[x_{k-1}, x_k]$, then the approximation is "good".

Digression: Lebesgue Integral.



$I = [a, b]$ $\qquad$ $J = [c, d]$.

$J = J_1 \cup J_2 \cup \ldots \cup J_n$ (disjoint union)

What Lebesgue did was:
instead of partitioning the domain I,
he partitioned the codomain J.

$I_k = \{x \in I : f(x) \in J_k\}$.

Clearly, I_k's are disjoint.

$\therefore$ $\{ I_k : k \in \{1, 2, \ldots, n\}\}$ automatically gives a partition of I.

i.e., disjoint partition of codomain set (J) induces a disjoint partition on domain set (I).

Pick $y_k \in J_k$, $k = 1, 2, \ldots, n$

$$\sum_k y_k \cdot \underbrace{\text{length}(I_k)}.$$

$\sum_k U_k \quad \underbrace{\hspace{6em}}$

(?problem?) How to assign this length?

(the problem here is similar to the problem we're facing, ie, of assigning probabilities when sample space is uncountable.)

"<u>σ- field</u>" — a class of subsets of Ω is called a σ- field if:

- it contains the whole set.
- it is closed under complementation.
- it is closed under countable union.

<u>Consequence</u>: this class will be closed under any countable set operation.

[<u>Read</u>:
* Borel σ- fields.
* Carotheodory extension theorem.]