

# CS6210 - Homework/Assignment-6

Arnab Das(u1014840)

December 7, 2016

---

**Question-2: Chapter-14, question-2**

---

Let  $f(x)$  be a given function that can be evaluated at points  $x_0 \pm jh, j = 0, 1, 2, \dots$  for any fixed value of  $h$ ,  $0 < h \ll 1$ .

(a) Using Taylor series expansion about  $x_0$ , we can write the following expansions about  $\pm h$ :

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(x_0) + \frac{h^5}{120}f^{(5)}(\zeta_1)$$

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{(4)}(x_0) - \frac{h^5}{120}f^{(5)}(\zeta_2)$$

where,  $x_0 \leq \zeta_1 \leq x_0 + h$ , and  $x_0 - h \leq \zeta_2 \leq x_0$  The difference of the above two equations gives the following:

$$f(x_0 + h) - f(x_0 - h) = 2\{hf'(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^5}{120}f^{(5)}(\zeta_3)\} \quad (1)$$

where,  $x_0 - h \leq \zeta_3 \leq x_0 + h$

Similarly, we can get the following expansions using nearby points at  $x_0 \pm 2h$ :

$$f(x_0 + 2h) = f(x_0) + 2hf'(x_0) + \frac{4h^2}{2}f''(x_0) + \frac{8h^3}{6}f'''(x_0) + \frac{16h^4}{24}f^{(4)}(x_0) + \frac{32h^5}{120}f^{(5)}(\zeta_4)$$

$$f(x_0 - 2h) = f(x_0) - 2hf'(x_0) + \frac{4h^2}{2}f''(x_0) - \frac{8h^3}{6}f'''(x_0) + \frac{16h^4}{24}f^{(4)}(x_0) - \frac{32h^5}{120}f^{(5)}(\zeta_5)$$

where,  $x_0 \leq \zeta_4 \leq x_0 + 2h$ , and  $x_0 - 2h \leq \zeta_5 \leq x_0$  The difference of the above two equations gives the following:

$$f(x_0 + 2h) - f(x_0 - 2h) = 2\{2hf'(x_0) + \frac{8h^3}{6}f'''(x_0) + \frac{32h^5}{120}f^{(5)}(\zeta_6)\} \quad (2)$$

where,  $x_0 - 2h \leq \zeta_6 \leq x_0 + 2h$

Then multiplying (1) by 2 and subtracting it from (2), we get:

$$f(x_0 + 2h) - f(x_0 - 2h) - 2f(x_0 + h) + 2f(x_0 - h) = 2h^3f'''(x_0) + \frac{h^5}{2}f^{(5)}(\zeta)$$

where,  $x_0 - 2h \leq \zeta \leq x_0 + 2h$  Rearranging the terms, we get:

$$f'''(x_0) = \frac{f(x_0 + 2h) - f(x_0 - 2h) - 2f(x_0 + h) + 2f(x_0 - h)}{h^3} + \frac{-h^2}{4}f^{(5)}(\zeta) \quad (3)$$

(3) provides the formula for approximating the third derivative  $f'''(x_0)$ , with the bold test showing the component of the truncation error: Thus, truncation error =  $-\frac{h^2}{4}f^{(5)}(\zeta)$

(c) To derive the expression for the round-off error, we denote the floating point evaluations of  $f$  by  $F$ . Thus the expression for round off error will be  $|F'''(x_0) - f'''(x_0)|$

$$\begin{aligned} &= \left| \frac{F(x_0 + 2h) - F(x_0 - 2h) - 2F(x_0 + h) + 2F(x_0 - h)}{2h^3} - \frac{f(x_0 + 2h) - f(x_0 - 2h) - 2f(x_0 + h) + 2f(x_0 - h)}{2h^3} \right| \\ &= \left| \frac{e_r(x_0 + 2h) - e_r(x_0 - 2h) - 2e_r(x_0 + h) + 2e_r(x_0 - h)}{2h^3} \right| \end{aligned}$$

where, the round-off error term,  $e_r(x) = F(x) - f(x)$

$$\leq \left| \frac{e_r(x_0 + 2h)}{2h^3} \right| + \left| \frac{e_r(x_0 - 2h)}{2h^3} \right| + \left| \frac{2e_r(x_0 + h)}{2h^3} \right| + \left| \frac{2e_r(x_0 - h)}{2h^3} \right|$$

Assuming the each round-off error term is bounded by the machine precision,  $\epsilon$ , then we get:

$$|F'''(x_0) - f'''(x_0)| \leq \frac{6\epsilon}{2h^3} = \frac{3\epsilon}{h^3}$$

Thus, the round-off error grows inversely proportional to  $h$ . Hence, as  $h$  decreases, the round-off error increases, thus with  $h \rightarrow 0$ , the round-off error will tend to  $\infty$ .

(d) We can obtain a fourth order formula in a similar manner as we obtained the third order formula with some minor changes. First evaluate  $S1 : f(x_0 + h) + f(x_0 - h)$ , then evaluate  $S2 : f(x_0 + 2h) + f(x_0 - 2h)$ . Then subtract  $S2 - 4S1$ , to extract the fourth order formula.

### Question-3: Chapter-14: question-8

Derivation of an approximate formulae for the second derivatives  $f''(x_0)$  of a smooth function  $f(x)$  using the three points  $x_{-1}, x_0 = x_{-1} + h_0, x_1 = x_0 + h_1$ , where  $h_0 \neq h_1$ .

(a) To prove both the given approximations are the same, we need to show:  $2[f_{x-1,x_0,x_1}] = \frac{g_{1/2} - g_{-1/2}}{(h_0 + h_1)/2}$ .

**Proof:**

Using divided difference, we can write:

$$2[f_{x-1,x_0,x_1}] = 2 \times \frac{\frac{f(x_1) - f(x_0)}{x_1 - x_0} - \frac{f(x_0) - f(x_{-1})}{x_0 - x_{-1}}}{x_1 - x_{-1}}$$

$$2[f_{x-1,x_0,x_1}] = 2 \times \frac{\frac{f(x_1) - f(x_0)}{h_1} - \frac{f(x_0) - f(x_{-1})}{h_0}}{h_1 + h_0}$$

Given, that:  $g_{1/2} = \frac{f(x_1) - f(x_0)}{h_1}$  and  $g_{-1/2} = \frac{f(x_0) - f(x_{-1})}{h_0}$ , we can replace them to get:

$$2[f_{x-1,x_0,x_1}] = 2 \times \frac{g_{1/2} - g_{-1/2}}{h_1 + h_0}$$

$$2[f_{x-1,x_0,x_1}] = \frac{g_{1/2} - g_{-1/2}}{\frac{h_1 + h_0}{2}}$$

(b) To show that the method is only first order accurate, we use the Taylor series expansion upto second order with third order error term to derive the formulation:

$$f(x_0 + h_1) = f(x_0) + h_1 f'(x_0) + \frac{h_1^2}{2} f''(x_0) + \frac{h_1^3}{3!} f'''(\zeta_1)$$

$$f(x_0 - h_0) = f(x_0) - h_0 f'(x_0) + \frac{h_0^2}{2} f''(x_0) - \frac{h_0^3}{3!} f'''(\zeta_2)$$

Which can be further simplified to:

$$\frac{f(x_1) - f(x_0)}{h_1} = f'(x_0) + \frac{h_1}{2} f''(x_0) + \frac{h_1^2}{3!} f'''(\zeta_1)$$

$$\frac{f(x_{-1}) - f(x_0)}{h_0} = -f'(x_0) + \frac{h_0}{2} f''(x_0) - \frac{h_0^2}{3!} f'''(\zeta_2)$$

Adding these two, we get:

$$\frac{f(x_1) - f(x_0)}{h_1} - \frac{f(x_0) - f(x_{-1})}{h_0} = \frac{(h_0 + h_1)}{2} f''(x_0) + \frac{h_1^2}{3!} f'''(\zeta_1) - \frac{h_0^2}{3!} f'''(\zeta_2)$$

For the error term, replacing with a  $\zeta$ , such that  $x_{-1} \leq \zeta \leq x_1$ :

$$\frac{f(x_1) - f(x_0)}{h_1} - \frac{f(x_0) - f(x_{-1})}{h_0} = \frac{(h_0 + h_1)}{2} f''(x_0) + f'''(\zeta) \left( \frac{h_1^2}{3!} - \frac{h_0^2}{3!} \right)$$

Rearranging to get the second order term:

$$f''(x_0) = \frac{\frac{f(x_1) - f(x_0)}{h_1} - \frac{f(x_0) - f(x_{-1})}{h_0}}{(h_0 + h_1)/2} - \frac{\mathbf{f'''(\zeta)(h_1 - h_0)}}{\mathbf{3}}$$

The error term shown in bold is thus only first order,  $O(h_0 + h_1)$ , accurate.

(c) Will be back soon

#### Question-6: Chapter-15, question-4

(a) **Prove:** Error in basic corrected trapezoidal rule in the interval  $[a, b]$  can be estimated by:

$$E(f) = \frac{f''''(\eta)}{720} (b - a)^5$$

**Proof:** The osculating polynomial formula for the basic corrected trapezoidal rule is written as:

$$p_3(x) = f(a) + f'(a)(x - a) + f[a, a, b](x - a)^2 + f[a, a, b, b](x - a)^2(x - b)$$

The error in the polynomial interpolation in that case will be given by:

$$f[a, a, b, b, x](x - a)(x - a)(x - b)(x - b)$$

Then, to find the error in the integral of the polynomial, we can integrate the error of polynomial described above, in the interval  $[a, b]$

$$E(f) = \int_a^b f[a, a, b, b, x] \psi(x)$$

where  $\psi(x) = \prod_{i=0}^3 (x - x_i) = (x - a)(x - a)(x - b)(x - b)$

Notice that, since  $x$  lies in the interval  $[a, b]$ , hence  $(x - a \geq 0)$  and  $(x - b \leq 0)$ . In any case, the square of the terms will be greater than equal to 0. So, in the given interval  $\psi(x) \geq 0$  always. Because  $\psi(x)$  does not changes sign in the interval, then using the intermediate value theorem, there exists  $a \leq \eta \leq b$ , such that:

$$E(f) = \int_a^b f[a, a, b, b, x] \psi(x) = \int_a^b f[a, a, b, b, \eta] \psi(x)$$

where,

$$f[a, a, b, b, \eta] = \frac{f''''(\eta)}{4!}$$

which is a constant, say  $K$ .

Then we can write the error integral as:

$$E(f) = K \int_a^b (x - a)^2 (x - b)^2$$

Doing integration by parts:

$$E(f) = K \left[ \frac{(x-a)^2(x-b)^3}{3} - \frac{(x-a)(x-b)^4}{6} + \frac{(x-b)^5}{30} \right]_a^b = \frac{-(a-b)^5}{30}$$

Replacijng back K,we get:

$$E(f) = \frac{f''''(\eta)}{4!} \frac{(-(a-b)^5)}{30} = \frac{f''''(\eta)(b-a)^5}{720}$$

(b) The integral for the corrected trapezoidal is written is:

$$I_f \approx \int_a^b p_3(x)dx = \frac{(b-a)}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(a) - f'(b)]$$

**part-1:**For the integral  $\int_0^1 e^x dx$ , thus  $a = 0, b = 1$ , and  $f(x) = e^x, f'(x) = e^x$ . So,  $f(a) = 1, f(b) = e, f'(a) = 1, f'(b) = e$

Using the basic corrected trapezoidal, we get:

$$\int_0^1 e^x dx = 1.71595$$

the acutal evaluation is 1.7183... while the basic trapezoidal evaluation from Example-15.2 is 1.7183.... Hence, the evaluation using the basic corrected trapezoidal is more accurate than the basic trapezoidal.

**part-2:** For the integral  $\int_{0.9}^1 e^x dx$ , thus  $a = 0.9, b = 1$ , and  $f(x) = e^x, f'(x) = e^x$ . So,  $f(a) = e^{0.9}, f(b) = e, f'(a) = e^{0.9}, f'(b) = e$

Using the basic corrected trapezoidal, we get:

$$\int_{0.9}^1 e^x dx = 0.258678$$

The actual evaluation is 0.2586787171..., while the basic trapezoidal evaluation from Example-15.2 is 0.258894.... hence, the evaluation using the basic corrected trapezoidal is more accurate than the basic trapezoidal.

---

### Question-7: Chapter-15, question-5

---

(a) In the interval  $[a,b]$ , the basic midpoint rule is given as :

$$I_f \approx (b-a)f\left(\frac{a+b}{2}\right) \quad (4)$$

For the composite midpoint rule, we consider  $r$  subintervals in the original interval  $[a,b]$  and apply the basic midpoint rule to each subinterval and then sum over all the subintervals to get the composite integral. The rule applied to an interval  $[t_{i-1}, t_i]$ , such that the interval widths are uniform and  $t_i - t_{i-1} = h = \frac{b-a}{r}$ , will be:

$$\int_{t_{i-1}}^{t_i} f(x)dx \approx hf\left(\frac{t_{i-1} + t_i}{2}\right)$$

Summing over all the subintervals to get the complete composite integral:

$$\int_a^b f(x)dx = h \sum_{i=1}^r f\left(\frac{t_{i-1} + t_i}{2}\right)$$

For,  $r$  equispaced intervals over  $[a, b]$ , we have the interval width as  $h = \frac{b-a}{r}$ . Then,  $t_0 = a, t_1 = a + h, t_2 = a + 2h, \dots, t_i = a + ih$ . So,

$$\frac{t_{i-1} + t_i}{2} = \frac{a + (i-1)h + a + ih}{2} = a + (i - \frac{1}{2})h$$

Replacing it in the original integral, we get the final form for the composite midpoint as:

$$\int_a^b f(x)dx \approx h \sum_{i=1}^r f(a + (i - \frac{1}{2})h)$$

From the above expression, we can see that there is one function evaluation per subinterval. Hence, the number of function evaluations is  $r = \frac{b-a}{h}$

### (b) Derive an expression for the error in composite midpoint rule

For the basic midpoint rule, the error expression in the interval  $[a, b]$  is given by:

$$E(f) = \frac{f''(\eta)}{24} (b-a)^3$$

This comes from doing the following integral:

$$\frac{f''}{2!}(\eta) \int_a^b (x - \frac{a+b}{2})(x - \frac{a+b}{2}) dx$$

The reason for adding the second term of  $(x - \frac{a+b}{2})$  even though there is only a single point, is that  $(x - \frac{a+b}{2})$  can change signs within the interval  $[a, b]$ , and hence we cannot apply the intermediate value theorem to take out  $f''(\eta)$  as a constant, for  $a \leq \eta \leq b$ . So, we duplicate the point  $\frac{a+b}{2}$  as a dummy interpolation point, since it does not change the area evaluated and hence the error term should remain the same.

Next, we come to the derivation of the error for the composite midpoint rule. In the composite cases, we have divided the original interval,  $[a, b]$ , into  $r$  equispaced sub-intervals of width  $h$ , such that  $r = \frac{b-a}{h}$ . For evaluating the composite integral using mid-point we applied the basic midpoint to each of these sub-intervals and summed them. Similarly, now each evaluation of the basic midpoint in these intervals will result in an error, which we can further sum up to get the expression for the error in composite mid-point. The error term in the interval  $[t_{i-1}, t_i]$  will be  $\frac{f''(\eta_i)}{24} h^3$ , where  $\frac{f''(\eta_i)}{24}$  is a constant in the interval  $[t_{i-1}, t_i]$

Hence, the expression of error for the composite midpoint will be:

$$E_{CM}(f) = \sum_{i=1}^r \frac{f''(\eta_i)}{24} h^3$$

Since,  $f''(\eta_i)$  is a constant in the interval  $[t_{i-1}, t_i]$ , we can generalize it with an appropriate constant  $f''(\eta)$  where  $a \leq \eta \leq b$ . Then the error expression comes to be:

$$E_{CM}(f) = \frac{f''(\eta)}{24} \sum_{i=1}^r h \cdot h^2$$

$$E_{CM}(f) = \frac{f''(\eta)}{24} h^2 \cdot rh$$

Since,  $h = \frac{b-a}{r}$ , so replacing  $(b-a) = rh$ , we get:

$$E_{CM}(f) = \frac{f''(\eta)}{24}(b-a)h^2 \quad (5)$$

This is the final expression for the error in the composite mid point rule.

(5) Suggests that the error varies proportional to  $h^2$ , hence it is second order accurate.

### Question-8: Chapter-15, question-13

Given that the interval of integration,  $[a,b]$ , is divided into equal sub-intervals of length  $h$ , such that  $r = \frac{b-a}{h}$

**Composite Simpson:**

$$\int_a^b f(x)dx \approx \frac{h}{3} \left[ f(a) + 2 \sum_{k=1}^{\frac{r}{2}-1} f(t_{2k}) + 4 \sum_{k=1}^{\frac{r}{2}} f(t_{2k-1}) + f(b) \right] \quad (6)$$

The expression for composite trapezoidal with step size  $h$  is given by:

**R2: Composite trapezoidal rule of step size  $h$**

$$\int_a^b f(x)dx \approx \frac{h}{2} \sum_{i=1}^r f(t_{i-1}) + f(t_i)$$

$$R_2 = \frac{h}{2} [f(a) + 2f(t_1) + 2f(t_2) + \cdots + 2f(t_{r-1}) + f(b)]$$

**R1: Composite trapezoidal rule of step size  $2h$**  For step-size of  $2h$ , we require even number of subintervals. In the above expression for summation, thus we change the summing variable  $i$  to  $2k$ , and the limit become  $\frac{r}{2}$ . Hence, we have:

$$R_1 = \frac{2h}{2} \sum_{k=1}^{\frac{r}{2}} f(t_{2k-2}) + f(t_{2k})$$

$$R_1 = h[\{f(t_0) + f(t_2) + \cdots + f(t_{r-2})\} + \{f(t_2) + f(t_4) + \cdots + f(t_r)\}]$$

Since,  $t_0$  and  $t_r$  are the two extreme end points of the interval, hence  $t_0 = a$  and  $t_r = b$  Thus, we get:

$$R_1 = h[f(a) + 2f(t_2) + 2f(t_4) + \cdots + 2f(t_{r-2}) + f(b)]$$

Hence, evaluating  $S = \frac{4R_2 - R_1}{3}$

$$4R_2 - R_1 = h[2f(a) + 4f(t_1) + 4f(t_2) + \cdots + 4f(t_{r-1}) + 2f(b)] - h[f(a) + 2f(t_2) + 2f(t_4) + \cdots + f(b)]$$

$$4R_2 - R_1 = h[f(a) + \{2f(t_2) + 2f(t_4) + \cdots + 2f(t_{r-2})\} + \{4f(t_1) + 4f(t_3) + \cdots + 4f(t_{r-1})\} + f(b)]$$

$$4R_2 - R_1 = h \left[ f(a) + 2 \sum_{k=1}^{\frac{r}{2}-1} f(t_{2k}) + 4 \sum_{k=1}^{\frac{r}{2}} f(t_{2k-1}) + f(b) \right]$$

$$\frac{4R_2 - R_1}{3} = \frac{h}{3} \left[ f(a) + 2 \sum_{k=1}^{\frac{r}{2}-1} f(t_{2k}) + 4 \sum_{k=1}^{\frac{r}{2}} f(t_{2k-1}) + f(b) \right]$$

The rhs of the above is exactly the expression for the composite Simpson's rule (6)