Notes on Homodyne Measurement

Arnab Ghorui

January 15, 2025

1 Notations

- $\hat{\mathbf{x}} = (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_n, \hat{p}_n)^T$, vector of cannonical operators.
- $\Omega = \bigoplus_{j=1}^{n} \Omega_1$, where $\Omega_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
  Note that, for $n = 1$, $[\hat{x}_i, \hat{x}_j] = i[\Omega_1]_{ij}$. Compactly,
  $$[\hat{\mathbf{x}}, \hat{\mathbf{x}}^T] = i\Omega, \qquad \text{(Canonical Commutation Relation)}$$
  where, think the commutation relation as element wise commutator. Some properties of Ω:
  * $\Omega^T = -\Omega$(Anti-symmetric)
  * $\Omega^2 = \mathbb{1}_{2n} \implies \Omega^T\Omega = \mathbb{1}_{2n}$ (Orthogonal)
- Borrowing from the optical and field-theoretical terminologies, canonical degrees of freedom are also referred to as '*modes*'.
- $\hat{a}_j = \frac{\hat{x}_j + \hat{p}_j}{\sqrt{2}}$, annihilation operator.
- **BCH formula:** $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ for operators A, B if $[A,[A,B]] = [B,[B,A]] = 0$

2 Prerequisits

2.1 Displacement operators

Definition 1 (Weyl operators)**.**

$$\hat{D}_\xi = e^{i\xi^T\Omega\hat{\boldsymbol{x}}} = e^{i(\hat{x}_1\xi_2 - \hat{p}_1\xi_2)} \otimes \cdots \otimes e^{i(\hat{x}_n\xi_{2n} - \hat{p}_n\xi_{2n-1})}, \tag{1}$$

where, $\xi \in \mathbb{R}^{2n}$*.*

Properties:

- $\hat{D}_\xi^\dagger \hat{D}_\xi = \mathbb{1}$ (Unitary operator).
- $\hat{D}_\xi \hat{D}_\xi = \hat{D}_{2\xi}$.
- $\hat{D}_\xi \hat{D}_\eta = e^{-\frac{i}{2}\xi^T\Omega\eta}\ \hat{D}_{\xi+\eta}$. (**Prove!**)

- $\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}\hat{D}_{\bar{\xi}} = \hat{\mathbf{x}} - \bar{\xi}$

  Proof: For the k^{th} component of $\hat{\mathbf{x}}$ *i.e.* $\hat{x}_k$, we have,

$$\begin{aligned}\hat{D}_{-\xi}\hat{x}_k\hat{D}_\xi &= e^{-i\xi^T\Omega\hat{\mathbf{x}}}\hat{x}_k e^{i\xi^T\Omega\hat{\mathbf{x}}} \\ &= \hat{x}_k - i\left[\xi^T\Omega\hat{\mathbf{x}}, \hat{x}_k\right] + \frac{i^2}{2}\left[\xi^T\Omega\hat{\mathbf{x}}, \left[\xi^T\Omega\hat{\mathbf{x}}, \hat{x}_k\right]\right] + \cdots (\textit{using BCH}) \\ &= \hat{x}_k - \xi_k\end{aligned}$$

  From here the result follows directly.

- $\hat{D}_{-\bar{\xi}} = \hat{D}^\dagger_{\bar{\xi}}$.

2.2 Symplectic Group

TODO: Linear canonical transformation and Symplectic group, Canonical transformations are those which respect **CCR**.

Definition 2 (Symplectic group)**.**

$$S \in Sp_{2n,\mathbb{R}} \iff S\Omega S^T = \Omega \tag{2}$$

Evolution:

2.3 Normal Modes

TODO: Definition, etc.

3 Gaussian States

3.1 Quadratic Hamiltonian and Gaussian States

The most general quadratic/second-order hamiltonian can be written as follows.

$$\hat{H} = \frac{1}{2}\hat{\mathbf{x}}^T H\hat{\mathbf{x}} + \hat{\mathbf{x}}^T\xi. \tag{3}$$

Here, ξ is a $2n$-dimensional real vector. H is a $2n \times 2n$ symmetric matrix called *Hamiltonian matrix*, not to be confused with Hamiltonian. It can always be taken as a symmetric matrix because, the antisymmetric part with give a term proportional to identity matrix due to **CCR**, which can always be discarded. If we take $\bar{\xi} = H^{-1}\xi$, then $\hat{H}' = \frac{1}{2}(\hat{\mathbf{x}} - \bar{\xi})^T H(\hat{\mathbf{x}} - \bar{\xi})$ is equivalent to $\hat{H}$ up to some additive constant term. Using the fourth property from section 2.1 we can write,

$$\begin{aligned}\hat{H}' = \frac{1}{2}(\hat{\mathbf{x}} - \bar{\xi})^T H(\hat{\mathbf{x}} - \bar{\xi}) &= \frac{1}{2}\sum_{jk}(\hat{\mathbf{x}} - \xi)_j H_{jk}(\hat{\mathbf{x}} - \xi)_j \\ &= \frac{1}{2}\sum_{jk}(\hat{D}_{-\xi}\hat{x}_j\hat{D}_\xi)H_{jk}(\hat{D}_{-\xi}\hat{x}_k\hat{D}_\xi) \\ &= \frac{1}{2}\sum_{jk}(\hat{D}_{-\xi}\hat{x}_j H_{jk}\hat{x}_k\hat{D}_\xi) \\ \hat{H}' &= \frac{1}{2}\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}^T H\hat{\mathbf{x}}\hat{D}_{\bar{\xi}}\end{aligned} \tag{4}$$

Definition 3 (Gaussian State)**.** *Gaussian states are defined as all the ground and thermal states of second-order Hamiltonians [eq.3] with positive definite Hamiltonian matrix $H > 0$.*

Thus a *Gaussian state* can be written as,

$$\rho_G = \frac{e^{-\beta \hat{H}}}{\mathrm{Tr}\left[e^{-\beta \hat{H}}\right]}, \tag{5}$$

where, $\beta > 0$ and $\hat{H}$ is defined in Eq. 3. Ground state is the limiting value,

$$\rho_G = \lim_{\beta \to \infty} \frac{e^{-\beta \hat{H}}}{\mathrm{Tr}\left[e^{-\beta \hat{H}}\right]}. \tag{6}$$

Note:

- All Gaussian states are mixed state by construction, except for the ground state.
- Gaussian states are parametrized by β, ξ and H. Though β is redundant and can be absorbed into H, it allows one to single out pure Gausian states as a limiting case like in Eq. 6.
- Gaussian states can be generated First and second moment of quadrature. We'll talk about them later.

4 Gaussian operations

Gaussian operations are CP-maps those take Gaussian states to Gaussian states.

4.1 Gaussian Unitaries

One may write most general second order hamiltonian of n-modes as, $\hat{H} = \frac{1}{2}\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}^T H \hat{\mathbf{x}} \hat{D}_{\bar{\xi}}$. First note that, $(\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}\hat{D}_{\bar{\xi}})(\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}\hat{D}_{\bar{\xi}}) = \hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}^2\hat{D}_{\bar{\xi}}$. Then, it is clear that, $e^{i\hat{H}} = \hat{D}_{-\bar{\xi}} e^{\frac{i}{2}\hat{\mathbf{x}}^T H \hat{\mathbf{x}}} \hat{D}_{\bar{\xi}}$.