Girsanov’s Theorem and its implementation

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Overview

Girsanov's theorem also known as the change of measure method helps us to change the probability measure of an equation from one probability space to other i.e

$$(\Omega, \mathcal{F}, P) \longrightarrow (\Omega, \mathcal{F}, Q)$$

where

Ω Is the probability space

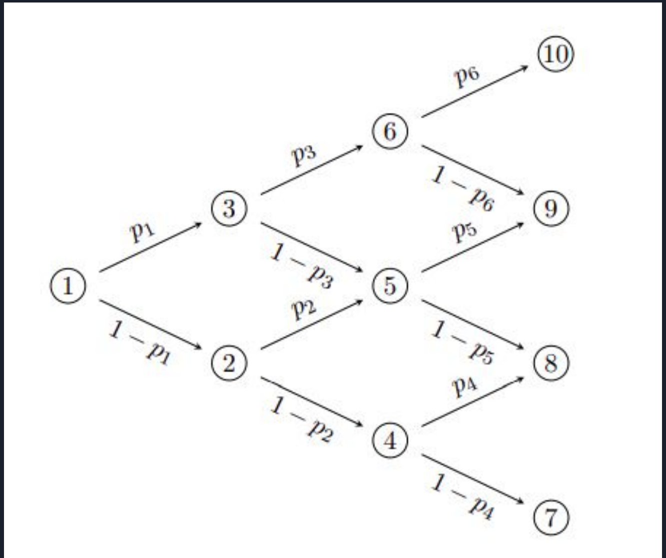
$\mathcal{F}$ Is a finite sigma algebra

P Is the probability measure

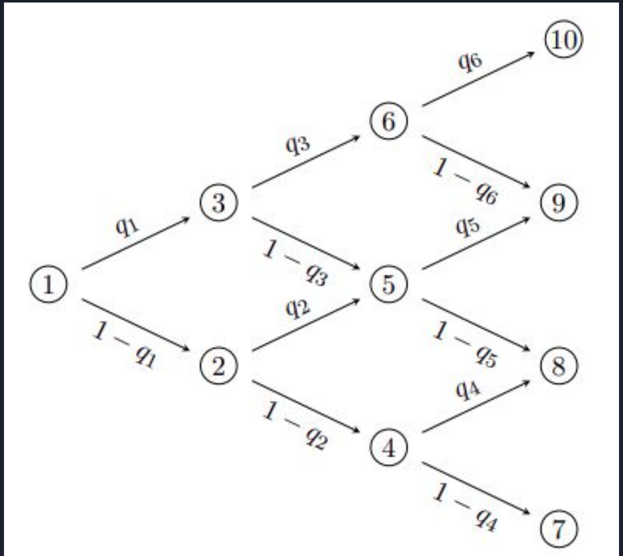
Binomial Tree analogy for change of measure in discrete world

To understand the working of girsanov theorem on continuous variable models let's first have a look on the following examples in the discrete world to understand better:

Lets say we are in a probability measure P and to reach from i to j in the following binomial tree the required probabilities are noted on the paths:



Similarly say in a probability measure Q we have the same binomial tree and nodes but now with different probabilities for each branch.



Say the probability to reach at node n is given by πn:

For instance: $$\pi_{10} = p_1 \cdot p_3 \cdot p_6$$

Similarly we can write the probabilities for all such nodes:

$$\pi_9 \quad = \quad p_1 \cdot p_3 \cdot (1 - p_6) + p_1 \cdot (1 - p_3) \cdot p_5 + (1 - p_1) \cdot p_2 \cdot p_5,$$

Now how can we write a relationship between the two?

Well quite naively as follows:

$$\tilde{\pi}_9 = \frac{\tilde{\pi}_9}{\pi_9} \cdot \pi_9.$$

Now under the condition of equivalence of the two measures, we have the following:

$$\mathbb{E}_{\mathbb{P}}(X) = \sum_i x_i \pi_i = \sum_i x_i \frac{\pi_i}{\tilde{\pi}_i} \tilde{\pi}_i = \mathbb{E}_{\mathbb{Q}}\left(X \frac{d\mathbb{P}}{d\mathbb{Q}}\right)$$

Statement of the theorem

Let (ψs) be in an adapted process

Next we define something called the Radon-Nikodym derivative which is

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T |\psi_s|^2 ds\right).$$

And in the new probability measure the GBM has the following equation:

$$\widehat{B}_t := B_t + \int_0^t \psi_s ds, \qquad 0 \leqslant t \leqslant T,$$

Making sense of the theorem

Lets for the sake of simplicity , let's take the brownian motion to be nothing but a series of infinitesimally small increments as follows:

$$\Delta B_t = \pm\sqrt{\Delta t},$$

With

$$\mathbb{P}(\Delta B_t = +\sqrt{\Delta t}) = \mathbb{P}(\Delta B_t = -\sqrt{\Delta t}) = \frac{1}{2},$$

Thus if we calculate the expectation for the process we have

$$\mathbb{E}[\Delta B_t] = \frac{1}{2}\sqrt{\Delta t} - \frac{1}{2}\sqrt{\Delta t} = 0.$$

Now for the changed process we have:

$$\widehat{B}_t := \nu t + B_t,$$

Now if we again calculate the expectation of the process:

$$\mathbb{E}\left[\widehat{B}_t\right] = \mathbb{E}[\nu t + B_t] = \nu t + \mathbb{E}[B_t] = \nu t \neq 0,$$

Thus now let's assume that the probabilities for each step are defined as follows so that we are able to make the above process a weiner again:

$$p^*(\nu\Delta t + \sqrt{\Delta t}) + q^*(\nu\Delta t - \sqrt{\Delta t}) = 0$$

$$p^* + q^* = 1.$$

Thus solving we get the following result:

$$p^* := \frac{1}{2}(1 - \nu\sqrt{\Delta t}) \quad \text{and} \quad q^* := \frac{1}{2}(1 + \nu\sqrt{\Delta t}).$$

Coming back to Brownian motion considered as a discrete random walk with independent increments $\pm\sqrt{\Delta t}$, we try to construct a new probability measure denoted $P*$ under which the drifted process $\widehat{B}_t := \nu t + B_t$ will be a standard Brownian motion.

Thus we define the likelihood ratios as follows:

$$\begin{aligned}\frac{d\mathbb{P}^*}{d\mathbb{P}} &:= \frac{\mathbb{P}^*(\Delta B_{t_1} = \epsilon_1\sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \epsilon_N\sqrt{\Delta t})}{\mathbb{P}(\Delta B_{t_1} = \epsilon_1\sqrt{\Delta t}, \ldots, \Delta B_{t_N} = \epsilon_N\sqrt{\Delta t})} \\ &= \frac{\mathbb{P}^*(\Delta B_{t_1} = \epsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}^*(\Delta B_{t_N} = \epsilon_N\sqrt{\Delta t})}{\mathbb{P}(\Delta B_{t_1} = \epsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}(\Delta B_{t_N} = \epsilon_N\sqrt{\Delta t})} \\ &= \frac{1}{(1/2)^N}\mathbb{P}^*(\Delta B_{t_1} = \epsilon_1\sqrt{\Delta t}) \cdots \mathbb{P}^*(\Delta B_{t_N} = \epsilon_N\sqrt{\Delta t}),\end{aligned}$$

Giving us:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} \simeq \frac{1}{(1/2)^N} \prod_{0<t<T} \left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{\Delta t}\right)$$

$$
\begin{aligned}
\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} &= 2^N \prod_{0<t<T} \left(\frac{1}{2} \mp \frac{1}{2}\nu\sqrt{\Delta t}\right) \\
&= \prod_{0<t<T} \left(1 \mp \nu\sqrt{\Delta t}\right) \\
&= \exp\left(\log \prod_{0<t<T} \left(1 \mp \nu\sqrt{\Delta t}\right)\right) \\
&= \exp\left(\sum_{0<t<T} \log\left(1 \mp \nu\sqrt{\Delta t}\right)\right) \\
&\simeq \exp\left(\nu \sum_{0<t<T} \mp\sqrt{\Delta t} - \frac{1}{2} \sum_{0<t<T} (\mp\nu\sqrt{\Delta t})^2\right) \\
&= \exp\left(-\nu \sum_{0<t<T} \pm\sqrt{\Delta t} - \frac{\nu^2}{2} \sum_{0<t<T} \Delta t\right) \\
&= \exp\left(-\nu \sum_{0<t<T} \Delta B_t - \frac{\nu^2}{2} \sum_{0<t<T} \Delta t\right) \\
&= \exp\left(-\nu B_T - \frac{\nu^2}{2} T\right),
\end{aligned}
$$

Martingale pricing

When we talk of european style options it is generally seen that solving underlying stochastic equations related pricing such options is well suited in a risk-neutral setting, thus making the martingale pricing model default method for these options. Usually the calculations are more often executed in terms of expected values.

Martingale pricing is a pricing approach based on the notions of martingale and risk neutrality.

The Girsanov theorem plays a crucial role in this pricing regime by helping convert risky assets which have non-martingale pricing to a risk-neutral setting along with being a martingale.

Thus the Girsanov theorem finds utility in many if not all martingale pricing models.

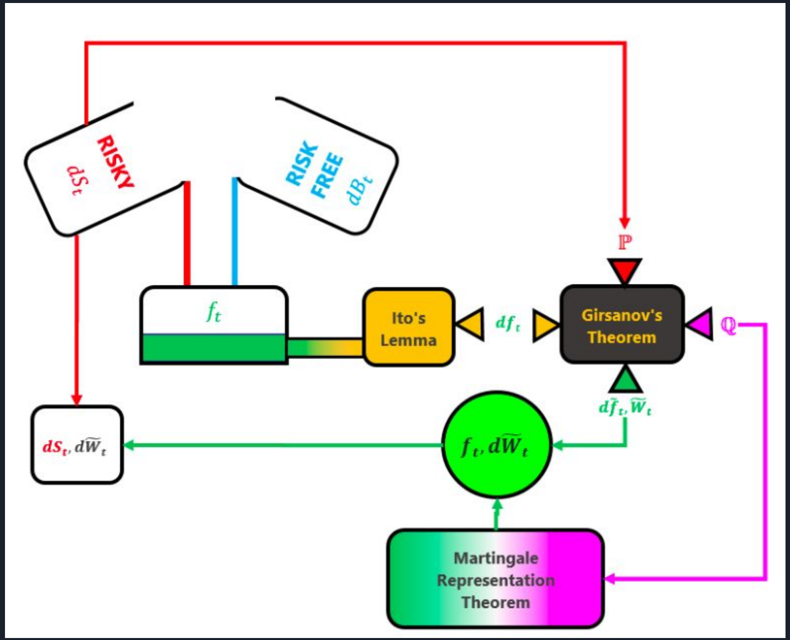
What’s a Martingale anyway?

In probability theory, a martingale is a sequence of random variables (i.e., a stochastic process) for which, at a particular time, the conditional expectation of the next value in the sequence is equal to the present value, regardless of all prior values.

I.e for a process with finite expectation $\mathbf{E}(|X_n|) < \infty$

We have the following statement $\mathbf{E}(X_{n+1} \mid X_1, \ldots, X_n) = X_n.$

The flow of the implementation



What is the Black-Scholes option pricing model

We will be applying our analysis to european style option which is also the option type usually traded in Indian markets.

Though the Black-Scholes model comes with a number of assumptions that aren’t true when talking of real markets but it has always been a good starting point for other different models.

The black-scholes model is an option pricing model that assumes that the price of a option follows a standard Geometric Brownian Motion.

The general structure of the GBM equation is as follows:

$$\text{Change in Price} = \text{Drift} + \text{Volatility}$$

Let us get a bit technical here:

A stochastic process St is said to follow a GBM if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

Here the mu represents the drift or the tilt/ direction of the movement of the motion.

And sigma denotes the volatility or deviation in the path of the motion.

Wt represents the equation of a brownian motion in a wiener process more specifically it is normally distributed with mean zero and variance t(time period of analysis)

$$W_t \sim N(0, t)$$

Complete market assumption and The risk neutral measure

As discussed earlier that the black scholes model comes with a number of assumptions and one of them being the complete market assumption which states that one can completely hedge against all types of market risks for his investments.

Let C be the function denoting the earning from a given option after time t. And it turns out that its takes two parameters as input time and the stock price at time t. And let the strike price of the option be denoted by k.

$$C(t, S(t)) = \max(S(t) - K, 0)$$

An important thing to keep in mind is that when we are talking about pricing a option today that will mature in the future at some time from now, and if the seller wants to hedge against the market his portfolio’s value should be equal to that of the call option buyer’s gain.

So let the seller's portfolio value be denoted by X(t) and thus by the discounting lemma we have the following relation.

$$X(t)e^{-rt} = C(t, S(t))e^{-rt}$$

Where the r represents the risk free rate of return.

After this we use ito's lemma multiple times and derive pretty useful equations, which are out of the scope of this presentation.

And we arrive at the following equation for black-scholes pricing regime where the function f signifies the pricing and r the risk free interest rate or the discounting rate

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 S^2 = rf$$

As we can see that the present equation do not make much sense at first glance and it really does not. Though there are methods to further solve the same through various ways like taking analogy with a heat equation but as we will see later that girsanov’s theorem help us to reach to a reasonable solution to the same at the same time averting a large portion of tedious calculation.

Making sense of girsanov theorem and its application

Lets again start with the GBM equation of option price that we had earlier seen.

$$S_t = S_0 \, \mathrm{e}^{\mu t + \sigma W_t}$$

Now let's make use of the risk neutral measure method to discount the stock price St with respect to a risk neutral interest rate r.

I.e

$$S^*(t) = \frac{S(t)}{B(t)} = \frac{S(t)}{e^{rt}}$$

The reason is the time value of money. Because of time value of money, the risk-free rate is already embedded inside the drift of all financial assets. So we want to remove it and consider the underlying asset's dynamics excluding the time value of money effect.

Giving us the following SDE:

$$dS^* = (\mu - r)S^* dt + \sigma S^* dX,$$

Now it is important to understand is that in the current probability measure P the process S* is not a martingale as it has non zero drift part and the whole point of the use of change of measure theorem is to make that happen.

Thus we are in search of process theta such that the drift term vanishes giving us a martingale in the process.

Why consider the discounted price rather than the current price?

Now let's assume that there exist a probability measure Q such that the following holds.

$$\frac{dS^*}{S^*} = (\mu - r)dt + \sigma\left(-\theta dt + dX^{\mathbb{Q}}\right)$$

$$\frac{dS^*}{S^*} = (\mu - r - \sigma\theta)dt + \sigma dX^{\mathbb{Q}}$$

Thus equating the drift to zero gives us the following result:

$$\theta = \frac{\mu - r}{\sigma}$$

When we talk about change of measure , we actually refer to two very important results :

1. The radon nikodym theorem

   If the measures P and Q share the same null sets, then, there exists a random variable Λ such that

$$\mathbb{Q} = \int_A \Lambda d\mathbb{P}$$

Where by Girsanov Theorem the following holds

$$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

The Novikov condition

A process satisfies novikov condition if the following holds true:

$$\mathbf{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)\right] < \infty$$

The catch with Girsanov is that the theorem stops short of identifying the process θ. We therefore need to have a process in mind and this process has to satisfy the Novikov condition if we want to use Girsanov.

This condition helps to analyse whether a process theta can be used in the girsanov's theorem.

Now the process theta here does satisfy the novikov condition as follows:

$$\mathbf{E}\left[\mathcal{E}\left(\int_0^T \theta_s dX_s\right)\right] = 1$$

Thus we have

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma} X_t - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} t\right)$$

Moreover, the Q-Brownian Motion, X Q, is defined as

$$X_t^{\mathbb{Q}} = X_t + \frac{\mu - r}{\sigma} t,$$

and the discounted asset process is effectively a Q-martingale:

$$\frac{dS^*}{S^*} = \sigma dX^{\mathbb{Q}}$$

Going back to our undiscounted asset price process, S, note that under the measure Q, we have

$$\frac{dS}{S} = r\,dt + \sigma\,dX^{\mathbb{Q}}$$

Integrating we get

$$S_T = S_t \exp\left\{ r\,(T-t) + \sigma \left(X_T^{\mathbb{Q}} - X_t^{\mathbb{Q}} \right) \right\}$$

Setting $$Y_T = \ln \frac{S_T}{S_t},$$

We have $$Y_T \sim \mathcal{N}\left(r\,(T-t), \sigma^2(T-t)\right)$$

Now as we know that payoff function of a call is given by

$$G(S_T) = \max\left[S_T - E, 0\right]$$

Thus we can rewrite our pricing equation as follows:

$$\chi(t, S_t) = e^{-r(T-t)} \int_{-\infty}^{\infty} G(S_0 e^y) p(y) dy$$

Solving we get the final black scholes equation as follows:

$$\chi(t, S_t) = S_t N(d_1) - E e^{-r(T-t)} N(d_2)$$

where

$$\begin{aligned} d_1 &= -z_0 + \sigma\sqrt{\tau} = \frac{\ln\left(\frac{S_t}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= -z_0 = \frac{\ln\left(\frac{S_t}{E}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$