CS6210 - Homework/Assignment-6

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December 7, 2016

Question-6: Chapter-15, question-4

(a) Prove: Error in basic corrected trapezoidal rule in the interval [a,b] can be estimated by:

$$E(f) = \frac{f'''' * (\eta)}{720}(b-a)^5$$

Proof: The osculating polynomial formula for the basic corrected trapezoidal rule is written as:

$$p_3(x) = f(a) + f'(a)(x-a) + f[a,a,b](x-a)^2 + f[a,a,b,b](x-a)^2(x-b)$$

The error in the polynomial interpolation in that case will be fiven by:

$$f[a,a,b,b,x](x-a)(x-a)(x-b)(x-b)$$

Then, to find the error in the intergral of the polynomial, we can integrate the error of polynomial described above, in the interval [a,b]

$$E(f) = \int_a^b f[a,a,b,b,x]\psi(x)$$

where $\psi(x) = \prod_{i=0} i = 3(x-x_i) = (x-a)(x-a)(x-b)(x-b)$
Notice that, since x lies in the interval [a,b], hence $(x-a \geq 0) and (x-b \leq 0)$. In any case, the square of the terms will be greater than equal to 0. So, in the given interval $\psi(x) \geq 0$ always. Because $\psi(x)$ does not changes sign in the interval, then using the intermediate value theorem, there exists $a \leq \eta \leq b$, such that:

$$E(f) = \int_a^b f[a,a,b,b,x]\psi(x) = \int_a^b f[a,a,b,b,\eta]\psi(x)$$

where,

$$f[a,a,b,b,\eta] = \frac{f''''(\eta)}{4!}$$

which is a constant, say K.
Then we can write the error integral as:

$$E(f) = K\int_a^b (x-a)^2(x-b)^2$$

Doing integration by parts:

$$E(f) = K\left[\frac{(x-a)^2(x-b)^3}{3} - \frac{(x-a)(x-b)^4}{6} + \frac{(x-b)^5}{30}\right]_a^b = \frac{-(a-b)^5}{30}$$

Replacijng back K,we get:

$$E(f) = \frac{f''''(\eta)}{4!}\frac{(-(a-b)^5)}{30} = \frac{f''''(\eta)(b-a)^5}{720}$$

(b) The integral for the corrected trapezoidal is written is:

$$I_f \approx \int_a^b p_3(x)dx = \frac{(b-a)}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(a)-f'(b)]$$

part-1: For the integral $\int_0^1 e^x dx$, thus $a=0, b=1$, and $f(x) = e^x$, $f'(x) = e^x$ .So, $f(a) = 1, f(b) = e, f'(a) = 1, f'(b) = e$
Using the basic corrected trapezoidal, we get:

$$\int_0^1 e^x dx = 1.71595$$

the acutal evaluation is $1.7183\ldots$ while the basic trapezoidal evaluation from Example-15.2 is $1.7183\ldots$. Hence, the evaluation using the basic corrected trapezoidal is more accurate than the basic trapezoidal.

part-2: For the integral $\int_{0.9}^{1} e^x dx$, thus $a = 0.9, b = 1$, and $f(x) = e^x$, $f'(x) = e^x$ .So, $f(a) = e^{0.9}, f(b) = e, f'(a) = e^{0.9}, f'(b) = e$
Using the basic corrected trapezoidal, we get:

$$\int_{0.9}^{1} e^x dx = 0.258678$$

The actual evaluation is $0.2586787171\ldots$, while the basic trapezoidal evaluation from Example-15.2 is $0.258894\ldots$. hence, the evaluation using the basic corrected trapezoidal is more accurate than the basic trapezoidal.

Question-7: Chapter-15, question-5

(a) In the interval [a,b], the basic midpoint rule is given as :

$$I_f \approx (b-a)f(\frac{(a+b)}{2}) \tag{1}$$

For the composite midpoint rule, we consider r subintervals in the original interval [a,b] and apply the basic midpoint rule to each subinterval and then sum over all the subintervals to get the composite integral. The rule applied to an interval $[t_{i-1}, t_i]$, such that the interval widths are uniform and $t_i - t_{i-1} = h = \frac{b-a}{r}$, will be:

$$\int_{t_{i-1}}^{t_i} f(x)dx \approx hf(\frac{t_{i-1}+t_i}{2})$$

Summing over all the subintervals to get the complete composite integral:

$$\int_a^b f(x)dx = h\sum_{i=1}^{r} f(\frac{t_{i-1}+t_i}{2})$$

For, r equispaced intervals over [a,b], we have the interval width as $h = \frac{b-a}{r}$ Then, $t_0 = a, t_1 = a+h, t_2 = a+2h, \ldots, t_i = a+ih$.So,

$$\frac{t_{i-1}+t_i}{2} = \frac{a+(i-1)h+a+ih}{2} = a+(i-\frac{1}{2})h$$

Replacing it in the orginal integral, we get the final form for the composite midpoint as:

$$\int_a^b f(x)dx \approx h\sum_{i=1}^{r} f(a+(i-\frac{1}{2})h)$$

From the above expression , we can see that there is one function evaluation per subinterval. Hence , the number of function evaluations is $r = \frac{b-a}{h}$

(b) Wait for Sourabh's response

Question-8: Chapter-15, question-13

Given that the interval of integration, [a,b], is divided into equal sub-intervals of length h, such that $r = \dfrac{b-a}{h}$

Composite Simpson:

$$\int_a^b f(x)dx \approx \frac{h}{3}\Bigg[f(a) + 2\sum_{k=1}^{\frac{r}{2}-1} f(t_{2k}) + 4\sum_{k=1}^{\frac{r}{2}} f(t_{2k-1}) + f(b)\Bigg] \tag{2}$$

The expression for composite trapezoidal with step size h is given by:

R1: Composite trapezoidal rule of step size h

$$\int_a^b \approx \frac{h}{2}\sum_{i=1}^{r} f(t_{i-1}) + f(t_i)$$

$$R_2 = \frac{h}{2}[f(a) + 2f(t_1) + 2f(t_2) + \cdots + 2f(t_{r-1}) + f(b)]$$

R2: Composite trapezoidal rule of step size 2h For step-size of 2h, we reuqire even number of subintervals. In the above expression for summation,thus we change the summing variable i to $2k$, and the limit become $\frac{r}{2}$. Hence, we have:

$$R_2 = \frac{2h}{2}\sum_{k=1}^{\frac{r}{2}} f(t_{2k-2}) + f(t_{2k})$$

$$R_2 = h[\{f(t_0) + f(t_2) + \cdots + f(t_{r-2})\} + \{f(t_2) + f(t_4) + \cdots + f(t_r)\}]$$

Since, t_0 and t_r are the two extreme end points of the interval, hence $t_0 = a$ and $t_r = b$ Thus, we get:

$$R_2 = h[f(a) + 2f(t_2) + 2f(t_4) + \cdots + 2f(t_{r-2}) + f(b)]$$

Hence, evaluating $S = \dfrac{4R_2 - R_1}{3}$

$$4R_2 - R_1 \ = h[2f(a) + 4f(t_1) + 4f(t_2) + \cdots + 4f(t_{r-1}) + 2f(b)] - h[f(a) - 2f(t_2) - 2f(t_4) - \cdots - f(b)]$$

$$4R_2 - R_1 = h[f(a) + \{2f(t_2) + 2f(t_4) + \cdots + 2f(t_{r-2})\} + \{4f(t_1) + 4f(t_3) + \cdots + 4f(t_{r-1})\} + f(b)]$$

$$4R_2 - R_1 = h[f(a) + 2\sum_{k=1}^{\frac{r}{2}-1} f(t_{2k}) + 4\sum_{k=1}^{\frac{r}{2}} f(t_{2k-1}) + f(b)]$$

$$\frac{4R_2 - R_1}{3} = \frac{h}{3}[f(a) + 2\sum_{k=1}^{\frac{r}{2}-1} f(t_{2k}) + 4\sum_{k=1}^{\frac{r}{2}} f(t_{2k-1}) + f(b)]$$

The rhs of the above is exactly the expression for the composite Simson's rule (2)