Research Statement

of Arnaja Mitra (PhD Student of Mathematics)

1. Current Research:

My research lies at the intersection of Dynamical Systems, Bifurcation Theory, and Nonlinear Analysis. My thesis primarily focuses on the equivariant degree theory and its use in symmetric dynamical systems, with or without memory. The **equivariant degree** is a component of Equivariant Analysis, addresses the qualitative study of nonlinear equations, exploring aspects such as the existence and multiplicity of solutions, stability, bifurcations, and the structure of solution sets, particularly in the presence of group symmetries. This can be viewed as a complement to classical methods based on singularity theory, particularly when applying the latter proves challenging or encounters significant hurdles due to a lack of smoothness and/or equivariant genericity. I developed a method based on equivariant degree theory for equations in abstract Hilbert spaces. This allows me to study the local and global symmetric Hopf bifurcation in abstract parabolic systems (see subsection 1.1.).

In an extension of the study of Hopf bifurcation in symmetric systems, my current research is focused on the systems of symmetrically coupled oscillators, with the phenomenon of hysteresis. **Hysteresis** is defined as the reliance of a system's state on its history. The hysteresis effect can be observed in diverse scientific domains, including magnetism, plasticity, tribology (which encompasses the study of friction and wear), materials science, and the investigation of fluid flows within porous media, among others (see subsection 1.2.).

1.1. Equivariant Hopf Bifurcation:

Given a parameterized family of dynamical systems, the Hopf bifurcation occurs when the parameter crosses a critical value, at which point the linearization admits a purely imaginary eigenvalue, resulting in the appearance of small amplitude periodic solutions. In order to study the Hopf bifurcation in symmetric systems of parabolic partial differential equations (PDEs), one employs suitable equivariant analysis tools ([2]). I have developed a general framework, which is founded upon the twisted equivariant degree theory ([1, 4]). In a nutshell, the twisted equivariant degree serves as a topological tool that enables the 'counting' of orbits of solutions to symmetric equations, taking into account the symmetric properties of the solutions. The method can answer the following paradigmatic question, which appears due to the presence of symmetries (the so-called equivariance): What is the impact of symmetries of a dynamical system on its dynamics? **For example, on the existence, multiplicity (the number) of bifurcating branches, global behavior of the branches, and symmetric properties of periodic solutions?** To the end, I have detected unbounded branches of non-constant periodic solutions bifurcating from equilibrium and provide a comprehensive description of their symmetric properties. This work has supported by an example of four identical parabolic systems coupled into a dihedral symmetric configuration.

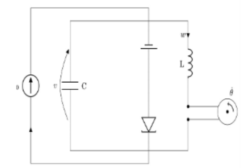
The study of Hopf bifurcation in symmetric systems of parabolic equations is crucial from a practical application standpoint across various domains. This includes biology (for modeling population dynamics and disease spread), physics (in fluid dynamics, to comprehend vortex dynamics), engineering (for designing control systems and averting instability-induced failures), medicine (to simulate conditions like atrial fibrillation), environmental science (for modeling pollutant dispersion and understanding ecosystem dynamics), economics (to assess the stability of economic systems), neuroscience (to grasp the dynamics of neural networks and the synchronization of neural oscillations), and more.

I presented my work [3] in a poster at both the "10th International Congress on Industrial and Applied Mathematics (ICIAM 2023)" and the "AWM Research Symposium (2023)". Furthermore, I was invited to present this work at two talks: the "SIAM Student Chapter Talk" at SUNY University at Buffalo and the "AWM Pittsburgh Chapter Seminar" at the University of Pittsburgh.

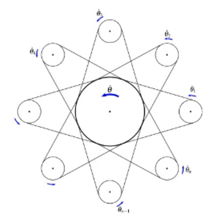
1.2. Dynamical Systems with Hysteresis:

The study of hysteresis has led to many models used in science and engineering [5]. Mark A. Krasnosel'skii and Alexei V. Pokrovskii established the foundational concepts for the operator approach to hysteresis models [6]. A key feature of hysteresis operators is rate-independence, i.e. these nonlinear operators are invariant with respect to the action of the group of affine transformations of the time scale. When one combine a dynamical system with a hysteresis operator, it can be challenging. This is because hysteresis operators aren't smooth due to their time rate feature. Also, this combined system lacks local linear structure. However, topological methods (here equivariant degree theory) proved to be efficient for obtaining existence results, which include conditions for existence of oscillations and localization of bifurcation points.

I have considered a symmetrically coupled system with one hysteresis operator. The system has obtained by n identical electromechanical coupled oscillators (motors) to a rotating inertial disk via elastic mechanical connections (see figure). I assume that the plasticity effect in the inertial disk produces hysteresis with the associated energy losses. This effect has taken into account by introducing the Prandtl-Ishlinskii hysteresis operator into the equations of motion of the disk. In this study, I analyze symmetries of branches of relative periodic solutions connected to the branch of relative equilibria of the system. My finding has further validated by a specific example where $n = 5$ (referred to as S_5 symmetry). For this example, using the degree theory approach, GAP, and numerical techniques, I have been categorized various branches with symmetries. This work [7] is in progress and very close to being ready with the preprint.



An S^1 -spatially equivariant electro-mechanical oscillator



S_n -symmetric coupling of n oscillators

1.3. Neural Network:

I am also interested in working with complex systems, primarily focusing on complexity and network science, with a particular emphasis on neural network modeling [8]. One of my main motivations for exploring neural network modeling and analysis is to understand human behavior and brain function. My objective is to construct a mathematical model and analyze networks of neurons from a dynamical systems perspective. The theory of dynamical systems permits the analysis of nonlinear systems of differential equations, including those models used in neuroscience and machine learning. Often, one can categorize the various behaviors of a model by observing how its solutions evolve as certain parameters change. Inspired by the paper [9], I have developed a mathematical model based on the Hopfield model. In this model, I consider a fully connected recurrent neural network with a complex weight matrix. I will also provide numerical validation for my findings. This research is currently in its early stages.

2. Proposed Research:

2.1. Systems with Hysteresis:

As a continuation of my current work (refer to subsection 1.2), I will concentrate on symmetrically coupled systems exhibiting plastic hysteresis. I plan to pursue a similar pattern but will consider six identical oscillators connected in an octahedral arrangement, with an inertial disc at the center. I will also attempt to modify the electrical circuit to accommodate the Van der Pol oscillators.

2.2. Bifurcation and Chaos:

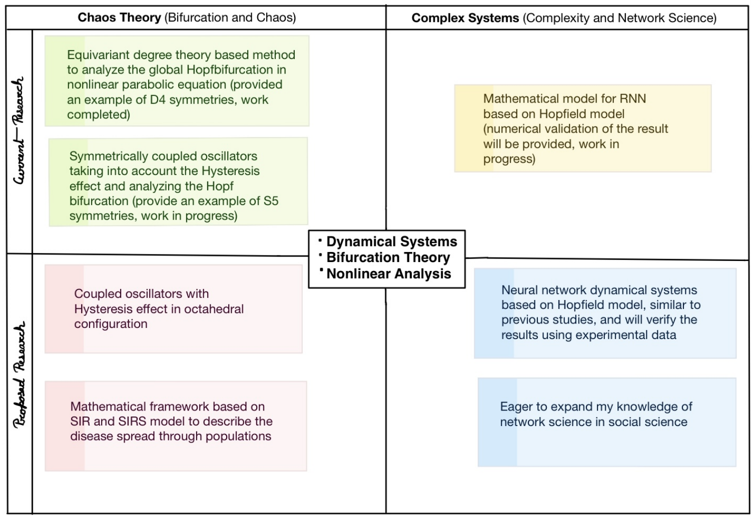
Bifurcation arises when a minor change in a system parameter results in a drastic change in the system's behavior. In contrast, chaos denotes the seemingly random behavior originating from complex nonlinear systems. Chaotic systems are primarily characterized by two features: unpredictable behavior and sensitivity to initial conditions. When a system undergoes a bifurcation, it often leads to a break in symmetries that can cause chaos to emerge. Chaos can also emerge spontaneously in certain dynamical systems due to their nonlinear nature. This implies that a system can display chaotic behavior even in the absence of a bifurcation point. In my subsequent work, I will concentrate on bifurcation and chaos, manage chaos via bifurcation, and engage with mathematical models and tools to explore bifurcations and chaos.

My work will focus on studying how infectious diseases affect populations, especially looking at the intersection between population dynamics and epidemiology. I will use the SIR and SIRS models, which are common mathematical frameworks used to describe how diseases spread through populations. In simple terms, these models help us understand how people move from being susceptible to a disease, to being infected, and finally to being recovered. By integrating theories of bifurcation and chaos with these models, I aim to explore the complex and unpredictable behaviors of disease spread in populations. This work will not only help to understand the mathematical aspects of disease modeling but also guide practical public health interventions to manage disease outbreaks more effectively.

2.3. Complex System and Data Science:

In the future, I plan to combine my data science expertise with dynamical systems to delve into a mathematical model inspired by the Hopfield network. This endeavor will be rooted in my understanding of how neurons, synapses, and neural circuits function, as well as how they change or adapt over time. I'll employ tools like phase space analysis to grasp how my models change and behave over time, ensuring their stability. A crucial aspect will be understanding how different disturbances influence these dynamics. Ultimately, I'll cross-check and confirm my findings with established experimental data. Moreover, I'm keen on expanding this approach to other network systems such as social network.

3. Summary: As a researcher, I enjoy exploring new directions and incorporating tools and ideas from other fields into bifurcation theory. During my postdoc, I plan to collaborate with researchers from different fields, namely biology, physics, and neuroscience, seeking inspiration for new challenges from a dynamical systems perspective. I hope to learn more and come up with new findings in future.



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