

# Notes on Homodyne Measurement

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January 15, 2025

## 1 Notations

- $\hat{\mathbf{x}} = (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_n, \hat{p}_n)^T$ , vector of canonical operators.

- $\Omega = \bigoplus_{j=1}^n \Omega_1$ , where  $\Omega_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ .

Note that, for  $n = 1$ ,  $[\hat{x}_i, \hat{x}_j] = i[\Omega_1]_{ij}$ . Compactly,

$$[\hat{\mathbf{x}}, \hat{\mathbf{x}}^T] = i\Omega, \quad (\text{Canonical Commutation Relation})$$

where, think the commutation relation as element wise commutator. Some properties of  $\Omega$ :

$$* \quad \Omega^T = -\Omega \text{ (Anti-symmetric)}$$

$$* \quad \Omega^2 = \mathbb{1}_{2n} \implies \Omega^T \Omega = \mathbb{1}_{2n} \text{ (Orthogonal)}$$

- Borrowing from the optical and field-theoretical terminologies, canonical degrees of freedom are also referred to as ‘*modes*’.
- $\hat{a}_j = \frac{\hat{x}_j + i\hat{p}_j}{\sqrt{2}}$ , annihilation operator.
- **BCH formula:**  $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$  for operators  $A, B$  if  $[A, [A, B]] = [B, [B, A]] = 0$

## 2 Prerequisites

### 2.1 Displacement operators

**Definition 1** (Weyl operators).

$$\hat{D}_\xi = e^{i\xi^T \Omega \hat{\mathbf{x}}} = e^{i(\hat{x}_1 \xi_2 - \hat{p}_1 \xi_2)} \otimes \dots \otimes e^{i(\hat{x}_n \xi_{2n} - \hat{p}_n \xi_{2n-1})}, \quad (1)$$

where,  $\xi \in \mathbb{R}^{2n}$ .

**Properties:**

- $\hat{D}_\xi^\dagger \hat{D}_\xi = \mathbb{1}$  (Unitary operator).
- $\hat{D}_\xi \hat{D}_\xi = \hat{D}_{2\xi}$ .
- $\hat{D}_\xi \hat{D}_\eta = e^{-\frac{i}{2}\xi^T \Omega \eta} \hat{D}_{\xi+\eta}$ . (**Prove!**)

- $\hat{D}_{-\bar{\xi}} \hat{\mathbf{x}} \hat{D}_{\bar{\xi}} = \hat{\mathbf{x}} - \bar{\xi}$

*Proof:* For the  $k^{\text{th}}$  component of  $\hat{\mathbf{x}}$  i.e.  $\hat{x}_k$ , we have,

$$\begin{aligned} \hat{D}_{-\xi} \hat{x}_k \hat{D}_{\xi} &= e^{-i\xi^T \Omega \hat{\mathbf{x}}} \hat{x}_k e^{i\xi^T \Omega \hat{\mathbf{x}}} \\ &= \hat{x}_k - i [\xi^T \Omega \hat{\mathbf{x}}, \hat{x}_k] + \frac{i^2}{2} [\xi^T \Omega \hat{\mathbf{x}}, [\xi^T \Omega \hat{\mathbf{x}}, \hat{x}_k]] + \dots (\text{using BCH}) \\ &= \hat{x}_k - \xi_k \end{aligned}$$

From here the result follows directly.

- $\hat{D}_{-\bar{\xi}} = \hat{D}_{\bar{\xi}}^\dagger$ .

## 2.2 Symplectic Group

**TODO:** Linear canonical transformation and Symplectic group, Canonical transformations are those which respect **CCR**.

**Definition 2** (Symplectic group).

$$S \in Sp_{2n, \mathbb{R}} \iff S \Omega S^T = \Omega \quad (2)$$

**Evolution:**

## 2.3 Normal Modes

**TODO:** Definition, etc.

# 3 Gaussian States

## 3.1 Quadratic Hamiltonian and evolution

The most general quadratic/second-order hamiltonian can be written as follows.

$$\hat{H} = \frac{1}{2} \hat{\mathbf{x}}^T H \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \xi. \quad (3)$$

Here,  $\xi$  is a  $2n$ -dimensional real vector.  $H$  is a  $2n \times 2n$  symmetric matrix called *Hamiltonian matrix*, not to be confused with Hamiltonian. It can always be taken as a symmetric matrix because, the antisymmetric part will give a term proportional to identity matrix due to **CCR**, which can always be discarded. If we take  $\bar{\xi} = H^{-1} \xi$ , then  $\hat{H}' = \frac{1}{2} (\hat{\mathbf{x}} - \bar{\xi})^T H (\hat{\mathbf{x}} - \bar{\xi})$  is equivalent to  $\hat{H}$  up to some additive constant term. Using the fourth property from section 2.1 we can write,

$$\begin{aligned} \hat{H}' &= \frac{1}{2} (\hat{\mathbf{x}} - \bar{\xi})^T H (\hat{\mathbf{x}} - \bar{\xi}) = \frac{1}{2} \sum_{jk} (\hat{\mathbf{x}} - \bar{\xi})_j H_{jk} (\hat{\mathbf{x}} - \bar{\xi})_k \\ &= \frac{1}{2} \sum_{jk} (\hat{D}_{-\xi} \hat{x}_j \hat{D}_{\xi}) H_{jk} (\hat{D}_{-\xi} \hat{x}_k \hat{D}_{\xi}) \\ &= \frac{1}{2} \sum_{jk} (\hat{D}_{-\xi} \hat{x}_j H_{jk} \hat{x}_k \hat{D}_{\xi}) \\ \hat{H}' &= \frac{1}{2} \hat{D}_{-\bar{\xi}} \hat{\mathbf{x}}^T H \hat{\mathbf{x}} \hat{D}_{\bar{\xi}} \end{aligned} \quad (4)$$

**Evolution of quadratures:**

- Heisenberg picture:  $\dot{\hat{O}} = i[\hat{H}, \hat{O}]$ .
- Using the above we get, time evolution of quadratures as,  $\dot{\hat{\mathbf{x}}} = \Omega H \hat{\mathbf{x}}$ . Solving it we get,  $\hat{\mathbf{x}}(t) = e^{\Omega H t} \hat{\mathbf{x}}(0)$ .

### 3.2 Gaussian state

**Definition 3** (Gaussian State). *Gaussian states are defined as all the ground and thermal states of second-order Hamiltonians [eq.3] with positive definite Hamiltonian matrix  $H > 0$ .*

Thus a *Gaussian state* can be written as,

$$\rho_G = \frac{e^{-\beta \hat{H}}}{\text{Tr} [e^{-\beta \hat{H}}]}, \quad (5)$$

where,  $\beta > 0$  and  $\hat{H}$  is defined in Eq. 3. Ground state is the limiting value,

$$\rho_G = \lim_{\beta \rightarrow \infty} \frac{e^{-\beta \hat{H}}}{\text{Tr} [e^{-\beta \hat{H}}]}. \quad (6)$$

**Note:**

- All Gaussian states are mixed state by construction, except for the ground state.
- Gaussian states are parametrized by  $\beta$ ,  $\xi$  and  $H$ . Though  $\beta$  is redundant and can be absorbed into  $H$ , it allows one to single out pure Gaussian states as a limiting case like in Eq. 6.
- Gaussian states can be generated First and second moment of quadrature. We'll talk about them later.

## 4 Gaussian operations

Gaussian operations are CP-maps those take Gaussian states to Gaussian states.

### 4.1 Gaussian Unitaries

One may write most general second order hamiltonian of  $n$ -modes as,  $\hat{H} = \frac{1}{2} \hat{D}_{-\xi} \hat{\mathbf{x}}^T H \hat{\mathbf{x}} \hat{D}_{\xi}$ . First note that,  $(\hat{D}_{-\xi} \hat{\mathbf{x}} \hat{D}_{\xi})(\hat{D}_{-\xi} \hat{\mathbf{x}} \hat{D}_{\xi}) = \hat{D}_{-\xi} \hat{\mathbf{x}}^2 \hat{D}_{\xi}$ . Then, it is clear that,  $e^{i\hat{H}} = \hat{D}_{-\xi} e^{\frac{i}{2} \hat{\mathbf{x}}^T H \hat{\mathbf{x}}} \hat{D}_{\xi}$ .