CS6210 - Homework/Assignment-5

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November 27, 2016

8: Chapter-11: Question-3

Let $f \in C^3[a, b]$ be given at equidistant points, $x_i = a + ih$, where $i = 0, 1, 2, \dots, n$ and $nh = b - a$. $f'(a)$ is given as well.

(a) Construct an algorithm for C^1 continuous quadratic interpolation:

$$v(x) = S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2; x_i \leq x \leq x_{i+1}$$

That is an algorithm to determine the 3n coefficients.

Algorithm: The polynomial $S_i(x)$ in the interval $x_i \leq x \leq x_{i+1}$ will match the function value exactly. So, we have our first condition as:

$$S_i(x_i) = a_i = f(x_i)$$

Thus, we have n equations in n variables due to this. This gives us all the n a_i values.

Also, since $S_i(x)$ is valid in the interval $x_i \leq x \leq x_{i+1}$, hence it should satisfy the function value at x_{i+1} as well. Thus we have our next condition as:

$$S_i(x_{i+1}) = f(x_{i+1})$$

$$a_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 = f(x_{i+1})$$

since, $x_{i+1} - x_i = h$, we get:

$$b_i h + c_i h^2 = a_{i+1} - a_i$$

This gives us n equations in 2n variables. In combination with the above we get 2n equations in 3n variables.

Next, since we want the interpolant to be C^1 continuous and the given fucntion is C^3, hence, the first derivative of $S_i(x_i)$ should match the first derivative of the function at x_i. Thus, we get:

$$S_i'(x_i) = f'(x_i)$$

$$b_i = f'(x_i)$$

Also, the derivatives should match at x_{i+1}. Hence,

$$S_i'(x_{i+1}) = f'(x_{i+1})$$

$$b_i + 2c_i(x_{i+1} - x_i) = f'(x_{i+1})$$

$$b_i + 2c_i h = f'(x_{i+1})$$

There is a certain pattern that emerges. If we know the derivative value at x_i, we can use it further to compute the derivative value at x_{i+1}. Since, we need to evaluate all the derivative values exceopt the first one, let us denote the derivative with the variable d_i.

Thus, from the above relations we have the following set of equations:

n equations for

$$a_i = f(x_i)$$

nequations for

$$b_i h + cih^2 - (a_{i+1} - a_i) = 0$$

n equations for

$$b_i - d_i = 0$$

(n-1) equations for

$$b_i + 2c_i h - d_{i+1} = 0$$

1 equations for the derivative given at x=a, that is d_0

$$d_0 = f(a)$$

Thus, in total we have 4n variables with 4n equations which can be solved to get the 3n coefficients $\{a_i, b_i, c_i\}$ and the derivatives d_i.

(b) The error estimate for an n-degree interpolate, $p_n(x)$, with respect to the original function, $f(x)$, is expressed as:

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

For piecewise quadratic interpolant , $v(x) = S_i(x)$, we have n=2. Hence, we get,

$$|f(x) - v(x)| \leq (x - x_{i-1})(x - x_i) \max_{a \leq \zeta \leq b} \frac{f^3(\zeta)}{3!} \tag{1}$$

That is the 3rd derivative is bounded by its max value at a point over the entire interval [a,b]. For the term, $(x - x_{i-1})(x - x_i)$, the maximum occurs at the midpoint of the interval , that is, at $\dfrac{x_{i-1} + x_i}{2}$. Thus, we get:

$$|(x - x_{i-1})(x - x_i)| \leq \left(\frac{x_i - x_{i-1})}{2}\right)^2$$

and since, $x_i - x_{i-1} = h$, we get

$$|(x - x_{i-1})(x - x_i)| \leq \left(\frac{h}{2}\right)^2$$

Plugging this back to (1), we get the following error bound:

$$ErrorBound = |f(x) - v(x)| \leq \frac{h^2}{24} \max_{a \leq \zeta \leq b} |f^3(\zeta)|$$

9: Chapter-11: Question-12

Derive a B-spline basis representation for piecewise linear interpolation and for piecewise cubic interpolation:

For (n+1) interpolating points and degree (k-1), the bspline representation is written as:

$$P(u) = \sum_{i=0}^{n} piN_{i,k}(u) \tag{2}$$

where $N_{i,k}$ are the bspline basis functions and (k-1) is the degree of the curve. Hence, for a piece-wise linear interpolant, the degree of the curve is 1, hence $k = 2$. Thus, we have (n+1) interpolating points and k=2. The idea of bspline formulation us to generate curve that are affected locally by the control points. So, if a point is changed, the curve formulation will change locally, and not cause changes in the entire curve unlike bezier curves. The entire interval of points $\{p_0, p_1, \ldots, p_n\}$ is divided into a sequence of knots $(t_0, t_1, \ldots, t_r)$, such that the effect of a point will have its control in a knot region $[t_{i-1}, t_i]$ and not anywhere else:

The basis function, $N_{i,k}$, is defined as :

$$N_{i,k}(u) = \frac{(u - t_i)N_{i,k-1}(u)}{t_{i+k} - t_i} + \frac{(t_{i+k} - u)N_{i+1,k-1}(u)}{t_{i+k} - t_{i+1}} \tag{3}$$

The base case of the basis functions is given as:

$$N_{i,1}(u) = 1; t_i \leq u < t_{i+1}$$

$$= 0; otherwise$$

This allows for the termination condition of the recursion defined in (3):

In general, the knot values, t_i are defined as:

$$t_i = 0; i < k$$

$$t_i = i - k + 1; k \leq i \leq n$$

$$t_i = n - k + 2; i > n$$

For the piecewise linear case, the value of $k = 2$, hence the above relations translate to:

$$t_i = 0; i < 2$$

$$t_i = i - 1; 2 \leq i \leq n$$

$$t_i = n; i > n$$

Thus , the set of $(n + k + 1) = (n + 3)$ knot values will be:

$$t = (0, 0, 1, 2, \ldots, n - 1, n, n)$$

(2) in this case translates to :

$$P(u) = \sum_{i=0}^{n} p_i N_{i,2}(u) = p_0 N_{0,2}(u) + p_1 N_{1,2}(u) + \cdots + p_i N_{i,2}(u) + \cdots + p_n N_{n,2}(u)$$

where, the basis function $N_{i,2}$ in this case will be :

$$N_{i,2}(u) = \frac{(u - t_i)N_{i,1}(u)}{t_{i+2} - t_i} + \frac{(t_{i+2} - u)N_{i+1,1}(u)}{t_{i+2} - t_{i+1}} \tag{4}$$

Here, the base cases of the recursion will become:
$N_{0,1} = 1$ if $t_0 \leq u < t_1$ or $0 \leq u < 0$
$N_{0,1} = 0, otherwise,$
$N_{0,1} = 0$

$N_{1,1} = 1$ if $t_1 \leq u < t2$ or $0 \leq u < 1$
$N_{1,1} = 0$, otherwise

$N_{2,1} = 1$ if $t_2 \leq u < t_3$ or $1 \leq u < 2$
$N_{2,1} = 0$, otherwise

...
$N_{n,1} = 1$ if $t_n \leq u < t_{n+1}$ or $n - 1 \leq u < n$

$N_{n,1} = 0$, otherwise

The above base cases plugged into the recursion of the $eqref eq : linbasis$ gives the representation for the bspline basis function for piecewise linear. (Answer).

For the case of the piecewise cubic hermiote interpolant, we have a degree 3 interpolant. Hence, (k-1) is 3, thus k=4. Hence, the relation for deriving the knot values will become:

$$t_i = 0; i < 4$$
$$t_i = i - 3; 4 \leq i \leq n$$
$$t_i = n - 2; i > n$$

Thus, the set of $(n + k + 1) = (n + 5)$ knot values will be:

$$t = (0, 0, 0, 0, 1, 2, \ldots, n - 3, n - 2, n - 2, n - 2, n - 2)$$

(2) in this case translates to :

$$P(u) = \sum_{i=0}^{n} p_0 N_{0,4}(u) + p_1 N_{1,4}(u) + \cdots + p_i N_{i,4}(u) + \cdots + p_n N_{n,4}(u) \tag{5}$$

Where the recurssions over the basis functions can be further expanded following the relation(1), to reach the terminal base cases which in these case will be defined as follows:
$N_{0,1} = 1$ if $t_0 \leq u < t_1$ or $0 \leq u < 0$
$N_{0,1} = 0$, otherwise,
So, $N_{0,1} = 0$

$N_{1,1} = 1$ if $t_1 \leq u < t_2$ or $0 \leq u < 0$
$N_{1,1} = 0$, otherwise,
So, $N_{1,1} = 0$

$N_{2,1} = 1$ if $t_2 \leq u < t_3$ or $0 \leq u < 0$
$N_{2,1} = 0$, otherwise,
So, $N_{2,1} = 0$

$N_{3,1} = 1$ if $t_3 \leq u < t_4$ or $0 \leq u < 1$
$N_{3,1} = 0$, otherwise,

$N_{4,1} = 1$ if $t_4 \leq u < t_5$ or $1 \leq u < 2$
$N_{4,1} = 0$, otherwise,

$N_{5,1} = 1$ if $t_5 \leq u < t_6$ or $2 \leq u < 3$
$N_{5,1} = 0$, otherwise,

...

$N_{n,1} = 1$ if $t_n \leq u < t_{n+1}$ or $(n-3) \leq u < (n-2)$
$N_{n,1} = 0$, otherwise,

Now consider (5), the basis functions recursively depends exactly on 4 base cases if we break them down as follows:

$$N_{0,4} : (N_{0,1}, N_{1,1}, N_{2,1}, N_{3,1})$$
$$N_{1,4} : (N_{1,1}, N_{2,1}, N_{3,1}, N_{4,1})$$
$$N_{2,4} : (N_{2,1}, N_{3,1}, N_{4,1}, N_{5,1})$$
$$N_{3,4} : (N_{3,1}, N_{4,1}, N_{5,1}, N_{6,1})$$

and so on,

In the region , $0 \leq u < 1$, only $N_{3,1} = 1$, other $N'_{i,1}s$ are zero. Now , $N_{3,1}$ is present only in the recursive list of the basis functions associated with (p_0, p_1, p_2, P_3). Hence, in the region $0 \leq u < 1$, only these 4 points serve as the control points with local control over the curve. Other points do not have any effect here. Similarly, in the region $1 \leq u < 2$, only $N_{4,1}$ is 1, which is present in the recursive list of the basis functions associated with (p_1, p_2, p_3, p_4). Hence, in the region $1 \leq u < 2$, only these 4 points serve as the local control points.

Thus plugging in the base cases , $N'_{i,1}s$, present in the recursive list of respective basis functions, we obtain the representation of the bspline basis functions. (Answer).