Notes on Homodyne Measurement

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1 Notations

- $\hat{\mathbf{x}} = (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_n, \hat{p}_n)^T$, vector of cannonical operators.
- $\Omega = \bigoplus_{j=1}^{n} \Omega_1$, where $\Omega_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
  Note that, for $n = 1$, $[\hat{x}_i, \hat{x}_j] = i[\Omega_1]_{ij}$. Compactly,
  $$[\hat{\mathbf{x}}, \hat{\mathbf{x}}^T] = i\Omega, \qquad \text{(Canonical Commutation Relation)}$$
  where, think the commutation relation as element wise commutator.
- Borrowing from the optical and field-theoretical terminologies, canonical degrees of freedom are also referred to as '*modes*'.
- $\hat{a}_j = \frac{\hat{x}_j + \hat{p}_j}{\sqrt{2}}$, annihilation operator.
- **BCH formula:** $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ for operators A, B if $[A, [A, B]] = [B, [B, A]] = 0$

2 Gaussian States

2.1 Quadratic Hamiltonian and Gaussian States

The most general quadratic/second-order hamiltonian can be written as follows.

$$\hat{H} = \frac{1}{2}\hat{\mathbf{x}}^T H \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \xi. \tag{1}$$

Here, ξ is a $2n$-dimensional real vector. H is a $2n \times 2n$ symmetric matrix called *Hamiltonian matrix*, not to be confused with Hamiltonian. It can alsways be taken as a symmetric matrix because, the antisymmetric part with give a term proportional to identity matrix due to **CCR**, which can always be discarded. If we take $\bar{\xi} = H^{-1}\xi$, then $\hat{H}' = \frac{1}{2}(\hat{\mathbf{x}} - \bar{\xi})^T H (\hat{\mathbf{x}} - \bar{\xi})$ is equivalent to $\hat{H}$ up to some additive constant term.

Definition 1 (Gaussian State)**.** *Gaussian states are defined as all the ground and thermal states of second-order Hamiltonians [eq.1] with positive definite Hamiltonian matrix $H > 0$.*

Thus a *Gaussian state* can be written as,

$$\rho_G = \frac{e^{-\beta \hat{H}}}{\mathrm{Tr}\left[e^{-\beta \hat{H}}\right]}, \tag{2}$$

where, $\beta > 0$ and $\hat{H}$ is defined in Eq. 1. Ground state is the limiting value,

$$\rho_G = \lim_{\beta\to\infty} \frac{e^{-\beta \hat{H}}}{\mathrm{Tr}\left[e^{-\beta \hat{H}}\right]}. \tag{3}$$

Note:

- All Gaussian states are mixed state by construction, except for the ground state.
- Gaussian states are parametrized by β, ξ and H. Though β is redundant and can be absorbed into H, it allows one to single out pure Gausian states as a limiting case like in Eq. 3.
- Gaussian states can be generated First and second moment of quadrature. We'll talk about them later.

2.2 Displacement operators

Definition 2 (Weyl operators)**.**

$$\hat{D}_\xi = e^{i\xi^T \Omega \hat{\boldsymbol{x}}} = e^{i(\hat{x}_1\xi_2 - \hat{p}_1\xi_2)} \otimes \cdots \otimes e^{i(\hat{x}_n\xi_{2n} - \hat{p}_n\xi_{2n-1})}, \tag{4}$$

where, $\xi \in \mathbb{R}^{2n}$*.*

Properties:

- $\hat{D}_\xi^\dagger \hat{D}_\xi = \mathbb{1}$ (Unitary operator).
- $\hat{D}_\xi \hat{D}_\xi = \hat{D}_{2\xi}$.
- $\hat{D}_\xi + \hat{D}_\eta = e^{-\frac{i}{2}\xi^T \Omega \eta}\ \hat{D}_{\xi+\eta}$. (**Prove!**)
- $\hat{D}_{-\bar{\xi}} \hat{\mathbf{x}} \hat{D}_{\bar{\xi}} = \hat{\mathbf{x}} - \bar{\xi}$ (**Prove!**)
- $\hat{D}_{-\bar{\xi}} = \hat{D}_{\bar{\xi}}^\dagger$.

Using the above fourth property we can write,

$$\hat{H}' = \frac{1}{2}(\hat{\mathbf{x}} - \bar{\xi})^T H (\hat{\mathbf{x}} - \bar{\xi}) = \frac{1}{2}(\hat{D}_{-\bar{\xi}} \hat{\mathbf{x}} \hat{D}_{\bar{\xi}})^T H (\hat{D}_{-\bar{\xi}} \hat{\mathbf{x}} \hat{D}_{\bar{\xi}}) \tag{5}$$

$$= \frac{1}{2} \hat{D}_{-\bar{\xi}} \hat{\mathbf{x}}^T H \hat{\mathbf{x}} \hat{D}_{\bar{\xi}} \tag{6}$$

See Serafini (eq. 3.17) for proof.

2.3 Symplectic Group

TODO: Linear canonical transformation and Symplectic group, Canonical transformations are those which respect **CCR**.

Definition 3 (Symplectic group)**.**

$$S \in Sp_{2n,\mathbb{R}} \iff S\Omega S^T = \Omega \tag{7}$$

2.4 Normal Modes

TODO: Definition, etc.