CS6210 - Homework/Assignment-6

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Question-1: Chapter-13

l2-norm of quad = 0.53115307 l2-norm of cubic = 0.25287738

Question-2: Chapter-14, question-2

Let f(x) be a given function that can be evaluated at points $x_o \pm jh, j = 0, 1, 2, \ldots$ for any fixed value of h, $0 < h << 1$.

(a) Using Taylor series expansion about x_0, we can write the following expansions about $\pm h$:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f''''(x_0) + \frac{h^5}{120}f'''''(\zeta_1)$$

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f''''(x_0) - \frac{h^5}{120}f'''''(\zeta_2)$$

where, $x_0 \le \zeta_1 \le x_0 + h$, and $x_0 - h \le \zeta_2 \le x_0$ The difference of the above two equations gives the following:

$$f(x_0 + h) - f(x_0 - h) = 2\{hf'(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^5}{120}f'''''(\zeta_3)\} \tag{1}$$

where, $x_0 - h \le \zeta_3 \le x_0 + h$

Similarly, we can get the following expansions using nearby points at $x_0 \pm 2h$:

$$f(x_0 + 2h) = f(x_0) + 2hf'(x_0) + \frac{4h^2}{2}f''(x_0) + \frac{8h^3}{6}f'''(x_0) + \frac{16h^4}{24}f''''(x_0) + \frac{32h^5}{120}f'''''(\zeta_4)$$

$$f(x_0 - 2h) = f(x_0) - 2hf'(x_0) + \frac{4h^2}{2}f''(x_0) - \frac{8h^3}{6}f'''(x_0) + \frac{16h^4}{24}f''''(x_0) - \frac{32h^5}{120}f'''''(\zeta_5)$$

where, $x_0 \le \zeta_4 \le x_0 + 2h$, and $x_0 - 2h \le \zeta_5 \le x_0$ The difference of the above two equations gives the following:

$$f(x_0 + 2h) - f(x_0 - 2h) = 2\{2hf'(x_0) + \frac{8h^3}{6}f'''(x_0) + \frac{32h^5}{120}f'''''(\zeta_6)\} \tag{2}$$

where, $x_0 - 2h \le \zeta_6 \le x_0 + 2h$

Then multiplying (1) by 2 and subtracting it from (2), we get:

$$f(x_0 + 2h) - f(x_0 - 2h) - 2f(x_0 + h) + 2f(x_0 - h) = 2h^3 f'''(x_0) + \frac{h^5}{2}f'''''(\zeta)$$

where, $z_0 - 2h \le \zeta \le x_0 + 2h$ Rearranging the terms, we get:

$$f'''(x_0) = \frac{f(x_0 + 2h) - f(x_0 - 2h) - 2f(x_0 + h) + 2f(x_0 - h)}{2h^3} + \frac{-\mathbf{h^2}}{\mathbf{4}}\mathbf{f'''''}(\zeta) \tag{3}$$

(3) provides the formula for approximating the third derivative $f'''(x_0)$, with the bold test showing the component of the truncation error: Thus, truncation error $= \frac{-\mathbf{h^2}}{\mathbf{4}}\mathbf{f'''''}(\zeta)$

(c) To derive the expression for the round-off error, we denote the floating point evaluations of f by F. Thus the expression for round off error will be $|F'''(x_0) - f'''(x_0)|$

$$= \left| \frac{F(x_0 + 2h) - F(x_0 - 2h) - 2F(x_0 + h) + 2F(x_0 - h)}{2h^3} - \frac{f(x_0 + 2h) - f(x_0 - 2h) - 2f(x_0 + h) + 2f(x_0 - h)}{2h^3} \right|$$

$$= \left| \frac{e_r(x_0 + 2h) - e_r(x_0 - 2h) - 2e_r(x_0 + h) + 2e_r(x_0 - h)}{2h^3} \right|$$

where, the round-off error term, $e_r(x) = F(x) - f(x)$

$$\leq \left|\frac{e_r(x_0+2h)}{2h^3}\right| + \left|\frac{e_r(x_0-2h)}{2h^3}\right| + \left|\frac{2e_r(x_0+h)}{2h^3}\right| + \left|\frac{2e_r(x_0-h)}{2h^3}\right|$$

Assuming the each round-off error term is bounded by the machine precision, ϵ, then we get:

$$|F'''(x_0) - f'''(x_0)| \leq \frac{6\epsilon}{2h^3} = \frac{3\epsilon}{h^3}$$

Thus, the round-off error grows inversely proportional to h. Hence, as h decreases, the round-off error increases, thus with $h- > 0$, the round-off error will tend to ∞ .

(d) We can obtain a fourth order formula in a similar manner as we obtained the third order formula with some minor changes. First evaluate $S1 : f(x_0+h) + f(x_0-h)$, then evaluate $S2 : f(x_0+2h) + f(x_0-2h)$. Then subtract $S2 - 4S1$, to extract the fourth order formula.

Question-3: Chapter-14: question-8

Derivation of an approximate formulae for the second derivatives $f''(x_0)$ of a smooth function $f(x)$ using the three points $x_{-1}, x_0 = x_{-1} + h_0, x_1 = x_0 + h_1$, where $h_0 \neq h_1$.

(a) To prove both the given approximations are the same, we need to show: $2[f_{x-1,x_0,x_1}] = \dfrac{g_{1/2} - g_{-1/2}}{(h_0+h_1)/2}$.

Proof:
Using divided difference, we can write:

$$2[f_{x-1,x_0,x_1}] = 2 \times \frac{\dfrac{f(x_1) - f(x_0)}{x_1 - x_0} - \dfrac{f(x_0) - f(x_{-1})}{x_0 - x_{-1}}}{x_1 - x_{-1}}$$

$$2[f_{x-1,x_0,x_1}] = 2 \times \frac{\dfrac{f(x_1) - f(x_0)}{h_1} - \dfrac{f(x_0) - f(x_{-1})}{h_0}}{h_1 + h_0}$$

Given, that: $g_{1/2} = \dfrac{f(x_1) - f(x_0)}{h_1}$ and $g_{-1/2} = \dfrac{f(x_0) - f(x_{-1})}{h_0}$, we can replace them to get:

$$2[f_{x-1,x_0,x_1}] = 2 \times \frac{g_{1/2} - g_{-1/2}}{h_1 + h_0}$$

$$2[f_{x-1,x_0,x_1}] = \frac{g_{1/2} - g_{-1/2}}{\dfrac{h_1 + h_0}{2}}$$

(b) To show that the method is only first order accurate, we use the Taylor series expansion upto second order with third order error term to derive the formualation:

$$f(x_0 + h_1) = f(x_0) + h_1 f'(x_0) + \frac{h_1^2}{2} f''(x_0) + \frac{h_1^3}{3!} f'''(\zeta_1)$$

$$f(x_0 - h_0) = f(x_0) - h_0 f'(x_0) + \frac{h_0^2}{2} f''(x_0) - \frac{h_0^3}{3!} f'''(\zeta_2)$$

Which can be further simplified to:

$$\frac{f(x_1) - f(x_0)}{h_1} = f'(x_0) + \frac{h_1}{2} f''(x_0) + \frac{h_1^2}{3!} f'''(\zeta_1)$$

$$\frac{f(x_{-1}) - f(x_0)}{h_0} = -f'(x_0) + \frac{h_0}{2} f''(x_0) - \frac{h_0^2}{3!} f'''(\zeta_2)$$

Adding these two, we get:

$$\frac{f(x_1) - f(x_0)}{h_1} - \frac{f(x_0) - f(x_{-1})}{h_0} = \frac{(h_0 + h_1)}{2} f''(x_0) + \frac{h_1^2}{3!} f'''(\zeta_1) - \frac{h_0^2}{3!} f'''(\zeta_2)$$

For the error term, replacing with a ζ, such that $x_{-1} \leq \zeta \leq x_1$:

$$\frac{f(x_1) - f(x_0)}{h_1} - \frac{f(x_0) - f(x_{-1})}{h_0} = \frac{(h_0 + h_1)}{2} f''(x_0) + f'''(\zeta)(\frac{h_1^2}{3!} - \frac{h_0^2}{3!})$$

Rearranging to get the second order term:

$$f''(x_0) = \frac{\dfrac{f(x_1) - f(x_0)}{h_1} - \dfrac{f(x_0) - f(x_{-1})}{h_0}}{(h_0 + h_1)/2} - \frac{\mathbf{f'''(\zeta)(h_1 - h_0)}}{\mathbf{3}}$$

The error term shown in bold is thus only first order,$O(h_0 + h_1)$, accurate.

(c) Will be back soon

Question-4: Chapter-14, question-13

Will be back soon

Question-5: Chapter-14, question-15

Consider the numerical differentiation of the function, $f(x) = c(x)e^{x/\pi}$, defined on $[0, \pi]$, where

$$c(x) = j; \frac{\pi}{4}(j-1) \leq x < \frac{\pi}{4} j$$

for $j = 1, 2, 3, 4, \ldots$ **(a)** We want a difference approximation with step size, $h = \frac{n}{\pi}$. Suppose, we have n as multiple of 4, that is, $n = 4l$, where l is an integer. Since, the points are equispaced, thus the entire $[0, \pi]$ region can be seen as partioned into 4 regions each having l points as described below:

L1 :$0 \leq x < \pi/4$: l points at $0, \frac{\pi}{4l}, \frac{2\pi}{4l}, \ldots, \frac{(l-1)\pi}{4l}$

L2 :$\pi/4 \leq x < \pi/2$: l points at $\frac{\pi}{4}, \frac{\pi}{4} + \frac{\pi}{4l}, \frac{\pi}{4} + \frac{2\pi}{4l}, \ldots, \frac{\pi}{4} + \frac{(l-1)\pi}{4l}$

L3 :$\pi/2 \leq x < 3\pi/4$: l points at $\frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{4l}, \frac{\pi}{2} + \frac{2\pi}{4l}, \ldots, \frac{\pi}{2} + \frac{(l-1)\pi}{4l}$

L4 :$3\pi/4 \leq x < \pi$: l points at $\frac{3\pi}{4}, \frac{3\pi}{4} + \frac{\pi}{4l}, \frac{3\pi}{4} + \frac{2\pi}{4l}, \ldots, \frac{3\pi}{4} + \frac{(l-1)\pi}{4l}$

The last point is at π

Now, $c(x)$ is discontinuous at the points of integer multiples of $\pi/4$. To make sure that the discontinuities do not affect out difference formulation it is beneficial to have some of the interval end points at these discontinuities. For example, if we want to approximate the difference value at a point t in the interval $[x_{i-1}, x_i]$, there will be two possibilities while taking the difference. If x_{i-1} and x_i lie in the same region described above, then $c(x)$ has the same values at both the points and can be approximated with the value of $c(x_{i-1})$. If on the other hand, x_{i-1} and x_i lie in two different regions, the only possibility is the x_{i-1} is the last point in one region while x_i is the first point of the other region and hence we have a discontinuity at x_i , however, t will be in the region of x_{i-1} and hence to avoid the discontinuity in the calculation, we can take the left continuous value at x_i, which will be equal to the value at t_i. Hence, with n being multiple of 4, we have the possibility of avoiding the discontinuities resulting in better values. Consider on the other hand, if n was not a multiple of 4, that is, we did not had this nice region partition, then the point t could itself have a discontinuity which we could not be able to derive from the sampling points $[x_{i-1}, x_i]$. Moreoever, suppose we have t in $[x_{i-1}, x_i]$, where $c(x_{i-1}) \neq c(x)$, and t lies not at the point of discontinuity, but either in the step with x_{i-1} or with x_i. All these points lying on either side of the step, will get the same difference value which will be highly inaccurate. These issues we can solve by having n has multiple of 4, and having the nice partitioning described above .

(b)Show that: $h^{-1}c(t_i)(e^{x_{i+1}/\pi} - e^{x_i/\pi})$ provides a second order approximation of $f'(t_i)$

Proof: $t_i = x_i + \frac{h}{2} = ih + \frac{h}{2} = (i + \frac{1}{2})h$; $i = 0, 1, \ldots, (n-1)$

Using Taylor series expansion around the point $t_i = x_i + \frac{h}{2}$, we can write:'

$$f(x_i + \frac{h}{2} - \frac{h}{2}) = f(x_i + \frac{h}{2}) - \frac{h}{2}f'(x_i + \frac{h}{2}) + \frac{h^2}{4}f''(x_i + \frac{h}{2}) - \frac{h^3}{8}f'''(\zeta_1)$$

and,

$$f(x_i + \frac{h}{2} + \frac{h}{2}) = f(x_i + \frac{h}{2}) + \frac{h}{2}f'(x_i + \frac{h}{2}) + \frac{h^2}{4}f''(x_i + \frac{h}{2}) + \frac{h^3}{8}f'''(\zeta_2)$$

Subtracting $f(x_{i+1} = x_i + h) - f(x_i)$ gives,

$$f(x_{i+1}) - f(x_i) = hf'(x_i + \frac{h}{2}) + \frac{h^3}{8}\{f'''(\zeta_1) + f'''(\zeta_2)\}$$

where $t_i - h/2 \leq \zeta_1 \leq t_i$ and $t_i \leq \zeta_2 \leq t_i + h/2$. Choosing a ζ, such that, $t_i - h/2 \leq \zeta \leq t_i + h/2$, we can then write:

$$f(x_{i+1}) - f(x_i) = hf'(x_i + \frac{h}{2}) + \frac{h^3}{4}\{f'''(\zeta)\}$$

Since, $t_i = x_i + h/2$,

$$f(x_{i+1}) - f(x_i) = hf'(t_i) + \frac{h^3}{4}\{f'''(\zeta)\}$$

$$f'(t_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{h^2}{4}f'''(\zeta)$$

$$f'(t_i) = \frac{c(x_{i+1})e^{x_{i+1}/\pi} - c(x_i)e^{x_i/\pi}}{h} - \frac{h^2}{4}f'''(\zeta)$$

As, described in first part of this question, if the end-points of the interval$[x_i, x_{i+1}]$, t_i belongs to falls completely in one of the regions, then we have $c(x_i) = c(t_i) = c(x_{i+1})$. If x_i and x_{i+1} fall in different region, then x_{i+1} falls in the point of discontinuity, so we take the left continuous value at x_{i+1} which will be equal to $c(t_i) = c(x_i)$. Thus, in the above expression we can replace $c(x_i), c(x_{i+1})$ with $c(t_i)$. Thus we get the following expression:

$$f'(t_i) = \frac{c(t_i)e^{x_{i+1}/\pi} - c(t_i)e^{x_i/\pi}}{h} - \frac{h^2}{4}f'''(\zeta)$$

$$f'(t_i) = \frac{c(t_i)(e^{x_{i+1}/\pi} - e^{x_i/\pi})}{h} - \mathbf{\frac{h^2}{4}f'''}(\zeta) \tag{4}$$

(4) shows the second order approximantion of $f'(t_i)$

Question-6: Chapter-15, question-4

(a) Prove: Error in basic corrected trapezoidal rule in the interval [a,b] can be estimated by:

$$E(f) = \frac{f'''' * (\eta)}{720}(b-a)^5$$

Proof: The osculating polynomial formula for the basic corrected trapezoidal rule is written as:

$$p_3(x) = f(a) + f'(a)(x-a) + f[a,a,b](x-a)^2 + f[a,a,b,b](x-a)^2(x-b)$$

The error in the polynomial interpolation in that case will be fiven by:

$$f[a,a,b,b,x](x-a)(x-a)(x-b)(x-b)$$

Then, to find the error in the intergral of the polynomial, we can integrate the error of polynomial described above, in the interval [a,b]

$$E(f) = \int_a^b f[a,a,b,b,x]\psi(x)$$

where $\psi(x) = \prod_{i=0}^{i=3}(x-x_i) = (x-a)(x-a)(x-b)(x-b)$
Notice that, since x lies in the interval [a,b], hence $(x-a \geq 0) and (x-b \leq 0)$. In any case, the square of the terms will be greater than equal to 0. So, in the given interval $\psi(x) \geq 0$ always. Because $\psi(x)$ does not changes sign in the interval, then using the intermediate value theorem, there exists $a \leq \eta \leq b$, such that:

$$E(f) = \int_a^b f[a,a,b,b,x]\psi(x) = \int_a^b f[a,a,b,b,\eta]\psi(x)$$

where,

$$f[a,a,b,b,\eta] = \frac{f''''(\eta)}{4!}$$

which is a constant, say K.
Then we can write the error integral as:

$$E(f) = K\int_a^b (x-a)^2(x-b)^2$$

Doing integration by parts:

$$E(f) = K\left[\frac{(x-a)^2(x-b)^3}{3} - \frac{(x-a)(x-b)^4}{6} + \frac{(x-b)^5}{30}\right]_a^b = \frac{-(a-b)^5}{30}$$

Replacijng back K,we get:

$$E(f) = \frac{f''''(\eta)}{4!}\frac{(-(a-b)^5)}{30} = \frac{f''''(\eta)(b-a)^5}{720}$$

(b) The integral for the corrected trapezoidal is written is:

$$I_f \approx \int_a^b p_3(x)dx = \frac{(b-a)}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(a)-f'(b)]$$

part-1: For the integral $\int_0^1 e^x dx$, thus $a = 0, b = 1$, and $f(x) = e^x$, $f'(x) = e^x$ .So, $f(a) = 1, f(b) = e, f'(a) = 1, f'(b) = e$
Using the basic corrected trapezoidal, we get:

$$\int_0^1 e^x dx = 1.71595$$

the acutal evaluation is $1.7183\ldots$ while the basic trapezoidal evaluation from Example-15.2 is $1.7183\ldots$. Hence, the evaluation using the basic corrected trapezoidal is more accurate than the basic trapezoidal.

part-2: For the integral $\int_{0.9}^{1} e^x dx$, thus $a = 0.9, b = 1$, and $f(x) = e^x$, $f'(x) = e^x$ .So, $f(a) = e^{0.9}, f(b) = e, f'(a) = e^{0.9}, f'(b) = e$
Using the basic corrected trapezoidal, we get:

$$\int_{0.9}^{1} e^x dx = 0.258678$$

The actual evaluation is $0.2586787171\ldots$, while the basic trapezoidal evaluation from Example-15.2 is $0.258894\ldots$. hence, the evaluation using the basic corrected trapezoidal is more accurate than the basic trapezoidal.

Question-7: Chapter-15, question-5

(a) In the interval [a,b], the basic midpoint rule is given as :

$$I_f \approx (b-a)f(\frac{(a+b)}{2}) \tag{5}$$

For the composite midpoint rule, we consider r subintervals in the original interval [a,b] and apply the basic midpoint rule to each subinterval and then sum over all the subintervals to get the composite integral. The rule applied to an interval $[t_{i-1}, t_i]$, such that the interval widths are uniform and $t_i - t_{i-1} = h = \frac{b-a}{r}$, will be:

$$\int_{t_{i-1}}^{t_i} f(x)dx \approx hf(\frac{t_{i-1}+t_i}{2})$$

Summing over all the subintervals to get the complete composite integral:

$$\int_a^b f(x)dx = h\sum_{i=1}^{r} f(\frac{t_{i-1}+t_i}{2})$$

For, r equispaced intervals over [a,b], we have the interval width as $h = \frac{b-a}{r}$ Then, $t_0 = a, t_1 = a+h, t_2 = a+2h, \ldots, t_i = a+ih$.So,

$$\frac{t_{i-1}+t_i}{2} = \frac{a+(i-1)h+a+ih}{2} = a+(i-\frac{1}{2})h$$

Replacing it in the orginal integral, we get the final form for the composite midpoint as:

$$\int_a^b f(x)dx \approx h\sum_{i=1}^{r} f(a+(i-\frac{1}{2})h)$$

From the above expression , we can see that there is one function evaluation per subinterval. Hence , the number of function evaluations is $r = \frac{b-a}{h}$

(b) Derive an expression for the error in composite midpoint rule
For the basic midpoint rule, the error expression in the interval [a,b] is given by:

$$E(f) = \frac{f''(\eta)}{24}(b-a)^3$$

This comes from doing the following integral:

$$\frac{f''}{2!}(\eta)\int_a^b (x-\frac{a+b}{2})(x-\frac{a+b}{2}))$$

The reason for adding the second term of $(x-\dfrac{a+b}{2})$ even though there is only a single point, is that $(x-\dfrac{a+b}{2})$ can change signs within the interval [a,b], and hence we cannot apply the indermediate value theorem to take out $f''(\eta)$ as a constant, for $a \le \eta \le b$. So, we duplicate the point $\dfrac{a+b}{2}$ as a dummy interpolation point, since it does not changes the area evaluated and hence the error term should remain the same.

Next, we come to the derivation of the error for the composite midpoint rule. In the composite cases, we have divided the original interval, [a,b], into r equispaced sub-intervals of width h, such that $r = \dfrac{b-a}{h}$. For evaluating the composite integral using mid-point we applied the basic midpoint to each of these sub-intervals and summed them. Similarly, now each evaluation of the basic midpoint in these intervals will result in an error , which we can further sum up to get the expression for the error in composite mid-point. The error term in the interval $[t_{i-1}, t_i]$ will be $\dfrac{f''(\eta_i)}{24}h^3$, where $\dfrac{f''(\eta_i)}{24}$ is a contstant in the interval $[t_{i-1}, t_i]$

Hence, the expression of error for the composite midpoint will be:

$$E_{CM}(f) = \sum_{i=1}^{r} \frac{f''(\eta_i)}{24} h^3$$

Since, $f''(\eta_i)$ is a constant in the interval $[t_{i-1}, t_i]$, we can generalize it with an appropriate constant $f''(\eta)$ where $a \le \eta \le b$. Then the error expression comes to be:

$$E_{CM}(f) = \frac{f''(\eta)}{24} \sum_{i=1}^{r} h.h^2$$

$$E_{CM}(f) = \frac{f''(\eta)}{24} h^2.rh$$

Since, $h = \dfrac{b-a}{r}$, so replacing $(b-a) = rh$, we get:

$$E_{CM}(f) = \frac{f''(\eta)}{24}(b-a)h^2 \tag{6}$$

This is the final expression for the error in the composite mid point rule.
(6) Suggests that the error varies proportional to h^2, hence it is second order accurate.

Question-8: Chapter-15, question-13

Given that the interval of integration, [a,b], is divided into equal sub-intervals of length h, such that $r = \dfrac{b-a}{h}$

Composite Simpson:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 2\sum_{k=1}^{\frac{r}{2}-1} f(t_{2k}) + 4\sum_{k=1}^{\frac{r}{2}} f(t_{2k-1}) + f(b)\right] \tag{7}$$

The expression for composite trapezoidal with step size h is given by:

R2: Composite trapezoidal rule of step size h

$$\int_a^b \approx \frac{h}{2}\sum_{i=1}^{r} f(t_{i-1}) + f(t_i)$$

$$R_2 = \frac{h}{2}[f(a) + 2f(t_1) + 2f(t_2) + \cdots + 2f(t_{r-1}) + f(b)]$$

R1: Composite trapezoidal rule of step size 2h For step-size of 2h, we reuqire even number of subintervals. In the above expression for summation,thus we change the summing variable i to $2k$, and the limit become $\frac{r}{2}$. Hence, we have:

$$R_1 = \frac{2h}{2} \sum_{k=1}^{\frac{r}{2}} f(t_{2k-2}) + f(t_{2k})$$

$$R_1 = h[\{f(t_0) + f(t_2) + \cdots + f(t_{r-2})\} + \{f(t_2) + f(t_4) + \cdots + f(t_r)\}]$$

Since, t_0 and t_r are the two extreme end points of the interval, hence $t_0 = a$ and $t_r = b$ Thus, we get:

$$R_1 = h[f(a) + 2f(t_2) + 2f(t_4) + \cdots + 2f(t_{r-2}) + f(b)]$$

Hence, evaluating $S = \frac{4R_2 - R_1}{3}$

$$4R_2 - R_1 \; = h[2f(a) + 4f(t_1) + 4f(t_2) + \cdots + 4f(t_{r-1}) + 2f(b)] - h[f(a) - 2f(t_2) - 2f(t_4) - \cdots - f(b)]$$

$$4R_2 - R_1 = h[f(a) + \{2f(t_2) + 2f(t_4) + \cdots + 2f(t_{r-2})\} + \{4f(t_1) + 4f(t_3) + \cdots + 4f(t_{r-1})\} + f(b)]$$

$$4R_2 - R_1 = h[f(a) + 2\sum_{k=1}^{\frac{r}{2}-1} f(t_{2k}) + 4\sum_{k=1}^{\frac{r}{2}} f(t_{2k-1}) + f(b)]$$

$$\frac{4R_2 - R_1}{3} = \frac{h}{3}[f(a) + 2\sum_{k=1}^{\frac{r}{2}-1} f(t_{2k}) + 4\sum_{k=1}^{\frac{r}{2}} f(t_{2k-1}) + f(b)]$$

The rhs of the above is exactly the expression for the composite Simson's rule (7)