# Notes on Homodyne Measurement

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## 1 Notations

- $\hat{\mathbf{x}} = (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_n, \hat{p}_n)^T$, vector of cannonical operators.
- $\Omega = \bigoplus_{j=1}^{n} \Omega_1$, where $\Omega_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
  Note that, for $n = 1$, $[\hat{x}_i, \hat{x}_j] = i[\Omega_1]_{ij}$. Compactly,

  $$[\hat{\mathbf{x}}, \hat{\mathbf{x}}^T] = i\Omega, \qquad \text{(Canonical Commutation Relation)}$$

  where, think the commutation relation as element wise commutator. Some properties of $\Omega$:

  * $\Omega^T = -\Omega$(Anti-symmetric)
  * $\Omega^2 = \mathbb{1}_{2n} \implies \Omega^T\Omega = \mathbb{1}_{2n}$ (Orthogonal)
- Borrowing from the optical and field-theoretical terminologies, canonical degrees of freedom are also referred to as '*modes*'.
- $\hat{a}_j = \frac{\hat{x}_j + \hat{p}_j}{\sqrt{2}}$, annihilation operator.
- **BCH formula:** $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ for operators $A, B$ if $[A,[A,B]] = [B,[B,A]] = 0$

## 2 Prerequisits

### 2.1 Displacement operators

**Definition 1** (Weyl operators)**.**

$$\hat{D}_\xi = e^{i\xi^T\Omega\hat{\boldsymbol{x}}} = e^{i(\hat{x}_1\xi_2 - \hat{p}_1\xi_2)} \otimes \cdots \otimes e^{i(\hat{x}_n\xi_{2n} - \hat{p}_n\xi_{2n-1})}, \tag{1}$$

*where,* $\xi \in \mathbb{R}^{2n}$.

**Properties:**

- $\hat{D}_\xi^\dagger \hat{D}_\xi = \mathbb{1}$ (Unitary operator).
- $\hat{D}_\xi \hat{D}_\xi = \hat{D}_{2\xi}$.
- $\hat{D}_\xi \hat{D}_\eta = e^{-\frac{i}{2}\xi^T\Omega\eta}\ \hat{D}_{\xi+\eta}$. (**Prove!**)

- $\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}\hat{D}_{\bar{\xi}} = \hat{\mathbf{x}} - \bar{\xi}$
  *Proof:* For the $k^{\text{th}}$ component of $\hat{\mathbf{x}}$ *i.e.* $\hat{x}_k$, we have,

  $$\begin{aligned}\hat{D}_{-\xi}\hat{x}_k\hat{D}_{\xi} &= e^{-i\xi^T\Omega\hat{\mathbf{x}}}\hat{x}_k e^{i\xi^T\Omega\hat{\mathbf{x}}} \\ &= \hat{x}_k - i\left[\xi^T\Omega\hat{\mathbf{x}}, \hat{x}_k\right] + \frac{i^2}{2}\left[\xi^T\Omega\hat{\mathbf{x}}, \left[\xi^T\Omega\hat{\mathbf{x}}, \hat{x}_k\right]\right] + \cdots (\textit{using BCH}) \\ &= \hat{x}_k - \xi_k\end{aligned}$$

  From here the result follows directly.

- $\hat{D}_{-\bar{\xi}} = \hat{D}^{\dagger}_{\bar{\xi}}$.

### 2.2 Symplectic Group

**TODO:** Linear canonical transformation and Symplectic group, Canonical transformations are those which respect **CCR**.

**Definition 2** (Symplectic group)**.**

$$S \in Sp_{2n,\mathbb{R}} \iff S\Omega S^T = \Omega \tag{2}$$

**Evolution:**

### 2.3 Normal Modes

**TODO:** Definition, etc.

## 3 Gaussian States

### 3.1 Quadratic Hamiltonian and evolution

The most general quadratic/second-order hamiltonian can be written as follows.

$$\hat{H} = \frac{1}{2}\hat{\mathbf{x}}^T H\hat{\mathbf{x}} + \hat{\mathbf{x}}^T\xi. \tag{3}$$

Here, $\xi$ is a $2n$-dimensional real vector. $H$ is a $2n \times 2n$ symmetric matrix called *Hamiltonian matrix*, not to be confused with Hamiltonian. It can always be taken as a symmetric matrix because, the antisymmetric part with give a term proportional to identity matrix due to **CCR**, which can always be discarded. If we take $\bar{\xi} = H^{-1}\xi$, then $\hat{H}' = \frac{1}{2}(\hat{\mathbf{x}} - \bar{\xi})^T H(\hat{\mathbf{x}} - \bar{\xi})$ is equivalent to $\hat{H}$ up to some additive constant term. Using the fourth property from section 2.1 we can write,

$$\begin{aligned}\hat{H}' = \frac{1}{2}(\hat{\mathbf{x}} - \bar{\xi})^T H(\hat{\mathbf{x}} - \bar{\xi}) &= \frac{1}{2}\sum_{jk}(\hat{\mathbf{x}} - \xi)_j H_{jk}(\hat{\mathbf{x}} - \xi)_j \\ &= \frac{1}{2}\sum_{jk}(\hat{D}_{-\xi}\hat{x}_j\hat{D}_{\xi})H_{jk}(\hat{D}_{-\xi}\hat{x}_k\hat{D}_{\xi}) \\ &= \frac{1}{2}\sum_{jk}(\hat{D}_{-\xi}\hat{x}_j H_{jk}\hat{x}_k\hat{D}_{\xi}) \\ \hat{H}' &= \frac{1}{2}\hat{D}_{-\bar{\xi}}(\hat{\mathbf{x}}^T H\hat{\mathbf{x}})\hat{D}_{\bar{\xi}} \qquad (4)\end{aligned}$$

One takeaway from the above is that, it's enough to study the property of the Hamiltonian, $\hat{H} = \frac{1}{2}\hat{\mathbf{x}}^T H\hat{\mathbf{x}}$.

**Evolution of quadratures under free Hamiltonian:**

- Heisenberg picture of evolution:
$$\dot{\hat{O}} = i[\hat{H}, \hat{O}]. \tag{5}$$
- Using the above we get, time evolution of quadratures as, $\dot{\hat{\mathbf{x}}} = \Omega H\hat{\mathbf{x}}$. Solving which one gets,
$$\hat{\mathbf{x}}(t) = e^{\Omega H}\hat{\mathbf{x}}(0) \tag{6}$$
Since it is an unitary the **CCR** must be preserved.
$$i\Omega = [\hat{\mathbf{x}}(0), \hat{\mathbf{x}}(0)^T] = [\hat{\mathbf{x}}(t), \hat{\mathbf{x}}(t)^T] = [e^{\Omega Ht}\hat{\mathbf{x}}(0), \left(e^{\Omega Ht}\hat{\mathbf{x}}(0)\right)^T] \tag{7}$$
$$= e^{\Omega Ht}[\hat{\mathbf{x}}, \hat{\mathbf{x}}^T]\left(e^{\Omega Ht}\right)^T = ie^{\Omega Ht}\Omega\left(e^{\Omega Ht}\right)^T \tag{8}$$
So, we see, that for if **CCR** is to be preserved, we must have, $e^{\Omega H} \in Sp_{2n,\mathbb{R}}$. See eq. [2].
- We can also write , $\hat{\mathbf{x}}(t) = e^{i\hat{H}t}\hat{\mathbf{x}}(0)e^{-i\hat{H}t}$, because unitary evolution. Denote, $\hat{S}_H = e^{i\hat{H}} = e^{\frac{i}{2}\hat{\mathbf{x}}^T H\hat{\mathbf{x}}}$ and $S = e^{\Omega H} \in Sp_{2n,\mathbb{R}}$. Combining the above two result we write,
$$\hat{S}_H\hat{\mathbf{x}}\hat{S}_H^\dagger = S\hat{\mathbf{x}} \tag{9}$$

## 3.2 Gaussian state

**Definition 3** (Gaussian State)**.** *Gaussian states are defined as all the ground and thermal states of second-order Hamiltonians [eq.3] with positive definite Hamiltonian matrix $H > 0$.*

Thus a *Gaussian state* can be written as,
$$\rho_G = \frac{e^{-\beta\hat{H}}}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]}, \tag{10}$$
where, $\beta > 0$ and $\hat{H}$ is defined in Eq. 3. Ground state is the limiting value,
$$\rho_G = \lim_{\beta\to\infty}\frac{e^{-\beta\hat{H}}}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]}. \tag{11}$$

**Note:**

- All Gaussian states are mixed state by construction, except for the ground state.
- Gaussian states are parametrized by $\beta$, $\xi$ and $H$. Though $\beta$ is redundant and can be absorbed into $H$, it allows one to single out pure Gausian states as a limiting case like in Eq. 11.
- Gaussian states can be generated First and second moment of quadrature. We'll talk about them later.

# 4 Gaussian operations

Gaussian operations are CP-maps those take Gaussian states to Gaussian states.

## 4.1 Gaussian Unitaries

One may write most general second order hamiltonian of $n$-modes as, $\hat{H} = \frac{1}{2}\hat{D}_{-\bar{\xi}}(\hat{\mathbf{x}}^T H\hat{\mathbf{x}})\hat{D}_{\bar{\xi}}$. First note that, $(\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}\hat{D}_{\bar{\xi}})(\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}\hat{D}_{\bar{\xi}}) = \hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}^2\hat{D}_{\bar{\xi}}$. Then, it is clear that, $e^{i\hat{H}} = \hat{D}_{-\bar{\xi}}e^{\frac{i}{2}\hat{\mathbf{x}}^T H\hat{\mathbf{x}}}\hat{D}_{\bar{\xi}}$.

Let's introduce some notations and relations:

- $\hat{S} = e^{\frac{i}{2}\hat{\mathbf{x}}^T H\hat{\mathbf{x}}}$ (Free Hamiltonian unitary)
- $S = e^{\Omega H} \in Sp_{2n,\mathbb{R}}$ (Evolution of quadratures, Eq. 6)
- $\hat{S}\hat{\mathbf{x}}\hat{S}^\dagger = S\hat{\mathbf{x}}$ (Eq. 9)

Since $\hat{S}$ is unitary,

$$\hat{S}(i\xi^T\Omega\hat{\mathbf{x}})^k\hat{S}^\dagger = \hat{S}(i\xi^T\Omega\hat{\mathbf{x}})\hat{S}^\dagger\hat{S}(i\xi^T\Omega\hat{\mathbf{x}})\hat{S}^\dagger\cdots\hat{S}^\dagger\hat{S}(i\xi^T\Omega\hat{\mathbf{x}})\hat{S}^\dagger \tag{12}$$

Now each term can be simplified as,

$$\begin{aligned}
\hat{S}(i\xi^T\Omega\hat{\mathbf{x}})\hat{S}^\dagger &= i\sum_{jk}\hat{S}(\xi_j\Omega_{jk}\hat{\mathbf{x}}_k)\hat{S}^\dagger && (13)\\
&= i\sum_{jk}\xi_j\Omega_{jk}(\hat{S}\hat{\mathbf{x}}_k\hat{S}^\dagger) && (14)\\
&= i\xi^T\Omega(\hat{S}\hat{\mathbf{x}}\hat{S}^\dagger) && (15)\\
&= i\xi^T\Omega S\hat{\mathbf{x}} && (16)
\end{aligned}$$

Thus we can write, $\hat{S}e^{i\xi^T\Omega\hat{\mathbf{x}}}\hat{S}^\dagger = e^{i\xi^T\Omega S\hat{\mathbf{x}}} = e^{i\xi^T S^{-T}\Omega\hat{\mathbf{x}}} = \hat{D}_{S^{-T}\xi}$.

**TODO:** To show, that any arbitrary quadratic unitary transformation can be generated by Displacement and Symplectic rotations.