Corrupted Multidimensional Binary Search: Learning in the Presence of Irrational Agents

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Abstract

Standard game-theoretic formulations for settings like contextual pricing and security games assume that agents act in accordance with a specific behavioral model. In practice however, some agents may not prescribe to the dominant behavioral model or may act in ways that are arbitrarily inconsistent. Existing algorithms heavily depend on the model being (approximately) accurate for all agents and have poor performance in the presence of even a few such arbitrarily irrational agents. How do we design learning algorithms that are robust to the presence of arbitrarily irrational agents?

We address this question for a number of canonical game-theoretic applications by designing a robust algorithm for the fundamental problem of multidimensional binary search. The performance of our algorithm degrades gracefully with the number of corrupted rounds, which correspond to irrational agents and need not be known in advance. As binary search is the key primitive in algorithms for contextual pricing, Stackelberg Security Games, and other game-theoretic applications, we immediately obtain robust algorithms for these settings.

Our techniques draw inspiration from learning theory, game theory, high-dimensional geometry, and convex analysis, and may be of independent algorithmic interest.

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1 Introduction

A common assumption in game-theoretic settings such as pricing and Stackelberg Security Games (SSGs) is that agents best-respond according to some (parametric) behavioral model. While models studied in the literature vary in complexity, no behavioral assumption perfectly captures reality, and, in practice, algorithms will interact with agents that deviate from the assumed model. As the behavioral model is the underlying theory of rationality, such deviations seem arbitrarily irrational from the perspective of the algorithm. Unfortunately, existing algorithms for learning in strategic settings rely heavily on behavioral assumptions and can fail dramatically even with a few irrational agents. This naturally leads to the following question:

How do we design learning algorithms that are robust to the presence of arbitrarily irrational agents?

We tackle this question by focusing on binary search, a key ingredient for several online game-theoretic settings, including contextual pricing. In the standard formulation for contextual pricing, the behavioral model is governed by an unknown parameter $\boldsymbol{\theta}^* \in \mathbb{R}^d$, which determines the significance of each product feature: A customer purchases an item with features \boldsymbol{v} at price p if and only if $\langle \boldsymbol{\theta}^*, \boldsymbol{v} \rangle \geq p$. When customers behave consistently according to this model, the seller can use multidimensional binary search techniques to learn the underlying parameter $\boldsymbol{\theta}^*$ and subsequently make pricing decisions that maximize revenue [CLPL19, LPLV18].

In practice, this behavioral assumption may hold for most customers, but some customers may not make purchase decisions according to θ^* . For example, some customers may make impulsive purchase decisions or otherwise exhibit irrational behavior that is not captured by the parametric model. Unfortunately, algorithms based on binary search techniques can completely break down in the presence of such irrational customers; in the sequel we provide examples where even a single irrational agent leads to vacuous performance guarantees.

In this paper, we design algorithms that are robust to violations of the behavioral model, by viewing irrational agents as *completely adversarial*. We present a robust multidimensional binary search algorithm that degrades gracefully in the presence of adversarial corruptions, with no assumptions on how many corrupted agents might appear. Since binary search is a fundamental algorithmic primitive for learning in strategic settings even beyond contextual pricing, our results offer a methodology to create mechanisms that are robust to the presence of an unknown number of irrational agents.

1.1 Our contributions

We make contributions on three fronts. First, from a modeling perspective, we provide a framework to view arbitrarily irrational agents via the lens of adversarial corruptions. We study settings in which the learner interacts with a sequence of agents, where most agents follow a (parametric) behavioral model, but an unknown number of agents act completely adversarially. We seek algorithms with performance guarantees that degrade gracefully with the number of adversarial agents. This perspective contrasts with previous work assuming bounded or stochastic deviations from the behavioral model, which provides a complementary form of robustness, but does not capture some kinds of model violations that may occur in practice. Our approach provides a general framework for designing mechanisms that are robust to arbitrarily irrational agents, and we hope our formulation will prove useful in other game-theoretic settings.

Second, our main algorithmic contribution is a corruption-robust multidimensional binary search procedure (Algorithm 4). We measure performance via a notion of regret, and we prove (Theorem 4.1) that the algorithm has regret $\mathcal{O}(d^3(C + \log T)\log(T)\log(d/\varepsilon))$ where C is the number of

corrupted rounds, d is the dimensionality of the problem, T is the total number of rounds, and ε is the accuracy to which we want to localize the ground truth parameter (for the uncorrupted rounds). This guarantee is similar to previous results for non-robust algorithms [LPLV18], with a favorable nearly-linear scaling with C, demonstrating a minimal overhead for robustness. On the computational side, the algorithm runs in quasipolynomial time. In addition, it is robust to an unknown amount corruption, so the parameter C above need not be known a priori. Finally, and perhaps most importantly, the algorithm or closely related ideas apply to different dynamic pricing formulations as well as Stackelberg Security Games (SSGs), so we seamlessly obtain corruption-robust algorithms and guarantees for these settings as corollaries of our main results.

Our third contribution is technical: our analysis for the corruption-robust multidimensional binary search algorithm requires elegant technical innovations and machinery from several mathematical disciplines. The core technical argument involves finding a hyperplane passing through the centroid of a given convex body, so that a certain *non-convex* region is contained entirely in one halfspace. To do so, the algorithm and proof use classical techniques from learning theory (the perceptron analysis), ideas from convex analysis (Carathéodory's theorem), as well as arguments from high dimensional geometry (Grünbaum's theorem and volumes of spherical caps). Elsewhere we also use modern techniques from learning theory, specifically we extend a multi-layering technique from the stochastic bandit literature [LMPL18] to continuous decision spaces.

1.2 Related work

Our work is squarely situated at the interface of the areas of learning in the presence of strategic agents and online learning robust to adversarial corruptions.

Learning in the presence of strategic agents. A recent thread of research stemming from the Algorithmic Game Theory and Machine Learning communities considers settings where a learner interacts with a number of strategic agents and wishes to obtain pristine data from them, despite their incentives to game her. Among these problems, our work is most closely related to dynamic pricing, which was originally introduced as a single-dimensional problem by Kleinberg and Leighton [KL03]. Subsequently, Amin et al. [ARS14] studied the problem in its multidimensional form. but focused on iid contexts. Cohen et al. [CLPL19] studied¹ contextual pricing with adversarial contexts and subsequently, Lobel et al. [LPLV18] and Paes Leme and Schneider [PLS18] obtained improved regret guarantees. Mao et al. [MPLS18] studied a variant of the standard contextual pricing problem, where the buyers have utilities that are no longer linear in the contexts, but are instead Lipschitz. Recently, Shah et al. [SJB19] also studied the problem of contextual dynamic pricing having under semi-parametric model assumptions. Our work is also related to the works studying learning of optimal commitment strategies in SSGs. It is nearly impossible to survey the vast literature in SSGs (see e.g., [SFA⁺18] for a survey). Closer to our work, Blum et al. [BHP14] study the sample complexity of learning the optimal strategy to commit to, when the utility of the attacker is unknown, and more recently, Peng et al. [PSTZ19] obtained strengthened guarantees. The fundamental difference between these works and ours is that we focus on cases where we might encounter a number of irrational agents, for which all previous works fail to provide meaningful guarantees. That said, these works (e.g., [LPLV18], [PSTZ19], [MPLS18]) serve as starting points for our algorithms.

¹Cohen et al. also extended their algorithm to a noise-tolerant variant, for stochastic noise models. Models of stochastic noise have also been considered for the problem of binary search in graphs in [EZKS16].

Online learning robust to adversarial corruptions. Lykouris et al. [LMPL18] introduced a variant of the standard stochastic multi-armed bandit problem, where an adversary can corrupt a number of samples, and provided algorithms with learning rates that degrade according to the number of corruptions. The guarantees for stochastic multi-armed bandits were subsequently strengthened by Gupta et al. [GKT19] and Zimmert and Seldin [ZS19], and the concept of adversarial corruptions has also been extended to several other settings including dynamic assortment optimization [CKW19], linear bandits [LLS19] and reinforcement learning [LSSS19]. Our work differs from these in that we use adversarial corruptions as a modeling tool to capture arbitrarily irrational agent behavior in game-theoretic settings.² On a technical level, our setting involves a continuous action space, while all of the prior results involve discrete (potentially large) action spaces. More generally, as our setting differs considerably from these prior works, achieving corruption-robustness requires quite different algorithmic and analytical techniques.

Finally, Feng et al. [FPX19] study the problem of whether stochastic multi-armed bandit algorithms are inherently robust to particular forms of strategic manipulation. This is orthogonal to our work, where we know that agents' irrational responses can have catastrophic effects on standard algorithms, so we propose algorithms that are robust to them.

Further related work. Distantly related to our work are papers that study variants of bounded rationality, especially in SSGs (see e.g., [PJT⁺10, HFN⁺16]), and other utility models from the attacker's perspective (e.g., [GGTT⁺19]). What sets apart these works from ours is that the authors assume that the behavior of the agents still subscribes to some specific utility model, e.g., quantal responses, or responses that inflict the highest loss to the learner, despite not attacking the target that gives them the higher value at the current round. In our paper, however, agents can give arbitrarily different responses from their dominant behavioral model, which could prove to be detrimental for the previously studied settings. Finally, the binary feedback model that our work considers has similarities with the feedback model considered by two recent papers in online learning in auctions ([WPR16, FPS18]). Apart from working in a regime without any corruptions, the techniques presented in [WPR16, FPS18] are tailored to auctions and do not address multidimensional binary search, as we consider in this work.

2 Preliminaries

In this section, we describe our model, and subsequently, we briefly review the algorithm of Lobel et al. [LPLV18], which achieves regret $\mathcal{O}(d\log(d/\varepsilon))$ (for target accuracy $\varepsilon > 0$) for the problem of multidimensional binary search without corruptions, where d is the dimension of the problem. This algorithm serves as a building block in our corrupted multidimensional binary search algorithm. Without loss of generality, we assume that the accuracy parameter is less than $1/\sqrt{d}$, i.e., $\varepsilon < 1/\sqrt{d}$.

2.1 Model

We consider the following repeated interaction between the learner and an opponent⁴ (also referred to as *nature*). There is a ground truth vector $\boldsymbol{\theta}^{\star} \in \mathbb{R}^d$ that parameterizes the behavioral model for rational agents. Initially, the opponent chooses $\boldsymbol{\theta}^{\star} \in K_0 = \{\boldsymbol{\theta} \in \mathbb{R}^d : \|\boldsymbol{\theta}\|_2 \leq 1\}$. The learning process proceeds for T rounds, and in round t:

²Chen et al. [CKW19] also consider a behavioral setting, but they do not make a connection to irrationality.

³Since our results scale logarithmically with ε , running our algorithm with smaller target accuracy $\varepsilon' < \varepsilon/\sqrt{d}$ guarantees that this assumption is satisfied and contributes only an extra logarithmic term on d.

⁴To differentiate between the two, we refer to the learner as "she" and to the opponent as "he."

- 1. The opponent chooses (potentially adaptively) context v_t from set $\mathcal{V} = \{ \mathbf{v} \in \mathbb{R}^d : ||\mathbf{v}||_2 = 1 \}$.
- 2. The learner observes v_t and forms a distribution \mathcal{D}_t over potential, real-valued query points.
- 3. The opponent decides whether to corrupt $(c_t = 1)$ or not $(c_t = 0)$. If the round is not corrupted $(c_t = 0)$, the parameter for the round is $\theta_t = \theta^*$; otherwise, θ_t is set arbitrarily.
- 4. The learner queries point $\omega_t \sim \mathcal{D}_t$ and observes feedback $y_t = \text{sign}(\langle \boldsymbol{v}_t, \boldsymbol{\theta}_t \rangle \omega_t) \in \{-1, 1\}$ where sign(x) = 1 if $x \geq 0$ and -1 otherwise. To ease exposition, we assume without loss of generality that $y_t = 1, \forall t \in [T]$ by appropriately adjusting \boldsymbol{v}_t .

Intuitively, θ_t corresponds to the behavioral model according to which the agent at round t acts. When there is no corruption, the agent is rational with behavior dictated by the ground truth parameter θ^* . On the other hand, in corrupted rounds, the opponent may return arbitrary answers potentially not aligned with the ground truth θ^* . The opponent knows the learner's algorithm, along with the realization of all randomness up to and including round t-1 (i.e., he knows all $\omega_{\tau}, \forall \tau \leq t-1$) but does not have access to the learner's randomness for the current round t (see discussion for this modeling decision in Section 5).

We measure the performance of the learner via a notion of regret, defined as follows. In round t, the instantaneous loss is $\ell_t(\omega_t, \boldsymbol{\theta}_t) = \mathbb{1}\{|\omega_t - \langle \boldsymbol{v}_t, \boldsymbol{\theta}_t \rangle| \geq \varepsilon\}$, so the learner incurs unit loss for a prediction ω_t that is ε far from the target $\langle \boldsymbol{v}_t, \boldsymbol{\theta}_t \rangle$. For ε -accurate predictions, the learner incurs no loss. The cumulative loss incurred by the learner over T rounds is: $R(T) = \sum_{t=1}^T \ell_t(\omega_t, \boldsymbol{\theta}_t)$. We refer to this quantity as $regret^6$ to align with prior work, and seek algorithms that minimize the regret.

We note that the learner does not observe the loss function $\ell_t(\cdot, \boldsymbol{\theta}_t)$ or even her own loss $\ell_t(\omega_t, \boldsymbol{\theta}_t)$. In the online learning terminology, the feedback is neither full-information nor bandit. Instead, the learner only observes the binary variable $y_t = \text{sign}(\langle \boldsymbol{v}_t, \boldsymbol{\theta}_t \rangle - \omega_t)$ as described in Step 4 of the protocol. In Section 5, we extend our results to two related loss functions: the *symmetric* and the *pricing* loss, which are more relevant for some game theoretic settings (e.g., dynamic pricing).

Notation. We use bold letters to denote multi-dimensional quantities, and we define hyperplanes in terms of their normal vectors, which we always take to have unit ℓ_2 norm. To simplify notation, we use the tuple (\mathbf{h}, ω) to denote the hyperplane with normal vector \mathbf{h} and intercept ω , i.e., $\{\mathbf{x} : \langle \mathbf{h}, \mathbf{x} \rangle = \omega\}$. For normal vector $\mathbf{h} \in \mathbb{R}^d$ and scalar ω , we use $\mathbf{H}^+(\mathbf{h}, \omega)$, $\mathbf{H}^-(\mathbf{h}, \omega)$ to denote the positive and negative halfspaces it creates with intercept ω , i.e., $\{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{h}, \mathbf{x} \rangle \geq \omega\}$ and $\{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{h}, \mathbf{x} \rangle \leq \omega\}$, respectively. For a convex body $K \subset \mathbb{R}^d$ we denote by $w(K, \mathbf{u}) = \max_{\mathbf{p}, \mathbf{q} \in K} \langle \mathbf{u}, \mathbf{p} - \mathbf{q} \rangle$ the width of K along the direction of unit vector \mathbf{u} . For a subspace L of K we denote by $\Pi_L K$ the projection of K onto L, formally this is $\{\pi_L(x) : x \in K\}$ where $\pi_L(x)$ is the orthogonal projection of K onto the subspace K. We define the distance of a point K from a hyperplane K as $K \in \mathbb{R}^d = \mathbb{R}^d =$

2.2 Multidimensional Binary Search without corruptions

At a high level, PROJECTEDVOLUME tries to identify θ^* . At all rounds $t \in [T]$ the algorithm maintains a convex body, called the *knowledge set* and denoted by $K_t \in \mathbb{R}^d$, which corresponds to all values θ that are not ruled out based on the information until round t. It also maintains a

⁵This is without loss of generality since the algorithm can negate v_t and ω_t to force $y_t = 1$.

⁶The terminology is apt, since with no corruptions, there is a benchmark linear prediction rule (namely $v \mapsto \langle \theta^*, v \rangle$) that incurs zero cumulative loss. Therefore, this performance measure captures the additional loss incurred for not knowing the ground truth θ^* and which rounds are corrupted.

set of orthonormal vectors $S_t = \{\mathbf{s}_1, \dots, \mathbf{s}_{|S_t|}\}$, spanning a subspace \mathcal{V}_t of dimensionality $|S_t|$, such that K_t has small width along any of those directions, i.e., $\forall \mathbf{s} \in S_t : w(K_t, \mathbf{s}) \leq \delta'$. The algorithm "ignores" a dimension of K_t , once it becomes small, and focuses on the projection of K_t onto a set L_t of dimensions that are orthogonal to S_t and have larger width, i.e., $\forall \mathbf{l} \in L_t : w(K_t, \mathbf{l}) \geq \delta'$. The so-called $Cylindrification \, Cyl(K_t, S_t)$, guarantees that the projection of K_t onto L_t is the same while also regularizing the projection of K_t onto S_t .

Definition 2.1 (Cylindrification, Definition 4.1 of [LPLV18]). Given a set of orthonormal vectors $S = \{\mathbf{s}_1, \dots, \mathbf{s}_n\}$, let $L = \{\mathbf{u} | \langle \mathbf{u}, \mathbf{s} \rangle = 0; \forall \mathbf{s} \in S\}$ be a subspace orthogonal to $\operatorname{span}(S)$ and $\Pi_L K$ be the projection of convex set $K \subseteq \mathbb{R}^d$ onto L. We define:

$$\mathtt{Cyl}(K,S) := \left\{ \mathbf{x} + \sum_{i=1}^n y_i \mathbf{s}_i \middle| \mathbf{x} \in \Pi_L K \text{ and } \min_{\boldsymbol{\theta} \in K} \langle \boldsymbol{\theta}, \mathbf{s}_i \rangle \leq y_i \leq \max_{\boldsymbol{\theta} \in K} \langle \boldsymbol{\theta}, \mathbf{s}_i \rangle \right\}.$$

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ALGORITHM 1: PROJECTED VOLUME [LPLV18]
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\overline{\text{Initialize } S_0 \leftarrow \emptyset, K_0 \leftarrow \{\boldsymbol{\theta} \in \mathbb{R}^d : \|\boldsymbol{\theta}\|_2 \leq 1\}}.
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for $t \in [T]$ do

Observe context v_t , chosen by the opponent.

Query point $\omega_t = \langle \boldsymbol{v}_t, \boldsymbol{\kappa}_t \rangle$, where $\boldsymbol{\kappa}_t \leftarrow \text{apx-centroid}(\text{Cyl}(K_t, S_t))$.

From opponent's response, $K_{t+1} \leftarrow K_t \cap \mathbf{H}^+(\boldsymbol{v}_t, \omega_t)$ or $K_{t+1} \leftarrow K_t \cap \mathbf{H}^-(\boldsymbol{v}_t, \omega_t)$.

Add all directions \mathbf{u} orthogonal to S_t with $w(K_{t+1}, \mathbf{u}) \leq \delta' = \frac{\varepsilon^2}{16d(d+1)^2}$ to S_t .

7 Set $S_{t+1} = S_t$

At round t, after observing \boldsymbol{v}_t , the algorithm aims to query point $\omega_t^* = \langle \boldsymbol{v}_t, \boldsymbol{\kappa}_t^* \rangle$, where $\boldsymbol{\kappa}_t^*$ is the *centroid* of $\operatorname{Cyl}(K_t, S_t)$. Computing $\boldsymbol{\kappa}_t^*$ exactly is a #P-hard problem but Lobel et al. [LPLV18] show that an approximate centroid $\boldsymbol{\kappa}_t$ can be computed in polynomial time and suffices for the performance guarantee. The algorithm therefore queries point $\omega_t = \langle \boldsymbol{v}_t, \boldsymbol{\kappa}_t \rangle$.

Definition 2.2 (Centroid). The centroid κ^* of a convex set K is $\kappa^* = \frac{1}{\text{vol}(K)} \int_K \mathbf{x} d\mathbf{x}$, where $\text{vol}(\cdot)$ denotes the volume of a set.

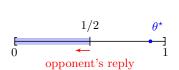
Based on the opponent's response, the algorithm eliminates one halfspace of hyperplane (v_t, ω_t) . The analysis uses the volume of $\Pi_{L_t}K_t$, denoted by vol $(\Pi_{L_t}K_t)$, as a potential function. After each query either the set of small dimensions S_t increases thus making vol $(\Pi_{L_t}K_t)$ increase by a bounded amount (which can happen at most d times) or vol $(\Pi_{L_t}K_t)$ decreases by a factor of $(1 - 1/e^2)$. This potential function argument leads to a regret of at most $\mathcal{O}(d \log(d/\varepsilon))$.

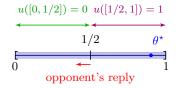
3 Multidimensional Binary Search with known corruption level

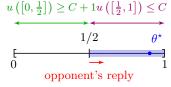
In this section, we focus on the task of making multidimensional binary search robust to a known corruption level C. Our main contribution here is the Corpv.K algorithm (Algorithm 2). We state its performance guarantee in the following theorem.

Theorem 3.1. For a known corruption level C and target accuracy $\varepsilon > 0$, the regret guarantee of CORPV.K is $\mathcal{O}\left(\left(d^2C + 1\right)d\log\left(\frac{d}{\varepsilon}\right)\right)$. Its expected runtime is $\mathcal{O}\left(\left(d^2C\right)^C \cdot \operatorname{poly}\left(d\log\left(\frac{d}{\varepsilon}\right), C\right)\right)$.

At first glance, CORPV.K seems to suffer from two fundamental problems. First, it assumes knowledge of the corruption level, which is unrealistic in our game-theoretic applications (see Section 5). Second, its expected runtime is exponential in C. In Section 4, we address both of these problems, by showing how CORPV.K can be extended to become agnostic to the corruption level C and have quasi-polynomial in T expected runtime.







- (a) Standard binary search after 1 round.
- (b) Undesirability levels after 1 round.
- (c) Undesirability levels after 2C + 1 round.

Figure 1: Single dimensional binary search. Opaque band denotes the knowledge set after each query.

Warm-up: Single-dimension. In the classical single dimensional binary search without corruptions, there exists a value $\theta^* \in \mathbb{R}$ that the learner wishes to identify by making a number of queries/guesses. For each of the queries, the opponent replies whether the queried scalar is *greater* or *smaller* than θ^* , in a way that is consistent with all past replies. In this sense, from the opponent's replies, the learner can keep a diminishing knowledge interval, where she thinks that θ^* lies.⁷ After $\log(1/\varepsilon)$ queries, this knowledge interval has shrunk enough (i.e., has length less than ε), so that any point in it is within ε precision from θ^* .

Imagine now that in this standard setting, the opponent is not always consistent with the ground truth; rather, some of his replies are *corrupted*. Interestingly, even a single corruption suffices to significantly mislead the algorithm. One such example is shown in Figure 1a, where the learner queries point 1/2, and the opponent with only 1 corruption, sends the learner to the opposite direction. This results in the learner keeping the interval [0, 1/2], instead of [1/2, 1], as her current knowledge interval. The adverse effect that this has becomes even more apparent in the dynamic pricing setting (Section 5).

However, if the learner knows that the maximum number of corruptions is C, then, by repeatedly querying the same value, she can guarantee that if she observes the same answer from the opponent for at least C+1 times, then this answer is definitely uncorrupted. In fact, if the learner repeats each query 2C+1 times, she can run the binary search procedure and incur regret at most $(2C+1)\log(1/\varepsilon)$. Unfortunately, this approach does not directly extend to multiple dimensions, since the opponent provides different contexts and hence, the learner cannot repeat the exact same query.

3.1 Overview of the approach

To generalize to higher dimensions, it is helpful to rephrase the one-dimensional solution with some additional terminology. We maintain an undesirability level for each point in the knowledge set, and we eliminate points from the knowledge set, only when their undesirability level strictly exceeds C. Thus, the algorithm proceeds in epochs: in each epoch ϕ the knowledge set K_{ϕ} remains fixed, and we always respond to queries using the (approximate) centroid κ_{ϕ} of K_{ϕ} . In one dimension, we see that after 2C + 1 queries, we must have that half of the knowledge set has undesirability at least C + 1, so we eliminate it, shrink our knowledge set, and move to the next epoch.

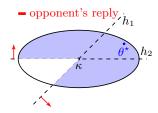
In higher dimensions, the concept of undesirability levels is still extremely useful, but the rest of the argument is substantially more intricate. We encounter 4 main challenges. We start with a high-level description of these challenges and how we overcome them.

Challenge 1: Non-convexity of the desirable region. In the one-dimensional case, as soon as a region has undesirability $\geq C+1$, we remove it and advance to the next epoch. In higher dimensions, this strategy may result in a non-convex knowledge set, for which it is much more

⁷This corresponds to the *knowledge set* we mentioned in Section 2.

analytically challenging to measure our progress (e.g., the convex hull or enclosing ellipsoid may not shrink). An example where we obtain a non-convex knowledge set is illustrated in Figure 2. The reason for this is that the query vectors v_t observed during the epoch may all be different, and so different regions of the knowledge set accumulate very different undesirability levels. In contrast, in one dimension all query vectors are the same (up to sign) so we only have two distinct regions of the knowledge set.

The first challenge that we face is to address this non-convexity that may arise when updating the knowledge set. Our solution avoids non-convexity altogether: we always update the knowledge set by taking intersection with some halfspace passing near the centroid of the knowledge set (which we call a hyperplane cut). By preserving convexity, hyperplane cuts allow us to apply potential function arguments to track progress.



Challenge 2: Existence of a hyperplane cut. Hyperplane cuts preserve convexity, but does one even exist? It is easy to see that with 2C+1 queries (as was sufficient in one-dimension), the answer to this question is "no" (consider Figure 2 where a third, different hyperplane h_3 is shown and corrupted). With more queries, we might expect that one of the context vectors \boldsymbol{v}_t we have seen during the epoch is a valid

Figure 2: Non-convex undesirable region when C = 1. Blue opaque region denotes knowledge set for next round.

hyperplane cut, separating the region with high undesirability from the rest. Indeed, as we show in Appendix A.1, this is the case for d = 2 (Proposition A.2) but is *not* true in higher dimensions (Proposition A.3).

Therefore, the second challenge is to show that a hyperplane cut even exists. We address this challenge by searching outside of the presented contexts v_t . Via an existential argument using tools from convex analysis (in particular, Carathéodory's theorem), we prove that if epochs have 2dC(d+1)+1 rounds, then a hyperplane cut is guaranteed to exist. To compare, in one dimension we are guaranteed that a context vector is a hyperplane cut if epochs have 2C+1 rounds. Similarly, with no corruptions, we are guaranteed that every single context vector is a hyperplane cut.

Challenge 3: Computing the hyperplane cut. The above argument establishes existence of a hyperplane cut, and while this is certainly helpful, our algorithm actually needs to compute it, so that it can update the knowledge set for the next epoch. The algorithmic question here is to find a hyperplane passing near the centroid of the knowledge set, such that all points with low undesirability lie in one halfspace.

We address this challenge with the Perceptron algorithm, using the fact that we can characterize the points with low undesirability. First, we randomly sample a point \mathbf{q} from a small ball around the centroid of the knowledge set. Then, we run Perceptron with \mathbf{q} as the negative example, and with the centroid and all the points with low undesirability as the positive examples (we can efficiently search over the latter via linear programming). If \mathbf{q} has large margin with respect to the hyperplane whose existence we already established, then via Perceptron's finite mistake bound, we are guaranteed to find a separating hyperplane. Finally, using tools from high-dimensional geometry, we prove that \mathbf{q} has large margin with good probability, so the procedure need not be restarted many times.

Challenge 4: Ensuring volume reduction. The final issue is that the knowledge set may have small width in the direction of the hyperplane cut that we have found, which implies that we do not make sufficient progress in this epoch. This was not an issue in Lobel et al., since (a) they

ALGORITHM 2: CORRUPTEDPROJECTEDVOLUME-KNOWN (CORPV.K)

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Initialize \phi = 1, A_{\phi} \leftarrow \emptyset, K_{\phi} \leftarrow K_0, S_{\phi} \leftarrow \emptyset, L_{\phi} \leftarrow \text{orthonormal-basis}(\mathbb{R}^d), T_{\phi} \leftarrow \emptyset, 
\kappa_{\phi} \leftarrow \text{apx-centroid}(\text{Cyl}(K_{\phi}, S_{\phi})), \text{ and threshold } \delta = \frac{\varepsilon}{4(d+\sqrt{d})}
      for t \in [T] do
                Observe context v_t, compute query point \omega_t = \langle v_t, \kappa_\phi \rangle and observe opponent's response y_t.
  3
                if L_{\phi} \neq \emptyset and w(\operatorname{Cyl}(K_{\phi}, S_{\phi}), \boldsymbol{v}_t) > \varepsilon then
                                                                                                                                                                                                                 Update T_{\phi} \leftarrow T_{\phi} \cup \{t\} and A_{\phi} \leftarrow A_{\phi} \bigcup \{(\Pi_{L_{\phi}} \boldsymbol{v}_{t}, \omega_{t}, y_{t})\}.
                        if |T_{\phi}| \geq 2d \cdot C(d+1) + 1 then
                                                                                                                                                                                                               \triangleright see Section 3.2
  6
                                Compute: ((\mathbf{h}_{\phi}, \psi_{\phi}), \mathbf{H}_{\phi}(\mathbf{h}_{\phi}, \psi_{\phi})) \leftarrow \text{ELIM}(A_{\phi}, S_{\phi}).
                                Update K_{\phi+1} \leftarrow K_{\phi} \cap \mathbf{H}_{\phi}(\mathbf{h}_{\phi}, \psi_{\phi}), and temporary sets \widetilde{S} \leftarrow S_{\phi} and \widetilde{L} \leftarrow L_{\phi}.
  8
                                if w(\Pi_{L_{\phi}}K_{\phi}, \mathbf{h}_{\phi}) \leq \delta then
  9
                                         Add hyperplane to small dimensions \widetilde{S} \leftarrow S_{\phi} \bigcup \{\mathbf{h}_{\phi}\}.
10
                                        Compute orthonormal basis for new large dimensions \widetilde{L} (without \widetilde{S}).
11
                                Update L_{\phi+1} \leftarrow \widetilde{L} \setminus \{e_i \in \widetilde{L} : w(K_{\phi+1}, e_i) \leq \delta\} and S_{\phi+1} \leftarrow \widetilde{S} \bigcup (\widetilde{L} \setminus L_{\phi+1}).
12
                                Move to next epoch: \phi \leftarrow \phi + 1, T_{\phi} \leftarrow \emptyset, A_{\phi} \leftarrow \emptyset, \kappa_{\phi} \leftarrow \text{apx-centroid}(\text{Cyl}(K_{\phi}, S_{\phi})).
13
```

always make cuts corresponding to context vectors, and (b) they can bound the regret incurred by the width in the direction of the hyperplane cut. However, with corruptions, these two steps are not possible. As we have discussed, no context vector may induce a valid hyperplane cut. More insidiously, all context vectors may correspond to large width directions (so we incur large regret), but the hyperplane found by Perceptron may have small width (so we make no volumetric progress).

We address this challenge by proving that the hyperplane found by Perceptron is orthogonal to all of the small dimensions (recall set S_t in Projected Volume). Then, if the knowledge set has large width in the direction of the found hyperplane, we make volumetric progress as in Lobel et al. [LPLV18]. On the other hand, if the width is small, then we can add this direction to the set of small dimensions. Either way we make sufficient progress.

Our algorithm. We conclude this overview by formally presenting our algorithm, CORPV.K, with pseudocode displayed in Algorithm 2. The algorithm proceeds in epochs, and in epoch ϕ , we maintain two sets of orthogonal directions S_{ϕ} and L_{ϕ} , as in PROJECTEDVOLUME. S_{ϕ} corresponds to the learned directions and is initially empty, while L_{ϕ} corresponds to the unknown ones, so it is initialized to an orthonormal basis for \mathbb{R}^d . In addition, we maintain the knowledge set K_{ϕ} .

At the beginning of each epoch, we compute an approximate centroid κ_{ϕ} for $\mathrm{Cyl}(K_{\phi}, S_{\phi})$ such that $\|\kappa_{\phi} - \kappa_{\phi}^{\star}\| \leq \bar{\nu}$ for a scalar $\bar{\nu}$ to be instantiated below. For the duration of the epoch ϕ , we always respond to queries with κ_{ϕ} , that is with context v_t , our response is $\omega_t = \langle v_t, \kappa_{\phi} \rangle$. We collect all the rounds for which $w(\mathrm{Cyl}(K_{\phi}, S_{\phi}, v_t)) > \varepsilon$, and once we have seen $2d \cdot C(d+1) + 1$ such contexts, we execute the ELIM subroutine to find a hyperplane cut $(\mathbf{h}_{\phi}, \psi_{\phi})$. The ELIM routine runs the Perceptron algorithm with a randomly sampled point as the negative example and using κ_{ϕ} and all regions with low undesirability as the positive examples. It returns a halfspace that is guaranteed to contain θ^{\star} and we update the knowledge set by taking intersection with the halfspace. Then if the updated knowledge set has small width in any large direction (for computational reasons, we only check \mathbf{h}_{ϕ} and the basis for L_{ϕ}), we can add that direction to the set of small dimensions and advance to the next epoch.

3.2 Existence of Appropriately Undesirable Hyperplane

In this subsection, we prove that after $|T_{\phi}| = 2dC(d+1) + 1$ rounds, there exists a hyperplane $(\mathbf{h}_{\phi}, \psi_{\phi})$ with undesirability at least C+1 in the entirety of one of its halfspaces. Formally, this is stated in Lemma 3.2. The results of this subsection hold for any scalar $0 < \delta < \frac{\varepsilon}{2\sqrt{d}+4d}$, so we defer the exact tuning of δ for Section 3.4.

Lemma 3.2. For any epoch ϕ , scalar $\delta \in \left(0, \frac{\varepsilon}{2\sqrt{d}+4d}\right)$, and scalar $\bar{\nu} = \frac{\varepsilon-2\sqrt{d}\cdot\delta}{4\sqrt{d}}$, after $|T_{\phi}| = 2d \cdot C(d+1) + 1$ rounds, there exists a hyperplane $(\mathbf{h}_{\phi}, \psi_{\phi})$ orthogonal to all small dimensions S_{ϕ} such that $\operatorname{dist}(\kappa_{\phi}, (\mathbf{h}_{\phi}, \psi_{\phi})) \leq \bar{\nu}$, and the resulting halfspace $\mathbf{H}_{\phi}(\mathbf{h}_{\phi}, \psi_{\phi})$ always contains $\boldsymbol{\theta}^{\star}$.

At a high level, the tuning of $\bar{\nu}$ depends on two factors. First, in order to make sure that we make enough progress in terms of volume elimination, despite the fact that we do not make a cut through κ_{ϕ} , we need $\bar{\nu}$ close enough to κ_{ϕ} (formally stated in Lemma 3.17). The second factor that affects the tuning of $\bar{\nu}$ is that we need to guarantee that there exists at least one point with a very high undesirability level. Formally, this is established in Lemma 3.10. To assist us in the analysis, we define the ν -margin projected undesirability levels, which we later apply for some $\nu < \bar{\nu}$:

Definition 3.3 (ν -Margin Projected Undesirability Level). Consider an epoch ϕ , a scalar ν , and a point \mathbf{p} in K_{ϕ} . Given the set $A_{\phi} = \{(\Pi_{L_{\phi}} \mathbf{v}_t, \omega_t), y_t\}_{t \in T_{\phi}}$, we define \mathbf{p} 's ν -margin projected undesirability level, denoted by $u_{\phi}(\mathbf{p}, \nu)$, as the number of rounds within epoch ϕ , for which

$$u_{\phi}(\mathbf{p},
u) = \sum_{t \in T_{\phi}} \mathbb{1}\left\{\left(\left\langle \mathbf{p} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle +
u\right) < 0\right\}$$

Recall that, to simplify presentation, we assume that $y_t = 1$ for all contexts (by appropriately adjusting \mathbf{v}_t and ω_t). With this simplification, we can show the following lemma:

Lemma 3.4. If $\nu > \underline{\nu}$, where $\underline{\nu} = \sqrt{d\delta}$, for any point **p**, we have that:

$$u_{\phi}(\mathbf{p}, \nu) = \sum_{t \in T_{\phi}} \mathbb{1}\left\{ \left(\left\langle \mathbf{p} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu \right) \cdot \left(\left\langle \boldsymbol{\theta}_{t} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu \right) < 0 \right\}$$

Proof. In order to prove the lemma, we argue that $\langle \boldsymbol{\theta}_t - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_t \rangle + \nu > 0$. Since $y_t = 1$, then $\langle \boldsymbol{\theta}_t - \boldsymbol{\kappa}_{\phi}, \boldsymbol{v}_t \rangle \geq 0$. Expanding the latter, we obtain that:

$$\langle \boldsymbol{\theta}_t - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_t \rangle + \langle \boldsymbol{\theta}_t - \boldsymbol{\kappa}_{\phi}, \Pi_{S_{\phi}} \boldsymbol{v}_t \rangle \ge 0$$
 (1)

We proceed by upper bounding the quantity $\langle \boldsymbol{\theta}_t - \boldsymbol{\kappa}_{\phi}, \Pi_{S_{\phi}} \boldsymbol{v}_t \rangle$. Let S be a matrix with columns corresponding to an orthonormal basis for S_{ϕ} , so that $\Pi_{S_{\phi}} = SS^{\top}$. Then, we obtain:

$$\langle \Pi_{S_{\phi}} \boldsymbol{v}_{t}, \mathbf{p} - \boldsymbol{\kappa}_{\phi} \rangle = \langle \Pi_{S_{\phi}} \boldsymbol{v}_{t}, \Pi_{S_{\phi}} (\mathbf{p} - \boldsymbol{\kappa}_{\phi}) \rangle \leq |\langle \Pi_{S_{\phi}} \boldsymbol{v}_{t}, \Pi_{S_{\phi}} (\mathbf{p} - \boldsymbol{\kappa}_{\phi}) \rangle|$$

$$\leq \|\Pi_{S_{\phi}} \boldsymbol{v}_{t}\|_{2} \cdot \|\Pi_{S_{\phi}} (\mathbf{p} - \boldsymbol{\kappa}_{\phi})\|_{2} \qquad \text{(Cauchy-Schwarz inequality)}$$

$$= \|\Pi_{S_{\phi}} \boldsymbol{v}_{t}\|_{2} \cdot \|S^{\top} (\mathbf{p} - \boldsymbol{\kappa}_{\phi})\|_{2}$$

$$\leq \|\boldsymbol{v}_{t}\|_{2} \cdot \sqrt{d} \cdot \|S^{\top} (\mathbf{p} - \boldsymbol{\kappa}_{\phi})\|_{\infty} \qquad (\|\mathbf{x}\|_{2} = \sqrt{\sum_{i \in [d]} \mathbf{x}_{i}^{2}} \leq \sqrt{d} \cdot \|\mathbf{x}\|_{\infty}^{2})$$

$$\leq 1 \cdot \delta \sqrt{d}. \qquad (\|\boldsymbol{v}_{t}\|_{2} = 1 \text{ and } w(K_{\phi}, \mathbf{s}) \leq \delta, \forall \mathbf{s} \in S_{\phi})$$

Using the latter to relax (1) along with $\underline{\nu} = \sqrt{d}\delta$ we get that: $\langle \boldsymbol{\theta}_t - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_t \rangle \geq -\underline{\nu}$. Since $\nu > \underline{\nu}$, it follows that $\langle \boldsymbol{\theta}_t - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_t \rangle + \nu \geq -\underline{\nu} + \nu > 0$ which concludes the lemma.

We next define the C-Protected Region in Large Dimensions, which in words is the set of points in K_{ϕ} with ν -Margin Projected Undesirability Level at most C.

Definition 3.5 (C-Protected Region in Large Dimensions). Given set $A_{\phi} = \{(\Pi_{L_{\phi}} \boldsymbol{v}_{t}, \omega_{t}), y_{t}\}_{t \in T_{\phi}}$ we define the C-Protected Region in Large Dimensions to be $\mathcal{P}(C, \nu) = \{\mathbf{p} : u_{\phi}(\mathbf{p}, \nu) \leq C\}$.

The next lemma establishes that if we keep set $\mathcal{P}(C,\nu)$ intact in the convex body formed for the next epoch $K_{\phi+1}$, then we are guaranteed to not eliminate point $\boldsymbol{\theta}^{\star}$.

Lemma 3.6. If $\nu > \underline{\nu}$, where $\underline{\nu} = \sqrt{d\delta}$, the ground truth θ^* belongs in set $\mathcal{P}(C, \nu)$.

Proof. By Lemma 3.4, $u_{\phi}(\mathbf{p}, \nu) = \sum_{t \in T_{\phi}} \mathbb{1}\left\{\left(\left\langle \mathbf{p} - \kappa_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t}\right\rangle + \nu\right) \cdot \left(\left\langle \boldsymbol{\theta}_{t} - \kappa_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t}\right\rangle + \nu\right) < 0\right\}$. For the uncorrupted rounds $\boldsymbol{\theta}^{\star} = \mathbf{p} = \boldsymbol{\theta}_{t}$; as a result, the corresponding summands are non-negative: $\left(\left\langle \mathbf{p} - \kappa_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t}\right\rangle + \nu\right) \cdot \left(\left\langle \boldsymbol{\theta}_{t} - \kappa_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t}\right\rangle + \nu\right) \geq 0$. Hence, the only rounds for which $\boldsymbol{\theta}^{\star}$ can incur undesirability are the corrupted rounds, of which there are at most C. As a result, $u_{\phi}(\boldsymbol{\theta}^{\star}, \nu) \leq C$ and $\boldsymbol{\theta}^{\star} \in \mathcal{P}(C, \nu)$ by Definition 3.5.

We next show that there exists a hyperplane that cuts orthogonally to all small dimensions in a way that guarantees that set $\mathcal{P}(C,\nu)$ is preserved in $K_{\phi+1}$. Note that due to Lemma 3.6 this is enough to guarantee that we have preserved $\boldsymbol{\theta}^*$ in $K_{\phi+1}$. However, $\mathcal{P}(C,\nu)$ is generally non-convex and it is not easy to directly make claims about it. Instead, we focus on its convex hull, denoted by $\operatorname{conv}(\mathcal{P}(C,\nu))$. Interestingly, we can upper bound the undesirability of any point in $\operatorname{conv}(\mathcal{P}(C,\nu))$, by applying Carathéodory's Theorem, which says that any point in the convex hull of a (possibly non-convex) set can be written as a convex combination of at most d+1 points of that set. Using this result, we can bound the ν -margin projected undesirability levels of all the points in $\operatorname{conv}(\mathcal{P}(C,\nu))$.

Lemma 3.7. For any scalar ν , epoch ϕ and any point $\mathbf{p} \in \text{conv}(\mathcal{P}(C,\nu))$, its ν -margin projected undesirability level is at most C(d+1), i.e., $u_{\phi}(\mathbf{p},\nu) \leq C(d+1)$.

Proof. From Carathéodory's Theorem, since $\mathbf{p} \in \mathbb{R}^d$ and is inside $\mathsf{conv}(\mathcal{P}(C,\nu))$, it can be written as the convex combination of at most d+1 points in $\mathcal{P}(C,\nu)$. Denoting these points by $\{\mathbf{x}_1,\ldots,\mathbf{x}_{d+1}\}$ such that $\mathbf{x}_i \in \mathcal{P}(C,\nu), \forall i \in [d+1]$, \mathbf{p} can be written as $\mathbf{p} = \sum_{i=1}^{d+1} a_i \mathbf{x}_i$ where $a_i \geq 0, \forall i \in [d+1]$ and $\sum_{i=1}^{d+1} a_i = 1$. Hence, the ν -margin projected undesirability level of \mathbf{p} in epoch ϕ is:

$$\begin{split} u_{\phi}(\mathbf{p},\nu) &= \sum_{t \in T_{\phi}} \mathbbm{1} \left\{ \left(\left\langle \mathbf{p} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu \right) < 0 \right\} \\ &= \sum_{t \in T_{\phi}} \mathbbm{1} \left\{ \sum_{i \in [d+1]} a_{i} \underbrace{\left(\left\langle \mathbf{x}_{i} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu \right)}_{Q_{i}} < 0 \right\} \\ &\leq \sum_{t \in T_{\phi}} \sum_{i \in [d+1]} \mathbbm{1} \left\{ \left(\left\langle \mathbf{x}_{i} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu \right) < 0 \right\} \\ &\leq \sum_{i \in [d+1]} \mathbbm{1} \left\{ \left(\left\langle \mathbf{x}_{i} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu \right) < 0 \right\} \\ &\leq \sum_{i \in [d+1]} \mathbbm{1} \left\{ \left(\left\langle \mathbf{x}_{i} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu \right) < 0 \right\} \end{split}$$

$$(Carathéodory Theorem)$$

$$\leq \sum_{i \in [d+1]} \mathbbm{1} \left\{ \left(\left\langle \mathbf{x}_{i} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu \right) < 0 \right\}$$

$$\leq \sum_{i \in [d+1]} \mathbbm{1} \left\{ \left(\left\langle \mathbf{x}_{i} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu \right) < 0 \right\}$$

$$\leq \sum_{i \in [d+1]} \mathbbm{1} \left\{ \left(\left\langle \mathbf{x}_{i} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu \right) < 0 \right\}$$

The argument for the first inequality is that if $Q_i \geq 0$ for all $\mathbf{x}_i, i \in [d+1]$, then the corresponding summand contributes 0 points to $u_{\phi}(\mathbf{p}, \nu)$, since $a_i \geq 0$ as this is a convex combination. As a result each undesirability point of the left hand side of the latter inequality can be attributed to at least one \mathbf{x}_i from the right hand side.

Next, we prove that there exists some $\mathbf{q} \in K_{\phi}$ such that $u_{\phi}(\mathbf{q}, \nu) \geq C(d+1) + 1$. Note that by the previous lemma, we know that $\mathbf{q} \notin \text{conv}(\mathcal{P}(C, \nu))$. As a result, any hyperplane separating \mathbf{q} from $\text{conv}(\mathcal{P}(C, \nu))$ preserves $\mathcal{P}(C, \nu)$ (and as a result, $\boldsymbol{\theta}^{\star}$) for $K_{\phi+1}$. To make sure that we also make progress in terms of volume elimination, we show below that there exists a separating hyperplane in the space of large dimensions (i.e., orthogonal to all small dimensions). For our analysis, we introduce the notion of landmarks.

Definition 3.8 (Landmarks). Let basis $E_{\phi} = \{e_1, \dots, e_{d-|S_{\phi}|}\}$ such that E_{ϕ} is orthogonal to S_{ϕ} , any scalar $\delta \in \left(0, \frac{\varepsilon}{2\sqrt{d}+4d}\right)$, and a scalar $\bar{\nu} = \frac{\varepsilon - 2\sqrt{d}\delta}{4\sqrt{d}}$. We define the $2(d-|S_{\phi}|)$ landmarks to be the points such that $\mathcal{L}_{\phi} = \{\kappa_{\phi} \pm \bar{\nu} \cdot e_i, \forall e_i \in E_{\phi}\}$.

Landmarks possess the convenient property that at every round where the observed context v_t was such that $w(K_{\phi}, v_t) \geq \varepsilon$, at least one of them accumulates a ν -margin projected undesirability point, when $\nu < \bar{\nu}$. Before proving this result (formally stated in Lemma 3.10) we need the following technical lemma.

Lemma 3.9. Let basis $E_{\phi} = \{e_1, \dots, e_{d-|S_{\phi}|}\}$ orthogonal to S_{ϕ} . For all $\{(\boldsymbol{v}_t, \omega_t)\}_{t \in T_{\phi}}$ such that $w(\text{Cyl}(K_{\phi}, S_{\phi}), \boldsymbol{v}_t) \geq \varepsilon$, there exists i such that: $|\langle e_i, \boldsymbol{v}_t \rangle| \geq \bar{\nu}$, where $\bar{\nu} = \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4\sqrt{d}}$.

Proof. We first show that $\|\Pi_{L_{\phi}} \boldsymbol{v}_{t}\| \geq \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4}$. Since for the contexts $\{\boldsymbol{v}_{t}\}_{t \in T_{\phi}}$ that we consider in epoch ϕ it holds that: $w(\operatorname{Cyl}(K_{\phi}, S_{\phi}), \boldsymbol{v}_{t}) \geq \varepsilon$, then there exists a point $\mathbf{p} \in \operatorname{Cyl}(K_{\phi}, S_{\phi})$ such that $|\langle \boldsymbol{v}_{t}, \mathbf{p} - \boldsymbol{\kappa}_{\phi} \rangle| \geq \frac{\varepsilon}{2}$. Applying triangular inequality:

$$\left| \left\langle \Pi_{L_{\phi}} \boldsymbol{v}_{t}, \Pi_{L_{\phi}} (\mathbf{p} - \boldsymbol{\kappa}_{\phi}) \right\rangle \right| + \left| \left\langle \Pi_{S_{\phi}} \boldsymbol{v}_{t}, \Pi_{S_{\phi}} (\mathbf{p} - \boldsymbol{\kappa}_{\phi}) \right\rangle \right| \ge \left| \left\langle \boldsymbol{v}_{t}, \mathbf{p} - \boldsymbol{\kappa}_{\phi} \right\rangle \right| \ge \frac{\varepsilon}{2}$$
 (2)

Along the directions in S_{ϕ} the following is true:

 $|\langle \Pi_{S_{\phi}} \boldsymbol{v}_t, \Pi_{S_{\phi}} (\mathbf{p} - \boldsymbol{\kappa}_{\phi}) \rangle| \leq \|\Pi_{S_{\phi}} \boldsymbol{v}_t\|_2 \cdot \|\Pi_{S_{\phi}} (\mathbf{p} - \boldsymbol{\kappa}_{\phi})\|_2 \leq \|\boldsymbol{v}_t\|_2 \cdot \sqrt{d} \cdot \|\Pi_{S_{\phi}} (\mathbf{p} - \boldsymbol{\kappa}_{\phi})\|_{\infty} \leq 1 \cdot \delta \sqrt{d}$ Using the latter, (2) now becomes:

$$\left|\left\langle \Pi_{L_{\phi}} \boldsymbol{v}_{t}, \Pi_{L_{\phi}} (\mathbf{p} - \boldsymbol{\kappa}_{\phi}) \right\rangle\right| \geq \frac{\varepsilon}{2} - \sqrt{d} \cdot \delta$$
 (3)

We next focus on upper bounding term $|\langle \Pi_{L_{\phi}} \boldsymbol{v}_{t}, \Pi_{L_{\phi}}(\mathbf{p} - \boldsymbol{\kappa}_{\phi}) \rangle|$. By applying the Cauchy-Schwarz inequality, (3) becomes:

$$\|\Pi_{L_{\phi}} \boldsymbol{v}_{t}\|_{2} \|\Pi_{L_{\phi}} (\mathbf{p} - \boldsymbol{\kappa}_{\phi})\|_{2} \ge \left| \left\langle \Pi_{L_{\phi}} \boldsymbol{v}_{t}, \Pi_{L_{\phi}} (\mathbf{p} - \boldsymbol{\kappa}_{\phi}) \right\rangle \right| \ge \frac{\varepsilon}{2} - \sqrt{d} \cdot \delta$$
(4)

For $\|\Pi_{L_{\phi}}(\mathbf{p} - \kappa_{\phi})\|_2$, observe that \mathbf{p} and κ_{ϕ} are inside $\mathrm{Cyl}(K_{\phi}, S_{\phi})$, and K_{ϕ} has radius at most 1. By the fact that $\mathbf{p} \in \mathrm{Cyl}(K_{\phi}, S_{\phi})$ and Definition 2.1, we can write it as $\mathbf{p} = \mathbf{x} + \sum_{i=1}^{|S_{\phi}|} y_i \mathbf{s}_i$ where \mathbf{s}_i form a basis for S_{ϕ} (which, recall, is orthogonal to L_{ϕ}) and $\mathbf{x} \in \Pi_{L_{\phi}} K_{\phi}$. Since K_{ϕ} is contained in the unit ℓ_2 ball, we also have that $\Pi_{L_{\phi}} K_{\phi}$ is contained in the unit ℓ_2 ball. Hence $\|\Pi_{L_{\phi}} \mathbf{p}\|_2 = \|\mathbf{x}\|_2 \leq 1$. The same holds for κ_{ϕ} , and so by the triangle inequality, we have $\|\Pi_{L_{\phi}}(\mathbf{p} - \kappa_{\phi})\|_2 \leq 2$. Hence, from Equation (4) we get that: $\|\Pi_{L_{\phi}} \mathbf{v}_t\| \geq \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4}$.

Assume now for contradiction that there does not exist i such that $|\langle e_i, \boldsymbol{v}_t \rangle| \geq \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4\sqrt{d}}$. This means that for all $j \in [d - |S_{\phi}|]$ and all contexts $\{\boldsymbol{v}_t\}_{t \in T_{\phi}} : \langle e_i, \boldsymbol{v}_t \rangle < \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4\sqrt{d}}$. Denoting by $(E\boldsymbol{v}_t)_j$ the j-th coordinate of $E\boldsymbol{v}_t$ we have that $(E\boldsymbol{v}_t)_j = \langle e_j, \boldsymbol{v}_t \rangle$. Hence, if $|\langle \boldsymbol{v}_t, e_j \rangle| < \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4\sqrt{d}}$ then:

$$||E\boldsymbol{v}_t||_2 = ||\Pi_{L_{\phi}}\boldsymbol{v}_t||_2 \le \sqrt{\sum_{i=1}^d (\langle \boldsymbol{v}_t, e_i \rangle)^2} < \sqrt{d\left(\frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4\sqrt{d}}\right)^2} < \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4}$$

which contradicts the fact that $\|\Pi_{L_{\phi}} \boldsymbol{v}_t\| \geq \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4}$ established above.

The tuning of $\bar{\nu}$ also helps us explain the constraint imposed on δ , i.e., $\delta < \frac{\varepsilon}{2\sqrt{d}+4\delta}$. This constraint is due to the fact that since $\nu > \underline{\nu}$ and $\nu < \bar{\nu}$, then it must be the case that $\underline{\nu} < \bar{\nu}$, where $\underline{\nu} = \sqrt{d}\delta$ and $\bar{\nu} = \frac{\varepsilon - 2\sqrt{d}}{4\sqrt{d}}$.

Lemma 3.10. For every round $t \in T_{\phi}$, any scalar $\delta \in (0, \frac{\varepsilon}{2\sqrt{d}+4d})$, any scalar $\nu < \bar{\nu}$, at least one of the landmarks in \mathcal{L}_{ϕ} gets one ν -margin projected undesirability point, i.e.,

$$\exists \mathbf{p} \in \mathcal{L}_{\phi} : (\langle \mathbf{p} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \rangle + \nu) < 0$$

Proof. By Lemma 3.9, there exists a direction $e_i \in E$ such that $|\langle e_i, v_t \rangle| \ge \bar{\nu} = \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4\sqrt{d}}$. The proof then follows by showing that for $\nu < \bar{\nu}$ landmark points $\mathbf{q}_+ = \kappa_\phi + \nu \cdot e_i$ and $\mathbf{q}_- = \kappa_\phi - \nu \cdot e_i$ get different signs in the undesirability point definition. This is shown by the following derivation:

$$\begin{aligned}
&\left(\left\langle \mathbf{q}_{+} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu\right) \cdot \left(\left\langle \mathbf{q}_{-} - \boldsymbol{\kappa}_{\phi}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu\right) \\
&= \left(\left\langle \bar{\nu} \cdot e_{i}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu\right) \cdot \left(\left\langle -\bar{\nu} \cdot e_{i}, \Pi_{L_{\phi}} \boldsymbol{v}_{t} \right\rangle + \nu\right) \\
&= \nu^{2} - \left(\bar{\nu} \cdot \left|\left\langle e_{i}, \boldsymbol{v}_{t} \right\rangle\right|\right)^{2} \leq \nu^{2} - \bar{\nu}^{2} < 0
\end{aligned}$$

where the last inequality comes from the fact that $\nu \in (\underline{\nu}, \overline{\nu})$. As a result there exists $\mathbf{p} \in \{\mathbf{q}_+, \mathbf{q}_-\} \subseteq \mathcal{L}_{\phi}$ satisfying the condition in the lemma statement.

Since Lemma 3.10 establishes that at every round at least one of the landmarks gets a ν -margin projected undesirability point, then if we make $|T_{\phi}|$ sufficiently large, by the pigeonhole principle at least one of the landmarks has ν -margin projected undesirability at least C(d+1)+1, which allows us to differentiate it from points in $conv(\mathcal{P}(C))$. Formally:

Lemma 3.11. For scalar $\nu \in (\underline{\nu}, \overline{\nu})$, after $|T_{\phi}| = 2d \cdot C(d+1) + 1$ rounds in epoch ϕ , there exists a landmark $\mathbf{p}^{\star} \in \mathcal{L}_{\phi}$ such that $\mathbf{p}^{\star} \notin \text{conv}(\mathcal{P}(C, \nu))$.

Proof. At each round $t \in T_{\phi}$, at least one of the landmarks gets a ν -margin projected undesirability point (Lemma 3.10). Since there are at most 2d landmarks, by the pigeonhole principle after $|T_{\phi}|$ rounds, there exists at least one of them with ν -margin projected undesirability $u_{\phi}(\mathbf{p}^{\star}, \nu) \geq C(d+1) + 1$. Since all points \mathbf{q} inside $\operatorname{conv}(\mathcal{P}(C, \nu))$ have $u_{\phi}(\mathbf{q}, \nu) \leq C(d+1)$, then $\mathbf{p} \notin \operatorname{conv}(\mathcal{P}(C, \nu))$. \square

We are now ready to prove the main lemma of this section. We clarify that we assume that during the computation of \mathbf{h}_{ϕ} we incur no additional regret.

Proof of Lemma 3.2. By Lemma 3.11, for $\bar{\nu} = \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4\sqrt{d}}$ and $\delta \in \left(0, \frac{\varepsilon}{2\sqrt{d} + 4d}\right)$, there exists a landmark $\mathbf{p}^{\star} \in \mathcal{L}_{\phi}$ that lies outside of $\mathrm{conv}(\mathcal{P}(C, \nu))$. As a result, there exists a hyperplane separating \mathbf{p}^{\star} from the convex hull. We denote this hyperplane by $(\mathbf{h}^{\star}, \psi^{\star})$. Recall that since $\mathbf{p}^{\star} \in \mathcal{L}_{\phi}$ then by definition $\|\boldsymbol{\kappa}_{\phi} - \mathbf{p}^{\star}\| = \bar{\nu}$. As the hyperplane separates $\boldsymbol{\kappa}_{\phi}$ from \mathbf{p}^{\star} , we immediately have that $\mathrm{dist}(\boldsymbol{\kappa}_{\phi}, (\mathbf{h}^{\star}, \psi^{\star})) \leq \bar{\nu}$ as well. The fact that $\boldsymbol{\theta}^{\star}$ is always in the preserved halfspace $\mathbf{H}_{\phi}(\mathbf{h}_{\phi}, \psi_{\phi})$ follows directly from Lemma 3.6.

3.3 Computation of appropriately undesirable hyperplane

In this subsection we show how to compute an appropriate hyperplane cut, applying ideas from computational learning theory and, in particular, the Perceptron algorithm [Ros58]. The corresponding subroutine (Algorithm 3) is randomized and has an expected running time that is polynomial in T and exponential⁸ in C. For what follows in this subsection, let $\mathcal{B}(\mathbf{p}^*,\zeta)$ denote the ball of radius ζ around \mathbf{p}^* , where \mathbf{p}^* is the landmark such that $u_{\phi}(\mathbf{p}^*,\nu) = C(d+1)+1$, whose existence we proved in Lemma 3.11. Note that although we do not a priori know the landmark corresponding to \mathbf{p}^* , this can be efficiently identified since $u_{\phi}(\mathbf{p},\mathbf{u})$ can be computed in time linear in $|T_{\phi}|$.

$\overline{\mathbf{ALGORITHM}}$ 3: $\mathrm{ELIM}(A_{\phi}, S_{\phi})$

```
\triangleright \bar{\nu} = \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4\sqrt{d}}, \, \mathbf{p}^* \in \mathcal{L}_{\phi}
 1 Fix a ball of radius \zeta = \bar{\nu} around \mathbf{p}^*: \mathcal{B}(\mathbf{p}^*, \zeta) \subseteq \text{Cyl}(K_{\phi}, S_{\phi}).
     while true do
           Initialize perceptron hyperplane to (\mathbf{h}, \psi) and mistake counter to M = 0.
 3
           Sample a point q from \mathcal{B}(\mathbf{p}^*,\zeta) uniformly at random.
 4
           while M < \frac{d-1}{\zeta^2 \cdot \ln^2(3/2)} do
 5
                 Set m \leftarrow 0.
                 if \mathbf{q} \in \mathbf{H}^+(\mathbf{h}, \psi) then make perceptron update of (\mathbf{h}, \psi) on \mathbf{q} and set m \leftarrow m+1.
                 if \kappa_{\phi} \in \mathbf{H}^{-}(\mathbf{h}, \psi) then make perceptron update of (\mathbf{h}, \psi) on \kappa_{\phi} and set m \leftarrow m + 1.
                 for subsets D_{\phi} \subseteq A_{\phi} such that |D_{\phi}| = C do
 9
                       Let P be the polytope created by hyperplanes of A_{\phi} \setminus D_{\phi} and \mathbf{H}^{-}(\mathbf{h}, \psi).
10
                       if P \neq \emptyset then make perceptron update of (\mathbf{h}, \psi) on \mathbf{z} \in P and set m \leftarrow m + 1.
11
12
                 if m \neq 0 then increase mistake counter M \leftarrow M + m.
                 else return final (\mathbf{h}, \psi)
```

We prove the following result for ELIM.

Lemma 3.12. For any ϕ , scalar $\delta \in \left(0, \frac{\varepsilon}{2\sqrt{d}+4d}\right)$, and scalar $\bar{\nu} = \frac{\varepsilon - 2\sqrt{d} \cdot \delta}{4\sqrt{d}}$, after $|T_{\phi}| = 2d \cdot C(d+1) + 1$ rounds, algorithm ELIM computes hyperplane $(\mathbf{h}_{\phi}, \psi_{\phi})$ orthogonal to all small dimensions S_{ϕ} such that $\mathbf{dist}(\kappa_{\phi}^{\star}, (\mathbf{h}_{\phi}, \psi_{\phi})) \leq 3\bar{\nu}$, and the resulting halfspace $\mathbf{H}_{\phi}(\mathbf{h}_{\phi}, \psi_{\phi})$ always contains $\boldsymbol{\theta}^{\star}$. The expected runtime of this computation is:

$$\mathcal{O}\left(\frac{(d-1)^{3/2}}{\bar{\nu}^2} \cdot \left(d^2C\right)^C \cdot O(\operatorname{LP}(d, C(2d(d+1)-1)+1))\right)$$

where O(LP(n, m)) denotes the computational complexity of solving a Linear Program (LP) with n variables and m constraints.

Note that we do not consume any contexts during the execution of Elim.

Overview of ELIM. At a high level Algorithm 3 does the following. In Lemma 3.2 we showed that there exists a hyperplane $(\mathbf{h}^{\star}, \psi^{\star})$ such that $\operatorname{dist}(\kappa_{\phi}, (\mathbf{h}^{\star}, \psi^{\star})) \leq \bar{\nu}$ and its associated half-space $\mathbf{H}^{\star}(\mathbf{h}^{\star}, \psi^{\star})$ with undesirability at least C+1 on one of its sides. To be more precise, if $\operatorname{conv}(\mathcal{P}(C, \nu)) \subseteq \mathbf{H}^{+}(\mathbf{h}^{\star}, \psi^{\star})$, then $\mathbf{H}^{\star}(\mathbf{h}^{\star}, \psi^{\star}) = \mathbf{H}^{+}(\mathbf{h}^{\star}, \psi^{\star})$, while if $\operatorname{conv}(\mathcal{P}(C)) \subseteq \mathbf{H}^{-}(\mathbf{h}^{\star}, \psi^{\star})$, then $\mathbf{H}^{\star}(\mathbf{h}^{\star}, \psi^{\star}) = \mathbf{H}^{-}(\mathbf{h}^{\star}, \psi^{\star})$. In Lemma 3.11 we also showed that landmark \mathbf{p}^{\star} , which is found at distance at most $\bar{\nu}$ from κ_{ϕ} , is separated from $\operatorname{conv}(\mathcal{P}(C, \nu))$ by $(\mathbf{h}^{\star}, \psi^{\star})$ and has $u_{\phi}(\mathbf{p}^{\star}, \nu) \geq C(d+1)+1$. Instead of identifying $(\mathbf{h}^{\star}, \psi^{\star})$ precisely, we are going to identify hyperplane $(\mathbf{h}^{\star}, \psi^{\star}+\bar{\nu})$.

 $^{^{8}}$ We remind the reader that, as we show in Section 4, for the cases where the corruption level is unknown our algorithm becomes quasi-polynomial in T, via the same technique that handles the unknown amount of corruption.

We therefore focus on a ball of radius $\zeta = \bar{\nu}$ around \mathbf{p}^* denoted by $\mathcal{B}(\mathbf{p}^*,\zeta)$ and identify a point $\mathbf{q}^* \in \mathcal{B}(\mathbf{p}^*,\zeta)$ such that $\langle \mathbf{h}^*, \mathbf{q}^* - \mathbf{p}^* \rangle \geq \frac{\zeta \cdot \ln(3/2)}{\sqrt{d-1}}$; this process is discussed below. Such a \mathbf{q}^* establishes that the *margin* between $\operatorname{conv}(\mathcal{P}(C,\nu))$ and \mathbf{q}^* is at least $\frac{\zeta \cdot \ln(3/2)}{\sqrt{d-1}}$, and we can apply the perceptron algorithm to find a hyperplane that separates them. For every returned hyperplane (\mathbf{h},ψ) from perceptron, we check whether $\mathbf{q}^* \in \mathbf{H}^-(\mathbf{h},\psi)$ and $\kappa_\phi \in \mathbf{H}^+(\mathbf{h},\psi)$, and we increase the inner mistake counter m every time these do not hold.

Next, for every candidate hyperplane (\mathbf{h}, ψ) , we check whether there exists a point in $\mathcal{P}(C, \nu)$ that lies inside $\mathbf{H}^-(\mathbf{h}^*, \psi^*)$. Such a point must have been deemed undesirable by at most C hyperplanes (Definition 3.5 and Lemma 3.4). As a result, if we remove these C hyperplanes, the polytope P formed by the remaining $|A_{\phi}| - C$ hyperplanes (where |A| denotes the cardinality of a set A) should have a non-empty intersection with $\mathbf{H}^-(\mathbf{h}^*, \psi^*)$ (point $\mathbf{z} \in P$). If we find such a point \mathbf{z} we again make an update of perceptron, and increase the inner mistake counter m. To make sure that we check all the possible ways to remove C hyperplanes, we take all the $\binom{|A_{\phi}|}{C}$ hyperplanes to be removed. If no such point is found for any combination, all of $\mathcal{P}(C, \nu)$ lies on the correct side. Finally, the global mistake counter M and the process repeats.

Identifying a point with a large enough margin. The previous overview relies on identifying a point $\mathbf{q}^{\star} \in \mathcal{B}(\mathbf{p}^{\star}, \zeta)$ such that $\langle \mathbf{h}^{\star}, \mathbf{q}^{\star} - \mathbf{p}^{\star} \rangle \geq \frac{\zeta \cdot \ln(3/2)}{\sqrt{d-1}}$. If such \mathbf{q}^{\star} has been found, and since $\mathcal{B}(\mathbf{p}^{\star}, \zeta)$ is a subset of the unit ball in \mathbb{R}^d , a separating hyperplane is identified after at most $\frac{d-1}{\zeta^2 \cdot \ln^2(3/2)}$, thanks to the seminal result of Novikoff [Nov63] on the mistake bound of the Perceptron:

Lemma 3.13 (Perceptron Mistake Bound [Nov63]). Let a dataset D such that $(\mathbf{x}, y) \in D : \|\mathbf{x}\|_2 \le R$ and $y = \{-1, +1\}$ for a scalar R > 0. Assume that points in D are linearly separable with a margin of $\gamma > 0$, i.e., there exists some $\mathbf{w} \in \mathbb{R}$ such that $\|\mathbf{w}\|_2 = 1$ and for all $(\mathbf{x}, y) \in D$: $y \cdot \langle \mathbf{w}, \mathbf{x} \rangle > \gamma$. Then, on input dataset D, the Perceptron algorithm makes at most R^2/γ^2 updates until it returns a separating hyperplane.

To identify this \mathbf{q}^* we repeatedly sample points \mathbf{q} uniformly at random⁹ from $\mathcal{B}(\mathbf{p}^*,\zeta)$ and apply perceptron until $\frac{d-1}{\zeta^2 \cdot \ln^2(3/2)}$ mistakes have been made. If by then the candidate hyperplane returned from perceptron is separating (i.e., m=0), we return it; otherwise we update the global mistake counter M and continue with a new sampled point. The following lemma lower bounds the probability of identifying a point \mathbf{q} such that $\langle \mathbf{h}^*, \mathbf{q} - \mathbf{p}^* \rangle \geq \psi^* + \frac{\zeta \cdot \ln(3/2)}{\sqrt{d-1}}$ by $\frac{1}{20\sqrt{d-1}}$. Its proof follows arguments related to measuring the volume of the part of a sphere which is sufficiently away from an equator [BHK16] and is provided in Appendix A.2 for completeness.

Lemma 3.14 (Volume of Cap). With probability at least $\frac{1}{20\sqrt{d-1}}$ a point randomly sampled from a ball of radius ζ around \mathbf{p}^{\star} , $\mathcal{B}(\mathbf{p}^{\star}, \zeta)$, lies on the following halfspace: $\mathbf{H}^{+}\left(\mathbf{h}^{\star}, \langle \mathbf{h}^{\star}, \mathbf{p}^{\star} \rangle + \frac{\zeta \cdot \ln(3/2)}{\sqrt{d-1}}\right)$.

This lower bound on the probability that a randomly sampled point has the large enough margin that the Perceptron requires for efficient convergence, suffices for us to guarantee that after a polynomial number of rounds, such a \mathbf{q}^* has been identified (Lemma 3.15).

Lemma 3.15. In expectation, after $N = 20\sqrt{d-1}$ number of samples from $\mathcal{B}(\mathbf{p}^*, \zeta)$, at least one of the samples lies in halfspace $\mathbf{H}^+\left(\mathbf{h}^*, \langle \mathbf{h}^*, \mathbf{p}^* \rangle + \frac{\zeta \cdot \ln(3/2)}{\sqrt{d-1}}\right)$.

⁹This can be done in a computationally efficient way by normalizing $\mathcal{B}(\mathbf{p}^*,\zeta)$ to a unit ball, and then using the techniques presented in [BHK16, Section 2.5]. Naturally, the sampled point needs to be scaled back to be inside $\mathcal{B}(\mathbf{p}^*,\zeta)$.

Proof. From Lemma 3.14 the probability that a point randomly sampled from $\mathcal{B}(\mathbf{p}^*,\zeta)$ lies on halfspace $\mathbf{H}^+\left(\mathbf{h}^*,\langle\mathbf{h}^*,\mathbf{p}^*\rangle+\frac{\zeta\cdot\ln(3/2)}{\sqrt{d-1}}\right)$ is at least $\frac{1}{20\sqrt{d-1}}$. Hence, in expectation after $20\sqrt{d-1}$ samples we have identified one such point by union bound.

We are now ready to prove Lemma 3.12.

Proof of Lemma 3.12. From Lemma 3.2, there exists a hyperplane $(\mathbf{h}^{\star}, \psi^{\star})$ with distance at most $\bar{\nu}$ from κ_{ϕ} that has all of $\mathcal{P}(C, \nu)$ on $\mathbf{H}^{+}(\mathbf{h}^{\star}, \psi^{\star})$. By Lemma 3.15, after $N = 20\sqrt{d-1}$ samples in expectation the algorithm samples a point \mathbf{q} with margin of at least $\frac{\zeta \cdot \ln(3/2)}{\sqrt{d-1}}$ from $(\mathbf{h}^{\star}, \psi^{\star})$. As a result, for the iteration of this point, the perceptron algorithm can indentify a hyperplane \mathbf{h}_{ϕ} separating \mathbf{q} and $\mathcal{P}(C, \nu)$ after $\frac{d-1}{\zeta^2 \cdot \ln^2(3/2)}$ samples (Lemma 3.13). Since $\mathbf{q} \in \mathcal{B}(\mathbf{p}^{\star}, \zeta)$, then,

$$\|\mathbf{q} - \boldsymbol{\kappa}_{\phi}^{\star}\| = \|\mathbf{q} - \boldsymbol{\kappa}_{\phi} + \boldsymbol{\kappa}_{\phi} - \boldsymbol{\kappa}_{\phi}^{\star}\|$$

$$\leq \|\mathbf{q} - \boldsymbol{\kappa}_{\phi}\| + \|\boldsymbol{\kappa}_{\phi} - \boldsymbol{\kappa}_{\phi}^{\star}\|$$

$$\leq \|\mathbf{q} - \boldsymbol{\kappa}_{\phi}\| + \bar{\nu}$$

$$\leq \|\mathbf{q} - \boldsymbol{\kappa}_{\phi}\| + \bar{\nu}$$
(approximation of $\boldsymbol{\kappa}_{\phi}^{\star}$ in polynomial time)
$$= \|\mathbf{q} - \mathbf{p}^{\star} + \mathbf{p}^{\star} - \boldsymbol{\kappa}_{\phi}\| + \bar{\nu}$$

$$\leq \|\mathbf{q} - \mathbf{p}^{\star}\| + \|\mathbf{p}^{\star} - \boldsymbol{\kappa}_{\phi}\| + \bar{\nu}$$
(triangular inequality)
$$\leq \bar{\nu} + \bar{\nu} + \bar{\nu}$$
($\mathbf{q} \in \mathcal{B}(\mathbf{p}^{\star}, \zeta)$ and Definition 3.8)

Hence, $\operatorname{dist}(\kappa_{\phi}^{\star}, (\mathbf{h}_{\phi}, \psi_{\phi})) \leq 3\bar{\nu}$.

The expected runtime of ELIM is N times the complexity for each iteration of the outer while loop. At every iteration of the inner while loop, the algorithm checks whether \mathbf{q} and κ_{ϕ} are on the correct side of \mathbf{h} and possibly updates the perceptron (this is done in time $\mathcal{O}(1)$). The most computationally demanding part is Steps 8–10 where we check whether the hyperplane cuts some part of $\mathcal{P}(C)$. For that, we check all the possible $\binom{|A_{\phi}|}{C} \leq (2dC(d+1)+1)^C = \Theta((d^2C)^C)$ combinations of which C hyperplanes to disregard and solve an LP for each such combination. These LPs have at most d variables (since $K_{\phi} \subseteq \mathbb{R}^d$) and $|A_{\phi}| - C = C(2d(d+1)-1)+1$) constraints. Denoting the complexity of solving an LP with n variables and m constraints as $O(\operatorname{LP}(n,m))$ we therefore obtain that Steps 8–10 have computational complexity $\mathcal{O}\left(O(\operatorname{LP}(d,C(2d(d+1)-1)+1)))\cdot (d^2C)^C\right)$. This concludes our proof.

3.4 Bounding Regret via Volume Progress

Having established the essential properties about the hyperplane \mathbf{h}_{ϕ} , we now proceed with proving the regret guarantee. Our proof follows the analysis of Lobel et al. [LPLV18] with a crucial difference: when Projected Volume observes a context such that $w(\text{Cyl}(K_{\phi}, S_{\phi}), \mathbf{v}_t) \leq \varepsilon$, the latter can be automatically discarded, since it does not contribute to the regret. In our epoch-based setting, this would translate into ignoring a returned hyperplane by Elim $(\mathbf{h}_{\phi}, \psi_{\phi})$ if $w(\text{Cyl}(K_{\phi}, S_{\phi}), \mathbf{h}_{\phi}) \leq \varepsilon$. However, when Elim returns a hyperplane \mathbf{h}_{ϕ} with $w(K_{\phi}, \mathbf{h}_{\phi}) < \varepsilon$, we cannot relate this information to the regret of epoch ϕ . This is due to the fact that for all rounds comprising the epoch, the width of K_{ϕ} in the direction of the observed context was greater than ε .

What we do know is that $(\mathbf{h}_{\phi}, \psi_{\phi})$ is orthogonal to *all* the small dimensions S_{ϕ} . So, we set a threshold of δ and if $w(K_{\phi}, \mathbf{h}_{\phi}) < \delta$ (Step 11), then we have identified a dimension in $\Pi_{L_{\phi}} K_{\phi}^{10}$ with width at most δ . We can therefore include it in the set of small dimensions for the next epoch $S_{\phi+1}$.

¹⁰Since \mathbf{h}_{ϕ} is orthogonal to the small dimensions S_{ϕ} , then it must lie in the space of large dimensions L_{ϕ} .

Auxiliary lemmas. First, we list several lemmas that are useful for the proof of Theorem 3.1, starting with the Directional Grünbaum which relates the width of $K \cap \{\mathbf{x} | \langle \mathbf{u}, \mathbf{x} - \boldsymbol{\kappa} \rangle \geq 0\}$ (where \mathbf{u} is any unit vector and $\boldsymbol{\kappa}$ is K's centroid) with the width of K towards any unit vector \mathbf{v} .

Lemma 3.16 (Directional Grünbaum [LPLV18, Theorem 5.3]). If K is a convex body and κ is its centroid, then, for *every* unit vector $\mathbf{u} \neq 0$, the set $K_+ = K \cap \{\mathbf{x} | \langle \mathbf{u}, \mathbf{x} - \kappa \rangle \geq 0\}$ satisfies:

$$\frac{1}{d+1}w(K,\mathbf{v}) \le w(K_+,\mathbf{v}) \le w(K,\mathbf{v})$$

for all unit vectors \mathbf{v} .

The Approximate Grünbaum relates the volume of a set $K_+^{\mu} = \{ \mathbf{x} \in K : \langle \mathbf{u}, \mathbf{x} - \kappa \rangle \ge \mu \}$ with the volume of set K, when $\mu \le 1/d$ for any unit vector \mathbf{u} .

Lemma 3.17 (Approximate Grünbaum). Let K be a convex body and κ be its centroid. For an arbitrary unit vector \mathbf{u} and a scalar μ such that $0 < \mu < \frac{1}{d}$, let $K_+^{\mu} = \{\mathbf{x} \in K : \langle \mathbf{u}, \mathbf{x} - \kappa \rangle \geq \mu\}$. Then: $\operatorname{vol}(K_+^{\mu}) \geq \frac{1}{2e^2} \operatorname{vol}(K)$.

The proof of this lemma (provided in Appendix A.3 for completeness) is similar to the proof of [LPLV18, Lemma 5.5] with the important difference that μ is no longer $w(K, \mathbf{u})/(d+1)^2$, but rather, $\mu < 1/d$.

The cylindrification lemma relates the volume of the convex body to the volume of its projection onto a subspace.

Lemma 3.18 (Cylindrification [LPLV18, Lemma 6.1]). Let K be a convex body in \mathbb{R}^d such that $w(K, \mathbf{u}) \geq \delta'$ for every unit vector \mathbf{u} . Then, for every (d-1)-dimensional subspace L it holds that $\text{vol}(\Pi_L K) \leq \frac{d(d+1)}{\delta'} \text{vol}(K)$

The next lemma states that a convex body K with width at least δ in every direction must fit a ball of diameter δ/d inside it.

Lemma 3.19 ([LPLV18, Lemma 6.3]). If $K \subset \mathbb{R}^d$ is a convex body such that $w(K, \mathbf{u}) \geq \delta$ for every unit vector \mathbf{u} , then K contains a ball of diameter δ/d .

Putting everything together. Using the approximate Grünbaum, we can prove our epoch based projected variant, presented formally below.

Lemma 3.20 (Epoch Based Projected Grünbaum). Setting $\delta = \frac{\varepsilon}{4(d+\sqrt{d})}$, then for $K_{\phi+1} = K_{\phi} \cap \mathbf{H}_{\phi}(\mathbf{h}_{\phi}, \psi_{\phi})$, where $\mathbf{H}_{\phi}(\mathbf{h}_{\phi}, \psi_{\phi})$ was the halfspace returned from ELIM it holds that:

$$\operatorname{vol}\left(\Pi_{L_{\phi}}K_{\phi+1}
ight) \leq \left(1-rac{1}{2e^2}
ight)\operatorname{vol}(\Pi_{L_{\phi}}K_{\phi})$$

Proof. By Lemma 3.12, we know that ELIM returned hyperplane $(\mathbf{h}_{\phi}, \psi_{\phi})$ orthogonal to all small dimensions, such that $\operatorname{dist}(\kappa_{\phi}^{\star}, (\mathbf{h}_{\phi}, \psi_{\phi})) \leq 3\bar{\nu} = 3 \cdot \frac{\varepsilon - 2\sqrt{d}\delta}{4\sqrt{d}}$. Substituting $\delta = \frac{\varepsilon}{4(d+\sqrt{d})}$ we get that:

$$\mathrm{dist}(\boldsymbol{\kappa}_{\phi}^{\star},(\mathbf{h}_{\phi},\psi_{\phi})) \leq \frac{(2\sqrt{d}+1)\varepsilon}{2\sqrt{d}(\sqrt{d}+1)} \leq \frac{1}{d}$$

where the last inequality uses the fact that $\varepsilon \leq 1/\sqrt{d}$ and that $\frac{2\sqrt{d}+1}{\sqrt{d}+1} \leq 2$. Hence, the clause in the approximate Grünbaum lemma (Lemma 3.17) holds and as a result, applying the approximate Grünbaum lemma with $K = \prod_{L_{\phi}} K_{\phi}$, the lemma follows.

We are now ready to present the proof for the main result of this section.

Proof of Theorem 3.1. We use $\Gamma_{\phi} = \text{vol}\left(\Pi_{L_{\phi}}K_{\phi}\right)$ as our potential function. Note that when L_{ϕ} becomes empty, S_{ϕ} must be an orthonormal basis for which $w(K_{\phi}, \mathbf{s}) \leq \delta, \forall \mathbf{s} \in S_{\phi}$. For any received context \boldsymbol{v} after that point we have the following. First, $\boldsymbol{v} = \sum_{i \in [|S_{\phi}|]} a_i \cdot s_i$, and for any vector \mathbf{a} it holds that: $\|\mathbf{a}\|_1 \leq \sqrt{d} \cdot \|\mathbf{a}\|_2$. So, the width of K_{ϕ} in the direction of \boldsymbol{v} when $|S_{\phi}| = d$ is:

$$\begin{split} w(K_{\phi}, \boldsymbol{v}) &= \max_{\mathbf{p}, \mathbf{q} \in K_{\phi}} \langle \boldsymbol{v}, \mathbf{p} - \mathbf{q} \rangle & \text{(definition of width)} \\ &\leq \sum_{i \in [|S_{\phi}|]} a_i \cdot \max_{\mathbf{p}, \mathbf{q} \in K_{\phi}} \langle s_i, \mathbf{p} - \mathbf{q} \rangle & (\boldsymbol{v} = \sum_{i \in [|S_{\phi}|]} a_i \mathbf{s}_i \text{ and properties of max}(\cdot)) \\ &\leq \sum_{i \in [|S_{\phi}|]} a_i \cdot \delta & \text{(definition of small dimensions)} \\ &\leq \|\mathbf{a}\|_1 \cdot \delta \leq \sqrt{d} \delta \|\mathbf{a}\|_2 & (\|\mathbf{a}\|_1 \leq \sqrt{d} \cdot \|\mathbf{a}\|_2) \end{split}$$

Substituting $\delta = \frac{\varepsilon}{4(d+\sqrt{d})}$ the latter becomes: $w(K_{\phi}, \boldsymbol{v}) \leq \frac{\varepsilon}{4(\sqrt{d}+1)} \leq \varepsilon$. Therefore, when $L_{\phi} = \emptyset$, then CORPV.K incurs no additional regret for any context it might receive. For that, we analyze the potential function Γ_{ϕ} until $L_{\phi} = \emptyset$.

We first focus on rounds $t \in T_{\phi}$ such that $w(\text{Cyl}(K_{\phi}, S_{\phi}), v_t) \leq \varepsilon$ (checked in Step 3), for which we incur no additional regret if the round is not corrupted. If these rounds were corrupted, then this adds an extra C term to the cumulative regret upper bound that we prove.

We present the lower bound for $\Gamma_{\phi} \geq \Omega \left(\frac{\delta}{d}\right)^{2d}$. First, we note that Step 11 of CorPV.K ensures that for all $\mathbf{u} \in L_{\phi}$ it holds that $w\left(\Pi_{L_{\phi}}K_{\phi},\mathbf{u}\right) \geq \delta$. As a result, from Lemma 3.19, $\Pi_{L_{\phi}}K_{\phi}$ contains a ball of radius $\frac{\delta}{|L_{\phi}|}$, so vol $\left(\Pi_{L_{\phi}}K_{\phi}\right) \geq V(d) \left(\frac{\delta}{|L_{\phi}|}\right)^{|L_{\phi}|}$, where by V(d) we denote the volume of the d-dimensional unit ball. Using the fact that $|L_{\phi}| \leq d$ and $V(d) \geq \Omega \left(\frac{1}{d}\right)^{d}$, the latter can be lower bounded by $\Omega \left(\frac{\delta}{d}\right)^{2d}$. Hence, $\Gamma_{\phi} = \text{vol}\left(\Pi_{L_{\phi}}K_{\phi}\right) \geq \Omega \left(\frac{\delta}{d}\right)^{2d}$.

We split our analysis of the upper bound of Γ_{ϕ} in two parts. In the first part, we study the potential function between epochs where the set of large dimensions L_{ϕ} does not change. In the second part, we study the potential function between where the set of large dimensions L_{ϕ} becomes smaller. For both cases, we use the fact that from Lemma 3.20 we have that:

$$\operatorname{vol}\left(\Pi_{L_{\phi}} K_{\phi+1}\right) \le \left(1 - \frac{1}{2e^2}\right) \operatorname{vol}\left(\Pi_{L_{\phi}} K_{\phi}\right) \tag{5}$$

For the case where the set L_{ϕ} does not change between epochs ϕ , $\phi+1$ (i.e., $L_{\phi}=L_{\phi+1}$) Equation (5) becomes:

$$\operatorname{vol}\left(\Pi_{L_{\phi+1}} K_{\phi+1}\right) \le \left(1 - \frac{1}{2e^2}\right) \operatorname{vol}\left(\Pi_{L_{\phi}} K_{\phi}\right) \tag{6}$$

For the other case, the set of small dimensions increases from S_{ϕ} to $S_{\phi+1}$. By the definition of large dimensions, $w(K_{\phi}, \mathbf{u}) \geq \delta$, $\forall \mathbf{u} \in L_{\phi}$. So, if we were to cut K_{ϕ} with a hyperplane that passes precisely from the centroid κ_{ϕ}^{\star} then, from Lemma 3.16 $w(K_{\phi+1}, \mathbf{u}) \geq \frac{\delta}{d+1}$. Since, however, we make sure that we cut K_{ϕ} through the approximate centroid κ_{ϕ} , then $w(K_{\phi+1}, \mathbf{u}) \geq \frac{\delta}{d+1}$. This is due to the fact that the halfspace $\mathbf{H}_{\phi}(\mathbf{h}_{\phi}, \psi_{\phi})$ returned from ELIM always contains κ_{ϕ} (since $u_{\phi}(\kappa_{\phi}, \mathbf{u}) = 0$). Since $\|\kappa_{\phi} - \kappa_{\phi}^{\star}\| \leq \nu$ then $\kappa_{\phi}^{\star} \in \mathbf{H}_{\phi}(\mathbf{h}_{\phi}, \kappa_{\phi})$. Hence, applying Lemma 3.18 we obtain:

$$\operatorname{vol}\left(\Pi_{L_{\phi+1}} K_{\phi+1}\right) \le \frac{d(d+1)^2}{\delta} \operatorname{vol}\left(\Pi_{L_{\phi}} K_{\phi+1}\right) \tag{7}$$

Since we add at most d new directions to S_{ϕ} , the volume of K_0 is upper bounded by O(1), and using the lower bound for Γ_{ϕ} that we computed, we have that:

$$\Omega\left(\frac{\delta}{d}\right)^{2d} \leq \Gamma_{\Phi} = \operatorname{vol}\left(\Pi_{L_{\Phi}}K_{\Phi}\right) \leq O(1) \cdot \left(\frac{d(d+1)^2}{\delta}\right)^d \cdot \left(1 - \frac{1}{2e^2}\right)^{\Phi}$$

where by Φ we denote the total number of epochs. Solving the above in terms of Φ and substituting δ we obtain: $\Phi \leq O(d \log \frac{d}{\delta})$.

From Lemma 3.2 we have that in order to be able to guarantee that there exists an appropriate separating hyperplane $(\mathbf{h}_{\phi}, \psi_{\phi})$ for epoch ϕ we need at most 2dC(d+1)+1 rounds per epoch. Thus, the total regret incurred is upper bounded by:

$$\mathcal{O}\left(\left(2dC(d+1)+1\right)d\log\frac{d}{\varepsilon}+C\right).$$

4 Robustness to unknown corruption level

In this section, we introduce algorithm CORPV.A, which achieves similar regret guarantees as CORPV.K (up to logarithmic factors) but is also robust to an *unknown* level of corruptions. Our solution draws intuition from a framework developed by Lykouris et al. [LMPL18] and its byproduct is that CORPV.A has running time *quasi-polynomial* in T. Our main result here is the following.

Theorem 4.1. For target accuracy $\varepsilon > 0$, CORPV.A (Algorithm 4) is agnostic to the corruption level, has expected runtime $\mathcal{O}\left(\left(d^2\log T\right)^{\log T+3}\cdot\operatorname{poly}\left(d\log d/\varepsilon,\log T\right)\right)$, and incurs expected regret:

$$\mathcal{O}\left(d^3\log\left(d/\varepsilon\right)\log(T)\cdot\left(\log T+C\right)\right)$$

Our results shed light into novel features of the two techniques (*subsampling* and *global eliminations*) presented by Lykouris et al. [LMPL18]. Notably, we show that global eliminations can be extended to convex bodies, and that the subsampling technique can yield quasipolynomial running time when the base algorithm has running time scaling exponentially in the number of corruptions.

At a high level, our algorithm has the following properties. First, it maintains $\log T$ layers, each of which corresponds to a different corruption level; i.e., layer $\ell = i$ corresponds to corruption level $C = 2^i$. The layers run instances of CORPV.K in parallel so that we can be robust to any amount of corruptions. Second, at each round $t \in [T]$ the learner samples and acts according to layer ℓ with probability $2^{-\ell}$. As a result, layers with larger indices are slower (i.e., sampled less often), but more robust in terms of corruptions. Similarly to [LMPL18, Lemma 3.3] (whose statement we include below for completeness) each layer $\ell \geq \log C$ observes at most $\ln (1/\beta_{\ell}) + 3$ corruptions, with probability $1 - \beta_{\ell}$.

Lemma 4.2 ([LMPL18, Lemma 3.3]). For corruption level C, each layer $\ell \ge \log C$ observes a constant number of corruptions on expectation during the exploration phase. With probability at least $1 - \beta$, the corruption it observes is at most $\ln (1/\beta) + 3$.

For each round $t \in [T]$, each layer $\ell \in [\log T]$ can be at a different phase, denoted by $\phi(\ell)$ and as a result, each layer holds its own knowledge set, denoted by $K_{\ell,\phi(\ell)}$, which is potentially different from the knowledge sets of other layers. That said, the $\log T$ layers are *coupled* through a global eliminations technique extended for continuous spaces, so that their knowledge sets are *consistent*, i.e., for two layers $\ell' \leq \ell$ it is either non-empty such that $K_{\ell',\phi(\ell')} \subseteq K_{\ell,\phi(\ell)}$, or $K_{\ell',\phi(\ell')}$ is the empty

ALGORITHM 4: CORRUPTEDPROJECTEDVOLUME-AGNOSTIC(CORPV.A)

```
1 Initialize all layer-specific quantities: \forall \ell \in [\log T] : \phi(\ell) \leftarrow 1, A_{\ell,\phi(\ell)} \leftarrow \emptyset, K_{\ell,\phi(\ell)} \leftarrow K_0,
          S_{\ell,\phi(\ell)} \leftarrow \emptyset, L_{\ell,\phi(\ell)} \leftarrow \text{orthonormal-basis}(\mathbb{R}^d), T_{\ell,\phi(\ell)} \leftarrow \emptyset, \kappa_{\ell,\phi(\ell)} \leftarrow \text{apx-centroid}(\text{Cyl}(K_{\ell,\phi(\ell)})), \text{ empty}_{\ell} \leftarrow 0,
         threshold \delta = \frac{\varepsilon}{4(\sqrt{d}+d)} and \widetilde{C} = \log\left(\min\left\{T, \frac{1}{\varepsilon}\right\} \cdot T\right).
 \mathbf{2} \ \mathbf{for} \ \mathit{rounds} \ t \in [T] \ \mathbf{do}
 3
               Observe context \boldsymbol{v}_t.
               Sample layer \ell \in [\log T] with probability 2^{-\ell}. With remaining probability, sample \ell = 1.
 4
               if not empty, then
                       Compute query point \omega_{\ell,t} = \langle \boldsymbol{v}_t, \boldsymbol{\kappa}_{\ell,\phi(\ell)} \rangle and observe opponent's response y_t.
  6
                       if L_{\ell,\phi(\ell)} \neq \emptyset and w(\text{Cyl}(K_{\ell,\phi(\ell)}, S_{\ell,\phi(\ell)}), v_t) > \varepsilon then
  7
                                Update T_{\ell,\phi(\ell)} \leftarrow T_{\ell,\phi(\ell)} + \{t\}.
 8
                                Update set of hyperplanes: A_{\ell,\phi(\ell)} \leftarrow A_{\ell,\phi(\ell)} \bigcup \{ (\Pi_{L_{\ell,\phi(\ell)}} v_t, \omega_{\ell,t}), y_t \}.
 9
                                if |T_{\ell,\phi(\ell)}| \geq 2d \cdot \tilde{C} \cdot (d+1) + 1 then
                                                                                                                                                                                                         \triangleright end of \ell's epoch
10
                                        Compute: ((\mathbf{h}_{\ell,\phi},\psi_{\ell,\phi(\ell)}),\mathbf{H}_{\ell,\phi(\ell)}(\mathbf{h}_{\ell,\phi(\ell)},\psi_{\ell,\phi(\ell)})) \leftarrow \text{ELIM}(A_{\ell,\phi(\ell)},S_{\ell,\phi(\ell)}).
11
                                        Update K_{\ell,\phi(\ell)+1} \leftarrow K_{\ell,\phi(\ell)} \cap \mathbf{H}_{\ell,\phi(\ell)}(\mathbf{h}_{\ell,\phi(\ell)},\psi_{\ell,\phi(\ell)}), \widetilde{S} \leftarrow S_{\ell,\phi(\ell)}, \widetilde{L} \leftarrow L_{\ell,\phi(\ell)}
12
                                        if w(\Pi_{L_{\ell,\phi(\ell)}}K_{\ell,\phi(\ell)}, \mathbf{h}_{\ell,\phi(\ell)}) \leq \delta then
13
                                                 Add hyperplane to small dimensions \widetilde{S} \leftarrow S_{\ell,\phi(\ell)} \bigcup \{\mathbf{h}_{\ell,\phi(\ell)}\}.
14
                                                 Compute orthonormal basis for new large dimensions \widetilde{L} (without \widetilde{S}).
15
                                        Update L_{\ell,\phi(\ell)+1} \leftarrow \widetilde{L} \setminus \{e_i \in \widetilde{L} : w(K_{\ell,\phi(\ell)+1}, e_i) \leq \delta\}, S_{\ell,\phi(\ell)+1} \leftarrow \widetilde{S} \bigcup (\widetilde{L} \setminus L_{\ell,\phi(\ell)+1}).
16
                                        for all layers \ell' \leq \ell in decreasing order do
17
                                                 K_{\ell',\phi(\ell')} \leftarrow K_{\ell',\phi(\ell')} \bigcap_{\ell''>\ell'} K_{\ell'',\phi(\ell'')}
18
                                                 empty_{\ell'} \leftarrow \mathbb{1}\left\{K_{\ell',\phi(\ell')} = \emptyset\right\}
19
                                                 Move to next epoch \phi(\ell') \leftarrow \phi(\ell') + 1, T_{\ell',\phi(\ell')} \leftarrow \emptyset, A_{\ell',\phi(\ell')} \leftarrow \emptyset.
20
                                                 if not \text{ empty}_{\ell'} then
21
                                                         Update L_{\ell',\phi(\ell')} \leftarrow L_{\ell',\phi(\ell')-1} \setminus \{e_i \in L_{\ell',\phi(\ell')-1} : w(K_{\ell',\phi(\ell')},e_i) \leq \delta\} \&
22
                                                                                                         S_{\ell',\phi(\ell')} \leftarrow S_{\ell',\phi(\ell')-1} \bigcup (L_{\ell',\phi(\ell)-1} \setminus L_{\ell',\phi(\ell')})
23
                                                         Compute \kappa_{\ell',\phi(\ell')} \leftarrow \text{apx-centroid}(\text{Cyl}(K_{\ell',\phi(\ell')}, S_{\ell',\phi(\ell')})).
24
25
                       Find first layer \tilde{\ell} \geq \ell such that K_{\tilde{\ell}, \phi(\tilde{\ell})} \neq \emptyset.
26
                        Query a random point from K_{\tilde{\ell},\phi(\tilde{\ell})}.
```

set. If a layer ℓ' is completely eliminated (i.e., $K_{\ell',\phi(\ell')} = \emptyset$), then, every time that ℓ' is sampled our algorithm makes a query in a way consistent to the smaller layer non-empty knowledge set. More formally, the algorithm queries any point $\mathbf{x} \in \text{Cyl}\left(K_{\ell,\phi(\ell)}, S_{\ell,\phi(\ell)}\right)$, where ℓ is the smallest layer $\ell \geq \ell'$ such that $K_{\ell,\phi(\ell)} \neq \emptyset$, and none of the knowledge sets gets updated.

Global Eliminations for continuous spaces. Assume that layer ℓ has determined that the separating hyperplane should be $(\mathbf{h}_{\ell,\phi(\ell)}^{\star}, \psi_{\ell,\phi(\ell)}^{\star})$ and the halfspace to be preserved is $\mathbf{H}^{\star}(\mathbf{h}_{\ell,\phi(\ell)}^{\star}, \psi_{\ell,\phi(\ell)}^{\star})$. Then, since layer ℓ is by definition more robust to corruptions than all layers $\ell' \leq \ell$, it is in our best interest to update $K_{\ell',\phi(\ell')}$ for all layers $\ell' \leq \ell$, such that $K_{\ell',\phi(\ell')} \leftarrow K_{\ell',\phi(\ell')} \cap_{\ell'' \geq \ell'} K_{\ell'',\phi(\ell'')}$ and have layer ℓ' move to the next epoch. In words, this ensures the consistency property of our layers. After every update of a layer's knowledge set, the sets of small and large dimensions are re-computed. Note here that since we always cut the knowledge sets with hyperplanes, the remaining knowledge sets are always convex bodies.

Proof of Theorem 4.1. We separate the layers into two categories: layers $\ell \ge \log C$ are corruption-tolerant, and layers $\ell < \log C$ are corruption-intolerant. Every layer ℓ , if it were to run in isolation,

would spend Φ_{ℓ} epochs until converging to a knowledge set with width at most ε in all the directions. However, in Corp. A layer ℓ 's epoch potentially gets increased every time that a layer $\ell' \geq \ell$ changes epoch. Since there are at most $\log T$ layers, this results in an added $\log T$ multiplicative overhead for the epochs of each layer. This overhead is suffered by the corruption-tolerant layers.

We first study the performance of the corruption-tolerant layers. From Lemma 4.2 we have that with probability at least $1 - \beta_{\ell}$, the actual corruption experienced by the tolerant layers is at most

$$\widetilde{C} = \ln\left(\frac{1}{\beta_{\ell}}\right) + 3 \le \log\left(\min\left\{T, \frac{1}{\varepsilon}\right\} \cdot \frac{1}{\beta_{\ell}}\right).$$
 (8)

Applying the regret guarantee of Theorem 3.1 for all rounds that this corruption-tolerant layer was sampled, the regret incurred by each of the tolerant layers ℓ is upper bounded by:

$$R_{\ell} \le \mathcal{O}\left(\left(d^2\widetilde{C} + 1\right)d\log\left(\frac{d}{\varepsilon}\right)\right)$$
 (9)

Since there are at most log T such layers, then with failure probability β_{ℓ} we have that the expected regret incurred by all the corruption-tolerant layers in upper bounded by:

$$R_{\text{tolerant}} \le \sum_{\ell=1}^{\log(T)} R_{\ell}$$
 (10)

We now move to the analysis of the corruption-intolerant layers. Let ℓ^* denote the smallest corruption-tolerant layer, i.e., $\ell^* = \min_{\ell} \{\ell \ge \log C\}$. Observe that each layer $\ell \le \ell^*$ is played until layer ℓ^* identifies the target knowledge set having directional width at most ε in every direction. If ℓ^* was run in isolation, from Equation (9) it would incur regret R_{ℓ^*} . When a context is not costly for ℓ^* , it is also not costly for layers $\ell < \ell^*$ (this follows because we have consistent sets across layers). As a result, whenever a context may cause regret for a corruption-intolerant layer, with probability 1/C, ℓ^* is selected and it makes progress towards identifying the target. Hence, the total regret from corruption-intolerant layers can be bounded by the total regret incurred by the first corruption-tolerant layer times C (as the latter layer is selected every 1/C costly contexts in expectation). This means that we incur a regret of $\mathcal{O}\left(C(2d(d+1)\log C+1)d(\log d/\varepsilon)\log T\right)$ until the appropriately small knowledge set is constructed for ℓ^* ; subsequently this knowledge set dictates the behavior of the intolerant layers. Since we do not know which layer is ℓ^* , taking a union bound, we have that the expected regret incurred by the intolerant layers is:

$$R_{\text{intolerant}} \le C \cdot R_{\ell^*}$$
 (11)

Putting everything together, the regret experienced by CORPV.A is comprised by the regret of the corruption-tolerant (Equation (10)) and the corruption-intolerant layers (Equation (11)). As a result, combined with Equations (8) and (9), the total expected regret is upper bounded by:

$$R = R_{\text{tolerant}} + R_{\text{intolerant}} \le \mathcal{O}\left(\left(\log T + C\right) \cdot \left(d^2 \tilde{C} + 1\right) d \log\left(\frac{d}{\varepsilon}\right)\right)$$

$$\le \mathcal{O}\left(d^3 \cdot \log\left(\min\left\{T, \frac{d}{\varepsilon}\right\} \cdot \frac{1}{\beta_{\ell}}\right) \cdot \log\left(\frac{d}{\varepsilon}\right) \cdot (\log(T) + C)\right)$$

Setting $\beta_{\ell} = 1/T$ gives us the desired guarantee.

Now we turn to the computational complexity of CORPV.A. Note that the complexity is dictated by the choice of $\widetilde{C} = \log(T) + 3$. As a result, from Lemma 3.12, substituting C with $\log T$, we get that CORPV.A has expected runtime: $\mathcal{O}\left(\left(d^2\log T\right)^{\log T+3}\cdot\operatorname{poly}\left(d\log\frac{d}{\varepsilon},\log T\right)\right)$.

5 Applications to game theory

In this section, we study the implications of our results for learning in the presence of irrational agents in three game-theoretic settings: dynamic pricing for linear buyers, Stackelberg Security Games (SSGs), and dynamic pricing for Lipschitz buyers ([MPLS18]). Naturally, in such settings we do not know a priori the total number of irrational agents C. For ease of exposition, we present the intuition of our algorithms assuming that C is known, given that one can easily then extend them to the case of unknown C using variants of Corpv.A. Our result statements, however, reflect the unknown C case. At a high level, the algorithms for learning in all these game-theoretic settings are based on variants of binary search. Our approach is hence to ensure that the binary-search type modules are robust to irrational responses. We also find it useful to note that the "decision" of whether a round was corrupted or not is not a strategic one. For example, in "online" dynamic pricing, where we can think of agents arriving one at a time, the irrationality intuition is that some of the agents themselves may be irrational, which is not a function of the price we post. Finally, we note that in SSGs and dynamic pricing with Lipschitz buyers a single dimensional variant of our results suffices, and the dependence on d that appears in Corollaries 5.2 and 5.3 is inherent to the non-corrupted versions of these problems.

5.1 Dynamic Pricing for Linear Buyers

The close connection between the problem of contextual multi-dimensional binary search and dynamic pricing had already been observed by several works (e.g., [CLPL19, LPLV18, PLS18]). In the language of dynamic pricing, v_t is the item's feature vector, the learner's query ω_t corresponds to the price that a seller sets, and the opponent's reply corresponds to whether the buyer buys the item at the queried price or not. If the opponent is rational, he buys the item if its price is less than his valuation for the item, i.e., if $\langle \boldsymbol{\theta}^*, \boldsymbol{v}_t \rangle \geq \omega_t$. The difference between the settings of multidimensional binary search and dynamic pricing is the learner's loss function: in dynamic pricing, if the buyer purchases the item at price ω_t , the loss that the learner incurs is $\langle \boldsymbol{\theta}^*, \boldsymbol{v}_t \rangle - \omega_t$, otherwise the learner's loss is $\langle \boldsymbol{\theta}^*, \boldsymbol{v}_t \rangle$. This loss function, which accounts for the loss in revenue that the seller suffers at each round, is called the pricing loss.

Albeit different from the loss function we consider in Sections 3 and 4, the results of those sections carry over for the pricing loss with a small modification. For every round t and according to context v_t we distinguish two cases: if $w(\operatorname{Cyl}(K_\phi, S_\phi), v_t) \leq \varepsilon$ then, the learner queries price $\underline{\omega}_t = \langle v_t, \kappa_\phi \rangle - \varepsilon$. This price guarantees that a sale occurs at round t, and the learner gets revenue $\underline{\omega}_t$. In other words, in these rounds the pricing loss is only ε . If $w(\operatorname{Cyl}(K_\phi, S_\phi), v_t) \geq \varepsilon$ then, the learner queries price $\omega_t = \langle v_t, \kappa_\phi \rangle$. Putting all these together and accounting for the fact that there are at most C corrupted rounds: $R_{\operatorname{pricing}}(T) \leq C + R_{\operatorname{explore}} + R_{\operatorname{exploit}} \leq C + R_{\operatorname{CorPV},K}(T) + \varepsilon \cdot T$. This equation also explains the role of the target accuracy ε in a dynamic pricing setting: a target accuracy of $\varepsilon = 1/T$ is enough to guarantee that for the known C case $R_{\operatorname{pricing}}(T) \leq C + \mathcal{O}(R_{\operatorname{CorPV},K}(T))$. Note that this idea directly extends to the unknown C case too, as every exploit price for layer ℓ is also a valid exploit price for all layers $\ell' \leq \ell$ due to their knowledge sets being consistent.

Corollary 5.1 (Dynamic Pricing with Irrational Responses Regret). CORPV. A applied to dynamic pricing with an unknown number of irrational responses and target accuracy $\varepsilon = d/T$ yields regret $R_{\text{pricing}}(T) = \mathcal{O}\left(d^3 \log^2(T) \left(\log T + C\right)\right)$.

¹¹In fact any deterministic algorithm for learning in the presence of irrational agents would yield linear regret, even with a minimal number of irrational responses.

¹²This corresponds to the quantity called "exploit" price by Cohen et al. [CLPL19].

5.2 Stackelberg Security Games (SSGs)

Let $D = \{1, \ldots, d\}$ denote the set of targets, and $p_{i,t} \in [0, 1], i \in [d]$ the probability that the learner protects target i at round t. We refer to each vector $\mathbf{p}_t = (p_{1,t}, \ldots, p_{d,t})$ as the coverage probabilities. The utilities of both the attacker and the learner are linear in the coverage probabilities that the learner deploys. The interaction protocol between the learner and the opponent is the following. First, the learner commits to coverage probability distribution $\mathbf{p}_t \in [0, 1]^d$. Next, the opponent observes \mathbf{p}_t , and chooses to attack target $i_t(\mathbf{p}_t)$, which is ultimately observed by the learner. We note that in SSGs the ground truth $\boldsymbol{\theta}^*$ corresponds to the optimal coverage probability that the learner can commit to, based on the responses from the opponent. In standard SSGs, the attacker is always best responding, i.e., $i_t(\mathbf{p}_t) = \arg \max_{i \in D} U_o(i, \mathbf{p}_t)$, where by $U_o(i, \mathbf{p}_t)$ we denote the attacker's utility for attacking target i under coverage probability \mathbf{p}_t from the learner.

In SSGs where the opponent can make completely irrational choices, target $i_t(\mathbf{p}_t)$ can be any target in D (i.e., not necessarily $\arg\max_{i\in D}U_o(i,\mathbf{p}_t)$) during at most C rounds. For the case where C=0 (i.e., fully rational opponent) the currently known best-performing algorithm¹⁴ is due to [PSTZ19]. The core idea behind the algorithm of Peng et al. [PSTZ19] is that in the d-simplex, the learner's probabilities for which the opponent attacks a specific target form polytopes (see e.g., Figure 3 for an example with 3 targets).

The algorithm of Peng et al. [PSTZ19] is prone to being manipulated by irrational responses every time that it has to decide whether to calibrate \mathbf{p}_{t+1} according to $i_t(\mathbf{p}_t)$ or not. As we observe, in order to be robust to irrational agents, the algorithm should calibrate \mathbf{p}_{t+1} according to $i_t(\mathbf{p}_t)$ only when the learner is certain that $i_t(\mathbf{p}_t)$ would be attacked by a rational agent. In the presence of irrational agents, we maintain an undesirability level for each target $i \in D$ and increment it each time target i is attacked. Once a target has reached undesirability level C+1, then we know for sure that it is one of the targets that the fully rational opponent would attack. More details can be found in Appendix B.1.

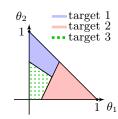


Figure 3: Learner's simplex.

Corollary 5.2 (SSGs with Irrational Responses). The regret incurred by the aforementioned algorithm which is robust against an unknown number of C irrational responses is: $\widetilde{\mathcal{O}}((2C \log C)d^3 \log T)$, where $\widetilde{\mathcal{O}}(\cdot)$ hides dependencies in the representation precision.

5.3 Learning a Lipschitz Function with Binary Feedback

In the problem of learning a Lipschitz function with binary feedback, recently studied by [MPLS18], a learner tries to learn a Lipschitz function $f:[0,1]^d \to [0,1]$ repeatedly over T rounds. In each round $t \in [T]$, the learner observes an adversarially chosen context \mathbf{x}_t . The learner then submits a guess about $f(\mathbf{x}_t)$, which we denote by y_t , and the opponent responds whether $y_t > f(\mathbf{x}_t)$ or $y_t < f(\mathbf{x}_t)$. For round $t \in [T]$ the learner incurs loss $\ell(f(\mathbf{x}_t) - y_t) = |f(\mathbf{x}_t) - y_t|$, i.e., symmetric.

Our algorithm for learning in the presence of such irrational agents extends the algorithm of Mao et al. [MPLS18], which at a high level works as follows. The learner maintains a partitioning of the domain $[0,1]^d$ into d-dimensional "boxes". For each box X_j the learner maintains an estimated

¹³We note that [BBHP15] study online learning algorithms in SSGs against multiple, known types of opponents. Although we believe that the our technique could also be useful in these settings, in such a setting there is no inherent, consistent "ground truth" similar to a single θ^* which is essential for any binary-type setting and for which CorPV.K readily applies.

¹⁴The results of their algorithm are stated in terms of sample complexity, we note that one can easily turn them into a no-regret learning algorithm (see e.g., Algorithm 5) with regret guarantees $\mathcal{O}(d^3)$.

interval of where $f(\mathbf{x})$ for $\mathbf{x} \in X_j$ lies, denoted by Y_j . For every \mathbf{x}_t queried, after the learner observes whether $y_t > f(\mathbf{x}_t)$ or $y_t < f(\mathbf{x}_t)$, the learner updates the estimated Y_j . Once Y_j has been estimated to a satisfactory degree, the interval X_j gets partitioned in 2^d smaller boxes and the process repeats.

For robustness, we must ensure that for each X_j the associated Y_j always contains the interval $[\min_{\mathbf{x} \in X_j} f(\mathbf{x}), \max_{\mathbf{x} \in X_j} f(\mathbf{x})]$. As before, we do this by tracking undesirability levels on subintervals of Y_j . Specifically, if $y_t > f(\mathbf{x}_t)$, then we increase the undesirability of interval $Y_j \setminus [0, y_t + L\ell_j]$, where $\ell_j = \text{diam}(X_j)$, otherwise we increase the undesirability of interval $Y_j \setminus [y_t - L\ell_j, 1]$. After 2C + 1 queries in interval X_j , there exists a subinterval of Y_j with undesirability level at least C + 1 and we can safely eliminate it from Y_j . Our algorithm is presented in Appendix B.2 for completeness.

Corollary 5.3. Algorithm 9 applied to a setting of learning an L-Lipschitz function with binary feedback, symmetric¹⁵ loss functions and with an unknown number of irrational responses yields regret: $R(T) = \mathcal{O}\left((2C\log C + 1)LT^{\frac{d-1}{d}\log T}\right)$.

¹⁵Our approach presented in Algorithm 9 can be easily adapted if the learner measures her regret in terms of pricing loss.

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A Supplementary material for Section 3

A.1 Supplementary material for Section 3.1

Definition A.1 (Undesirability Level). For an epoch $\phi \in \Phi$, let \mathbf{p} be a point in K_{ϕ} . We define \mathbf{p} 's undesirability level for epoch ϕ , denoted by $u_{\phi}(\mathbf{p})$, as the number of rounds within epoch ϕ , for which

$$u_{\phi}(\mathbf{p}) = \sum_{t \in T_{\phi}} \mathbb{1} \left\{ \langle \mathbf{p} - \boldsymbol{\kappa}_{\phi}, \boldsymbol{v}_{t} \rangle \cdot y_{t} < 0 \right\}$$

We next present two useful propositions regarding the number of contexts needed in order to guarantee that we have found an appropriately undesirable hyperplane, for the cases of d = 2 and d = 3 respectively.

Proposition A.2. For d=2 and any corruption level C, after 3C+1 rounds within an epoch, there exists a hyperplane $\boldsymbol{v}^*(\omega_{\phi})$ among $\{\boldsymbol{v}_t(\omega_{\phi})\}_{t\in T_{\phi}}$ with undesirability level at least C+1 in the entirety of one of its halfspaces.

Proof. Since we know that there exist at most C corrupted rounds among the 3C+1 rounds of epoch ϕ , then at least 2C+1 are uncorrupted. We say that these rounds are part of the set U_{ϕ} . For all $t \in U_{\phi}$, the learner's $\{v_t(\omega_t)\}_{t \in U_{\phi}}$ pass from the same centroid κ_{ϕ} and they all protect the region where θ^* lies. In other words, none among $\{v_t\}_{t\in U_{\phi}}$ adds an undesirability point to θ^{\star} . See Figure 4 for a depiction for the case where C=1 and each context appears only once. Since all hyperplanes point towards the same direction (i.e., the region containing θ^* never gets an undesirability point), starting from the region where θ^* lies and moving counter clockwise the undesirability levels of the formed regions first increase (moving from 0 to 2C + 1) and then decrease (moving from 2C + 1 to 0). Due to this being a concave function, it is clear to see that there always exists a hyperplane with undesirability level at least C+1in the entirety of one of its halfspaces.

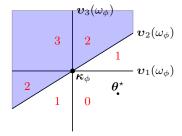


Figure 4: Sketch of the undesirability levels for epoch ϕ , after 2C+1 uncorrupted rounds (i.e., $c_t=0$), assuming that each context appears once. Red numbers denote undesirability of each region. Blue opaque region denotes the knowledge set for epoch $\phi+1$.

Proposition A.3. For d=3, any corruption level C, any centroid κ and any number of rounds N during epoch ϕ , there exists a θ^* and a sequence of N contexts $\{v_t\}_{t=1}^N$, such that there does not exist a hyperplane $v^* \in \{v_t\}_{t=1}^N$ with either of its halfspaces having undesirability at least C+1.

Proof. For any convex body K with centroid κ , we show how to construct a problematic instance of a θ^* and N contexts. Fix the corruption level to be C=1, and $c_t=0, \forall t\in [N]$. However, the learner does not know that none of the rounds is corrupted. Construct a sequence of contexts $\{v_t\}_{t=1}^N$ such that no two are equal and for $\omega_t=\langle v_t,\kappa\rangle, \forall t\in [N]$ we have that:

$$\{oldsymbol{v}_{t_1}(\omega_{t_1})\}igcap \{oldsymbol{v}_{t_2}(\omega_{t_2})\} = oldsymbol{\kappa}$$

and the smallest region r^* that contains $\boldsymbol{\theta}^*$ is defined by all $\{\boldsymbol{v}_t\}_{t=1}^N$ Intuitively, these hyperplanes form a conic hull.

Take any hyperplane $h \in \mathbb{R}^3$ neither parallel nor orthogonal with any hyperplane $\{v_t(\omega_t)\}_{t=1}^N$ such that $h \cap r^* = q \neq \emptyset$. Take q's projection in \mathbb{R}^2 . Observe that we have constructed an instance

where no matter how big N is, there does not exist any hyperplane with undesirability level at least C+1 (i.e., 2 in our example for C=1) in either one of its halfspaces. This problematic instance easily generalizes for any C>1.

A.2 Supplementary Material for Section 3.3

Proof of Lemma 3.14. We want to compute the probability that a point randomly sampled from $\mathcal{B}(\mathbf{p}^{\star},\zeta)$ falls in the following halfspace:

$$\mathbf{H}' \equiv \left\{ \mathbf{x} : \langle \mathbf{h}^{\star}, \mathbf{x} - \mathbf{p}^{\star} \rangle \ge \frac{\zeta \cdot \ln(3/2)}{\sqrt{d-1}} \right\}$$

Hence, we want to bound the following probability: $\mathbb{P}\left[\mathbf{x} \in \mathbf{H}' | \mathbf{x} \in \mathcal{B}(\mathbf{p}^*, \zeta)\right]$. If we normalize $\mathcal{B}(\mathbf{p}^*, \zeta)$ to be the unit ball B, then this probability is equal to:

$$\mathbb{P}\left[\mathbf{x} \in \mathbf{H}' | \mathbf{x} \in \mathcal{B}(\mathbf{p}^{\star}, \zeta)\right] = \mathbb{P}\left[\mathbf{x} \in \mathbf{H}^{1} | \mathbf{x} \in B\right] = \frac{\operatorname{vol}\left(B \cap \mathbf{H}^{1}\right)}{\operatorname{vol}(B)}$$
(12)

where \mathbf{H}^1 is the halfspace such that $\mathbf{H}^1 \equiv \left\{ \mathbf{x} : \langle \mathbf{h}^{\star}, \mathbf{x} \rangle \geq \frac{\ln(3/2)}{\sqrt{d-1}} = \rho \right\}$, and the last equality is due to the fact that we are sampling uniformly at random.

Similar to the steps in [BHK16, Section 2.4.2], in order to compute $\operatorname{vol}(B \cap \mathbf{H}^1)$ we integrate the incremental volume of a disk with width dx_1 , with its face being a (d-1)-dimensional ball of radius $\sqrt{1-x_1^2}$. Let V(d-1) denote the volume of the (d-1)-dimensional unit ball. Then, the surface area of the aforementioned disk is: $(1-x_1^2)^{\frac{d-1}{2}} \cdot V(d-1)$.

$$\operatorname{vol}\left(B\bigcap\mathbf{H}^{1}\right) = \int_{\rho}^{1} (1-x_{1}^{2})^{\frac{d-1}{2}} \cdot V(d-1) dx_{1} = V(d-1) \cdot \int_{\rho}^{1} (1-x_{1}^{2})^{\frac{d-1}{2}} dx_{1} \qquad (V(d-1) \text{ is a constant})$$

$$\geq V(d-1) \cdot \int_{\rho}^{\sqrt{\frac{\ln 2}{d-1}}} (1-x_{1}^{2})^{\frac{d-1}{2}} dx_{1} \qquad (\sqrt{\frac{\ln 2}{d-1}} < 1, \forall d \geq 2)$$

$$\geq V(d-1) \cdot \int_{\rho}^{\sqrt{\frac{\ln 2}{d-1}}} \left(e^{-2x_{1}^{2}}\right)^{\frac{d-1}{2}} dx_{1} \qquad (1-x^{2} \geq e^{-2x^{2}}, x \in [0,0.8], \frac{\ln 2}{d-1} \leq 0.8, \forall d \geq 2)$$

$$= V(d-1) \cdot \int_{\rho}^{\sqrt{\frac{\ln 2}{d-1}}} e^{-x_{1}^{2}(d-1)} dx_{1} \qquad (x_{1} \leq \sqrt{\frac{\ln 2}{d-1}})$$

$$\geq V(d-1) \cdot \int_{\rho}^{\sqrt{\frac{\ln 2}{d-1}}} \sqrt{\frac{d-1}{\ln 2}} \cdot x_{1} \cdot e^{-x_{1}^{2}(d-1)} dx_{1} \qquad (x_{1} \leq \sqrt{\frac{\ln 2}{d-1}})$$

$$\geq -\frac{V(d-1)}{2\sqrt{(d-1) \cdot \ln 2}} \left[e^{-(d-1)x^{2}}\right]_{\rho}^{\sqrt{\frac{\ln 2}{d-1}}}$$

$$= \frac{V(d-1)}{2\sqrt{(d-1) \cdot \ln 2}} \left(e^{-\ln(3/2)} - e^{-\ln 2}\right) = \frac{V(d-1)}{2\sqrt{(d-1) \cdot \ln 2}} \left(\frac{2}{3} - \frac{1}{2}\right)$$

$$= \frac{V(d-1)}{12\sqrt{(d-1) \cdot \ln 2}}$$

$$(13)$$

Next we show how to upper bound the volume of the unit ball B. First we compute the volume of one of the ball's hemispheres, denoted be vol(H). Then, the volume of the ball is vol(B) = 2vol(H). The volume of a hemisphere is at most the volume of a cylinder of height 1 and radius 1, i.e., $V(d-1) \cdot 1$. Hence, $vol(B) \leq 2V(d-1)$. Combining this with Equation (13), Equation (12) gives the following ratio:

$$\frac{\operatorname{vol}\left(B\bigcap\mathbf{H}^{1}\right)}{\operatorname{vol}(B)}\geq\frac{1}{24\sqrt{(d-1)\cdot\ln2}}\geq\frac{1}{20\sqrt{d-1}}.$$

This concludes our proof.

A.3 Supplementary material for Section 3.4

Lemma A.4 (Grünbaum Theorem). Let K denote a convex body and κ its centroid. Given an arbitrary non-zero vector \mathbf{u} , let $K_+ = \{\mathbf{x} | \langle \mathbf{u}, \mathbf{x} - \kappa \rangle \geq 0\}$. Then:

$$\frac{1}{e} \mathrm{vol}(K) \leq \mathrm{vol}\left(K_{+}\right) \leq \left(1 - \frac{1}{e}\right) \mathrm{vol}(K)$$

Lemma A.5 (Brunn's Theorem). For convex set K if g(x) is the (d-1)-dimensional volume of the section $K \cap \{\mathbf{y} | \langle \mathbf{y}, e_i \rangle = x\}$, then the function $r(x) = g(x)^{\frac{1}{d-1}}$ is concave in x over its support.

Proof of Lemma 3.17. For this proof we assume without loss of generality that $\mathbf{u} = e_1$, and that the projection of K onto e_1 is interval [a,1]. We are interested in comparing the following two quantities: vol(K) and $vol(K^{\mu}_{+})$. By definition:

$$vol(K_{+}) = \int_{0}^{1} r(x)^{d-1} dx \quad and \quad vol(K_{+}^{\mu}) = \int_{\mu}^{1} r(x)^{d-1} dx$$
 (14)

where $r(x) = g(x)^{\frac{1}{d-1}}$ and g(x) corresponds to the volume of the (d-1)-dimensional section $K_x = K \bigcap \{\mathbf{x} | \langle \mathbf{x}, e_i \rangle = x\}$. We now prove that $\operatorname{vol}(K_+^{\mu}) \geq \frac{1}{e} \operatorname{vol}(K_+)$. Combining this with Grünbaum Theorem (Lemma A.4) gives the result. We denote by ρ the following ratio:

$$\rho = \frac{\int_{\mu}^{1} r(x)^{d-1} dx}{\int_{0}^{1} r(x)^{d-1} dx} \ge \frac{\int_{1/d}^{1} r(x)^{d-1} dx}{\int_{0}^{1} r(x)^{d-1} dx}$$
(15)

We approximate function r(x) with function \tilde{r} :

$$\tilde{r}(x) = \begin{cases} r(x) & \text{if } 0 \le x \le \delta \\ (1-x) \cdot \frac{r(\delta)}{1-\delta} & \text{if } \delta < x \le 1 \end{cases}$$

Note that since $0 = \tilde{r}(1) \le r(1)$ (because r(x) is a non-negative function) and r(x) is concave from Brunn's theorem (Lemma A.5), for functions r(x) and $\tilde{r}(x)$ it holds that $r(x) \ge \tilde{r}(x)$. Using this approximation function $\tilde{r}(x)$ along with the fact that function $f(z) = \frac{z}{y+z}$ is increasing for any scalar y > 0, we can relax Equation (15) as follows:

$$\rho \ge \frac{\int_{1/d}^{1} \tilde{r}(x)^{d-1} dx}{\int_{0}^{1/d} \tilde{r}(x)^{d-1} dx + \int_{1/d}^{1} \tilde{r}(x)^{d-1} dx}$$
(16)

Next, we use another approximation function $\hat{r}(x) = (1-x) \cdot \frac{r(\delta)}{1-\delta}$, $0 \le x \le 1$; this time in order to approximate function $\tilde{r}(x)$. For $x \in [\delta, 1]$: $\tilde{r}(x) = \hat{r}(x)$. For $x \in [0, \delta]$ and since $\tilde{r}(0) = r(0) = 0$ and $\tilde{r}(x)$ is concave in $x \in [0, \delta]$, $\hat{r}(x) \ge \tilde{r}(x) = r(x)$, $x \in [0, \delta]$. Hence, Equation (16) can be relaxed to:

$$\rho \ge \frac{\int_{1/d}^{1} \hat{r}(x)^{d-1} dx}{\int_{0}^{1/d} \hat{r}(x)^{d-1} dx + \int_{1/d}^{1} \hat{r}(x)^{d-1} dx} = \frac{\int_{1/d}^{1} (1-x)^{d-1} \cdot \left(\frac{r(\delta)}{1-\delta}\right)^{d-1} dx}{\int_{0}^{1} (1-x)^{d-1} \cdot \left(\frac{r(\delta)}{1-\delta}\right)^{d-1} dx}$$
$$= \frac{\int_{1/d}^{1} (1-x)^{d-1} dx}{\int_{0}^{1} (1-x)^{d-1} dx} = \frac{-\frac{1}{d} \left(0 - \left(1 - \frac{1}{d}\right)^{d}\right)}{-\frac{1}{d} (0-1)} = \left(1 - \frac{1}{d}\right)^{d} \ge \frac{1}{2e}$$

This concludes our proof.

B Supplementary material for Section 5

B.1 Stackelberg Security Games

B.1.1 Offline-to-Online Algorithm

ALGORITHM 5: OFFLINETOONLINE(ALG)

- 1 Let ALG be the algorithm that identifies the learner's optimal \mathbf{p}^{\star} with O(s) samples.
- 2 for $1 \le t \le s$ do Learner commits to \mathbf{p}_t^{\star} suggested by Alg.

 \triangleright exploration phase

3 for $s+1 \le t \le T$ do Learner commits to \mathbf{p}^* .

▷ exploitation phase

Regarding the regret guarantee, we note that for all the exploration rounds the learner suffers a regret of at most 1, while for the exploitation rounds, if \mathbf{p}^* has been identified correctly, the regret is 0. The latter happens with probability at least $1-\delta$ and hence, the regret of incurred by the exploitation rounds is at most δT . As a result, tuning $\delta = 1/T$, from [PSTZ19] combined with Algorithm 5 we get an upper bound of $\widetilde{\mathcal{O}}(d^3)$ for the regret of the learner, where the $\widetilde{\mathcal{O}}(\cdot)$ notation hides dependencies in the representation precision.

B.1.2 Missing Details

ALGORITHM 6: SEPARATESEARCH ([PSTZ19])

1 $\mathcal{U}_i = \overline{\mathcal{P}} = [0, 1]^d, i \in [d]$

▷ upper bound of the polytopes

2 $\mathcal{L}_i = \emptyset, i \in [d]$

- \triangleright lower bound of the polytopes
- $\mathcal{F} = \emptyset$ \triangleright effective actions set
- 4 Randomly choose a probability distribution θ from \mathcal{P} .
- 5 Observe opponent's response $\hat{r}(\boldsymbol{\theta}) \in [d]$ and $\mathcal{F} \leftarrow \{\hat{r}(\boldsymbol{\theta})\}.$
- 6 while $\exists \mathcal{U}_i \neq \mathcal{L}_i \ and \bigcup_{i \in [d]} \mathcal{L}_i \neq \mathcal{P} \ \mathbf{do}$
- while $\exists i, j \in [d]$ such that $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ and $i, j \in \mathcal{F}$ do FindSeparatingPlane $(\mathcal{U}_i, \mathcal{U}_j, \mathcal{L}_i, \mathcal{L}_j)$
- s if $\exists \mathcal{U}_i \neq \mathcal{L}_i$ and $i \in \mathcal{F}$ then SecuritySearch($\mathcal{U}_i, \mathcal{L}_i$)
- 9 return $\mathcal{L}_1, \dots, \mathcal{L}_d$

Fix target $i \in [d]$. If there exists a coverage probability distribution such that target j is attacked by the opponent, then we call target j an *effective* action for the opponent. For each identified effective action $i \in [d]$, the learner maintains an upper and a lower bound on the size of \mathcal{P}_i . The estimated

size shrinks as new separating hyperplanes between actions are identified. Interestingly, a module of the algorithm of [PSTZ19] (called SeparateSearch) makes sure that given two different effective actions j, k with overlapping bounds, then the module either finds their separating hyperplane, or a new effective action. For completeness, we include the SeparateSearch Algorithm for the learner above. Let \mathcal{P}_i the polytope that corresponds to target i being attacked. The learner's target thus becomes to precisely find the frontiers of these polytopes.

ALGORITHM 7: SECURITYSEARCH($\mathcal{U}_i, \mathcal{L}_i$) ([PSTZ19])

```
<sup>1</sup> Find \boldsymbol{\theta} = (\theta_1, \dots, \theta_d) such that \boldsymbol{\theta} \in \mathcal{U}_i and \max \theta_i.
```

- 2 $\forall j \in [d] \text{ set } \boldsymbol{\theta}_t \leftarrow \boldsymbol{\theta}_j \text{ if } j \in \mathcal{F} \text{ and } \boldsymbol{\theta}_t \leftarrow 0 \text{ otherwise.}$
- **3** Learner commits to $\boldsymbol{\theta}_t$ and observes the opponent's response: $\hat{r}(\boldsymbol{\theta}_t)$.
- 4 if $r(\boldsymbol{\theta}_t) = i$ then $\mathcal{L}_i \leftarrow \mathcal{U}_i$
- 5 else $\mathcal{L}_{\hat{r}(\boldsymbol{\theta}_t)} \leftarrow \boldsymbol{\theta}_t$ and $\mathcal{F} \leftarrow \mathcal{F} \bigcup \{\hat{r}(\boldsymbol{\theta}_t)\}.$

We also include a sketch of the FINDSEPARATINGHYPERPLANE Algorithm of [LCM09], which is called by SEPARATESEARCH.

$\overline{\text{ALGORITHM 8: FINDSEPARATINGHYPERPLANE}}(\mathcal{U}_i, \mathcal{U}_j, \mathcal{L}_i, \mathcal{L}_j) \ ([\text{LCM09}])$

- 1 Form $\mathcal{P}_i, \mathcal{P}_j$.
- **2** Find any point $\mathbf{q} \in \mathcal{P}_i \cap \mathcal{P}_i$.
- **3** Query **q** and observe that opponents response $\hat{r}(\mathbf{q})$.
- 4 Draw line between **q** and some θ such that $\hat{r}(\theta) \neq \hat{r}(\mathbf{q}), \hat{r}(\theta) \in \{i, j\}$.
- 5 Binary search on this line to find a single point on a hyperplane that you have not yet discovered, h.
- 6 Find set of d linearly independent points on h (i.e., reconstruct it).
- 7 Update $\mathcal{P}_{\hat{r}(\boldsymbol{\theta})}$ to include \mathbf{h} .

Observe that Algorithm SeparateSearch is prone to being manipulated by irrational responses from the opponent essentially when it observes the opponent's response and it has to decide whether the target should be added in the effective actions set or not. At a high level, the learner would be robust against irrational replies from the opponent, if she had a way to guarantee that she has observed a truly effective action. Here is how our solution with the undesirability levels can be proven useful for this task: every time that the opponent attacks a specific target i, increase its undesirability level by 1, but do not make it part of the effective action set yet. After 2C+1 rounds, the learner's best-response is the target i with $u(i) \geq C+1$, since the learner has only C corruptions available.

For the case of Algorithm 8 this is even more straightforward, as it actually includes a *single* dimensional binary search as a subroutine, which we showed in Section 3 that it can be robustified by asking the same query 2C + 1 times. Putting all these pieces together, we get the following.

B.2 Algorithm Corpv.K for Lipschitz buyers.

As pointed out earlier, our algorithm uses a lot of insights from the algorithm of Mao et al. [MPLS18].

ALGORITHM 9: CORPV.K for Lipschitz Buyers

```
for t \in [T] do

Learner observes context \mathbf{x}_t \in [0,1]^d chosen by the opponent.

Learner finds j s.t., \mathbf{x}_t \in X_j. Let \ell_j = \operatorname{diam}(X_j). \triangleright [MPLS18, Algorithm 3, Step 6]

Learner guesses y_t = (\max(Y_j) + \min(Y_j))/2 \triangleright [MPLS18, Algorithm 3, Step 7]

if y_t > f(x_t) then Increase undesirability of interval Y_j \setminus [0, y_t + L\ell_j] by one.

else Increase undesirability of interval Y_j \setminus [y_t - L\ell_j, 1] by one.

if interval X_j has been queried at least 2C + 1 times then

Update Y_j to not include the interval that was undesirable at least C + 1 times.

if h_j = \operatorname{length}(Y_j) < 4L\ell_j then

\triangleright [MPLS18, Algorithm 3, Steps 13–15]

Bisect each side of X_j to form 2^d new boxes with the same Y_j.
```