# Regression Model

**Jaya Sil**

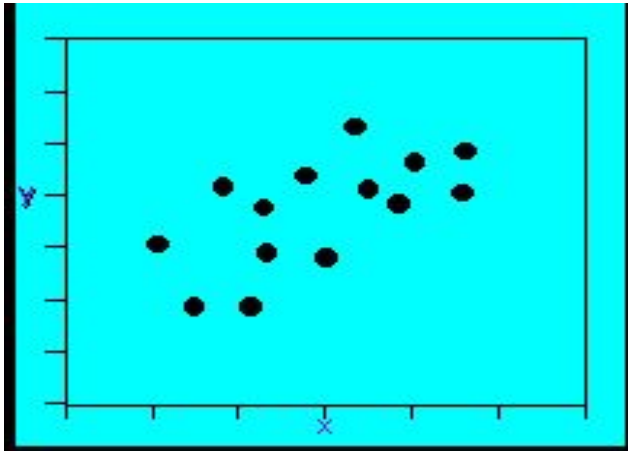
**Indian Institute of Engineering Science & Technology, Shibpur**

**Howrah**

# Linear Model

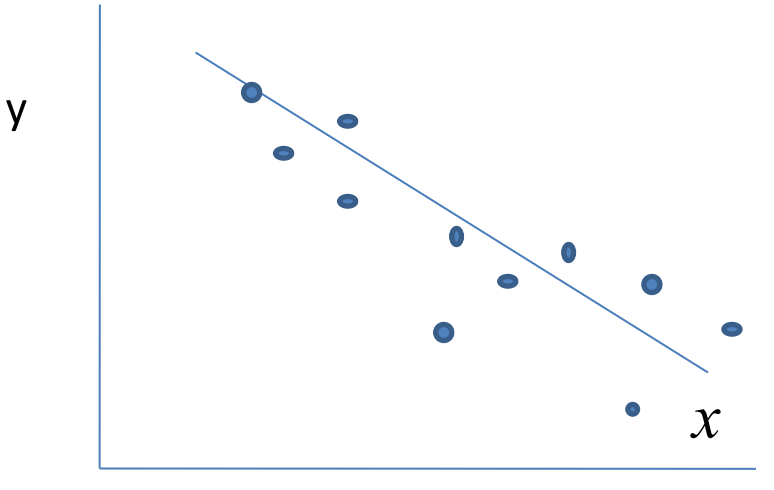
- **Linear models** describe a continuous response variable as a function of one or more predictor variables.

- Learning a linear relationship between the input attributes (predictor variables) and target values (response variable) values.

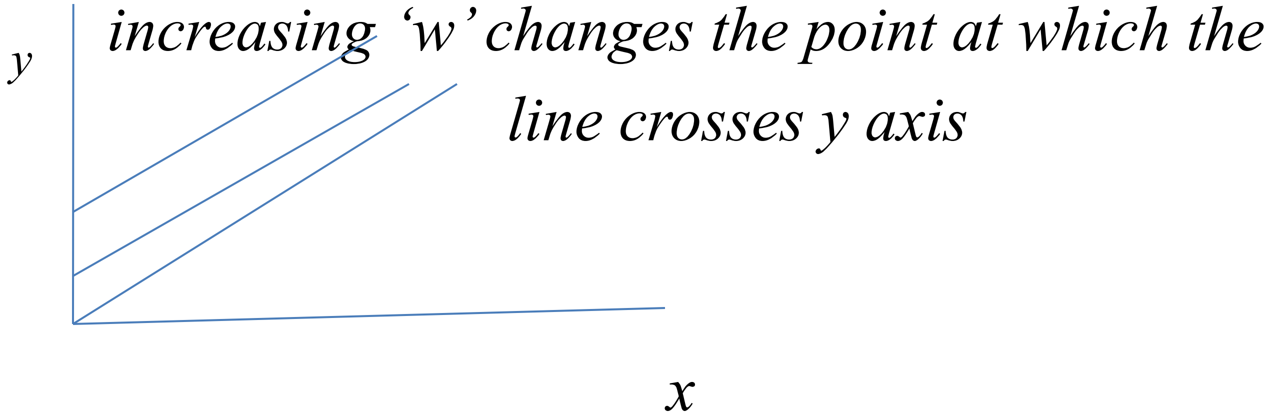


# Linear Model

- Instead of evaluating h($x$) as a function of $x$, we make it more flexible using a set of associated parameters.
- $y = wx$ or $y = h(x; w)$ and the relationship between $x$ and $y$ is linear.
- *Assumption: The data could be adequately modeled with a straight line*



- The assumption is not perfectly satisfied in the Figure.



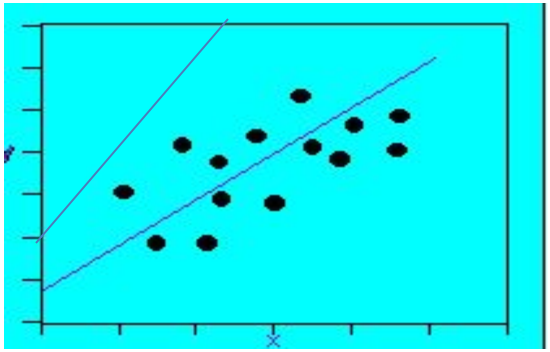

- *Add a single parameter as* $y = wx$ or $y = \mathrm{h}(x;\, w)$; enhancing the model with any gradient using the choice of $w$.

- But it is not realistic at $x = 0$ ; $y = w \times 0$ is zero.

- Adding one more parameter to the model overcomes the problem; $y = \mathrm{h}(x;\, w_0,\, w_1) = w_0 + w_1 x$

# Supervised Machine Learning

- Increasing $w_1$ changes the gradient



- There are many functions which could be used to define the mapping.
- The ultimate goal is to develop a finely tuned predictor function h($x$) such that $y$ = h($x$)
- The learning task now involves using the data in figure choose two suitable values of $w_0$ and $w_1$

# Supervised Machine Learning

- We decide to approximate y as a linear function of x:

  $h(x) = w_0 + w_1 x_1 + w_2 x_2 + \ldots\ldots + w_n x_n$

- The $w_i$'s are the parameters (also called weights) parameterizing the space of linear functions mapping from X to Y.
- simple predictor: $y = h(x; w_0, w_1) = w_0 + w_1 x$

  Where $w_0$ and $w_1$ are constants.

- Our goal is to find the perfect values of $w_0$ and $w_1$ in order to make our predictor work as *best* as possible.

- We need to define what is the meaning of *best*.

# Defining a Good Model

- The best solution consists of the values of $w_0$ and $w_1$ that produce a line that passes as close as possible to *all* of the data points.

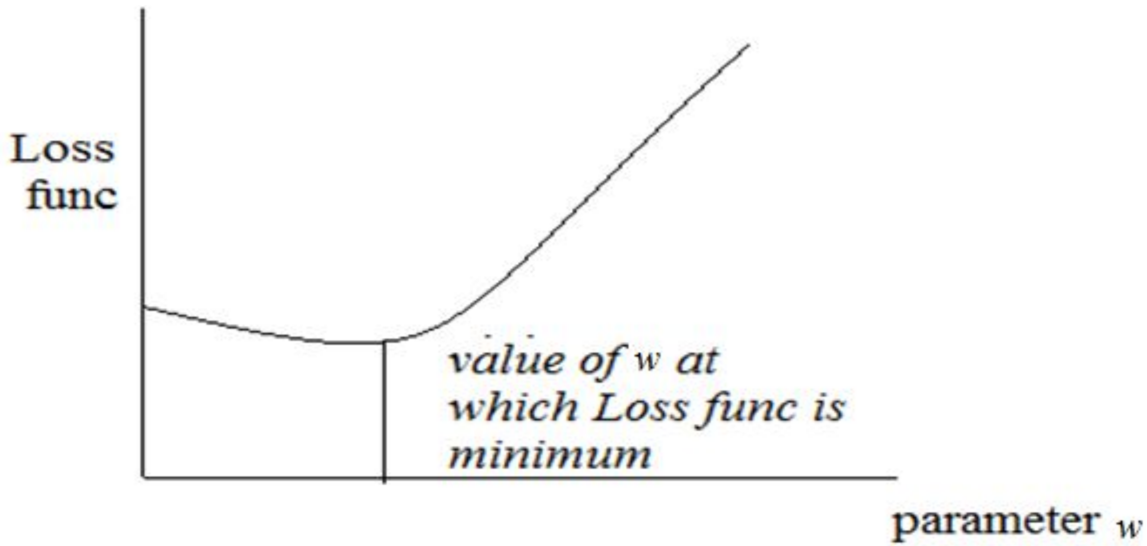
- The minimum squared difference between the target value and the predicted value is a measure of how good is the model.

- The squared difference is defined as: $(t_n - h(x_n; w_0, w_1))^2$ for $n$-th pattern and known as the *squared loss function or cost function* $L_n()$

- $L_n(t_n, h(x_n; w_0, w_1)) = (t_n - h(x_n; w_0, w_1))^2$

# Loss function

"Learning" optimizes the loss function so that, given input data $x$ accurately predict value $h(x)$.




- Loss is always positive and lower the loss better the function describes the data.
- Average loss function: $L = 1/N \sum_{n=1}^{N} L_n (t_n - h(x_n; w_0, w_1))$

# Loss Function

- Tune $w_0$ and $w_1$ to produce the model that results lowest value of the average Loss function.

- $$L = \arg\min_{w_0, w_1} 1/N \sum_{n=1}^{N} L_n (t_n - h(x_n; w_0, w_1))$$

- Minimization of the squared loss function is the basis of Least Mean Square Error (LMSE) method of function approximation.

- Other loss functions, like Absolute Loss function

# Gradient Descent Algorithm

- We want to choose w so as to minimize Loss function.
- Use a search algorithm that starts with some "initial guess" for w, and that repeatedly changes w to make Loss smaller.
- Hopefully we converge to a value of w that minimizes Loss.
- Weight updating: $w_j := w_j - \eta \frac{\partial L}{\partial w_j}$

•Weight update is simultaneously performed for all values of *j*. Here, η is called the learning rate.

• Gradient Descent algorithm repeatedly takes a step in the direction of steepest decrease of L.

# LMS Algorithm

Tune $w_0$ and $w_1$ to produce the model that results lowest value of the average Loss function for a single training pattern.

$$\frac{\partial L}{\partial w_0} = \frac{\partial}{\partial w_0} \frac{1}{2} (\mathrm{t}_n - \mathrm{h}(x_n; w_0, w_1))^2$$

$$\frac{\partial L}{\partial w_0} = (\mathrm{t}_n - \mathrm{h}(x_n; w_0, w_1)) \cdot \frac{\partial}{\partial w_0} (\mathrm{t}_n - \mathrm{h}(x_n; w_0, w_1))$$

$$= - (\mathrm{t}_n - \mathrm{h}(x_n; w_0, w_1)) . \frac{\partial}{\partial w_0} ( \textstyle\sum_{n=1}^{d} w_n x_n - \mathrm{t}_n )$$

$$\frac{\partial L}{\partial w_1} = (\mathrm{t}_n - \mathrm{h}(x_n; w_0, w_1)) \cdot \frac{\partial}{\partial w_1} (\mathrm{t}_n - \mathrm{h}(x_n; w_0, w_1))$$

$$= - (\mathrm{t}_n - \mathrm{h}(x_n; w_0, w_1)) . \frac{\partial}{\partial w_1} ( \textstyle\sum_{n=1}^{d} w_n x_n - \mathrm{t}_n )$$

$$\frac{\partial L}{\partial w_{\mathbf{n}}} = ( \mathrm{h}(x_n; w_0, w_1) - \mathrm{t}_n ) x_n$$

# LMS UPDATE RULE

- For a single training example, the update rule is:

$$w_{\mathbf{n}} := w_{\mathbf{n}} - \eta\,(\mathrm{h}(x_n; w_0, w_1) - \mathrm{t}_n)\,x_n$$

- The magnitude of weight updating is proportional to error i.e.

$$(\mathrm{h}(x_n; w_0, w_1) - \mathrm{t}_n)$$

For N number of training patterns weight update rule:

$$w_{\mathbf{n}} := w_{\mathbf{n}} - \eta \sum_{n=1}^{\mathbf{N}} (\mathrm{h}(x_n; w_0, w_1) - \mathrm{t}_n)\,x_n$$

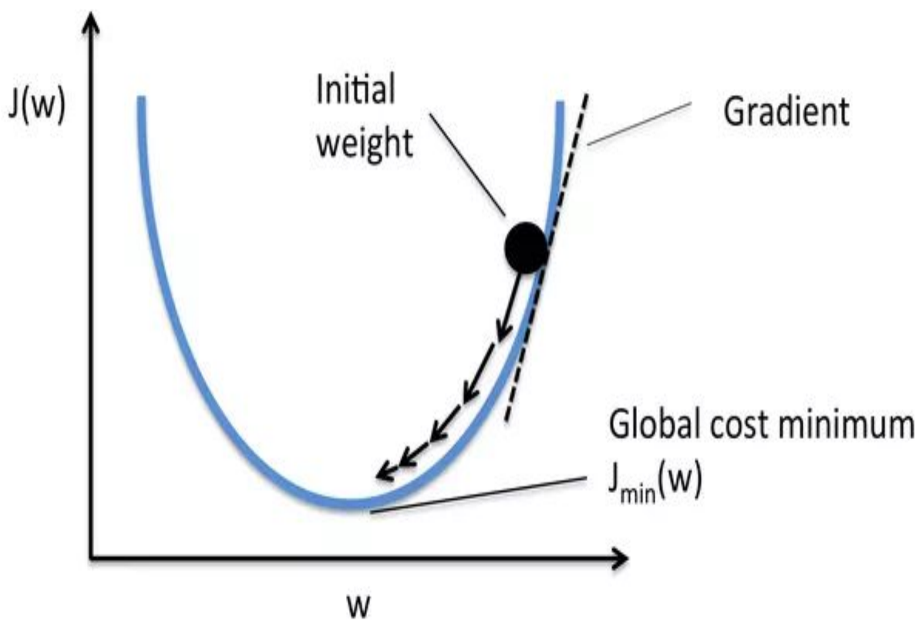
The algorithm will converge when there no weight update takes place in case it is performed iteratively.

# Gradient Descent Algorithm

- The update rule is gradient descent when summation is substituted by $\frac{\partial L}{\partial w_j}$ i.e. gradient of cost or loss function.
- L is a convex quadratic function, so converges at global minima/maxima.
- When updating is performed for each training example, called Batch Gradient Descent.
- When updating is performed for a set of training example, called Stochastic Gradient descent.

- Searching for points where the gradient of a function is zero, called minima.

- To determine the value of the zero gradient point (minima, maxima) we examine the second derivative



# Analytical Solution

$L$=1/N $\sum L_n$ ($t_n$- h($x_n$; $w_0$, $w_1$)); L is average Loss function

=1/N $\sum$ ($t_n$- h($x_n$; $w_0$, $w_1$))$^2$

=1/N $\sum$ ($t_n$- ($w_0$+$w_1 x_n$))$^2$

- Differentiating $L$ by calculating the partial derivatives with respect to $w_0$ and $w_1$ and equating them to zero to obtain $w_0$ and $w_1$
- Differentiating again w.r.t. $w_0$ and $w_1$ we find the point at which loss is minimum.

# Turning points

- $w_0 = 1/N\,(\sum t_n)\ - w_1(1/N(\sum x_n))$ when $\dfrac{\partial^2 L}{\partial w_0^2} = 2$
- $w_0^{av} = t^{av} - w_1 x^{av}$

There is one turning point that correspond to minimum loss

$$w_1^{av} = \frac{\frac{1}{N}\left(\sum_{n=1}^{N} x_n\, t_n\right) - t^{av} x^{av}}{\left(\frac{1}{N}\sum_{n=1}^{N} x_n^{\,2}\right) - x^{av} x^{av}}$$

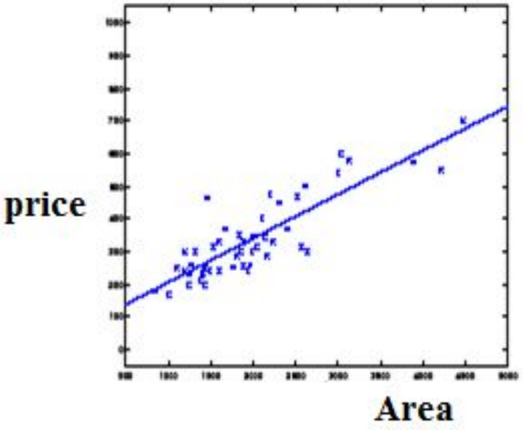
when

$$\frac{\partial^2 L}{\partial w_1^2} = \frac{2}{N} \sum_{n=1}^{N} x_n^2$$

Now we can compute the best parameter values

# Prediction

- Based on linear regression model we predict the output for some input.
- A simple linear model can fit a small dataset and used for prediction.
- $w_0 = 71.27$, $w_1 = 0.1345$




- Linear model can be extended to larger sets of attributes, modeling complex relationship between input and output.

# Vector-Matrix Notation

- Each data point is described by a set of attributes.
- Solving partial derivatives for each parameter associated with the attributes are time consuming affair.
- Representing attributes of each data point into vector form.
- For example $n$-th data point by $\mathbf{x}_n$ and with two attributes $\mathbf{x}_n = [x_{n1}, x_{n2}]^{\mathrm{T}}$
- Column vectors $\mathbf{w}$ and $\mathbf{x}_n$ is defined as $\mathrm{h}(x_n ; w_0\, \mathrm{w}_1) = \mathbf{w}^{\mathrm{T}}\, \mathbf{x}_n = w_{0} + w_1\, x_n$

- L = =1/N $\sum$ ($t_n$- ($w_0$+$w_1x_n$))$^2$ = =1/N $\sum$ ($t_n$ - $\mathbf{w}^T\mathbf{x}_n$ )$^2$
- $(\mathbf{t} - \mathbf{Xw})^T(\mathbf{t} - \mathbf{Xw})$ is used to write the loss function.
- $L = 1/N\ (\mathbf{t} - \mathbf{Xw})^T(\mathbf{t} - \mathbf{Xw})$
- Differentiating loss in vector/matrix form to obtain the vector **w** corresponding to the point where *L* is minimum.

$$\frac{\partial L}{\partial \mathbf{w}} = \begin{bmatrix} \frac{\partial L}{\partial w_0} \\ \frac{\partial L}{\partial \mathbf{w_1}} \end{bmatrix}$$

# Making Prediction

- Given a new vector of attributes, $x_{new}$, the prediction using the model as $t_{new} = \mathbf{W}^T \mathbf{X}_{new}$

- Linear model of the form with multiple attributes:

$$h(x_1, x_2, \ldots, x_n; w_{0,} w_1, \ldots w_n);$$

$$t_n = w_0 + w_1 x_{n1} + w_2 x_{n2} + \ldots + ..$$

- *Prediction from such model is very precise but not always sensible.*

# Learning Task

- Learning using **training examples :** statistically significant random sample.
- If the training set is too small (law of large numbers), we won't learn enough and may even reach inaccurate conclusions.
- For each training example, an input value *x*_train, and corresponding output, *y or target* is known in advance.
- For each example, we find the squared difference between the *target,* and predicted value h(*x*_train).
- With enough training examples, these differences give us a useful way to measure the "wrongness" of h(*x*).

# Learning Task

- Find parameter values so that the difference makes it “less wrong”.
- This process is repeated over and over until the system has converged on the best values.
- In this way, the predictor becomes trained, and is ready to do some real-world predicting.

# Linear Regression

- Get familiar with objective functions, computing their gradients and optimizing the objectives over a set of parameters.

- Goal is to predict a target value $y$ using a vector of input values $x \in \Re^n$ where the elements $x_j$ of $x$ represent "features" that describe the output $y$.

- Suppose many examples of houses where the features for the $i^{th}$ house are denoted $x^{(i)}$ and the price is $y^{(i)}$.

- Find a function $y = h(x)$

- If we succeed in finding a function h(x) and we have seen enough examples of houses and their prices, we hope that the function h(x) will also be a good predictor of the house price when we are given the features for a new house where the price is not known.

# Linear Regression

$h_w(x) = \sum_j w_j x_j = w^\top x$); functions parametrized by the choice of w.

- Task is to find *w* so that $h_w(x^{(i)})$ is as close as possible to y(*i*).
- In particular, we search for a w that minimizes:

  $L(w) = 1/2\sum_i(h_w(x^{(i)}) - y^{(i)})^2 = 1/2\sum_i(w^\top x^{(i)} - y^{(i)})^2$
- This function is the "cost function" which measures how much error is incurred in predicting $y^{(i)}$ for a particular choice of w.
- This may also be called a "loss", "penalty" or "objective" function.

- Find the choice of *w* that minimizes L(*w*).
- The optimization procedure finds the best choice of *w*
- The gradient $\nabla_{\mathbf{w}}L(w)$ of a differentiable function L is a vector that points in the direction of steepest increase as a function of w
- It is easy to see how an optimization algorithm could use this to make a small change to w that decreases (or increase) L(*w*).

# Optimization Method

- Compute the gradient:

- $$\nabla_w L(w) = \begin{bmatrix} \partial L(w)/\partial w_1 \\ \partial L(w)/\partial w_2 \\ \vdots \\ \partial L(w)/\partial w_n \end{bmatrix}$$

- Differentiating the cost function L(w) with respect to a particular parameter $w_j$ :
- $\partial L(w)/\partial w_j = \sum_i x^{(i)}_j (h_w(x^{(i)}) - y^{(i)})$

# Non-Linear Response from a Linear Model

- The linear model in terms of $w$ and $x$: $h(x; w) = w_0 + w_1 x$
- The model is linear in term of $w$ only:
  $h(x; w) = w_0 + w_1 x + w_2 x^2$ but the function is quadratic in terms of data.

- We can add as many power we like to get a polynomial function of any order.

- The general form for a K-th order polynomial:
- $h(x; \mathbf{w}) = \sum_{k=0}^{K} w_k x^k$ *OR* $h(x; w) = w_0 + w_1 x_1^2 + w_2 x_1 x_2 + w_3 x_2^2 + \ldots.$

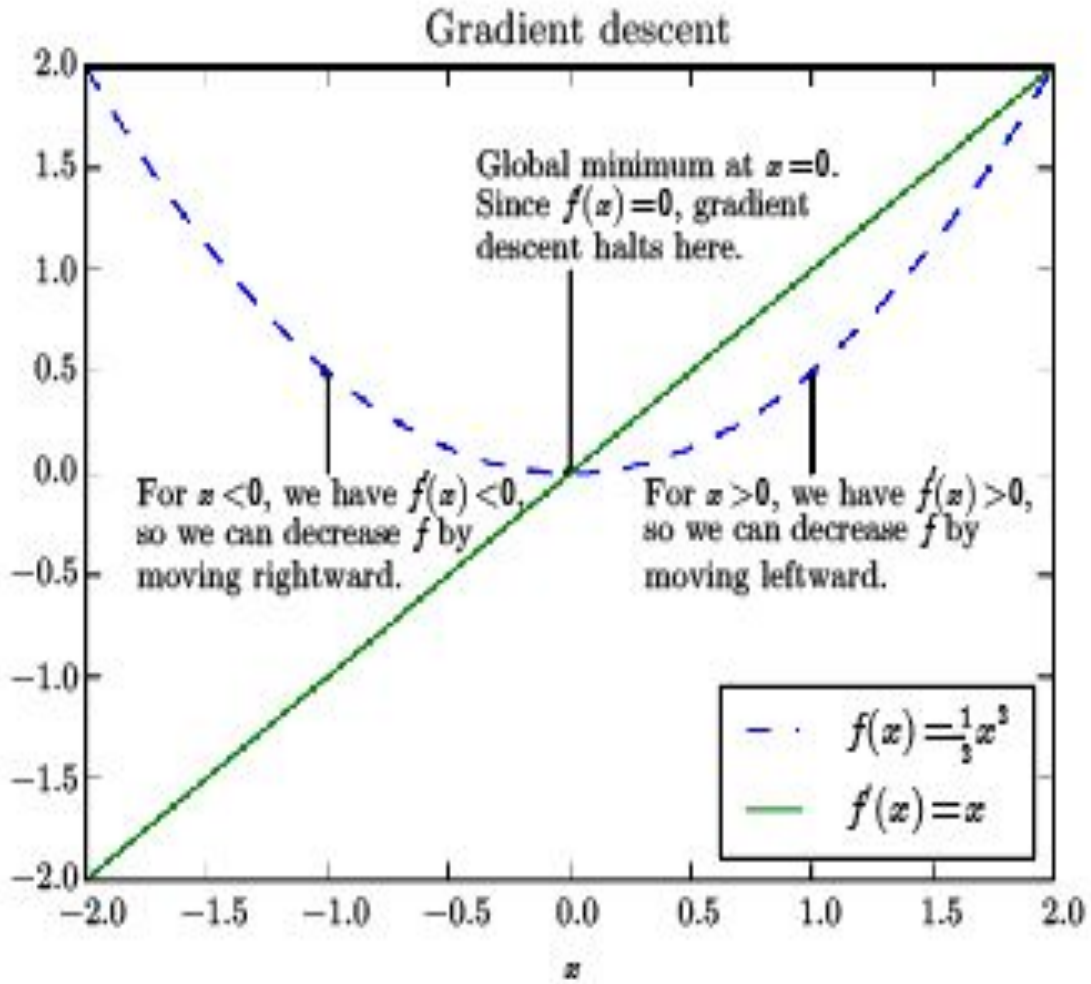
# Gradient Descent

$y = f(x)$

$x$ and $y$ are real numbers.

$dy/dx = f'(x)$

$f'(x)$ says how to change $x$ for a small improvement of y



# Critical Points

When d$y$/d$x$ = $f'(x)$ = 0, the derivative provides no information about which direction to move, points are called critical points.

- A local minima is a point where $y = f(x)$ is lower than at all the neighbouring points. So it is no longer possible to decrease $f(x)$ by infinitesimal steps.

- A local maxima is a point where f($x$) is higher than neighboring points, so not possible to increase $f(x)$