Notes on Homodyne Measurement

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1 Notations

- $\hat{\mathbf{x}} = (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_n, \hat{p}_n)^T$, vector of cannonical operators.

- $\Omega = \bigoplus_{j=1}^{n} \Omega_1$, where $\Omega_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
  Note that, for $n = 1$, $[\hat{x}_i, \hat{x}_j] = i[\Omega_1]_{ij}$. Compactly,

$$[\hat{\mathbf{x}}, \hat{\mathbf{x}}^T] = i\Omega, \qquad \text{(Canonical Commutation Relation)}$$

  where, think the commutation relation as element wise commutator. Some properties of Ω:

  * $\Omega^T = -\Omega$ (Anti-symmetric)
  * $\Omega^2 = \mathbb{1}_{2n} \implies \Omega^T \Omega = \mathbb{1}_{2n}$ (Orthogonal)

- Borrowing from the optical and field-theoretical terminologies, canonical degrees of freedom are also referred to as '*modes*'.

- $\hat{a}_j = \frac{\hat{x}_j + \hat{p}_j}{\sqrt{2}}$, annihilation operator.

- **BCH formula:** $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ for operators A, B if $[A, [A, B]] = [B, [B, A]] = 0$

2 Prerequisits

2.1 Displacement operators

Definition 1 (Weyl operators).

$$\hat{D}_\xi = e^{i\xi^T \Omega \hat{\boldsymbol{x}}} = e^{i(\hat{x}_1 \xi_2 - \hat{p}_1 \xi_2)} \otimes \cdots \otimes e^{i(\hat{x}_n \xi_{2n} - \hat{p}_n \xi_{2n-1})}, \tag{1}$$

where, $\xi \in \mathbb{R}^{2n}$.

Properties:

- $\hat{D}_\xi^\dagger \hat{D}_\xi = \mathbb{1}$ (Unitary operator).

- $\hat{D}_\xi \hat{D}_\xi = \hat{D}_{2\xi}$.

- $\hat{D}_\xi \hat{D}_\eta = e^{-\frac{i}{2}\xi^T \Omega \eta} \, \hat{D}_{\xi+\eta}$. (**Prove!**)

- $\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}\hat{D}_{\bar{\xi}} = \hat{\mathbf{x}} - \bar{\xi}$

  Proof: For the kth component of $\hat{\mathbf{x}}$ *i.e.* $\hat{x}_k$, we have,

  $$\hat{D}_{-\xi}\hat{x}_k\hat{D}_\xi = e^{-i\xi^T\Omega\hat{\mathbf{x}}}\hat{x}_k e^{i\xi^T\Omega\hat{\mathbf{x}}}$$
  $$= \hat{x}_k - i\left[\xi^T\Omega\hat{\mathbf{x}}, \hat{x}_k\right] + \frac{i^2}{2}\left[\xi^T\Omega\hat{\mathbf{x}}, \left[\xi^T\Omega\hat{\mathbf{x}}, \hat{x}_k\right]\right] + \cdots (using\ BCH)$$
  $$= \hat{x}_k - \xi_k$$

  From here the result follows directly.

- $\hat{D}_{-\bar{\xi}} = \hat{D}^\dagger_{\bar{\xi}}$.

2.2 Symplectic Group

TODO: Linear canonical transformation and Symplectic group, Canonical transformations are those which respect **CCR**.

Definition 2 (Symplectic group).

$$S \in Sp_{2n,\mathbb{R}} \iff S\Omega S^T = \Omega \tag{2}$$

Evolution:

2.3 Normal Modes

TODO: Definition, etc.

3 Gaussian States

3.1 Quadratic Hamiltonian and evolution

The most general quadratic/second-order hamiltonian can be written as follows.

$$\hat{H} = \frac{1}{2}\hat{\mathbf{x}}^T H\hat{\mathbf{x}} + \hat{\mathbf{x}}^T\xi. \tag{3}$$

Here, ξ is a $2n$-dimensional real vector. H is a $2n \times 2n$ symmetric matrix called *Hamiltonian matrix*, not to be confused with Hamiltonian. It can always be taken as a symmetric matrix because, the antisymmetric part with give a term proportional to identity matrix due to **CCR**, which can always be discarded. If we take $\bar{\xi} = H^{-1}\xi$, then $\hat{H}' = \frac{1}{2}(\hat{\mathbf{x}} - \bar{\xi})^T H(\hat{\mathbf{x}} - \bar{\xi})$ is equivalent to $\hat{H}$ up to some additive constant term. Using the fourth property from section 2.1 we can write,

$$\hat{H}' = \frac{1}{2}(\hat{\mathbf{x}} - \bar{\xi})^T H(\hat{\mathbf{x}} - \bar{\xi}) = \frac{1}{2}\sum_{jk}(\hat{\mathbf{x}} - \xi)_j H_{jk}(\hat{\mathbf{x}} - \xi)_j$$
$$= \frac{1}{2}\sum_{jk}(\hat{D}_{-\xi}\hat{x}_j\hat{D}_\xi)H_{jk}(\hat{D}_{-\xi}\hat{x}_k\hat{D}_\xi)$$
$$= \frac{1}{2}\sum_{jk}(\hat{D}_{-\xi}\hat{x}_j H_{jk}\hat{x}_k\hat{D}_\xi)$$
$$\hat{H}' = \frac{1}{2}\hat{D}_{-\bar{\xi}}(\hat{\mathbf{x}}^T H\hat{\mathbf{x}})\hat{D}_{\bar{\xi}} \tag{4}$$

One takeaway from the above is that, it's enough to study the property of the hamiltonian, $\hat{H} = \frac{1}{2}\hat{\mathbf{x}}^T H \hat{\mathbf{x}}$.

Evolution of quadratures:

- Heisenberg picture evolution:

$$\dot{\hat{O}} = i[\hat{H}, \hat{O}]. \tag{5}$$

- Using the above we get, time evolution of quadratures as, $\dot{\hat{\mathbf{x}}} = \Omega H \hat{\mathbf{x}}$. Solving it gets,

$$\hat{\mathbf{x}}(t) = e^{\Omega H} \hat{\mathbf{x}}(0) \tag{6}$$

3.2 Gaussian state

Definition 3 (Gaussian State). *Gaussian states are defined as all the ground and thermal states of second-order Hamiltonians [eq.3] with positive definite Hamiltonian matrix H > 0.*

Thus a *Gaussian state* can be written as,

$$\rho_G = \frac{e^{-\beta \hat{H}}}{\text{Tr}\left[e^{-\beta \hat{H}}\right]}, \tag{7}$$

where, $\beta > 0$ and $\hat{H}$ is defined in Eq. 3. Ground state is the limiting value,

$$\rho_G = \lim_{\beta \to \infty} \frac{e^{-\beta \hat{H}}}{\text{Tr}\left[e^{-\beta \hat{H}}\right]}. \tag{8}$$

Note:

- All Gaussian states are mixed state by construction, except for the ground state.

- Gaussian states are parametrized by β, ξ and H. Though β is redundant and can be absorbed into H, it allows one to single out pure Gausian states as a limiting case like in Eq. 8.

- Gaussian states can be generated First and second moment of quadrature. We'll talk about them later.

4 Gaussian operations

Gaussian operations are CP-maps those take Gaussian states to Gaussian states.

4.1 Gaussian Unitaries

One may write most general second order hamiltonian of n-modes as, $\hat{H} = \frac{1}{2}\hat{D}_{-\bar{\xi}}(\hat{\mathbf{x}}^T H \hat{\mathbf{x}})\hat{D}_{\bar{\xi}}$. First note that, $(\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}\hat{D}_{\bar{\xi}})(\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}\hat{D}_{\bar{\xi}}) = \hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}^2\hat{D}_{\bar{\xi}}$. Then, it is clear that, $e^{i\hat{H}} = \hat{D}_{-\bar{\xi}}e^{\frac{i}{2}\hat{\mathbf{x}}^T H \hat{\mathbf{x}}}\hat{D}_{\bar{\xi}}$.

Let's introduce some notations and relations:

- $\hat{S} = e^{\frac{i}{2}\hat{\mathbf{x}}^T H \hat{\mathbf{x}}}$ (Free Hamiltonian unitary)

- $S = e^{\Omega H}$ (Evolution of quadratures, Eq. 6)

- $\hat{S}\hat{\mathbf{x}}\hat{S}^\dagger = S\hat{\mathbf{x}}$ (**Check!**)

- Now using the above we can write

$$\hat{S} \tag{9}$$