Notes on Homodyne Measurement

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1 Notations

- $\hat{\mathbf{x}} = (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_n, \hat{p}_n)^T$, vector of cannonical operators.
- $\Omega = \bigoplus_{j=1}^{n} \Omega_1$, where $\Omega_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
  Note that, for $n = 1$, $[\hat{x}_i, \hat{x}_j] = i[\Omega_1]_{ij}$. Compactly,
  $$[\hat{\mathbf{x}}, \hat{\mathbf{x}}^T] = i\Omega, \qquad \text{(Canonical Commutation Relation)}$$
  where, think the commutation relation as element wise commutator.
- Borrowing from the optical and field-theoretical terminologies, canonical degrees of freedom are also referred to as '*modes*'.
- $\hat{a}_j = \frac{\hat{x}_j + \hat{p}_j}{\sqrt{2}}$, annihilation operator.

2 Gaussian States

2.1 Quadratic Hamiltonian and Gaussian States

The most general quadratic/second-order hamiltonian can be written as follows.

$$\hat{H} = \frac{1}{2}\hat{\mathbf{x}}^T H \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \xi. \tag{1}$$

Here, ξ is a $2n$-dimensional real vector. H is a $2n \times 2n$ symmetric matrix called *Hamiltonian matrix*, not to be confused with Hamiltonian. It can alsways be taken as a symmetric matrix because, the antisymmetric part with give a term proportional to identity matrix due to **CCR**, which can always be discarded. If we take $\bar{\xi} = H^{-1}\xi$, then $\hat{H}' = \frac{1}{2}(\hat{\mathbf{x}} - \bar{\xi})^T H (\hat{\mathbf{x}} - \bar{\xi})$ is equaivalent to $\hat{H}$ upto some additive constant term.

Definition 1 (Gaussian State)**.** *Gaussian states are defined as all the ground and thermal states of second-order Hamiltonians [eq.1] with positive definite Hamiltonian matrix $H > 0$.*

Thus a *Gaussian state* can be written as,

$$\rho_G = \frac{e^{-\beta \hat{H}}}{\mathrm{Tr}\left[e^{-\beta \hat{H}}\right]}, \tag{2}$$

where, $\beta > 0$ and $\hat{H}$ is defined in Eq. 1. Ground state is the limiting value,

$$\rho_G = \lim_{\beta \to \infty} \frac{e^{-\beta \hat{H}}}{\mathrm{Tr}\left[e^{-\beta \hat{H}}\right]}. \tag{3}$$

Note:

- All Gaussian states are mixed state by construction, except for the ground state.
- Gaussian states are parametrized by β, ξ and H. Though β is redundant and can be absorbed into H, it allows one to single out pure Gausian states as in Eq. 3.

2.2 Displacement operators

Definition 2 (Weyl operators)**.**

$$D \tag{4}$$