CS6210 - Homework/Assignment-3

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Question-1: Chapter-4: Exercise-12

The condition number of an eigen value, λ , of a matrix A is defined as

$$s(\lambda) = \frac{1}{\mathbf{x}^T\mathbf{w}}$$

Referencing from example 4.7/4.6, let is define the two matrices as:
$A_1 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

Both the matrices have eigen value of 4 with algebraic multiplicaity 2, that is both its eigen values are 4,4. Now first let us consider A_1. Its eigen vectors corresponding to eigen valuesm of 4 are $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus the geometric multiplicity is also 2. The left eigen vectors, w^T, will be $w_1^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $w_2^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

Thus, for each of the above eigen vectors for A_1, the inner product $\frac{1}{\mathbf{x}^T\mathbf{w}}$ is 1, hence the condition number turns out to be,

$$S(\lambda = 4)_{A_1} = 1 \tag{1}$$

Let us consider A_2 for now. The eigen values for A_2 is 4 with algebraic multiplicity 2. However , it has only one right eigen vector, $x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and only one left eigen vector, $w_1^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

Thus, for the above pair of left and right eigen vector for A_2, the inner product $\frac{1}{\mathbf{x}^T\mathbf{w}}$ is $\frac{1}{0} = \infty$. Hence the condition number turns out to be

$$S(\lambda = 4)_{A_2} = \infty \tag{2}$$

Condition numbers indicate how stable the computation is expected to be, such that lower computation numbers indicate more stability. If we refer to example-4.7, where the experiment was done with small perturbation to the matrix, A_1 came to be well-conditioned while A_2 came to be ill-conditioned. Our evaluation of the condition number also suggests that since condition number for A_1 is small it is numerically more stable and hence well conditioned while A_2 has condition number number of ∞ and hence ill-conditioned.

Question-2: Chapter-5: Exercise-2

For an $n \times n$ matrix A, and a vector vector b, the pseudoCode for the Gauss-Jordan elimination method for Solving $Ax = b$, is as described below(Assuming no pivoting):

(a) PseudoCode:

for k=1 : $n-1$
    for i=1 : n
        if (i $\neq$ k)
            $l_{i,k} = \frac{a_{i,k}}{a_{k,k}}$
            for $j = k+1 : n$
                $a_{i,j} = a_{i,j} - l_{i,k}a_{k,j}$
            $b_i = b_i - l_{i,k}b_k$

Since, it does the update for all rows except the row k, one **if** condition is introduced to check for $i \neq k$, and the row traversal instead of $k+1$ to n, has been increased as 1 to n.

(b) The cost of the Gauss-Jordan algorithm in terms of operation count(or flop count) is as follows:

$$
\begin{aligned}
&\sum_{k=1}^{n-1} 2(n-1)(n-k) + 2(n-1) + (n-1) \\
&\qquad = \sum_{k=1}^{n-1} 2(n-1)(n-k+1) + (n-1) \\
&\qquad = (n-1)\sum_{k=1}^{n-1} 2(n-k+1) + 1 \\
&\qquad = (n-1)\sum_{k=1}^{n-1} (2n-2k+3) \\
&\qquad = (n-1)\Bigg(2n(n-1) - 2\frac{n(n-1)}{2} + 3(n-1)\Bigg) \\
&\qquad = (n-1)^2\Bigg(2n - n + 3\Bigg) \\
&\qquad = (n-1)^2(n+3) \\
&\qquad = n^3 - 2n^2 + n + 3n^2 - 6n + 3 \\
&\qquad = n^3 + O(n^2)
\end{aligned}
$$

(Proved).

Question-3: Chapter-5: Exercise-3

Let A and T be two non-singular, $n \times n$ matrices. Furthermore, we are given two matrices, L and U such that L is unit lower triangular and U is upper triangular and the following relation holds:

$$TA = LU \tag{3}$$

The algorithm to find the solution for $Ax = b$ is detailed below

Algorithm: :
Step-1: Perform a matvec operation to evaluate **Tb**.
Step-2: Solve for y: Ly = Tb ; //By forward substitution
Step-3: Solve for x: Ux = y ; //By backward substituion

Explanation: To evaluate Ax=b, from the given conditions, we first perform a matvec of T and b which is $O(n^2)$, since it invloves a matrix-vector multiplication of $n \times n$ matrix T, and a $n \times 1$ vector b. In second step we solve for y using forward substitution since there is a lower triangular matrix. This step also involves $O(n^2)$ operations. The third step involves the evaluation of the final solution **x**, and since there is an upper triangular matrix, we use backward substitution which is again $O(n^2)$. Thus total flops required is in order of $O(n^2)$.

PseudoCode: :