

Permutations

Hoàn v!

Permutations represent the number of different possible ways we can **arrange** a number of elements.

$$P(n) = n \times (n - 1) \times (n - 2) \times \cdots \times 1$$

The diagram illustrates the formula for permutations, $P(n) = n \times (n - 1) \times (n - 2) \times \cdots \times 1$. Four blue arrows point from descriptive text below to specific parts of the formula: one from 'Permutations' to $P(n)$, one from 'Options for who we put first' to n , one from 'Options for who we put second' to $(n - 1)$, and one from 'Options for who we put last' to 1 .

Characteristics of Permutations:

- Arranging **all** elements within the sample space.
- No repetition.
- $P(n) = n \times (n - 1) \times (n - 2) \times \cdots \times 1 = n!$ (Called "n factorial")

Example:

- If we need to arrange 5 people, we would have $P(5) = 120$ ways of doing so.

Factorials

Giai thừa

Factorials express the **product** of all integers from 1 to n and we denote them with the “!” symbol.

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 1$$

Key Values:

- $0! = 1$.
- If $n < 0$, $n!$ does not exist.

Rules for factorial multiplication. (For $n > 0$ and $n > k$)

- $(n + k)! = n! \times (n + 1) \times \cdots \times (n + k)$
- $(n - k)! = \frac{n!}{(n - k + 1) \times \cdots \times (n - k + k)} = \frac{n!}{(n - k + 1) \times \cdots \times n}$
- $\frac{n!}{k!} = \frac{k! \times (k + 1) \times \cdots \times n}{k!} = (k + 1) \times \cdots \times n$

Examples: $n = 7, k = 4$

- $(7 + 4)! = 11! = 7! \times 8 \times 9 \times 10 \times 11$
- $(7 - 4)! = 3! = \frac{7!}{4 \times 5 \times 6 \times 7}$
- $\frac{7!}{4!} = 5 \times 6 \times 7$

Variations

Chỉnh hợp \Rightarrow Có thứ tự

Variations represent the number of different possible ways we can pick and arrange a number of elements.

Variations with repetition $\rightarrow \bar{V}(n, p) = n^p$

lặp

Number of different elements available \uparrow

Number of elements we are arranging \leftarrow

Variations without repetition $\rightarrow V(n, p) = \frac{n!}{(n-p)!}$

Number of different elements available \uparrow

Number of elements we are arranging \leftarrow

Intuition behind the formula. (With Repetition)

- We have n -many options for the first element.
- We **still have** n -many options for the second element because repetition is allowed.
- We have n -many options for each of the p -many elements.
- $n \times n \times n \dots n = n^p$

Intuition behind the formula. (Without Repetition)

- We have n -many options for the first element.
- We **only have** $(n-1)$ -many options for the second element because we cannot repeat the value for we chose to start with.
- We have less options left for each additional element.
- $n \times (n-1) \times (n-2) \dots (n-p+1) = \frac{n!}{(n-p)!}$

Combinations

Tổ hợp \Rightarrow k thứ tự

Combinations represent the number of different possible ways we can pick a number of elements.

$$C(n, p) = \frac{n!}{(n-p)! p!}$$

Combinations \nearrow

\uparrow Total number of elements in the sample space

\nwarrow Number of elements we need to select

repetition

$$C_p^n = \frac{(n+p-1)!}{p! (n-1)!}$$

Characteristics of Combinations:

- Takes into account double-counting. (Selecting Johnny, Kate and Marie is the same as selecting Marie, Kate and Johnny)
- All the different permutations of a single combination are different variations.
- $C = \frac{V}{P} = \frac{n!/(n-p)!}{p!} = \frac{n!}{(n-p)!p!}$
- Combinations are symmetric, so $C_p^n = C_{n-p}^n$, since selecting p elements is the same as omitting n-p elements.

đối xứng

Combinations with separate sample spaces

Combinations represent the number of different possible ways we can pick a number of elements.

$$C = n_1 \times n_2 \times \cdots \times n_p$$

Diagram illustrating the formula for combinations with separate sample spaces:

- Combinations (points to C)
- Size of the first sample space. (points to n_1)
- Size of the second sample space. (points to n_2)
- Size of the last sample space. (points to n_p)

Characteristics of Combinations with separate sample spaces:

- The option we choose for any element does not affect the number of options for the other elements.
- The order in which we pick the individual elements is arbitrary.
- We need to know the size of the sample space for each individual element. $(n_1, n_2 \dots n_p)$

Winning the Lottery

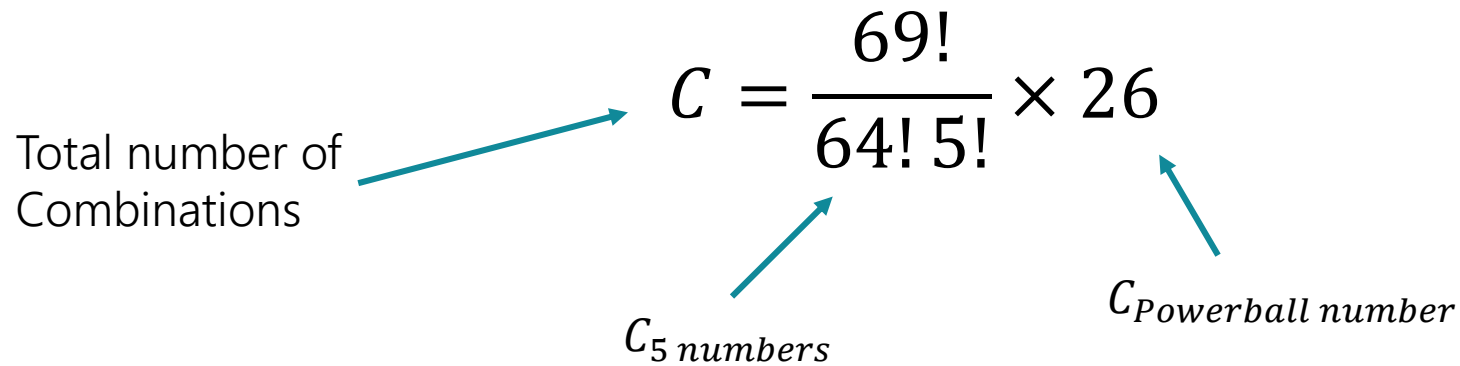
To win the lottery, you need to satisfy two distinct independent events:

- Correctly guess the “Powerball” number. (From 1 to 26)
- Correctly guess the 5 regular numbers. (From 1 to 69)

Total number of Combinations

$$C = \frac{69!}{64! 5!} \times 26$$

$C_{5 \text{ numbers}}$ $C_{\text{Powerball number}}$



Intuition behind the formula:

- We consider the two distinct events as a combination of two elements with different sample sizes.
 - One event has a sample size of 26, the other has a sample size of C_5^{69} .
- Using the “favoured over all” formula, we find the probability of any single ticket winning equals $1 / (\frac{69!}{64! 5!} \times 26)$.