

Notes on Homodyne Measurement

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1 Notations

- $\hat{\mathbf{x}} = (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_n, \hat{p}_n)^T$, vector of canonical operators.

- $\Omega = \bigoplus_{j=1}^n \Omega_1$, where $\Omega_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Note that, for $n = 1$, $[\hat{x}_i, \hat{x}_j] = i[\Omega_1]_{ij}$. Compactly,

$$[\hat{\mathbf{x}}, \hat{\mathbf{x}}^T] = i\Omega, \quad (\text{Canonical Commutation Relation})$$

where, think the commutation relation as element wise commutator. Some properties of Ω :

$$* \quad \Omega^T = -\Omega \text{ (Anti-symmetric)}$$

$$* \quad \Omega^2 = \mathbb{1}_{2n} \implies \Omega^T \Omega = \mathbb{1}_{2n} \text{ (Orthogonal)}$$

- Borrowing from the optical and field-theoretical terminologies, canonical degrees of freedom are also referred to as ‘*modes*’.
- $\hat{a}_j = \frac{\hat{x}_j + i\hat{p}_j}{\sqrt{2}}$, annihilation operator.
- **BCH formula:** $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ for operators A, B if $[A, [A, B]] = [B, [B, A]] = 0$

2 Prerequisites

2.1 Displacement operators

Definition 1 (Weyl operators).

$$\hat{D}_\xi = e^{i\xi^T \Omega \hat{\mathbf{x}}} = e^{i(\hat{x}_1 \xi_2 - \hat{p}_1 \xi_2)} \otimes \dots \otimes e^{i(\hat{x}_n \xi_{2n} - \hat{p}_n \xi_{2n-1})}, \quad (1)$$

where, $\xi \in \mathbb{R}^{2n}$.

Properties:

- $\hat{D}_\xi^\dagger \hat{D}_\xi = \mathbb{1}$ (Unitary operator).
- $\hat{D}_\xi \hat{D}_\xi = \hat{D}_{2\xi}$.
- $\hat{D}_\xi \hat{D}_\eta = e^{-\frac{i}{2}\xi^T \Omega \eta} \hat{D}_{\xi+\eta}$. (**Prove!**)

- $\hat{D}_{-\bar{\xi}}\hat{\mathbf{x}}\hat{D}_{\bar{\xi}} = \hat{\mathbf{x}} - \bar{\xi}$

Proof: For the k^{th} component of $\hat{\mathbf{x}}$ i.e. \hat{x}_k , we have,

$$\begin{aligned}\hat{D}_{-\xi}\hat{x}_k\hat{D}_{\xi} &= e^{-i\xi^T\Omega\hat{\mathbf{x}}}\hat{x}_ke^{i\xi^T\Omega\hat{\mathbf{x}}} \\ &= \hat{x}_k - i[\xi^T\Omega\hat{\mathbf{x}}, \hat{x}_k] + \frac{i^2}{2}[\xi^T\Omega\hat{\mathbf{x}}, [\xi^T\Omega\hat{\mathbf{x}}, \hat{x}_k]] + \dots (\text{using BCH}) \\ &= \hat{x}_k - \xi_k\end{aligned}$$

From here the result follows directly.

- $\hat{D}_{-\bar{\xi}} = \hat{D}_{\bar{\xi}}^{\dagger}$.

2.2 Symplectic Group

TODO: Linear canonical transformation and Symplectic group, Canonical transformations are those which respect **CCR**.

Definition 2 (Symplectic group).

$$S \in Sp_{2n, \mathbb{R}} \iff S\Omega S^T = \Omega \quad (2)$$

Evolution:

2.3 Normal Modes

TODO: Definition, etc.

3 Gaussian States

3.1 Quadratic Hamiltonian and evolution

The most general quadratic/second-order hamiltonian can be written as follows.

$$\hat{H} = \frac{1}{2}\hat{\mathbf{x}}^T H \hat{\mathbf{x}} + \hat{\mathbf{x}}^T \xi. \quad (3)$$

Here, ξ is a $2n$ -dimensional real vector. H is a $2n \times 2n$ symmetric matrix called *Hamiltonian matrix*, not to be confused with Hamiltonian. It can always be taken as a symmetric matrix because, the antisymmetric part will give a term proportional to identity matrix due to **CCR**, which can always be discarded. If we take $\bar{\xi} = H^{-1}\xi$, then $\hat{H}' = \frac{1}{2}(\hat{\mathbf{x}} - \bar{\xi})^T H (\hat{\mathbf{x}} - \bar{\xi})$ is equivalent to \hat{H} up to some additive constant term. Using the fourth property from section 2.1 we can write,

$$\begin{aligned}\hat{H}' &= \frac{1}{2}(\hat{\mathbf{x}} - \bar{\xi})^T H (\hat{\mathbf{x}} - \bar{\xi}) = \frac{1}{2} \sum_{jk} (\hat{\mathbf{x}} - \bar{\xi})_j H_{jk} (\hat{\mathbf{x}} - \bar{\xi})_k \\ &= \frac{1}{2} \sum_{jk} (\hat{D}_{-\xi}\hat{x}_j\hat{D}_{\xi}) H_{jk} (\hat{D}_{-\xi}\hat{x}_k\hat{D}_{\xi}) \\ &= \frac{1}{2} \sum_{jk} (\hat{D}_{-\xi}\hat{x}_j H_{jk} \hat{x}_k \hat{D}_{\xi}) \\ \hat{H}' &= \frac{1}{2} \hat{D}_{-\bar{\xi}}(\hat{\mathbf{x}}^T H \hat{\mathbf{x}}) \hat{D}_{\bar{\xi}}\end{aligned} \quad (4)$$

One takeaway from the above is that, it's enough to study the property of the hamiltonian, $\hat{H} = \frac{1}{2}\hat{\mathbf{x}}^T H \hat{\mathbf{x}}$.

Evolution of quadratures:

- Heisenberg picture: $\dot{\hat{O}} = i[\hat{H}, \hat{O}]$.
- Using the above we get, time evolution of quadratures as, $\dot{\hat{\mathbf{x}}} = \Omega H \hat{\mathbf{x}}$. Solving it we get, $\hat{\mathbf{x}}(t) = e^{\Omega H} \hat{\mathbf{x}}(0)$.

3.2 Gaussian state

Definition 3 (Gaussian State). *Gaussian states are defined as all the ground and thermal states of second-order Hamiltonians [eq.3] with positive definite Hamiltonian matrix $H > 0$.*

Thus a *Gaussian state* can be written as,

$$\rho_G = \frac{e^{-\beta \hat{H}}}{\text{Tr} \left[e^{-\beta \hat{H}} \right]}, \quad (5)$$

where, $\beta > 0$ and \hat{H} is defined in Eq. 3. Ground state is the limiting value,

$$\rho_G = \lim_{\beta \rightarrow \infty} \frac{e^{-\beta \hat{H}}}{\text{Tr} \left[e^{-\beta \hat{H}} \right]}. \quad (6)$$

Note:

- All Gaussian states are mixed state by construction, except for the ground state.
- Gaussian states are parametrized by β , ξ and H . Though β is redundant and can be absorbed into H , it allows one to single out pure Gaussian states as a limiting case like in Eq. 6.
- Gaussian states can be generated First and second moment of quadrature. We'll talk about them later.

4 Gaussian operations

Gaussian operations are CP-maps those take Gaussian states to Gaussian states.

4.1 Gaussian Unitaries

One may write most general second order hamiltonian of n -modes as, $\hat{H} = \frac{1}{2}\hat{D}_{-\xi}\hat{\mathbf{x}}^T H \hat{\mathbf{x}}\hat{D}_{\xi}$. First note that, $(\hat{D}_{-\xi}\hat{\mathbf{x}}\hat{D}_{\xi})(\hat{D}_{-\xi}\hat{\mathbf{x}}\hat{D}_{\xi}) = \hat{D}_{-\xi}\hat{\mathbf{x}}^2\hat{D}_{\xi}$. Then, it is clear that, $e^{i\hat{H}} = \hat{D}_{-\xi}e^{\frac{i}{2}\hat{\mathbf{x}}^T H \hat{\mathbf{x}}}\hat{D}_{\xi}$.

- Define, $\hat{S} = e^{\frac{i}{2}\hat{\mathbf{x}}^T H \hat{\mathbf{x}}}$ and $S = e^{\Omega H}$.
- $\hat{S}_H \hat{\mathbf{x}} \hat{S}_H^\dagger = S_H \hat{\mathbf{x}}$.