Ambrose Lo

Derivative Pricing

A Problem-Based Primer

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Derivative Pricing

A Problem-Based Primer

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Ambrose Lo

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Preface

Derivatives, which are financial instruments whose value depends on or is "derived" from (hence the name "derivatives") other more basic underlying variables, have become commonplace in financial markets all over the world. The proliferation of these relatively new financial innovations, options in particular, has underscored the ever-increasing importance of derivative literacy among a wide range of users that span students, practitioners, regulators, and researchers, all of whom are in need of a fundamental understanding of the mechanics, typical uses, and pricing theory of derivatives, though to different extents. Despite the diversity of such users, existing books on the subject have predominantly catered to only a very specific group of users and gone to two extremes. They either adopt a mostly descriptive approach to the intrinsically technical subject of derivatives, with occasional number crunching and slavish applications of pricing formulas taken without proof, or are preoccupied with sophisticated mathematical techniques from such areas as random processes and stochastic calculus, which can be inaccessible to students or practitioners lacking the necessary background and undesirably obscure the underlying conceptual ideas. Neither the "black box" approach nor the "purely mathematical" approach is of much pedagogical value.

Being an outgrowth of my lecture notes for a course entitled *ACTS:4380 Mathematics of Finance II* offered at the University of Iowa for advanced undergraduate and beginning graduate students in actuarial science, this book is a solid attempt to strike a balance between the two aforementioned methods to teach and learn derivatives, and to meet the needs of different types of readers. Adopting a mathematically rigorous yet widely accessible approach that will appeal to a wide variety of audience, the book is conceptually driven and strives to demystify the mechanics of typical derivatives and the fundamental mechanism of derivative pricing methodologies that should be part of the toolkit of every professional these days. This is accomplished by a combination of lucid explanations of the theory and assumptions behind common derivative pricing models, repeated emphasis on a small set of core ideas (e.g., no-arbitrage principle, replication, risk-neutral pricing), and a careful selection of fully worked-out illustrative examples and end-of-chapter problems. Readers of this book will leave with a firm understanding of "what" derivatives are, "how" and, more importantly, "why" derivatives are used and derivative pricing works.

Here is the skeleton of this book, divided into three parts.

- Part I (Chapters 1 to 3) lays the conceptual groundwork of the whole book by setting up the terminology of derivatives commonly encountered in the literature and introducing the definition, mechanics, typical use, and payoff structures of the two primary groups of derivatives, namely, forwards and options, which bestow upon their holders an obligation and a right to trade an underlying asset at a fixed price on a fixed date, respectively. Particular emphasis is placed on how and why a derivative works in a given scenario of interest. In due course, we also present the all-important *no-arbitrage assumption* and the method of *pricing by replication*. In loose terms, the former says that the prices of derivatives should be such that the market does not admit "free lunches," and the latter implements the former using the common-sense idea that if two derivatives possess the same payoff structure at expiration, they must enjoy the same initial price. Underlying the pricing and hedging of derivatives throughout this book, these two vehicles are applied in this part to determine the fair price of a forward, where "fair" is meant in the sense that the resulting price permits no free lunch opportunities.

- Whereas the pricing of forwards is model-independent in that it works for any asset price distribution, the pricing of options depends critically on the probabilistic behavior of the future asset price. In Part II (Chapters 4 to 8), which is the centerpiece of this book, we build upon the background material in Part I and tackle option pricing in two stages—first in the discrete-time binomial tree model (Chapter 4), which is simple, intuitive, and easy to implement, then in the technically more challenging continuous-time Black-Scholes model (Chapters 5 to 8). In this part, the no-arbitrage assumption and the method of replication continue to play a vital role in valuing options and lead to the celebrated *risk-neutral pricing formula*, which asserts that the price of a (European) derivative can be computed as its expected payoff at expiration in a risk-neutral sense, discounted at the risk-free interest rate. The implementation and far-reaching implications of the method of risk-neutral valuation for the pricing and hedging of derivatives are explored in Chapters 6 to 8.
- Finally, we end in Part III (Chapter 9) with a description of some general properties satisfied by option prices when no asset price model is prescribed. Even in this model-free framework setting, there is a rich theory describing the no-arbitrage properties universally satisfied by option prices. Although this part can be read prior to studying Part II, you will find that what you learn from Part II, especially the notion of an exchange option in Chapter 8, will provide you with surprisingly useful insights into the connections between different options.

It deserves mention that this book, as a primer, is indisputably not encyclopedic in scope. The choice of topics is geared towards the derivatives portion (Topics 6 to 10) of the Society of Actuaries' *Investment and Financial Markets* (IFM) Exam,[i] which is typically taken by advanced undergraduate students in actuarial science and allied disciplines. The theory of random processes and stochastic calculus, while conducive to understanding the pricing theory of derivatives in full but often an insurmountable barrier to first-time learners, is not covered in the book, neither are credit and interest rate derivatives (which, without doubt, are important in practice). By concentrating on the most essential conceptual ideas, we realize the huge "payoff" of being able to disseminate these core ideas to readers with minimal mathematical background; it is understandable that individuals interested in using and pricing derivatives nowadays come from a wide variety of background. To be precise, readers are only assumed to have taken a calculus-based probability and statistics course at the level of Hogg, Tanis and Zimmerman (2014) or Hogg, McKean and Craig (2013), where the basic notions of random variables, expectations, variances, are taught, and be able to perform simple discounted cash flow calculations as covered in a theory of interest or corporate finance course. With these modest prerequisites, this book is self-contained, with the necessary mathematical ideas presented progressively as the book unfolds. For readers interested in more advanced aspects of the use and pricing of derivatives, this book will provide them with a springboard for performing further studies in this burgeoning field.

It is widely acknowledged that the best way to learn a subject deeply is to test your understanding with a number of meaningful exercises. With this in mind, this primer lives up to its name and features an abundance of illustrative in-text examples and end-of-chapter problems (to be precise, 177 examples and 209 problems) on different aspects of derivatives. These problems are of a diverse nature and varying levels of difficulty (harder ones

[i] Section 4.5, Section 7.3 (the portion on the Black-Scholes equation), Subsection 8.1.2, and Section 9.1 are beyond the scope of the Exam IFM syllabus.

are labeled as [**HARDER!**]); while many emphasize calculating quantities such as payoffs, prices, profits, a primitive skill that most students in a derivatives course need to acquire, at least for exam purposes (in this respect, this book is an ideal exam preparation aid for students who will write Exam IFM), some concern more theoretical aspects of using and pricing derivatives, and consist of true-or-false items or derivations of formulas. All of these problems can be worked out in a pen-and-paper environment with the aid of a scientific calculator and a standard normal distribution function calculator (an example is `https://www.prometric.com/en-us/clients/soa/pages/mfe3f_calculator.aspx`, which is designed for students who will take Exam IFM.). If you do not have Internet access, you may use the less precise standard normal distribution table provided in Appendix A of this book. Readers who attempt these examples and problems seriously will benefit from a much more solid understanding of the relevant topics. To help you check your answers, full solutions to all odd-numbered end-of-chapter problems are provided in Appendix B. A solutions manual with solutions to all problems is available to qualified instructors.

It would be remiss of me not to thank my past ACTS:4380 students for personally class testing earlier versions of the book manuscript and many of the end-of-chapter problems, as well as my esteemed colleague, Professor Elias S.W. Shiu, at the University of Iowa, for sharing with me his old ACTS:4380 notes and questions, from which some of the examples and problems in this book were motivated. I am also grateful to the Society of Actuaries and Casualty Actuarial Society for kindly allowing me to reproduce their past and sample exam questions, of which they own the sole copyright, and which have proved instrumental in illustrating ideas in derivative pricing. Doctoral student Zhaofeng Tang at the University of Iowa merits a special mention for his professional assistance with some of the figures in this book and for meticulously proofreading part of the book manuscript. All errors that remain, typographical or otherwise, are solely mine. To help improve the content of the book, I would deeply appreciate it if you could bring any potential errors you have identified to my attention; my email address is `ambrose-lo@uiowa.edu`. For readers' benefits, an erratum and updates to the book will be maintained on my web page at `https://sites.google.com/site/ambroseloyp/publications/derivative-pricing`.

It is my sincere hope that this book will not only introduce you to the fascinating world of derivatives, but also to instill in you a little of the enthusiasm I have for this subject since my undergraduate studies. Welcome and may the fun begin!

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Iowa City, IA
May 2018

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Symbols

Symbol Description

S or $S(0)$	time-0 price of an underlying asset
$S(t)$	time-t price of an underlying asset
$S^a(0)$	time-0 ask price of an underlying asset
$S^b(0)$	time-0 bid price of an underlying asset
$X(t)$	time-t exchange rate
K	strike price of an option
K_1	strike price of a gap option
K_2	payment trigger of a gap option
r	continuously compounded risk-free interest rate (per annum)
r^b	continuously compounded borrowing rate (per annum)
r^l	continuously compounded lending rate (per annum)
i	effective annual interest rate (per annum)
$\mathrm{PV}_{t,T}$	time-t (present) value of cash flows between time t and time$T$
$\mathrm{FV}_{t,T}$	time-T (future) value of cash flows between time t and time$T$
T	maturity time of a generic derivative
T_f	maturity time of a futures contract
T_1	maturity time of a compound option
T_2	maturity time of the underlying option of a compound option
$F^P_{t,T}$	time-t price of a prepaid forward maturing at time T
$F_{t,T}$	time-t price of a forward maturing at time T
$F^{\mathrm{obs}}_{t,T}$	time-t observed price of a forward maturing at time T
$F^{\mathrm{fair}}_{t,T}$	time-t fair price of a forward maturing at time T
V	time-0 price of a generic derivative
$V^{\max}$	time-0 price of a maximum contingent claim
$V^{\min}$	time-0 price of a minimum contingent claim
C	time-0 price of a generic call
C^E	time-0 price of a generic European call
C^A	time-0 price of a generic American call
$C(K,T)$	time-0 price of a K-strike T-year call
$C^{\mathrm{gap}}(K_1,K_2)$	time-0 price of a K_1- strike K_2-trigger generic gap call
$C(S(t),K,t,T)$	time-t price of a K-strike call maturing at time T when the time-t stock price is $S(t)$
P	time-0 price of a generic put
P^E	time-0 price of a generic European put
P^A	time-0 price of a generic American put
$P^{\mathrm{gap}}(K_1,K_2)$	time-0 price of a K_1-strike K_2-trigger generic gap put
$P(S(t),K,t,T)$	time-t price of a K-strike put maturing at time T when the time-t stock price is $S(t)$
BS	Black-Scholes pricing function
σ	volatility of an underlying asset
σ_{option}	volatility of an option
Δ	delta of a generic derivative

Δ_C	delta of a generic call
Δ_P	delta of a generic put
Γ	gamma of a generic derivative
Γ_C	gamma of a generic call
Γ_P	gamma of a generic put
Ω	elasticity of a generic derivative
Ω_C	elasticity of a generic call
Ω_P	elasticity of a generic put
$\mathbb{E}[X]$	expectation of random variable X
$\mathrm{Var}(X)$	variance of random variable X
$\mathrm{Cov}(X, Y)$	covariance between random variables X and Y
$\rho(X, Y)$	correlation coefficient between random variables X and Y
$N(\cdot)$	distribution function of the standard normal distribution
$N'(\cdot)$	density function of the standard normal distribution
$:=$	defined as
LHS	left-hand side
RHS	right-hand side
x_+	positive part of real number x
1_A	indicator function of event A
ZCB	zero-coupon bond

Part I

Conceptual Foundation on Derivatives

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Parity arbitrage.

Put-call parity is a mathematical equation that relates the *fair* prices of European call and put options having the same strike price, maturity date and underlying asset. If it is violated, then it is possible to design an arbitrage strategy, called a *parity arbitrage* in this context, to earn risk-free profits. The next example illustrates how this can be done.

Example 3.2.9. (Parity arbitrage) You are given:

(i) The price of a nondividend-paying stock is \$31.

(ii) The continuously compounded risk-free interest rate is 10%.

(iii) The price of a 3-month 30-strike European call option is \$3.

(iv) The price of a 3-month 30-strike European put option is \$2.25. Construct a trading strategy that will generate risk-free arbitrage profits at time 0.

Solution. It is easy to see that the call and put prices violate put-call parity:

- LHS:
$$C(30, 0.25) - P(30, 0.25) = 3 - 2.25 = 0.75,$$
- RHS:
$$F_{0,T}^P(S) - F_{0,T}^P(K) = S(0) - Ke^{-rT} = 31 - 30e^{-(0.1)(0.25)} = 1.7407.$$

To exploit arbitrage profits, we "buy the LHS" ("low") and "sell the RHS" ("high") by engaging in the following transactions:

Transaction	Cash Flows	
	Time 0	**Time 0.25**
Buy a 3-month 30-strike call	-3	$(S(0.25) - 30)_+$
Sell a 3-month 30-strike put	$+2.25$	$(30 - S(0.25))_+$
Short sell one share of the stock	$+31$	$-S(0.25)$
Lend $30e^{-(0.1)(0.25)} = 29.2593$	-29.2593	30
Total	0.9907	0

□

3.3 Spreads and Collars

This section and the next continue the spirit of Sections 3.1 and 3.2, and present several common strategies involving two or more options of possibly different types and strike prices. We shall examine the composition and payoff structure of and the motivation underlying each option strategy.

3.3.1 Spreads

A *spread* is a position typically consisting of only call or only put options (with the exception of the box spread), some purchased and some written, all having the same underlying asset, time to maturity, exercise style, but different strike prices. We present four most common spreads in this subsection.

Spread #1: Bull spreads.

Motive.

We have learnt several strategies to exploit the belief that an asset will appreciate, including: (1) A long forward; (2) A long call; (3) A short put. A *bull spread* is a strategy which enables you to take advantage of a rise in the asset price, but at a lower financing cost than a long call. As a trade-off, you are forced to forgo some of the upside payoff when the terminal asset price indeed rises substantially.

Composition.

A bull spread can be set up by buying a low-strike call and selling an otherwise identical high-strike call. We denote the two strike prices by K_1 and K_2, where $K_1 < K_2$, and refer to the resulting position as a K_1-K_2 *(call) bull spread*. Its payoff is

$$\underbrace{(S(T)-K_1)_+}_{\text{Long } K_1\text{-strike call}} \quad \underbrace{-(S(T)-K_2)_+}_{\text{Short } K_2\text{-strike call}} = \begin{cases} 0, & \text{if } S(T) < K_1, \\ S(T)-K_1, & \text{if } K_1 \le S(T) < K_2, \\ K_2 - K_1, & \text{if } K_2 \le S(T), \end{cases}$$

and is plotted in Figure 3.3.1. Here are some tips on sketching the payoff diagram more easily:

> In drawing the payoff diagram of the bull spread, it will be more convenient for you to work *from left to right* and adjust the slope of the payoff function as we pass through each strike price. This is because the payoff function of a call option is always zero to the left of the strike price. Therefore, the payoff function of the call bull spread must be zero to the left of the lower strike price K_1, then direct upward with a slope of $+1$ to the right of K_1 owing to the long K_1-strike call. Beyond K_2, the slope of the payoff function is adjusted downward by 1 as a result of the short K_2-strike call, becoming zero. This means that the payoff function levels off thereafter.

Coupling the long K_1-strike call with a short K_2-strike call has two effects. On the positive side, a K_1-K_2 long bull spread requires only $C(K_1) - C(K_2)$ as the initial investment whereas a K_1-strike long call costs $C(K_1)$. The K_2-strike call premium serves as a source of extra income that counteracts the original investment of $C(K_1)$. On the negative side, the maximum payoff of the bull spread is capped at $K_2 - K_1$, however high the terminal asset price is. When $S(T) \geq K_2$, every extra unit of payoff we gain from the long K_1-strike call is exactly offset by the additional loss on the short K_2-strike call, resulting in a flat payoff function.

Mnemonics.

Observe that:

(a) The payoff is a non-decreasing function of the terminal stock price. This explains why the strategy is called "bull," or long with respect to the underlying asset.

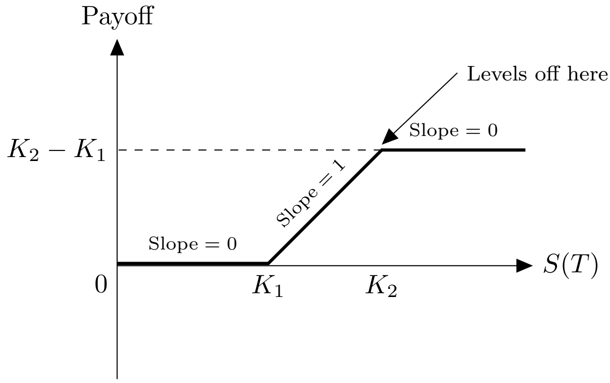


FIGURE 3.3.1
Payoff diagram of a call K_1-K_2 bull spread.

(b) The payoff increases only over a "spread" (i.e., the interval $[K_1, K_2]$), outside which the payoff becomes level.

These two observations may help you remember why a "bull spread" is called so.

Construction by put options.

Interestingly, a K_1-K_2 bull spread can also be created by buying a K_1-strike put and selling an otherwise identical K_2-strike put. Example 3.3.1 below explores the similarities and differences between a *call* bull spread and a *put* bull spread.

Example 3.3.1. (Call bull spread vs put bull spread) Student A constructs a K_1-K_2 bull spread using call options, while Student B constructs a K_1-K_2 bull spread using put options. All options are European with the same underlying asset and time to expiration.

Determine which of the following statements about these two bull spreads is/are correct.

I. Students A and B both have a short position with respect to the underlying asset.

II. Students A and B have the same payoff at expiration.

III. Students A and B have the same profit.

(A) I only

(B) II only

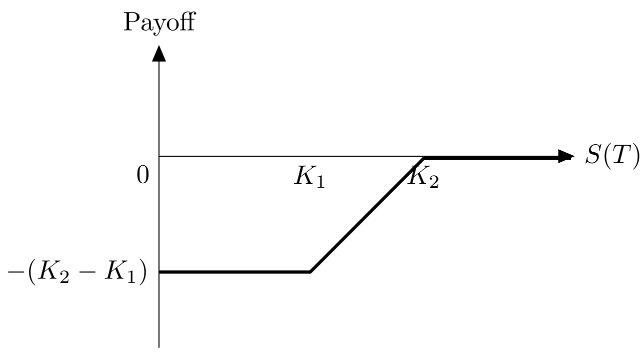
(C) III only

(D) I and III only

(E) The correct answer is not given by (A), (B), (C), or (D)

Solution. Central to the comparison between the call bull spread and the put bull spread is the payoff diagram of the latter. Because put options are used and the payoff of each is zero to the right of the strike price, we find it convenient to work from right to left. To the right of K_2, none of the put options pay off, so the payoff of the put bull spread is zero. To the left of K_2, the K_2-strike put starts to pay off. Because we are short the K_2-strike put, the payoff function traverses *downward* until it hits the lower strike price K_1 and becomes flat as a result of the long K_1-strike put.

The payoff diagram of the put bull spread is shown below.



Given the payoff diagram of the put bull spread, we are now ready to determine the truth value of each statement.

I. False. Both bull spreads have a higher payoff as the terminal price of the underlying asset increases and thus are long with respect to the underlying asset.

II. False. The payoff of the call bull spread is always higher than that of the put bull spread by $K_2 - K_1$. The former is always non-negative while the latter is always non-positive.

III. True. Because the two payoff functions are parallel, they must also share the same profit function because of the "parallel payoffs, identical profit" observation in Subsection 3.1.4.

(Answer: (C)) □

Example 3.3.2. (SOA Exam IFM Introductory Derivatives Sample Question 50: Break-even price for a put bull spread) An investor bought a 70-strike European put option on an index with six months to expiration. The premium for this option was 1.

The investor also wrote an 80-strike European put option on the same index with six months to expiration. The premium for this option was 8.

The six-month interest rate is 0%.

Calculate the index price at expiration that will allow the investor to break even.

(A) 63

(B) 73

(C) 77

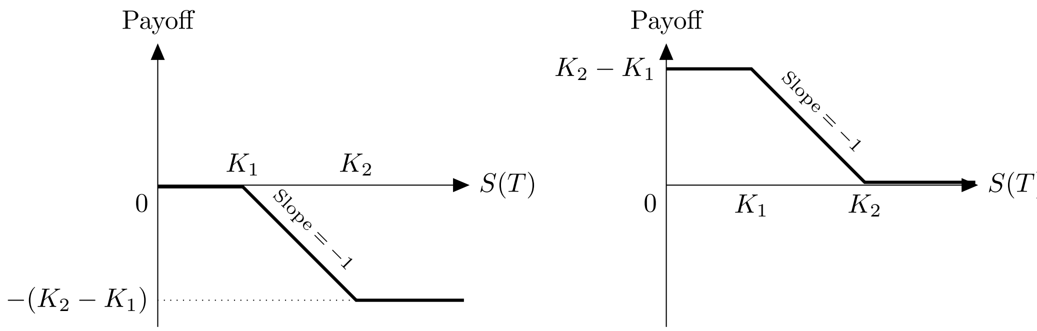


FIGURE 3.3.2
Payoff diagram of K_1-K_2 bear spreads constructed by calls (left) and by puts (right).

(D) 80

(E) 87

Solution. Buying a low-strike put and selling a high-strike put, the investor has set up a put bull spread. The initial investment is $1 - 8 = -7$, or the investor receives 7 at the outset. Because the interest rate is zero, to break even the investor has to lose 7 (i.e., the payoff has to be -7) from the put bull spread. From the payoff diagram, this is attained when $S(0.5) = 80 - 7 = \boxed{73}$. **(Answer: (B))** □

Spread #2: Bear Spreads.

As its name suggests, a K_1-K_2 *bear spread* is the "bearish" counterpart of a bull spread[ii] and can be constructed by selling a low-strike option and buying a high-strike option of the same type. Like a bull spread, there are two possible modes of construction:

(a) Selling a K_1-strike call and buying an otherwise identical K_2-strike call.

(b) Selling a K_1-strike put and buying an otherwise identical K_2-strike put.

The payoff diagrams are given in Figure 3.3.2. As an exercise, try to sketch these two diagrams using the slope adjustment tips given on page 75.

Example 3.3.3. (Investment of a put bear spread given call prices) You are given:

(i) The current price of a stock is 70.

(ii) The continuously compounded risk-free interest rate is 5%.

(iii) The price of a 70-strike 1-year European call option is 5.

(iv) The price of a 75-strike 1-year European call option is 3.

[ii] Note that a bull spread from the perspective of the purchaser is a bear spread from the perspective of the writer.

Calculate the amount of investment required to create a 1-year 70-75 European put bear spread.

Ambrose's comments:

This question nicely combines put-call parity discussed in Section 3.2 with bear spread.

Solution. The 70-75 put bear spread is set up by a short 70-strike put and a long 75-strike put. The investment required is $P(75) - P(70)$. By put-call parity, we have

$$\begin{cases} C(70) - P(70) = F_{0,1}^P - \mathrm{PV}_{0,1}(70) \\ C(75) - P(75) = F_{0,1}^P - \mathrm{PV}_{0,1}(75) \end{cases}.$$

Subtracting the second equation from the first yields

$$[P(75) - P(70)] + \underbrace{[C(70) - C(75)]}_{5-3} = \mathrm{PV}_{0,1}(5) = 5\mathrm{e}^{-0.05(1)},$$

which implies $P(75) - P(70) = \boxed{2.7561}$. □

Spread #3: Box Spreads.

A *box spread* is an option strategy whose payoff and profit do not vary with the terminal price of the underlying asset. It is a synthetic risk-free investment created using options.

Construction 1: By synthetic forwards.

The standard way to construct a box spread is to combine a synthetic long forward at one forward price K_1 and a synthetic short forward at another forward price K_2. This effectively means that at expiration you pay K_1 and receive K_2.

To create a box spread this way, you:

Buy a K_1-strike call, sell a K_1-strike put *(synthetic long forward)*, and

sell a K_2-strike call, buy a K_2-strike put *(synthetic short forward)*.

The initial investment is

$$C(K_1) - P(K_1) - C(K_2) + P(K_2),$$

which, by put-call parity, equals $\mathrm{PV}_{0,T}(K_2 - K_1)$. The overall payoff at expiration is

$$\text{Payoff} = \underbrace{(S(T) - K_1)}_{\text{Long } K_1\text{-strike synthetic forward}} + \underbrace{(K_2 - S(T))}_{\text{Short } K_2\text{-strike synthetic forward}} = K_2 - K_1,$$

which is a constant. Therefore, a box spread has no asset price risk and is financially equivalent to lending when $K_1 < K_2$ (in which case the initial investment and the payoff are both fixed and positive) and borrowing when $K_1 > K_2$ (in which case the initial investment and the payoff are both fixed and negative).

Example 3.3.4. (SOA Exam IFM Introductory Derivatives Sample Question 17: Which one is a short position?) The current price for a stock index is 1,000. The following premiums exist for various options to buy or sell the stock index six months from now:

Strike Price	Call Premium	Put Premium
950	120.41	51.78
1,000	93.81	74.20
1,050	71.80	101.21

Strategy I is to buy the 1,050-strike call and to sell the 950-strike call.
Strategy II is to buy the 1,050-strike put and to sell the 950-strike put.
Strategy III is to buy the 950-strike call, sell the 1,000-strike call, sell the 950-strike put, and buy the 1,000-strike put.
Assume that the price of the stock index in 6 months will be between 950 and 1,050.
Determine which, if any, of the three strategies will have greater payoffs in six months for lower prices of the stock index than for relatively higher prices.

(A) None

(B) I and II only

(C) I and III only

(D) II and III only

(E) The correct answer is not given by (A), (B), (C), or (D)

Solution. We need to judge which strategy is short with respect to the underlying asset.

- Strategy I is a call bear spread and bear spreads perform better when the price of the underlying asset is relatively low
- Strategy II is also a bear spread – it is a put bear spread.
- Strategy III is a box spread, which has no price risk. The payoff is constant at $1,000 - 950 = 50$, regardless of the price of the underlying asset in 6 months.

Only Strategies I and II are short. **(Answer: (B))** □

Remark. The option premiums given in the question seem completely redundant...

Construction 2: By bull and bear spreads.

The positions taken in the K_1-strike and K_2-strike options can be summarized in the following table or "box" (this is the reason why "box" spreads are called so):

Strike price	Call	Put
K_1	Long	Short
K_2	Short	Long

Reading the table horizontally gives Construction 1. Traversing the table vertically and assuming that $K_1 < K_2$, we obtain a second way of constructing a box spread:

A long K_1-K_2 call bull spread and a long K_1-K_2 put bear spread.

Example 3.3.5. (SOA Exam IFM Introductory Derivatives Sample Question 55: To create a box spread) Box spreads are used to guarantee a fixed cash flow in the future. Thus, they are purely a means of borrowing or lending money, and have no stock price risk.

Consider a box spread based on two distinct strike prices (K, L) that is used to lend money, so that there is a positive cost to this transaction up front, but a guaranteed positive payoff at expiration.

Determine which of the following sets of transactions is equivalent to this type of box spread.

(A) A long position in a (K, L) bull spread using calls and a long position in a (K, L) bear spread using puts.

(B) A long position in a (K, L) bull spread using calls and a short position in a (K, L) bear spread using puts.

(C) A long position in a (K, L) bull spread using calls and a long position in a (K, L) bull spread using puts.

(D) A short position in a (K, L) bull spread using calls and a short position in a (K, L) bear spread using puts.

(E) A short position in a (K, L) bull spread using calls and a short position in a (K, L) bull spread using puts.

Solution. To result in a positive initial cost and a guaranteed positive payoff at expiration, we should take a long position in a synthetic forward (long call and short put) and a short position in a synthetic forward at a higher forward price (short call and long put). If $K < L$, this requires that a K-L call bull spread be long and a K-L put bear spread be long. **(Answer: (A))** □

Application of box spreads: Arbitrage mispricing.

If the end result of a box spread is pure borrowing or lending, why do people take all the fuss to use a box spread in the first place? Buying or selling so many options may incur huge transaction costs. Because a box spread is a replication of a risk-free investment, an important practical application of box spread is to exploit the mispricing of options arising from the discrepancy between the interest rate implicit in a box spread and the interest rate observed in the market. The following example illustrates how this can be done given a set of real data.

Example 3.3.6. (Arbitrage by box spread) The following table shows 1-year European call and put option premiums at two strike prices:

Strike Price	Call Premium	Put Premium
40	2.58	1.79
45	1.17	5.08

The continuously compounded risk-free interest rate is 6%.

Describe actions you could take to construct an arbitrage strategy that results in profits at the end of one year using the above options and/or zero-coupon bonds only.

Solution. We begin by determining the risk-free interest rate implied by a box spread and see if it is higher or lower than the interest rate in the market. The initial investment of the box spread is $2.58 + 5.08 - 1.17 - 1.79 = 4.7$ while its 1-year payoff is $45 - 40 = 5$. The implicit interest rate satisfies $e^r = 5/4.7$, so $r = 6.19\%$, which is higher than the observed r.

To construct an arbitrage strategy, we buy the more attractive position, which is the box spread, and sell the less attractive position, which is the risk-free investment in the market. The following table summarizes the positions to be taken in different derivatives:

Derivative	**Position**
40-strike call	Buy
40-strike put	Sell
45-strike call	Sell
45-strike put	Buy
1-year zero-coupon bond	Sell one with face value $\$4.7e^{0.06}$ (equivalently, borrow \$4.7)

By taking this set of transactions, the initial investment is zero, and the 1-year payoff is $45 - 40 - 4.7e^{0.06} = \boxed{0.009368}$. □

Spread #4: Ratio Spreads.

Composition.

A *ratio spread* is constructed by buying and selling unequal numbers of options of the same type (call or put) and time to maturity but at different strikes, e.g., buying m calls at one strike and selling n calls at another strike with $n \neq m$.

Motives.

The reason for trading a ratio spread depends on which type of options is used, and the relative values of m and n. By varying the type and number of options, a wide variety of payoff structures can be accommodated, making it difficult to give a unified motive behind a ratio spread. Also, it is possible to adjust the ratio $m : n$ so that the initial investment of a ratio spread is zero.

Example 3.3.7. (SOA Exam IFM Introductory Derivatives Sample Question 39: How to create a ratio spread) Determine which of the following strategies creates a ratio spread, assuming all options are European.

(A) Buy a one-year call, and sell a three-year call with the same strike price.

(B) Buy a one-year call, and sell a three-year call with a different strike price.

(C) Buy a one-year call, and buy three one-year calls with a different strike price.

(D) Buy a one-year call, and sell three one-year puts with a different strike price.

(E) Buy a one-year call, and sell three one-year calls with a different strike price.

Solution. A ratio spread involves buying and selling options of the same type (call or put), same time to maturity, but different strike prices. Only Answer (E) fulfills these requirements. **(Answer: (E))** □

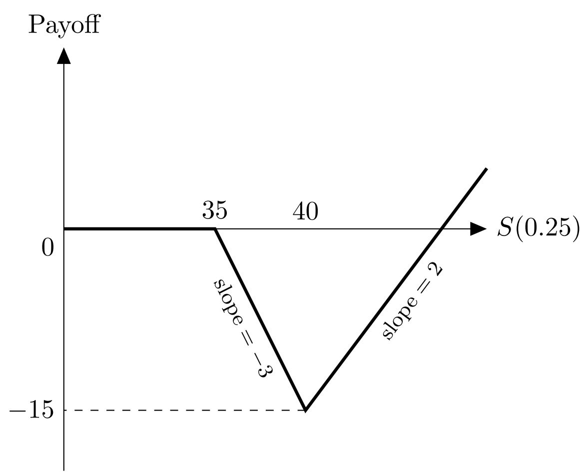
Example 3.3.8. (SOA Exam IFM Introductory Derivatives Sample Question 15: Maximum profit and loss of a ratio spread) The current price of a nondividend-paying stock is 40 and the continuously compounded risk-free interest rate is 8%. You enter into a short position on 3 call options, each with 3 months to maturity, a strike price of 35, and an option premium of 6.13. Simultaneously, you enter into a long position on 5 call options, each with 3 months to maturity, a strike price of 40, and an option premium of 2.78.

All 8 options are held until maturity.

Calculate the maximum possible profit and the maximum possible loss for the entire option portfolio.

	Maximum Profit	Maximum Loss
(A)	3.42	4.58
(B)	4.58	10.42
(C)	Unlimited	10.42
(D)	4.58	Unlimited
(E)	Unlimited	Unlimited

Solution. The initial cost to establish this position is $5 \times 2.78 - 3 \times 6.13 = -4.49$. Thus, you receive 4.49 upfront. This grows to $4.49e^{0.08(0.25)} = 4.58$ after 3 months. Working from left to right (because we are dealing with calls) and adjusting the slope of the payoff function as we pass through each strike price,, we can sketch the payoff diagram of the ratio spread as follows:



The maximum payoff and minimum payoff are unlimited and -15 (attained at $S(0.25) = 40$), respectively. It follows that the maximum profit and minimum profit are $\boxed{\text{unlimited}}$ and $-15 + 4.58 = \boxed{-10.42}$, respectively. **(Answer: (C))** □

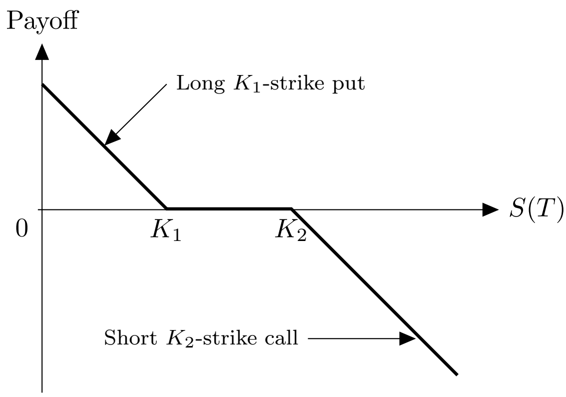


FIGURE 3.3.3
Payoff diagram of a long K_1-K_2 collar.

3.3.2 Collars

Composition.

A *collar* comprises a *long put* with strike price K_1 and a *short call* with a higher strike price K_2, where $K_1 \leq K_2$, and both options have the same underlying asset and maturity date. The difference between the strike prices of the call and put options, $K_2 - K_1$, is termed the *collar width*. If the positions in the put and call are reversed (i.e., a short K_1-strike put plus a long K_2-strike call), then a short collar is created.

Payoff.

The payoff of a long K_1-K_2 collar is given by

$$\underbrace{(K_1 - S(T))_+}_{\text{Long } K_1\text{-strike put}} \underbrace{-(S(T) - K_2)_+}_{\text{Short } K_2\text{-strike call}} = \begin{cases} K_1 - S(T), & \text{if } S(T) < K_1, \\ 0, & \text{if } K_1 \leq S(T) < K_2, \\ K_2 - S(T), & \text{if } K_2 \leq S(T). \end{cases}$$

Depicted in Figure 3.3.3, the payoff diagram of a collar shows that it is short with respect to the underlying asset. A short synthetic forward is a special case of a long collar with forward price $K_1 = K_2$.

To construct a collar, we need to pay $P(K_1) - C(K_2)$ at time 0. In general, the financing cost can be positive or negative, depending on the relative locations of K_1 and K_2, and the model that is used to price the call and put (see the "Zero-cost collars" paragraph on page 89).

Collared stock.

In practice, collars are often used to effect an insurance strategy. Suppose that we are holding a stock and would like to limit the range of possible payoffs to a specific range, say between K_1 and K_2, where $K_1 < K_2$. In this case, we can couple the long position in the underlying stock with a long collar. The resulting portfolio, called a *collared stock*, has a total payoff of

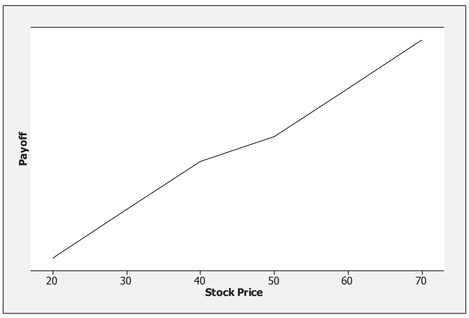
$$\text{Payoff} = S(T) + \underbrace{(K_1 - S(T))_+ - (S(T) - K_2)_+}_{\text{Long collar}} = \begin{cases} K_1, & \text{if } S(T) < K_1, \\ S(T), & \text{if } K_1 \leq S(T) < K_2, \\ K_2, & \text{if } K_2 \leq S(T), \end{cases} \quad (3.3.1)$$

which always lies between K_1 and K_2. Compared to a mere long stock position, a collared stock insures us against downside risk because of the long put position. Meanwhile, the sale of a call helps finance the purchase of the put, but undesirably rules out the possibility of a huge gain when $S(T)$ is large enough.

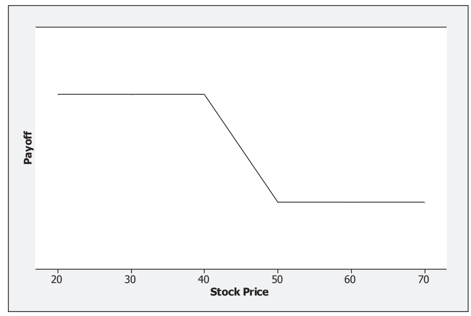
Example 3.3.9. (SOA Exam IFM Introductory Derivatives Sample Question 59: Payoff diagram of a long collared stock) An investor has a long position in a nondividend-paying stock, and, additionally, has a long collar on this stock consisting of a 40-strike put and 50-strike call.

Determine which of these graphs represents the payoff diagram for the overall position at the time of expiration of the options.

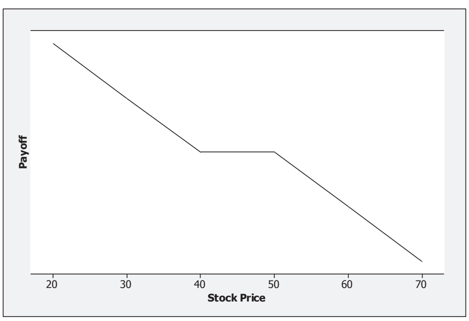
(A)



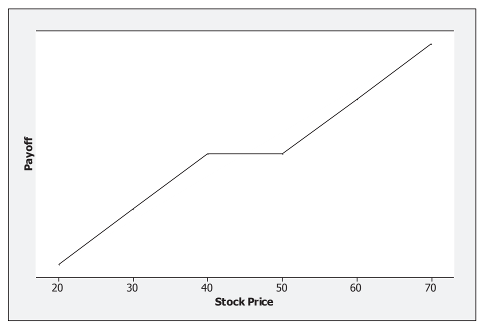
(B)



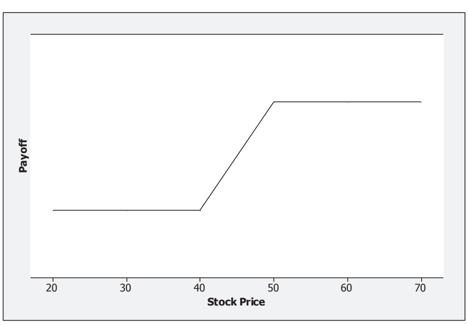
(C)



(D)



(E)



Solution. We determine the shape of the collared stock by looking at the slope of the payoff function over different stock price ranges:

Stock price range	Slope $[0, 40)$	$[40, 50)$	$[50, \infty)$
Long stock	$+1$	$+1$	$+1$
Long 40-strike put	-1	0	0
Short 50-strike call	0	0	-1
Total	0	$+1$	0

Only (E) is consistent with the shape of the payoff function. **(Answer: (E))** □

Remark. The payoff of a long collared stock has the same shape and therefore the same profit as that of a long bull spread.

Example 3.3.10. (Stock vs collared stock) Determine which of the following statements about a long stock and a long K_1-K_2 collared stock is/are always correct.

I. Both are long with respect to the underlying stock.

II. The long collared stock requires a lower amount of initial investment than the long stock.

III. The long collared stock has a higher payoff than the long stock when the stock price at expiration is sufficiently high.

(A) I only

(B) II only

(C) III only

(D) I and III only

(E) The correct answer is not given by (A), (B), (C), or (D)

Solution. I. True. Both benefit from increases in the stock price.

II. False. The initial investment of a long stock is $S(0)$, while that of a collared stock is $S(0)+P(K_1)-C(K_2)$. In general, $P(K_1)-C(K_2)$ can be positive, zero, or negative, so we are not sure which costs more in terms of initial investment.

III. False. When $S(T) \geq K_2$, the payoff of the collared stock is constant at K_2, which is less than the payoff of the long stock being $S(T)$.
(Answer: (A)) □

Example 3.3.11. (SOA Exam IFM Introductory Derivatives Sample Question 60: A collared stock in a business context – I) Farmer Brown grows wheat, and will be selling his crop in 6 months. The current price of wheat is 8.50 per bushel. To reduce the risk of fluctuation in price, Brown wants to use derivatives with a 6-month expiration date to sell wheat between 8.60 and 8.80 per bushel. Brown also wants to minimize the cost of using derivatives.

The continuously compounded risk-free interest rate is 2%.
Which of the following strategies fulfills Farmer Brown's objectives?

(A) Short a forward contract

(B) Long a call with strike 8.70 and short a put with strike 8.70

(C) Long a call with strike 8.80 and short a put with strike 8.60

(D) Long a put with strike 8.60

(E) Long a put with strike 8.60 and short a call with strike 8.80

Solution. In the absence of any hedging strategy, Brown's payoff per bushel of wheat is $S(0.5)$, the 6-month price of wheat. By (3.3.1), coupling his position with a long 8.6-8.8 collar (i.e., a long 8.6-strike put and a short 8.8-strike call) allows Brown to lock in a selling price between 8.6 and 8.8. **(Answer: (E))** □

Remark. The current price of wheat and the continuously compounded risk-free interest rate play no role in this question.

Example 3.3.12. (SOA Exam IFM Introductory Derivatives Sample Question 3: A collared stock in a business context – II) Happy Jalapenos, LLC has an exclusive contract to supply jalapeno peppers to the organizers of the annual jalapeno eating contest. The contract states that the contest organizers will take delivery of 10,000 jalapenos in one year at the market price. It will cost Happy Jalapenos 1,000 to provide 10,000 jalapenos and today's market price is 0.12 for one jalapeno. The continuously compounded risk-free interest rate is 6%.

Happy Jalapenos has decided to hedge as follows:

Buy 10,000 0.12-strike put options for 84.30 and sell 10,000 0.14-stike call options for 74.80. Both options are one-year European.

Happy Jalapenos believes the market price in one year will be somewhere between 0.10 and 0.15 per jalapeno.

Determine which of the following intervals represents the range of possible profit one year from now for Happy Jalapenos.

(A) −200 to 100

(B) −110 to 190

(C) −100 to 200

(D) 190 to 390

(E) 200 to 400

Solution. The future value of the long collar investment is $(84.3 - 74.8)e^{0.06} = 10.09$.

- The minimum profit is $\underbrace{0.12}_{\because\text{Floor}} \times 10{,}000 - \underbrace{1{,}000}_{\text{Cost}} - \underbrace{10.09}_{\text{FV of investment}} = \boxed{189.91}$
- The maximum profit is $\underbrace{0.14}_{\because\text{Covered call}} \times 10{,}000 - 1{,}000 - 10.09 = \boxed{389.91}$

(Answer: (D)) □

Example 3.3.13. (SOA Exam IFM Introductory Derivatives Sample Question 43: Short collared stock) You are given:

(i) An investor short-sells a nondividend-paying stock that has a current price of 44 per share.

(ii) This investor also writes a collar on this stock consisting of a 40-strike European put option and a 50-strike European call option. Both options expire in one year.

(iii) The prices of the options on this stock are:

Strike Price	Call option	Put option
40	8.42	2.47
50	3.86	7.42

(iv) The continuously compounded risk-free interest rate is 5%.

(v) Assume there are no transaction costs.

Calculate the maximum profit for the overall position at expiration.

(A) 2.61

(B) 3.37

(C) 4.79

(D) 5.21

(E) 7.39

Solution 1 (Standard). The written collar consists of a short 40-strike put and a long 50-strike call. The initial cost of the short collared stock is $-S(0) - P(40) + C(50) = -44 - 2.47 + 3.86 = -42.61$, which grows to $-42.61\mathrm{e}^{0.05} = -44.7947$ in one year.

Because the short collared stock is a short position, its maximum profit is attained at the smallest 1-year stock price which is $S(1) = 0$. With the maximum payoff being $-0-(40-0)_+ + (0-50)_+ = -40$, the maximum profit is $-40-(-44.7947) = \boxed{4.7947}$.
(Answer: (C)) □

Solution 2. In terms of profit, the short collared stock is equivalent to a 40-50 put bear spread. Hence the maximum profit is achieved when $S(1) \leq 40$ and equals

$$10 - (\underbrace{7.42}_{\text{Buy 50-strike put}} \quad \underbrace{-2.47}_{\text{Sell 40-strike put}}) \times \mathrm{e}^{0.05} = \boxed{4.7962}. \quad \textbf{(Answer: (C))}$$

□

Remark. (i) Solution 2 is shorter and does not require the knowledge of the values of $S(0)$ and the call premiums.

(ii) The current stock price and the continuously compounded interest rate r can be deduced from the table in (iii) via two applications of put-call parity.

Zero-cost collars.

A *zero-cost collar* is a special collar for which the prices of the purchased put and the written higher-strike call exactly offset each other, resulting in a zero net investment. Two questions concerning a zero-cost collar arise naturally. First, does a zero-cost collar always exist? That is, for any option pricing model, can one always find some strike prices K_1 and K_2 with $K_1 \leq K_2$ such that the K_1-K_2 collar is costless? Second, is the design of a zero-cost collar unique, given that one exists? That is, can we find one or more pairs of strike prices (K_1, K_2) satisfying the zero-cost requirement? We answer these two questions in turn.

Existence. With respect to the existence of zero-cost collars, the answer is simple. A T-year collar with $K_1 = K_2 = F_{0,T}$ is trivially a zero-cost collar. This is a direct consequence of put-call parity:

$$C(F_{0,T}, T) - P(F_{0,T}, T) = \mathrm{PV}_{0,T}(F_{0,T} - F_{0,T}) = 0,$$

or $C(F_{0,T}, T) = P(F_{0,T}, T)$. In fact, such a collar is identical to a short genuine forward contract, which entails zero investment by definition.

Uniqueness. The uniqueness issue is more complex. An important result is the following inequality relating the put strike and call strike of a zero-cost collar, and the forward price:

$$\boxed{K_1 \leq F_{0,T} \leq K_2.} \tag{3.3.2}$$

To prove this, we assume by way of contradiction that $K_1 > F_{0,T}$. Then put-call parity implies that

$$C(K_1) - P(K_1) = \mathrm{PV}_{0,T}(F_{0,T} - K_1) < 0.$$

Assuming for the time being that call option prices are non-increasing in the strike price (see page 348 in Chapter 9), as intuition suggests, we have

$$P(K_1) > C(K_1) \geq C(K_2) \quad \text{for all } K_2 \geq K_1.$$

In other words, it is impossible to find $K_2 \geq K_1$ such that $P(K_1) = C(K_2)$. Similarly, if $K_2 < F_{0,T}$, then $P(K_1) = C(K_2)$ cannot be true for all $K_1 \leq K_2$. With (3.3.2), we can first fix the put strike $K_1 (\leq F_{0,T})$, then find the call strike $K_2 (\geq F_{0,T})$ such that $C(K_2) = P(K_1)$ (see Figure 3.3.4). The existence of K_2 is guaranteed by the continuity of the call price with respect to the strike price (see Property 2 in Subsection 9.3.1), the intermediate value theorem in calculus, as well as the fact that $C(F_{0,T}) = P(F_{0,T}) \geq 0$ and $\lim_{K\to\infty} C(K) = 0$. As K_1 can be any number less than $F_{0,T}$, there are infinitely many zero-cost collars that can be constructed using different pairs of strike prices (K_1, K_2).

Example 3.3.14. (SOA Exam IFM Introductory Derivatives Sample Question 1: Some facts about a zero-cost collar) Determine which statement about zero-cost purchased collars is FALSE.

(A) A zero-width, zero-cost collar can be created by setting both the put and call strike prices at the forward price.

(B) There are an infinite number of zero-cost collars.

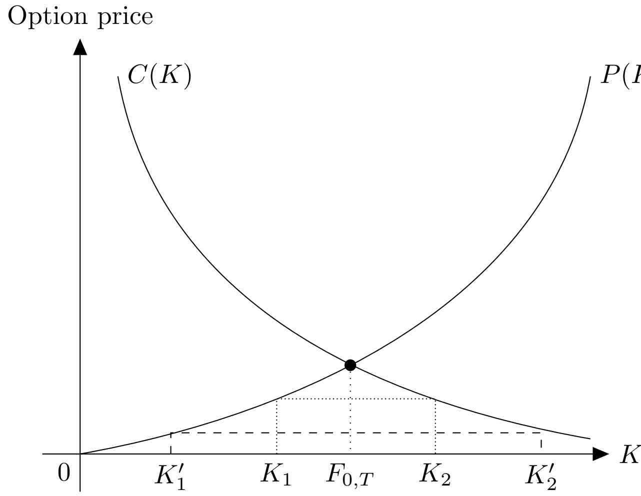


FIGURE 3.3.4
Illustration of the construction of two different zero-cost collars: a K_1-K_2 zero-cost collar and a K_1'-K_2' zero-cost collar.

(C) The put option can be at-the-money.

(D) The call option can be at-the-money.

(E) The strike price on the put option must be at or below the forward price.

Solution. Answers (A), (B), and (E) have been discussed above. If we assume that the underlying asset pays no dividends, then (D) must be false. This is because $K_2 \geq F_{0,T} = S(0)e^{rT} > S(0)$, i.e., $K_2 = S(0)$ is impossible. □

Remark. If the underlying asset pays dividends, then (C) and (D) are both *possible* (not necessarily true). This is because there is no definite order between $F_{0,T}$ and $S(0)$—it is possible for $F_{0,T}$ to be less than $S(0)$ if the present value of the dividends is large enough.

3.4 Volatility Speculation

The option strategies in Section 3.3 (with the exception of box spread) are all directional in nature in the sense that they benefit from movements of the price of the underlying asset in a particular direction. In this section, we study how options can be fused so that the resulting portfolio is non-directional with respect to the underlying asset. These option strategies are valuable to investors who would be adversely affected or would like to speculate on vigorous movements in the price of the underlying asset in either direction in the future.

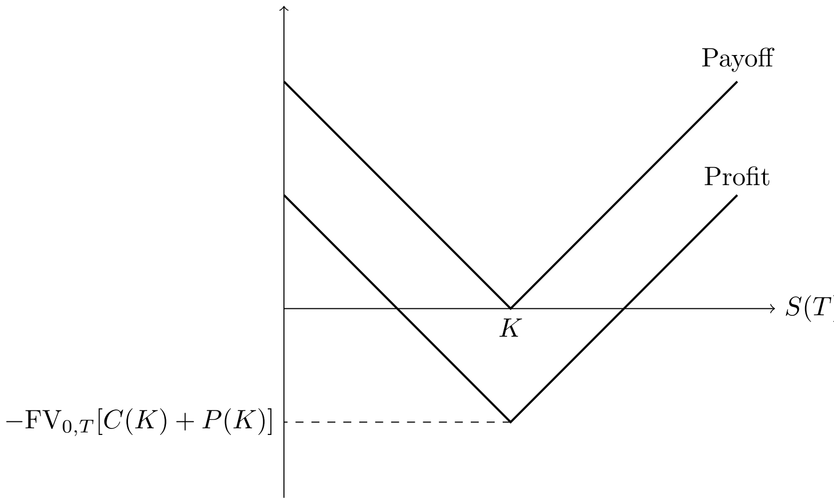


FIGURE 3.4.1
The payoff and profit diagrams of a long K-strike straddle.

3.4.1 Straddles

Composition.

A *straddle* is the combination of a long call and a long put with the same underlying asset, strike price K, and expiration time T. Often the call and put are at-the-money. This position is non-directional in nature, because if the asset price rises, the straddle benefits from the purchased call; if the asset price drops, the purchased put yields a positive payoff. This two-sided protection is achieved at the cost of paying both the call premium and put premium upfront.

Payoff and profit.

The payoff of a long K-strike straddle is

$$\begin{aligned}\underbrace{(S(T)-K)_+}_{\text{Long call}}+\underbrace{(K-S(T))_+}_{\text{Long put}} &= \begin{cases} K-S(T), & \text{if } S(T)<K \\ S(T)-K, & \text{if } S(T)\geq K \end{cases} \\ &= \boxed{|S(T)-K|}.\end{aligned}$$

As shown in Figure 3.4.1, the payoff of a straddle is V-shaped, with the tip situated at the common strike price K. The two straight lines emanating from K can be thought of as the two legs of a man who "straddles" on the plane, hence the name "straddle" for this position. The more the terminal asset price $S(T)$ differs from the strike price K, the higher the payoff of the straddle will be. Holding a straddle can therefore be viewed as a bet on the volatility of the underlying asset being higher than that perceived by the market. The profit diagram is obtained by subtracting from the payoff diagram the future value of the total initial investment, which is the sum of the call price and the put price.

Example 3.4.1. (Break-even asset prices) In Figure 3.4.1, identify all possible asset price(s) at expiration such that the profit of a long K-strike straddle is zero.

Solution. Let $\Delta = \mathrm{FV}_{0,T}[C(K) + P(K)]$. Then the two values of $S(T)$ such that the profit of a long K-strike straddle is zero are $\boxed{K - \Delta}$ and $\boxed{K + \Delta}$. □

Example 3.4.2. (Calculating the price of a straddle by put-call parity) You are given the following information:

(i) The current price of the stock is 45 per share.

(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

(iii) The price of a 9-month at-the-money European call option on the stock is 3.

(iv) The continuously compounded risk-free interest rate is 4%.

Calculate the current price of a 9-month 45-strike European straddle.

Solution. The (long) 9-month 45-strike straddle comprises a long 9-month 45-strike call and a long 9-month 45-strike put. By put-call parity, the price of the put is

$$P(45) = C(45) - S(0)\mathrm{e}^{-\delta T} + K\mathrm{e}^{-rT} = 3 - 45\mathrm{e}^{-0.02(0.75)} + 45\mathrm{e}^{-0.04(0.75)} = 2.340012.$$

The price of the straddle is $C(45) + P(45) = \boxed{5.3400}$. □

Written straddle.

A short straddle is the opposite of a long straddle. It involves simultaneously selling a call and a put with the same underlying asset, strike price, and expiration date. The payoff has an inverted V-shape. The profit is limited to the premiums of the call and put accumulated with interest, but the loss can be huge if the price of the underlying asset goes up or down substantially. Therefore, a short straddle position is highly risky due to unlimited potential loss. An investor may take a short straddle position if he/she strongly believes that the volatility of the underlying asset is smaller than the market's assessment.

3.4.2 Strangles

Motivation.

If one believes that the market is highly volatile so that $S(T)$ would be far away from K, a long straddle can be used. An obvious objection to the use of a straddle is the high investment required. A strangle is the more economical alternative to a straddle involving the purchase of out-of-the-money calls and puts in place of at-the-money ones as are typically used in a straddle. More specifically, a long K_1-K_2 *strangle* is established by a long put with the lower strike price K_1 and a long call with the higher strike price K_2, with $K_1 \leq S(0) \leq K_2$. The K_1-strike put is cheaper than the at-the-money put, and the K_2-strike call costs less than the at-the-money call, so a strangle requires a lower initial investment to set up than that of a straddle while allowing its holder to protect himself from or to speculate on the volatility of the underlying asset.

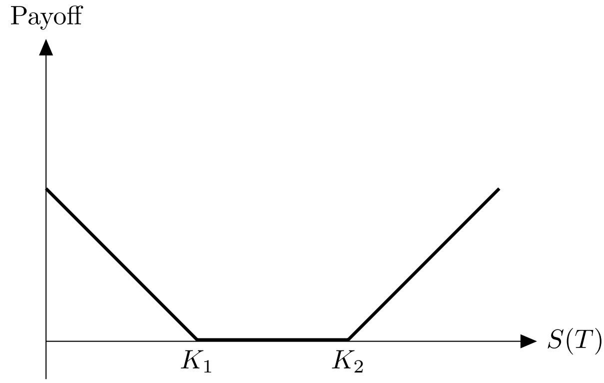


FIGURE 3.4.2
Payoff diagram of a long K_1-K_2 strangle.

Payoff.

The payoff of a strangle, sketched in Figure 3.4.2, is akin to a horrendous "strangle" (imagine that the base between K_1 and K_2 is a cord to be put around someone's throat!) shares characteristics similar to that of a straddle in the sense that both increase when $S(T)$ deviates significantly from K, but the payoff of a strangle has a flattened base on the $S(T)$-axis when $S(T)$ is between the two strike prices.

Example 3.4.3. (SOA Exam IFM Introductory Derivatives Sample Question 16: Straddle vs strangle) The current price of a nondividend-paying stock is 40 and the continuously compounded risk-free interest rate is 8%. The following table shows call and put option premiums for three-month European options of various exercise prices:

Exercise price	Call premium	Put premium
35	6.13	0.44
40	2.78	1.99
45	0.97	5.08

A trader interested in speculating on volatility in the stock price is considering two investment strategies. The first is a 40-strike straddle. The second is a strangle consisting of a 35-strike put and a 45-strike call.

Determine the range of stock prices in 3 months for which the strangle outperforms the straddle.

(A) The strangle never outperforms the straddle.

(B) $33.56 < S(T) < 46.44$

(C) $35.13 < S(T) < 44.87$

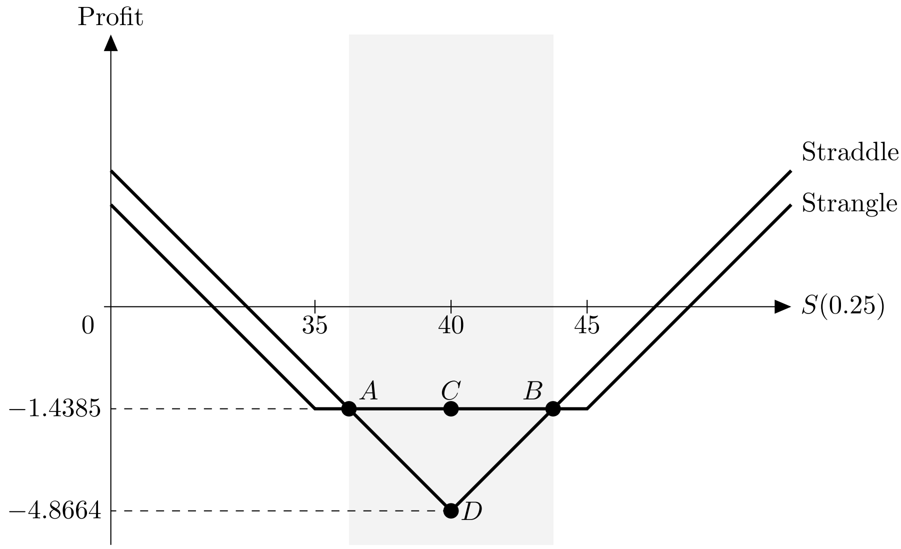
(D) $36.57 < S(T) < 43.43$

(E) The strangle always outperforms the straddle.

Solution. To begin with, the 40-strike straddle comprises a long 40-strike call and a long 40-strike put. It costs $2.78 + 1.99 = 4.77$, which grows to $4.77e^{0.08/4} = 4.8664$ in 3 months. The 35-45 strangle consists of a long 35-strike put and a long 45-strike call. It costs $0.97 + 0.44 = 1.41$, which grows to $1.41e^{0.08/4} = 1.4385$ in 3 months.

We present two solutions for solving this example.

- *Geometric solution:* The payoff diagrams of the straddle and strangle are sketched below (the diagrams are not drawn to scale):



By inspection, the strangle outperforms the straddle when $S(0.25)$ lies in the interval AB. To find the horizontal coordinates of the two points, A and B, observe that the slope of line BD is 1, so that triangle BCD is an isosceles triangle with $BC = CD = -1.4385 - (-4.8664) = 3.4279$. Thus the horizontal coordinate of point B is $40+3.4279 = 43.4279$. Similarly, the horizontal coordinate of point A is $40-3.4279 = 36.5721$. The required stock price range is $S(0.25) \in \boxed{(36.5721, 43.4279)}$. **(Answer: (D))**

- *Algebraic solution:* Comparing the profit of the strangle to that of the straddle, we solve the inequality

$$(S(0.25) - 45)_+ + (35 - S(0.25))_+ - 1.4385 > |S(0.25) - 40| - 4.8664. \qquad (3.4.1)$$

and distinguish three cases:

Case 1. If $S(0.25) \geq 45$, then LHS $= S(0.25) - 45 - 1.4385 = S(0.25) - 46.4385$, while RHS $= S(0.25) - 44.8664 >$ LHS. Therefore, (3.4.1) has no solution.

Case 2. If $S(0.25) \leq 35$, then LHS $= 35 - S(0.25)$, while RHS $= (40 - S(0.25)) - 4.8664 = 35.1336 - S(0.25) >$ LHS. Again, (3.4.1) has no solution.

Case 3. It remains to investigate inequality (3.4.1) when $S(0.25) \in (35, 45)$, in which case it reduces to

$$-1.4385 > |S(0.25) - 40| - 4.8664,$$

or $S(0.25) \in \boxed{(36.5721, 43.4279)}$. **(Answer: (D))** □

Remark. (i) The current stock price is not used anywhere in the solution.

(ii) If I were the setter of this exam question, I might not be so kind to give you the value of r—You can deduce it from the given table of option prices!

(iii) This example reveals that there are relative merits and demerits of a strangle and straddle. While a strangle entails a lower amount of initial investment and carries a higher profit when the terminal stock price stays near $S(0)$, the terminal price at expiration has to move further away from K in order to result in a positive profit.

Example 3.4.4. (How serious is the student's mistake?) In the Midterm Exam of your favorite derivatives pricing course, a really "clever" student erroneously constructed a K_1-K_2 strangle by using *in-the-money* options. Specifically, he bought a call option with strike price K_1, and bought a put option with strike price K_2, where $K_1 < S(0) < K_2$.

Determine which of the following statements about the student's "strangle" and a correct K_1-K_2 strangle is/are correct.

I. Both positions are long with respect to the underlying asset.

II. Both positions have the same payoff at expiration.

III. Both positions have the same profit.

(A) I only

(B) II only

(C) III only

(D) II and III only

(E) The correct answer is not given by (A), (B), (C), or (D)

Solution. We first determine the payoff function of the student's "strangle":

$$\text{Payoff} = (K_2 - S(T))_+ + (S(T) - K_1)_+ = \begin{cases} K_2 - S(T), & \text{if } S(T) < K_1, \\ K_2 - K_1, & \text{if } K_1 \leq S(T) < K_2, \\ S(T) - K_1, & \text{if } K_2 \leq S(T). \end{cases}$$

The payoff diagram is sketched in Figure 3.4.3. It can be seen that the payoff of the student's "strangle" is always greater than that of a genuine K_1-K_2 strangle by $K_2 - K_1$. However, because the two payoff functions are parallel, the two positions actually share the same profit. In other words, if the student was asked to calculate the profit of the genuine strangle at a particular $S(T)$ in the Midterm Exam, the student would get the correct answer based on his own "strangle!" **(Answer: (C))** □

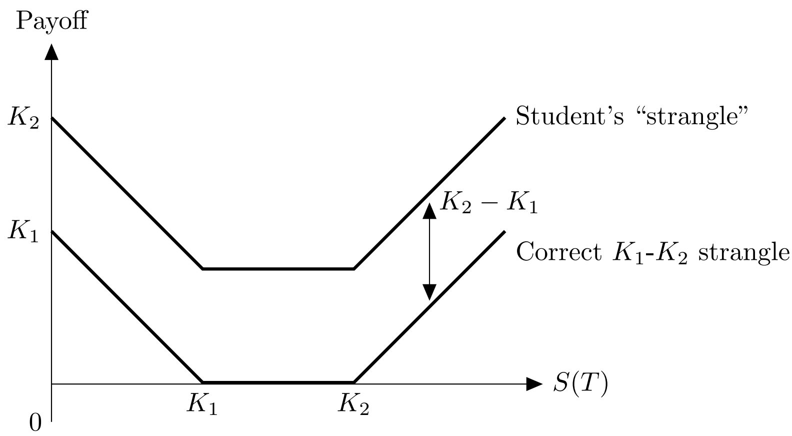


FIGURE 3.4.3
The payoff diagrams of the student's "strange" and a genuine strangle in Example 3.4.4.

3.4.3 Butterfly Spreads

Motivation.

If you believe that the market volatility is low so that $S(T)$ would be close to a certain level, say K_2, you may take a short position in a K_2-strike straddle. As we have seen in Subsection 3.4.1, the most obvious objection to a written straddle is that you will be exposed to an unlimited loss potential. To limit your losses in case you are wrong, you may insure your written straddle with:

- A long out-of-the-money, say K_3-strike call, which provides protection on the upside.
- A long out-of-the-money, say K_1-strike put, which offers protection on the downside.

The resulting portfolio consisting of a short K_2-strike straddle and a long K_1-K_3 strangle is called a K_1-K_2-K_3 *butterfly spread*, which derives its name from the fact that its payoff function resembles a butterfly (exercise some imagination, please!). Graphed in Figure 3.4.4, the payoff is the greatest when the terminal stock price is closest to K_2. Even if your belief about low volatility turns out to be incorrect and the stock price does change inexorably, your loss is still limited.

Example 3.4.5. (SOA Exam IFM Introductory Derivatives Sample Question 8) Joe believes that the volatility of a stock is higher than indicated by market prices for options on that stock. He wants to speculate on that belief by buying or selling at-the-money options.

Determine which of the following strategies would achieve Joe's goal.

(A) Buy a strangle

(B) Buy a straddle

(C) Sell a straddle

(D) Buy a butterfly spread

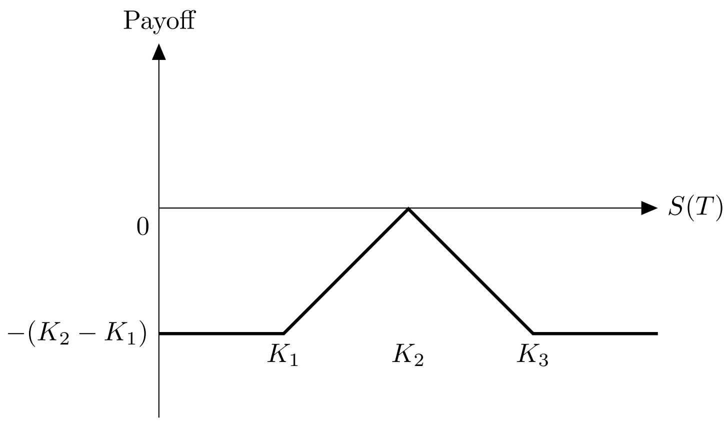


FIGURE 3.4.4
Payoff diagram of a long K_1-K_2-K_3 butterfly spread constructed by a short K_2-strike straddle coupled with a long K_1-K_3 strangle.

(E) Sell a butterfly spread

Solution. Only straddles use at-the-money options and buying is correct for this speculation. **(Answer: (B))** □

Other methods of construction.

As discussed in the preceding paragraph, a long K_1-K_2-K_3 butterfly spread can be constructed by a short K_2-strike straddle along with a long K_1-K_3 strangle. Alternative, indeed more popular, ways of construction using solely calls and puts include:

1. Long 1 K_1-strike call, 2 short K_2-strike calls and long 1 K_3-strike call
2. Long 1 K_1-strike put, 2 short K_2-strike puts and long 1 K_3-strike put
3. Long 1 K_1-K_2 (call/put) bull spread and long 1 K_2-K_3 (call/put) bear spread

The payoff functions of all of these constructions have the same general shape as Figure 3.4.4. As a consequence, all of them lead to the same profit.

As an illustration, let's examine the construction solely by calls and proceed from left to right (why not from right to left?).

Case 1. If $S(T) < K_1$, then none of the three call options pay off. The payoff of the entire portfolio is constant at zero.

Case 2. If $K_1 \leq S(T) < K_2$, then only the K_1-strike call pays off. As we are long 1 K_1-strike call, the portfolio payoff increases by 1 for every unit increase in $S(T)$. In other words, the payoff function after passing through K_1 is upward tilting with a slope of $+1$.

Case 3. If $K_2 \leq S(T) < K_3$, then the K_2-strike call, in addition to the K_1-strike call, also pays off. Due to the short 2 K_2-strike calls, the slope of the payoff function in this range of stock price is decreased by 2 to $-1\,(= 1 - 2)$. The payoff starts to become downward sloping.

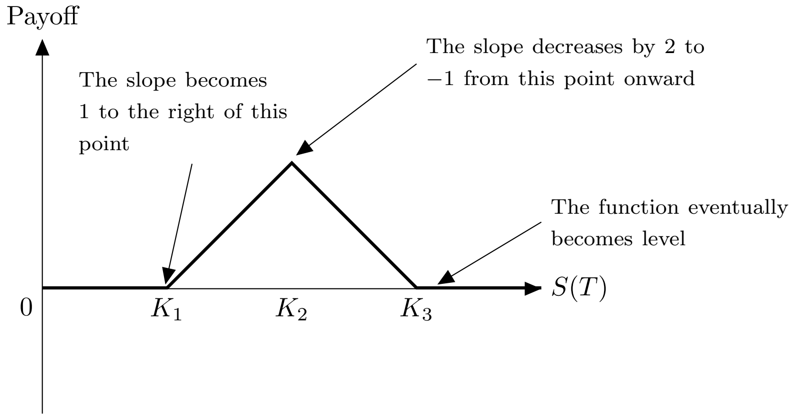


FIGURE 3.4.5
Payoff diagram of a long K_1-K_2-K_3 call (or put) butterfly spread.

Case 4. If $K_3 \leq S(T)$, then all three kinds of call options pay off. In this range of stock price, the slope of the payoff function is increased from -1 to $-1+1=0$. Thus the payoff eventually levels off and becomes constant at zero again.

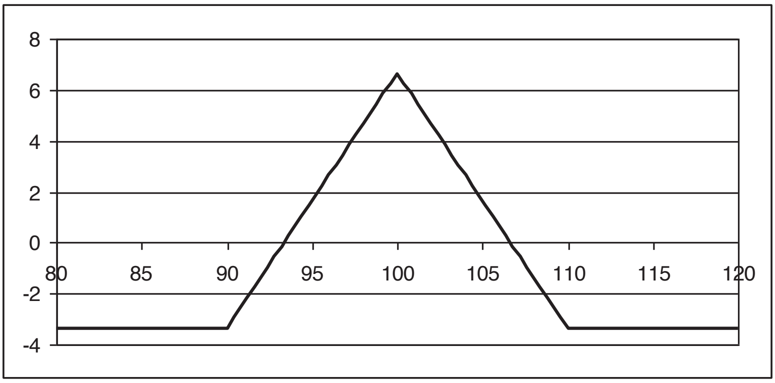
These considerations are visualized in Figure 3.4.5, which displays the payoff diagram of such a long K_1-K_2-K_3 call butterfly spread. Observe that unlike Figure 3.4.4, the payoff function in Figure 3.4.5 is always non-negative and is strictly positive when $S(T) \in (K_1, K_3)$. By the no-arbitrage principle, the initial cost of setting up the call butterfly spread, which is $C(K_1)+C(K_3)-2C(K_2)$, must be non-negative. Because K_1 and K_3 are arbitrary numbers, this non-negativity condition is a remarkably strong no-arbitrage restriction on call prices. In Subsection 9.3.1, we will see that this condition is a special form of the so-called convexity of option prices.

Example 3.4.6. (SOA Exam IFM Introductory Derivatives Sample Question 9: Possible composition of a butterfly spread) Stock ABC has the following characteristics:

- The current price to buy one share is 100.
- The stock does not pay dividends.
- European options on one share expiring in one year have the following prices:

Strike Price	Call option price	Put option price
90	14.63	0.24
100	6.80	1.93
110	2.17	6.81

A butterfly spread on this stock has the following profit diagram.



The continuously compounded risk-free interest rate is 5%.
Determine which of the following will NOT produce this profit diagram.

(A) Buy a 90 put, buy a 110 put, sell two 100 puts

(B) Buy a 90 call, buy a 110 call, sell two 100 calls

(C) Buy a 90 put, sell a 100 put, sell a 100 call, buy a 110 call

(D) Buy one share of the stock, buy a 90 call, buy a 110 put, sell two 100 puts

(E) Buy one share of the stock, buy a 90 put, buy a 110 call, sell two 100 calls

Ambrose's comments:

(A) and (B) refer to the standard ways to construct a butterfly spread by calls and puts. (C) corresponds to a 90-100 long put bull spread and a 100-110 long call bear spread. What about (D) and (E)?

Solution. Consider the shape of the payoff function of Position (D):

	Slope			
Stock price range	$[0, 90)$	$[90, 100)$	$[100, 110)$	$[110, \infty)$
Long stock	$+1$	$+1$	$+1$	$+1$
Long one 90-strike call	0	$+1$	$+1$	$+1$
Long one110-strike put	-1	-1	-1	0
Short two 100-strike puts	$+2$	$+2$	0	0
Total	$+2$	$+3$	$+1$	$+2$

This payoff function is always increasing, not in the shape of a butterfly spread. **(Answer: (D))** □

Example 3.4.7. (SOA Exam IFM Introductory Derivatives Sample Question 67: Payoff calculations) Consider the following investment strategy involving put options on a stock with the same expiration date.

(i) Buy one 25-strike put

(ii) Sell two 30-strike puts

(iii) Buy one 35-strike put

Calculate the payoffs of this strategy assuming stock prices (i.e., at the time the put options expire) of 27 and 37, respectively.

(A) –2 and 2

(B) 0 and 0

(C) 2 and 0

(D) 2 and 2

(E) 14 and 0

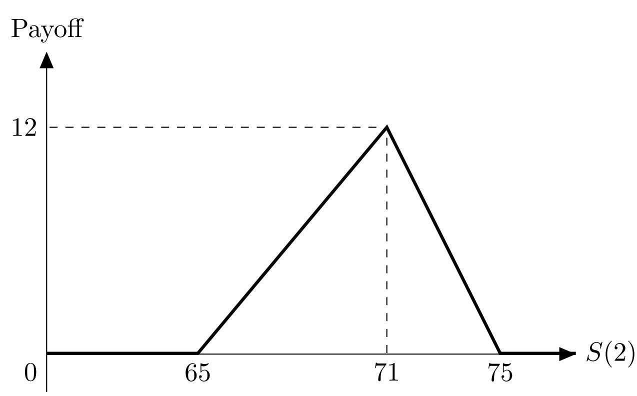
Solution. When $S(T) = 27$, the payoff is $-2(30-27)+(35-27) = \boxed{2}$. When $S(T) = 37$, the payoff is $\boxed{0}$ because none of the puts will be exercised. **(Answer: (C))** □

Remark. You may also draw a payoff diagram to answer this question.

Asymmetric butterfly spreads.

It is not necessary for the peak of a butterfly spread to lie mid-way between the two extreme strike prices, K_1 and K_3. By varying the number of options you buy at K_1 and K_3, it is possible to make the peak tilted to the left or right and the butterfly spread *asymmetric*. The slope adjustment technique featured in this chapter continues to be useful for understanding the composition and payoff structure of such asymmetric butterfly spreads.

Example 3.4.8. (Given its payoff diagram, find the cost of an asymmetric butterfly spread) The payoff diagram of a certain investment strategy involving 2-year European put options on a stock is shown on the right.



You are given:

(i)

Strike Price	Put Option Price
65	1.5
71	4.0
75	6.0

(ii) The continuously compounded risk-free interest rate is 1.5%.

Calculate the profit on the investment strategy assuming that the stock price at expiration is 70.

Solution. To determine how the investment strategy, which is just an asymmetric butterfly spread, can be constructed by the given puts, observe that the slope of the upward sloping straight line is $(12-0)/(71-65) = 2$ and the slope of the downward sloping straight line is $(0-12)/(75-71) = -3$. Going from right to left, we deduce that three 75-strike puts should be bought (so that the slope of the payoff function between 71 and 75 is -3), five 71-strike puts should be sold (so that the slope between 65 and 71 is $-3+5=2$), and two 65-strike puts should be bought (so that the slope to the left of 65 is $2-2=0$) to construct the given butterfly spread. The initial investment required is

$$3P(75) - 5P(71) + 2P(65) = 3(6.0) - 5(4.0) + 2(1.5) = 1.$$

When the 2-year stock price is 70, the payoff of the butterfly spread, by linear interpolation, is $12 \times 5/6 = 10$. Therefore, the profit on the butterfly spread is $10 - \mathrm{e}^{0.015(2)} = \boxed{8.9695}$. □

Remark. For any butterfly spread, symmetric or asymmetric, the number of options to buy ($2+3=5$ in this example) must equal the number of options to sell (equal to 5 in this example). This ensures that the slope of the overall payoff function must be zero when the terminal asset price is sufficiently small (less than 65 in this example) or sufficiently large (greater than 75 in this example).

3.5 Problems

Problem 3.5.1. (Warm-up true-or-false items) Determine whether each of the following positions has an unlimited loss potential from adverse price movements in the underlying asset, regardless of the initial premium received.

(A) Long forward

(B) Short naked call

(C) Long collared stock

(D) Short straddle

(E) Long butterfly spread

Basic insurance strategies

Problem 3.5.2. (Simple profit calculation for a floor) You buy a stock at \$300 and buy an at-the-money 9-month European put option on the stock at a price of \$15.

The continuously compounded risk-free interest rate is 5%.

Calculate your 9-month profit if the 9-month stock price is \$280.

Problem 3.5.3. (Considerations that go into the construction of a floor) You are selecting among various put options with different strike prices to hedge a long asset position.

Which of the following statements is true? Give your reasoning.

(A) Higher-strike puts cost more and provide higher floors.

(B) Higher-strike puts cost less and provide higher floors.

(C) Lower-strike puts cost more and provide higher floors.

(D) Lower-strike puts cost less and provide higher floors.

(E) The strike price does not matter at all.

Problem 3.5.4. (Hedging an implicit short position) Supway is a sandwich shop, one of its main production inputs being wheat.

Determine whether each of the following risk management techniques can hedge the financial risk faced by Supway arising from the price of wheat that it *buys*.

(A) Long forward on wheat

(B) Long call option on wheat

(C) Short put option on wheat

(D) Long bear spread on wheat

(E) Short collar on wheat

Problem 3.5.5. (Combining different insurance strategies – I) Assume the same underlying stock, same time to expiration, and same strike price for all derivatives in this problem.

Which of the following must have the same profit as a floor coupled with a cap? Give your reasoning.

(A) Long stock

(B) Short stock

(C) Long straddle

(D) Short straddle

(E) None of the above

Problem 3.5.6. (Combining different insurance strategies – II) Assume the same underlying stock, same time to expiration, and same strike price for all derivatives in this problem.

Which of the following must have the same profit as a floor coupled with a written covered call? Give your reasoning. There can be more than one answer.

(A) Long stock

(B) Short stock

(C) Long forward

(D) Short forward

(E) Long straddle

(F) Short straddle

Put-call parity

Problem 3.5.7. (Synthetic short forward) The current price of a nondividend-paying stock is 1,000 and the continuously compounded risk-free interest rate is 5%.

Richard wants to lock in the ability to *sell* a unit of this stock in six months for a price of 1,020. He can do this by buying or selling 6-month 1,020-strike European put and call options on the stock.

Determine the positions in the put and call options that Richard should take to achieve his objective and calculate the cost today of establishing this position.

Problem 3.5.8. (Long/short synthetic forward with discrete dividends) The current price of a stock is 100, and the continuously compounded risk-free interest rate is 10%. A \$2.5 dividend will be paid every quarter, with the first dividend occurring 2 months from now.

Roger uses a K-strike European call option and a K-strike European put option on the same stock to create a synthetic six-month short forward. The initial investment is 5.3.

Calculate K.

Problem 3.5.9. (Comparing a short call with a long synthetic forward) The current price of a nondividend-paying stock is 60 and the continuously compounded risk-free interest rate is 6%.

Actuary A writes a 1-year 70-strike call option whose price is 1.50. Actuary B enters into a 1-year synthetic long forward which permits him to buy the stock for 65 in one year.

It is known that Actuary B's profit from his synthetic long forward is twice as large as Actuary A's profit from his written call option.

Determine all possible 1-year price(s) of the stock.

Problem 3.5.10. (The strike price such that the call and put prices coincide) You are given that one-year 15-strike European call and put premiums on a share of Iowa Inc. are 6.46 and 0.75, respectively.

The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2%. The effective annual interest rate is 5%.

Determine the strike price at which the call and put premiums on a share of Iowa Inc. will be equal.

Problem 3.5.11. (Given a table of option prices) Stock X pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

You are given the following option prices for European puts and calls, all written on stock X and with two years to expiration:

Strike Price	Put Price	Call Price
98	0.4394	14.3782
100	0.6975	12.7575

Calculate the current price of stock X.

Problem 3.5.12. (Profit comparison) The current price of stock ABC is 40. Stock ABC pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

You are given the following premiums of one-year European call options on stock ABC for various strike prices:

Strike	Call premium
35	7.24
40	4.16
45	2.62

The effective annual risk-free interest rate is 8%.
Let $S(1)$ be the price of the stock one year from now.
Determine the range for $S(1)$ such that a 35-strike short put produces a higher profit than a 45-strike short put, but a lower profit than a 40-strike short put.

(Note: All put positions being compared are short.)

Problem 3.5.13. (Absolute value of the difference between call/put premiums) Let $C(K)$ and $P(K)$ be the premiums of three-month K-strike European call and put options on the same stock, respectively. You are given that $|C(60) - C(65)| = 3$ and the continuously compounded risk-free interest rate is 5%.

Calculate $|P(60) - P(65)|$.

Problem 3.5.14. [HARDER!] (Parity arbitrage with discrete dividends) You are given the following information:

(i) The current price of stock Y is 30.

(ii) Dividends of 1 per unit of stock will be paid in two months and in eight months.

(iii) The continuously compounded risk-free interest rate is 6%.

(iv) The price of a 1-year 32-strike European call option on stock Y is 3.

(v) The price of a 1-year 32-strike European put option on stock Y is 4.

Describe how you could earn arbitrage profits with actions taken *at time 0 only*.

Spreads and collars

Problem 3.5.15. (How to create a bear spread by calls and puts?) Determine whether each of the following strategies creates a long bear spread, assuming that all options are European and on the same underlying asset.

(A) Buy a 45-strike call and sell a 50-strike call.

(B) Buy a 45-strike put and sell a 50-strike put.

(C) Buy a call with a price of 6 and sell a call with a price of 10.

(D) Buy a put with a price of 6 and sell a put with a price of 10.

Problem 3.5.16. (Comparing a bull spread with a bear spread) The current price of a nondividend-paying stock is 40 and the continuously compounded risk-free interest rate is 8%. The following table shows call option premiums for 3-month European options of various exercise prices:

Exercise Price	Call Premium
35	6.13
40	2.78
45	0.97

Student A constructs a 35-45 bull spread using call options.
Student B constructs a 40-45 bear spread using put options.
Determine the three-month stock price such that Student A and Student B have the same profit, and the value of the common profit.

Problem 3.5.17. (Call/put bear spread) You are given that the price of a 70-strike call option is 8.3 and the price of a 80-strike call option is 2.7, where both options expire in one year and have the same underlying asset.

The continuously compounded risk-free interest rate is 6%.

You create a one-year 70-80 long bear spread using put options. If the amount of profit from the bear spread is 4, calculate the 1-year stock price.

(Hint: Does it matter whether the bear spread is a call bear spread or a put bear spread?)

Problem 3.5.18. (Break-even price for a call bear spread) An investor wrote a 45-strike European call option on an index with three years to expiration. The premium for this option was 4.

The investor also bought a 55-strike European call option on the same index with three years to expiration. The premium for this option was 2.5.

The continuously compounded risk-free interest rate is 2%.

Calculate the index price at expiration that will allow the investor to break even.

Problem 3.5.19. (Break-even price for a bear spread) The following table shows the premiums of European call and put options having the same nondividend-paying stock, the same time to expiration but different strike prices:

Strike Price	Call Premium	Put Premium
50	8.4	0.8
60	2.6	4.7

An investor constructs a 50-60 long bear spread using the above options and breaks even at expiration.

Calculate the amount that the stock price at expiration should move from its current level.

Problem 3.5.20. (Different ways of constructing box spreads) Consider a box spread based on two distinct strike prices K and L with $K < L$ that is used to lend money, so that there is a positive cost to this transaction up front, but a guaranteed positive payoff at expiration.

Determine which of the following sets of transactions is equivalent to this type of box spread.

I. Buy a K-strike put, buy a L-strike call, sell a K-strike call, and sell a L-strike put

II. Buy a K-L call bull spread and sell a K-L put bull spread

III. Buy a K-L strangle and sell a K-L strangle

(A) I only

(B) II only

(C) III only

(D) I, II, and III

(E) The correct answer is not given by (A), (B), (C), or (D)

Problem 3.5.21. (Does the option box permit arbitrage?) You are given the following information about four European options on the same underlying asset:

(i) The price of a 25-strike 1-year call option is 6.85.

(ii) The price of a 35-strike 1-year call option is 1.77.

(iii) The price of a 25-strike 1-year put option is 0.63.

(iv) The price of a 35-strike 1-year put option is 5.06.

The continuously compounded risk-free interest rate is 6%.

Describe actions you could take at time 0 using only appropriate bull/bear spread(s) and/or zero-coupon bond(s) to earn arbitrage profits at time 0. Specify the contractual details of the bull/bear spread(s) and zero-coupon bond(s) you use clearly.

Problem 3.5.22. (Profit of a collared stock) The following table shows the prices of European call and put options with the same underlying asset, time to expiration, but different strike prices:

Strike Price	Call Premium	Put Premium
17	5.16	1.35
23	2.45	4.36

Calculate the profit on a 17-23 long collared stock position if the ending stock price is 22.

Problem 3.5.23. [HARDER!] (Comparing a bear spread with a collar in terms of profit) The current price of a stock is 50. The stock will pay a single dividend of 0.75 in one month. The continuously compounded risk-free interest rate is 6%.

The following table shows the premiums of 6-month European call options on the stock:

Strike Price	Call Premium
50	5.19
60	1.96

Let S be the price of the stock six months from now.

Determine the range of values of S such that a long 50-60 6-month European bear spread outperforms (in terms of profit) a long 50-60 6-month European collar.

Problem 3.5.24. (Zero-cost collar: Forward price inequality) Apple expects to sell pork bellies 3 months from now. The current 3-month forward price for pork belly is 4 per ton. Apple has decided to buy a zero-cost collar to reduce his exposure to the price of pork belly.

Determine whether each of the following options could be a *possible* component of Apple's collar.

(A) A written call with a strike price of 4.2

(B) A written put with a strike price of 3.9

(C) A purchased call with a strike price of 4.1

(D) A purchased put with a strike price of 3.9

(E) A purchased put with a strike price of 4.0

Problem 3.5.25. (Some basic facts about zero-cost collars: Motivation, existence, uniqueness, and moneyness of call and put) Determine whether each of the following statements about zero-cost purchased collars on stocks is true or false. Assume that options at all positive strike prices are available for trading. Note that you are not given whether the underlying stock pays dividends or not.

(A) A long zero-cost collar is a bet on the price of the underlying stock going up in the future.

(B) A zero-cost collar can always be constructed by some pair of call and put options.

(C) A zero-cost collar, if it exists, can be constructed by one and only one pair of call and put options.

(D) The put option can be at-the-money.

(E) The call option must be out-of-the-money.

Problem 3.5.26. [HARDER!] (The fair price of a strange derivative – I) The payoff of a derivative contract maturing in one year is given by

$$\text{Payoff} = \begin{cases} 3S(1) - 40, & \text{if } 0 \leq S(1) < 10, \\ 4S(1) - 50, & \text{if } 10 \leq S(1) < 20, \\ S(1) + 10, & \text{if } S(1) \geq 20, \end{cases}$$

where $S(1)$ is the one-year price of the underlying nondividend-paying stock.

You are given:

(i) The continuously compounded risk-free interest rate is 5%.

(ii) The current stock price is 15.

(iii) The price of a 10-strike 1-year call option is 5.52.

(iv) The price of a 20-strike 1-year call option is 0.38.

Calculate the fair price of the above derivative.

(Hint: It will help if you first sketch the payoff diagram of the derivative. Then work from left to right and deduce how many call options should be bought/sold at the two strike prices.)

Problem 3.5.27. [HARDER!] (The fair price of a strange derivative – II) The payoff of a special 1-year derivative on a nondividend-paying stock is described by the following piecewise linear function:

$$\text{Payoff} = \begin{cases} -S(1) + 25, & \text{if } 0 \leq S(1) < 5, \\ 2S(1) + 10, & \text{if } 5 \leq S(1) < 10, \\ 4S(1) - 10, & \text{if } 10 \leq S(1) < 15, \\ -2S(1) + 80, & \text{if } 15 \leq S(1), \end{cases}$$

where $S(1)$ is the one-year price of the stock.

You are given:

(i) The continuously compounded risk-free interest rate is 4%.

(ii) The current stock price is 10.

(iii) The price of a 5-strike 1-year European put option is 0.11.

(iv) The price of a 10-strike 1-year European put option is 1.75.

(v) The price of a 15-strike 1-year European put option is 9.52.

(vi) The price of the above derivative is 3.60.

Describe actions you could take to exploit an arbitrage opportunity using only the above put options, stocks and zero-coupon bonds, and calculate the present value of the arbitrage profit (per unit of stock).

(Hint: Decide whether you should work from left to right or from right to left, and deduce how many put options should be bought/sold at the three strike prices.)

Volatility speculation

Problem 3.5.28. (Given the profit/loss of a straddle) The stock of Iowa Actuarial Corporation has been trading in a narrow range around its current price of 45 per share for months. Dividends of 2 are payable quarterly, with the first dividend payable one month from now. The continuously compounded risk-free rate of interest is 6%.

You are convinced that the stock price will remain in a narrow range around 45 over the next three months. To take advantage of your belief, you speculate on the volatility of the stock through an appropriate position in a 3-month 45-strike straddle.

It turns out that you realize a loss (i.e., your profit is negative) if the 3-month stock price moves by more than 8.52 in either direction from its current level of 45.

Calculate the price of a 3-month 45-strike put option.

Problem 3.5.29. (How volatile does the stock have to be?) The price of a stock has hovered around its current price of 70 for several months, but you believe that the stock price will break far out of that range over the next 4 months, without knowing whether it will go up or down. You have decided to take advantage of your conviction through an appropriate position in a 4-month 70-strike European straddle.

You are given:

(i) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

(ii) The price of a 4-month at-the-money European call option on the stock is 7.

(iii) The continuously compounded risk-free interest rate is 8%.

Determine how far the price of the stock has to move in either direction from its current level so that you make a profit on your straddle after 4 months.

Problem 3.5.30. (Bull spread vs straddle) The current price of stock ABC is 40. Stock ABC pays dividends continuously at a rate proportional to its price. The dividend yield is 3%. The continuously compounded risk-free interest rate is 6%.

The following table shows the premiums of two-year put options on stock ABC of various strike prices:

Strike Price	Put Premium
35	0.44
40	1.99
45	5.08

Let $S(2)$ be the price of stock ABC two years from now.
Determine the range of values of $S(2)$ for which a 35-45 long bull spread outperforms a 40-strike long straddle, both of which are on stock ABC and expire in two years.

(Suggestion: Sketch the payoff diagrams of the bull spread and the straddle.)

Problem 3.5.31. [HARDER!] (Butterfly spread, the bloodshed – I! Some theoretical questions) Determine whether each of the following statements about butterfly spread is true or false.

(A) A long butterfly spread is a bet on the volatility of the underlying asset being higher than that perceived by the market.

(B) Combining a long K_1-K_2 bear spread combined with a long K_2-K_3 bull spread produces a short K_1-K_2-K_3 butterfly spread.

(C) A long butterfly spread created using call options has the same cost as an otherwise identical long butterfly spread created using put options.

(D) The maximum profit of a butterfly spread can be made arbitrarily large by appropriately scaling up the number of options to buy and sell.

(Note: This problem, which is not as easy as it may seem, tests the theoretical understanding of butterfly spread, particularly its motive and construction. The result of part (c) will be useful in 9.3.1.)

Problem 3.5.32. [HARDER!] (Butterfly spread, the bloodshed – II!) The current price of stock XYZ is 1,000, and the continuously compounded risk-free interest rate is 8%. A dividend will be paid at the end of every 2 months over the next year, with the first dividend occurring 2 months from now. The amount of the first dividend is $1^2 = 1$, that of the second dividend is $2^2 = 4$. In general, the amount of the n^{th} dividend is n^2, for $n = 1, 2, \ldots, 6$.

The following table shows the premiums of 1-year call options on stock XYZ of various exercise prices:

Exercise Price	Call Premium
950	120.41
1,020	80.28
1,050	71.80

You create a one-year 950-1,020-1,050 long asymmetric butterfly spread with the following characteristics:

- The maximum payoff of 210 is attained when the stock price at expiration is 1,020.
- The payoff is strictly positive only when the stock price at expiration is strictly between 950 and 1,050.
- Only put options are used in the construction of the butterfly spread.

Calculate all possible 1-year stock price(s) that result(s) in a profit of 15.

(Hint: The calculations can be tedious, but...they need not be, if you make good use of the results in the previous problem.)

Problem 3.5.33. (Butterfly spread: Given its desired characteristics) The following table shows the premiums of European call and put options having the same underlying stock, the same time to expiration but different strike prices:

Strike price	Call premium	Put premium
20	3.59	2.64
23	2.45	4.36
25	1.89	5.70

You use the above call and put options to construct an asymmetric butterfly spread with the following characteristics:

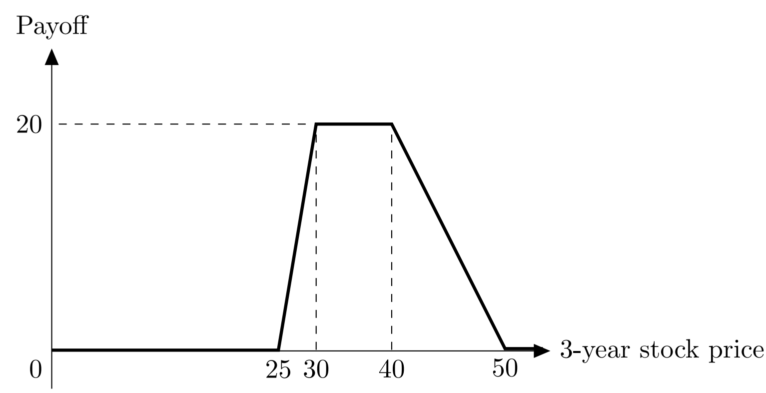
(i) The maximum payoff of 6 is attained when the stock price at expiration is 23.

(ii) The payoff is strictly positive as long as the stock price at expiration is strictly between 20 and 25.

Calculate your profit from the asymmetric butterfly spread if the stock price at expiration is 21.

(Hint: First sketch the payoff diagram of the butterfly spread from the descriptions in (i) and (ii). Then construct the butterfly spread using any method you please — why is this possible?)

Problem 3.5.34. (Given its payoff diagram, find the maximum profit of an unfamiliar derivative) The payoff diagram of an investment strategy involving 3-year European put options on a stock is shown below:



You are given:

(i)

Strike Price	Put Option Price
25	0.26
30	0.87
40	3.98
50	9.66

(ii) The continuously compounded risk-free interest rate is 2.5%.

Calculate the maximum possible profit of this investment strategy.

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Part II

Pricing and Hedging of Derivatives

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4

Binomial Option Pricing Models

Chapter overview: This chapter marks the beginning of *option pricing*, which is the central theme of Part II of this book. Unlike the pricing of prepaid forwards and forwards, option pricing is inherently model-dependent, i.e., different models of the stock price give rise to different option prices. Consider, for instance, the pricing of a K-strike call. Loosely speaking, the price of the call depends on the thickness of the right tail of the stock price distribution beyond the strike price K. The more likely the stock stays above K at expiration, the higher the call price. Similar considerations apply to a put, for which the left tail of the stock price distribution plays a pivotal role.

In this book, you will learn two broad classes of option pricing models, namely, the discrete-time binomial option pricing model (Chapter 4 and part of Chapter 8) and the continuous-time Black-Scholes model (Chapters 5 to 8). The discrete-time binomial model is intuitively easy to understand and, albeit simplistic to some extent, gives you the valuable intuition that carries over to the more complicated Black-Scholes model. More importantly, all derivatives can be priced and hedged in this discrete-time framework, at least in theory. The binomial model therefore forms the ideal starting point for understanding option pricing.

This chapter describes the mechanics of valuing options in the binomial option pricing model. Section 4.1 sets the stage, introducing the one-period binomial model and illustrating many of the essential ideas that underlie option pricing in general. In particular, the conceptually and practically important methods of pricing by replication and risk-neutral pricing are presented with brief discussions on their pros and cons. The valuation task is continued in Section 4.2, where we extend the simple one-period model to multiple periods, and the method of risk-neutral pricing remains to thrive. American options are treated in Section 4.3, in which we will see that pricing European options and pricing American options share essentially the same fundamental ideas. Options on other underlying assets such as exchange rates and futures are studied in Section 4.4. Finally, Section 4.5 reconciles the apparent inconsistency between risk-neutral valuation and traditional discounted cash flow valuation.

4.1 One-period Binomial Trees

4.1.1 Pricing by Replication

Model setting.

Suppose that the current time is 0. Consider a primitive market comprising a risky stock and a risk-free (zero-coupon) bond (also known as a bank account) earning a continuously compounded risk-free interest rate of r. The stock has a current price of S_0 and pays dividends continuously at a rate proportional to its price, with a dividend yield[i] of δ.

In a one-period binomial stock price model whose duration is h[ii] (in years), the stock price at the end of the period is assumed to take only two possible values (hence the term "binomial"):

$$\boxed{S_u := u \times S_0 \quad \text{or} \quad S_d := d \times S_0,}$$

where the subscripts "u" and "d" on "S" suggest "up" and "down,"[iii] respectively, and the proportional constants u and d are the multiplicative factors applied to the initial stock price to form the time-h stock price. Between time 0 and time h, the stock price stays put at the initial stock price. Because the stock price in h years is a two-point (hence binomial) random variable and the evolution of the stock price, depicted in Figure 4.1.1, resembles a tree[iv], we also refer to a binomial stock price model as a *binomial tree model*. To visualize the development of future stock prices, tree diagrams such as the one in Figure 4.1.1 will be intensively used in this chapter. For expository convenience, we say that we are in the u *node* (resp. d *node*) of the binomial tree when the time-h stock price is S_u (resp. S_d).[v] The initial node is nothing but a synonym for the initial time.

In the context of this one-period binomial model, we are interested in a generic European derivative (e.g., call, put, bull spread, straddle, etc.) maturing at time h written on the above stock with the following payoff structure:

- If the stock price at time h is S_u, then the payoff of the derivative is V_u.
- If the stock price at time h is S_d, then the payoff of the derivative is V_d.

Our principal objective is to determine the time-0 price (or current price) of this derivative, denoted by V_0.

Pricing by replication.

As with forwards, the fundamental idea behind pricing options is replication: to invest in assets available in the market to replicate the payoffs of the security of interest. In the current binomial tree setting, we have two assets at our disposal: (1) the risky stock; and (2) the risk-free bond. If we can invest in these two assets in such a way that the payoff of the resulting portfolio is always identical to that of the concerned derivative, then the no-arbitrage principle mandates that the fair price of the derivative coincides with the cost of setting up this portfolio.

Mathematically, suppose that our portfolio consists of Δ shares of the stock and \$$B$ in the risk-free bond (i.e., lending \$$B$; you may regard B as "bonds"). The cost of this

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[i] In the binomial tree framework, we will only treat stocks paying continuous proportional dividends. Binomial tree models involving stocks that pay discrete dividends are studied, for example, in Section 11.4 of McDonald (2013).

[ii] In binomial tree models, the symbol h typically denotes the length of each period. In a one-period model, it also equals the entire duration of the tree T.

[iii] Associating "d" with "down" can occasionally be misleading. As we will see in (4.1.6), it is possible that $d \geq 1$ if r is substantially larger than δ.

[iv] The tree diagram, although aesthetically appealing, may encourage the misguided impression that the evolution of the stock price is uniform over time. In reality, the stock price stays put at S_0 and moves momentarily to either S_u or S_d at the end of the period.

[v] In a break with the notation in other parts of this book, where we write $S(t)$ as the time-t price of the underlying stock, in binomial tree models we distinguish stock prices by their locations in the binomial tree (e.g., S_0, S_u, S_d), with our attention paid to not only the time point of interest, but also the particular stock node in question.

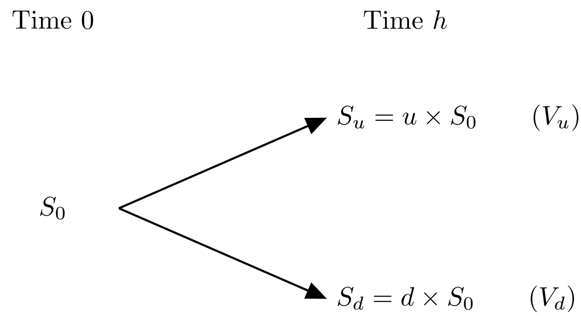


FIGURE 4.1.1
A generic one-period binomial stock price model. The derivative payoffs are shown in parentheses.

portfolio is $\Delta S_0 + B$, and its payoff at time h is

$$\begin{cases} (\Delta e^{\delta h})S_u + Be^{rh}, & \text{if the time-}h\text{ stock price is } S_u, \\ (\Delta e^{\delta h})S_d + Be^{rh}, & \text{if the time-}h\text{ stock price is } S_d. \end{cases}$$

Note that Δ shares of the stock at time 0 grow, because of the reinvestment of dividends, to $\Delta e^{\delta h}$ shares at time h. We try to choose Δ and B so that the payoff of the portfolio matches that of the derivative under all circumstances, i.e., Δ and B satisfy

$$\begin{cases} \Delta S_u e^{\delta h} + Be^{rh} = V_u \\ \Delta S_d e^{\delta h} + Be^{rh} = V_d \end{cases}.$$

This 2×2 linear system in the two variables Δ and B can be easily solved to yield[vi]

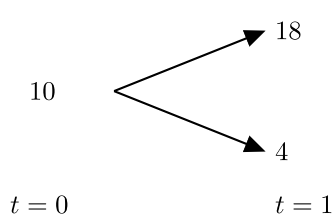
$$\boxed{\Delta = e^{-\delta h}\left(\frac{V_u - V_d}{S_u - S_d}\right) = e^{-\delta h}\left[\frac{V_u - V_d}{S_0(u-d)}\right], \quad B = e^{-rh}\left(\frac{uV_d - dV_u}{u-d}\right).} \tag{4.1.1}$$

With these choices of Δ and B, the portfolio successfully replicates the derivative in terms of payoff regardless of the level of the time-h stock price. For this reason, we call this portfolio the *replicating portfolio*, represented by (Δ, B). Consequently, the fair time-0 price of the derivative should equal the cost of the replicating portfolio, which in turn is

$$\boxed{V_0 = \Delta S_0 + B,} \tag{4.1.2}$$

with Δ and B given in (4.1.1). Our pricing task is thus completed!

Example 4.1.1. (CAS Exam 3 Spring 2007 Question 16: Calculating Δ) A nondividend-paying stock, S, is modeled by the binomial tree shown below.



[vi]We assume that $S_u \neq S_d$, so that the stock is genuinely risky.

A European call option on S expires at $t = 1$ with strike price $K = 12$.
Calculate the number of shares of stock in the replicating portfolio for this option.

(A) Less than 0.3

(B) At least 0.3, but less than 0.4

(C) At least 0.4, but less than 0.5

(D) At least 0.5, but less than 0.6

(E) At least 0.6

Solution. Note that the payoff of the call is either $C_u = (18 - 12)_+ = 6$ or $C_d = (4 - 12)_+ = 0$. By (4.1.1), the number of shares of stock in the replicating portfolio for the call is

$$\Delta = e^{-\delta h}\left(\frac{C_u - C_d}{S_u - S_d}\right) = \frac{6-0}{18-4} = \boxed{0.4286}. \quad \textbf{(Answer: (C))}$$

□

Example 4.1.2. (CAS Exam 8 Spring 2004 Question 27: Calculating call price) The price of a nondividend-paying stock is currently \$50.00. It is known that at the end of two months, it will be either \$54.00 or \$46.00. The risk-free interest rate is 9.0% per annum with continuous compounding.

Calculate the value of a two-month European call option with a strike price of \$48 on this stock. Show all work.

Solution. Note that the tree parameters are $u = 54/50 = 1.08$ and $d = 46/50 = 0.92$, and the payoff of the call is either $C_u = (54 - 48)_+ = 6$ or $C_d = (46 - 48)_+ = 0$. By (4.1.1),

$$\Delta = e^{-\delta h}\left(\frac{C_u - C_d}{S_u - S_d}\right) = \frac{6-0}{54-46} = 0.75$$

and

$$B = e^{-rh}\left(\frac{uC_d - dC_u}{u-d}\right) = e^{-0.09(2/12)}\left[\frac{-0.92(6)}{1.08-0.92}\right] = -33.9864.$$

The call price is

$$C_0 = \Delta S_0 + B = 0.75(50) - 33.9864 = \boxed{3.5136}.$$

□

Example 4.1.3. (SOA Exam MFE Spring 2007 Question 14: Our old friend from Chapter 3 – straddle!) For a one-year straddle on a nondividend-paying stock, you are given:

(i) The straddle can only be exercised at the end of one year.

(ii) The payoff of the straddle is the absolute value of the difference between the strike price and the stock price at expiration date.

(iii) The stock currently sells for \$60.

(iv) The continuously compounded risk-free interest rate is 8%.

(v) In one year, the stock will either sell for \$70 or \$45.

(vi) The option has a strike price of \$50.00.

Calculate the current price of the straddle.

(A) \$0.90

(B) \$4.80

(C) \$9.30

(D) \$14.80

(E) \$15.70

Ambrose's comments:

Using a binomial tree, it is easy to value any derivatives, not necessarily calls or puts, however complicated their payoffs are. Even a straddle can be easily handled!

Solution. Consider a replicating portfolio consisting of Δ shares of stock and B amount of cash. Matching the payoffs in the up and down scenarios, we solve

$$\begin{cases} 70\Delta + e^{0.08}B = |50 - 70| = 20, \\ 45\Delta + e^{0.08}B = |50 - 45| = 5, \end{cases}$$

which gives $\Delta = 0.6$ and $B = -20.3086$. The current price of the derivative is the same as the portfolio value at time 0, which is

$$60(0.6) + (-20.3086) = \boxed{15.6914}. \quad \textbf{(Answer: (E))}$$

□

Remark. We will revisit this example in the next subsection using an alternative method.

4.1.2 Risk-neutral Pricing

Pricing formula viewed as a discounted expectation.

Pricing via replication is the fundamental, economically correct way of determining the fair price of a derivative. The actual computations, as you experienced in Subsection 4.1.1, can be cumbersome and lack insights. It turns out that the pricing formula given in equation (4.1.2) can be rearranged into a form that admits a very useful probabilistic interpretation.

To see this, we plug in the expressions for Δ and B into the pricing formula, yielding

$$\begin{aligned} V_0 &= \Delta S_0 + B \\ &= \mathrm{e}^{-\delta h}\left[\frac{V_u - V_d}{S_0(u-d)}\right] S_0 + \mathrm{e}^{-rh}\left(\frac{uV_d - dV_u}{u-d}\right) \\ &= \mathrm{e}^{-rh}\left[\left(\frac{\mathrm{e}^{(r-\delta)h} - d}{u-d}\right) V_u + \left(\frac{u - \mathrm{e}^{(r-\delta)h}}{u-d}\right) V_d\right]. \end{aligned}$$

If we define

$$\boxed{p^* = \frac{\mathrm{e}^{(r-\delta)h} - d}{u-d},} \tag{4.1.3}$$

then the time-0 price of the derivative can be written more compactly as

$$\boxed{V_0 = \mathrm{e}^{-rh}[p^* V_u + (1-p^*)V_d].} \tag{4.1.4}$$

We will show shortly that p^* lies between 0 and 1 for the binomial model to make sense (see Subsection 4.1.3). For now, it is enough to notice that: If we "regard" p^* as the "probability" that the stock price will go up in h years, then $p^* V_u + (1-p^*)V_d$ resembles the "expected value" of the derivative payoff at expiration. (4.1.4) then says that the price of the derivative can be interpreted as the *expected derivative payoff, discounted at the risk-free interest rate*:

$$V_0 = \mathrm{e}^{-rh}\mathbb{E}^*[\text{Payoff at expiration}]. \tag{4.1.5}$$

Here we use the asterisk ($*$) to emphasize that the expectation $\mathbb{E}^*[\cdot]$ is computed under the assumption that p^* is the probability that the stock price will increase at the end of the period. It should be firmly kept in mind that p^* is generally not the same as the probability that the stock will go up in price *in the real world*. It is an artificial, mathematical quantity which, when thought of as a probability, allows one to associate equation (4.1.4) with a discounted expectation.

What is so special about p^*?

The probability-looking quantity p^* is commonly designated as the *risk-neutral probability* (also known as pseudo-probabilities) of an increase in the stock price (or simply an up move), and equations (4.1.4) or (4.1.5) as the *risk-neutral pricing formula*. To explain the terminology, it will be useful to look at how a *risk-neutral* investor behaves. Such an investor cares only about the expected return on an asset, not on its other characteristics such as riskiness. He/she will be indifferent between the risky stock and the risk-free bond if both of them earn the same rate of r. In fact, p^* and $1-p^*$ are the unique "probabilities" such that this indifference prevails, with the initial price of the risky stock equal to the discounted expected stock price *with dividends*:

$$S_0 = \mathrm{e}^{-rh} \times \mathrm{e}^{\delta h}[p^* S_u + (1-p^*)S_d] = \mathrm{e}^{-rh} \times \mathrm{e}^{\delta h}\mathbb{E}^*[\text{time-}h \text{ stock price}].$$

This can be readily checked by

$$\begin{aligned} \mathrm{e}^{-rh} \times \mathrm{e}^{\delta h}[p^* S_u + (1-p^*)S_d] &= S_0 \mathrm{e}^{-(r-\delta)h}[p^* u + (1-p^*)d] \\ &= S_0 \mathrm{e}^{-(r-\delta)h}\left[u\left(\frac{\mathrm{e}^{(r-\delta)h} - d}{u-d}\right) + d\left(\frac{u - \mathrm{e}^{(r-\delta)h}}{u-d}\right)\right] \\ &= S_0 \mathrm{e}^{-(r-\delta)h} \times \frac{(u-d)\mathrm{e}^{(r-\delta)h}}{u-d} \\ &= S_0. \end{aligned}$$

At all other probabilities, the investor would strictly prefer the risky stock or the risk-free bond. This unique association with risk-neutrality makes p^* be termed the risk-neutral probability.

Again, it is imperative to emphasize that we are not assuming that investors in practice are risk-neutral. Rather, risk-neutral pricing provides an interpretation of the pricing formula presented in (4.1.4), which in turn arises from the no-arbitrage hedging argument.

To recap, the use of the risk-neutral pricing formula involves a simple three-step procedure:

Step 1. Identify the risk-neutral probability p^* that the stock price will go up.

Step 2. Calculate the expected payoff of the derivative under the probabilities p^* and $1-p^*$.

Step 3. Discount the expected payoff in Step 2 at the risk-free interest rate.

Example 4.1.4. (Example 4.1.3 revisited) Rework Example 4.1.3 using risk-neutral valuation.

Solution. The risk-neutral probability of an "up" movement is

$$p^* = \frac{60e^{0.08} - 45}{70 - 45} = 0.7999.$$

Using risk-neutral valuation, the current price of the straddle is

$$e^{-0.08}[20p^* + 5(1-p^*)] = \boxed{15.6914}. \quad \textbf{(Answer: (E))}$$

□

An intuitive explanation of why risk-neutral pricing works.

Risk-neutral pricing is one of the deepest results in modern finance. Not only does it have no obvious connections to replication (which is the origin of risk-neutral pricing), it also transforms the pricing problem to the computation of the *expected present value* of the derivative payoff, a concept that is familiar to actuaries (perhaps you have computed expected present values in a life contingencies course!). However, this expected present value is puzzling in two respects:

(i) The expectation is taken with respect to the so-called risk-neutral probability measure, which is generally different from the real probability measure.

(ii) The discounting is performed using the risk-free interest rate, not a risk-adjusted rate that reflects the riskiness of the derivative.

Useful and elegant as it is, risk-neutral pricing is at the same time very perplexing and counterintuitive. The inevitable question is:

> Why can we simply "pretend" that investors are risk-neutral and that the derivative earns the risk-free rate r? Why?

An intuitive explanation of the "validity" of risk-neutral pricing, though not possibly tested in a multiple-choice exam environment, is clearly in order from an educational point of view.

To better understand why risk-neutral pricing works, consider the following two worlds, both of which consist of the same set of assets having the same current prices and realizable future prices, but with different probabilities of attaining these future prices:

World 1. This is the real world in which we live. The real probability that the stock price will go up is p (see Section 4.5 for more discussions on how real probabilities can be used in the valuation procedure).

World 2. This is the risk-neutral world in which investors care only about expected returns, the probability that the stock price will go up is the risk-neutral probability p^*, and the stock earns the risk-free rate.

Now we perform the replication procedure described in Subsection 4.1.1 and obtain the two replicating portfolios in the two worlds. Because the composition of the replicating portfolios does not depend on the probabilities of attaining different future stock prices, the two worlds actually share the same replicating portfolio. The price of the derivative, which is equal to the cost of setting up the common replicating portfolio, turns out to be the same in the two worlds! The important implication is that to price the derivative in the real world, we may instead look at how much the derivative would cost in the risk-neutral world. This risk-neutral price takes the form of the discounted risk-neutral expectation

$$(V_0^{\text{real}} =) V_0^{\text{risk-neutral}} = \mathrm{e}^{-rT}\mathbb{E}^*[\text{Derivative terminal payoff}]$$

by the very definition of the risk-neutral world.

Can we forget about the method of replication?

Armed with the risk-neutral interpretation of the pricing formula, it becomes unnecessary to determine the replicating portfolio (Δ, B) using the non-intuitive formulas given in (4.1.1). All we have to do is to calculate the risk-neutral probability p^* using equation (4.1.3), the risk-neutral expected derivative payoff, and discount it at the risk-free rate. However, it does not mean the method of replicating portfolio has no value. In fact, the uses of replicating portfolios are (at least) three-fold:

(1) From a theoretical point of view, replicating portfolios form the basis for risk-neutral valuation. Without the method of replicating portfolios, the validity of risk-neutral pricing may become ill-founded.

(2) Practically, the method of replicating portfolios provides a *synthetic construction* of any derivative of interest. To be precise, any derivative is financially equivalent to buying Δ shares of the stock and investing $\$B$ in risk-free bonds, where Δ and B are given in (4.1.1). This synthetic construction provides the recipe for an arbitrage strategy when the observed market price deviates from the fair derivative price.

(3) The method of replicating portfolios works even for multinomial trees, in which the stock price can take multiple values at the end of the model period, whereas it is not immediately clear what the risk-neutral probabilities should be.

Example 4.1.5 below illustrates the second use and Example 4.1.6 the third use.

Example 4.1.5. (SOA Exam MFE Spring 2009 Question 3: Arbitraging a mispriced call) You are given the following regarding stock of Widget World Wide (WWW):

(i) The stock is currently selling for \$50.

(ii) One year from now the stock will sell for either \$40 or \$55.

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 10%.

The continuously compounded risk-free interest rate is 5%.

While reading the *Financial Post*, Michael notices that a one-year at-the-money European call written on stock WWW is selling for \$1.90. Michael wonders whether this call is fairly priced. He uses the binomial option pricing model to determine if an arbitrage opportunity exists.

What transactions should Michael enter into to exploit the arbitrage opportunity (if one exists)?

(A) No arbitrage opportunity exists.

(B) Short shares of WWW, lend at the risk-free rate, and buy the call priced at \$1.90.

(C) Buy shares of WWW, borrow at the risk-free rate, and buy the call priced at \$1.90.

(D) Buy shares of WWW, borrow at the risk-free rate, and short the call priced at \$1.90.

(E) Short shares of WWW, borrow at the risk-free rate, and short the call priced at \$1.90.

Solution. The replicating portfolio of the given call is defined by

$$\Delta = \mathrm{e}^{-0.1}\left(\frac{5-0}{55-40}\right) = 0.3016$$

and

$$B = \mathrm{e}^{-0.05}\left(\frac{1.1\times 0 - 0.8\times 5}{1.1-0.8}\right) = -12.6831.$$

It follows that the fair price of the call is $C_0^{\text{fair}} = \Delta S_0 + B = 2.3969$, which is higher than the observed price of 1.9. This means that the observed call is underpriced.

To exploit the arbitrage opportunity, Michael should "buy low and sell high," i.e., purchase the call option at the observed price of \$1.9 and short sell the replicating portfolio, which means shorting 0.3016 shares of the stock and lending \$12.6831 at the risk-free rate, for \$2.3969 **(Answer: (B))**. Currently, he receives \$0.4969. One year from now, his overall payoff will be constant at

$$\underbrace{(S(1)-50)_+}_{\text{long observed call}} - \underbrace{(S(1)-50)_+}_{\text{short replicating portfolio}} = 0.$$

□

Remark. A two-period version of this example is Problem 4.6.15.

We end this subsection with an example involving a multinomial tree model, in which the asset price at expiration can take multiple values.

Example 4.1.6. (SOA Exam MFE Advanced Derivatives Sample Question 27: A three-state world) You are given the following information about a securities market:

- There are two nondividend-paying stocks, X and Y.
- The current prices for X and Y are both \$100.
- The continuously compounded risk-free interest rate is 10%.
- There are three possible outcomes for the prices of X and Y one year from now:

Outcome	X	Y
1	\$200	\$0
2	\$50	\$0
3	\$0	\$300

Let C_X be the price of a European call option on X, and P_Y be the price of a European put option on Y. Both options expire in one year and have a strike price of \$95.

Calculate $P_Y - C_X$.

(A) \$4.30

(B) \$4.45

(C) \$4.59

(D) \$4.75

(E) \$4.94

Ambrose's comments:

Although there are three possible outcomes, the philosophy of the method of replicating portfolio is exactly the same as that in the binomial model.

Solution. Instead of calculating P_Y and C_X separately, we view regard $P_Y - C_X$ as the time-0 price of the portfolio consisting of a long put on Y and a short call on X. The payoff of this "put-less-call" portfolio in one year is simply the payoff of the put on Y less the payoff of the call on X:

Outcome	Payoff of Put on Y	Payoff of Call on X	Total Payoff
1	$(95-0)_+ = 95$	$(200-95)_+ = 105$	-10
2	$(95-0)_+ = 95$	$(50-95)_+ = 0$	95
3	$(95-300)_+ = 0$	$(0-95)_+ = 0$	0

We now replicate the payoff of the portfolio using Δ_X shares of X, Δ_Y shares of Y, and \$$B$ in the risk-free bond. Solving the 3×3 linear system

$$\begin{cases} e^{0.1}B + 200\Delta_X = -10, \\ e^{0.1}B + 50\Delta_X = 95, \\ e^{0.1}B + 300\Delta_Y = 0, \end{cases}$$

gives $\Delta_X = -0.7$, $B = 117.6289$ and $\Delta_Y = -13/30$. The cost of the portfolio then equals the cost of the replicating portfolio, which in turn is

$$P_Y - C_X = B + 100\Delta_X + 100\Delta_Y = \boxed{4.2955}. \quad \textbf{(Answer: (A))}$$

□

Remark. Stocks X and Y can be said to be *mutually exclusive*, in that whenever one of them has a positive payoff, then the payoff of the other must be zero.

4.1.3 Constructing a Binomial Tree

In the previous subsections, we take the binomial tree as given and perform the valuation task. We now discuss how a binomial tree should be constructed (i.e., how the growth factors u and d should be appropriately chosen) taking stock market data into account.

No-arbitrage restrictions on u and d.

To preclude arbitrage opportunities, any reasonable binomial tree model should satisfy

$$d < \mathrm{e}^{(r-\delta)h} < u. \tag{4.1.6}$$

In other words, u cannot be too small whereas d cannot be too large. To make sense of these restrictions, we first multiply every quantity by the initial stock price, giving

$$S_d = S_0 \times d < S_0\mathrm{e}^{(r-\delta)h} < S_0 \times u = S_u.$$

Putting the factor $\mathrm{e}^{-\delta h}$ in the middle to both sides, we further rearrange (4.1.6) equivalently as

$$\underbrace{S_d\mathrm{e}^{\delta h}}_{② \text{ in } d \text{ node}} < \underbrace{S_0\mathrm{e}^{rh}}_{①} < \underbrace{S_u\mathrm{e}^{\delta h}}_{② \text{ in } u \text{ node}}. \tag{4.1.7}$$

This chain of inequalities then imply that neither of the following two strategies with the same initial investment of S_0 in the binomial stock market is always dominant over another:

① Using $\$S_0$ to buy a risk-free bond that earns the risk-free interest rate r.

② Using $\$S_0$ to buy the risky stock that pays dividends continuously at the rate of δ.

The time-h payoff of Strategy ① is always $S_0\mathrm{e}^{rh}$, while that of Strategy ② is $S_u\mathrm{e}^{\delta h}$ in the u node or $S_d\mathrm{e}^{\delta h}$ in the d node. Then (4.1.6), or equivalently, (4.1.7), says that Strategy ① is inferior to Strategy ② in the u node but superior to Strategy ② in the d node. In a fair market, no strategy is universally the winner!

The no-arbitrage conditions (4.1.6) effectively mean that the risk-neutral probability p^* is valued between 0 and 1:

$$0 = \frac{d-d}{u-d} < p^* = \frac{\mathrm{e}^{(r-\delta)h} - d}{u-d} < \frac{u-d}{u-d} = 1.$$

This justifies viewing p^* as a probability.

Example 4.1.7. (SOA Course 6 Spring 2001 Multiple-Choice Question 3: What can r possibly be?) An arbitrage-free securities market model consists of a bank account and one security. The security price today is 100. The security price one year from now will be either 104 or 107.

Determine which of the following can be the bank account interest rate.

(A) 0%

(B) 3%

(C) 5%

(D) 8%

(E) 10%

Solution. Since the securities market model is arbitrage-free, we can use (4.1.6), which says that (assume no dividends)

$$1.04 < \mathrm{e}^r < 1.07,$$

or $\boxed{3.92\% < r < 6.77\%}$. **(Answer: (C))** □

Volatility.

The tree parameters u and d control the size of the "jaw" of the binomial tree and their selection naturally depends on the riskiness of the stock price. A formal measure of the variability of the stock price is given by its *volatility*, which is defined as the *annualized* standard deviation of its continuously compounded returns. It is given mathematically by

$$\boxed{\sigma = \sqrt{\frac{1}{h}\mathrm{Var}\left[\ln\left(\frac{S(h)}{S(0)}\right)\right]} \overset{\text{(why?)}}{=} \sqrt{\frac{\mathrm{Var}[\ln S(h)]}{h}},}$$

where $S(h)$ is the time-h (random) stock price (equal to either S_u or S_d). Note that the division by $\sqrt{h}$ serves to annualize the volatility. In general, under the assumption of independent and identically distributed continuously compounded returns over disjoint time periods, an h-year volatility σ_h and the annual volatility σ are related via

$$\sigma = \frac{\sigma_h}{\sqrt{h}}.$$

A common method of construction: Forward tree.

One popular way of constructing a binomial tree is that of the *forward tree*,[vii] which requires setting u and d to be

$$\boxed{u = \mathrm{e}^{(r-\delta)h+\sigma\sqrt{h}} \quad \text{and} \quad d = \mathrm{e}^{(r-\delta)h-\sigma\sqrt{h}}.} \tag{4.1.8}$$

These choices of u and d naturally satisfy the no-arbitrage conditions (4.1.6).

[vii] The term "forward tree" comes from page 303 of McDonald (2013).

To understand why a forward tree is so called, recall from Section 2.3 that the h-year forward price of the stock is $F_{0,h} = S(0)\mathrm{e}^{(r-\delta)h}$. Since

$$
\begin{aligned}
S_u &= S(0)\mathrm{e}^{(r-\delta)h+\sigma\sqrt{h}} = F_{0,h}\mathrm{e}^{\sigma\sqrt{h}},\\
S_d &= S(0)\mathrm{e}^{(r-\delta)h-\sigma\sqrt{h}} = F_{0,h}\mathrm{e}^{-\sigma\sqrt{h}},
\end{aligned}
$$

uncertainty about the future stock price is introduced using the forward price $F_{0,h}$ as the benchmark, with $\mathrm{e}^{\sigma\sqrt{h}}$ and $\mathrm{e}^{-\sigma\sqrt{h}}$ serving as the multiplicative increment and decrement factors.

The forward tree has an inadvertent implication for the risk-neutral probability. We plug (4.1.8) into (4.1.3), yielding

$$
p^* = \frac{\mathrm{e}^{(r-\delta)h} - d}{u-d} = \frac{\mathrm{e}^{(r-\delta)h} - \mathrm{e}^{(r-\delta)h-\sigma\sqrt{h}}}{\mathrm{e}^{(r-\delta)h+\sigma\sqrt{h}} - \mathrm{e}^{(r-\delta)h-\sigma\sqrt{h}}} = \boxed{\frac{1}{1+\mathrm{e}^{\sigma\sqrt{h}}}}. \tag{4.1.9}
$$

(Warning: $p^* \neq (1+\mathrm{e}^{-\sigma\sqrt{h}})^{-1}$!) This simplified form of the risk-neutral probability means that it is possible to find p^* without knowing u, d, r and δ. This can bring huge computational convenience in an exam setting. However, the fact that $\sigma > 0$ forces $p^* < 1/2 < 1 - p^*$, which may be viewed as a built-in bias in the forward tree model.

Example 4.1.8. (CAS Exam 3 Spring 2007 Question 15: Calculation of p^* in a forward tree) For a binomial option pricing model, you are given the following information:

- The current stock price is \$110.
- The strike price is \$100.
- The interest rate is 5% (continuously compounded)
- The continuous dividend yield is 3.5%.
- The volatility is 0.30.
- The time to expiration is 1 year.
- The length of period is 4 months.

Compute the risk-neutral probability of an increase in the stock price over one period.

(A) Less than 0.425

(B) At least 0.425, but less than 0.445

(C) At least 0.445, but less than 0.465

(D) At least 0.465, but less than 0.485

(E) At least 0.485

Solution. Although the question doesn't state it, a tree based on forward prices is assumed. By (4.1.9), the risk-neutral probability of an up move is

$$
p^* = \frac{1}{1+\mathrm{e}^{\sigma\sqrt{h}}} = \frac{1}{1+\mathrm{e}^{0.3\sqrt{4/12}}} = \boxed{0.4568}. \quad \textbf{(Answer: (C))}
$$

□

Remark. There is no need to compute u and d.

Example 4.1.9. (CAS Exam 3 Fall 2007 Question 19: Calculation of call price in a forward tree) A three-month European call is modeled by a single period binomial tree using the following parameters:

- Continuously compounded risk-free rate = 4%
- Dividend = 0
- Annual volatility = 15%
- Current stock price = 10
- Strike price = 10.5

Calculate the value of the call option.

(A) Less than 0.15

(B) At least 0.15, but less than 0.30

(C) At least 0.30, but less than 0.45

(D) At least 0.45, but less than 0.60

(E) At least 0.60

Solution. The tree parameters are

$$\begin{aligned} u &= \mathrm{e}^{0.04(0.25)+0.15\sqrt{0.25}} = 1.088717, \\ d &= \mathrm{e}^{0.04(0.25)-0.15\sqrt{0.25}} = 0.937067. \end{aligned}$$

The risk-neutral probability of an up move is

$$p^* = \frac{1}{1+\mathrm{e}^{\sigma\sqrt{h}}} = \frac{1}{1+\mathrm{e}^{0.15\sqrt{0.25}}} = 0.481259.$$

As $C_u = (10 \times 1.088717 - 10.5)_+ = 0.387170$ and $C_d = (10 \times 0.937067 - 10.5)_+ = 0$, the risk-neutral pricing formula says that

$$C_0 = \mathrm{e}^{-0.04(0.25)}(0.481259)(0.387170) = \boxed{0.1845}. \quad \textbf{(Answer: (B))}$$

□

Example 4.1.10. (CAS Exam 3 Fall 2007 Question 23: Given the call price, what is σ?) A one-year European call option is currently valued at 0.9645. The following parameters are given.

- Current stock price = 10
- Continuously compounded risk-free rate = 6%
- Continuously compounded dividend rate = 1%
- Strike price = 10

Using a single period binomial tree, calculate the implied volatility of the stock, assuming that it is greater than 5%.

(A) Less than 0.10

(B) At least 0.10, but less than 0.20

(C) At least 0.20, but less than 0.30

(D) At least 0.30, but less than 0.40

(E) At least 0.40

Solution. Because $\sigma > 0.05$, we have $d = \mathrm{e}^{0.05-\sigma} < 1$, so the call does not pay off at the d node. Using the risk-neutral pricing formula,

$$0.9645 = \mathrm{e}^{-rh} p^*(C_u - K)_+ = \frac{\mathrm{e}^{-0.06}}{1+\mathrm{e}^{\sigma}}(10\mathrm{e}^{0.05+\sigma} - 10),$$

which results in $\mathrm{e}^{\sigma} = 1.161834$ and $\sigma = \boxed{0.15}$. **(Answer: (B))** □

Sidebar: How to estimate volatility given past stock price data?[viii]

From the above discussions, it goes without saying that σ is a very important input for constructing a binomial tree. As σ is not a directly observable quantity, how can the value of σ be selected?

Suppose that $n+1$ stock prices, $S(ih)$ for $i = 0, 1, 2, \ldots, n$ are observed, where h is length of time (in years) between adjacent observations. For example, if $h = 1/12$, then the stock prices are observed at consecutive monthly intervals. These $n + 1$ successive stock prices are neither independent nor identically distributed—knowledge of the preceding stock price dictates the possible values that can be assumed by the succeeding stock price. However, the *observed* (non-annualized) continuously compounded returns, denoted by $r_1, r_2, \ldots, r_n$ and defined as

$$r_i = \ln \frac{S(ih)}{S[(i-1)h]}, \quad \text{for } i = 1, 2, \ldots, n,$$

are independent and identically distributed (i.i.d.)—they are either $\ln u$ or $\ln d$, independently of each other—with common variance

$$\mathrm{Var}(r_1) = \mathrm{Var}\left[\ln\left(\frac{S(h)}{S(0)}\right)\right] = \sigma^2 h. \tag{4.1.10}$$

Using these n i.i.d. observed returns, $r_1, r_2, \ldots, r_n$, we are back to the standard i.i.d. setting in statistics for estimating the unknown variance of a population.

Method-of-moments estimate of σ.

In statistics, there are a multitude of methods for estimating the value of an unknown parameter from observed data. The current setting is no exception. One conceptually very simple estimation method, based on Equation (4.1.10), is the *method of moments* (or MOM in short). The idea of MOM is to consider the sample version of (4.1.10), with the population

[viii] The estimation procedure here also applies to the Black-Scholes framework.

variance $\mathrm{Var}(r_1)$ replaced by its sample counterpart computable from the returns data, and solve for the resulting value of σ. In essence, we pick the value of σ in order to match the sample variance (this is why MOM is also known as moment matching) and designate such a value the estimate of σ.

To determine the MOM estimate of σ, we equate, by virtue of Equation (4.1.10), the sample variance of $r_1, r_2, \ldots, r_n$ and $\sigma^2 h$. The sample variance of the r_i's is

$$S_r^2 = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2 = \frac{n}{n-1}\left(\overline{r^2} - \bar{r}^2\right),$$

where $\bar{r}$ and $\overline{r^2}$ are the sample mean and sample second moment given by

$$\bar{r} = \frac{\sum_{i=1}^{n} r_i}{n} = \frac{1}{n}\ln\frac{S(nh)}{S(0)} \quad \text{and} \quad \underbrace{\overline{r^2} = \frac{\sum_{i=1}^{n} r_i^2}{n}}_{\text{not to be confused with } (\bar{r})^2!}.$$

Hence the MOM estimate of σ is

$$\boxed{\hat{\sigma} = \sqrt{\frac{S_r^2}{h}}.}$$

Note that:

- The division by $n-1$ makes the sample variance of the r_i's unbiased for their true variance.
- The division by h serves to annualize the estimate of σ.

The estimate of σ obtained this way is also known as *historical volatility*, because it is based on historical stock prices and stock returns.

Example 4.1.11. (SOA Exam IFM Advanced Derivatives Sample Question 17: Estimating σ) You are to estimate a nondividend-paying stock's annualized volatility using its prices in the past nine months.

Month	Stock Price ($/share)
1	80
2	64
3	80
4	64
5	80
6	100
7	80
8	64
9	80

Calculate the historical volatility for this stock over the period.

(A) 83%

(B) 77%

(C) 24%

(D) 22%

(E) 20%

Solution. Let r_i be the continuously compounded monthly returns for the i^{th} month. Then:

Month i	$r_i = \ln\{S(ih)/S[(i-1)h]\}$
1	—
2	$\ln(64/80) = \ln 0.8$
3	$\ln(80/64) = \ln 1.25$
4	$\ln(64/80) = \ln 0.8$
5	$\ln(80/64) = \ln 1.25$
6	$\ln(100/80) = \ln 1.25$
7	$\ln(80/100) = \ln 0.8$
8	$\ln(64/80) = \ln 0.8$
9	$\ln(80/64) = \ln 1.25$

Note that four of the r_i's are $\ln 1.25$ and the other four are $\ln 0.8 = -\ln 1.25$. In particular, their mean $\bar{r}$ is zero.

The (unbiased) sample variance of the non-annualized monthly returns is

$$\hat{\sigma}^2_{1/12} = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2 = \frac{1}{7}\sum_{i=1}^{8} r_i^2 = \frac{8}{7}(\ln 1.25)^2,$$

and the estimated annualized volatility is

$$\hat{\sigma} = \frac{\hat{\sigma}_{1/12}}{\sqrt{1/12}} = \sqrt{12} \times \sqrt{\frac{8}{7}(\ln 1.25)^2} = \boxed{82.64\%}. \quad \textbf{(Answer: (A))}$$

□

4.2 Multi-period Binomial Trees

An obvious objection to the single-period binomial tree model presented in the preceding section is that the stock price at expiration can only take two possible values. This restriction can be easily overcome by dividing the time to expiration into several periods, in each of which the stock price evolves according to a one-period binomial model. This process generates a more realistic multi-period binomial tree where the asset price at expiration can take any finite number of values, at the expense of substantially more intensive computations. In this section, we build upon the foundation laid in Section 4.1 and discuss the valuation of options on assets whose price movements are driven by multi-period binomial trees.

How to represent a multi-period binomial tree?

Figure 4.2.1 shows a two-period binomial stock price tree, in which the length of each period is h and the duration of the whole tree is $T = 2h$. At the end of the first period, there are two possible stock prices, namely S_u and S_d. At the end of the second period, each of these

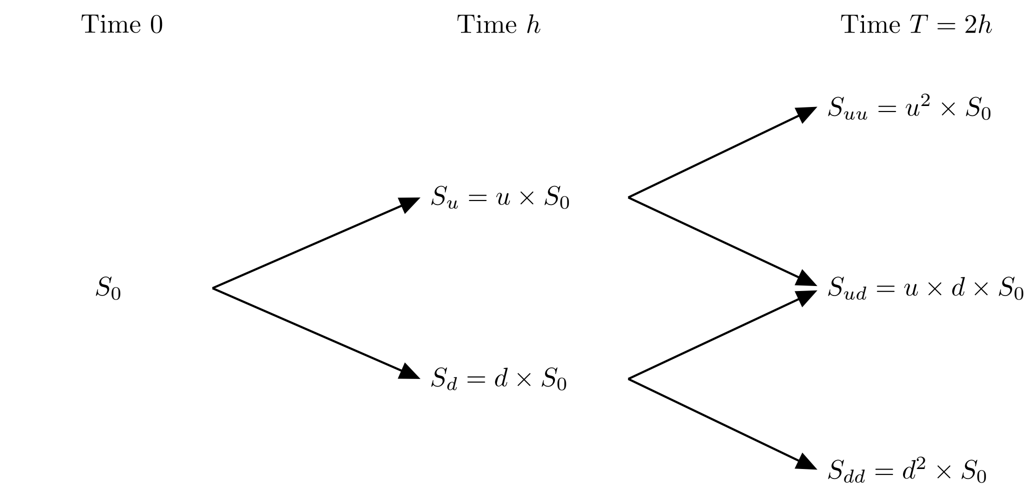


FIGURE 4.2.1
A generic two-period binomial stock price tree.

two prices can itself go up by u or down by d. In this two-period setting, the generic stock price symbol $S_{(\cdot)(\cdot)}$ is decorated by a subscript that represents chronologically the precise path through which the stock price passes over the two periods. For example, S_{uu} is the two-period stock price corresponding to the uu path (i.e., two consecutive up moves), and S_{ud} is the two-period stock price corresponding to an up move followed by a down move. Such notation will carry over to a general multi-period binomial tree.

One conspicuous feature of the stock price tree in Figure 4.2.1 is that the stock prices move up or down by the same factors u and d at the end of each period. The effect is $S_{ud} = S_{du}$, so there are only three distinct two-period stock prices. In general, a binomial tree in which an up move followed by a down move leads to the same price as a down move followed by an up move is said to be *recombining*. The use of a recombining binomial tree provides huge computational ease—an n-period binomial model only has $n+1$ distinct terminal prices, in contrast to 2^n for a non-recombining one, and is an attempt to balance the flexibility and tractability of the stock price model. Unless otherwise stated, in this book we will always use a recombining binomial tree with the same growth factors u and d in each period.

The method of backward induction.

The key to valuing options using a multi-period binomial tree is to *work backward through the binomial tree*. We start from the option payoffs at expiration and determine the possible values of the option one period before expiration corresponding to different stock price paths. This valuation task involves only one-period binomial trees and therefore can be accomplished by what we learnt in Section 4.1. Then we use these option values one period before expiration to compute the option values two periods before expiration. This process is repeated recursively until we reach the initial node and obtain the price of the option at time 0. The essence of this backward induction procedure lies in reducing the multi-period valuation problem to a series of one-period valuation problems.

Let's use the following CAS exam question to illustrate the intricate ideas.

Example 4.2.1. (CAS Exam 3 Spring 2007 Question 14: First encounter with a two-period binomial tree) Consider the following information about a European call option on stock ABC:

- The strike price is \$95.
- The current stock price is \$100.
- The time to expiration is 2 years.
- The continuously compounded risk-free rate is 5% annually.
- The stock pays no dividends.
- The price is calculated using a 2-step binomial model where each step is one year in length.

The stock price tree is shown below:

		121
	110	
100		99
	90	
		81

Calculate the price of the call on stock ABC.

(A) Less than 13.50

(B) At least 13.50, but less than 14.00

(C) At least 14.00, but less than 14.50

(D) At least 14.50, but less than 15.00

(E) At least 15.00

Ambrose's comments:

This is the very first time we handle a multi-period binomial tree, so we will do this example from basics. Later we will introduce shortcuts that will save much of the work.

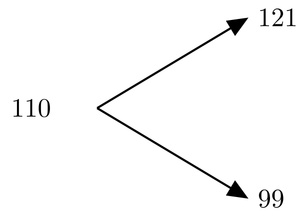
Solution. We break down our solution into three steps.

- *Step 1:* The first step of the solution is to identify the possible payoffs of the call option at expiration. They are

$$C_{uu} = (121-95)_+ = 26, \quad C_{ud} = (99-95)_+ = 4, \quad C_{dd} = (81-95)_+ = 0.$$

- *Step 2:* The second step, also the key step of the solution, lies in transforming the given two-period call into an equivalent one-period derivative (not a call). To this end, consider what happens in one year:

Case 1. Suppose that the stock price goes up in one year to 110. From the perspective of the 110 node, the stock price can be either \$121 or \$99 one year thereafter. This constitutes a single-period binomial tree model with $u = 1.1$ and $d = 0.9$:



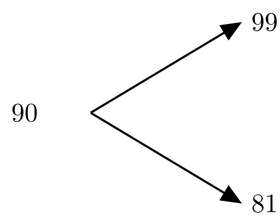
Since the risk-neutral probability of an up move is

$$p^* = \frac{e^{0.05(1)} - 0.9}{1.1 - 0.9} = 0.756355481,$$

by risk-neutral pricing the value of the call option at the 110 node is

$$C_u = e^{-0.05}[p^* C_{uu} + (1 - p^*)C_{ud}] = 19.6332.$$

Case 2. Similarly, at the 90 node, we again have a single-period binomial tree model with the same parameters $u = 1.1$, $d = 0.9$ and $p^* = 0.756355$, but with different beginning and ending stock prices:

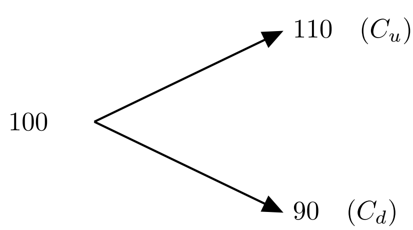


By risk-neutral pricing again, the value of the call option at the 90 node is

$$C_d = e^{-0.05}[p^*(99 - 95)_+ + (1 - p^*)(81 - 95)_+] = 2.8779.$$

At this point, we have reduced the two-year \$95-strike call option to a *one-year derivative* which pays $C_u = 19.6332$ if the stock price goes up in one year, and pays $C_d = 2.8779$ if the stock price goes down in one year.

- *Step 3:* In the third and final step of the solution, we price the equivalent one-year derivative using the single-period binomial model emanating from time 0:



The required call price is

$$C_0 = e^{-0.05}[p^* C_u + (1 - p^*)C_d] = \boxed{14.7924}. \quad \textbf{(Answer: (D))}$$

□

Remark. If you love calculations, you might choose to determine the replicating portfolio at each node of the binomial tree:

Node	(Δ, B)
u	$(1, -90.3668)$
d	$(2/9, -17.1221)$
0	$(0.8378, -68.9841)$

You can check that the value of the replicating portfolio at time 0 is also equal to the call price you have just computed. Moreover, the composition of the replicating portfolio varies as the stock price tree unfolds.

An important shortcut for European options: From maturity date directly to time 0.

As Example 4.2.1 demonstrates, pricing an option in a multi-period binomial tree involves computing the value of the option at all intermediate nodes. These calculations can be computationally burdensome (although easily implemented by an Excel spreadsheet), especially when the binomial tree has three or more periods. Such pain, however, can be completely avoided in the case of *European* options, for which early exercise is not an "option." Instead of working backward through the entire binomial tree and computing all the intermediate option values, we simply calculate the expected payoff at expiration using *binomial risk-neutral probabilities* and discount this expected payoff directly back to time 0 at the risk-free interest rate without bothering with the interim stock prices.

To see this more concretely, consider a two-period binomial model in which the length of each period is h. The possible values of the option at the end of the first period are

$$V_u = \mathrm{e}^{-rh}[p^* V_{uu} + (1-p^*)V_{ud}] \quad \text{and} \quad V_d = \mathrm{e}^{-rh}[p^* V_{ud} + (1-p^*)V_{dd}].$$

Discounting the "risk-neutral" expectations of these two values back to time 0 yields

$$\begin{aligned} V_0 &= \mathrm{e}^{-rh}[p^* V_u + (1-p^*)V_d] \\ &= \mathrm{e}^{-rh}\{p^* \mathrm{e}^{-rh}[p^* V_{uu} + (1-p^*)V_{ud}] + (1-p^*)\mathrm{e}^{-rh}[p^* V_{ud} + (1-p^*)V_{dd}]\} \\ &= \mathrm{e}^{-2rh}[(p^*)^2 V_{uu} + 2p^*(1-p^*)V_{ud} + (1-p^*)^2 V_{dd}] \\ &\overset{(T=2h)}{=} \boxed{\mathrm{e}^{-rT}[(p^*)^2 V_{uu} + 2p^*(1-p^*)V_{ud} + (1-p^*)^2 V_{dd}]}. \end{aligned} \tag{4.2.1}$$

Since the risk-neutral probabilities of entering the uu node, ud node (Note: There are two paths to reach the ud node, namely up-down and down-up) and dd node are

$$(p^*)^2, \qquad 2p^*(1-p^*), \quad \text{and} \quad (1-p^*)^2,$$

respectively, the preceding expression can be conveniently cast as

$$\boxed{V_0 = \mathrm{e}^{-rT}\mathbb{E}^*[\text{Payoff at expiration}],} \tag{4.2.2}$$

where the asterisk again signifies a risk-neutral expectation. By induction, it can be shown that (4.2.2) is true even for a general n-period binomial tree.

PRACTICAL NOTE

Given that (4.2.2) allows you to calculate the current price of the option directly from the terminal payoffs along with the binomial probabilities, this is the usual way you value a European option by hand calculation.

Example 4.2.2. (CAS Exam 3 Spring 2007 Question 17: A two-period binomial tree for a European put) For a two-year European put option, you are given the following information:

- The stock price is \$35.
- The strike price is \$32.
- The continuously compounded risk-free rate is 5%.
- The stock price volatility is 35%.

Using a binomial tree with annual valuations, calculate the price of this option.

(A) Less than \$3.00

(B) At least \$3.00, but less than \$3.40

(C) At least \$3.40, but less than \$3.80

(D) At least \$3.80, but less than \$4.20

(E) At least \$4.20

Solution. Prelude: The fact that the length of each period of the binomial tree is one year ($h = 1$) means that the binomial model is two-period ($n = 2$). Since there is no information about the dividends of the stock and how the binomial tree is constructed, we simply assume no dividends and a forward tree.

For a forward tree, we have

$$u = \mathrm{e}^{0.05(1)+0.35\sqrt{1}} = 1.491825 \quad \text{and} \quad d = \mathrm{e}^{0.05(1)-0.35\sqrt{1}} = 0.740818,$$

and the risk-neutral probability of an up move is

$$p^* = \frac{1}{1 + \mathrm{e}^{0.35\sqrt{1}}} = 0.413382.$$

The possible payoffs at expiration of the put option are:

Terminal Stock Price	Terminal Put Payoff
$S_{uu} = 35u^2 = 77.8940$	$P_{uu} = (32 - 77.8940)_+ = 0$
$S_{ud} = 35ud = 38.6810$	$P_{ud} = (32 - 38.6810)_+ = 0$
$S_{dd} = 35d^2 = 19.2084$	$P_{dd} = (32 - 19.2084)_+ = 12.7916$

Using (4.2.1), we have

$$\begin{aligned} P_0 &= \mathrm{e}^{-rT}[(p^*)^2 P_{uu} + 2p^*(1-p^*)P_{ud} + (1-p^*)^2 P_{dd}] \\ &= \mathrm{e}^{-0.05(2)}[0 + 0 + (1 - 0.413382)^2(12.7916)] \\ &= \boxed{3.9830}. \quad \textbf{(Answer: (D))} \end{aligned}$$

□

Example 4.2.3. (A three-period binomial tree) You use the following information to construct a binomial forward tree for modeling the price movements of a stock:

(i) The length of each period is 4 months.

(ii) The current stock price is 41.

(iii) The stock's volatility is 30%.

(iv) The stock pays no dividends.

(v) The continuously compounded risk-free interest rate is 8%.

Calculate the price of a one-year at-the-money European call option on the stock.

Ambrose's comments:

The three-period risk-neutral pricing formula takes the form

$$\mathrm{e}^{-rT}[(p^*)^3V_{uuu} + 3(p^*)^2(1-p^*)V_{uud} + 3p^*(1-p^*)^2V_{udd} + (1-p^*)^3V_{ddd}].$$

Don't forget the factor of 3, which comes from $\binom{3}{1} = \binom{3}{2}$, in the second and third terms!

Solution. The tree parameters are

$$u = \mathrm{e}^{0.08(1/3)+0.3\sqrt{1/3}} = 1.221246 \quad \text{and} \quad d = \mathrm{e}^{0.08(1/3)-0.3\sqrt{1/3}} = 0.863693.$$

Then the terminal stock prices are

$$\begin{aligned} S_{uuu} &= 41u^3 = 74.678110, \\ S_{uud} &= 41u^2d = 52.814061, \\ S_{udd} &= 41ud^2 = 37.351308, \end{aligned}$$

and there is no need to compute S_{ddd} (why?). With the risk-neutral probability of an up move being

$$p^* = \frac{1}{1+\mathrm{e}^{0.3\sqrt{1/3}}} = 0.456807,$$

the time-0 price of the required European call is

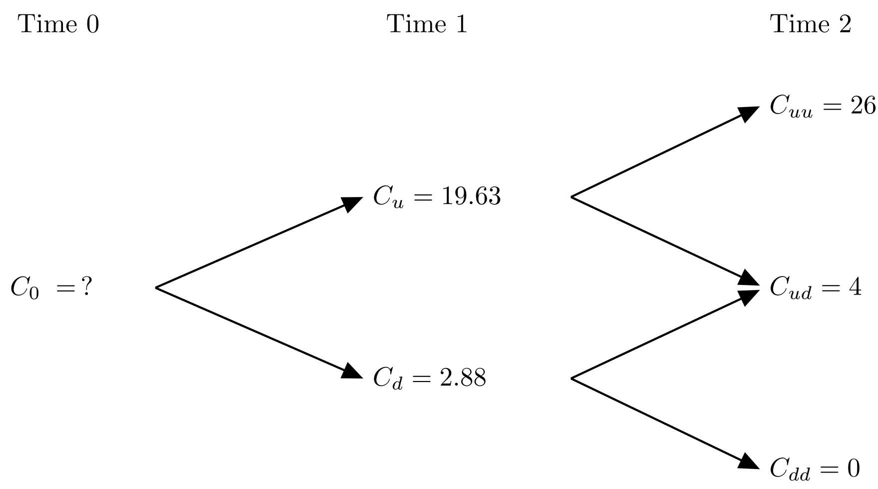
$$\begin{aligned} & \mathrm{e}^{-0.08}[0.456807^3(74.678110-41) \\ & \qquad +3(0.456807)^2(1-0.456807)(52.814061-41)] \\ = \; & \boxed{6.6720}. \end{aligned}$$

□

Example 4.2.4. (Given the call price tree, not the stock price tree!) For a two-period binomial stock price model, you are given:

(i) The length of each period is 1 year.

(ii) The (incomplete) price evolution of a 2-year European call option on the stock:



Calculate the current price of the call option.

Ambrose's comments:

We are not told anything about the stock (does it pay dividends?) or the call option (what is its strike price?). However, the given call price tree contains enough information for us to solve this problem.

Solution. Let's deduce the values of the risk-neutral probability p^* and the discount factor e^{-r} per period. To this end, we apply the risk-neutral pricing formula to C_d and C_u, yielding

$$\begin{cases} \mathrm{e}^{-r}[p^*(4) + (1-p^*)(0)] = 2.88 \\ \mathrm{e}^{-r}[p^*(26) + (1-p^*)(4)] = 19.63 \end{cases}.$$

The first equation implies that $\mathrm{e}^{-r}p^* = 0.72$, which, when plugged into the second equation, leads to $\mathrm{e}^{-r}(1-p^*) = 0.2275$. Then it follows from a further application of the risk-neutral pricing formula that

$$C_0 = \mathrm{e}^{-r}[p^*(19.63) + (1-p^*)(2.88)] = 0.72(19.63) + 0.2275(2.88) = \boxed{14.79}.$$

□

Remark. Although we do not need the individual values of p^* and e^{-r}, they can be found via $\mathrm{e}^{-r} = \mathrm{e}^{-r}p^* + \mathrm{e}^{-r}(1-p^*) = 0.9475$ and $p^* = 0.72/0.9475 = 0.7599$.

It is hard to conceive a pen-and-paper problem involving an n-period binomial tree with $n \geq 4$. If such catastrophic cases happen, chances are that the option will pay off only at a few nodes.

Example 4.2.5. (A 10-period tree!!) For a 10-period binomial stock price model, you are given:

(i) The length of each period is one year.

(ii) The current stock price is 1,000.

(iii) At the end of every year, the stock price will either increase by 5% or decrease by 5% in proportion.

(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 1%.

(v) The continuously compounded risk-free interest rate is 2%.

Calculate the price of a 10-year 1,400-strike European call option on the stock.

Ambrose's comments:

A 10-period binomial model!? WTF!? (WOW, THAT'S FANTASTIC!)

Solution. Although there are a total of 11 possible stock prices after 10 years, observe carefully that the call will be in-the-money only at two nodes. With $u = 1.05$, $d = 0.95$, and $K = 1,400$, note that the call will pay off only at

the u^{10} node, where $S_{u^{10}} = 1,000(1.05)^{10} = 1,628.8946 > 1,400$, and

the u^9d node, where $S_{u^9d} = 1,000(1.05)^9(0.95) = 1,473.7618 > 1,400$,

but not at the u^8d^2 node, where $S_{u^8d^2} = 1,000(1.05)^8(0.95)^2 = 1,333.4035 < 1,400$, and other lower nodes.

The risk-neutral probability of an up move is

$$p^* = \frac{e^{(0.02-0.01)(1)} - 0.95}{1.05 - 0.95} = 0.600502.$$

By the risk-neutral pricing formula,

$$\begin{aligned} C &= e^{-rT}[(p^*)^{10}C_{u^{10}} + 10(p^*)^9(1-p^*)C_{u^9d}] \\ &= e^{-0.02(10)}[0.600502^{10}(1,628.8946 - 1,400) \\ &\qquad\qquad +10(0.600502)^9(1-0.600502)(1,473.7618 - 1,400)] \\ &= \boxed{3.592396}. \end{aligned}$$

□

Remark. What if you are asked to price the otherwise identical put? It pays off at $11 - 2 = 9$ nodes, which means that the risk-neutral pricing formula will consists of 9 terms! See Problem 4.6.10.

Sidebar: A general representation of European option prices.

It is possible to give a closed-form expression for a European derivative in a general n-period recombining binomial tree model with constant u and d. For any fixed i between 0 and n, the probability of i up moves (and necessarily $n - i$ down-moves) over the n periods is

$$P_i := \binom{n}{i}(p^*)^i(1-p^*)^{n-i}.$$

Then by risk-neutral pricing, the time-0 price of a K-strike European call option is

$$\begin{aligned} C &= \mathrm{e}^{-rT} \sum_{i=0}^{n} P_i(i)(S(0)u^i d^{n-i} - K)_+ \\ &= \mathrm{e}^{-rT} \sum_{i=0}^{n} \binom{n}{i} (p^*)^i (1-p^*)^{n-i} (S(0)u^i d^{n-i} - K)_+. \end{aligned}$$

This expression can be represented in a form that mimics the Black-Scholes pricing formula we shall learn in Chapter 6. We denote by n^* the smallest number of up moves that makes the call option in-the-money at expiration. Then the call price can be rewritten as

$$\begin{aligned} C &= \mathrm{e}^{-rT} \sum_{i=n^*}^{n} P_i(i)(S(0)u^i d^{n-i} - K) \\ &= S(0)\mathrm{e}^{-rT} \sum_{i=n^*}^{n} P_i(i) u^i d^{n-i} - K\mathrm{e}^{-rT} \sum_{i=n^*}^{n} P_i(i). \end{aligned}$$

The second term is the anticipated time-0 *cost* of exercising the call: the present value of the strike price K multiplied by the probability that the call will finish in-the-money. The first term measures the anticipated *benefit* from exercising the option and owning the stock. As you will see later, the Black-Scholes pricing formula takes a similar cost-benefit form:

$$C = S(0) \times \text{term } 1 - K\mathrm{e}^{-rT} \times \text{term } 2,$$

where "term 2" is again the risk-neutral probability that the call expires in-the-money.

4.3 American Options

Arguably the most powerful feature of the binomial tree models among all option pricing models is that it readily accommodates American options. Because of the possibility of early exercise, (4.2.2) based on discounting the payoffs at expiration is no longer applicable. As such, the options need not be held until expiration, making the term "terminal payoff" not well defined. In this section, we will illustrate how the method of backward induction can be employed to price American options.

Backward induction is inevitable...

To value American options using a binomial model, the end-of-tree shortcut as discussed on page 135 does not work and we have no choice but to work backward through the whole binomial tree and check whether early exercise is optimal at any intermediate node. The reason why we travel backward but not forward over time is that an early exercise decision made at a later node will increase the value of the option at an earlier node and may create a ripple effect along the tree, inducing early exercise at an earlier node.

In the presence of early exercise, the value of the option at a particular node is given by the *maximum* of:

(1) The *holding value* (i.e., the value of holding the American option until the end of the current period), which can be determined by risk-neutral valuation (or the method of replication, if you insist).

(2) The immediate exercise value, which is given by $(S-K)_+$ in the case of a call, and $(K-S)_+$ in the case of a put, where S is the price of the stock corresponding to that node of the binomial tree.

If (1) < (2), then early exercise at that node is optimal, and the value of the option therein will become (2).

Example 4.3.1. (SOA Exam IFM Sample Question 4: American call I) For a two-period binomial model, you are given:

(i) Each period is one year.

(ii) The current price for a nondividend-paying stock is 20.

(iii) $u = 1.2840$, where u is one plus the rate of capital gain on the stock per period if the stock price goes up.

(iv) $d = 0.8607$, where d is one plus the rate of capital loss on the stock per period if the stock price goes down.

(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of an American call option on the stock with a strike price of 22.

(A) 0

(B) 1

(C) 2

(D) 3

(E) 4

Solution. The two-period binomial tree is constructed in Figure 4.3.1. The risk-neutral probability for the stock price to go up each year is

$$p^* = \frac{e^{0.05} - 0.8607}{1.2840 - 0.8607} = 0.450203.$$

Then we start from the end of the tree and travel backward to see if it pays to exercise the American call early.

- *At the u node:* The value of the call option is

$$\max\left\{\underbrace{e^{-0.05}[p^*(10.9731) + (1-p^*)(0.1028)]}_{=4.752922}, \underbrace{(25.680-22)_+}_{=3.680}\right\} = 4.752922.$$

  Because the holding value of 4.752922 is greater than the exercise value of 3.680, it is more worthwhile to hold than to exercise the American call.

- *At the d node:* The value of the call option becomes

$$\max\left\{\underbrace{e^{-0.05}[p^*(0.1028) + (1-p^*)(0)]}_{=0.044023}, \underbrace{(17.214-22)_+}_{=0}\right\} = 0.044023.$$

  Early exercise is again not optimal.

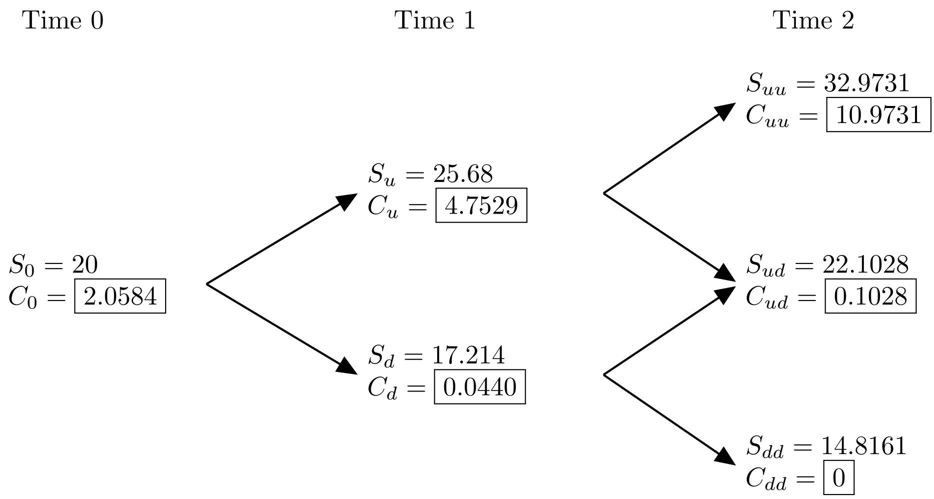


FIGURE 4.3.1
The two-period binomial tree for Example 4.3.1.

- *At the initial node:* Finally, the call price is

$$\max\left\{\underbrace{e^{-0.05}[p^*(4.752922) + (1-p^*)(0.044023)]}_{=2.0584}, \underbrace{(20-22)_+}_{=0}\right\} = \boxed{2.0584}.$$

(Answer: (C))

□

Remark. (i) Strictly speaking, point (iv) of the question should be written as "$d = 0.8607$, where d is one plus the *percentage change* in the stock price per period if the stock price goes down." When $d < 1$, the stock price drops and the rate of capital loss on the stock is a positive quantity.

(ii) It turns out that in the absence of dividends, early exercise of an American call is never optimal (see Section 9.4). Accordingly, the American call can be priced in the same way as its European counterpart.

Example 4.3.2. (SOA Exam MFE Spring 2009 Question 1: American call II) You use the following information to construct a binomial forward tree for modeling the price movements of a stock. (This tree is sometimes called a forward tree.)

(i) The length of each period is one year.

(ii) The current stock price is 100.

(iii) The stock's volatility is 30%.

(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 5%.

(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of a two-year 100-strike American call option on the stock.

(A) 11.40

(B) 12.09

(C) 12.78

(D) 13.47

(E) 14.16

Ambrose's comments:

This is a more involved version of the preceding example. You are required to construct the binomial tree, which is a forward tree, yourself. Also, early exercise is shown to be optimal at some point.

Solution.

- *Step 1 (Constructing the forward tree):* In a forward tree, we have

$$\begin{aligned} u &= \exp[(r-\delta)h + \sigma\sqrt{h}] = \exp[(0.05-0.05)\times 1 + 0.3\sqrt{1}] = \mathrm{e}^{0.3}, \\ d &= \exp[(r-\delta)h - \sigma\sqrt{h}] = \exp[(0.05-0.05)\times 1 - 0.3\sqrt{1}] = \mathrm{e}^{-0.3}. \end{aligned}$$

  The risk-neutral probability of an up move is

$$p^* = \frac{\mathrm{e}^{(r-\delta)h} - d}{u-d} = \frac{\mathrm{e}^{(0.05-0.05)\times 1} - \mathrm{e}^{-0.3}}{\mathrm{e}^{0.3} - \mathrm{e}^{-0.3}} = 0.425557$$

  or

$$p^* = \frac{1}{1+\mathrm{e}^{\sigma\sqrt{h}}} = \frac{1}{1+\mathrm{e}^{0.3}} = 0.425557.$$

- *Step 2 (Computing the time-1 option values):* The resulting stock prices are depicted in Figure 4.3.2. Now we calculate the option values at time 1, paying special attention to whether early exercise is optimal:

Node	Holding Value (Do the calculations!)	Exercise Value	Early Exercise?
u	$\mathrm{e}^{-0.05}[p^*C_{uu} + (1-p^*)C_{ud}] = 33.2796$	34.9859	Yes
d	$\mathrm{e}^{-0.05}[p^*C_{ud} + (1-p^*)C_{dd}] = 0$	0	No

  Therefore, the values of the call options at time 1 are

$$C_u = 34.9859 \quad \text{and} \quad C_d = 0.$$

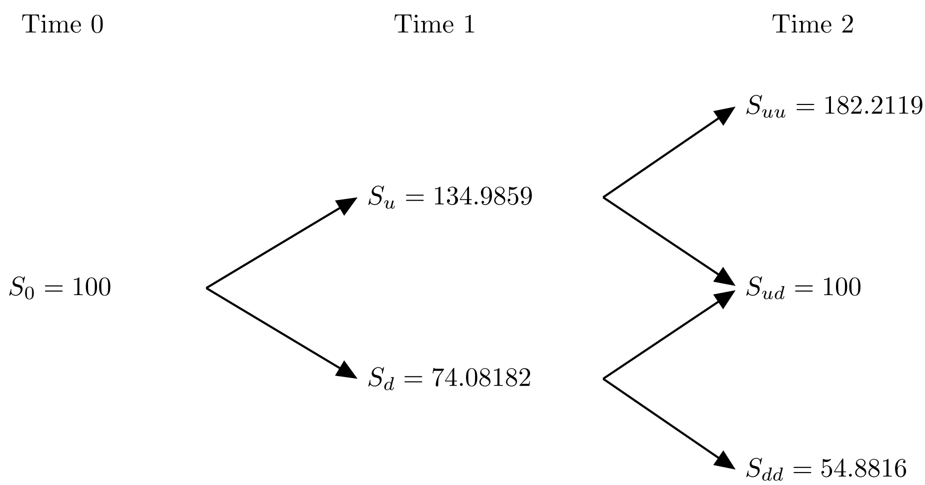


FIGURE 4.3.2
The two-period binomial forward tree for Example 4.3.2.

- *Step 3 (Time-0 price):* Now that the values of the call option at time 1 are available, the time-0 price of the call can be easily calculated as

$$\begin{aligned} C_0 &= \max\{e^{-0.05}[p^* C_u + (1-p^*)C_d], 100-100\} \\ &= \max\{e^{-0.05}[0.425557(34.9859) + (1-0.425557)(0)], 0\} \\ &= \boxed{14.1624}. \quad \textbf{(Answer: (E))} \end{aligned}$$

□

Remark. Don't forget to check whether early exercise is optimal at time 0.

Example 4.3.3. (SOA Exam MFE Spring 2007 Question 11: American put I) For a two-period binomial model for stock prices, you are given:

(i) Each period is 6 months.

(ii) The current price for a nondividend-paying stock is \$70.00.

(iii) $u = 1.181$, where u is one plus the rate of capital gain on the stock per period if the price goes up.

(iv) $d = 0.890$, where d is one plus the rate of capital loss on the stock per period if the price goes down.

(v) The continuously compounded risk-free interest rate is 5%.

Calculate the current price of a one-year American put option on the stock with a strike price of \$80.00.

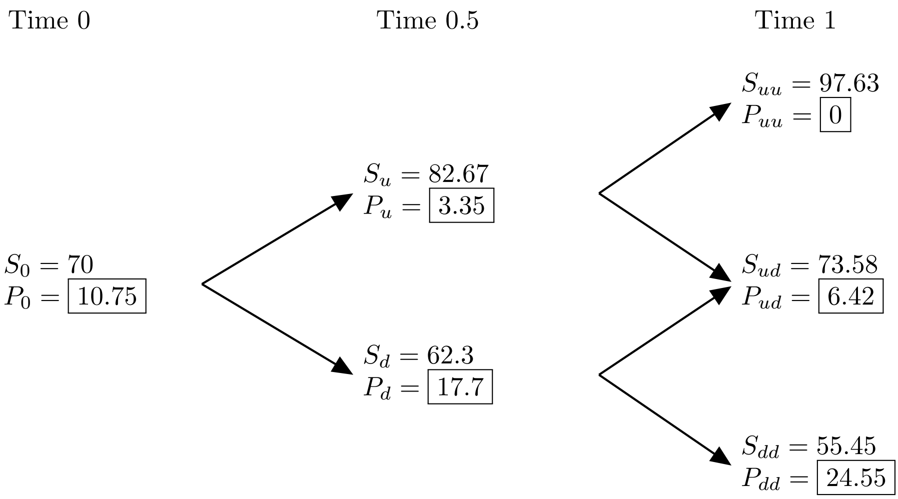


FIGURE 4.3.3
The two-period binomial tree for Example 4.3.3.

(A) \$9.75

(B) \$10.15

(C) \$10.35

(D) \$10.75

(E) \$11.05

Solution. With $u = 1.181$, $d = 0.89$, $h = 0.5$, $n = 2$ (i.e., two-period model) and $\delta = 0$, the risk-neutral probability of an increase in the stock price at the end of a period is

$$p^* = \frac{e^{0.05(0.5)} - 0.89}{1.181 - 0.89} = 0.4650.$$

The stock prices and put values are depicted in Figure 4.3.3. Note that early exercise is optimal at the d node, because the exercise value of $80 - 62.3 = 17.7$ exceeds the holding value of $e^{-0.05(0.5)}[p^*(6.42) + (1 - p^*)(24.55)] = 15.72$. The time-0 price of the American option is

$$P_0 = \max\{e^{-0.05(0.5)}[p^*(3.35) + (1 - p^*)(17.7)], 80 - 70\} = \boxed{10.75}. \quad \textbf{(Answer: (D))}$$

□

Example 4.3.4. (CAS Exam 3 Fall 2007 Question 18: American put II) You are given the following information about American options on a stock:

- The current stock price is 72.

- The strike price of the options is 80.
- The continuously compounded risk-free rate is 5%.
- Time to expiration is 1 year.
- Every six months, the stock price either increases by 25% or decreases by 15%.

Using a two-period binomial tree, calculate the price of an American put option.

(A) Less than 8

(B) At least 8, but less than 9

(C) At least 9, but less than 10

(D) At least 10, but less than 11

(E) At least 11

Solution. With $u = 1.25$, $d = 0.85$, $h = 0.5$, $n = 2$ (i.e., two-period model) and $\delta = 0$, the risk-neutral probability of an increase in the stock price at the end of one year is

$$p^* = \frac{e^{0.05(0.5)} - 0.85}{1.25 - 0.85} = 0.438288.$$

The stock prices and put values are depicted in Figure 4.3.4. Note that early exercise is optimal at the d node, because the exercise value of $80 - 61.2 = 18.8$ exceeds the holding value of $e^{-0.05(0.5)}[p^*(3.5) + (1-p^*)(27.98)] = 16.8248$. The time-0 price of the American option is

$$P_0 = \max\{e^{-0.05(0.5)}[p^*(1.9175) + (1-p^*)(18.8)], 80 - 72\} = \boxed{11.12}. \quad \textbf{(Answer: (E))}$$

□

Example 4.3.5. (SOA Exam IFM Advanced Derivatives Sample Question 49: How high should the put strike be to induce immediate exercise?) You use the following information to construct a one-period binomial forward tree for modeling the price movements of a nondividend-paying stock. (The tree is sometimes called a forward tree.)

(i) The period is 3 months.

(ii) The initial stock price is \$100.

(iii) The stock's volatility is 30%.

(iv) The continuously compounded risk-free interest rate is 4%.

At the beginning of the period, an investor owns an American put option on the stock. The option expires at the end of the period.

Determine the smallest integer-valued strike price for which an investor will exercise the put option at the beginning of the period.

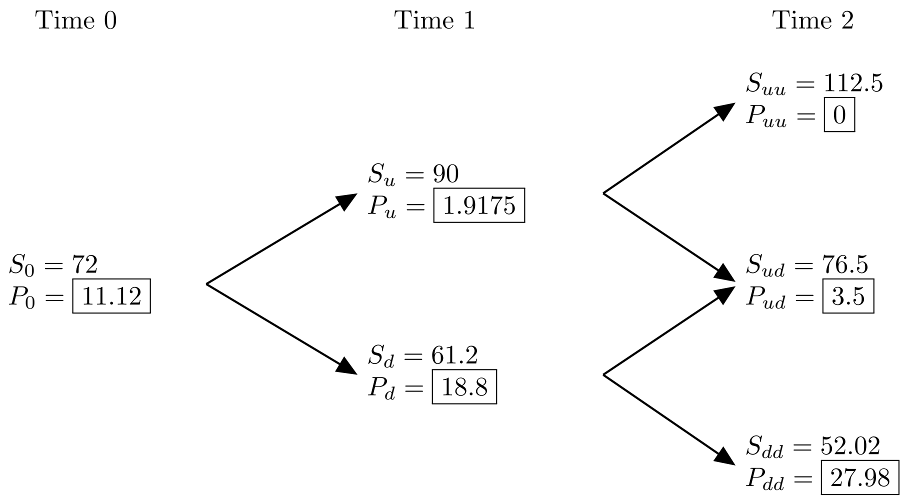


FIGURE 4.3.4
The two-period binomial tree for Example 4.3.4.

(A) 114

(B) 115

(C) 116

(D) 117

(E) 118

Solution. For a forward tree, we have

$$\begin{aligned} u &= \exp[(r-\delta)h + \sigma\sqrt{h}] = \exp[(0.04)(0.25) + 0.3\sqrt{0.25}] = 1.173511, \\ d &= \exp[(r-\delta)h - \sigma\sqrt{h}] = \exp[(0.04)(0.25) - 0.3\sqrt{0.25}] = 0.869358, \\ p^* &= 1/(1+\mathrm{e}^{\sigma\sqrt{h}}) = 1/(1+\mathrm{e}^{0.3\sqrt{0.25}}) = 0.462570, \end{aligned}$$

so that $S_u = S_0 \times u = 117.3511$ and $S_d = S_0 \times d = 86.9358$. In terms of the strike price K, the possible payoffs of the American put option are

$$P_u = (K - 117.3511)_+ \quad \text{and} \quad P_d = (K - 86.9358)_+.$$

In order that the investor exercises the put option at the beginning of the period, the strike price K should be such that

$$\underbrace{(K - S_0)_+}_{\text{Exercise value}} > \underbrace{\mathrm{e}^{-rh}[p^* P_u + (1-p^*)P_d]}_{\text{Holding value}},$$

or

$$(K - 100)_+ > 0.457968(K - 117.3511)_+ + 0.532082(K - 86.9358)_+. \tag{4.3.1}$$

There are different good ways to solve this inequality:

- Try the five answer choices one by one and find the smallest K such that Inequality (4.3.1) is true.
- Distinguish various ranges of values of K so that Inequality (4.3.1) holds. Because the right-hand side of Inequality (4.3.1) is non-negative, we can restrict our attention to $K > 100$.

  Case 1. If $K > 117.3511$, then the left-hand side of Inequality (4.3.1) is $K - 100$, whereas its right-hand side is $e^{-0.01}K - 110 = 0.99005K - 110$, so Inequality (4.3.1) is true.

  Case 2. If $100 < K \leq 117.3511$, then we solve

  $$K - 100 > 0.532082(K - 86.9325),$$

  resulting in $K > 114.8594$. The smallest integer-valued K is therefore $\boxed{115}$. **(Answer: (B))** □

Remark. It can be shown that if an American put on a nondividend-paying stock in a one-period binomial tree (not necessarily a forward tree) pays off at both the u and d nodes, then it must be optimally exercised at the beginning of the period. To see this, we determine the holding value of the put option, which is

$$e^{-rT}[p^*(K - S_u) + (1 - p^*)(K - S_d)] = e^{-rT}\{K - [p^*u + (1 - p^*)d]S_0\}.$$

The weighted average $p^*u + (1 - p^*)d$ can be evaluated as

$$\frac{e^{(r-\delta)h} - d}{u - d} \times u + \frac{u - e^{(r-\delta)h}}{u - d} \times d = e^{(r-\delta)h}.$$

Hence

$$\text{Holding value} = e^{-rT}[K - e^{(r-\delta)h} \times S_0] = Ke^{-rT} - S_0e^{-\delta h} \overset{(\delta=0)}{=} Ke^{-rT} - S_0,$$

which is always less than the exercise value of $K - S_0$ for any positive r.

4.4 Options on Other Assets

The fundamental ideas underlying the binomial option pricing model have been sketched in the previous three sections. In this section, we extend our analysis to options on currencies and options on futures, and explore these options' similarities and peculiarities compared with stock options. It is shown that the term "stock" is essentially a label and that as soon as the correct "dividend yields" (in a general sense) applicable to currencies and futures are identified, options on these assets can be priced in technically the same way as stock options.

4.4.1 Case Study 1: Currency Options

Currency options are options whose underlying assets are currencies. For concreteness, consider two given currencies, say dollars (\$) and euros (€). We are interested in options on euros (underlying asset).

Suppose that

the current dollar-euro exchange rate is $\$X(0)/€$,

the exchange rate at any future time t is $\$X(t)/€$ (random),

the continuously compounded risk-free interest rate on euros is $r_€$,

the continuously compounded risk-free interest rate on dollars is $r_\$$.

Notice that although the numbers of the two currencies will grow without uncertainty in the future (e.g., \$1 will grow to $\$\mathrm{e}^{r_\$ t}$ and €1 will grow to $€\mathrm{e}^{r_€ t}$ at time t), their relative values (i.e., the exchange rate) in the future are random. Such randomness provides the fundamental motivation for trading currency options, which allow one to hedge against currency risk, also known as exchange rate risk.

Consider a dollar-denominated $\$K$-strike T-year call option on (1 unit of) euro. By "dollar-denominated," we mean that the strike price and premium of the option are both expressed in dollars. This call option gives you the right to give up $\$K$ in return for €1 at time T, with a payoff of

$$\text{Currency call payoff} = (€1 - \$K)_+ = (\underbrace{\$X(T) - \$K}_{\text{same denomination}})_+.$$

It allows its holder to benefit from an appreciation of euros against dollars. An otherwise identical put option allows you to give up €1 in return for $\$K$, with a time-$T$ payoff of

$$\text{Currency put payoff} = (\$K - €1)_+ = (\$K - \$X(T))_+,$$

and offers protection against the depreciation of euros against dollars.

Valuing currency options in the binomial tree framework is in principle no different than valuing options on stocks, as soon as you recognize the same roles that $r_€$ and $r_\$$ play as the dividend yield of the underlying asset and the continuously compounded risk-free interest rate, respectively. With this identification, the risk-neutral probability of an increase in the *dollar-euro exchange rate* at the end of a binomial period is

$$\boxed{p^* = \frac{\mathrm{e}^{(r_\$ - r_€)h} - d}{u - d}.}$$

The value of a currency option can then be recursively calculated at each node of a binomial tree using risk-neutral pricing as usual.

Example 4.4.1. (SOA Exam IFM Advanced Derivatives Sample Question 5: Currency option) Consider a 9-month dollar-denominated American put option on British pounds. You are given that:

(i) The current exchange rate is 1.43 US dollars per pound.

(ii) The strike price of the put is 1.56 US dollars per pound.

(iii) The volatility of the exchange rate is $\sigma = 0.3$.

(iv) The US dollar continuously compounded risk-free interest rate is 8%.

(v) The British pound continuously compounded risk-free interest rate is 9%.

Using a three-period binomial model, calculate the price of the put.

(A) 0.23

(B) 0.25

(C) 0.27

(D) 0.29

(E) 0.31

Solution. The method of constructing the binomial tree is not specified, so we use a forward tree, which has parameters

$$\begin{aligned} u &= \exp[(r_\$ - r_£)h + \sigma\sqrt{h}] = \exp[(0.08 - 0.09)(0.25) + 0.3\sqrt{0.25}] = 1.158933, \\ d &= \exp[(r_\$ - r_£)h - \sigma\sqrt{h}] = \exp[(0.08 - 0.09)(0.25) - 0.3\sqrt{0.25}] = 0.858559. \end{aligned}$$

The risk-neutral probability of an up move is

$$p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}} = \frac{1}{1 + e^{0.3\sqrt{0.25}}} = 0.462570.$$

The three-period binomial tree for the exchange rate (not stock price) is shown in Figure 4.4.1, along with the values of the put option. Note that early exercise is optimal at the *dd* node. **(Answer: (A))** □

4.4.2 Case Study 2: Options on Futures

Options on futures are options that easily cause confusion and deserve special attention. One of the reasons is that there are two derivatives, the concerned option maturing at time T, and the underlying futures maturing at a later time T_f with $T \leq T_f$. If we repeat the one-period replicating procedure discussed in Subsection 4.1.1, with Δ now being the number of futures to buy, then the two replicating equations become

$$\begin{cases} \Delta \times (F_u - F_0) + Be^{rh} & = V_u, \\ \Delta \times (F_d - F_0) + Be^{rh} & = V_d, \end{cases}$$

where $F_u - F_0$ and $F_d - F_0$ are the mark-to-market payments (recall what you learnt in Section 2.4) of the futures in the u node and d node, respectively. The solutions for Δ and B are

$$\Delta = \frac{V_u - V_d}{F_u - F_d} \quad \text{and} \quad B = e^{-rh}\left[V_u\left(\frac{1-d}{u-d}\right) + V_d\left(\frac{u-1}{u-d}\right)\right].$$

Because no investment is required to enter a futures contract, the price of the option on the futures is simply

$$V_0 = \Delta \times 0 + B = e^{-rh}\left[V_u\left(\frac{1-d}{u-d}\right) + V_d\left(\frac{u-1}{u-d}\right)\right]. \tag{4.4.1}$$

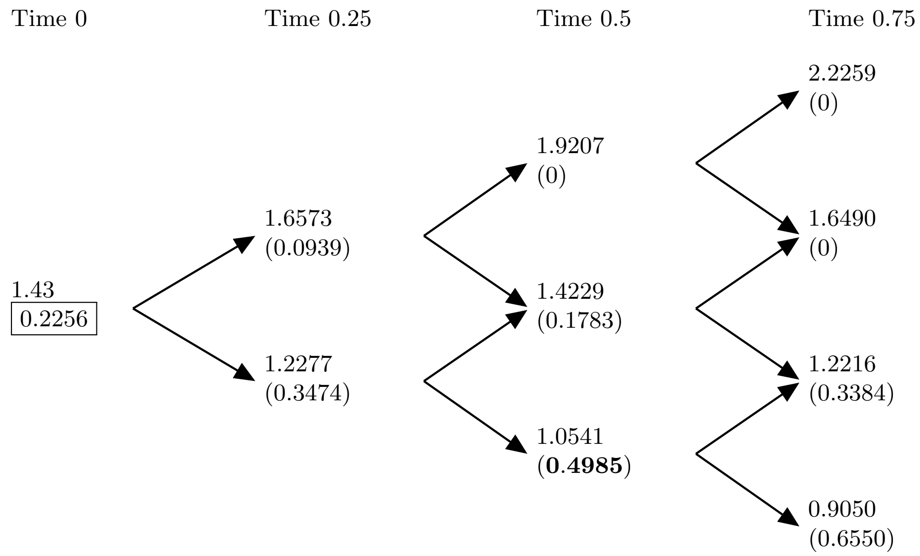


FIGURE 4.4.1
The exchange rate evolution in Example 4.4.1.

Viewed from the risk-neutral pricing perspective, this formula implies that the risk-neutral probability of an up move *in the futures price* in the current setting is

$$\boxed{p^* = \frac{1-d}{u-d}.} \tag{4.4.2}$$

The one-period pricing formula (4.4.1) extends easily to a multi-period setting, with (4.4.2) serving as the risk-neutral probability.

Parenthetically, the risk-neutral probability given in equation (4.4.2) can be obtained by equating the dividend yield δ of the futures with the continuously compounded risk-free interest rate r. Heuristic justification of this somewhat mysterious fact can be found on pages 324, 327, and 328 of Sundaram and Das (2016).

Example 4.4.2. (SOA Exam IFM Advanced Derivatives Sample Question 46: Futures option) You are to price options on a futures contract. The movements of the futures price are modeled by a binomial tree. You are given:

(i) Each period is 6 months.

(ii) $u/d = 4/3$, where u is one plus the rate of gain on the futures price if it goes up, and d is one plus the rate of loss if it goes down.

(iii) The risk-neutral probability of an up move is 1/3.

(iv) The initial futures price is 80.

(v) The continuously compounded risk-free interest rate is 5%.

Let C_I be the price of a 1-year 85-strike European call option on the futures contract, and C_{II} be the price of an otherwise identical American call option.

Determine $C_{II} - C_I$.

(A) 0

(B) 0.022

(C) 0.044

(D) 0.066

(E) 0.088

Solution. To find the individual values of u and d, consider

$$p^* = \frac{1-d}{u-d} = \frac{1/d-1}{u/d-1} = \frac{1/d-1}{4/3-1} = \frac{1}{3},$$

which gives $d = 0.9$, and so $u = 1.2$. Now we can construct the binomial futures price tree in Figure 4.4.2.

To calculate the difference $C_{II} - C_I$, we can compute C_I and C_{II} separately. Alternatively, one can observe that $C_{II} - C_I$ returns the current value of the early exercise right carried by the American futures call. Prior to maturity, the only node at which the call is in-the-money is the u node, where the holding value is

$$C_u = \mathrm{e}^{-0.05(0.5)} \left[\frac{1}{3}(30.2) + \frac{2}{3}(1.4)\right] = 10.7284,$$

whereas the exercise value is $(96-85)_+ = 11$, which is higher. Thus the right to early exercise brings us an extra value of $11 - 10.7284 = 0.2716$ realized at the u node. Then $C_{II} - C_I$ equals this extra value weighed by the risk-neutral probability of entering the u node, discounted for half a year, or

$$C_{II} - C_I = \mathrm{e}^{-0.05(0.5)} \times (1/3) \times 0.2716 = \boxed{0.0883}. \quad \textbf{(Answer: (E))}$$

□

4.5 Epilogue: Pricing by Real Probabilities of Stock Price Movements

The risk-neutral pricing formula in the form of (4.2.2) is simple, amazing yet somehow perplexing. One of its far-reaching implications is that the true probability distribution of the stock price plays no role in the pricing problem – nowhere in the pricing formula do we have the true probability of the stock price going up. The only important ingredients are the risk-neutral probability of an increase in the stock price and the risk-free interest rate.

One would wonder:

> Shouldn't the true stock price probability distribution be an important input for determining option prices? If the stock price has a higher probability of moving up, shouldn't a call option be more liable to end up in-the-money and hence become more expensive?

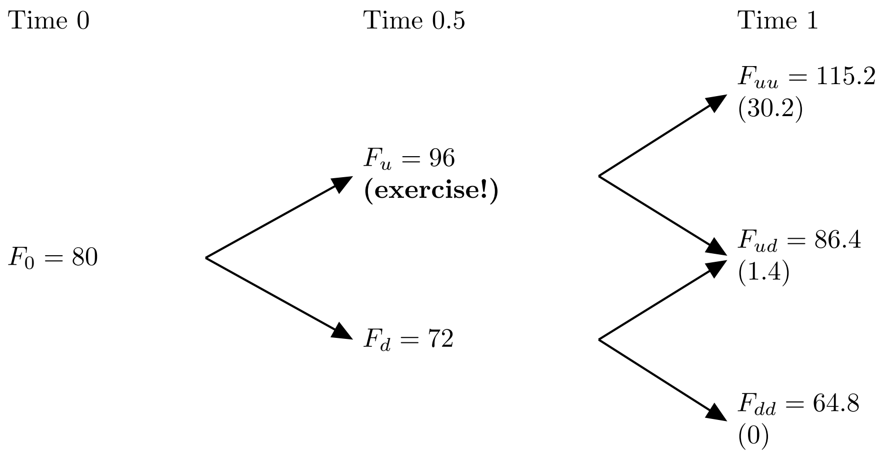


FIGURE 4.4.2
The binomial futures price tree in Example 4.4.2.

More generally, two interrelated questions arise naturally:

Question 1: Is risk-neutral pricing, albeit technically correct, consistent with standard discounted cash flow calculations, for which we discount the *genuine* expected values of cash flows at a rate of return that reflects the expected rate of return on an asset of equivalent risk?

Question 2: How can pricing be performed using the true probability distribution of stock prices?

We shall address these two theoretically and practically important questions in this section.

True probability of an up move in stock price.

Let α be the continuously compounded expected return on the stock that pays continuous proportional dividends at a rate of δ and p be the *true* probability (also known as the *real* or *physical* probability) of the stock price going up. By definition,

$$S_0 e^{\alpha h} = \mathbb{E}[\text{Stock price}] = p(e^{\delta h} S_u) + (1-p)(e^{\delta h} S_d) = e^{\delta h}[p(u \times S_0) + (1-p)(d \times S_0)].$$

Canceling S_0 on both sides and solving for p, we have

$$\boxed{p = \frac{e^{(\alpha-\delta)h} - d}{u - d}.} \tag{4.5.1}$$

Comparing (4.1.3) and (4.5.1), one notices that the risk-free rate r goes with the risk-neutral probability p^*, whereas the true rate of return of the stock α stays in company with the true probability p.

Example 4.5.1. (SOA Exam MFE Spring 2007 Question 2: True probability of an up move) For a one-period binomial model for the price of a stock, you are given:

(i) The period is one year.

(ii) The stock pays no dividends.

(iii) $u = 1.433$, where u is one plus the rate of capital gain on the stock if the price goes up.

(iv) $d = 0.756$, where d is one plus the rate of capital loss on the stock if the price goes down.

(v) The continuously compounded annual expected return on the stock is 10%.

Calculate the true probability of the stock price going up.

(A) 0.52

(B) 0.57

(C) 0.62

(D) 0.67

(E) 0.72

Solution. Directly applying equation (4.5.1) yields

$$p = \frac{\mathrm{e}^{(\alpha-\delta)h} - d}{u - d} = \frac{\mathrm{e}^{(0.1-0)(1)} - 0.756}{1.433 - 0.756} = \boxed{0.5158}. \quad \textbf{(Answer: (A))}$$

□

Discounted rate for the option.

Using the true probability p, we can compute the actual expected payoff of the option as

$$pV_u + (1-p)V_d.$$

Denote by $\boxed{\gamma}$ the *discount rate* for (or the *expected rate of return* on) the option. To determine γ in terms of α and r, we use the following familiar fact from corporate finance:

> The rate of return on a portfolio is the weighted average of the constituent rates of return.

With this fact, we can relate γ, α and r via

$$\mathrm{e}^{\gamma h} = \underbrace{\frac{S_0\Delta}{S_0\Delta + B}}_{\text{weight on stock}} \mathrm{e}^{\alpha h} + \underbrace{\frac{B}{S_0\Delta + B}}_{\substack{\text{weight on}\\\text{risk-free bond}}} \mathrm{e}^{rh}. \tag{4.5.2}$$

(Note: It is *not* true that $\mathrm{e}^{-\gamma h} = \dfrac{S_0\Delta}{S_0\Delta + B}\mathrm{e}^{-\alpha h} + \dfrac{B}{S_0\Delta + B}\mathrm{e}^{-rh}$.) The price of the option can then be evaluated as

$$\boxed{V_0^{\text{real}} = \mathrm{e}^{-\gamma h}[pV_u + (1-p)V_d].} \tag{4.5.3}$$

At this point, we have two "prices": One computed by risk-neutral valuation given in equation (4.1.4), and another computed by actual probabilities given in (4.5.3). If these two prices are not the same, then risk-neutral pricing is defective and there is no reason for us to dwell on it (and this book should not have been published!). Fortunately, we are reassured that these two prices are indeed identical with each other. Here is an algebraic proof.

Proof. We are to show

$$e^{-\gamma h}[pV_u + (1-p)V_d] = e^{-rh}[p^* V_u + (1-p^*)V_d]. \tag{4.5.4}$$

Due to (4.1.1) and (4.5.1), the left-hand side equals

$$\begin{aligned}
& \frac{\Delta S_0 + B}{\Delta S_0 e^{\alpha h} + B e^{rh}} [pV_u + (1-p)V_d] \\
= \quad & \frac{\Delta S + B}{e^{(\alpha-\delta)h}\left(\frac{V_u - V_d}{u-d}\right) + \frac{uV_d - dV_u}{u-d}} \left[\frac{e^{(\alpha-\delta)h} - d}{u-d} \times V_u + \frac{u - e^{(\alpha-\delta)h}}{u-d} \times V_d\right] \\
= \quad & \frac{\Delta S + B}{\frac{e^{(\alpha-\delta)h} - d}{u-d} \times V_u + \frac{u - e^{(\alpha-\delta)h}}{u-d} \times V_d} \left[\frac{e^{(\alpha-\delta)h} - d}{u-d} \times V_u + \frac{u - e^{(\alpha-\delta)h}}{u-d} \times V_d\right] \\
= \quad & \Delta S + B \\
\overset{(4.1.2)}{=} \quad & e^{-rh}[p^* V_u + (1-p^*)V_d] \\
= \quad & V_0.
\end{aligned}$$

□

We are now in a position to give complete answers to the two questions posted at the beginning of this section.

Question 1: Risk-neutral pricing is compatible with traditional discounted cash flow calculations. More precisely, these two pricing methods give identical results.

Question 2: To perform standard discounted cash flow calculations, if your course instructor stubbornly insists, then the following three-step procedure motivated from equation (4.5.3) can be adopted:

Step 1. Find the *true* probability p of an increase in the stock price using (4.5.1). The expected rate of return on the stock α is needed.

Step 2. Compute the appropriate discount rate γ using (4.5.2).

Step 3. Use (4.5.3) to calculate the derivative price.

In comparison, risk-neutral pricing is considerably simpler because it makes Step 2 unnecessary. There is no need to identify the correct discount rate at all—the discount rate is simply the risk-free interest rate, which is common among all derivatives.

Incidentally, a by-product of the equality (4.5.4) of risk-neutral valuation and physical valuation is the following alternative and easier equation for γ:

$$\boxed{\underbrace{e^{-\gamma h}[pV_u + (1-p)V_d]}_{\text{involving } (p,\gamma)} = \underbrace{e^{-rh}[p^* V_u + (1-p^*)V_d]}_{\text{involving } (p^*,r)}.}$$

The use of this equation obviates the need for determining the replicating portfolio (Δ, B). More importantly, determining the rate of return on the derivative this way is valid even for *multi-period* binomial trees.

Example 4.5.2. (CAS Exam 8 Spring 2003 Question 36: Rate of return in a one-period tree) A price of a nondividend-paying stock is currently \$40.

It is known that at the end of one month the stock's price will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding.

(a) Determine the value of a one-month European call option with a strike price of \$39.

(b) Assume that the expected return on the stock is 10% as opposed to the risk-free rate.

What is the correct discount rate to be applied to the payoff in the real world?

Show all work.

Solution. (a) With $u = 42/40 = 1.05$ and $d = 38/40 = 0.95$, the risk-neutral probability of an up move is

$$p^* = \frac{e^{0.08/12} - 0.95}{1.05 - 0.95} = 0.566889.$$

Hence

$$C_0 = e^{-0.08/12}[0.566889(42-39)_+ + (1-0.566889)(38-39)_+] = \boxed{1.6894}.$$

(b) The true probability of an up move is

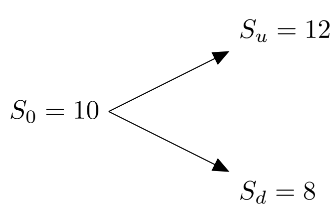
$$p = \frac{e^{0.1/12} - 0.95}{1.05 - 0.95} = 0.583682.$$

Setting

$$e^{-\gamma/12}[p(42-39)_+ + (1-p)(38-39)_+] = C_0 = 1.6894$$

yields $\gamma = \boxed{0.4301}$. □

Example 4.5.3. (SOA Exam MFE Spring 2009 Question 7: Modification to a binomial tree and its implications) The following one-period binomial stock price model was used to calculate the price of a one-year 10-strike call option on the stock.



You are given:

(i) The period is one year.

(ii) The true probability of an up move is 0.75.

(iii) The stock pays no dividends.

(iv) The price of the one-year 10-strike call is \$1.13.

Upon review, the analyst realizes that there was an error in the model construction and that S_d, the value of the stock on a down-move, should have been 6 rather than 8. The true probability of an up move does not change in the new model, and all other assumptions were correct.

Recalculate the price of the call option.

(A) \$1.13

(B) \$1.20

(C) \$1.33

(D) \$1.40

(E) \$1.53

Ambrose's comments:

This is one of my favorite MFE questions. It shows a counter-intuitive but interesting result about the use of risk-neutral pricing and physical valuation.

Solution. We can perform risk-neutral valuation to deduce the value of e^{-r}, which remains the same after the model review:

$$\begin{aligned}
\mathrm{e}^{-r}[p^*(2)+(1-p^*)(0)] &= 1.13 \\
\mathrm{e}^{-r}\left[2\left(\frac{10\mathrm{e}^{r}-8}{12-8}\right)\right] &= 1.13 \\
\mathrm{e}^{-r} &= 0.9675.
\end{aligned}$$

After the model review, the risk-neutral probability becomes

$$(p^*)' = \frac{10\mathrm{e}^{r}-6}{12-6} = 0.7226.$$

The new price of the call option is then $\mathrm{e}^{-r}[2(p^*)'] = \boxed{1.3983}$. **(Answer: (D))** □

Remark. (i) Because the call option pays off only at the u node, and S_u and p are both unchanged after the model review, it is true that the expected payoff of the call is unchanged as well after the model review. This seems to imply no change in the call price, as Answer A suggests! However, we have no reason to believe that the expected rate of return on the call, γ, remains the same. In fact, a by-product of this question is that γ has indeed altered. (Exercise: Try to calculate the change.)

(ii) The true probability of an up move given in (ii) of the question is designed by the SOA to distract you!

Example 4.5.4. (Expected rate of return as a function of the scaled strike price) Consider a one-period binomial tree. The length of the period is h years. The stock price moves from S to $S \times u$ or to $S \times d$, $0 < d < u$.

The stock pays no dividends.

Let α be the continuously compounded expected rate of return on the stock.

Consider a one-period put option on the stock with strike price $S \times k$, for some $k > 0$.

Let $\gamma(k)$ denote the continuously compounded expected rate of return on the put option, considered as a function of k.

Determine $\gamma(k)$. Your answer should involve k, r, α, h, u and d. Note that k ranges from 0 to ∞.

Solution. The key equation is

$$e^{-\gamma(k)h}[pP_u + (1-p)P_d] = e^{-rh}[p^*P_u + (1-p^*)P_d],$$

or, upon canceling the initial stock price S_0,

$$e^{-\gamma(k)h}[p(k-u)_+ + (1-p)(k-d)_+] = e^{-rh}[p^*(k-u)_+ + (1-p^*)(k-d)_+]. \quad (4.5.5)$$

There are three cases:

Case 1. $0 < k < d$

Here, (4.5.5) becomes $e^{-\gamma(k)h}(0) = e^{-rh}(0)$. Hence $\gamma(k)$ is not well defined; it can be any real number.

Case 2. $d < k < u$

Here, (4.5.5) becomes $e^{-\gamma(k)h}[(1-p)(k-d)] = e^{-rh}[(1-p^*)(k-d)]$. Thus

$$\gamma(k) = r - \frac{1}{h}\ln\left(\frac{1-p^*}{1-p}\right) = r - \frac{1}{h}\ln\left(\frac{u-e^{rh}}{u-e^{\alpha h}}\right).$$

Case 3. $u < k < \infty$

Here, (4.5.5) becomes

$$e^{-\gamma(k)h}[p(k-u) + (1-p)(k-d)] = e^{-rh}[p^*(k-u) + (1-p^*)(k-d)].$$

As $p(k-u) + (1-p)(k-d) = k - [pu + (1-p)d] = k - e^{\alpha h}$ and $p^*(k-u) + (1-p^*)(k-d) = k - e^{rh}$,

$$\gamma(k) = r - \frac{1}{h}\ln\left(\frac{k-e^{rh}}{k-e^{\alpha h}}\right).$$

□

Example 4.5.5. (Rate of return in a two-period tree) You use the following information to construct a binomial forward tree for modeling the price movements of a stock.

(i) The length of each period is one year.

(ii) The current stock price is 82.

(iii) The stock's volatility is 30%.

(iv) The stock pays no dividends.

(v) The continuously compounded risk-free interest rate is 5%.

(vi) The continuously compounded expected return on the stock is 10%.

Calculate the continuously compounded expected rate of return on a two-year 80-strike European call option on the stock.

Solution. The forward tree parameters are

$$u = \mathrm{e}^{0.05(1)+0.3\sqrt{1}} = \mathrm{e}^{0.35} = 1.419068 \quad \text{and} \quad d = \mathrm{e}^{0.05(1)-0.3\sqrt{1}} = \mathrm{e}^{-0.25} = 0.778801.$$

The risk-neutral probability of an up move is

$$p^* = \frac{1}{1+\mathrm{e}^{0.3\sqrt{1}}} = 0.425557.$$

With $S_{uu} = 165.1277$, $S_{ud} = 90.6240$ and $S_{dd} = 49.7355$, we have $C_{uu} = 85.1277$, $C_{ud} = 10.6240$ and $C_{dd} = 0$. The price of the call option is

$$\begin{aligned} & \mathrm{e}^{-2r}[(p^*)^2 C_{uu} + 2p^*(1-p^*)C_{ud} + (1-p^*)^2 C_{dd}] \\ = \ & \mathrm{e}^{-0.05(2)}[(0.425557)^2(85.1277) + 2(0.425557)(1-0.425557)(10.6240)] \\ = \ & 18.6494. \end{aligned}$$

With the true probability of an up move being

$$p = \frac{\mathrm{e}^{0.1} - 0.778801}{1.419068 - 0.778801} = 0.509740,$$

we solve

$$\mathrm{e}^{-2\gamma}[\underbrace{p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}}_{(0.509740)^2(85.1277)+2(0.509740)(1-0.509740)(10.6240)}] = 18.6494,$$

resulting in $\gamma = \boxed{0.1929}$. □

4.6 Problems

One-period binomial trees

Problem 4.6.1. (Valuing a strangle) Consider a 50-65 1-year strangle strategy. You are given:

(i) The stock currently sells for \$55.

(ii) In one year, the stock will either sell for \$70 or \$45.

(iii) The *effective annual* risk-free interest rate is 10%.

Calculate the price you now pay for the strangle.

Problem 4.6.2. (Valuing a derivative that pays the square of terminal stock price) The current price of a nondividend-paying stock is $S(0) = 100$. The price of the stock at the end of one year, $S(1)$, is either 90 or 120. The continuously compounded risk-free interest rate is 10%.

Consider a derivative security that pays $[S(1)]^2$ at the end of one year.

(a) Determine the replicating portfolio for the derivative, i.e., find B and Δ.

(b) Calculate the current price of the derivative.

Problem 4.6.3. (Valuing a strange derivative) Suppose that the current stock price is \$30 per share. At the end of 6 months, the stock price will be either \$25 or \$38. The 6-month effective risk-free interest rate is 10%.

A European-type derivative written on this stock has its payoff in 6 months equal to

$$[S(T) - 32]_+^2 + [28 - S(T)]_+^3,$$

where $S(T)$ is the stock price at maturity.

Calculate the current price of this derivative.

Problem 4.6.4. (Arbitraging a mispriced bear spread in a one-period binomial model) You are given:

(i) The current price of a stock is \$65.

(ii) One year from now the stock will sell for either \$60 or \$70.

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 4%.

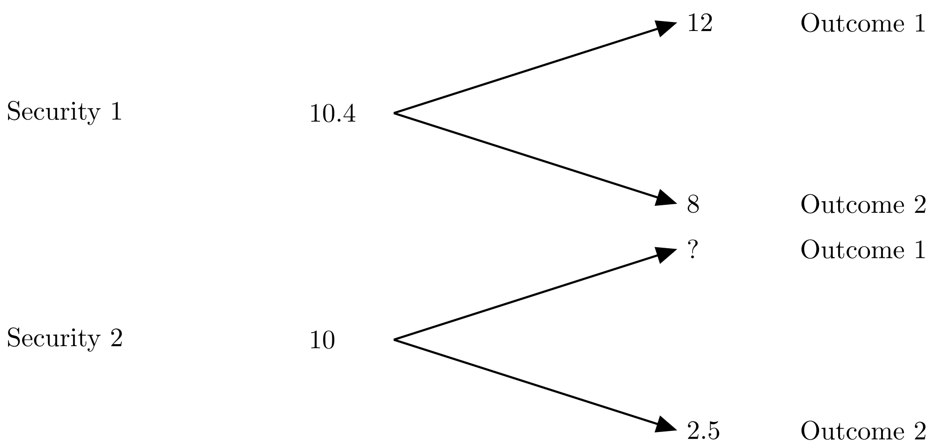
(iv) The continuously compounded risk-free interest rate is 6%.

(v) The current price of a one-year 65-75 European put bear spread on the above stock is \$6.50.

Describe transactions (i.e., what to buy/sell/borrow/lend) that one should enter into to exploit an arbitrage opportunity (if one exists). Show your work.

Problem 4.6.5. [HARDER!] (A market consisting only of risky securities – I) For a one-period arbitrage-free binomial model with two nondividend-paying securities, you are given:

(i) The following price evolution of the two securities:



(ii) The following information about two European call options:

Call Option	Underlying Asset	Strike Price	Current Price
A	Security 1	9	1.8
B	Security 2	11	??

Calculate the current price of call option B.

(Hints:

(i) This market does not have a risk-free bond, at least in its current form. Without the risk-free rate, the risk-neutral probability cannot be defined. If you use risk-neutral valuation, you need to explain in your solution how the risk-free bond can be constructed and what the risk-free interest rate is. If these prove too hard, go back to basics and use the method of replication instead.

(ii) In any case, you may find it useful to realize that a replicating portfolio can consist of any securities at your choosing. Why not replicate using the given securities and/or call options with known prices and/or payoffs!?)

Problem 4.6.6. (A market consisting only of risky securities – II) In an arbitrage-free securities market, there are two nondividend-paying stocks, A and B, both with current price \$90. There are two possible outcomes for the prices of A and B one year from now:

Outcome	A	B
1	\$100	\$80
2	\$60	\$$x$

The current price of a one-year 100-strike European put option on B is \$15. Determine all possible values of x.

Problem 4.6.7. (Calculation of p^* given volatility information) You are to use a binomial forward tree model to model the price movements of a stock that pays dividends continuously at a rate proportional to its price. The length of each period is three months.

If $\mathrm{Var}[\ln[S(t)]] = 0.25t$ for $t > 0$, find p^*, the risk-neutral probability of an up move.

Multi-period binomial trees (European options)

Problem 4.6.8. [HARDER!] (A two-period binomial model with varying u, d, and i) You are given the following with respect to a public company:

- The common shares of the company were trading at 100 as of December 31, 2015.
- No dividends are paid.
- An industry analyst has projected the possible stock prices over the next two years as a function of the performance of the US economy:

US Economy		Share Price of Company	
2016	2017	December 31, 2016	December 31, 2017
Expansion	Expansion	110	120
Expansion	Recession	110	100
Recession	Expansion	95	110
Recession	Recession	95	90

- The effective annual risk-free interest rate that prevails in Year 2016 (i.e., from January 1, 2016 to December 31, 2016) is 5%.
- The effective annual risk-free interest rate that prevails in Year 2017 is 6% if 2016 has seen an expansion, and 4% if 2016 has seen a recession.

Using the analyst's projections, determine the price, as of December 31, 2015, of a two-year European strangle constructed by buying a 100-strike European put option and a 110-strike European call option.

(Remember that a multi-period binomial tree is essentially a series of one-period binomial trees. In the context of this problem, these one-period trees have different u, d, i, and p^*.)

Problem 4.6.9. (Pricing a European strangle in a three-period binomial forward tree) You use the following information to construct a binomial forward tree for modeling the price movements of a stock:

(i) The length of each period is 1 year.

(ii) The current stock price is 190.

(iii) The stock's volatility is 30%.

(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 5%.

(v) The continuously compounded risk-free interest rate is 3%.

Calculate the price of a 3-year 150-250 European strangle.

Problem 4.6.10. [HARDER!] (Valuing a straddle in a 10-period binomial tree) For a 10-period binomial stock price model, you are given:

(i) The length of each period is one year.

(ii) The current stock price is 1,000.

(iii) At the end of every year, the stock price will either increase by 5% or decrease by 5% in proportion.

(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 1%.

(v) The continuously compounded risk-free interest rate is 2%.

Calculate the current price of a 10-year 1,400-strike European straddle on the stock.

(Hint: Think twice before you discount the eleven possible 10-year payoffs of the straddle back to time 0! That could take you half an hour!)

Problem 4.6.11. (A derivative with payoff dependent on interim stock price) Let $S(t)$ be the time-t price of a nondividend-paying stock. For a three-period binomial stock price model, you are given:

(i) The length of each period is one year.

(ii) $S(0) = 100$.

(iii) $u = 1.1$, where u is one plus the percentage change in the stock price per period if the price goes up.

(iv) $d = 1/1.1$, where d is one plus the percentage change in the stock price per period if the price goes down.

(v) The continuously compounded risk-free interest rate is 5%.

Consider a special derivative which pays, at the end of three years,

$$\max\{S(2) - 100, 0\} + \max\{S(3) - 100, 0\}.$$

Calculate the current price of this derivative.

(Note: Note that the time-2 stock price $S(2)$ that constitutes the payoff formula is paid at time 3. This derivative, whose payoff depends on intermediate stock prices, is an example of a *path-dependent* derivative, more examples of which will be given in Sections 8.4 to 8.7.)

Problem 4.6.12. [HARDER!] (Choosing between a call or a put) For a two-period binomial model for stock prices, you are given:

(i) The length of each period is one year.

(ii) The current price of a nondividend-paying stock is \$150.

(iii) $u = 1.25$, where u is one plus the percentage change in the stock price per period if the price goes up.

(iv) $d = 0.80$, where d is one plus the percentage change in the stock price per period if the price goes down.

(v) The continuously compounded risk-free interest rate is 6%.

Consider a *chooser option* (also known as an as-you-like-it option) on the stock. At the end of the first year, its holder will choose, to his/her advantage, whether it becomes a European call option or a European put option, both of which will expire at the end of the second year with a strike price of \$150.

Calculate the current price of the chooser option.

(Hint: If you were the holder of the chooser option and you were rational, which option, the call or the put, would you choose at the end of one year?)

Problem 4.6.13. [HARDER!] (Valuing an option on another option in the binomial setting) For a two-period binomial model for stock prices, you are given:

(i) The length of each period is one year.

(ii) The current price of a nondividend-paying stock is $40.

(iii) $u = 1.05$, where u is one plus the percentage change in the stock price per period if the price goes up.

(iv) $d = 0.9$, where d is one plus the percentage change in the stock price per period if the price goes down.

(v) The continuously compounded risk-free interest rate is 3%.

Consider Derivative X, which gives its holder the right, but not the obligation, to buy a $38-strike European put option at the end of the first year for $0.5. This put option is written on the stock and will mature at the end of the second year.

(a) Calculate the current price of Derivative X.

(Hint: What is the underlying asset of Derivative X? The stock, or something else? Does the price of this asset evolve according to a binomial tree model?)

(b) Using the result of part (a), calculate the current price of Derivative Y, which is identical to Derivative X, except that it gives its holder the right to *sell* the same put option for $0.5 at the end of the first year.

Problem 4.6.14. [HARDER!] (Valuing a forward on a call in the binomial setting) For a binomial forward tree modeling the price movements of a stock, you are given:

(i) The length of each period is 6 months.

(ii) The current price of a nondividend-paying stock is $9,000.

(iii) The stock's volatility is 32%.

(iv) The continuously compounded risk-free interest rate is 20%.

Consider the following offer:

> By receiving this offer, 6 months from now you are *obligated* to buy a 6-month $9,000-strike European call option on the stock for $1,500. This call option expires one year from now.

Calculate the current fair price of this offer.

Problem 4.6.15. [HARDER!] (Arbitraging a mispriced option in a two-period binomial tree) You are given the following regarding the stock of Iowa Actuarial Association (IAA):

(i) The stock is currently selling for \$100.

(ii) $u = 1.1$, where u is one plus the percentage change in the stock price per period if the price goes up.

(iii) $d = 0.9$, where d is one plus the percentage change in the stock price per period if the price goes down.

(iv) The stock pays no dividends.

The effective annual risk-free interest rate is 2%.

While reading the *Well Street Journal*, Peter notices that a two-year at-the-money European call written on the stock of IAA is selling for \$7.5. Peter wonders whether this call is fairly priced. He uses the binomial option pricing model to determine if an arbitrage opportunity exists.

Construct the trading strategies for Peter to exploit the arbitrage opportunity (if one exists).

(Hint:

- This problem is a two-period version of Example 4.1.5. Now "Michael" becomes "Peter" and "Financial Post" becomes "*Well Street Journal*"!
- Peter may not just sit on the sofa and rest leisurely as soon as the strategies are set up at time 0. He may need to "do something" in response to the stock market environment. In your solution, describe clearly what Peter needs to do at what time and at what stock price level. Check, at least mentally, that your strategies indeed lead to an arbitrage.
- This problem is another illustration of the fundamental value of the method of replication.)

Problem 4.6.16. (Swapping parameters: Binomial tree setting) Two actuaries, A and B, use a two-period binomial forward tree to compute the prices of a European call and a European put using different parameters.

You are given:

Actuary	Option	Underlying Stock Price	Strike Price	Dividend Yield	Risk-free Interest Rate	Stock Volatility	Option Maturity
A	Call	190	200	5%	3%	30%	1
B	Put	200	190	3%	5%	30%	1

Describe the relationship between the call price computed by Actuary A and the put price computed by Actuary B.

Multi-period binomial trees (American options)

Problem 4.6.17. (European vs American prices: Two-period version) You use the following information to construct a binomial forward tree for modeling the price movements of a nondividend-paying stock:

(i) The length of each period is 6 months.

(ii) The current stock price is 100.

(iii) The stock's volatility is 20%.

(iv) The continuously compounded risk-free interest rate is 6%.

Let P_I be the price of a 120-strike 1-year European put option on the stock, and P_{II} be the price of an otherwise identical American put option.

Calculate $P_{II} - P_I$.

(Hint: Be sure to check whether early exercise is optimal at ALL possible nodes, including...)

Problem 4.6.18. (European vs American prices: Three-period version) You use the following information to construct a binomial forward tree for modeling the price movements of a stock.

(i) The length of each period is 4 months.

(ii) The current stock price is 100.

(iii) The stock's volatility is 30%.

(iv) The stock pays no dividends.

(v) The continuously compounded risk-free interest rate is 8%.

Let P_I be the price of a 95-strike 1-year European put option on the stock, and P_{II} be the price of an otherwise identical American put option.

Calculate $P_{II} - P_I$.

Problem 4.6.19. (Three-period binomial tree for an American put) You use the following information to construct a binomial forward tree for modeling the price movements of a stock:

(i) The length of each period is 4 months.

(ii) The current stock price is 55.

(iii) The stock's volatility is 30%.

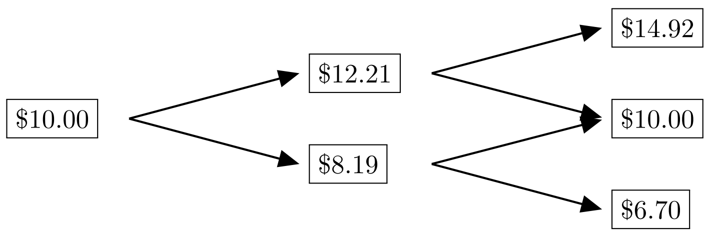
(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3.5%.

(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of a 1-year 50-strike American put option on the stock.

Problem 4.6.20. (SOA Exam FETE Fall 2011 Question 7 (c): Pricing a warrant as an American call with varying strike prices) The Ashwaubenon Company (Ash Co) needs to raise capital to support its rapidly growing business. One proposal is to publicly issue a certain number of equity units, each of which consists of one share of stock and a warrant to purchase one share of stock.

Assume that the price of the underlying asset follows a binomial tree with 1-year time steps as follows:



The warrant provides the right to purchase one share of the stock for \$9 at the first anniversary or, if not exercised, for \$10 at the second anniversary.

Assume further that:

- The stock pays no dividend.
- The risk-free interest rate is 4% per annum.

Calculate the value of the warrant using the binomial tree.

(Note: There are many reasons for structuring the warrant to have the exercise price change with the exercise date. One of them is to encourage earlier exercise of the warrant allowing for better alignment of cash flow needs for a rapidly growing company.)

Options on other assets

Problem 4.6.21. (Pricing an American currency put using a binomial forward tree) You use the following information to construct a two-period binomial forward tree for modeling the movements of the dollar-euro exchange rate:

(i) The current dollar-euro exchange rate is \$1.50/€.

(ii) The volatility of the exchange rate is 20%.

(iii) The continuously compounded risk-free interest rate on dollars is 4%.

(iv) The continuously compounded risk-free interest rate on euros is 5%.

Calculate the price of a 6-month \$1.80-strike dollar-denominated American put option on euros.

Problem 4.6.22. [HARDER!] (Four-period binomial tree for a Bermudan currency call) For a four-period binomial tree model for the dollar/pound exchange rate, you are given:

(i) The length of each period is 3 months.

(ii) The current dollar/pound exchange rate is 1.4.

(iii) $u = 1.1$ and $d = 0.9$, where u and d are one plus the percentage change in the dollar/pound exchange rate per period if the exchange rate goes up and if the exchange rate goes down, respectively.

(iv) The continuously compounded risk-free interest rate on dollars is 8%.

(v) The continuously compounded risk-free interest rate on pounds is 7%.

Calculate the price of a 1-year at-the-money dollar-denominated Bermudan call option on pounds, where exercise is allowed at any time following an initial 9-month period of call protection (i.e., exercise is allowed in 9 months and thereafter).

(Hint: There is no need to construct the whole 4-period binomial tree. Generate only as many exchange rates as needed.)

Problem 4.6.23. (Replicating portfolio for a futures call option) You use the following information to construct a binomial tree for modeling the price movements of a futures contract on the S&V 150:

(i) The length of each period is 6 months.

(ii) The initial futures price is 500.

(iii) $u = 1.2363$, where u is one plus the percentage change in the futures price per period if the price goes up.

(iv) $d = 0.8089$, where d is one plus the percentage change in the futures price per period if the price goes down.

(v) S&V 150 pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

(vi) The continuously compounded risk-free interest rate is 6%.

Consider a 1-year at-the-money *American* call option on the above S&V 150 futures contract.

Determine the replicating portfolio of the put option *at the initial node.*

Pricing by true probabilities

Problem 4.6.24. (Expected rate of return on an option in a one-period binomial model) You use the following information to construct a one-period binomial forward tree for modeling the price movements of a nondividend-paying stock:

(i) The current stock price is 82.

(ii) The stock's volatility is 30%.

(iii) The continuously compounded risk-free interest rate is 8%.

(iv) The continuously compounded expected return on the stock is 15%.

Calculate the expected rate of return on a 80-strike 1-year European call option.

Problem 4.6.25. (First-period expected rate of return on an option in a two-period binomial model) You use the following information to construct a binomial forward tree for modeling the price movements of a nondividend-paying stock:

(i) The length of each period is 4 months.

(ii) The current stock price is 50.

(iii) The stock's volatility is 30%.

(iv) The continuously compounded risk-free interest rate is 8%.

(v) The continuously compounded expected return on the stock is 15%.

Calculate the continuously compounded true discount rate for an 8-month 40-strike European call option during the first time period.

Problem 4.6.26. (Expected rate of return on an option in a three-period binomial model) You use the following information to construct a binomial forward tree for modeling the price movements of a nondividend-paying stock:

(i) The length of each period is 4 months.

(ii) The current stock price is 123.

(iii) The stock's volatility is 30%.

(iv) The continuously compounded risk-free interest rate is 8%.

(v) The continuously compounded expected return on the stock is 12%.

Calculate the continuously compounded expected rate of return on a 120-strike 1-year European call option *over its 1-year life.*

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5

Mathematical Foundations of the Black-Scholes Framework

Chapter overview: In this rudimentary chapter, we migrate from the discrete-time binomial tree model in the preceding chapter to the most celebrated continuous-time option pricing model commonly referred to as the *Black-Scholes option pricing model*, rightfully due to the two prominent mathematicians Fischer Black and Myron Scholes. In contrast to the binomial tree model, which postulates that future stock prices unfold in a binomial manner, the principal assumption of the Black-Scholes model is that future stock prices evolve continuously in accordance with a lognormal distribution. The primary objective of this chapter is to make this statement mathematically precise and set up the lognormal stock price model so as to pave the way for performing option pricing in Chapter 6. In Section 5.1, we introduce the lognormal distribution, which is the continuous probability distribution that underlies the Black-Scholes model and governs the probabilistic behavior of each future stock price, and formulate the so-called "Black-Scholes framework." In the Black-Scholes context, some simple probabilistic calculations are performed in Section 5.2, including the calculation of the exercise probability of a European option, unconditional and conditional expected stock prices, and the construction of lognormal prediction intervals for stock prices. These probabilistic quantities will be instrumental in deriving the Black-Scholes option pricing formula for European options in the next chapter.

5.1 A Lognormal Model of Stock Prices

The Black-Scholes option pricing model assumes that stock prices in the future obey a lognormal distribution, which can be seen as the limit of the binomial tree model as the number of periods n approaches infinity. Because the stock prices process is a collection of lognormal random variables lumped over time, we will occasionally use the non-standard term "*lognormal process*."[i] Given the close relevance of the lognormal distribution to continuous-time option pricing, we begin with a description of its key properties.

Definition of the lognormal distribution.

I assume that you are no strangers to the normal distribution, justifiably the most well-known continuous distribution, based on your learning in prior statistics courses. If you need to refresh your memory, you may refer to your favorite probability textbook.

The normal distribution gives rise to a closely related distribution, called the lognormal distribution, which is useful for modeling stock prices. A random variable Y follows a

[i]The mathematically impeccable assumption of the Black-Scholes model is that stock prices follow a geometric Brownian motion, which entails more than just the stock prices being marginally lognormally distributed. The notion of this stochastic process is beyond the scope of this book.

lognormal distribution with parameters m and v^2 if the natural logarithm of Y follows a normal distribution with the same parameters m and v^2 (mean and variance, respectively). The nomenclature can be explained by the fact that if we "log" a lognormal random variable, then it becomes "normal." We shall write $Y \sim \mathrm{LN}(m, v^2)$ to represent such a lognormal random variable. The probability density function of Y is

$$f_Y(y; m, v^2) = \frac{1}{\sqrt{2\pi v^2} y} \exp\left[-\frac{(\ln y - m)^2}{2v^2}\right], \quad y > 0,$$

although we have rather limited use of this expression in this book.

Moments of a lognormal random variable.

It is possible to give a direct but clumsy proof of the expectation of a lognormal random variable by brute-force integration. A more instructive way to obtain lognormal moments is to realize that if Y is lognormal with parameters m and v^2, then Y has the same distribution as e^X, where X is a normal random variable with mean m and variance v^2. Therefore, for any real k,

$$\mathbb{E}[Y^k] = \mathbb{E}[\mathrm{e}^{kX}] = M_X(k) = \underbrace{\exp\left(km + \frac{1}{2}k^2 v^2\right)}_{\text{remember?}},$$

which is the moment generating function of X evaluated at k. In particular,

$$\mathbb{E}[Y] = \mathrm{e}^{m+v^2/2}, \qquad \mathbb{E}[Y^2] = \mathrm{e}^{2(m+v^2)},$$

and the variance of Y is

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \mathrm{e}^{2m+v^2}(\mathrm{e}^{v^2} - 1).$$

Lognormal stock price model.

Armed with the basic facts about the lognormal distribution, we are now ready to formulate a lognormal model of stock prices, which lies at the heart of the Black-Scholes framework. For $t \geq 0$, we denote by $S(t)$ the time-t price of the underlying stock paying continuous proportional dividends at a rate of δ. In the Black-Scholes model, we assume that stock prices are configured by

$$\boxed{S(t) = S(0) \exp\left[\left(\alpha - \delta - \frac{1}{2}\sigma^2\right) t + \sigma\sqrt{t} Z\right],} \tag{5.1.1}$$

for some non-negative parameters α and σ, and some standard normal random variable Z[ii] governing the evolution of the stock prices. This is indeed a lognormal model because the logarithm of each stock price, given by $\ln S(t) = \ln S(0) + (\alpha - \delta - \sigma^2/2)t + \sigma\sqrt{t}Z$, is normally distributed as

$$\boxed{\ln S(t) \sim \mathrm{N}\left[\ln S(0) + \left(\alpha - \delta - \frac{1}{2}\sigma^2\right) t,\ \sigma^2 t\right],} \tag{5.1.2}$$

[ii] Rigorously speaking, the random variable Z varies with the time t. That is, for different choices of time, the stock prices will involve different Z's, which are dependent random variables. This subtle fact will not concern us because in this book we will mostly look at stock prices in isolation.

or equivalently,

$$\boxed{S(t) \sim \mathrm{LN}\left[\underbrace{\ln S(0)}_{\text{Don't omit this!}} + \left(\alpha - \delta - \frac{1}{2}\sigma^2\right)t,\ \sigma^2 t\right].}$$

Another way to express (5.1.1) is that the continuously compounded expected rate of return (also known as the capital gain) on the stock from time 0 to any future time t, $\ln[S(t)/S(0)]$, is normally distributed, with the following specification:

$$\ln\left[\frac{S(t)}{S(0)}\right] \sim \mathrm{N}\left[\left(\alpha - \delta - \frac{1}{2}\sigma^2\right)t,\ \sigma^2 t\right] \tag{5.1.3}$$

To put it simply, the *normality* of the continuously compounded rates of return on the stock is identical to the *lognormality* of the stock prices.

It is undeniably true that the configuration of the parameters α and σ given in (5.1.3) is rather difficult to remember and not intuitive at all. In particular, it is not immediately clear why $\sigma^2/2$ needs to be subtracted from the mean of the rate of return. In fact, α and σ enter in such a way that makes α the continuously compounded (annualized) expected *total* rate of return on the stock and σ the (annualized) volatility of the stock. To see this, we take expectation on both sides of (5.1.1) and get

$$\mathbb{E}[S(t)] = S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)t} \times \underbrace{\mathrm{e}^{\sigma^2 t/2}}_{\text{why?}} = S(0)\mathrm{e}^{(\alpha-\delta)t},$$

which is equivalent to

$$\underbrace{S(0)}_{\text{beginning value}}\mathrm{e}^{\alpha t} = \underbrace{\mathrm{e}^{\delta t}\mathbb{E}[S(t)]}_{\text{expected ending value}}.$$

This confirms that α is the continuously compounded expected rate of return on the stock. Moreover, it follows from (5.1.2) that

$$\mathrm{Var}[\ln S(t)] = \mathrm{Var}\left[\ln \frac{S(t)}{S(0)}\right] = \sigma^2 t,$$

showing that σ is the volatility of the stock price (recall the general definition of volatility on page 126). In brief, the seemingly complex arrangement of terms in (5.1.1) is meant to make the model parameters easily interpretable.

Figure 5.1.1 exhibits a typical path of the stock prices in the Black-Scholes framework over a 1-year horizon with the parameters $S(0) = 100$, $\alpha = 0.08$, and $\sigma = 0.3$. The path can be viewed as the superposition of infinitely many binomial trees to produce continuous evolution.

Example 5.1.1. (Specifying the distribution of $S(T)$) Assume the Black-Scholes framework. Let $S(t)$ denote the time-t price of a nondividend-paying stock. You are given:

(i) The current stock price is 38.

(ii) The stock's volatility is 35%.

(iii) The continuously compounded expected return on the stock is 16%.

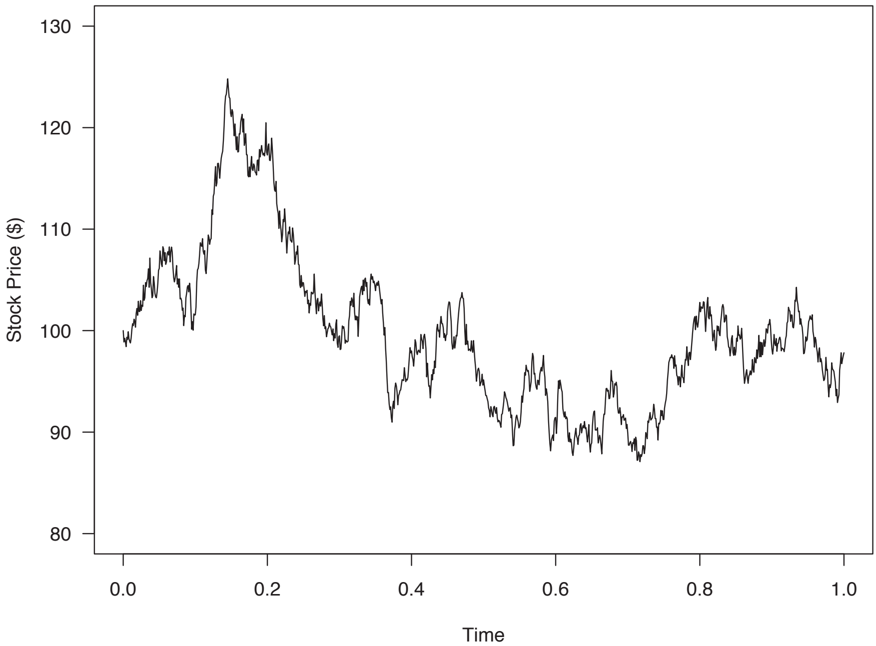


FIGURE 5.1.1
A stock price path in the Black-Scholes stock price model with $S(0) = 100$, $\alpha = 0.08$, and $\sigma = 0.3$.

(iv) The continuously compounded risk-free interest rate is 5%.

Determine the probability distribution of $S(0.5)$.

Solution. The 6-month stock price $S(0.5)$ is lognormal with parameters

$$m = \ln S(0) + \left(\alpha - \delta - \frac{1}{2}\sigma^2\right) t = \ln 38 + (0.16 - \frac{1}{2} \times 0.35^2)(0.5) = \boxed{3.6951}$$

and

$$v^2 = 0.35^2(0.5) = \boxed{0.06125}.$$

□

5.2 Lognormal-Based Probabilistic Quantities

The fact that each future stock price is lognormally distributed according to (5.1.1) allows us to easily calculate a number of distributional quantities of theoretical and practical interest. In this section, we concern ourselves with the exercise probability of a European option, unconditional and conditional expected stock prices, and lognormal prediction intervals for stock prices. Some of these quantities will provide us with the necessary ingredients to form the Black-Scholes option pricing formula for European options in Chapter 6.

Quantity 1: Exercise probabilities.

The first probabilistic quantity of interest is the (physical) probability that a T-year K-strike European call will be exercised at expiration:

$$\mathbb{P}\left(S(T) > K\right).$$

Using (5.1.1), we have

$$\begin{aligned}
\mathbb{P}\left(S(T) > K\right) &= \mathbb{P}\left(S(0)\exp\left[\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z\right] > \ln K\right)\\
&= \mathbb{P}\left(Z > \frac{\ln[K/S(0)]-(\alpha-\delta-\sigma^2/2)T}{\sigma\sqrt{T}}\right)\\
&= 1-N\left[\frac{\ln[K/S(0)]-(\alpha-\delta-\sigma^2/2)T}{\sigma\sqrt{T}}\right]\\
&= N\left[\frac{\ln[S(0)/K]+(\alpha-\delta-\sigma^2/2)T}{\sigma\sqrt{T}}\right],
\end{aligned}$$

where $N(\cdot)$ is the distribution function of the standard normal distribution (you may have used $\Phi(\cdot)$ in your prior statistics courses) and the last equality follows from the symmetry of $N(\cdot)$ about zero, i.e., $N(x)+N(-x)=1$ for all $x \in \mathbb{R}$. If we let

$$\hat{d}_2 = \frac{\ln[S(0)/K]+(\alpha-\delta-\sigma^2/2)T}{\sigma\sqrt{T}}, \tag{5.2.1}$$

then the above exercise probability can be compactly written as

$$\boxed{\mathbb{P}\left(S(T) > K\right) = N(\hat{d}_2).} \tag{5.2.2}$$

The reason why we decorate the symbol d by the subscript "2" and with a hat will become apparent in Chapter 6. Likewise, the exercise probability for an otherwise identical European put is

$$\boxed{\mathbb{P}\left(S(T) < K\right) = 1-\mathbb{P}\left(S(T) > K\right) = N(-\hat{d}_2).}$$

As you work out problems in this book, you will frequently need values of the standard normal distribution function $N(x)$ and standard normal quantiles $N^{-1}(p)$ for various $x \in \mathbb{R}$ and $p \in (0,1)$. For this purpose, it is strongly suggested that you have access to a standard normal distribution function calculator (e.g., actuarial students who will take Exam IFM, which is computerized, can use `https://www.prometric.com/en-us/clients/soa/pages/mfe3f_calculator.aspx`, which is capable of computing $N(x)$ and $N^{-1}(p)$ to five decimal places). For your convenience, a standard normal distribution table is provided in Appendix A of this book. In what follows we will follow the IFM standard normal calculator and display intermediate results to five decimal places.

Example 5.2.1. (CAS Exam 8 Spring 2000 Question 30: Probability that a call will be exercised – I) You are given the following information about a European call option.

- The current stock price is \$35.
- The exercise price is \$40.
- The option matures in 6 months.
- The expected return on the stock is 18% per annum.
- The volatility, σ, of the stock price is 24% per annum.
- The stock's price at each future time T, given its current price, is lognormally distributed.
- The stock pays no dividends.

What is the probability that the call option will be exercised?

Solution. Since

$$\hat{d}_2 = \frac{\ln(35/40) + (0.18 - 0 - 0.24^2/2)(0.5)}{0.24\sqrt{0.5}} = -0.34136,$$

the exercise probability for the call option is

$$\mathbb{P}\left(S(0.5) > 40\right) = N(\hat{d}_2) = 1 - \underbrace{N(0.34136)}_{0.63358} = \boxed{0.36642}.$$

□

Example 5.2.2. (SOA Exam MFE Spring 2009 Question 16: Probability that a call will be exercised II) You are given the following information about a nondividend-paying stock:

(i) The current stock price is 100.

(ii) Stock prices are lognormally distributed.

(iii) The continuously compounded expected return on the stock is 10%.

(iv) The stock's volatility is 30%.

Consider a nine-month 125-strike European call option on the stock.
Calculate the probability that the call will be exercised.

(A) 24.2%

(B) 25.1%

(C) 28.4%

(D) 30.6%

(E) 33.0%

Solution. With

$$\hat{d}_2 = \frac{\ln(100/125) + (0.1 - 0 - 0.3^2/2)(0.75)}{0.3\sqrt{0.75}} = -0.70011,$$

the exercise probability for the call option is

$$\mathbb{P}\left(S(0.75) > 125\right) = N(\hat{d}_2) = N(-0.70011) = \boxed{0.24193}. \quad \textbf{(Answer: (A))}$$

□

Quantity 2: Lognormal prediction intervals for stock prices.

The next probabilistic quantity we are interested in is a range of values in which the stock price has a high probability of lying. To be precise, given any $t \geq 0$ and $p \in (0, 1)$, we are to find a pair of constants (S^L, S^U) (depending on t and p) such that

$$\mathbb{P}(S^L < S(t) < S^U) = 1 - p.$$

We then call (S^L, S^U) a $100(1-p)\%$ *lognormal prediction interval* for $S(t)$. Quite often, we require the prediction interval to be equal-tailed in the sense that

$$\mathbb{P}(S(t) < S^L) = \mathbb{P}(S(t) > S^U) = \frac{p}{2}.$$

As you will see soon, using $1-p$ as the probability level instead of p makes the construction of the prediction interval easier.

A note about the term "prediction interval" is in order. Here we are forming two deterministic constants, S^L and S^U, which will bound the *random* variable $S(t)$ with the prespecified probability of $1 - p$. This is in contradistinction to the usual setting in mathematical statistics in which we use a pair of random variables, called a *confidence interval*, to estimate a *constant* parameter (e.g., the mean or variance of a normal distribution). The differences between prediction intervals and confidence intervals are summarized in the following table:

Type of interval	Nature of bounds of interval	Nature of target
<u>Confidence</u> interval	Random variables (i.e., estimators/statistics)	Constant parameter
<u>Prediction</u> interval	Can be random variables or constants	Random variable

To determine a $100(1-p)\%$ prediction interval for $S(t)$ for any $t \geq 0$, we first construct a $100(1-p)\%$ prediction interval for Z, the standard normal random variable "driving" $S(t)$. In terms of the inverse of the standard normal distribution function, we have

$$\mathbb{P}\left(N^{-1}(p/2) < Z < N^{-1}(1-p/2)\right) = 1 - p,$$

so the $100(1-p)\%$ equal-tailed prediction interval for Z is $\left(N^{-1}(p/2), N^{-1}(1-p/2)\right)$. Then we go from Z to $S(t)$ by

$$\mathbb{P}\left(S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)t+\sigma\sqrt{t}N^{-1}(p/2)} < S(t) < S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)t+\sigma\sqrt{t}N^{-1}(1-p/2)}\right) = 1 - p,$$

meaning that

$$\boxed{(S^L, S^U := \left(S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)t+\sigma\sqrt{t}N^{-1}(p/2)}, S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)t+\sigma\sqrt{t}N^{-1}(1-p/2)}\right)} \tag{5.2.3}$$

is the desired $100(1-p)\%$ equal-tailed prediction interval for $S(t)$. An easy way to remember (5.2.3) is that the two end-points in (5.2.3) are obtained by replacing Z in the stock price equation (5.1.1) by the two end-points of the corresponding prediction interval for Z, i.e., $N^{-1}(p/2)$ and $N^{-1}(1-p/2)$.

As a matter of fact, prediction intervals are not unique. For example,

$$\left(0, S(0)\exp\left[\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}N^{-1}(1-p)\right]\right)$$

and

$$\left(S(0)\exp\left[\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}N^{-1}(p)\right], \infty\right)$$

are also $100(1-p)\%$ prediction intervals for $S(t)$ (why do they work?). They are sometimes referred to as one-sided or one-tailed prediction intervals. However, unless otherwise stated, in this book we always use (5.2.3), which is a two-sided prediction interval based on a *symmetric* prediction interval for the corresponding normal random variable $\ln S(t)$ (although (5.2.3) is not symmetric about the mean of $S(t)$).

Example 5.2.3. (SOA Exam IFM Advanced Derivatives Sample Question 50: Upper limit of a lognormal prediction interval) Assume the Black-Scholes framework.

You are given the following information for a stock that pays dividends continuously at a rate proportional to its price.

(i) The current stock price is 0.25.

(ii) The stock's volatility is 0.35.

(iii) The continuously compounded expected rate of stock-price appreciation is 15%.

Calculate the upper limit of the 90% lognormal prediction interval for the price of the stock in 6 months.

(A) 0.393

(B) 0.425

(C) 0.451

(D) 0.486

(E) 0.529

Solution. We are given from (i), (ii), and (iii) that

$$S(0)=0.25, \quad \sigma=0.35, \quad \alpha-\delta=0.15.$$

With $t=0.5$ and $p=1-0.9=0.1$, the upper limit of the 90% lognormal prediction interval for the 6-month stock price is

$$\begin{aligned} S^U(0.5) &= S(0.5)\exp\left[\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t+N^{-1}\left(1-\frac{p}{2}\right)\sigma\sqrt{t}\right] \\ &= 0.25\exp\left\{\left[0.15-\frac{1}{2}(0.35)^2\right]\times 0.5+\underbrace{N^{-1}(0.95)}_{1.64485}0.35\sqrt{0.5}\right\} \\ &= \boxed{0.3926}. \qquad \textbf{(Answer: (A))} \end{aligned}$$

□

Example 5.2.4. (SOA Exam FETE Spring 2012 Question 7 (a)–(b): Distributional quantities of a lognormal stock price) You are given the following information on a stock:

Initial Price	\$25
Expected Annual Return per annum	8%
Estimated Annual Volatility	20%

Assume that the log returns are normally distributed.

(a) Calculate the 95% prediction interval for the stock price in one year.

(b) Calculate the expected stock price and the standard deviation of the stock price in one year.

Solution. (a) As

$$\ln S(1) \sim \mathrm{N}(\ln 25 + (0.08 - {}^1\!/\!_2 \times 0.2^2), 0.2^2) \equiv \mathrm{N}(3.278876, 0.04),$$

a 95% prediction interval for $S(1)$ is

$$\mathrm{e}^{3.278876 \pm 1.96\sqrt{0.04}} = \boxed{(17.94, 39.29)}.$$

(b)
- $\mathbb{E}[S(1)] = S(0)\mathrm{e}^{\alpha T} = 25\mathrm{e}^{0.08(1)} = \boxed{27.08}$
- $\mathrm{Var}[S(1)] = \mathrm{e}^{2(3.278876)+0.04}(\mathrm{e}^{0.04} - 1) = 29.93$, so $\sqrt{\mathrm{Var}[S(1)]} = \sqrt{29.93} = \boxed{5.47}$. □

Quantity 3: Conditional expected stock prices.

Apart from exercise probabilities for European options and lognormal prediction intervals, we are also interested in the expected value of the terminal stock price, given that the European option in question finishes in-the-money. Mathematically, we aim to determine the two conditional expectations

$$\mathbb{E}[S(T)|\underbrace{S(T) > K}_{\substack{\text{call expires}\\ \text{in-the-money}}}] \quad \text{and} \quad \mathbb{E}[S(T)|\underbrace{S(T) < K}_{\substack{\text{put expires}\\ \text{in-the-money}}}],$$

where T and K are the maturity time and strike price of the European option, respectively.

To evaluate these conditional expectations, we rewrite them in terms of unconditional ones, recalling the definition for any random variable X and event A with a non-zero probability:

$$\mathbb{E}[X|A] := \frac{\mathbb{E}[X 1_A]}{\mathbb{P}(A)},$$

where 1_A is the indicator function of A, i.e., $1_A = 1$ if A is true, and $1_A = 0$ otherwise. Applying this definition to $X = S(T)$ and $A = \{S(T) > K\}$, we have

$$\mathbb{E}[S(T)|S(T) > K] = \frac{\mathbb{E}[S(T)1_{\{S(T)>K\}}]}{\mathbb{P}(S(T) > K)}.$$

The denominator is precisely the exercise probability of the K-strike T-year European call studied earlier, i.e., $\mathbb{P}(S(T) > K) = N(\hat{d}_2)$. For the numerator, recall from page 175 that

$$S(T) > K \quad \Leftrightarrow \quad Z > \frac{\ln[K/S(0)] - (\alpha - \delta - \sigma^2/2)T}{\sigma\sqrt{T}} = -\hat{d}_2.$$

Regarding $S(T) = S(0)\exp[(\alpha - \delta - \sigma^2/2)t + \sigma\sqrt{t}Z]$ as a function of the standard normal random variable Z, we can evaluate

$$\begin{aligned}
\mathbb{E}[S(T)1_{\{S(T)>K\}}] &= \int_{-\hat{d}_2}^{\infty} S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)T+\sigma\sqrt{T}z} \times \underbrace{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}}_{\text{standard normal p.d.f. } \phi(z)} \,\mathrm{d}z \\
&= S(0)\mathrm{e}^{(\alpha-\delta)T}\int_{-\hat{d}_2}^{\infty} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(z-\sigma\sqrt{T})^2/2}\,\mathrm{d}z \\
&= S(0)\mathrm{e}^{(\alpha-\delta)T}\int_{-\hat{d}_2-\sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}\,\mathrm{d}z \quad \text{(change of variable)} \\
&= S(0)\mathrm{e}^{(\alpha-\delta)T}\int_{-\infty}^{\hat{d}_2+\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}\,\mathrm{d}z \quad \text{(symmetry of } \phi(z)\text{)} \\
&= S(0)\mathrm{e}^{(\alpha-\delta)T}N(\hat{d}_1),
\end{aligned}$$

where

$$\hat{d}_1 = \hat{d}_2 + \sigma\sqrt{T} = \frac{\ln[S(0)/K] + \left(\alpha - \delta \boxed{+} \sigma^2/2\right)T}{\sigma\sqrt{T}}. \tag{5.2.4}$$

Combining all pieces, we obtain

$$\boxed{\mathbb{E}[S(T)|S(T) > K] = S(0)\mathrm{e}^{(\alpha-\delta)T}\frac{N(\hat{d}_1)}{N(\hat{d}_2)}.} \tag{5.2.5}$$

Observe that such a conditional expected stock price equals the unconditional expected T-year stock price $\mathbb{E}[S(T)] = S(0)\mathrm{e}^{(\alpha-\delta)T}$ scaled by a factor of $N(\hat{d}_1)/N(\hat{d}_2)$, which, because $\hat{d}_1 > \hat{d}_2$, is always greater than 1. When $S(T)$ is *a priori* assumed to be greater than K, the conditional expected stock price exceeds the unconditional expected stock price, as expected.

To determine the expected stock price given that the put expires in-the-money, one can adapt the above derivations to the case of $S(T) < K$. Alternatively, we can appeal to the law of total probability, which says that

$$\mathbb{E}[S(T)] = \mathbb{E}[S(T)|S(T) > K]\mathbb{P}(S(T) > K) + \mathbb{E}[S(T)|S(T) < K]\mathbb{P}(S(T) < K),$$

or

$$S(0)\mathrm{e}^{(\alpha-\delta)T} = S(0)\mathrm{e}^{(\alpha-\delta)T}N(\hat{d}_1) + \mathbb{E}[S(T)|S(T) < K]N(-\hat{d}_2).$$

A rearrangement of terms readily gives

$$\boxed{\mathbb{E}[S(T)|S(T) < K] = S(0)\mathrm{e}^{(\alpha-\delta)T}\frac{N(-\hat{d}_1)}{N(-\hat{d}_2)},} \tag{5.2.6}$$

which is again in the form of the unconditional expected T-year stock price multiplied by an adjusted factor, this time being less than unity.

Example 5.2.5. (A "sandwiched" conditional stock price) Assume the Black-Scholes framework.

Determine an expression for $\mathbb{E}[S(T)|a < S(T) < b]$, where a and b are positive constants with $a < b$.

Solution. Mimicking the derivations above, we have

$$\mathbb{E}[S(T)|a < S(T) < b] = \frac{\mathbb{E}[S(T)1_{\{S(T)>a\}}] - \mathbb{E}[S(T)1_{\{S(T)>b\}}]}{\mathbb{P}(a < S(T) < b)},$$

where

$$\mathbb{P}(a < S(T) < b) = \mathbb{P}(S(T) > a) - \mathbb{P}(S(T) > b) = N(\hat{d}_2^{@K=a}) - N(\hat{d}_2^{@K=b})$$

and

$$\mathbb{E}[S(T)1_{\{S(T)>a\}}] - \mathbb{E}[S(T)1_{\{S(T)>b\}}] = S(0)\mathrm{e}^{(\alpha-\delta)T}[N(\hat{d}_1^{@K=a}) - N(\hat{d}_1^{@K=b})]$$

In conclusion,

$$\mathbb{E}[S(T)|a < S(T) < b] = \boxed{\frac{S(0)\mathrm{e}^{(\alpha-\delta)T}[N(\hat{d}_1^{@K=a}) - N(\hat{d}_1^{@K=b})]}{N(\hat{d}_2^{@K=a}) - N(\hat{d}_2^{@K=b})}}.$$

□

Remark. Letting $b \to \infty$ retrieves (5.2.5) with $K = a$ (because $\hat{d}_1^{@K=b} = \hat{d}_2^{@K=b} \to -\infty$ and $N(\hat{d}_1^{@K=b}) = N(\hat{d}_2^{@K=b}) \to 0$), while setting $a = 0$ recovers (5.2.6) with $K = b$ (because $\hat{d}_1^{@K=a} = \hat{d}_2^{@K=a} \to +\infty$ and $N(\hat{d}_1^{@K=a}) = N(\hat{d}_2^{@K=a}) \to 1$, and note that $N(x) = 1 - N(-x)$ for any $x \in \mathbb{R}$).

5.3 Problems

Lognormal probabilistic quantities

Problem 5.3.1. (Exercise probability of a call) Consider a nondividend-paying stock whose current price is 100.

The stock-price process is a lognormal process with volatility 30%.

The continuously compounded expected return on the stock is 10%.

The continuously compounded risk-free interest rate is 7%.

Calculate the probability that a nine-month 75-strike European call option on the stock will be exercised.

Problem 5.3.2. (Calculations of miscellaneous probabilistic quantities) Assume the Black-Scholes framework. You are given:

(i) The current stock price is 100.

(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2.5%.

(iii) The continuously compounded expected rate of return on the stock is 6%.

(iv) The stock's volatility is 30%.

Calculate:

(a) The probability that a 3-year at-the-money European put option on the stock is exercised

(b) The 95% percentile of the 3-year stock price

(c) The expected 3-year stock price

(d) The expected 3-year stock price, given that the put option in (a) pays off at maturity

(e) The variance of the 3-year stock price

Problem 5.3.3. (Lognormal prediction interval) Assume the Black-Scholes framework. You are given:

(i) The stock, whose current price is 100, pays dividends continuously at a rate proportional to its price.

(ii) The stock's volatility is 0.35.

(iii) The continuously compounded expected rate of stock-price appreciation is 15%.

(iv) The continuously compounded risk-free interest rate is 12%.

Construct a 95% lognormal prediction interval for the price of the stock in 3 months.

Problem 5.3.4. (Given one prediction interval, find another) Assume the Black-Scholes framework. Let $S(t)$ denote the time-t price of a stock, which pays dividends continuously at a rate proportional to its price.

You are given:

(i) $S(0) = 8$

(ii) The 90% lognormal prediction interval for $S(2)$ is $(13.10, 41.93)$.

Calculate the width of the 95% lognormal prediction interval for $S(4)$.

Problem 5.3.5. [HARDER!] (Given two prediction intervals, find one more) Assume the Black-Scholes framework. For $t \geq 0$, let $S(t)$ be the time-t price of a stock. You are given:

(i) The stock pays dividends continuously at a rate proportional to its price.

(ii) The 90% lognormal prediction interval for $S(2)$ is $(13.1072, 41.9448)$.

(iii) The 95% lognormal prediction interval for $S(4)$ is $(25.7923, 183.1083)$.

Calculate the width of the 99% lognormal prediction interval for $S(6)$.

Problem 5.3.6. (What can you deduce from an exercise probability and a lognormal prediction interval?) Assume the Black-Scholes framework. For a stock which pays dividends continuously at a rate proportional to its price, you are given:

(i) The probability that a 2-year at-the-money European put option on the stock will be exercised is 0.5279.

(ii) The 99% lognormal prediction interval for the 3-year stock price is $(9.4493, 524.0208)$.

Calculate the expected 4-year stock price.

Problem 5.3.7. (Going from unconditional to conditional expected stock price) Assume the Black-Scholes framework. You are given:

(i) The current price of a stock is 80.

(ii) The stock's volatility is 25%.

(iii) The stock pays dividends continuously at a rate proportional to its price.

(iv) The continuously compounded risk-free interest rate is 6%.

(v) The expected 1-year stock price is 84.2069.

Calculate the expected 1-year stock price, given that a 1-year at-the-money (when issued) European put option on the stock expires in-the-money.

Problem 5.3.8. (Given the unconditional and conditional expected stock prices – I) Assume the Black-Scholes framework. You are given:

(i) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is identical to the continuously compounded (total) expected rate of return on the stock.

(ii) The expected 6-month stock price is 150.

(iii) The expected 6-month stock price, given that an at-the-money (when issued) 6-month European put option expires in-the-money, is 125.21.

Determine the 95% lognormal prediction interval for the 1-year stock price.

Problem 5.3.9. (Given the unconditional and conditional expected stock prices – II) Assume the Black-Scholes framework. For a stock which pays dividends continuously at a rate proportional to its price, you are given:

(i) The probability that an 8-month European put option on the stock will be exercised is 0.512.

(ii) The expected 8-month stock price is 10.134.

(iii) The expected 8-month stock price, given that the put option in (i) expires in-the-money, is 8.483.

Determine the 98% lognormal prediction interval for the 8-month stock price.

(Note: The strike price of the put option is not given.)

6

The Black-Scholes Formula

Chapter overview: Now that the Black-Scholes framework and its technical underpinnings have been set up, in this chapter we are ready to derive, interpret, and apply the all-important Black-Scholes option pricing formula—one of the highlights of this book—for European call and put options. To begin with, Section 6.1, our first encounter with the Black-Scholes formula, revolves around the simple case of stocks paying continuous proportional dividends. Section 6.2 extends the analysis in Section 6.1 and presents the Black-Scholes formula for options on other underlying assets, including stocks that pay discrete dividends, currencies, and futures contracts. It is shown that options on these seemingly unrelated assets can all be priced by suitably imposing the Black-Scholes framework and cosmetically modifying the basic Black-Scholes formula in Section 6.1. With the Black-Scholes formula at our disposal, we proceed to investigate in Section 6.3 the sensitivity of the option price to various model parameters as quantified by option Greeks. They are the partial derivatives of the option price with respect to the option input under consideration and are widely used in practice to assess the risk of an option position. Further aspects and applications of the Black-Scholes formula and option Greeks will be studied in Chapters 7 and 8.

6.1 The Black-Scholes Formula for Stocks Paying Continuous Proportional Dividends

What is the Black-Scholes formula?

The *Black-Scholes option pricing formula*, or *Black-Scholes formula* in short, is an analytic (i.e., exact, closed-form) formula for the prices of *European*[i] call and put options in terms of parameters in the Black-Scholes framework. It is justifiably the best known option pricing formula—so famous that "Black-Scholes" has become a household item in modern finance. Its validity is predicated upon several assumptions, which can be broadly categorized into two groups:

1. *Behavior of future stock prices:* As you can expect, the Black-Scholes formula lives in the Black-Scholes framework introduced in Chapter 5. In the simplest setting, it is assumed that future stock prices are lognormally distributed over the life of the option in question.[ii] The stock pays continuous proportional dividends at the dividend yield of δ and its volatility is constant over time[iii] at σ.

2. *Economic environment:* It is also assumed that the (continuously compounded) risk-free

[i] The prices of American options in the Black-Scholes framework are beyond the scope of this book.

[ii] This assumption will be weakened in Section 6.2.

[iii] It is possible to generalize the Black-Scholes formula to the case with deterministic time-dependent risk-free interest rate and volatility. See, e.g., Exercise 5.4 on page 253 of Shreve (2004).

interest rate is known and constant at r and the market is frictionless, i.e., there are no taxes, transaction costs, bid/ask spreads or restrictions on short sales.

Example 6.1.1. (CAS Exam 3 Fall 2007 Question 20: Black-Scholes assumptions) Which of the following is an assumption of the Black-Scholes option pricing model?

(A) Stock prices are normally distributed.

(B) Stock price volatility is a constant.

(C) Changes in stock price are lognormally distributed.

(D) All transaction costs are included in stock returns.

(E) The risk-free interest rate is a random variable.

Solution. Only (B) is correct. For (A) and (C), the correct statements are that stock prices are *lognormally* distributed whereas changes in the stock price are *normally* distributed. There are no transaction costs and the risk-free interest rate is a constant, rendering (D) and (E) incorrect. **(Answer: (B))** □

Sidebar: (Very) Brief history of the Black-Scholes formula.

The Black-Scholes formula was first published in the *Journal of Political Economy* in 1973 by Fischer *Black* and Myron *Scholes* (hence the names "Black" and "Scholes"). Their paper, Black and Scholes (1973), along with related work by Robert Merton(the Black-Scholes formula is also known as the Black-Scholes-Merton formula), was said to revolutionize the theory and practice of finance. In recognition of their groundbreaking work on option pricing, Robert Merton and Myron Scholes won the Nobel Prize in Economics in 1997. Regrettably, Fischer Black was ineligible for the prize because he had already passed away.

Black-Scholes pricing formula for European calls and puts.

Without further ado, we now formally state the Black-Scholes formula for the price of a T-year K-strike European call option on a stock that pays continuous proportional dividends at a rate of δ. The formula reads

$$\boxed{C = \mathrm{BS}(S(0), \delta; K, r; \sigma, T) = S(0)\mathrm{e}^{-\delta T}\, N(d_1) - K\mathrm{e}^{-rT}\, N(d_2),} \tag{6.1.1}$$

where C is the (time-0) price of the call, $N(\cdot)$ is the standard normal distribution function, and the two extra parameters d_1 and d_2 are defined by

$$\begin{aligned} d_1 &= \boxed{\frac{\ln[S(0)/K] + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}},} \\ d_2 &= \boxed{d_1 - \sigma\sqrt{T} = \frac{\ln[S(0)/K] + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}.} \end{aligned}$$

Note that d_1 and d_2 are identical to $\hat{d}_1$ and $\hat{d}_2$ in (5.2.4) and (5.2.1), respectively, except that the stock's expected rate of return α is replaced by the risk-free interest rate r.

By put-call parity (which holds for any no-arbitrage option pricing model), the accompanying Black-Scholes European put price formula is

$$\boxed{P = \mathrm{BS}(K, r; S(0), \delta; \sigma, T) = K\mathrm{e}^{-rT} N(-d_2) - S(0)\mathrm{e}^{-\delta T} N(-d_1),} \tag{6.1.2}$$

with the same definitions of d_1 and d_2 above. When you apply (6.1.2), make sure that you *do not omit the negative signs in front of* d_1 *and* d_2*!*

At first sight, the Black-Scholes pricing formulas, (6.1.1) and (6.1.2), look intimidating and not particularly informative. The monstrous forms of (6.1.1), (6.1.2), and the mysterious quantities d_1 and d_2 seem to make it a daunting task to remember the formulas by heart. To your dismay, however, two principal and inter-related learning objectives of this chapter (indeed, of this book) are:

(1) Remember the Black-Scholes pricing formula impeccably and keep it ingrained in your mind; every piece of the formula must appear in the right place.[iv] You cannot afford making any mistakes in writing the pricing formula!

(2) Calculate Black-Scholes option prices proficiently.

With respect to (2), the best way to gain proficiency with the use of the Black-Scholes formula is to work out numerous computational problems. This will be deferred to page 190 after we have learned how to interpret the pricing formula. With respect to (1), two useful pedagogical vehicles are to view the Black-Scholes formula from a unifying function and to understand the economic meaning of the two terms that constitute the formula from a cost-benefit perspective.

General Black-Scholes pricing function.

To emphasize the symmetry and common structure obeyed by the Black-Scholes call price and put price in (6.1.1), (6.1.2), as well as many other Black-Scholes pricing formulas in the later part of this book, we have devised the generic Black-Scholes-type pricing function defined by

$$\boxed{\mathrm{BS}(s_1, \delta_1; s_2, \delta_2; \sigma, T) := s_1\mathrm{e}^{-\delta_1 T} N(d_1) - s_2\mathrm{e}^{-\delta_2 T} N(d_2),} \tag{6.1.3}$$

where

$$d_1 = \frac{\ln(s_1/s_2) + (\delta_2 - \delta_1 + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

All of the Black-Scholes-type pricing formulas[v] we encounter in this book can be expressed in the form of the "BS" function for some appropriate choices of the arguments specific to a given context. Further discussions on the meaning of the variables inside the "BS" function will be provided when the notion of exchange options is introduced in Section 8.2. For the time being, it suffices to say that s_1 (resp. s_2) is the current price of the asset you acquire (resp. lose) if you exercise the option, δ_1 (resp. δ_2) is the "dividend yield" (in a general sense) of this asset, σ is the volatility of the asset which is assumed to follow the Black-Scholes framework, and T is the time to maturity of the option.

If you exercise a call, you lose the cash of $\$K$ and receive the stock. The dividend yield of the stock is δ while the "dividend yield" of cash is the risk-free interest rate r—\$1 at time 0 grows at the rate of r to $\$\mathrm{e}^{rt}$ at any future time t. This suggests setting $s_1 = S(0)$, $\delta_1 = \delta$, $s_2 = K$, and $\delta_2 = r$ in (6.1.3) to obtain the Black-Scholes call price

$$C = \mathrm{BS}(S(0), \delta; K, r; \sigma, T),$$

[iv]If you misstate the BS (Black-Scholes) formula, it will become the BS (<u>B</u>xll<u>s</u>xit) formula...
[v]The only exception is the pricing formula for gap options in Section 8.1.

with

$$d_1^C = \frac{\ln[S(0)/K] + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2^C = \frac{\ln[S(0)/K] + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

(The superscript "C" on d_1 and d_2 suggests "call") A put is the opposite—you lose the stock and receive the cash of $\$K$ upon exercising a put. To get the Black-Scholes put price, we set $s_1 = K$, $\delta_1 = r$, $s_2 = S(0)$, and $\delta_2 = \delta$ in (6.1.3), leading to

$$P = \mathrm{BS}(K, r; S(0), \delta; \sigma, T),$$

with

$$d_1^P = \frac{\ln[K/S(0)] + (\delta - r + \sigma^2/2)T}{\sigma\sqrt{T}} = -d_2^C \quad \text{and} \quad d_2^P = d_1^P - \sigma\sqrt{T} = -d_1^C.$$

However, it is customary to use the d_1 and d_2 for calls and omit the superscripts "C" and "P" altogether.

Cost-benefit interpretation of the Black-Scholes formula.

There are different ways to make sense of the abstruse Black-Scholes formula. One instructive way is to view the cost and benefit (in an accounting sense) of exercising the option.

Call. The first term of the pricing formula, $S(0)\mathrm{e}^{-\delta T}N(d_1) = F_{0,T}^P(S)N(d_1)$, represents the (risk-neutral) expected present value of what you receive at maturity upon exercising the call, namely the stock with a random time-T value of $S(T)$. The appearance of the term $N(d_1)$, which is between zero and one, stems from the fact that the call will be exercised only when $S(T) > K$, an even with a probability of between zero and one. The second term, $K\mathrm{e}^{-rT}N(d_2)$, is the (risk-neutral) expected present value of what you *pay* at maturity if you exercise the call, namely the strike price of K, provided that $S(T) > K$. Here $N(d_2)$ is the risk-neutral probability that the call will be exercised (in contrast, $N(\hat{d}_2)$ is the *real* exercise probability).

Put. Analogously, the put price formula can be thought of as the difference between the (risk-neutral) expected present value of what you *receive* (the strike price of K, provided that $K < S(T)$) and that of what you *give up* (the random amount of $S(T)$, provided that $K < S(T)$) by exercising the put.

Regardless of whether it is a call or a put, the Black-Scholes formula expresses the price of the option as the (risk-neutral) expected present value of the benefit of exercising the option less the (risk-neutral) expected present value of the corresponding cost.

Deriving the Black-Scholes formula.

There are many possible derivations of the Black-Scholes formula in the option pricing literature, including but not limited to:

1. The method of risk-neutral pricing
2. The method of dynamic replication
3. The method of solving the Black-Scholes partial differential equation (see page 249)
4. The method of Esscher transform (see the celebrated paper by Gerber and Shiu (1994))

5. By the Capital Asset Pricing Model (CAPM) (see the original paper of Black and Scholes (1973))

6. Taking the limit of the binomial option pricing formula as the number of periods approaches infinity and the length of each period approaches zero (the binomial formula will converge to the Black-Scholes formula; see the Appendix of Chapter 13 of Hull (2015))

In this book, we will only discuss Method 1 as it is the least technically demanding but most germane to our treatment.

To kick-start the risk-neutral valuation, recall that under the risk-neutral probability distribution (whose existence is posited)[vi], the stock (with dividends) earns the risk-free interest rate. Upon the substitution of the expected rate of return α by the risk-free interest rate r, we have the following important distributional representation result:

$$S(T) \stackrel{d}{=} S(0)\exp\left[\left(r-\delta-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}Z\right],$$

where Z denotes a standard normal random variable *under the risk-neutral probability.* By risk-neutral pricing, the time-0 price of the European call option takes the discounted expectation in the form of

$$C=\mathbb{E}^*\left[\mathrm{e}^{-rT}\left(S(T)-K\right)_+\right].$$

Because $x_+=x$ if $x\geq 0$ and $x_+=0$ if $x<0$, we can appeal to double expectation (also known as iterated expectation) and simplify the preceding expectation as

$$\begin{aligned} C &= \mathbb{E}^*\left[\mathrm{e}^{-rT}\left(S(T)-K\right)_+ \middle| S(T)>K\right]\mathbb{P}^*(S(T)>K) \\ &\quad +\mathbb{E}^*\left[\mathrm{e}^{-rT}\underbrace{\left(S(T)-K\right)_+}_{0}\middle| S(T)\leq K\right]\mathbb{P}^*(S(T)\leq K) \\ &= \mathrm{e}^{-rT}\mathbb{E}^*\left[S(T)-K\,|\,S(T)>K\right]\mathbb{P}^*(S(T)>K). \end{aligned}$$

Now $\mathbb{P}^*(S(T)>K)$ is the *risk-neutral* exercise probability of the call, which, by (5.2.2) with $\hat{d}_2$ replaced by d_2, equals $N(d_2)$. Moreover, the linearity of conditional expectations yields

$$\begin{aligned} \mathbb{E}^*\left[S(T)-K\,|\,S(T)>K\right] &= \mathbb{E}^*\left[S(T)\,|\,S(T)>K\right]-K \\ &= S(0)\mathrm{e}^{(r-\delta)T}\frac{N(d_1)}{N(d_2)}-K, \end{aligned}$$

where the last equality follows from (5.2.5) with $\alpha,\hat{d}_1,\hat{d}_2$ changed to r,d_1,d_2, respectively. Combining all pieces, we have

$$C=\mathrm{e}^{-rT}\left[S(0)\mathrm{e}^{(r-\delta)T}\frac{N(d_1)}{N(d_2)}-K\right]N(d_2)=S(0)\mathrm{e}^{-\delta T}N(d_1)-K\mathrm{e}^{-rT}N(d_2),$$

which is the same as (6.1.1). We are done with deriving the Black-Scholes formula (and ready to clinch Scholes and Merton's Nobel Prize)!!

[vi] Indeed, a lot of work has to be done to justify the existence of and to define the risk-neutral probability distribution.

The Black-Scholes formula in action.

We end this introductory section with several concrete numerical examples that are designed to sharpen your computational proficiency with the use of the Black-Scholes formula. It is strongly suggested that you try these calculations out on a piece of paper and see how (un)interesting they are. It is time to get our hands dirty!

Example 6.1.2. (SOA Exam IFM Advanced Derivatives Sample Question 6: Black-Scholes call price) You are considering the purchase of 100 units of a 3-month 25-strike European call option on stock.

You are given:

(i) The Black-Scholes framework holds.

(ii) The stock is currently selling for 20.

(iii) The stock's volatility is 24%.

(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(v) The continuously compounded risk-free interest rate is 5%.

Calculate the price of the block of 100 options.

(A) 0.04

(B) 1.93

(C) 3.63

(D) 4.22

(E) 5.09

Solution. With $S = 20$, $K = 25$, $\sigma = 0.24$, $r = 0.05$, $\delta = 0.03$ and $T = 0.25$, we have

$$\begin{aligned} d_1 &= \frac{\ln(20/25) + (0.05 - 0.03 + 0.24^2/2)(0.25)}{0.24\sqrt{0.25}} = -1.75786, \\ d_2 &= d_1 - 0.24\sqrt{0.25} = -1.87786, \\ N(d_1) &= 0.03939, \\ N(d_2) &= 0.03020. \end{aligned}$$

The Black-Scholes price of each call is

$$\begin{aligned} C &= S(0)\mathrm{e}^{-\delta T}N(d_1) - K\mathrm{e}^{-rT}N(d_2) \\ &= 20\mathrm{e}^{-0.03(0.25)}(0.03939) - 25\mathrm{e}^{-0.05(0.25)}(0.03020) \\ &= 0.036292. \end{aligned}$$

Hence the price of the block of 100 options is $100C = \boxed{3.6292}$. **(Answer: (C))** □

Example 6.1.3. (SOA Exam MFE Spring 2007 Question 3: Black-Scholes put price) You are asked to determine the price of a European put option on a stock. Assuming the Black-Scholes framework holds, you are given:

(i) The stock price is \$100.

(ii) The put option will expire in 6 months.

(iii) The strike price is \$98.

(iv) The continuously compounded risk-free interest rate is $r = 0.055$.

(v) $\delta = 0.01$.

(vi) $\sigma = 0.50$.

Calculate the price of this put option.

(A) \$3.50

(B) \$8.60

(C) \$11.90

(D) \$16.00

(E) \$20.40

Solution. With

$$\begin{aligned} d_1 &= \frac{\ln(100/98) + (0.055 - 0.01 + 0.5^2/2)(0.5)}{0.5\sqrt{0.5}} = 0.29756, \\ d_2 &= d_1 - 0.5\sqrt{0.5} = -0.05600, \\ N(-d_1) &= 0.38302, \\ N(-d_2) &= 0.52233, \end{aligned}$$

the price of the put is

$$\begin{aligned} P &= K\mathrm{e}^{-rT} N(-d_2) - S(0)\mathrm{e}^{-\delta T} N(-d_1) \\ &= 98\mathrm{e}^{-0.055(0.5)}(0.52233) - 100\mathrm{e}^{-0.01(0.5)}(0.38302) \\ &= \boxed{11.6889}. \qquad \textbf{(Answer: (C))} \end{aligned}$$

□

6.2 Applying the Black-Scholes Formula to Other Underlying Assets

The Black-Scholes option pricing formula is by no means confined to European options on stocks that pay continuous proportional dividends. In this section, we broaden the applicability of the Black-Scholes formula by extending its scope to European options on more diverse underlying assets, including stocks that pay non-random, discrete dividends at known times, foreign currencies, and futures contracts. Despite the varied nature of these assets, it will be shown that with the right perspective, European options on each of these assets can be priced by a mostly cosmetic modification of the basic Black-Scholes formulas (6.1.1) and (6.1.2) with appropriate choices of the initial stock price $S(0)$, continuous dividend yield δ, and with σ being the volatility of an appropriate asset on which the Black-Scholes framework is imposed. The mathematical structure of the pricing formula is preserved.

6.2.1 Case study 1: Stocks paying non-random, discrete dividends.

The discrete-dividend case is an intriguing item that is often not given due attention in the literature and in the classroom. Pricing European options in this setting requires a subtle application of the standard Black-Scholes pricing formula for nondividend-paying stocks presented in the last section.

Pricing assumption.

To set the stage, consider a stock that, at each *known* time t_i, will make a dividend payment with a *known* amount of $D(t_i)$, where $0 < t_1 < t_2 < \cdots < t_n \le T$ and T is the maturity time of a European option written on such a stock. This is the same discrete-dividend setting in Subsection 2.2.2. In this case, the stock price process $\{S(t)\}$ cannot be a lognormal process, because there must be a downward jump in the stock price immediately after each dividend is paid, making the stock price path discontinuous. So, what process is lognormal here? It turns out that the Black-Scholes pricing formula in this discrete-dividend setting can be derived from the assumption that the stochastic process of the *prepaid forward prices* for the time-T delivery of one share of the stock, that is,

$$\{F^P_{t,T}(S)\}_{t\in[0,T]},$$

is a lognormal process, with σ now being the standard deviation per unit time of its natural logarithm:

$$\mathrm{Var}[\ln F^P_{t,T}(S)] = \sigma^2 t, \quad 0 \le t \le T.$$

Why does applying the Black-Scholes framework to $\{F^P_{t,T}(S)\}_{t\in[0,T]}$ make (practical or technical) sense? To see this, recall from Subsection 2.2.2 that the time-t price of a prepaid forward maturing at time T is the current stock price less the present value of the dividends payable over the remaining life of the option, that is,

$$F^P_{t,T}(S) = S(t) - \mathrm{PV}_{t,T}(\mathrm{Div}) = S(t) - \sum_{\{i:t_i \ge t\}} D(t_i)\mathrm{e}^{-r(t_i - t)}.$$

See (2.2.1) on page 32 with the valuation date changed to time t. Although the stock price decreases abruptly by the amount of each discrete dividend as the date of each dividend payment is approached from the left, so does the present value factor $\mathrm{PV}_{t,T}(\mathrm{Div})$—that particular dividend need not be discounted. This preserves the continuity of the prepaid forward price $F^P_{t,T}(S)$ as a function of time t.

Pricing formula and its proof.

Under the lognormality assumption on the prepaid forward prices on the above stock that pays discrete dividends, the Black-Scholes formula for the price of a T-year K-strike European call option on the stock is given by

$$\boxed{C = \mathrm{BS}\left(F^P_{0,T}(S), 0; K, r; \sigma, T\right) = F^P_{0,T}(S)\,N(d_1) - K\mathrm{e}^{-rT}\,N(d_2),} \tag{6.2.1}$$

where

$$\begin{aligned} d_1 &= \frac{\ln[F^P_{0,T}(S)/K] + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

The corresponding European put price is

$$\boxed{P = \text{BS}\left(K, r; F^P_{0,T}(S), 0; \sigma, T\right) = K\mathrm{e}^{-rT} N(-d_2) - F^P_{0,T}(S)\, N(-d_1).}$$

The proof of (6.2.1) is interesting in its own right because it introduces an unconventional but useful way to view options—any option can be considered an option on a prepaid forward written on the underlying asset. To derive (6.2.1), we appeal to risk-neutral valuation again, whence

$$C = \mathrm{e}^{-rT}\mathbb{E}^* \left[(S(T) - K)_+\right].$$

Because $F^P_{T,T}(S) = S(T)$, the preceding expectation can be written as

$$C = \mathrm{e}^{-rT}\mathbb{E}^* \left[\left(\boxed{F^P_{T,T}(S)} - K\right)_+\right],$$

which suggests that such a call on the stock can also be viewed as a K-strike T-year European call on a *prepaid forward contract* on the stock maturing at time T. Because $\{F^P_{t,T}(S)\}_{t\in[0,T]}$ is a lognormal process and the prepaid forward pays no dividends, we can apply the standard Black-Scholes formula (6.1.1) with

the time-0 price of the stock, $S(0)$, changed to $F^P_{0,T}(S)$ (i.e., the time-0 price of the T-year prepaid forward, which is the underlying asset),

the dividend yield on the underlying asset, δ, set to 0, and

the parameter, σ, being the volatility of the prepaid forward.

This yields the call price formula

$$C = \text{BS}\left(F^P_{0,T}(S), 0; K, r; \sigma, T\right) = F^P_{0,T}(S)N(d_1) - K\mathrm{e}^{-rT}N(d_2),$$

where

$$d_1 = \frac{\ln[F^P_{0,T}(S)/K] + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Example 6.2.1. (SOA Exam MFE Spring 2007 Question 15: Discrete dividend – I) For a six-month European put option on a stock, you are given:

(i) The strike price is \$50.00.

(ii) The current stock price is \$50.00.

(iii) The only dividend during this time period is \$1.50 to be paid in four months.

(iv) $\sigma = 0.30$.

(v) The continuously compounded risk-free interest rate is 5%.

Under the Black-Scholes framework, calculate the price of the put option.

(A) \$3.50

(B) \$3.95

(C) \$4.19

(D) \$4.73

(E) \$4.93

Solution. The 6-month prepaid forward price of the stock is

$$F^P_{0,1/2}(S) = S(0) - \mathrm{PV}_{0,1/2}(\mathrm{Div}) = 50 - 1.5\mathrm{e}^{-0.05/3} = 48.52479282.$$

Then

$$\begin{aligned} d_1 &= \frac{\ln(48.52479282/50) + (0.05 + 0.3^2/2)(0.5)}{0.3\sqrt{0.5}} = 0.08274, \\ d_2 &= d_1 - 0.3\sqrt{0.5} = -0.12939, \\ N(-d_1) &= 0.46703, \\ N(-d_2) &= 0.55148. \end{aligned}$$

The put option price is

$$\begin{aligned} P &= K\mathrm{e}^{-rT}N(-d_2) - F^P_{0,1/2}(S)N(-d_1) \\ &= 50\mathrm{e}^{-0.05(0.5)}(0.55148) - 48.52479282(0.46703) \\ &= \boxed{4.2307}. \qquad \textbf{(Answer: (C))} \end{aligned}$$

□

Remark. (i) The difference between the computed put price and the price in Answer Choice C is due to the way the normal c.d.f. $N(\cdot)$ is computed. Using the rounding rules in a standard normal distribution table, we have

$$N(-d_1) = 0.4681, \quad \text{and} \quad N(-d_2) = 0.5517,$$

and the final answer becomes

$$50\mathrm{e}^{-0.05(0.5)}(0.5517) - 48.52479282(0.4681) = \boxed{4.19}. \quad \textbf{(Answer: (C))}$$

(ii) Make sure you know the precise meaning of σ in this context. It is the volatility of the prepaid forward that calls for the delivery of the stock in six months.

Example 6.2.2. (SOA Exam MFE Spring 2009 Question 19: Discrete dividend – II) Consider a one-year 45-strike European put option on a stock S. You are given:

(i) The current stock price, $S(0)$, is 50.00.

(ii) The only dividend is 5.00 to be paid in nine months.

(iii) $\mathrm{Var}[\ln F_{t,1}^P(S)] = 0.01 \times t, \quad 0 \le t \le 1.$

(iv) The continuously compounded risk-free interest rate is 12%.

Under the Black-Scholes framework, calculate the price of 100 units of the put option.

(A) 1.87

(B) 18.39

(C) 18.69

(D) 19.41

(E) 23.76

Solution. This question is similar to the preceding one in terms of content and wording, except that we are not directly told the value of σ—we need to identify it from Point (iii): $\sigma = \sqrt{0.01} = 0.1$. The time-0 prepaid forward price for time-1 delivery of the stock is

$$F_{0,1}^P(S) = S(0) - \mathrm{PV}_{0,1}(\mathrm{Div}) = 50 - 5\mathrm{e}^{-0.12(0.75)} = 45.4303.$$

Thus

$$d_1 = \frac{\ln[F_{0,1}^P(S)/K] + (r + \sigma^2/2)(1)}{\sigma\sqrt{1}} = \frac{\ln(45.4303/45) + (0.12 + 0.1^2/2)(1)}{0.1\sqrt{1}} = 1.34518$$

and

$$d_2 = d_1 - \sigma\sqrt{1} = 1.24518.$$

The price of the one unit of the put option is

$$\begin{aligned} P &= K\mathrm{e}^{-rT}N(-d_2) - F_{0,1}^P(S)N(-d_1) \\ &= 45\mathrm{e}^{-0.12(1)}(0.10653) - 45.4303(0.08928) \\ &= 0.1957. \end{aligned}$$

For 100 units, the price is $100(0.1957) = \boxed{19.57}$. **(Answer: (D))** □

Remark. Using the rounding rules in a standard normal distribution table, we have

$$N(-d_1) = 1 - 0.9115 = 0.0885, \quad \text{and} \quad N(-d_2) = 1 - 0.8944 = 0.1056,$$

and the final answer becomes

$$P = 100[39.9114(0.1056) - 45.4303(0.0885)] = \boxed{19.41}. \quad \textbf{(Answer: (D))}$$

Remark.

The proof of (6.2.1) and the symbol $\mathrm{BS}\left(F_{0,T}^P(S), 0\,; \cdots\right)$ suggest that one can always treat the underlying asset of an option as a prepaid forward on the asset with the same time to maturity as the option. In fact, the lognormality of the prepaid forward prices $F_{t,T}^P(S)$ for $t \in [0,T]$ is a more general assumption for the Black-Scholes formula for dividend-paying stocks to hold true (see Subsection 8.2.2 for a further generalization), and (6.2.1) accommodates not only stocks paying discrete dividends of known amounts at known times, as discussed in this subsection, but also stocks paying dividends continuously at a rate proportional to its price, as in Section 6.1. If the stock pays continuous proportional dividends, then it can be shown that $\{F^P_{t,T}(S)\}_{0\leq t\leq T}$ is a lognormal process if and only if the stock price process $\{S(t)\}$ is a lognormal process with both stochastic processes sharing the same volatility σ. Furthermore, in this continuous-dividend case $F^P_{0,T}(S) = S(0)\mathrm{e}^{-\delta T}$, so (6.2.1) reduces to

$$C = S(0)\mathrm{e}^{-\delta T}\,N(d_1) - K\mathrm{e}^{-rT}\,N(d_2),$$

with

$$d_1 = \frac{\ln[S(0)\mathrm{e}^{-\delta T}/K] + (r+\sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln[S(0)/K] + (r-\delta+\sigma^2/2)T}{\sigma\sqrt{T}}$$

and $d_2 = d_1 - \sigma\sqrt{T}$. This call price formula is nothing but (6.1.1) on page 186.

6.2.2 Case Study 2: Currency options.

In addition to stocks paying discrete dividends, the Black-Scholes methodology can also be applied to price options on currencies, perhaps more easily than one would have imagined.

For specificity, let $X(t)$ be the time-t exchange rate, which is the value of one unit of the foreign currency in question measured in terms of the domestic currency at time t. Valuing a currency option in the Black-Scholes framework hinges upon the realization that the underlying asset of the currency option is the foreign risk-free asset, which plays the same role as the underlying for stock options. Just as the number of shares of a stock paying continuous proportional dividends grows exponentially via the reinvestment of dividends at the dividend yield δ, an investment in the foreign currency grows exponentially via the reinvestment of interest at the foreign risk-free interest rate r_f. With this resemblance (see Table 6.1) in mind, we can price currency options by assuming that future exchange rates (as opposed to stock prices) are lognormally distributed with constant volatility σ and applying the Black-Scholes formula in the form of (6.1.1) and (6.1.2), with

the current stock price $S(0)$ replaced by the current exchange rate $X(0)$,

the dividend yield δ by r_f, and

the (continuously compounded) risk-free interest rate r by the risk-free interest rate of the domestic currency, r_d.

This yields the following prices, expressed in the domestic currency, for a K-strike T-year European currency call and an otherwise identical currency put on the foreign currency:

$$\boxed{\begin{aligned} C &= \mathrm{BS}\,(X(0), r_f; K, r_d; \sigma, T) = X(0)\mathrm{e}^{-r_f T}N(d_1) - K\mathrm{e}^{-r_d T}N(d_2), \\ P &= \mathrm{BS}\,(K, r_d; X(0), r_f; \sigma, T) = K\mathrm{e}^{-r_d T}N(-d_2) - X(0)\mathrm{e}^{-r_f T}N(-d_1), \end{aligned}} \tag{6.2.2}$$

where

$$d_1 = \frac{\ln[X(0)/K] + (r_d - r_f + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}.$$

These pricing formulas for currency options are sometimes known as the *Garman-Kohlhagen* formula, in recognition of Garman and Kohlhagen (1983).

Example 6.2.3. (CAS Exam 3 Fall 2007 Question 21: Straightforward application of (6.2.2)) On January 1st, 2007, the following currency information is given:

	Stock Options	Currency Options
Underlying asset	Stock	Foreign currency
Time-t value of underlying	$S(t)$	$X(t)$
Dividend yield	δ	r_f
Pricing assumption	$S(t)$'s are lognormally distributed	$X(t)$'s are lognormally distributed
Meaning of σ	Volatility of the stock	Volatility of the exchange rate

TABLE 6.1
Comparing stock options and currency options in the Black-Scholes framework.

- Spot exchange rate = \$0.82/euro
- Dollar interest rate = 5.0% compounded continuously
- Euro interest rate = 2.5% compounded continuously
- Exchange rate volatility = 0.10

What is the price of 850 dollar-denominated euro call options with a strike exchange rate of \$0.80/euro that expire on January 1st, 2008?

(A) Less than \$10

(B) At least \$10, but less than \$20

(C) At least \$20, but less than \$30

(D) At least \$30, but less than \$40

(E) At least \$40

Solution. As

$$\begin{aligned} d_1 &= \frac{\ln(0.82/0.8) + (0.05 - 0.025 + 0.1^2/2)(1)}{0.1\sqrt{1}} = 0.54693, \\ d_2 &= d_1 - 0.1\sqrt{1} = 0.44693, \\ N(d_1) &= 0.70779, \\ N(d_2) &= 0.67254, \end{aligned}$$

the price of the euro call is

$$C = 0.82e^{-0.025}(0.70779) - 0.8e^{-0.05}(0.67254) = 0.05427,$$

so the total price is $850(0.05427) = \boxed{46.1262}$. **(Answer: (E))** □

Currency options bear particular relevance to multinational corporations, which export and import goods to and from international markets as part of their regular operating cycle. Because their cash inflows and outflows depend on the future exchange rates of foreign currencies, these companies are susceptible to exchange rate fluctuations. The following example illustrates how currency options can be leveraged to hedge against currency risk.

Example 6.2.4. (SOA Exam IFM Advanced Derivatives Sample Question 7: How to use currency options to hedge against currency risk?) Company A is a US international company, and Company B is a Japanese local company. Company A is negotiating with Company B to sell its operation in Tokyo to Company B. The deal will be settled in Japanese yen. To avoid a loss at the time when the deal is closed due to a sudden devaluation of yen relative to dollar, Company A has decided to buy at-the-money dollar-denominated yen puts of the European type to hedge this risk.

You are given the following information:

(i) The deal will be closed 3 months from now.

(ii) The sale price of the Tokyo operation has been settled at 120 billion Japanese yen.

(iii) The continuously compounded risk-free interest rate in the United States is 3.5%.

(iv) The continuously compounded risk-free interest rate in Japan is 1.5%.

(v) The current exchange rate is 1 US dollar = 120 Japanese yen.

(vi) The daily volatility of the yen per dollar exchange rate is 0.261712%.

(vii) 1 year = 365 days; 3 months = $^1/_4$ year.

Calculate Company A's option cost.

(A) 7.32 million

(B) 7.42 million

(C) 7.52 million

(D) 7.62 million

(E) 7.72 million

Solution. Prelude: Before computing the required option cost, we explain why currency puts are of value to Company A. In this example, Company A is a US-based company which has a business in Japan. Let $X(t)$ be the exchange rate of US dollar per Japanese yen at time t. That is, at time t, ¥1 = \$$X(t)$.

Time 0. According to (v), we have $X(0) = 1/120$.

Time 0.25. According to (ii), the sale price of the business received at time $t = 0.25$ is fixed at ¥120 billion. Being US-based, Company A naturally wants to convert the sale price back to US dollars. Then the sale price of the business in US dollars will be \$$120X(0.25)$ billion, which is a random amount because $X(0.25)$ is not known at time 0. If $X(0.25)$ turns out to be very low (i.e., Japanese yen depreciates substantially against the US dollars), then Company A will only receive a disappointingly low revenue in US dollars from selling its business. This is a great cause for concern to Company A.

To hedge such exchange rate risk, dollar-denominated currency puts on yen come to Company A's rescue. If at-the-money ones are used and the downside risk is to be completely eliminated, then Company A shall need to buy 120 billion yen puts, and its overall payoff in 3 months, in billion dollars, becomes

$$120\left[X(0.25)+\underbrace{\left(120^{-1}-X(0.25)\right)_+}_{\text{payoff of each yen put}}\right]$$
$$=\left[120X(0.25)+(1-120X(0.25))_+\right]=\max\{1,120X(0.25)\},$$

which is bounded from below by \$1 billion. In the language of Section 3.1, the 120 billion yen puts place a floor on the revenue Company A will receive in US dollars in 3 months. The cash flows between different parties are depicted in Figure 6.2.1.

Calculations: Using the Black-Scholes formula with $S(0)=K=\$1/120$, $r=0.035$ (interest rate in the US), $\delta=0.015$ (interest rate in Japan), $\sigma/\sqrt{365}=0.261712\%$, or $\sigma=0.05$, we have

$$\begin{aligned}
d_1 &= \frac{(r-\delta+\sigma^2/2)T}{\sigma\sqrt{T}}=\frac{(0.035-0.015+0.05^2/2)(0.25)}{0.05\sqrt{0.25}}=0.2125,\\
d_2 &= d_1-0.05\sqrt{0.25}=0.1875,\\
N(-d_1) &= 0.41586,\\
N(-d_2) &= 0.42563,
\end{aligned}$$

and the total cost of the put options in dollars is

$$\$120\text{ billion}\times\left[\frac{1}{120}\mathrm{e}^{-0.035(0.25)}(0.42563)-\frac{1}{120}\mathrm{e}^{-0.015(0.25)}(0.41586)\right]$$
$$=\boxed{\$0.00761854\text{ billions}},$$

or \$7.61854 millions. **(Answer: (D))** □

Remark. (i) As an alternative to currency puts, Company A could have locked in the sale price of its Tokyo operation via selling currency yen forwards with forward price $X(0)\mathrm{e}^{(r_\$-r_¥)/4}$ (analog of $S(0)\mathrm{e}^{(r-\delta)T}$ for forwards on stocks) from the market or synthetically creating short currency forwards. The latter can be accomplished by borrowing ¥$120\mathrm{e}^{-r_¥/4}$ billion at time 0, immediately converting this amount to \$$\mathrm{e}^{-r_¥/4}$, and depositing it in the risk-free account in the US. After repaying the yen loan with interest (i.e., ¥120), the overall 3-month payoff of Company A, in billion US dollars, becomes constant at

$$120X(0.25)+[\mathrm{e}^{(r_\$-r_¥)/4}-120X(0.25)]=\mathrm{e}^{(r_\$-r_¥)/4}.$$

The downside currency risk is eliminated. On the other hand, with the currency options, Company A has the opportunity to benefit from the appreciation of the yen against the dollar.

(ii) A more challenging version of this example is Problem 6.4.11, where you need to figure out whether currency calls or puts should be purchased. You cannot just calculate blindly!

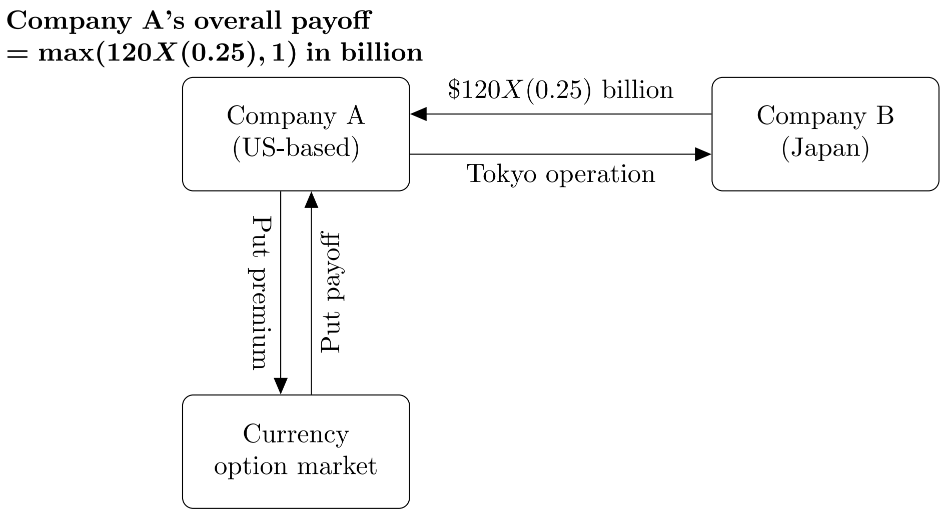


FIGURE 6.2.1
The cash flows between different parties in Example 6.2.4.

6.2.3 Case Study 3: Futures options.

As with currency options, futures options can be embedded in the standard Black-Scholes pricing framework by assuming that future futures prices (not a typo!) are lognormally distributed with constant volatility σ and treating the underlying futures as a stock whose "current price" is the current futures price and the "dividend yield" is the continuously compounded risk-free interest rate r (see the discussions in Subsection 4.4.2).[vii] This yields the following Black-Scholes pricing formulas for K-strike T-year European futures options:

$$\boxed{\begin{aligned} C &= \mathrm{BS}\left(F_{0,T_f}, r; K, r; \sigma, T\right) = F_{0,T_f}\mathrm{e}^{-rT}N(d_1) - K\mathrm{e}^{-rT}N(d_2), \\ P &= \mathrm{BS}\left(K, r; F_{0,T_f}, r; \sigma, T\right) = K\mathrm{e}^{-rT}N(-d_2) - F_{0,T_f}\mathrm{e}^{-rT}N(-d_1), \end{aligned}} \tag{6.2.3}$$

where

T is the time to maturity of the futures call and put,

T_f is the time to maturity of the underlying futures with $T \leq T_f$,

F_{0,T_f} is the current price of the T_f-year futures, and

$$d_1 = \frac{\ln(F_{0,T_f}/K) + \sigma^2 T/2}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

Example 6.2.5. (SOA Exam IFM Advanced Derivatives Sample Question 55: Put on futures) Assume the Black-Scholes framework. Consider a 9-month at-the-money European put option on a futures contract. You are given:

(i) The continuously compounded risk-free interest rate is 10%.

(ii) The strike price of the option is 20.

[vii] This can also be seen by the Itô's Lemma, which is beyond the scope of this book.

(iii) The price of the put option is 1.625.

If three months later the futures price is 17.7, what is the price of the put option at that time?

(A) 2.09

(B) 2.25

(C) 2.45

(D) 2.66

(E) 2.83

Solution. Since the put option is at-the-money, we have

$$d_1 = \frac{\overbrace{\ln(F_{0,T_f}/K)}^{0} + \sigma^2 T/2}{\sigma\sqrt{T}} = \frac{\sigma\sqrt{T}}{2} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T} = -d_1.$$

With $N(-d_2) = N(d_1) = 1 - N(-d_1)$, it follows from Point (iii) that

$$P = \mathrm{e}^{-rT}[K \times N(-d_2) - K \times N(-d_1)] = 20\mathrm{e}^{-0.1(0.75)}[1 - 2N(-d_1)] = 1.625,$$

whence $N(-d_1) = 0.45621$ and $d_1 = -N^{-1}(0.45621) = 0.10999$. Then $\sigma = 2d_1/\sqrt{0.75} = 0.25401$.

Three months later, the put option has 6 months to go before it expires. With the Black-Scholes parameters updated to

$$\begin{aligned} d_1 &= \frac{\ln(17.7/20) + (0.25401^2/2)(0.5)}{0.25401\sqrt{0.5}} = -0.59037, \\ d_2 &= d_1 - 0.25401\sqrt{0.5} = -0.76998, \\ N(-d_1) &= 0.72253, \\ N(-d_2) &= 0.77934, \end{aligned}$$

the price of the put option at that time is

$$P = 20\mathrm{e}^{-0.1(0.5)}(0.77934) - 17.7\mathrm{e}^{-0.1(0.5)}(0.72253) = \boxed{2.66156}. \quad \textbf{(Answer: (D))}$$

□

Remark. As this example shows, at-the-money futures options provide a rare instance in which observed option prices can be used to infer volatility analytically, given other model parameters (note that the Black-Scholes pricing formula is a highly non-linear function of the volatility; σ appears in both d_1 and d_2). Such a volatility is aptly named as the *implied volatility.*

Option Greek	Greek Symbol	Partial Derivative of the Option Price With Respect To
Delta	Δ	Stock price (first derivative)
Gamma	Γ	Stock price (second derivative)
Vega	N.A.	Volatility
Theta	θ	Passage of time (not time to maturity)
Rho	ρ	Interest rate
Psi	Ψ	Dividend yield

TABLE 6.2
The definitions and symbols of the six most common option Greeks.

6.3 Option Greeks

The explicit pricing formulas given in Sections 6.1 and 6.2 take the model parameters (e.g., stock price, time to maturity, risk-free interest rate, dividend yield, volatility) as input and provide the value of a European option at a single point of time. As time elapses, these model parameters, especially the stock price, are likely to change, which in turn alters the value of the option. An appeal of the Black-Scholes formula is that its explicitness easily allows us to quantify the sensitivity of the option price to changes in these model parameters. Such *sensitivity analysis* can be formally performed by investigating vehicles known as option Greeks. These metrics give a great deal of information about the risk exposure of an option trader and can be carefully exploited to optimize one's risk position. In this section, we study the definitions, computations, interpretation, and qualitative properties of option Greeks. Their practical uses in hedging and risk management will be examined in Chapter 7.

Definition.

Mathematically, option Greeks are partial derivatives of the option price with respect to the option parameter in question, holding other inputs fixed. The six most common option Greeks are given in Table 6.2.

Option Greeks inherit the "approximation" interpretation that partial derivatives carry in multi-variable calculus. For instance, if the delta of an option equals 0.8, then a unit increase in the *current* price of the underlying stock causes the price of the option to increase by *approximately* 0.8, provided that the values of other option parameters are held constant. With this way of interpretation, option Greeks are natural indicators of the sensitivity of option prices to changes in the option parameters. The larger the magnitude of an option Greek, the more sensitive the option price is to the underlying risk factor.

Questions of interest.

For each option Greek, the following questions are often of both theoretical and practical interest:

- Question 1: Is there a convenient computing formula for the Greek?

  The definitions of the six option Greeks, in terms of partial derivatives, are suitable mostly for making interpretation, but do not lend themselves to practical calculations.[viii] To obtain explicit expressions of the option Greeks, one can partially differentiate the

[viii]Actuarial students taking Exam IFM should know that one of the learning outcomes of Exam IFM is to *compute* and interpret Option Greeks

option price with respect to the relevant option parameters. The manipulations, as will be shown shortly, are not as straightforward as you may have thought.

- Question 2: How do the Greeks of otherwise identical European calls and puts compare?

  The key to this question is partially differentiating both sides of put-call parity. The difference between the call Greek and the put Greek is then related to the right-hand side of put-call parity differentiated with respect to the option parameter in question. Such a relation between the two Greeks makes it easy to translate our analysis for the call Greek to the put Greek, or the other way round.

- Question 3: Is the Greek always positive, always negative, or sometimes positive or negative? How can this behavior be explained?

  In most cases, an option Greek takes a definite sign, which can be justified algebraically using its closed-form expression or more easily and instructively by verbal reasoning.

- Question 4: How does the Greek vary with the moneyness and time to expiration of the option? Why does it exhibit this behavior?

  In addition to its sign, a Greek typically varies regularly with the price of the underlying asset (or equivalently, its moneyness) and the time to expiration of the option. These regular patterns, again, admit intuitive explanations.

To avoid repetition, the answers to these four questions will be epitomized by delta and gamma, which are arguably the Greeks of predominant interest. For concreteness, throughout this section we concentrate on European options on stocks that pay continuous proportional dividends, although option Greeks can be evaluated and studied for options on many other kinds of underlying asset (examples can be found in Section 6.2). We will also simplify notation and write S for $S(0)$ when no confusion arises.

6.3.1 Option Delta

Definition.

The *delta* of an option is defined as the partial derivative of the option price with respect to the current price of the underlying asset:

$$\Delta := \frac{\partial V}{\partial S}.$$

It serves to measure the sensitivity of the option price to changes in the current price of the underlying stock and is arguably the most important option Greek in theory as well as in practice.

Question 1: Closed-form expression for delta.

In the Black-Scholes model, we have a closed-form formula for the prices of European call and put options. Partially differentiating the call and put prices with respect to the current stock price yields

$$\boxed{\Delta_C := \frac{\partial C}{\partial S} = \mathrm{e}^{-\delta T}N(d_1) \quad \text{and} \quad \Delta_P := \frac{\partial P}{\partial S} = -\mathrm{e}^{-\delta T}N(-d_1).} \tag{6.3.1}$$

Proof of (6.3.1). First consider the call delta. At first glance, it seems that we directly have

$$\Delta_C = \frac{\partial}{\partial S}[S\mathrm{e}^{-\delta T}N(d_1) - K\mathrm{e}^{-rT}N(d_2)] = \mathrm{e}^{-\delta T}\,N(d_1).$$

However, this completely ignores the fact that d_1 and d_2 themselves are also functions of S! This should be firmly kept in mind in using the product rule of differentiation. It turns out, however, that the above naive differentiation is coincidentally true because of the following simple but useful auxiliary result:

$$F_{0,T}^P(S)N'(d_1) = F_{0,T}^P(K)N'(d_2). \tag{6.3.2}$$

This result can be remembered as the strikingly simple fact that if you "accidentally" mistake the standard normal distribution function $N(\cdot)$ as the standard normal probability density function $N'(\cdot)$ given by

$$N'(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2), \tag{6.3.3}$$

your Black-Scholes call price must be zero (don't make this fatal mistake in the exam!). To check (6.3.2), we use the identity

$$\mathrm{e}^{-(x+y)^2/2} = \mathrm{e}^{-(x-y)^2/2} \times \mathrm{e}^{-2xy}$$

for any real x and y and the prepaid forward versions of d_1 and d_2 to yield

$$\begin{aligned}
&F_{0,T}^P(S)N'(d_1)\\
&= F_{0,T}^P(S) \times \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\left(\overbrace{\ln[F_{0,T}^P(S)/F_{0,T}^P(K)]/\sigma\sqrt{T}}^{x}+\overbrace{\sigma\sqrt{T}/2}^{y}\right)^2}{2}\right]\\
&= F_{0,T}^P(S) \times \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\left(\ln[F_{0,T}^P(S)/F_{0,T}^P(K)]/\sigma\sqrt{T}\boxed{-}\sigma\sqrt{T}/2\right)^2}{2} - \ln\frac{F_{0,T}^P(S)}{F_{0,T}^P(K)}\right]\\
&= F_{0,T}^P(S) \times \frac{1}{\sqrt{2\pi}}\exp(-d_2^2/2) \times \frac{F_{0,T}^P(K)}{F_{0,T}^P(S)}\\
&= F_{0,T}^P(K)N'(d_2).
\end{aligned}$$

Using the chain rule, we have

$$\begin{aligned}
\Delta_C &= \frac{\partial}{\partial S}[S\mathrm{e}^{-\delta T}N(d_1) - K\mathrm{e}^{-rT}N(d_2)]\\
&= \mathrm{e}^{-\delta T}N(d_1) + S\mathrm{e}^{-\delta T}N'(d_1)\frac{\partial d_1}{\partial S} - K\mathrm{e}^{-rT}N'(d_2)\frac{\partial d_2}{\partial S}.\\
&= \mathrm{e}^{-\delta T}N(d_1) + \underbrace{[S\mathrm{e}^{-\delta T}N'(d_1) - K\mathrm{e}^{-rT}N'(d_2)]}_{0} \times \frac{\partial d_1}{\partial S}\\
&= \mathrm{e}^{-\delta T}N(d_1),
\end{aligned}$$

where the third equality is due to (6.3.2) and $\partial d_1/\partial S = \partial d_2/\partial S$ (because $d_2 = d_1 - \sigma\sqrt{T}$).

To derive the put delta, the easiest way is to use put-call parity, which says that

$$C - P = S\mathrm{e}^{-\delta T} - K\mathrm{e}^{-rT}.$$

Partially differentiating both sides with respect to S yields

$$\Delta_C - \Delta_P = e^{-\delta T},$$

whence

$$\Delta_P = \Delta_C - e^{-\delta T} = -e^{-\delta T} N(-d_1).$$

This addresses Question 2. □

Example 6.3.1. (Direct calculation of delta) Assume the Black-Scholes framework. You are given:

(i) The stock price is 100.

(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

(iii) The continuously compounded risk-free interest rate is 6%.

(iv) The stock's volatility is 40%.

Calculate the delta of a one-year 105-strike European call option.

Solution. As

$$d_1 = \frac{\ln(100/105) + (0.06 - 0.02 + 0.4^2/2)(1)}{0.4\sqrt{1}} = 0.17802,$$

the delta of the call option is

$$\Delta_C = \underbrace{e^{-0.02(1)}}_{\text{Don't omit this!}} \times \underbrace{N(d_1)}_{0.57065} = \boxed{0.55935}.$$

□

Example 6.3.2. (SOA Exam IFM Advanced Derivatives Sample Question 8) You are considering the purchase of a three-month 41.5-strike American call option on a nondividend-paying stock.

You are given:

(i) The Black-Scholes framework holds.

(ii) The stock is currently selling for 40.

(iii) The stock's volatility is 30%.

(iv) The current call option delta is 0.5.

Determine the current price of the option.

(A) $20 - 20.453 \int_{-\infty}^{0.15} e^{-x^2/2}\,dx$

(B) $20 - 16.138 \int_{-\infty}^{0.15} e^{-x^2/2}\,dx$

(C) $20 - 40.453\int_{-\infty}^{0.15} \mathrm{e}^{-x^2/2}\,\mathrm{d}x$

(D) $16.138\int_{-\infty}^{0.15} \mathrm{e}^{-x^2/2}\,\mathrm{d}x - 20.453$

(E) $40.453\int_{-\infty}^{0.15} \mathrm{e}^{-x^2/2}\,\mathrm{d}x - 20.453$

Solution. It can be shown that it is never optimal to exercise an American call option on a nondividend-paying stock before maturity (see Chapter 9), so the current price of the American call is the same as that of the otherwise identical European call.

Because $N(d_1) = 0.5$, we have $d_1 = 0$. It follows from

$$d_1 = \frac{\ln(40/41.5) + (r + 0.3^2/2)(0.25)}{0.3\sqrt{0.25}} = 0$$

that $r = 0.102256$. Then $d_2 = d_1 - 0.3\sqrt{0.25} = -0.15$. Finally, by (6.1.1), the price of the call option is

$$\begin{aligned} C &= S(0)\Delta - K\mathrm{e}^{-rT}N(d_2) \\ &= 40(0.5) - 41.5\mathrm{e}^{-(0.102256)(0.25)}N(-0.15) \\ &= 20 - 40.452540[1 - N(0.15)] \\ &= 40.452540\int_{-\infty}^{0.15} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\,\mathrm{d}x - 20.4525 \\ &= \boxed{16.138\int_{-\infty}^{0.15} \mathrm{e}^{-x^2/2}\,\mathrm{d}x - 20.453}. \quad \textbf{(Answer: (D))} \end{aligned}$$

□

Sidebar: Delta in the Black-Scholes framework vs delta in the binomial framework.

We have seen in the (one-period) binomial tree setting of Chapter 4 that the fair price of a derivative, V, equals the cost of setting up the replicating portfolio (Δ, B). Mathematically,

$$V = S\Delta + B, \tag{6.3.4}$$

where Δ is the number of stocks to buy and B is the amount of risk-free investment made initially so as to mimic the payoff of the derivative at the end of the period. Surprisingly, the Black-Scholes option pricing formula, when viewed in the correct light, shares the same structure as (6.3.4). In the case of a European call, for instance, we have

$$C = S\boxed{\mathrm{e}^{-\delta T}N(d_1)} - K\mathrm{e}^{-rT}N(d_2) = S\Delta_C + B,$$

with $\Delta_C := \mathrm{e}^{-\delta T}N(d_1)$, which is the delta of the call (see (6.3.1)), and $B := -K\mathrm{e}^{-rT}N(d_2)$. It can be shown that $(\Delta_C, B) = (\mathrm{e}^{-\delta T}N(d_1), -K\mathrm{e}^{-rT}N(d_2))$ provides the ingredients for the replicating portfolio of the call *at time 0*. To replicate the payoff of the call at maturity, it is necessary to adjust the values of Δ and B *continuously* in response to the emergence of information in the market (the technical details are beyond the scope of this book). The bottom line is that the Black-Scholes formula provides not only the fair price of an option, but also the *initial* replicating portfolio of the option.

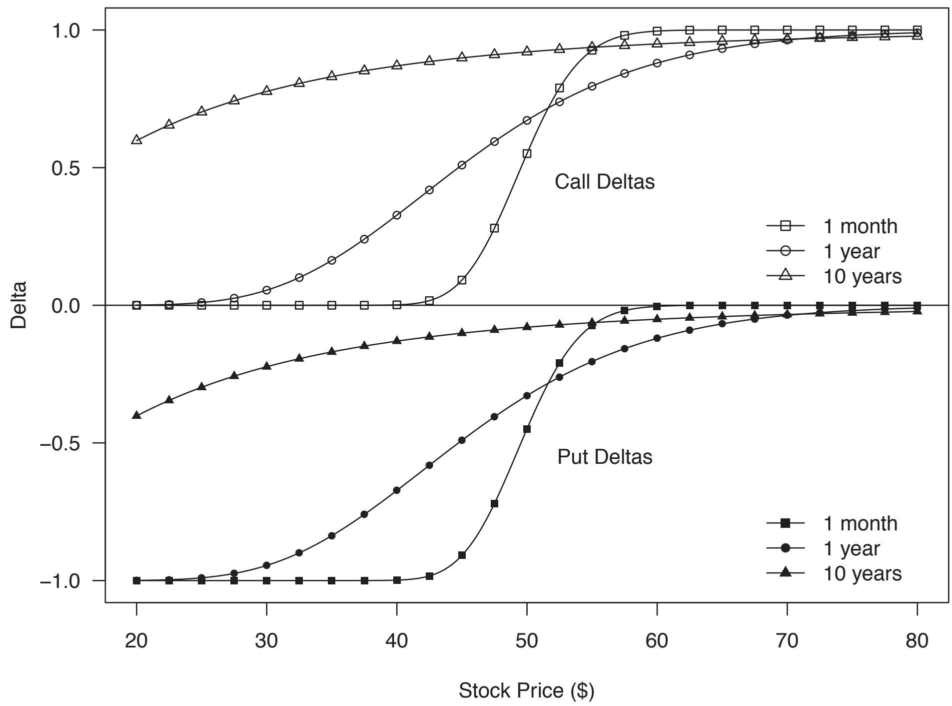


FIGURE 6.3.1
The variation of call and put deltas with the current stock price for different times to expiration.

Question 3: Signs of delta.

Intuition suggests and (6.3.1) confirms that call prices (resp. put prices) increase (resp. decrease) with the price of the underlying asset, because $\Delta_C > 0$ and $\Delta_P < 0$. As the stock price increases, the call (resp. put), which is a right to acquire (resp. give up) the stock, becomes more (resp. less) valuable. Moreover, the delta of a call is always less than 1 because the call price cannot increase by more than one dollar for every dollar increase in the price of the underlying asset. Similarly, the delta of a put is always greater than -1 because a unit decrease in the price of the underlying asset cannot lead to more than a dollar increase in the put price.

Question 4: Graphs of delta as a function of S.

Figure 6.3.1 shows how the deltas of a call option and a put option on a nondividend-paying stock varies with the current stock price S for different times to expiration. These curves (and other option curves in this section) are generated using the following common set of baseline parameters:

$$K = 40, \qquad \sigma = 30\%, \qquad r = 8\%, \qquad \delta = 0.$$

Some salient features of the figure deserve special attention:

1. The delta curve of a put is the delta curve of an otherwise identical call translated downward by 1.

Explanations: This can be easily seen by differentiating both sides of put-call parity, yielding

$$\Delta_C - \Delta_P = \mathrm{e}^{-\delta T} \quad \Rightarrow \quad \Delta_P = \Delta_C - \mathrm{e}^{-\delta T} \overset{(\text{if } \delta=0)}{=} \Delta_C - 1.$$

The fact that the put delta is simply a downward translation of the call delta means that we can easily understand the put delta if we can analyze the behavior of the call delta.

2. The *magnitude* of the delta depends closely on the option's *moneyness.* More specifically:
   - Deep in-the-money options have deltas close to 1 in absolute value (i.e., $\Delta_C \approx 1$ for very high S and $\Delta_P \approx -1$ for very low S).

     Explanations (for call): A deep in-the-money call is very likely to be exercised at expiration, with a terminal payoff of approximately $S(T) - K$. The current price is then approximately $S(0) - K\mathrm{e}^{-rT}$, leading to an approximate delta of 1. Such a call is very sensitive to changes in the stock price, as a unit increase in S will lead to an approximate unit increase in the call price.
   - Deep out-of-the-money options have deltas close to 0 (i.e., $\Delta_C \approx 0$ for very low S and $\Delta_P \approx 0$ for very high S).

     Explanations: A deep out-of-the-money call is very unlikely to be exercised at expiration and is very insensitive to changes in the stock price – even if S increases by one unit, the deep out-of-the-money call remains deep out-of-the-money, with a price close to zero.

   You can also see these phenomena mathematically by noticing that

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \begin{cases} \to \infty, & \text{as } S \to \infty, \\ \to -\infty, & \text{as } S \to 0, \end{cases}$$

   so

$$\Delta_C = N(d_1) = \begin{cases} \to 1, & \text{as } S \to \infty, \\ \to 0, & \text{as } S \to 0. \end{cases}$$

3. Comparing the three delta curves for each of the call and put options shows how the delta varies with the *time to expiration*, when the stock price is fixed. Typically, the delta of a deep out-of-the-money call becomes greater and the delta of a deep in-the-money call becomes lower as the time to expiration increases.

   Explanations: With greater time to expiration, it is more probable that a deep out-of-the-money (resp. in-the-money) option will expire in-the-money (out-of-the-money), thereby becoming more (resp. less) sensitive to the stock price.

6.3.2 Option Gamma

Definition.

The *gamma* of an option is defined as the second partial derivative of the option price with respect to the current price of the underlying asset. Equivalently, it is the partial derivative of the option delta with respect to the current price of the underlying asset:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}.$$

Corresponding to these two equivalent definitions, gamma can be thought of as a measure of the sensitivity of the delta to changes in the current stock price, and a measure of the curvature of the option price as a function of S.

Question 1: Closed-form expression for gamma.

Differentiating the expression for delta, by virtue of the chain rule again, shows that gamma is given by

$$\boxed{\Gamma_C = \Gamma_P = \mathrm{e}^{-\delta T} N'(d_1)\frac{\partial d_1}{\partial S} = \mathrm{e}^{-\delta T} \times \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-d_1^2/2} \times \frac{1}{S\sigma\sqrt{T}},}$$

where $N'(\cdot)$ is the probability density function of the standard normal distribution given in (6.3.3). Because every term that constitutes Γ is positive, so is Γ, answering Question 3. This comes as no surprise to us because delta as a function of S is increasing, as we have seen in the previous subsection. Mathematically, the positivity of gamma for both call and put options means that they are *convex* functions of the stock price. The convexity holds generally, even outside the Black-Scholes framework.

Example 6.3.3. (Direct calculation of gamma) Assume the Black-Scholes framework. You are given:

(i) The stock price is 100.

(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

(iii) The continuously compounded risk-free interest rate is 6%.

(iv) The stock's volatility is 40%.

Calculate the gamma of a 1-year 105-strike European call option.

Solution. As

$$d_1 = \frac{\ln(100/105) + (0.06 - 0.02 + 0.4^2/2)(1)}{0.4\sqrt{1}} = 0.17802,$$

the gamma of the call option is

$$\Gamma = \mathrm{e}^{-0.02} \times \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-0.17802^2/2} \times \frac{1}{100(0.4)\sqrt{1}} = \boxed{0.009622}.$$

□

Graphs of gamma.

Figure 6.3.2 shows how the gamma of a call and put on a nondividend-paying stock varies with the current stock price S for different times to expiration, using the same set of baseline parameters on page 207. Again, many features of the figure can be justified intuitively:

1. The gamma curve of a call and that of an otherwise identical put coincide. This can be seen by differentiating both sides of put-call parity with respect to S, yielding

$$\Gamma_C - \Gamma_P = 0 \quad \Rightarrow \quad \Gamma_C = \Gamma_P.$$

This settles Question 2 regarding the relationship between the gamma of otherwise identical calls and puts.

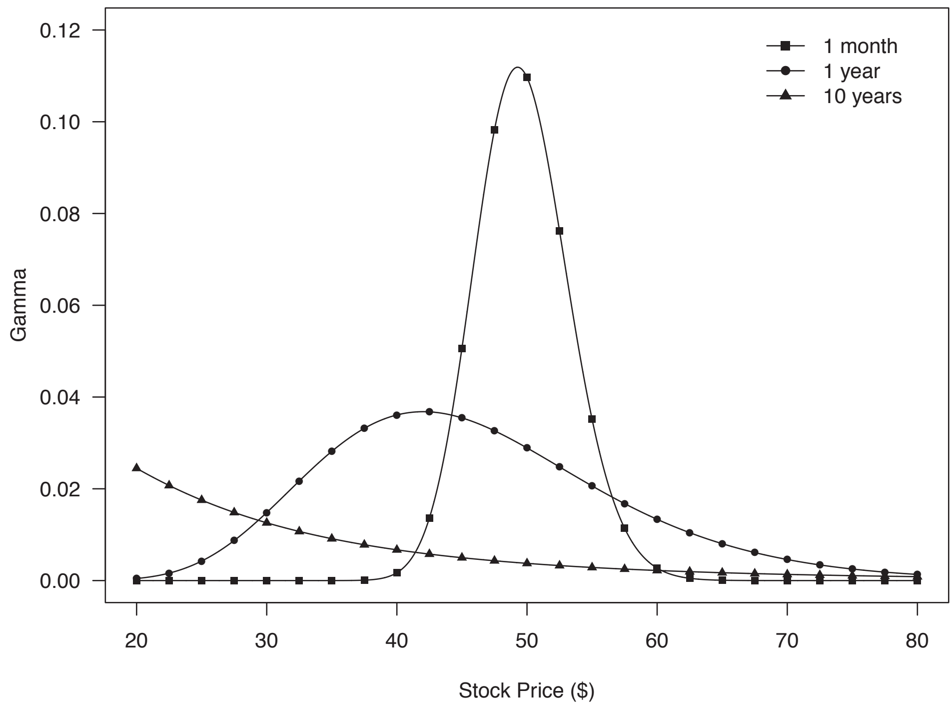


FIGURE 6.3.2
The variation of gamma with the current stock price for different times to expiration.

2. Gamma is almost zero when the option is deep in-the-money or deep out-of-the-money.

   Explanations: Without loss of generality, consider a call (since a put has the same gamma). When the call is deep in-the-money (resp. deep out-of-the-money), delta is very close to one (resp. zero) and does not change much as the stock price moves. Gamma, being the rate of change of delta, is then almost zero.

3. For short-lived options, gamma peaks prominently near the strike price ($40). For longer-lived options, gamma peaks further to the left of the strike price and less steeply.

   Explanations: For short-lived options, delta increases very abruptly from 0 or 1 around the strike price. Defined as the rate of change in delta, gamma takes very positive values in the same region. The increase in delta is more gradual for longer-lived options, so they have a lower gamma.

Example 6.3.4. (Highest gamma) Which of the following otherwise identical European options has the highest gamma?

(A) 1-day deep out-of-the-money call option

(B) 10-day deep in-the-money call option

(C) 1-year at-the-money put option

(D) 10-year at-the-money put option

(E) There is not enough information to determine the answer.

(Answer: (C))

6.3.3 Option Greeks of a Portfolio

Option Greeks are by no means confined to call or put options; they can be defined even for a portfolio of derivatives and evaluated easily. Since partial differentiation is a linear operator, the Greek of a portfolio equals the *sum* of the constituent Greeks. Symbolically, we have:

$$V_{\text{portfolio}} = \sum V_{\text{component}} \overset{\text{(by partial differentiation)}}{\Rightarrow} \text{Greek}_{\text{portfolio}} = \sum \text{Greek}_{\text{component}}. \tag{6.3.5}$$

The additivity of option Greeks is practically important because it means not only that the risk of a portfolio as reflected by an option Greek can be easily calculated from the component risks, but also that risk management and hedging at the portfolio level can be readily implemented. This will prove useful in the next chapter when we study delta-hedging and delta-gamma-hedging.

Example 6.3.5. (SOA Exam IFM Advanced Derivatives Sample Question 31: Delta of a bull spread) You compute the current delta for a 50-60 bull spread with the following information:

(i) The continuously compounded risk-free rate is 5%.

(ii) The underlying stock pays no dividends.

(iii) The current stock price is $50 per share.

(iv) The stock's volatility is 20%.

(v) The time to expiration is 3 months.

How much does delta change after 1 month, if the stock price does not change?

(A) increases by 0.04

(B) increases by 0.02

(C) does not change, within rounding to 0.01

(D) decreases by 0.02

(E) decreases by 0.04

Ambrose's comments:

This question involves the calculations of four d_1's! Be patient!

Solution. Assume without loss of generality that the bull spread is a call bull spread. The same answers would be obtained by assuming put options instead of call options. Then the delta of the bull spread equals

$$\Delta_{\text{bull spread}} = \Delta_C^{\text{50-strike}} - \Delta_C^{\text{60-strike}}.$$

Initially:

- For the 50-strike call,

$$\begin{aligned} d_1 &= \frac{\ln(50/50) + (0.05 + 0.2^2/2)(0.25)}{0.2\sqrt{0.25}} = 0.175, \\ \Delta_C^{\text{50-strike}} &= N(d_1) = 0.56946. \end{aligned}$$

- For the 60-strike call,

$$\begin{aligned} d_1 &= \frac{\ln(50/60) + (0.05 + 0.2^2/2)(0.25)}{0.2\sqrt{0.25}} = -1.64822, \\ \Delta_C^{\text{60-strike}} &= N(d_1) = 0.04965. \end{aligned}$$

The original delta is thus $0.56946 - 0.04965 = 0.51981$.

After 1 month (the remaining time to expiration is 2 months):

- For the 50-strike call,

$$\begin{aligned} d_1 &= \frac{\ln(50/50) + (0.05 + 0.2^2/2)/6}{0.2\sqrt{1/6}} = 0.14289, \\ \Delta_C^{\text{50-strike}} &= N(d_1) = 0.55681. \end{aligned}$$

- For the 60-strike call,

$$\begin{aligned} d_1 &= \frac{\ln(50/60) + (0.05 + 0.2^2/2)/6}{0.2\sqrt{1/6}} = -2.09009, \\ \Delta_C^{\text{60-strike}} &= N(d_1) = 0.01830. \end{aligned}$$

The new delta is $0.55681 - 0.01830 = 0.53851$. Therefore, the change in delta is $0.53851 - 0.51981 = \boxed{0.0187}$. **(Answer: (B))** □

Remark. The calculation of the change in the price of the bull spread by hand is much more cumbersome.

6.3.4 Option Elasticity

Motivation.

Whereas the delta of an option quantifies the approximate change in the option price for a dollar increase in the underlying stock, such a sensitivity measure can be criticized since *absolute* changes in the option price and *absolute* changes in the stock price are, in fact, not directly comparable. Call prices, for example, are constrained by the price of the stock, so that a \$1 change in the call price can mean a lot compared to a \$1 change in the stock price. A fairer and dimensionless sensitivity measure can be developed by weighing the *percentage* changes in the option price against the *percentage* changes in the stock price. This leads to the notion of option elasticity, which emanates from economics.

Computing formula.

Mathematically, let $V(S)$ be the price of a generic option (in fact, any derivative) when the current price of the underlying stock is S (other arguments, e.g., strike price, time to maturity, are suppressed and assumed to be fixed here). The *elasticity* of the option, denoted by Ω, measures the *percentage* change in the option price relative to the *percentage* change in the stock price. It provides answers to the question

> "If the price of the stock increases by 1%, by how much does the price of the option change in proportion?"

To derive the expression for the elasticity of an option, consider

$$\Omega(S) = \lim_{\epsilon \to 0} \frac{\overbrace{[V(S+\epsilon) - V(S)]/V(S)}^{\text{\% change in option price}}}{\underbrace{\epsilon/S}_{\text{\% change in stock price}}} = \frac{S}{V(S)} \lim_{\epsilon \to 0} \frac{V(S+\epsilon) - V(S)}{\epsilon} = \frac{SV'(S)}{V(S)},$$

or, more compactly,

$$\boxed{\Omega = \frac{S\Delta_V}{V}.} \tag{6.3.6}$$

This provides a convenient formula for computing the elasticity of an option. When viewed in the form

$$\Omega = \frac{\Delta_V/V}{1/S},$$

the formula confirms that the elasticity indeed captures the proportional changes in the option price and the stock price.

Example 6.3.6. (CAS Exam 3 Fall 2007 Question 22: Warm-up calculation of Ω) A call option is modeled using the Black-Scholes formula with the following parameters.

- $S = 25$
- $K = 24$
- $r = 4\%$
- $\delta = 0\%$
- $\sigma = 20\%$
- $T = 1$

Calculate the call option elasticity, Ω.

(A) Less than 5

(B) At least 5, but less than 6

(C) At least 6, but less than 7

(D) At least 7, but less than 8

(E) At least 8

Solution. With

$$\begin{aligned}
d_1 &= \frac{\ln(25/24) + (0.04 + 0.2^2/2)(1)}{0.2\sqrt{1}} = 0.50411, \\
d_2 &= d_1 - 0.2\sqrt{1} = 0.30411, \\
N(d_1) &= 0.69291, \quad (= \Delta_C) \\
N(d_2) &= 0.61948,
\end{aligned}$$

the Black-Scholes call price is

$$C = 25(0.69291) - 24e^{-0.04(1)}(0.61948) = 3.03819.$$

The call option elasticity is

$$\Omega_C = \frac{S\Delta_C}{C} = \frac{25(0.69291)}{3.03819} = \boxed{5.7017}. \quad \textbf{(Answer: (B))}$$

□

Example 6.3.7. (SOA Exam IFM Advanced Derivatives Sample Question 41: Elasticity of a capped stock) Assume the Black-Scholes framework. Consider a 1-year European contingent claim on a stock.

You are given:

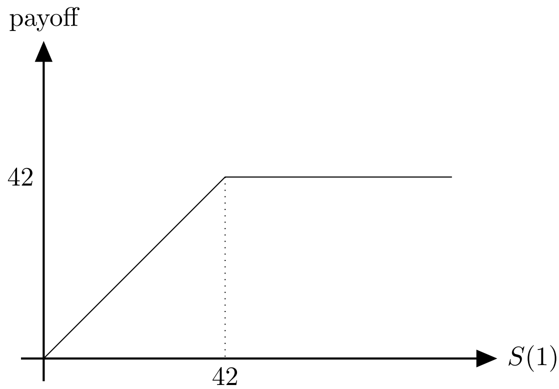
(i) The time-0 stock price is 45.

(ii) The stock's volatility is 25%.

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(iv) The continuously compounded risk-free interest rate is 7%.

(v) The time-1 payoff of the contingent claim is as follows:



Calculate the time-0 contingent-claim elasticity.

(A) 0.24

(B) 0.29

(C) 0.34

(D) 0.39

(E) 0.44

Solution. There are two ways to view the contingent claim. We discuss both below. As with option Greeks, we decorate Ω by the subscripts "C" and "P" to denote the elasticity of a call and that of a put, respectively.

- *Short put plus bond:* The first way is to consider the contingent claim as a short 42-strike 1-year put coupled with a long 1-year zero-coupon bond with a face value of 42. As

$$\begin{aligned} d_1 &= \frac{\ln(45/42) + (0.07 - 0.03 + 0.25^2/2)(1)}{0.25\sqrt{1}} = 0.56097, \\ d_2 &= d_1 - 0.25\sqrt{1} = 0.31097, \\ N(-d_1) &= 0.28741, \\ N(-d_2) &= 0.37791, \end{aligned}$$

the price of the put option is

$$P = 42e^{-0.07}(0.37791) - 45e^{-0.03}(0.28741) = 2.24795,$$

so the time-0 price of the contingent claim is

$$V = \mathrm{PV}_{0,1}(42) - P = 42e^{-0.07} - 2.24795 = 36.91259.$$

Because the zero-coupon bond has zero delta, the delta of the contingent claim is

$$\Delta_V = -\Delta_P = -[-e^{-0.03}(0.28741)] = 0.27892.$$

Thus the contingent-claim elasticity is

$$\Omega_V = \frac{S\Delta_V}{V} = \frac{45(0.27892)}{36.91259} = \boxed{0.34003}. \quad \textbf{(Answer: (C))}$$

- *Long stock plus short call:* Another way to view the contingent claim is to regard it as one unit of the stock *at time 1* (i.e., a 1-year prepaid forward on the stock) and a short 42-strike 1-year call. In the language of Section 3.1, the contingent claim is a written covered call.

  As $N(d_1) = 0.71259$ and $N(d_2) = 0.62209$, the price of the call is

  $$C = 45\mathrm{e}^{-0.03}(0.71259) - 42\mathrm{e}^{-0.07}(0.62209) = 6.75746,$$

  so the time-0 price of the contingent claim is

  $$V = F^P_{0,1}(S) - C = 45\mathrm{e}^{-0.03} - 6.75746 = 36.91259.$$

  The delta of the contingent claim is

  $$\Delta_V = \mathrm{e}^{-\delta T} - \Delta_C = \mathrm{e}^{-0.03} - \mathrm{e}^{-0.03}(0.71259) = 0.27892.$$

  Therefore, the contingent-claim elasticity is

  $$\Omega_V = \frac{S\Delta_V}{V} = \frac{45(0.27892)}{36.91259} = \boxed{0.34003}. \quad \textbf{(Answer: (C))}$$

  □

Elasticity for call and put options.

Like option delta, the elasticity of call and put options enjoys bounds that can be justified either algebraically or verbally, although there are no straightforward connections between the elasticity of a call and the elasticity of a put having the same strike price and time to expiration.

Call. For a call, $\Omega > 1$ always, because

$$\Omega_C = \frac{S\Delta_C}{C} = \frac{S\mathrm{e}^{-\delta T}N(d_1)}{S\mathrm{e}^{-\delta T}N(d_1) - K\mathrm{e}^{-rT}N(d_2)} > \frac{S\mathrm{e}^{-\delta T}N(d_1)}{S\mathrm{e}^{-\delta T}N(d_1)} = 1.$$

As a result, a call can be said to be riskier than the underlying stock in the sense that the call price increases more vigorously in proportion than the stock price.

Put. For a put, $\Omega \leq 0$, because the stock price and the put price exhibit changes in different directions, as reflected by the negativity of the delta of a put. It is possible, however, that $-1 \leq \Omega_P \leq 0$, which occurs when the put is deep in-the-money, i.e., when the current stock price is substantially below the strike price. In this case, a 1% percent increase in the stock price means a very small absolute increase in the stock price. As $\Delta_P > -1$, the absolute decrease in the put price is minuscule as well. This, together with the high put price, in turn causes a negligible percentage drop in the put price and explains the small magnitude of the put elasticity.

Elasticity of a portfolio.

Unlike the portfolio Greek, the elasticity of a portfolio is not the sum, but a *weighted average* of the constituent elasticities. To see this and identify the weights, consider a portfolio of

n_1 option 1,

n_2 option 2,

$\vdots$

and n_N option N,

all of which are on the same underlying stock. Denote by V_i and Δ_i the value and delta of the i^{th} option, for $i = 1, 2, \ldots, N$. Applying (6.3.6) to this portfolio treated as another asset and using the additivity of delta, we have

$$\Omega_{\text{portfolio}} = \frac{S\Delta_{\text{portfolio}}}{V_{\text{portfolio}}} = \frac{S\sum_{i=1}^N n_i\Delta_i}{\sum_{i=1}^N n_i V_i}, \tag{6.3.7}$$

which can be further rearranged as

$$\Omega_{\text{portfolio}} = \sum_{i=1}^N \left(\frac{n_i V_i}{\sum_{j=1}^N n_j V_j}\right)\left(\frac{S\Delta_i}{V_i}\right) = \sum_{i=1}^N \omega_i \Omega_i, \tag{6.3.8}$$

where $\omega_i = n_i V_i / \sum_{j=1}^N n_j V_j$ is the portion of the portfolio invested in option i.

As a matter of fact, (6.3.7) is what you usually use in a computational problem, while (6.3.8) is useful only when you are given minimal information, e.g., only constituent elasticities and prices are given, but not S, r, δ, T, σ, etc.

Example 6.3.8. (SOA Exam IFM Advanced Derivatives Sample Question 20: Elasticity of portfolio) Assume the Black-Scholes framework. Consider a stock, and a European call option and a European put option on the stock. The current stock price, call price, and put price are 45.00, 4.45, and 1.90, respectively.

Investor A purchases two calls and one put. Investor B purchases two calls and writes three puts.

The current elasticity of Investor A's portfolio is 5.0. The current delta of Investor B's portfolio is 3.4.

Calculate the current put-option elasticity.

(A) -0.55

(B) -1.15

(C) -8.64

(D) -13.03

(E) -27.24

Solution. Considering the elasticity of Investor A's portfolio, we get

$$\begin{aligned}\frac{S}{2C+P}(2\Delta_C + \Delta_P) = 5.0 \quad &\Rightarrow \quad \frac{45}{2(4.45)+1.90}(2\Delta_C + \Delta_P) = 5.0\\ &\Rightarrow \quad 2\Delta_C + \Delta_P = 1.2.\end{aligned}$$

Moreover, the current delta of Investor B's portfolio is

$$2\Delta_C - 3\Delta_P = 3.4.$$

Solving these two equations in Δ_C and Δ_P, we have $\Delta_P = -0.55$ (and $\Delta_C = 0.875$). It follows that the current put-option elasticity is

$$\Omega_P = \frac{S\Delta_P}{P} = \frac{45(-0.55)}{1.90} = \boxed{-13.03}. \quad \textbf{(Answer: (D))}$$

□

Example 6.3.9. (Elasticity of a bull spread) You are given the following information about 50-strike and 60-strike European put options with the same stock and time to expiration:

Strike price	Elasticity	Put premium
50	−4.9953	3.7295
60	−3.4267	9.5865

Calculate the elasticity of a 50-60 European put bull spread.

Solution. The 50-60 put bull spread is set up by buying the 50-strike put and selling the 60-strike put. By (6.3.8), the elasticity of the bull spread as a portfolio is

$$\begin{aligned}
\Omega_{\text{bull spread}} &= \frac{3.7295}{3.7295 - 9.5865}\Omega_P^{\text{50-strike}} + \left(\frac{-9.5865}{3.7295 - 9.5865}\right)\Omega_P^{\text{60-strike}} \\
&= \frac{3.7295}{3.7295 - 9.5865}(-4.9953) + \left(\frac{-9.5865}{3.7295 - 9.5865}\right)(-3.4267) \\
&= \boxed{-2.4279}.
\end{aligned}$$

Alternatively, from the two put elasticities, we deduce that

$$\begin{aligned}
&\begin{cases}
\Omega_P^{\text{50-strike}} = \dfrac{S\Delta_P^{\text{50-strike}}}{P^{\text{50-strike}}} = \dfrac{S\Delta_P^{\text{50-strike}}}{3.7295} = -4.9953 \\[2ex]
\Omega_P^{\text{60-strike}} = \dfrac{S\Delta_P^{\text{60-strike}}}{P^{\text{60-strike}}} = \dfrac{S\Delta_P^{\text{60-strike}}}{9.5865} = -3.4267
\end{cases} \\
\Rightarrow &\begin{cases}
S\Delta_P^{\text{50-strike}} &= -18.6300 \\
S\Delta_P^{\text{60-strike}} &= -32.8501
\end{cases}.
\end{aligned}$$

An application of (6.3.7) results in

$$\Omega_{\text{bull spread}} = \frac{S\Delta_{\text{bull spread}}}{P^{\text{50-strike}} - P^{\text{60-strike}}} = \frac{-18.6300 - (-32.8501)}{3.7295 - 9.5865} = \boxed{-2.4279}.$$

□

Remark. While a put bull spread has the same delta as a call bull spread (see Example 6.3.5), they do not share the same price. Accordingly, their elasticities also differ.

Volatility of an option in terms of its elasticity

There are other practically useful and theoretically appealing results surrounding the elasticity of an option. For example, in the Black-Scholes framework the volatility of the *option* and the volatility of the underlying stock are related via[ix]

$$\boxed{\sigma_{\text{option}} = |\Omega|\sigma_{\text{stock}}.} \tag{6.3.9}$$

In this respect, the elasticity serves as an amplifier that is applied to the stock volatility to form the option volatility. However, in contrast to the stock volatility, which is a constant, the volatility of the option, as with elasticity, generally varies with the current stock price and the remaining time to expiration. To be mathematically precise, we should state (6.3.9) as

$$\sigma_{\text{option}}(s,t) = |\Omega(s,t)|\sigma_{\text{stock}}$$

to emphasize the dependence of $\Omega(\cdot,\cdot)$ and $\sigma_{\text{option}}(\cdot,\cdot)$ on the current stock price s and current time t.

[ix]This can be shown by the Itô's Lemma, which is beyond the scope of this book.

6.4 Problems

The Black-Scholes formula for stocks paying continuous proportional dividends

Problem 6.4.1. (Based on SOA Exam MFE Spring 2009 Question 4: Direct calculation of put prices) Your company has just written a one-year European put option on an equity index fund.

The equity index fund is currently trading at 1000. It pays dividends continuously at a rate proportional to its price; the dividend yield is 2%. It has a volatility of 20%.

The strike price of the put option is set in order to insure against a reduction of more than 40% in the value of the equity index fund at the end of one year.

The continuously compounded risk-free interest rate is 2.5%.

Using the Black-Scholes model, determine the price of the put option.

Problem 6.4.2. (Given the price of an ATM option, deduce the current stock price) Assume the Black-Scholes framework. For an at-the-money, 8-month European put option on a stock, you are given:

(i) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

(ii) The continuously compounded risk-free interest rate is 5%.

(iii) $\mathrm{Var}[\ln S(t)] = 0.16t$ for all $t \geq 0$, where $S(t)$ is the time-t price of the stock.

(iv) The current price of this put option is 7.

Calculate the current stock price.

Problem 6.4.3. (Black-Scholes price of a strange claim) Assume the Black-Scholes framework. Consider a 3-year European contingent claim on a stock. For $t \geq 0$, let $S(t)$ be the time-t price of the stock.

You are given:

(i) $S(0) = 45$.

(ii) The stock's volatility is 20%.

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(iv) The continuously compounded risk-free interest rate is 6%.

(v) The 3-year payoff of the contingent claim is

$$\text{Payoff} = \max\{(S(3) - 45)_+, 2(S(3) - 60)\}.$$

Calculate the current price of the contingent claim.

(Hint: You may sketch the claim's payoff diagram, from which you may deduce that the claim equals the sum of two calls with different strike prices.)

Problem 6.4.4. (Given the true exercise probability of a put, find its price) Assume the Black-Scholes framework. You are given:

(i) The current price of a stock is 80.

(ii) The stock's volatility is 25%.

(iii) The stock pays dividends continuously at a rate proportional to its price.

(iv) The continuously compounded risk-free interest rate is 5%.

(v) The continuously compounded expected rate of return on the stock, α, is 8%.

(vi) The *true* probability that a 1-year at-the-money European put option on the stock will be exercised is 0.4681.

Calculate the current price of the put option in (vi).

Applying the Black-Scholes formula to other underlying assets

Problem 6.4.5. (Stock paying discrete dividends – I) Consider a stock with current price \$50. You are given:

(i) There will be only one dividend; \$2 will be paid in three months.

(ii) $\sigma = 0.30$.

(iii) The continuously compounded risk-free interest rate is 5%.

Use the Black-Scholes methodology to price a nine-month at-the-money European put option on the stock. What is the meaning of σ in this context?

Problem 6.4.6. (Stock paying discrete dividends – II) Assume the Black-Scholes framework. You are given:

(i) The current stock price is \$82.

(ii) The stock's volatility is 30%.

(iii) The stock pays no dividends.

(iv) The continuously compounded risk-free interest rate is 8%.

Using the above information, you calculate the price of a 3-month 80-strike European call.

Immediately after your valuation, it is publicly announced that the stock will pay a dividend of \$6 in 1 month, and no other payouts over the life of the call. Using the Black-Scholes methodology with the same volatility parameter of 30%, you recalculate the price of the call.

Calculate the change in the price of the call.

Problem 6.4.7. (Valuing a bear spread in the presence of discrete dividends) Assume the Black-Scholes framework. For a 9-month 45-55 put bear spread on a stock, you are given:

(i) The current stock price is 50.

(ii) The only dividends during this time period are 2.50 to be paid in two months and five months.

(iii) $\mathrm{Var}[\ln F^P_{t,0.75}(S)] = 0.16t$ for $0 \le t \le 0.75$.

(iv) The continuously compounded risk-free interest rate is 8%.

Calculate the current price of the bear spread.

Problem 6.4.8. (Profit on a put in the presence of discrete dividends) Assume the Black-Scholes framework. For a 3-month at-the-money European put option on a stock, you are given:

(i) The stock is currently selling for 50.

(ii) The stock will pay a single dividend of 1.5 in two months.

(iii) $\mathrm{Var}[\ln F^P_{t,0.25}(S)] = 0.09t$, for $0 \le t \le 0.25$.

(iv) The continuously compounded risk-free interest rate is 10%.

The stock price when the put option expires is 45.

Calculate the 3-month profit on the put option.

Problem 6.4.9. (Profit on a cap on a stock paying discrete dividends) Assume the Black-Scholes framework. Consider a 3-month European contingent claim on a stock. You are given:

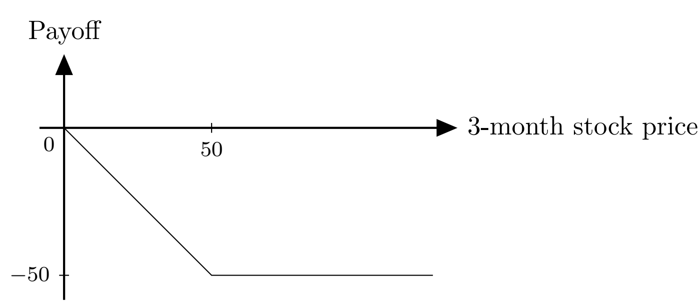
(i) The stock is currently selling for 50.

(ii) The stock will pay a single dividend of 1.5 in two months.

(iii) $\mathrm{Var}[\ln F^P_{t,0.25}(S)] = 0.09t$, for $0 \le t \le 0.25$.

(iv) The continuously compounded risk-free interest rate is 10%.

(v) The 3-month payoff of the contingent claim is as follows:



Calculate the profit on the contingent claim if the 3-month stock price is 45.

Problem 6.4.10. (How do a binomial price and a Black-Scholes price compare with each other?) You are given:

(i) The current dollar-euro exchange rate is 1.50\$/€.

(ii) The volatility of the exchange rate is 20%.

(iii) The continuously compounded risk-free interest rate on dollars is 3%.

(iv) The continuously compounded risk-free interest rate on euros is 4%.

Consider a 6-month at-the-money dollar-denominated European put option on euros.

Actuary A values the put option using a binomial forward tree, where the length of each period is 3 months, to model the movements of the dollar-euro exchange rate. Actuary B values the same put option assuming the Black-Scholes framework.

Calculate the absolute value of the difference between the prices computed by Actuary A and Actuary B.

Problem 6.4.11. (Use of currency options in a daily life context – "rich" version) You have ordered a Rolls Royce car for the price of 200,000 British pounds, which you will pay when the car is delivered to you in three months. The current exchange rate is 1.60 US dollars per British pound, and your insanely rich mom will give you US \$320,000 three months from now. Because the US dollar may lose value, you now buy appropriate 3-month at-the-money currency options of the European type to exactly cover the shortfall in case it occurs.

You are given:

(i) The continuously compounded risk-free interest rate in the United States is 1%.

(ii) The continuously compounded risk-free interest rate in the United Kingdom is 2%.

(iii) The volatility of the dollar/pound exchange rate is 20%.

Under the Black-Scholes framework, determine the total cost of the currency options in US dollars.

(Hint: Are you long or short with respect to the 3-month dollar/pound exchange rate? Are you worried about the exchange rate going up or down in three months? Should you set up a floor or a cap? Hopefully these questions will help you decide whether to buy a pound call or a pound put.)

Problem 6.4.12. (Use of currency options in a daily life context – "poor" version) To settle an urgent debt of US \$300,000 payable in three months, you have decided to (reluctantly!) sell your favorite Rolls Royce car for the price of 200,000 British pounds, which you will receive when the car is delivered to the buyer in three months. Because the British pound may lose value relative to the US dollar, you decide to buy appropriate 3-month at-the-money European currency options now to exactly cover the shortfall in case it occurs.

You are given:

(i) The current dollar/pound exchange rate is 1.5.

(ii) The continuously compounded risk-free interest rate in the United States is 4%.

(iii) The continuously compounded risk-free interest rate in the United Kingdom is 8%.

(iv) The future exchange rates of dollar per pound are lognormally distributed with a volatility of 30%.

Calculate the total cost of the currency options in US dollars.

Problem 6.4.13. (Pricing and using a currency put) Assume the Black-Scholes framework. You are given:

(i) The current dollar/euro exchange rate is 1.50\$/€.

(ii) The volatility of the exchange rate is 20%.

(iii) The continuously compounded risk-free interest rate on dollars is 4%.

(iv) The continuously compounded risk-free interest rate on euros is 5%.

Consider a 6-month at-the-money dollar-denominated European put option on euros.

(a) Calculate the price of the put option.

(b) Determine whether the euro put above can be used to hedge against exchange rate risk faced by each of the following individuals living in the United States.

  (1) Apple (Ambrose's twin brother), a famous chef in Iowa City, regularly imports food raw materials from Europe, with the next order made in 6 months and settled in euros.

  (2) Ambrosio (Ambrose's another twin brother), fed up with his newly purchased iPhone XX, has decided to sell it to his aunt living in Europe for €500 in 6 months.

(c) The continuously compounded expected rate of *appreciation* of the dollar/euro exchange rate is 1.5%.

Calculate the true probability that the put option will be exercised.

Problem 6.4.14. (Pricing a futures option – I) Assume the Black-Scholes framework. Consider a 95-strike 9-month European call option on an S&V 150 futures contract which matures one year from now. You are given:

(i) The current price of the S&V 150 index is 100.

(ii) The volatility of the S&V 150 index price is 30%.

(iii) The volatility of the futures price is 30%.

(iv) S&V 150 pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(v) The continuously compounded risk-free interest rate is 8%.

Calculate the price of the call option.

(Note: You may assume the fact that when the interest rate is constant, futures prices and forward prices agree.)

Problem 6.4.15. (Pricing a futures option – II) Assume the Black-Scholes framework. You are given:

(i) The current price of the P&K 777 index is 500.

(ii) The P&K 777 index pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

(iii) The continuously compounded risk-free interest rate is 6%.

(iv) The current prices and volatility of futures contracts on P&K 777 of various maturities:

Maturity (in Years)	1	2	3	4
Current Price	520.41	541.64	563.75	586.76
Volatility	30%			

Calculate the price of a 2-year 550-strike European put option on a 1-year futures contract (i.e., the futures matures at the end of 3 years).

Option Greeks

Problem 6.4.16. (What can you say given the price and delta of a put?) Assume the Black-Scholes framework. The current price of a nondividend-paying stock is \$60 and the continuously compounded risk-free interest rate is 5%.

Your boss has asked you to quote a price for Put A, which is a 6-month at-the-money European put option on the stock. Although the market price for Put A is not available, the market price of Put B, which is a 6-month 65-strike European put option on the same stock, is observed to be \$6.2514, and its delta is -0.5882.

Calculate the price of Put A.

Problem 6.4.17. (Gamma of a futures option) Assume the Black-Scholes framework. Consider a 6-month 90-strike European put option on a futures contract. You are given:

(i) The price of the underlying futures contract is 95.

(ii) The delta of the put option is -0.3382.

(iii) The continuously compounded risk-free interest rate is 3%.

(iv) The volatility of the futures is less than 50%.

Calculate the gamma of the futures put option.

Problem 6.4.18. (Signs of option Greeks) Assume the Black-Scholes framework. Consider European call and put options on a stock that pays dividends continuously at a rate proportional to its price.

Determine the signs of the following twelve Greeks:

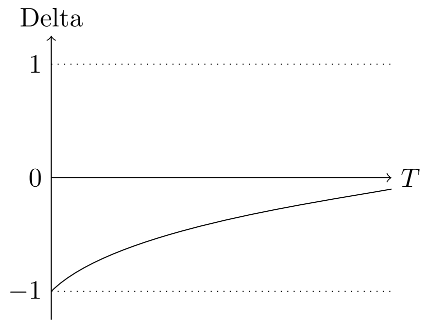
Call delta, call gamma, call theta, call vega, call rho, call psi,

put delta, put gamma, put theta, put vega, put rho, and put psi.

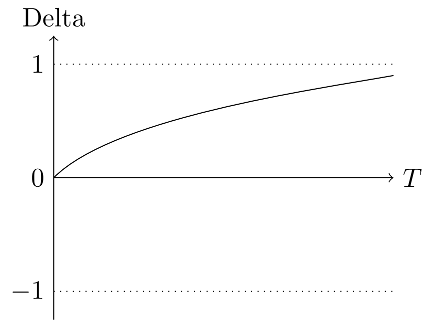
That is, for each of these Greeks, answer "Always positive," "Always negative," or "Sometimes positive and sometimes negative." Explain your answers briefly.

Problem 6.4.19. (Graphical question) Assume the Black-Scholes framework. Which of the following graphs best represents the relationship between the delta of a deep out-of-the-money European call option on a nondividend-paying stock and the time to maturity T?

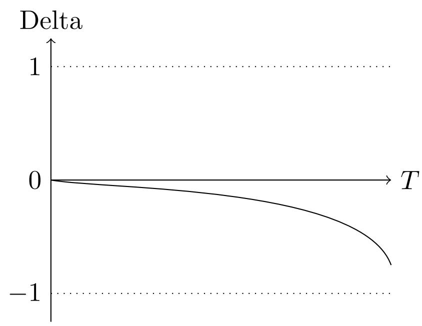
(A)



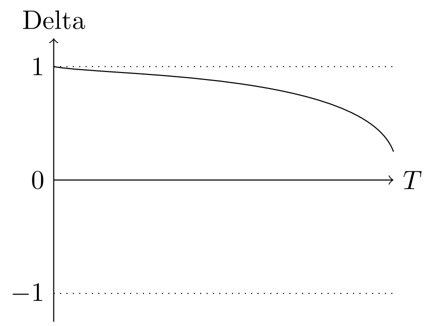
(B)



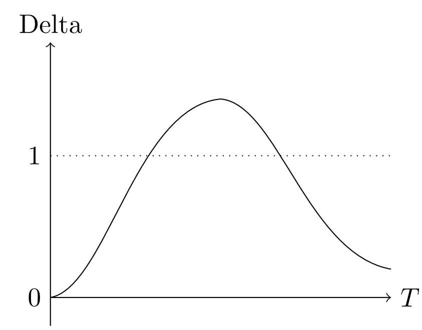
(C)



(D)



(E)



Problem 6.4.20. (Limiting behavior of gamma) Determine the limiting value of gamma as we approach maturity (i.e., find $\lim_{T\to 0}\Gamma$) for each of the following options:

(a) A deep in-the-money European call option

(b) A deep out-of-the-money European call option

(c) An at-the-money European call option

For each option, provide an intuitive explanation and a mathematical explanation.

(Note: It may not be easy to examine the global monotonicity of gamma as a function of the time to maturity. Determining the limiting values of gamma is not as hard.)

Problem 6.4.21. (Relationship between the option price and the strike price)

(a) If the strike price of a European call option decreases, then how will the price of the call change, holding everything else constant?

(b) Verify your conclusion in part (a) by general reasoning.

(c) Verify your conclusion in part (a) by the Black-Scholes formula.

Problem 6.4.22. (Greeks of a straddle) Assume the Black-Scholes framework. Consider a long K-strike European straddle.

For each of delta and gamma, discuss intuitively how the option Greek of the straddle varies with S, the current stock price.

(Hint: Pay special attention to how the characteristics of the call and put Greeks combine to form the overall picture.)

Problem 6.4.23. (Extension of Example 6.3.5: Gamma of a bull spread) You compute the current gamma for a 50-60 bull spread with the following information:

(i) The continuously compounded risk-free interest rate is 5%.

(ii) The underlying stock pays no dividends.

(iii) The current stock price is \$50 per share.

(iv) The stock's volatility is 20%.

(v) The time to expiration is 3 months.

Calculate the change in gamma after 1 month, if the stock price does not change.

Problem 6.4.24. (Calculations of Greeks and elasticity for currency options) Assume the Black-Scholes framework. You are given:

(i) The current dollar/euro exchange rate is 1.2.

(ii) The continuously compounded risk-free interest rate in the United States is 2%.

(iii) The continuously compounded risk-free interest rate in Europe is 3%.

(iv) The volatility of the dollar/euro exchange rate is 15%.

For a dollar-denominated nine-month at-the-money European call option on euro, calculate and interpret the values of:

(a) Delta and gamma

(b) Vega, theta, and rho

(c) Elasticity

Problem 6.4.25. (Direct calculation of elasticity) Assume the Black-Scholes framework. You are given:

(i) The current stock price is 82.

(ii) The stock's volatility is 30%.

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(iv) The continuously compounded risk-free interest rate is 8%.

Calculate the elasticity of a 3-month 80-strike European call option.

Problem 6.4.26. (Given elasticity, what is σ?) Assume the Black-Scholes framework. Consider a one-year at-the-money European put option on a nondividend-paying stock.

You are given:

(i) The ratio of the current put option price to the current stock price is 0.073445.

(ii) The current put-option elasticity is -5.941861.

(iii) The continuously compounded risk-free interest rate is 1.2%.

Determine the stock's volatility.

Problem 6.4.27. (Elasticity of a straddle) Assume the Black-Scholes framework. For a 3-month 32-strike European straddle on a stock, you are given:

(i) The stock currently sells for $30.

(ii) The stock's volatility is 30%.

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

(iv) The continuously compounded risk-free interest rate is 5%.

Calculate the current elasticity of the straddle.

Problem 6.4.28. (Given the constituent prices, deltas, and elasticities, find the portfolio elasticity) Assume the Black-Scholes framework. Consider a portfolio consisting of three European options, X, Y, and Z, on the same stock. You are given:

	X	Y	Z
Option price	6.8268	?	1.9299
Option delta	?	−0.4269	0.3537
Option elasticity	5.6496	−6.8755	9.1627

Calculate the elasticity of the portfolio.

Problem 6.4.29. (Given the gamma of a derivative, find its elasticity) Assume the Black-Scholes framework. For $t \geq 0$, let $S(t)$ be the time-t price of a stock.

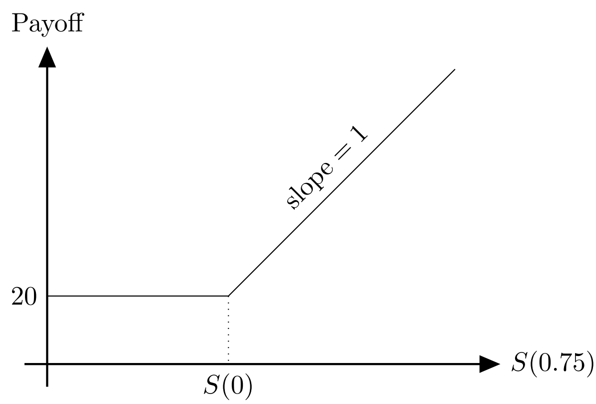
Consider a 9-month European contingent claim on the stock. You are given:

(i) The stock's volatility is 35%.

(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

(iii) The continuously compounded risk-free interest rate is 6%.

(iv) The 9-month payoff of the contingent claim is as follows:



(v) The current gamma of the contingent claim is 0.0314.

Calculate the time-0 contingent-claim elasticity.

Problem 6.4.30. (Option volatility) Assume the Black-Scholes framework. For $t \geq 0$, let $S(t)$ be the time-t price of a nondividend-paying stock. You are given:

(i) $S(0) = 100$.

(ii) $\mathrm{Var}[\ln S(t)] = 0.16t$, for $t \geq 0$.

(iii) The continuously compounded risk-free interest rate is 6%.

Calculate the current volatility of a 105-strike 1-year European call option.

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7

Option Greeks and Risk Management

Chapter overview: On the basis of the Black-Scholes pricing formula and option Greeks studied in the previous chapter, we now examine the risk management aspect of the Black-Scholes framework and migrate from the *measurement* to the *management* of risks inherent in options. The central question we address is: How can we implement a hedging strategy using the underlying stock and related options to reduce, if not eliminate, the stock price risk we are exposed to? This problem, which practitioners confront on a day-to-day basis, is addressed in Section 7.1, where the important technique of delta-hedging is formally introduced. Attention is paid to the motivation behind and mechanics of delta-hedging and the calculation of the holding profit of a delta-hedged portfolio. Section 7.2 extends the discussion in Section 7.1 to hedging multiple Greeks, most notably delta and gamma, and demonstrates how delta-gamma-hedging remedies the deficiencies of delta-hedging. Section 7.3 presents a further application of option Greeks to the approximation of option prices, leading naturally to the development of the celebrated Black-Scholes equation (not formula!). Intriguingly, delta-hedging, besides being a widely used hedging strategy in practice, gives rise to new results concerning the pricing of options in theory.

7.1 Delta-hedging

In this chapter, we expand the notation of delta from Δ to $\Delta(S(t), t)$ to indicate its dependence on the current time t and the current stock price $S(t)$. When only one of the two arguments is of interest, the other argument will be suppressed. The same applies to other option Greeks. At minimal technical cost, such a notational expansion is desirable because it emphasizes the dynamic nature of risk management—things change as life moves on!

Why delta-hedge?

To motivate the concept of delta-hedging, let's put ourselves in the shoes of an option trader who has *sold* a European call option on a nondividend-paying stock and has therefore incurred a delta of $-\Delta_C(0)$, where $\Delta_C(0)$ is the time-0 delta of the call option. Because the trader's delta is negative, he is exposed to upside stock price risk, i.e., the risk that the stock price rises in the future. To protect himself against a decline in the option value should the stock price increase, the trader can buy $\Delta_C(0)$ shares[i] of the stock to *delta-hedge* his position. Because the Greek of a portfolio is given by the sum of the Greeks of its components (see (6.3.5) in Subsection 6.3.3), the delta of the trader's delta-hedged portfolio is

$$\underbrace{-\Delta_C(0)}_{\text{short call}} + \underbrace{\Delta_C(0)}_{\text{long stock}} = 0,$$

[i]We assume that trading fractional units of the stock is allowed in the market.

meaning that the investor is *locally* protected against stock price risk. The trader's portfolio is said to be *delta-neutral* because its delta is neutralized to exactly zero. In general, *delta-hedging* entails entering into a set of offsetting transactions (such as buying $\Delta_C(0)$ shares in the above example) in such a way that your overall delta is zero.

Example 7.1.1. (SOA Exam FETE Fall 2008 Question 10: Delta-hedging for currency options) Your company has just sold a European put option on 10,000 USD for a premium paid in Japanese yen.

You are given the following:

- The time to maturity $T = 180$ days
- Yen risk-free interest rate $r_j = 0.4988\%$ per annum, continuously compounded
- USD risk-free interest rate $r_u = 4.97\%$ per annum, continuously compounded
- The current exchange rate is $Q = 120$ yen/USD
- The strike price of the option is 117 yen/USD
- The volatility of the exchange rate is 10%
- Assume there are 365 days in a year

(a) Calculate the option premium received, based on the Black-Scholes currency option formula.

Your company plans to delta-hedge the short put position using the option premium received.

A hedge portfolio will be purchased such that its value in Yen will approximately equal the value of the put option at each point in time.

To hedge the put option, you buy Y units of USD and invest the balance of the portfolio at Yen money market rate.

(c) Calculate Y using the Black-Scholes currency option formula.

(Note: Parts (b) and (d) are not included here.)

Solution. (a) Note that the foreign currency in this example is USD while the domestic currency is Japanese Yen. With

$$\begin{aligned} d_1 &= \frac{\ln(120/117) + (0.4988\% - 4.97\% + 0.1^2/2)(180/365)}{0.1\sqrt{180/365}} = 0.08165, \\ d_2 &= d_1 - 0.1\sqrt{180/365} = 0.01143, \\ N(-d_1) &= 0.46746, \\ N(-d_2) &= 0.49544, \end{aligned}$$

the price of the currency put is

$$10{,}000[117\mathrm{e}^{-0.4988\%(180/365)}(0.49544) - 120\mathrm{e}^{-4.97\%(180/365)}(0.46746)] = \boxed{30{,}870}.$$

(c) The delta of each currency put option is (Note: Don't omit the negative sign!)

$$\Delta_P = -e^{-4.97\%(180/365)}N(-d_1) = -0.45614.$$

Shorting 10,000 of these put options means that the delta of the company is

$$-10,000(-0.45614) = 4,561,$$

which suggests short selling 4,561 units of USD, i.e., $\boxed{Y = -4,561}$.

□

Calculations of holding profits.

Our next task is to investigate the profit at any future time t before expiration of a trader who has delta-hedged his portfolio. Our analysis holds true not only for a call, but also for a general European derivative V. Recall that we are assuming no dividends are payable over the life of the option.

- *At time 0:* Suppose that the trader sells the derivative for an initial income of $V(0)$ and buys $\Delta(0)$ shares of the stock for the purpose of delta-hedging for a cost of $\Delta(0)S(0)$. The net investment made at time 0 equals

$$\text{Time-0 investment} = \Delta(0)S(0) - V(0).$$

- *At time t:* To close his/her positions, the trader should sell the $\Delta(0)$ shares of the stock he/she is holding for a cash inflow of $\Delta(0)S(t)$ and cover the short position in the derivative by buying it back at the time-t price of $V(t)$. Overall, the time-t payoff is (recall the definition of a payoff in Chapter 1)

$$\text{Time-}t\text{ payoff} = \Delta(0)S(t) - V(t).$$

As the profit is defined simply as the payoff less the *future value* of the previous investments, the holding profit of the trader is given by

$$\boxed{\text{Holding profit} = \underbrace{[\Delta(0)S(t) - V(t)]}_{\text{time-}t\text{ payoff}} - e^{rt}\underbrace{[\Delta(0)S(0) - V(0)]}_{\text{time-0 investment}}.} \tag{7.1.1}$$

When t is 1 day (i.e., $t = 1/365$), the holding profit is referred to as the *overnight profit*—you are computing the profit "overnight."

Note that:

- The same delta of $\Delta(0)$ is used in both the time-t payoff term and the time-0 investment term. Do not use (and there is no need to calculate!) the time-t delta $\Delta(t)$ for the payoff term!

- To calculate the holding profit, you need the option price at time 0 (i.e., $V(0)$), and that at the new time t (i.e., $V(t)$). They require two rounds of Black-Scholes calculations.

Example 7.1.2. (SOA Exam IFM Advanced Derivatives Sample Question 47: Calculation of holding profit) Several months ago, an investor sold 100 units of a one-year European call option on a nondividend-paying stock. She immediately delta-hedged the commitment with shares of the stock, but has not ever re-balanced her portfolio. She now decides to close out all positions.

You are given the following information:

(i) The risk-free interest rate is constant.

(ii)

	Several months ago	Now
Stock price	\$40.00	\$50.00
Call option price	\$8.88	\$14.42
Put option price	\$1.63	\$0.26
Call option delta	0.794	

The put option in the table above is a European option on the same stock and with the same strike price and expiration date as the call option.

Calculate her profit.

(A) \$11

(B) \$24

(C) \$126

(D) \$217

(E) \$240

Solution. Denote the date several months ago by time 0 and the current date by time t. Through delta-hedging, the net investment of the investor at time 0 was

$$100[\Delta(0)S(0) - C(0)] = 100[0.794(40) - 8.88] = 2,288.$$

After closing out all positions at time t, her profit will be

$$\begin{aligned} 100[\Delta(0)S(t) - C(t)] - 2,288\mathrm{e}^{rt} &= 100[0.794(50) - 14.42] - 2,288\mathrm{e}^{rt} \\ &= 2,528 - 2,288\mathrm{e}^{rt}. \end{aligned}$$

To find the accumulation factor e^{rt}, put-call parity can be applied to the two pairs of call and put prices:

$$\begin{cases} C(0) - P(0) = S(0) - K\mathrm{e}^{-rT} \\ C(t) - P(t) = S(t) - K\mathrm{e}^{-r(T-t)} \end{cases} \Rightarrow \begin{cases} 8.88 - 1.63 = 40 - K\mathrm{e}^{-rT} \\ 14.42 - 0.26 = 50 - K\mathrm{e}^{-r(T-t)} \end{cases}$$
$$\Rightarrow \mathrm{e}^{rt} = 1.094351145.$$

It follows that the profit of the investor is $2,528 - 2,288(1.094351145) = \boxed{24.12}$. **(Answer: (B))** ☐

Remark. The problem can still be solved if the short-rate is a deterministic function $r(\cdot)$. Then, the accumulation factor e^{rt} is replaced by $\exp\left[\int_0^t r(s)\,\mathrm{d}s\right]$, which can be determined using the put-call parity formulas

$$\begin{cases} C(0) - P(0) = S(0) - K\exp\left[-\displaystyle\int_0^T r(s)\,\mathrm{d}s\right] \\ C(t) - P(t) = S(t) - K\exp\left[-\displaystyle\int_t^T r(s)\,\mathrm{d}s\right] \end{cases}.$$

If interest rates are stochastic, the problem as stated cannot be solved.

A cautionary note is in order. (7.1.1) and (7.1.2) are developed under the assumption that the trader *sells* the derivative and the stock pays no dividends. If you keep the definitions of payoff and profit in mind, you should not have difficulties in developing the holding profit formula when an investor *buys* a derivative and delta-hedges his/her position in the presence of dividends.

Example 7.1.3. (Calculation of the holding profit for a long put position) Assume the Black-Scholes framework.

One month ago, Tidy *bought* 1,000 units of a 9-month 60-strike European put option on a stock. He immediately delta-hedged the commitment with shares of the stock, but has not ever re-balanced his portfolio. He now decides to close out all positions.

You are given:

(i) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 4%.

(ii) The stock's volatility is 45%.

(iii) The continuously compounded risk-free interest rate is 10%.

(iv) The stock price one month ago was \$60.

(v) The current stock price is \$65.

Calculate Tidy's profit.

Ambrose's comments:

In this example, we deal with a long position in a put. Also, the stock pays dividends, which are (assumed to be) reinvested, leading to a change in the number of shares in the future.

Solution. We need the put price one month ago and the current put price.

One month ago, we have

$$\begin{aligned} d_1 &= \frac{\ln(60/60) + (0.1 - 0.04 + 0.45^2/2)(0.75)}{0.45\sqrt{0.75}} = 0.31033, \\ d_2 &= d_1 - 0.45\sqrt{0.75} = -0.07939, \\ N(-d_1) &= 0.37816, \\ N(-d_2) &= 0.53164, \end{aligned}$$

so the put price was

$$P(0) = 60\mathrm{e}^{-0.1(0.75)}(0.53164) - 60\mathrm{e}^{-0.04(0.75)}(0.37816) = 7.57451.$$

The put delta was

$$\Delta_P(0) = \underbrace{-\mathrm{e}^{-0.04(0.75)}}_{\text{Don't miss this!}}(0.37816) = -0.36698.$$

To delta-hedge his position, the Tidy should *buy* $100(0.36698) = 36.698$ shares to hedge his *long* position in the puts.

Currently, the put has 8 months to live before it expires. With

$$\begin{aligned} d_1 &= \frac{\ln(65/60) + (0.1 - 0.04 + 0.45^2/2)(2/3)}{0.45\sqrt{2/3}} = 0.51043, \\ d_2 &= d_1 - 0.45\sqrt{2/3} = 0.14300, \\ N(-d_1) &= 0.30488, \\ N(-d_2) &= 0.44315, \end{aligned}$$

the current put price is

$$P(1/12) = 60\mathrm{e}^{-0.1(2/3)}(0.44315) - 65\mathrm{e}^{-0.04(2/3)}(0.30488) = 5.57847.$$

Finally, Tidy's profit is given by

$$\begin{aligned} \text{Holding profit} &= \text{Payoff} - \text{FV(Investment)} \\ &= 1,000\{[-\Delta_P(0)\boxed{\mathrm{e}^{\delta t}}S(1/12) + P(1/12)] \\ &\qquad - [-\Delta_P(0)S(0) + P(0)]\mathrm{e}^{rt}\} \\ &= 1,000\{[0.36698\mathrm{e}^{0.04/12}(65) + 5.57847] \\ &\qquad - [0.36698(60) + 7.57451]\mathrm{e}^{0.1/12}\} \\ &= \boxed{-329}. \end{aligned}$$

□

Remark. (i) Note that Tidy bought the put option as well as the stock for delta-hedging, incurring a rather high initial investment.

(ii) Because of the reinvestment of dividends, 36.698 shares bought one month ago grow to $36.698\mathrm{e}^{0.04/12}$ shares currently. .

In practice, a marker-maker engaged in delta-hedging needs to rebalance his/her portfolio periodically in response to changes in the value of delta as time elapses and the price of the underlying stock changes. Such a multi-period delta-hedging strategy can be decomposed into a series of one-period delta-hedging strategies, each of which can be analyzed by techniques introduced thus far and evaluated by (7.1.1). Here is an illustrative example.

Example 7.1.4. (CAS Exam 8 Spring 2005 Question 36: Two-period delta-hedging) Assume you have purchased European put options for 100,000 shares of a nondividend-paying stock and you are given the following information.

- Price of stock = \$49.16
- Strike price = \$50.00
- Continuously compounded risk-free interest rate = 5% per annum
- Volatility = 20% per annum
- There are 20 weeks remaining until maturity.

(a) Determine the initial position you should take in the underlying stock to implement a delta hedging strategy.

(b) You now have the following information.

T (weeks)	Stock Price	d_1
1	\$49.33	0.10
2	\$49.09	0.05

You decide to readjust the delta hedging strategy on a weekly basis.

Calculate the cumulative cost, including interest, of the hedge at the end of week 2.

Solution. (a) Since

$$\begin{aligned} d_1 &= \frac{\ln(49.16/50) + (0.05 + 0.2^2/2)(20/52)}{0.2\sqrt{20/52}} = 0.08046, \\ N(-d_1) &= 0.46794, \end{aligned}$$

the delta of each put option is $\Delta_P = -N(-d_1) = -0.46794$. In total, we need to $\boxed{\text{buy}}$ $100,000(0.46794) = \boxed{46,794}$ shares of the stock to implement a delta-hedging strategy.

(b)
- Initially, the delta-hedging strategy costs $46,794(49.16) = 2,300,393.04$.
- At the end of week 1, the delta becomes $\Delta_P = -N(-d_1) = -0.46017$, so to remain delta-neutral, we need to hold $100,000(0.46017) = 46,017$ shares in the portfolio. Thus we sell $46,794 - 46,017 = 777$ shares for an income of $777(49.33) = 38,329.41$.
- At the end of week 2, the delta becomes $\Delta_P = -N(-d_1) = -0.48006$, so we need to hold $100,000(0.48006) = 48,006$ shares in the portfolio. Thus we buy $48,006 - 46,017 = 1,989$ shares for a cost of $1989(49.09) = 97,640.01$.

After taking time value of money into account, the cumulative cost of the hedge at the end of week 2 is

$$2,300,393.04e^{0.05(2/52)} - 38,329.41e^{0.05(1/52)} + 97,640.01 = \boxed{2,364,095}.$$

□

An alternative view on the holding profit formula.

The formula of the delta-hedged trader's holding profit as given in (7.1.1) can be recast in an alternative way to cast light on the factors contributing to the holding profit. This can be achieved by rearranging (7.1.1) as (assuming no dividends)

$$\begin{aligned}\text{Profit} &= [\Delta(0)S(t) - V(t)] - \mathrm{e}^{rt}[\Delta(0)S(0) - V(0)] \\ &= [\Delta(0)S(t) - V(t)] - [\Delta(0)S(0) - V(0)] - (\mathrm{e}^{rt} - 1)[\Delta(0)S(0) - V(0)] \\ &= \underbrace{\Delta(0)[S(t) - S(0)]}_{(1)} - \underbrace{[V(t) - V(0)]}_{(2)} - \underbrace{(\mathrm{e}^{rt} - 1)[\Delta(0)S(0) - V(0)]}_{(3)}. \end{aligned} \tag{7.1.2}$$

This formula decomposes the profit into three components:

(1) Gain on the shares (*not* taking time value of money into account)

(2) Gain on the derivative (again, *not* taking time value of money into account; also, recall that the trader is short with respect to the derivative, hence the negative sign before (2))

(3) Interest expense on the initial investment

The difference (1) – (2) is sometimes called the *capital gain*.

Which holding profit formula to use, (7.1.1) or (7.1.2)?

As a matter of fact, (7.1.2) is mainly of theoretical significance, and you will mostly use (7.1.1) for practical computations for two reasons:

- The derivations of (7.1.2) are unnecessarily complicated and rather difficult to follow, whereas the ideas underlying (7.1.1) are not only substantially more transparent, but also consistent with the usual definitions of payoff and profit. When asked to calculate the holding profit in a specific problem, you need not dwell on (7.1.2). Rather, simply apply (7.1.1), which is much more convenient, taking care of dividends if necessary.
- Apart from its relative simplicity, an additional merit of (7.1.1) compared to (7.1.2) is its wider applicability. Not only does the rationale underlying (7.1.2) hold true when the trader employs no hedging (see Example 7.1.5 below), it can also be applied to more complex hedging strategies such as delta-gamma-hedging, while the analogs of (7.1.2) in these cases are not immediately clear.

Example 7.1.5. (SOA Exam MFE Spring 2009 Question 13: Holding profit of a plain long call position) Assume the Black-Scholes framework.

Eight months ago, an investor borrowed money at the risk-free interest rate to purchase a one-year 75-strike European call option on a nondividend-paying stock. At that time, the price of the call option was 8.

Today, the stock price is 85. The investor decides to close out all positions.

You are given:

(i) The continuously compounded risk-free interest rate is 5%.

(ii) The stock's volatility is 26%.

Calculate the eight-month holding profit.

(A) 4.06

(B) 4.20

(C) 4.27

(D) 4.33

(E) 4.47

Solution. Eight months after the purchase of the call, the remaining time to expiration is 4 months. With

$$\begin{aligned} d_1 &= \frac{\ln(85/75) + (0.05 + 0.26^2/2)(4/12)}{0.26\sqrt{4/12}} = 1.01989, \\ d_2 &= d_1 - 0.26\sqrt{4/12} = 0.86978, \\ N(d_1) &= 0.84611, \\ N(d_2) &= 0.80779, \end{aligned}$$

the current call price is

$$C = 85(0.84611) - 75\mathrm{e}^{-0.05(4/12)}(0.80779) = 12.33647.$$

The 8-month holding profit is today's call price (payoff) less the future value of the old call price (initial investment), or $12.33647 - 8\mathrm{e}^{0.05(8/12)} = \boxed{4.0653}$. **(Answer: (A))** □

Remark. Because we are not doing delta-hedging in this example, there is no need to know the stock price eight months ago.

Example 7.1.6. (CAS Exam 3 Spring 2007 Question 32: Given the change in option price) A market-maker has sold 100 call options, each covering 100 shares of a dividend-paying stock, and has delta-hedged by purchasing the underlying stock.

You are given the following information about the market-maker's investment:

- The current stock price is \$40.
- The continuously compounded risk-free rate is 9%.
- The continuous dividend yield of the stock is 7%.
- The time to expiration of the options is 12 months.
- $N(d_1) = 0.5793$
- $N(d_2) = 0.5000$

The price of the stock quickly jumps to \$41 before the market-maker can react. This change causes the price of one call option to increase by \$56.08. Calculate the net profit on the market-maker's investment associated with this price move.

(A) Less than −\$1,600

(B) At least −\$1,600, but less than −\$800

(C) At least −\$800, but less than \$0

(D) At least \$0, but less than \$800

(E) At least \$800

Solution. Because $\Delta(0) = \mathrm{e}^{-\delta T}N(d_1) = \mathrm{e}^{-0.07(1)}(0.5793) = 0.540136$ for each call, $0.540136(100)(100) = 5,401.36$ shares of the stock are bought at time 0. If we ignore the interest expense because everything happens instantaneously, then the overnight profit, by (7.1.1) (or (7.1.2)), is

$$5,401.36(41-40) - 56.08(100) = \boxed{-206.64}. \quad \textbf{(Answer: (C))}$$

□

Profit of an unhedged position vs profit of a delta-hedged position.

We close this section by turning the algebraic results we have developed thus far for delta-hedging into a diagram that visualizes the financial effects of delta-hedging. Specifically, we contrast the profit of an unhedged written option with the profit of a delta-hedged written option for a range of values of stock prices in an attempt to show not only the superiority of delta-hedging, but also its potential deficiencies. For concreteness, we consider a written European call option with the following parameters:

$$S(0) = K = 50, \quad r = 0.08, \quad \delta = 0, \quad \sigma = 0.25, \quad T = 1.$$

The time-0 delta of the call is $\Delta_C(50, 0) = 0.67184$, which means that a seller of the call should buy 0.67184 shares of the stock for the purpose of delta-hedging.

In Figure 7.1.1, we assume that the stock price changes from its initial value of 50 to any value between \$0 to \$100 in one day, say $S(^1/_{365})$, and plot the overnight profit of the unhedged written call, equal to the future value of initial call price less the new call price evaluated at $S(^1/_{365})$, as well as the overnight profit of the delta-hedged written call (i.e., the written call coupled with 0.67184 shares of the stock) computed according to (7.1.1) with $t = 1/365$ (i.e., 1 day). That the change of the stock price from $S(0) = 50$ to $S(^1/_{365})$ occurs in one day allows us to concentrate mostly on the effect of stock price changes alone on the holding profit of the two positions. It can be seen that as the call becomes more in-the-money, the profit of the (unhedged) written call, which appears as a smoothed version of the short call payoff, decreases inexorably, seriously jeopardizing the financial well-being of the call seller. This corroborates our concern that the call seller is exposed to significant upside stock price risk and justifies the need for delta-hedging. In contrast, the profit of the delta-hedged written call, albeit still exhibiting a decreasing trend as the stock price rises above the initial price of 50, declines at a much more controllable rate. Meanwhile, when the call becomes more out-of-the-money, the profit of the delta-hedged written call decreases quite sharply, due to the shrinking value of the stock holding, and is much below the profit of the unhedged written call. This is another manifestation of the phenomenon we first observed in Subsection 3.1.4 that no position can always outperform another in terms of profit, in the absence of arbitrage opportunities. Parenthetically, although it is not clearly evident in Figure 7.1.1, the profit of the delta-hedged written call is positive when the 1-day stock price is close to 50.

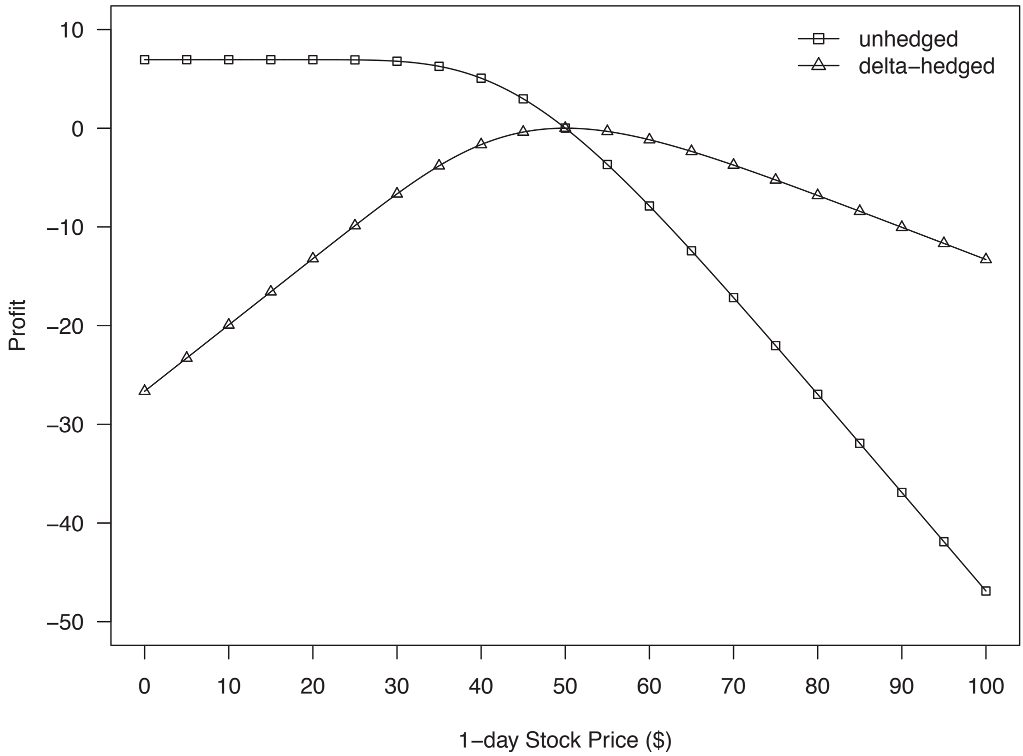

FIGURE 7.1.1
Comparison of the overnight profit of an unhedged written call and the overnight profit of a delta-hedged written call.

You may wonder:

> Shouldn't the delta-hedged written call, with a zero delta, be shielded from the stock price risk? Why does the profit of the delta-hedged written call still vary quite vigorously with the current stock price?

The sad truth is that the delta-hedged written call is only *locally* protected against stock price movements—the tangent to the profit function of the delta-hedged written call at $S(1/365) = 50$ is almost[ii] horizontal. As the stock price changes, however, so does delta itself (recall from Subsection 6.3.1 that the delta of a call increases with the current stock price, or equivalently, the gamma of a call is positive). When $S(1/365) > 50$ (resp. $S(1/365) \leq 50$), having $\Delta_C(50, 0) = 0.67184$ shares at hand is no longer sufficient (resp. more than sufficient) to offset the new delta of the call. In fact, the overall delta of the delta-hedged written is, ignoring the passage of one day,

$$-\Delta_C(S(1/365), 1/365) + \Delta_C(S(0), 0) \begin{cases} < 0, & \text{if } S(1/365) < S(0), \\ = 0, & \text{if } S(1/365) = S(0), \\ > 0, & \text{if } S(1/365) > S(0), \end{cases}$$

[ii] The tangent is not exactly horizontal because one day has passed, i.e., $\Delta_C(50, 1/365) \neq \Delta_C(50, 0)$.

which explains the downward-opening shape of the profit of the delta-hedged written call in Figure 7.1.1.

The above discussions suggest that a delta-hedged written call suffers from huge losses owing to large stock price moves in either direction. These undesirable characteristics of the delta-hedged written call motivate us to bring in additional option Greeks to our portfolio in an effort to hedge against the stock price risk more effectively. This will be the subject of the next section.

7.2 Hedging Multiple Greeks

Why hedge multiple Greeks?

We have illustrated in Figure 7.1.1 that a delta-hedged portfolio does not eliminate risk and can sustain large losses in the case of large stock price moves. In fact, the susceptibility to changes in the stock price is expected from the negativity of the *gamma* of the delta-hedged written call. Its delta is decreasing in $S(1/365)$, being strictly positive when $S(1/365) < 50$, (almost) zero at $S(1/365) = 50$, and strictly negative when $S(1/365) > 50$. This suggests neutralizing not only the delta of a portfolio, but also its gamma, to eliminate any systematic behavior in the portfolio delta and to further control the stock price risk. Because the gamma of the underlying stock is always zero ($\partial^2 S/\partial S^2 = 0$), a *delta-gamma-neutral* portfolio cannot be constructed with the stock alone. Other financial instruments such as options need to be traded to offset the gamma of the prior position.

Furthermore, delta and gamma are measures of the sensitivity of the option price to only one source of uncertainty, namely the stock price. A delta-gamma-hedged position can remain vulnerable to many external factors, including but not limited to stock price volatility, changes in interest rates, the passage of time, etc. To formulate a prudent risk management system, it therefore makes sense to hedge several option Greeks that are of most practical concern simultaneously.

Hedging multiple Greeks: Solving a system of linear equations.

Consider a given position which is to be hedged with respect to m option Greeks. Hedging these m Greeks in general requires m additional financial instruments, the number of units to buy or sell in each of which is to be determined. Since the Greek of a portfolio is simply the sum of the individual Greeks (recall (6.3.7) on page 217), zeroing the m Greeks boils down to solving a system of m simultaneous linear equations, one for each Greek, for the compositions in the m instruments. In a conceivable pen-and-paper problem, you are most likely asked to neutralize $m = 2$ option Greeks via solving two linear equations.

Example 7.2.1. (SOA Exam MFE Spring 2007 Question 10: Delta-gamma-hedging I) For two European call options, Call-I and Call-II, on a stock, you are given:

Greek	Call-I	Call-II
Delta	0.5825	0.7773
Gamma	0.0651	0.0746
Vega	0.0781	0.0596

Suppose you just sold 1000 units of Call-I.

Determine the numbers of units of Call-II and stock you should buy or sell in order to both delta-hedge and gamma-hedge your position in Call-I.

(A) Buy 95.8 units of stock and sell 872.7 units of Call-II

(B) Sell 95.8 units of stock and buy 872.7 units of Call-II

(C) Buy 793.1 units of stock and sell 692.2 units of Call-II

(D) Sell 793.1 units of stock and buy 692.2 units of Call-II

(E) Sell 11.2 units of stock and buy 763.9 units of Call-II

Ambrose's comments:

Be sure to adopt a consistent sign convention. If you buy (resp. sell) an asset, the contribution of that asset to the overall portfolio Greek will be positive (resp. negative) of the constituent option Greek.

Solution. Let x and y be, respectively, the number of units of Call-II and stock to be *bought* (if x or y turns out to be negative, this means that you *sell*). To maintain delta-neutrality and gamma-neutrality, we solve

$$\begin{cases} -1000(0.0651) + 0.0746x = 0 & \text{(gamma-neutrality)} \\ -1000(0.5825) + 0.7773x + y = 0 & \text{(delta-neutrality)} \end{cases}.$$

This gives $x = \boxed{872.6542}$ and $y = \boxed{-95.8141}$. In other words, 872.7 units of Call-II should be bought and 95.8 units of the stock should be sold. **(Answer: (B))** □

Remark. (i) It is enough to calculate the value of x to conclude that (B) is the correct answer.

(ii) In this example, the values of x and y are determined in turn. We first choose to buy $x = 872.7$ units of Call-II solely to neutralize the gamma of the 1000 units of the short Call-I, then buy $y = -95.8141$ units of the stock to offset the delta of the resulting portfolio.

(iii) The row of "vega" is redundant in this example.

Example 7.2.2. (CAS Exam 3 Fall 2007 Question 24: Delta-gamma-hedging II) An investor has a portfolio consisting of 100 put options on stock A, with a strike price of 40, and 5 shares of stock A. The investor can write put options on stock A with a strike price of 35. The deltas and gammas of the options are listed below:

	Put (Strike = 35)	Put (Strike = 40)
Delta	−0.10	−0.05
Gamma	0.50	0.25

Which one of the following actions would delta and gamma neutralize this portfolio?

(A) Write 100 put options with a strike price of 35.

(B) Write 50 put options with a strike price of 35.

(C) Write 100 put options with a strike price of 35, and buy 5 shares of stock.

(D) Write 100 put options with a strike price of 35, and sell 5 shares of stock.

(E) Write 50 put options with a strike price of 35, and sell 5 shares of stock.

Solution. To gamma-hedge the portfolio, we need to *sell* $100(0.25)/0.50 = 50$ put options. The overall delta is

$$100\Delta_P^{\text{40-strike}} + 5 - 50\Delta_P^{\text{35-strike}} = 100(-0.05) + 5 - 50(-0.10) = 5,$$

so to delta-hedge the resulting portfolio, we also need to sell 5 shares of stock. **(Answer: (E))** □

Remark. The numbers in this example seem pathological. It appears that the 35-strike put is equivalent to two units of the 40-strike put. Thus for delta-gamma-hedging, the 100 units of the 40-strike put, it suffices to sell $100/2 = 50$ units of the 35-strike put without recourse to the stock.

Delta-gamma-hedging vs delta-hedging.

Figure 7.2.1 depicts the overnight profits of the delta-hedged written call and the delta-gamma-hedged written call using the same set of parameters as Figure 7.1.1. The delta-gamma-hedged written call is set up by coupling the written call with 0.94308 units of an otherwise identical 60-strike call and 0.30583 shares of the stock (check!). Whereas stock price risk, be it upside or downside, is not completely eliminated even with delta-gamma-hedging, the fluctuation of the overnight profit with the 1-day stock price is much milder than that of the delta-hedged written call. This suggests that delta-gamma-hedging is a much more prudent risk management strategy than delta-hedging alone. In fact, the delta-gamma-hedged written call presents a decent amount of profit if the stock price rises. (Question: How to compute the overnight profit of the delta-gamma-hedged written call? See Problem 7.4.9)

7.3 Delta-Gamma-Theta Approximation

One of the usefulness of option Greeks is that they indicate the sensitivity of the option price to different risk factors, especially the prevailing stock price and time to maturity, both of which inevitably change over the life of the option. Such sensitivity, once gauged, can be reduced by appropriate hedging strategies, as the previous two sections show. Another application of option Greeks is to approximate option prices computed at new stock prices and at new points of time by virtue of a Taylor series approximation. Interestingly, this simple approximation tool emanating from multi-variable calculus, when applied to delta-hedging, a strategy driven by practical considerations, has surprisingly far-reaching implications for the theory of derivative pricing in general.

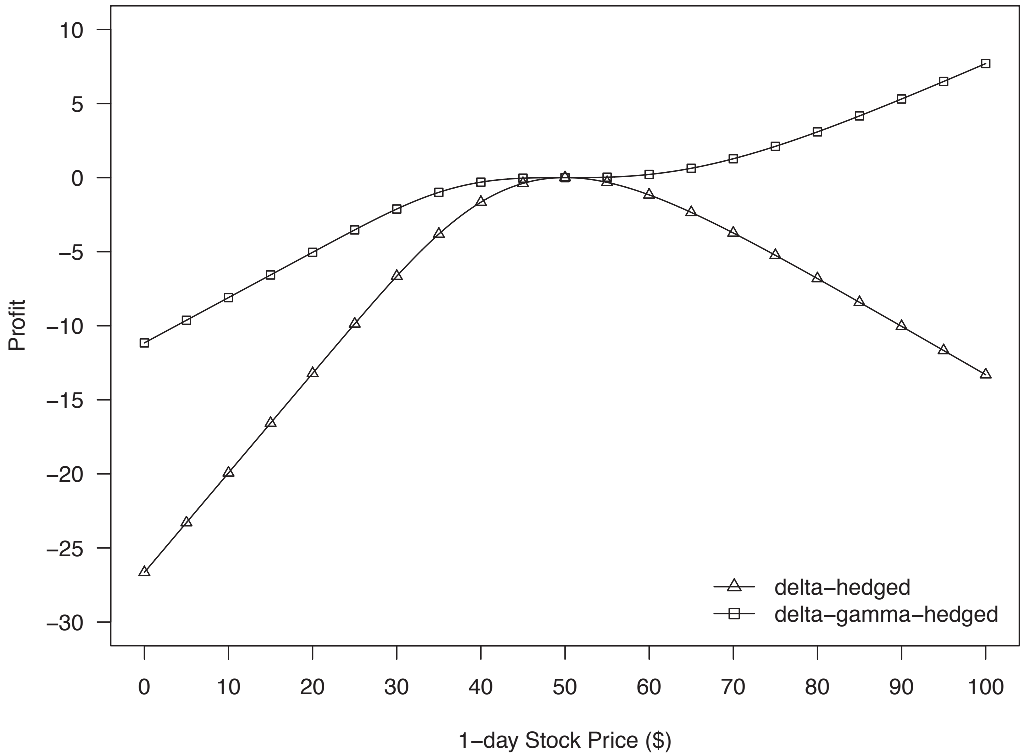

FIGURE 7.2.1
Comparison of the overnight profit of a delta-hedged written call and the overnight profit of a delta-gamma-hedged written call.

Taylor series expansion.

For $t \geq 0$, let $V(S(t), t)$ be the time-t price of a generic option corresponding to the time-t stock price of $S(t)$. Suppose that the time-0 derivative price, $V(S(0), 0)$, and the associated option Greeks are known, and we are interested in the time-h derivative price for some positive h. One solution is to repeat the Black-Scholes valuation procedure at time h and at the new realized stock price $S(h)$. This can be cumbersome (especially in a pen-and-paper exam environment!). A somewhat crude but easier alternative is to approximate the new option price by a Taylor series expansion of the bivariate function $V(s, t)$ at the time-0 arguments $(S(0), 0)$ and capitalizing on the time-0 information about the option price and option Greeks, yielding

$$\boxed{\underbrace{V(S(h), h)}_{\text{new arguments}} \approx \underbrace{V(S(0), 0)}_{\text{old arguments}} + \underbrace{\Delta(S(0), 0)}_{\text{old arguments}} \epsilon + \frac{1}{2} \underbrace{\Gamma(S(0), 0)}_{\text{old arguments}} \epsilon^2 + \underbrace{\theta(S(0), 0)}_{\text{old arguments}} h,} \tag{7.3.1}$$

where $\epsilon = S(h) - S(0)$ is the change in the stock price, and the option price and option Greeks on the right-hand side are all evaluated at the original stock price $S(0)$ and time 0. With the use of the "approximation trinity," namely delta, gamma, and theta of the option, (7.3.1), is naturally termed the *delta-gamma-theta approximation* of the new option price (at the new stock price $S(h)$ and new time h). It is a Taylor series approximation of second order (because of the use of the first and second partial derivatives) in the variable s and first order in the variable t. Note that the units of h and θ should always be consistent: If θ is expressed in days, then so should h be. The two truncated versions,

$$V(S(h), h) \approx V(S(0), 0) + \Delta(S(0), 0)\epsilon, \tag{7.3.2}$$

and

$$\boxed{V(S(h), h) \approx V(S(0), 0) + \Delta(S(0), 0)\epsilon + \frac{1}{2}\Gamma(S(0), 0)\epsilon^2,} \tag{7.3.3}$$

are referred to as the *delta approximation* and *delta-gamma approximation*, respectively.

Example 7.3.1. (SOA Exam MFE Spring 2007 Question 19: Delta-gamma approximation) Assume that the Black-Scholes framework holds. The price of a nondividend-paying stock is \$30.00. The price of a put option on this stock is \$4.00.

You are given:

(i) $\Delta = -0.28$

(ii) $\Gamma = 0.10$

Using the delta-gamma approximation, determine the price of the put option if the stock price changes to \$31.50.

(A) \$3.40

(B) \$3.50

(C) \$3.60

(D) \$3.70

(E) \$3.80

Solution. Using the delta-gamma approximation with $\epsilon = 31.50 - 30 = 1.50$, we estimate the price of the put option if the stock price changes to \$31.50 as

$$\begin{aligned} P(31.5) &= P(30) + \Delta(30)\epsilon + \frac{1}{2}\Gamma(30)\epsilon^2 \\ &= 4.00 + (-0.28)(1.50) + \frac{1}{2}(0.10)(1.50)^2 \\ &= \boxed{3.6925}. \qquad \textbf{(Answer: (D))} \end{aligned}$$

□

Example 7.3.2. (SOA Exam MFE Spring 2009 Question 20: Given the estimated price, deduce $S(0)$) Assume that the Black-Scholes framework holds. Consider an option on a stock.

You are given the following information at time 0:

(i) The stock price is $S(0)$, which is greater than 80.

(ii) The option price is 2.34.

(iii) The option delta is -0.181.

(iv) The option gamma is 0.035.

The stock price changes to 86.00. Using the delta-gamma approximation, you find that the option price changes to 2.21.

Determine $S(0)$.

(A) 84.80

(B) 85.00

(C) 85.20

(D) 85.40

(E) 85.80

Ambrose's comments:

Unlike the previous example, this time you are directly given the delta-gamma approximation, which is quadratic in $\epsilon = S^{\text{new}} - S^{\text{old}}$, and asked to back out ϵ via solving a quadratic equation.

Solution. The delta-gamma approximation says that

$$2.21 = 2.34 + (-0.181)\epsilon + \frac{1}{2}(0.035)\epsilon^2 \text{ or } 0.0175\epsilon^2 - 0.181\epsilon + 0.13 = 0.$$

The solutions to this quadratic equation are

$$\epsilon = \frac{0.181 \pm \sqrt{(-0.181)^2 - 4(0.0175)(0.13)}}{2(0.0175)} = 9.5663 \text{ or } 0.7765.$$

To see which one(s) is/are acceptable values of ϵ, recall that $\epsilon = S^{\text{new}} - S^{\text{old}} = 86 - S^{\text{old}}$, which is less than 6, because $S^{\text{old}} > 80$. Thus we can only take $\epsilon = 0.7765$, which in turn implies that $S^{\text{old}} = 86 - \epsilon = \boxed{85.2235}$. **(Answer: (C))** □

Example 7.3.3. (SOA Exam FETE Fall 2010 Question 16 (c)–(d)) You hold a trading book consisting of many long/short positions in Tempranillo Corp. stock and options. You are analyzing two possible scenarios for Tempranillo Corp. stock and want to make money by adjusting your trading book.

Scenario	Market Conditions
1	Swift downward price movement; rising implied volatility
2	No price movement; falling implied volatility

(c) For each of the above scenarios, determine whether a positive or negative delta, gamma, or vega in your trading book would produce a profit.

Now consider the table below which summarizes your positions at the prior day's close of trading:

Option	Position (# of units)	\$delta/\$	\$gamma/\$	\$theta/day	\$vega/point
Call totals	−600	−736	−19.6	+0.5	−0.8
Put totals	400	−618	+2.2	+0.2	−0.1
Stock	1,200	1,200	0.0	0.0	0.0

where \$delta/\$ is the change in the \$ value of your portfolio per \$1 change in the underlying asset or index.

Today Tempranillo Corp.'s stock price dropped \$4 and implied volatility on all options rose 2 percentage points.

(d) Estimate the value change in your trading book at closing today.

Solution. (c) The signs of delta, gamma and vega leading to a profit are tabulated below:

Scenario	Delta	Gamma	Vega
1	Negative	Positive	Positive
2	Does not matter	Does not matter	Negative

(d) The portfolio Greeks are given by:

Option	Position	\$delta/\$	\$gamma/\$	\$theta/day	\$vega/point
Call totals	−600	−736	−19.6	+0.5	−0.8
Put totals	400	−618	+2.2	+0.2	−0.1
Stock	1,200	1,200	0.0	0.0	0.0
Total	?	−154	−17.4	+0.7	−0.9

Hence the estimated change in the value of the trading book at closing today is

$$-154(-4) + \frac{1}{2}(-17.4)(-4)^2 + 0.7(1) + (-0.9)(2) = \boxed{475.7}.$$

□

Geometric meaning of delta- and delta-gamma approximations.

Geometrically, the delta approximation is represented by a straight line which is tangent to the option price function at the current stock price, as illustrated in Figure 7.3.1 in the case of a call with the same parameters as page 240. Because the delta of a call increases with the current stock price, the delta approximation as represented by the tangent will understate the increase in the call price when the stock price rises and overstate the decrease in the call price when the stock price declines. In other words, the tangent will be always lying below the call price function. The delta-gamma approximation improves the accuracy of the delta approximation by approximating the call price function by a quadratic function that shares the same first and second derivatives at the current stock price. Incorporating the gamma correction term $\Gamma(S(0), 0)\epsilon^2/2$, which is always positive regardless of the direction of the stock price move, the delta-gamma approximation will always be larger in value than the delta approximation. As shown in Figure 7.3.1, the delta-gamma approximation dramatically enhances the precision of the delta approximation and is much closer to the true call price function.

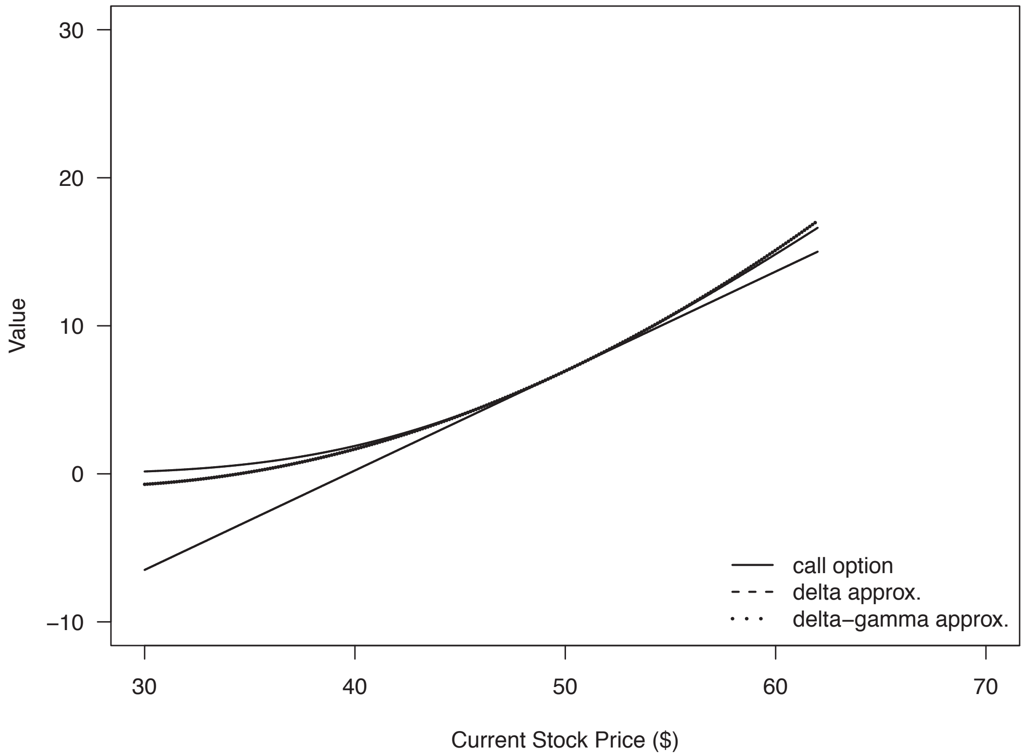

FIGURE 7.3.1
Geometric meaning of the delta- and delta-gamma approximations of the option price.

Application of delta-gamma-theta approximation: The Black-Scholes equation.

Warning!

The material in what follows, which may be skipped on a first reading without losing continuity, is of a more advanced nature and is only given a semi-formal treatment. Actuarial students taking Exam IFM should know that a "one standard deviation move" (see (7.3.5) below) in the stock price causes the overnight profit of a delta-hedged position to be (approximately) zero. Then study Example 7.3.4. The Black-Scholes equation, (7.3.7), despite being a fundamental result in the theory of option pricing, is not required for Exam IFM.

A cursory glance at (7.1.2) shows that three option Greeks play a role in determining the holding profit of a delta-hedged market-maker who sells a European derivative on a nondividend-paying stock. They are delta and gamma (associated with stock price moves), and theta (quantifying the passage of time) of the concerned derivative. Via the use of the delta-gamma-theta approximation to derivative prices, it will be shown that these three seemingly unrelated Greeks are governed by a partial differential equation.

To begin with, we consider a generic time t before the expiration of a T-year derivative and specialize (7.1.2) to the two time points, t and $t+\mathrm{d}t$ for some infinitesimally small $\mathrm{d}t$. The holding profit formula then reads

$$\text{Profit} = \Delta(t)[S(t+\mathrm{d}t) - S(t)] - [V(t+\mathrm{d}t) - V(t)] - (\mathrm{e}^{r\,\mathrm{d}t} - 1)[\Delta(t)S(t) - V(t)]. \quad (7.3.4)$$

When $\mathrm{d}t$ is small enough, the following approximations work reasonably well:

- *Interest:* $\mathrm{e}^{r\,\mathrm{d}t} - 1 \approx (1 + r\,\mathrm{d}t) - 1 = r\,\mathrm{d}t$.
- *Delta-gamma-theta approximation:* By (7.3.1),

$$V(t+\mathrm{d}t) - V(t) \approx \Delta(t)[S(t+\mathrm{d}t) - S(t)] + \frac{1}{2}\Gamma(t)[S(t+\mathrm{d}t) - S(t)]^2 + \theta(t)\,\mathrm{d}t.$$

- *One-standard-deviation move:*[iii]

$$[S(t+\mathrm{d}t) - S(t)]^2 \approx \boxed{[\sigma S(t)]^2\,\mathrm{d}t}. \quad (7.3.5)$$

Inserting these approximations into (7.3.4), we have

$$\text{Profit} \approx -\left\{\frac{1}{2}[\sigma S(t)]^2\Gamma(t) + \theta(t) + r[\Delta(t)S(t) - V(t)]\right\}\mathrm{d}t. \quad (7.3.6)$$

When we stand at time t, the time-t stock price, $S(t)$, is known. Thus (7.3.6) is non-random. If there are no arbitrage opportunities, then the quantity within the braces in (7.3.6) must be zero—or else we must either always earn (positive) profits or always lose. Therefore,

$$\boxed{rS(t)\Delta(t) + \frac{1}{2}[\sigma S(t)]^2\Gamma(t) + \theta(t) = rV(t), \quad 0 \le t \le T.} \quad (7.3.7)$$

This equation is referred to as the celebrated *Black-Scholes partial differential equation*, or the Black-Scholes *equation* (not to be confused with the "formula!") in short, which must be satisfied by the time-t price of a derivative, $V = V(s,t)$ under the no-arbitrage assumption. Different derivatives are distinguished by different boundary and terminal conditions that the function $V = V(s,t)$ should satisfy as $s \downarrow 0$, $t \downarrow 0$ and $t \uparrow T$. In the case of a K-strike T-year European call option on the stock, the terminal and boundary conditions are

$$V(0,t) = 0 \text{ for all } 0 \le t \le T, \quad V(s,T) = (s-K)_+.$$

Note that α, the continuously compounded expected rate of return on the stock, does not enter (7.3.7), just as it has no role to play in the Black-Scholes pricing formula.

[iii] To understand the term "one standard deviation move," observe that

$$\mathrm{Var}[\ln S(t+\mathrm{d}t)|S(t)] = \sigma^2\mathrm{d}t.$$

Applying the delta method in statistics to the exponential function $f(x) = \exp(x)$, we have

$$\mathrm{Var}[S(t+\mathrm{d}t)|S(t)] = \mathrm{Var}\{\exp[\ln S(t+\mathrm{d}t)]|S(t)\} = \{\exp[\ln S(t)]\}^2 \times \sigma^2\,\mathrm{d}t = [\sigma S(t)]^2\,\mathrm{d}t.$$

Students with a background in stochastic calculus may also understand the one-standard-deviation move as

$$[S(t+\mathrm{d}t) - S(t)]^2 \approx [\mathrm{d}S(t)]^2 = [\alpha S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}Z(t)]^2 = [\sigma S(t)]^2\,\mathrm{d}t$$

using the multiplication rules $\mathrm{d}t\mathrm{d}t = \mathrm{d}t\mathrm{d}Z(t) = \mathrm{d}Z(t)\mathrm{d}t = 0$ and $[\mathrm{d}Z(t)]^2 = \mathrm{d}t$ for Brownian motion.

Example 7.3.4. (SOA Exam IFM Advanced Derivatives Sample Question 9: Zero-profit stock price) Consider the Black-Scholes framework. A market-maker, who delta-hedges, sells a three-month at-the-money European call option on a nondividend-paying stock.

You are given:

(i) The continuously compounded risk-free interest rate is 10%.

(ii) The current stock price is 50.

(iii) The current call option delta is 0.61791.

(iv) There are 365 days in the year.

If, after one day, the market-maker has zero profit or loss, determine the stock price move over the day.

(A) 0.41

(B) 0.52

(C) 0.63

(D) 0.75

(E) 1.11

Solution. By (7.3.5), the approximate change in the stock price move is given by $\pm\sigma S(0)\sqrt{h}$, where $h = 1/365$ (one day) and $S(0) = 50$. To deduce the value of σ, we use (iii) and get $\Delta = N(d_1) = 0.61791$, or $d_1 = N^{-1}(0.61791) = 0.3$. As

$$d_1 = \frac{\ln(50/50) + (0.1 + \sigma^2/2)(0.25)}{\sigma\sqrt{0.25}} = 0.3,$$

or

$$0.5\sigma^2 - 0.6\sigma + 0.1 = 0.5(\sigma - 1)(\sigma - 0.2) = 0.$$

The two roots are $\sigma = 1$ and $\sigma = 0.2$. We reject $\sigma = 1$ because such a volatility seems too large (and none of the five answers fit). Hence, the answer is $0.2(50)\sqrt{1/365} = \boxed{0.5234}$. **(Answer: (B))** □

Example 7.3.5. (SOA Exam MFE Sample Question 36: Given the form of $V(s,t)$) Assume the Black-Scholes framework. Consider a derivative security of a stock.

You are given:

(i) The continuously compounded risk-free interest rate is 0.04.

(ii) The volatility of the stock is σ.

(iii) The stock does not pay dividends.

(iv) The derivative security also does not pay dividends.

(v) $S(t)$ denotes the time-t price of the stock.

(vi) The time-t price of the derivative security is $[S(t)]^{-k/\sigma^2}$, where k is a positive constant.

Find k.

(A) 0.04

(B) 0.05

(C) 0.06

(D) 0.07

(E) 0.08

Solution. For notational convenience, let $a = -k/\sigma^2$ and the time-t price of the derivative security can be written as $V(s,t) = s^a$ when the stock price at that time is s. Then $V_t(s,t) = 0$, $V_s(s,t) = as^{a-1}$ and $V_{ss}(s,t) = a(a-1)s^{a-2}$. The derivative security is a tradable asset, so the function $V(s,t)$ must satisfy the Black-Scholes partial differential equation. Plugging the above partial derivatives into the Black-Scholes equation yields

$$rs(as^{a-1}) + \frac{1}{2}\sigma^2 s^2[a(a-1)s^{a-2}] = rs^a,$$

or, upon canceling s^a on both sides,

$$ra + \frac{1}{2}\sigma^2 a(a-1) = r,$$

which is a quadratic equation in a. The two solutions are $a = 1$ and $a = -2r/\sigma^2$. Thus $k = 2r = 2(0.04) = \boxed{0.08}$. **(Answer: (E))** □

7.4 Problems

Delta-hedging and calculations of holding profits

Problem 7.4.1. (Calculation of holding profit for long calls) Assume the Black-Scholes framework. Four months ago, Eric *bought* 100 units of a one-year 45-strike European call option on a nondividend-paying stock. He immediately delta-hedged his position with shares of the stock, but has not ever re-balanced his portfolio. He now decides to close out all positions.

You are given:

(i)

	Four Months Ago	Now
Stock price	\$40.00	\$50.00
Call option price	\$4.45539	?
Call option delta	?	0.73507

(ii) The continuously compounded risk-free interest rate is 5%.

(iii) The volatility of the stock is less than 50%.

(a) Calculate the volatility of the stock.

(b) Calculate the four-month holding profit for Eric.

Problem 7.4.2. [HARDER!] (Holding profit for a long straddle) One year ago, Jacky *bought* 10 units of a 2-year at-the-money European straddle on a nondividend-paying stock. He immediately delta-hedged his position with shares of the stock, but has not ever re-balanced his portfolio. He now decides to close out all positions.

You are given:

(i) The risk-free interest rate is a positive constant.

(ii) The current stock price and the stock price one year ago are the same.

(iii) The following information about the European call and put options constituting the straddle:

	One Year Ago	Now
Call option price	\$8.29391	\$5.59651
Put option price	\$4.85116	\$3.83641
Call option delta	0.66431	?
Put option delta	?	– 0.38209

Calculate Jacky's 1-year holding profit.

(Hint: The given table contains lots of useful information about the market parameters!)

Problem 7.4.3. (Given the delta-hedging strategy, find the price of the option) Assume the Black-Scholes framework. You are given:

(i) The current price of a nondividend-paying stock is 80.

(ii) An investor has sold 1,000 units of a one-year at-the-money European call option on the stock. He immediately delta-hedges the commitment with 750 shares of the stock.

(iii) The continuously compounded risk-free interest rate is 7%.

(iv) The volatility of the stock is less than 100%.

Calculate the price of each call option.

(Hint: Infer from (ii) the volatility of the stock.)

Problem 7.4.4. (Calculation of holding profit for short calls with dividends) Assume the Black-Scholes framework. You are given:

(i) The current stock price is 50.

(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(iii) The volatility of the stock is 16%.

(iv) The prices of 1-year at-the-money European call and put options are 4.348 and 1.981, respectively.

Timothy has just written 200 units of the call in (iv), and he delta-hedged his position immediately.

After 3 months, the stock price rises to 55 and the call price increases to 7.316.

Calculate the three-month holding profit for Timothy.

(Hint: Remember to take care of dividends.)

Problem 7.4.5. [HARDER!] (Delta-hedging a call using an otherwise identical put) Assume the Black-Scholes framework. You are given:

(i) The current price of a nondividend-paying stock is 20.

(ii) The stock's volatility is 28%.

(iii) The continuously compounded risk-free interest rate is 2%.

(iv) The following information about a 3-month at-the-money European call option on the stock:

Current Price	Current Delta	Current Gamma	Current Theta
1.16393	0.54210	0.14169	-2.41519

Suppose you have just sold 1,000 units of the call option above. You immediately delta-hedge your position by trading appropriate units of a European put option having the same underlying stock, strike price, and time to expiration as the call option.

Calculate the theta of your overall position.

Problem 7.4.6. (Testing your conceptual understanding of Figure 7.1.1) Assume the Black-Scholes framework. Yesterday, you sold a European call option on a nondividend-paying stock. You immediately delta-hedged the commitment with shares of the stock. Today, you decide to close out all positions.

Which of the following statements about your delta-hedged portfolio today is *incorrect*?

(A) Your delta when today's stock price is \$50 is (approximately) zero.

(B) Stock price risk is not completely eliminated.

(C) You lose from large stock price moves in either direction.

(D) The larger the stock price today, the smaller your delta.

(E) Your gamma is a negative constant.

Hedging multiple Greeks

Problem 7.4.7. (Implementing delta-gamma-hedging given raw information) Assume the Black-Scholes framework. You are given:

(i) The current price of a stock is 60.

(ii) The stock pays no dividends.

(iii) The stock's volatility is 30%.

(iv) The continuously compounded risk-free interest rate is 5%.

Suppose you have just *bought* 200 1-year 60-strike European call options.

Determine the numbers of units of a 1-year 65-strike European *put* option and the stock you should buy or sell in order to both delta-hedge and gamma-hedge your position in the 60-strike European calls.

Problem 7.4.8. (Based on Example 7.2.1: Delta-vega-hedging given summarized information) For two European call options, Call-I and Call-II, on a stock, you are given:

Greek	Call-I	Call-II
Delta	0.5825	0.7773
Gamma	0.0651	0.0746
Vega	0.0781	0.0596

Suppose you just sold 1,000 units of Call-I. You buy or sell appropriate units of the stock and Call-II in order to both delta-hedge and vega-hedge your position in Call-I.

Calculate the gamma of your hedged portfolio.

Problem 7.4.9. (Holding profit of a delta-gamma-hedged portfolio) Assume the Black-Scholes framework. You are given:

(i) The current price of a stock is \$50.

(ii) The stock pays no dividends.

(iii) The stock's volatility is 25%.

(iv) The continuously compounded risk-free interest rate is 5%.

Suppose you have just sold 1,000 1-year 50-strike European call options.

(a) Determine the numbers of units of a 1-year 60-strike European call option and the stock you should buy or sell in order to both delta-hedge and gamma-hedge your position in the 50-strike European calls.

(b) You are further given:

(v) The original (i.e., time-0) prices of the 1-year 50-strike call and 1-year 60-strike call are 6.1680 and 2.5127, respectively.

(vi) If the one-month stock price remains unchanged at \$50, then the one-month prices of the 50-strike call and 60-strike call (both of which will expire in 11 months) are 5.8611 and 2.2591, respectively.

Calculate your profit after one month if the delta-gamma-hedging strategy in part (a) is implemented and the one-month stock price remains unchanged at \$50.

(Hint: We don't have a formula of the holding profit for a delta-gamma-hedging strategy. Reason flexibly and adapt the rationale behind (7.1.1).)

Problem 7.4.10. (Delta-, gamma-, and theta-hedge given option Greeks table) Assume the Black-Scholes framework. You are given:

(i) The current price of a nondividend-paying stock is 50.

(ii) The stock's volatility is 30%.

(iii) The continuously compounded risk-free interest rate is 8%.

(iv) The following information about two European call options on the stock:

	Call A	Call B
Price	10.0618	6.0214
Delta	0.6951	0.5056
Gamma	0.0191	0.0217
Theta (Per Year)	-4.1201	-3.9835

In each of the following cases, calculate the amount of the net investment you make today (including the sale of the 1,000 options in the first place):

(a) You have just sold 1,000 units of Call A. You immediately delta-hedge your position with shares of the stock.

(b) You have just sold 1,000 units of Call A. You immediately delta-hedge and gamma-hedge your position with shares of the stock and Call B.

(c) You are now further given that Call A is 50-strike and Call B is 60-strike, and both of them are 18-month call options. You have just sold 1,000 units of a 50-strike put otherwise identical to Call A. You immediately delta-hedge and theta-hedge your position with Call A and Call B.

Delta-gamma-theta approximation

Problem 7.4.11. (Direct application of delta-gamma approximation) Assume the Black-Scholes framework. The current prices of a stock and a call option on the stock are \$10 and \$2, respectively.

You are given:

(i) $\Delta = 0.6$

(ii) $\Gamma = 0.2$

Use the delta-gamma approximation to estimate the option value if the stock price jumps to \$10.50.

Problem 7.4.12. (Given the delta approximation, find the delta-gamma approximation) Assume the Black-Scholes framework. For a 3-month 80-strike European put option on a nondividend-paying stock, you are given:

(i) The current price of the stock is 75.

(ii) The current price of the put option is 6.168.

(iii) The continuously compounded risk-free interest rate is 5%.

The price of the stock suddenly increases to 78. Using the delta approximation, you find that the put price decreases to 4.253.

Using the delta-gamma approximation, calculate the price of the put.

Problem 7.4.13. [HARDER!] (Given the delta and delta-gamma approximations, find the exact option price) Assume the Black-Scholes framework. Consider a 9-month at-the-money European put option on a futures contract.

The continuously compounded risk-free interest rate is 8%.

The futures price instantaneously decreases by 10. You are given:

(i) Using the delta approximation, you find that the option price increases by 4.148.

(ii) Using the delta-gamma approximation, you find that the option price increases by 4.231.

Calculate the *exact* price of the put option at the new futures price, i.e., after the initial futures price drops by 10.

Problem 7.4.14. (Parameter-dependent candidate price) Assume the Black-Scholes framework. Determine all value(s) of a, in terms of r, σ, γ, such that $V(S(t), t) := AS(t)^a e^{\gamma t}$ represents the time-t price of a derivative security on a nondividend-paying stock.

Problem 7.4.15. (Delta-theta-hedging) Assume the Black-Scholes framework. You are given:

(i) The current price of a nondividend-paying stock is 70.

(ii) The stock's volatility is 25%.

(iii) The continuously compounded risk-free interest rate is 5%.

(iv) The following information about two European put options on the stock:

Put	Current Price	Current Delta	Current Gamma	Current Theta
A	6.9389	−0.2867	0.0112	?
B	9.0062	−0.3433	0.0121	−0.2060

Suppose you have just sold 1,000 units of Put A. You immediately delta-hedge and theta-hedge your position by trading appropriate units of Put B and the stock.

Calculate the amount of net investment you make today (including the sale of the 1,000 Put A).

Problem 7.4.16. (Calculation of holding profit given various partial derivatives) Assume the Black-Scholes framework. For $t \geq 0$, let $S(t)$ be the time-t price of a nondividend-paying stock and $V(s,t)$ be the time-t price of a European derivative when the price of the underlying stock at that time is s. You are given:

(i) $S(0) = 10$ and $S(2) = 12$.

(ii) The continuously compounded risk-free interest rate is 5%.

(iii) The stock's volatility is 40%.

(iv) The following partial derivatives of $V(s,t)$ for various s and t:

s	t	$V_t(s,t)$	$V_s(s,t)$	$V_{ss}(s,t)$	$V(s,t)$
10	0	−0.5861	?	0.0491	3.2738
12	0	−0.6310	?	0.0341	4.7877
10	2	−0.9804	0.6274	0.0946	?
12	2	−1.0142	0.7825	0.0613	?

At time 0, Jason *bought* 100 units of the derivative. He immediately delta-hedged his position with shares of the stock, but has not ever re-balanced his portfolio. Two years later, he decided to close out all positions.

Calculate the two-year holding profit for Jason.

Problem 7.4.17. (Holding profit calculations given a table of option Greeks – I) Assume the Black-Scholes framework. One year ago, Kelvin bought 1,000 units of a European call option on a nondividend-paying stock. He immediately delta-hedged his position with appropriate number of shares of the stock, but has not ever re-balanced his portfolio. He now decides to close out all positions.

You are given:

(i) The stock's volatility is 20%.

(ii)

	One Year Ago	Now
Stock price	40	?
Call price	5.6295	5.7653
Call delta	?	0.7296
Call gamma	0.0331	0.0385
Call theta (per year)	−1.6569	−2.1911
Call elasticity	?	5.4417

Calculate Kelvin's 1-year holding profit.

Problem 7.4.18. [HARDER!] (Holding profit calculations given a table of option prices and Greeks – II) Assume the Black-Scholes framework. Three months ago, Tyler bought 100 units of an at-the-money European call option on a nondividend-paying stock. He immediately delta-hedged his position with appropriate units of an otherwise identical European put option, but has not ever re-balanced his portfolio. He now decides to close out all positions.

You are given:

(i) The stock's volatility is 36%.

(ii)

	Three Months Ago	Now
Stock price	45	42
Call option price	?	3.22341
Call delta	0.58082	0.4584
Call gamma	0.02785	0.03711
Call theta (per year)	−4.05975	−4.56265

Calculate Tyler's three-month holding profit.

(Hint: Make good use of the information presented in the table. First deduce the continuously compounded risk-free interest rate from the "Now" column. Then infer the time to maturity of the call from the original "Call delta." Finally, use put-call parity to find the prices and delta of the put. Be patient—you can do this problem!)

Problem 7.4.19. (Holding profit calculations given a table of option Greeks – III) Assume the Black-Scholes framework. Three months ago, you sold 1,000 units of a 1-year European put option on a nondividend-paying stock. You immediately delta-hedged your position with appropriate number of shares of the stock, but have not ever re-balanced his portfolio. You now decide to close out all positions.

You are given:

(i) The stock's volatility is 30%.

(ii) The put option is at-the-money *currently.*

(iii) Your careless secretary has provided you with the following values. However, she is not sure about whether these values are for the put option you sold or for a call option with the same strike, time to maturity, and underlying stock:

	Three Months Ago	Now
Stock price	50	55
Price	5.2121	4.5111
Delta	0.5129	−0.3809
Gamma	0.0266	0.0267
Theta (per year)	−4.2164	−2.1069

(Note: Values for each column are for the same option. The two columns, however, may or may not correspond to the same option.)

Calculate your three-month holding profit.

Appendix A

Standard Normal Distribution Table

Entries below represent the area under the standard normal distribution function from $-\infty$ to z, $N(z) = \mathbb{P}(Z \leq z)$. The value of z to the first decimal is given in the left column. The second decimal is given in the top row. Areas for negative values of z can be obtained by symmetry, i.e., $N(z) = 1 - N(-z)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

Values of z for selected values of $\mathbb{P}(Z \leq z)$							
z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
$\mathbb{P}(Z \leq z)$	0.800	0.850	0.900	0.950	0.975	0.990	0.995

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Appendix B

Solutions to Odd-Numbered End-of-Chapter Problems

B.1 Chapter 1

1. *Solution.* Solving the two payoff equations

$$\begin{cases} S - F_{0,T} = -5 \\ 1.1S - F_{0,T} = 1 \end{cases}$$

we get $S = 60$ and $F_{0,T} = 65$. If the spot price at expiration were 20% higher, then Aaron's profit, which is the same as his payoff, would be $1.2S - F_{0,T} = 1.2(60) - 65 = \boxed{7}$. □

3. *Solution.* Because Rose purchases a put and makes a profit, the put must be in-the-money at expiration, i.e., it must be the case that $S(0.5) < 50$. With this information available, Jack's loss is

$$-\text{Profit} = 8e^{0.04(0.5)} - \underbrace{(S(0.5) - 50)_+}_{0} = 8.1616,$$

and Rose's profit is

$$(50 - S(0.5))_+ - 6e^{0.04(0.5)} = (50 - S(0.5)) - 6e^{0.02} = 43.8788 - S(0.5).$$

As Rose's profit is twice as large as Jack's loss, we have $43.8788 - S(0.5) = 2(8.1616)$, which implies that $S(0.5) = \boxed{27.56}$. □

5. *Solution.* We begin by computing the future value of the put premiums:

Strike	FV of Put Premium
35	$0.44 \times 1.08 = 0.48$
40	$1.99 \times 1.08 = 2.15$
45	$5.08 \times 1.08 = 5.49$

The profit functions of the three short puts are sketched in Figure B.1.1. The 35-strike line crosses the 40-strike and 45-strike lines, respectively, at $40 - (2.15 - 0.48) = 38.33$ and $45 - (5.49 - 0.48) = 39.99$. Visually inspecting the profit diagram, we conclude that the 35-strike put produces a higher profit than the 45-strike put, but a lower profit than the 40-strike put, when $\boxed{38.33 < S(1) < 39.99}$. □

7. *Solution.* Only I and II are correct. For III, the short European put, being long in nature, results in an obligation to buy the underlying asset if the long put holder chooses to exercise his/her put. **(Answer: (B))** □

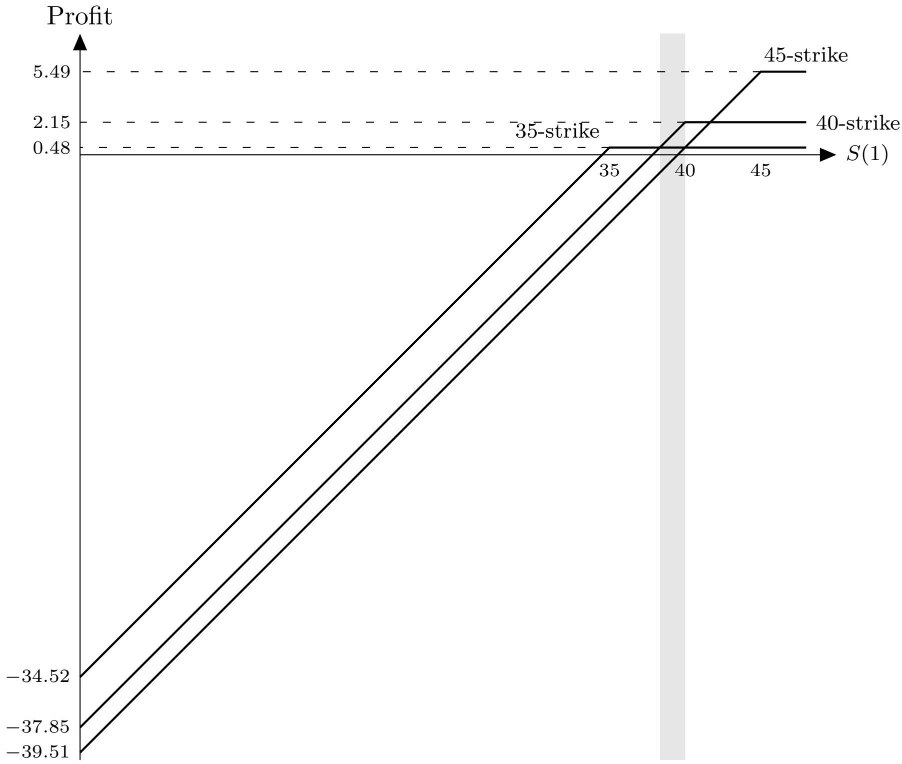


FIGURE B.1.1
Profit diagrams of the three puts in Problem 1.4.5.

B.2 Chapter 2

1. *Solution.* Using (2.2.1), we have

$$\begin{aligned} F^P_{0,1} &= 50 - \mathrm{PV}_{0,1}(\mathrm{Div}) \\ &= 50 - (\mathrm{e}^{-0.06(0.25)} + \mathrm{e}^{-0.06(0.5)} + \mathrm{e}^{-0.06(0.75)} + \mathrm{e}^{-0.06}) \\ &= \boxed{46.15} \end{aligned}$$

□

3. *Solution.* Let i be the effective annual risk-free rate of interest. By the first formula in (2.3.2), the 1-year forward price satisfies the equation

$$\mathrm{FV}_{0,1}(S(0)) - \mathrm{FV}_{0,1}(\mathrm{Div}) = F_{0,1},$$

or

$$80(1+i) - 2(1+i)^{1/2} - 3 = 84. \tag{B.2.1}$$

Let $x = (1+i)^{1/2}$. Then (B.2.1) becomes the quadratic equation (in x)

$$80x^2 - 2x - 87 = 0,$$

whose positive solution is

$$x = \frac{2 + \sqrt{2^2 - 4(80)(-87)}}{2(80)} = 1.0554.$$

Then $i = x^2 - 1 = \boxed{0.1139}$. □

5. *Solution.* By the first formula in (2.3.2), we have

$$\begin{aligned} F_{0,3} &= S(0)\mathrm{e}^{3r} - 1.5[\mathrm{e}^{2.75r} + \mathrm{e}^{2.5r}(1.01) + \cdots + (1.01)^{11}] \\ &= 200\mathrm{e}^{3(0.04)} - 1.5\mathrm{e}^{2.75(0.04)}\left[\frac{1-(1.01\mathrm{e}^{-0.25(0.04)})^{12}}{1-1.01\mathrm{e}^{-0.25(0.04)}}\right] \\ &= \boxed{205.4119}. \qquad \textbf{(Answer: (B))} \end{aligned}$$

□

Remark. In the second equality, we use the geometric series formula $a + ar + ar^2 + \cdots + ar^{n-1} = a(1-r^n)/(1-r)$ for $r \neq 1$.

7. *Solution.* By the second formula in (2.3.2), the fair 1-year forward price is

$$F_{0,1}^{\text{fair}} = S(0)\mathrm{e}^{(r-\delta)T} = 120\mathrm{e}^{(0.06-0.04)(1)} = 122.4242,$$

which is higher than the observed price of $F_{0,1}^{\text{obs}} = 121$. In other words, the observed forward in the market is underpriced. To exploit the arbitrage opportunity, we should buy the observed forward in the market and sell the synthetic forward. The table below shows the details and the associated cash flows:

Transaction	**Cash Flows**	
	Time 0	**Time 1**
Buy observed forward	0	$S(1) - 121$
Short sell $\mathrm{e}^{-0.04} = 0.9608$ units of Stock XYZ	$120\mathrm{e}^{-0.04} = 115.2947$	$-S(1)$
Lend 115.2947	-115.2947	122.4242
Total	0	$122.4242 - 121 = 1.4242$

□

9. *Solution.* There are two cash flows between November 1, 3016 and November 1, 3018:

- On November 1, 3016, you short sold 100 shares of stock X at the *bid* price of 95.00 per share, together with the commission, for a total cash inflow of

$$95.00 \times 100 \times 0.98 = 9,310.$$

- On November 1, 3018, due to the reinvestment of dividends you have to buy back $100\mathrm{e}^{0.04(2)} = 108.328707$ shares at the *ask* price of 100.50 per share, along with the commission, for a total cash outflow of

$$108.328707 \times 100.50 \times 1.02 = 11,104.77573.$$

Therefore, your profit measured as of November 1, 3018 equals

$$-11,104.77573 + 9,310e^{0.05(2)} = \boxed{-815.63}.$$

□

11. *Solution.* In symbols, we have

$$\begin{aligned}
(A) &= S(0),\\
(B) &= F_{0,1} = S(0)e^{r-\delta},\\
(C) &= F_{0,2} = S(0)e^{2(r-\delta)},\\
(D) &= S(0)e^{-2\delta},\\
(E) &= \mathbb{E}[S(2)].
\end{aligned}$$

Because $\delta < r$, the correct ranking is:

$$\boxed{(D) < (A) < (B) < (C) < (E)}.$$

□

13. *Solution.* The no-arbitrage interval of the 3-year forward price is

$$\left[(40 \underbrace{-1}_{\text{paid at time 0}})e^{(0.06-0.03)(3)} \underbrace{-2}_{\text{paid at time 3}}, (41+1)e^{(0.07-0.03)(3)} + 2\right] = [40.6728, 49.3549];$$

remember that market frictions enter the interval in such a way to make it as wide as possible. Since the observed forward price of 38 is lower than the lower end-point of the no-arbitrage interval, we engage in a reverse cash-and-carry arbitrage by performing the following actions:

Transaction	**Cash Flows**	
	Time 0	**Time 3**
Buy observed forward	0	$[S(3) - 38] - 2 = S(3) - 40$
Short sell $e^{-0.09} = 0.9139$ units of Stock Y	$(40-1)e^{-0.03(3)} = 35.6433$	$-S(3)$
Lend 35.6433	-35.6433	$35.6433e^{0.06(3)} = 42.6728$
Total	0	$42.6728 - 40 = 2.6728$

□

15. *Solution.* The notional value of Peter's futures contracts is

$$8 \times 250 \times 1,629 = 3,258,000.$$

The initial margin is 10% of the notional value, or $10\% \times 3,258,000 = 325,800$.

The maintenance margin is 80% of the initial margin, or $70\% \times 325,800 = 228,060$.

There will be a margin call one month from now if the new margin balance is less than the maintenance margin, i.e.

$$325,800\mathrm{e}^{0.06/12} + \overbrace{8 \times 250 \times (F_{\underbrace{1,3}_{\text{in months}}} - 1,629)}^{\text{Mark-to-market proceed}} \leq 228,060,$$

which results in $F_{1,3} \leq 1,579.31346$.

Solving

$$S(1)\mathrm{e}^{(0.06-0.02)(2/12)} = F_{1,3} \leq 1,579.31346,$$

we get $(0 \leq) \boxed{S(1) \leq 1,568.82}$. □

B.3 Chapter 3

1. *Solution.* (A) No
   (B) Yes
   (C) No (the answer would be "Yes" for a long collar.)
   (D) Yes
   (E) No

□

3. *Solution.* The final payoff as a result of using a floor is $\max(S(T), K)$, which increases with the floor level K. The higher the strike price, the higher the payoff and the more expensive the put is. **(Answer: (A))** □

5. *Solution.* You can combine the payoff diagrams of a floor and cap and see that the resulting diagram is that of a long straddle. Alternatively, recall that a floor consists of a long asset plus a long put, and that a cap comprises a short asset plus a long call. Adding a floor and a cap cancels the asset and leaves only the long put and the long call. This is simply a $\boxed{\text{long straddle}}$. **(Answer: (C))** □

7. *Solution.* The positions that Richard should take in the call and put options are:

   - $\boxed{\text{Short}}$ call
   - $\boxed{\text{Long}}$ put

   By put-call parity, the cost required to establish these two positions equals (recall that $F^P_{0,T}(S) = S(0)$ for a nondividend-paying stock)

$$P - C = \mathrm{PV}_{0,0.5}(1,020) - F^P_{0,1} = \mathrm{e}^{-0.05(0.5)}(1,020) - 1,000 = \boxed{-5.1839}.$$

□

9. *Solution.* • Actuary A, being short the 70-strike call, has a profit of

$$\mathrm{Pr}_A = 1.5\mathrm{e}^{0.06} - (S(1) - 70)_+.$$

- Actuary B, being long the synthetic forward, has the same profit as a genuine long forward, whose profit is (recall that the stock pays no dividends, so $F_{0,T} = S(0)\mathrm{e}^{rT}$)

$$\mathrm{Pr}_B = S(1) - F_{0,1} = S(1) - 60\mathrm{e}^{0.06}.$$

Since $\mathrm{Pr}_B = 2\,\mathrm{Pr}_A$, we solve

$$S(1) - 60\mathrm{e}^{0.06} = 2[1.5\mathrm{e}^{0.06} - (S(1) - 70)_+]. \tag{B.3.1}$$

Case 1. If $S(1) < 70$, then (B.3.1) implies that $S(1) - 60\mathrm{e}^{0.06} = 2(1.5\mathrm{e}^{0.06})$, or $S(1) = 66.90$.

Case 2. If $S(1) \geq 70$, then (B.3.1) results in

$$S(1) - 60\mathrm{e}^{0.06} = 2[1.5\mathrm{e}^{0.06} - (S(1) - 70)],$$

which can be solved to yield $S(1) = 68.97$, contradicting $S(1) \geq 70$.

The only possible 1-year stock price is $S(1) = \boxed{66.90}$. □

11. *Solution.* Two applications of put-call parity give

$$\begin{cases} 14.3782 - 0.4394 = S(0)\mathrm{e}^{-0.02(2)} - 98\mathrm{e}^{-2r} \\ 12.7575 - 0.6975 = S(0)\mathrm{e}^{-0.02(2)} - 100\mathrm{e}^{-2r} \end{cases}.$$

Solving the two simultaneous equations, we have $\mathrm{e}^{-2r} = 0.9394$ and $S(0) = \boxed{110.33}$ □

13. *Solution.* Since $C(K)$ is decreasing in K, we have $|C(60) - C(65)| = C(60) - C(65) = 3$. By put-call parity,

$$\begin{cases} C(60) - P(60) = F^P_{0,0.25} - 60\mathrm{e}^{-0.05(0.25)} \\ C(65) - P(65) = F^P_{0,0.25} - 65\mathrm{e}^{-0.05(0.25)} \end{cases}.$$

Subtracting the second equation from the first one yields

$$[C(60) - C(65)] + [P(65) - P(60)] = 5\mathrm{e}^{-0.05(0.25)},$$

and because $P(K)$ is increasing in K, we have $|P(60) - P(65)| = P(65) - P(60) = \boxed{1.9379}$. □

15. *Solution.* By definition, a bear spread is set up by selling a low-strike option and buying a high-strike otherwise identical option. Both (A) and (B) give rise to a bull spread instead. For (C) and (D), note that $C(K)$ is non-increasing in K whereas $P(K)$ is non-decreasing in K. It follows that the call (resp. put) with a premium of 6 has a higher (resp. lower) strike than the call (resp. put) with a premium of 10.

(A) No

(B) No

(C) Yes

(D) No

□

17. *Solution 1 (Preferred).* Note that a 70-80 bear spread constructed using puts has the same profit as one constructed using calls. The latter is constructed by selling a 70-strike call and buying a 80-strike call. The investment required is

$$C(80) - C(70) = 2.7 - 8.3 = -5.6.$$

For the profit to be 4, the payoff has to be

$$\text{Profit} + \text{FV}_{0,1}(\text{Investment}) = 4 + (-5.6e^{0.06}) = -1.9463,$$

which happens when the 1-year stock price is $S(1) = 70 + 1.9463 = \boxed{71.95}$. □

Solution 2 (Not preferred). A 70-80 put bear spread is constructed by constructed selling a 70-strike put and buying a 80-strike put. By put-call parity,

$$\begin{cases} C(70) - P(70) = \text{PV}_{0,1}(F_{0,1}) - \text{PV}_{0,1}(70) \\ C(80) - P(80) = \text{PV}_{0,1}(F_{0,1}) - \text{PV}_{0,1}(80) \end{cases}.$$

Hence

$$\begin{aligned} [C(70) - C(80)] + [P(80) - P(70)] &= \text{PV}_{0,1}(10) \\ (8.3 - 2.7) + [P(80) - P(70)] &= 10e^{-0.06} \\ P(80) - P(70) &= 3.817645. \end{aligned}$$

Setting $4 = \text{Profit} = \text{Payoff} - \text{FV}_{0,1}(\text{Investment})$, we get

$$\text{Payoff} = 4 + 3.817645e^{0.06} = 8.053715,$$

which is realized at $S(1) = 80 - 8.053715 = \boxed{71.95}$. □

19. *Solution.* From the given table of option premiums, it is possible to deduce the T-year present value factor $\text{PV}_{0,T}$. Specifically, put-call parity implies (recall that $F^P_{0,T} = S(0)$ if the stock pays no dividends)

$$\begin{cases} 8.4 - 0.8 = S(0) - 50\text{PV}_{0,T} \\ 2.6 - 4.7 = S(0) - 60\text{PV}_{0,T} \end{cases},$$

which gives $\text{PV}_{0,T} = 0.97$ and $S(0) = 56.1$. Since a bear spread constructed by calls has the same profit as one constructed by puts, with respect to profit we may assume without loss of generality that the bear spread is constructed by puts, or more precisely, by buying the 60-strike put and selling the 50-strike put. The initial investment in this case is

$$P(60) - P(50) = 4.7 - 0.8 = 3.9,$$

which accumulates to $3.9/0.97 = 4.0206$ at expiration. For the profit of the bear spread to be zero, the stock price at expiration has to be $S(T) = 60 - 4.0206 = 55.9794$, meaning that the stock price will move by $55.9794 - 56.1 = \boxed{-0.1206}$. □

21. *Solution.* A (long) box spread is created by buying a 25-strike call, selling a 25-strike put, selling a 35-strike call and buying a 35-strike put. The investment required is

$$C(25) - P(25) - C(35) + P(25) = 9.51,$$

and the payoff at expiration is $35 - 25 = 10$. The implicit 1-year accumulation factor is $10/9.51 = 1.0515$, whereas the 1-year accumulation factor in the market is $e^{0.06} = 1.0618$. In other words, the long box spread is worse than a risk-free investment in the market and should be short:

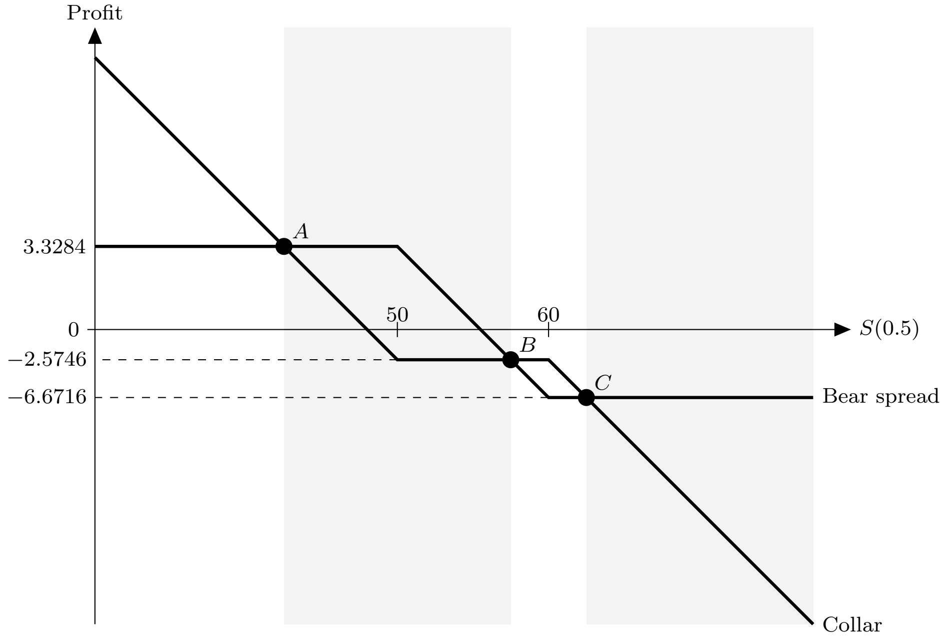


FIGURE B.3.1
The profit diagrams of the 50-60 bear spread and the 50-60 collar in Problem 3.5.23.

Strike	**Position in Call**	**Position in Put**
25	Short	Long
35	Long	Short

In the language of bull/bear spread(s), we should:

- Buy a 25-35 call bear spread (or equivalently sell a 25-35 call bull spread)
- Buy a 25-35 put bull spread (or equivalently sell a 25-35 put bear spread)

Together with a long 1-year zero-coupon bond with a face value of 10, the resulting profit at time 0 is $9.51 - 10e^{-0.06} = \boxed{0.0924}$. (At time 1, our payoff is constant at $\underbrace{-10}_{\text{from box spread}} \underbrace{+10}_{\text{from bond}} = 0$) □

23. *Solution.* Without loss of generality (why?), we may assume that the 50-60 bear spread is constructed by calls, i.e., a short 50-strike call coupled with a long 60-strike call. The initial investment is $C(60) - C(50) = 1.96 - 5.19 = -3.23$, which grows to $-3.23e^{0.06(0.5)} = -3.3284$ after 6 months. The 50-60 collar is set up by a long 50-strike put and a short 60-strike call. By put-call parity, the put price is

$$P(50) = 5.19 - 50 + 0.75e^{-0.06/12} + 50e^{-0.06(0.5)} = 4.4585.$$

The initial investment of the collar is $P(50) - C(60) = 4.4585 - 1.96 = 2.4985$, which accumulates to $2.4985e^{0.06(0.5)} = 2.5746$ after 6 months.

The profit functions of the 50-60 bear spread and the 50-60 collar are sketched in Figure B.3.1 (not drawn to scale). The horizontal coordinates of points A, B, and C are 44.0970, 55.9030, and 64.097, respectively. It follows that the bear spread's profit outweighs the collar's profit when and only when $\boxed{44.10 < S(0.5) < 55.90}$ and $\boxed{S(0.5) > 64.10}$. □

25. *Solution.* (A) False. It is a short with respect to the underlying stock and is a bet on the stock price going down in the future.
    (B) True. Simply set $K_1 = K_2 = F_{0,T}$.
    (C) False. There are infinitely many possible zero-cost collars.
    (D) True. Recall that K_1 and K_2 are restricted by $K_1 \leq F_{0,T} \leq K_2$. If $F_{0,T} \geq S(0)$, then K_1 can possibly take the value of $S(0)$.
    (E) False. If $F_{0,T} < S(0)$, which happens when the dividends are sufficiently large, then K_2 can possibly be less than $S(0)$, in which case the call is in-the-money.

□

27. *Solution.* By inspection (starting from the right because of put options), the payoff of the derivative equals

$$\text{Payoff} = 3(5 - S(1))_+ + 2(10 - S(1))_+ - 6(15 - S(1))_+ - 2S(1) + 80,$$

which shows that the derivative can be replicated by:

- 3 long 5-strike puts
- 2 long 10-strike puts
- 6 short 15-strike puts
- 2 short stocks
- A long zero-coupon bond with a face value of 80 (i.e. lend PV(80))

Therefore, the fair price of the derivative is

$$\begin{aligned} & 3P(5) + 2P(10) - 6P(15) - 2S(0) + 80\mathrm{e}^{-0.04} \\ = \; & 3(0.11) + 2(1.75) - 6(9.52) - 2(10) + 80\mathrm{e}^{-0.04} \\ = \; & 3.5732, \end{aligned}$$

which is lower than the observed price by 0.02684.

To exploit an arbitrage opportunity, we sell the observed derivative and buy the synthetic derivative by:

- 3 long 5-strike puts
- 2 long 10-strike puts
- 6 short 15-strike puts
- 2 short stocks
- A long zero-coupon bond with a face value of 80 (i.e. lend PV(80))

The amount of arbitrage profit per unit of stock is $\boxed{0.02684}$. □

29. *Solution.* Note that we should buy the 70-strike straddle. By put-call parity, the put price is

$$P(70) = C(70) - F_{0,T}^P + K\mathrm{e}^{-rT} = 7 - 70\mathrm{e}^{-0.02(4/12)} + 70\mathrm{e}^{-0.08(4/12)} = 5.6231.$$

Then the stock price has to move by $\mathrm{FV}_{0,4/12}[C(70) + P(70)] = 12.6231\mathrm{e}^{0.08(4/12)} = \boxed{12.96}$ in either direction after 4 months to result in a positive profit. □

31. *Solution.* (A) False.

    Explanations: The holder of a long butterfly spread benefits the most when the spot price at expiration is very close to the current price. A long butterfly spread is therefore a bet on the volatility of the underlying asset being *lower* than that perceived by the market.

    (B) True.

    Explanations: Adding the payoff functions of a long K_1-K_2 bear spread and a long K_2-K_3 bull spread yields a payoff structure having the same general shape (horizontal, down, up, horizontal, from left to right) as a short K_1-K_2-K_3 butterfly spread.

    (C) True.

    Explanations: Algebraically, this is a consequence of put-call parity. Write the middle strike price as a weighted average of the two extreme strike prices, i.e., $K_2 = \lambda K_1 + (1-\lambda)K_3$ for some $\lambda \in (0,1)$. Then the cost of a call (generally asymmetric) butterfly spread is

$$\begin{aligned}
& \lambda C(K_1) - C(K_2) + (1-\lambda)C(K_3) \\
= \; & \lambda[P(K_1) + \mathrm{PV}_{0,T}(F_{0,T} - K_1)] - [P(K_2) + \mathrm{PV}_{0,T}(F_{0,T} - K_2)] \\
& \quad + (1-\lambda)[P(K_3) + \mathrm{PV}_{0,T}(F_{0,T} - K_3)] \\
= \; & \lambda P(K_1) - P(K_2) + (1-\lambda)P(K_3),
\end{aligned}$$

    which is the cost of the otherwise identical put butterfly spread.

    Geometrically, you can also employ the slope adjustment technique and see that exactly the same payoff structure can be produced by call or put options. For example, try to construct Figure 3.4.5 by put options.

    (D) True.

    Explanations: Scaling up the constituent options of a butterfly spread by a common factor of k leads to the maximum profit multiplied by k, which can be any arbitrary positive number. □

33. *Solution.* From (i) and (ii), the asymmetric butterfly spread can be created by buying two 20-strike calls, selling five 23-strike calls and buying three 25-strike calls. From the given table, the net investment is

$$I = 2C(20) + 3C(25) - 5C(23) = 2(3.59) + 3(1.89) - 5(2.45) = 0.6.$$

    To find the present value factor, we apply put-call parity at $K = 20$ and $K = 23$ (or any other pair):

$$\begin{cases} 3.59 - 2.64 = F_{0,T}^P - 20\mathrm{PV}_{0,T} \\ 2.45 - 4.36 = F_{0,T}^P - 23\mathrm{PV}_{0,T} \end{cases},$$

    which gives $\mathrm{PV}_{0,T} = 143/150$ (and $F_{0,T}^P = 1,201/60$). If $S(T) = 21$, then the payoff is 2 (by slope considerations), so the profit equals $2 - 0.6(150/143) = \boxed{1.37}$. □

B.4 Chapter 4

1. *Solution.* It turns out that the strangle pays $(50-70)_+ + (70-65)_+ = 5$ in the up state and $(50-45)_+ + (45-65)_+ = 5$ in the down state as well. It is the same as a risk-free

bond with a face value of 5. The current price of the strangle is therefore the present value of 5, or

$$V_0 = \frac{5}{1.1} = \boxed{4.5455}.$$

□

Remark. There is no need to determine the tree parameters and the risk-neutral probability of an up move. For your information, they are $u = 70/55 = 14/11$, $d = 45/55 = 9/11$ and

$$p^* = \frac{(1+0.1) - 9/11}{14/11 - 9/11} = 0.62.$$

3. *Solution.* The payoff of the special derivative is $V_u = (38-32)^2 = 36$ in the up-state and $V_d = (28-25)^3 = 27$ in the down-state. With the risk-neutral probability of an up move being

$$p^* = \frac{30(1.1) - 25}{38 - 25} = \frac{8}{13},$$

the current price of the derivative is

$$V_0 = \frac{1}{1.1}[36p^* + 27(1-p^*)] = \boxed{29.5804}.$$

□

Remark. For your information, the replicating portfolio consists of $\Delta = 9/13$ shares and $B = 1,260/143$ in the risk-free bond.

5. This is a hard but interesting question on replication. Two solutions are provided, both of which are instructive.

Solution 1 (Replication). Call option A pays $(12-9)_+ = 3$ in Outcome 1, and $(8-9)_+ = 0$ in Outcome 2. To determine the value of Security 2 in Outcome 1, let's use α units of Security 1 and β units of call option A to replicate Security 2. Matching the time-0 prices and the payoffs in Outcome 2, we are prompted to solve the equations (see the remark below)

$$\begin{cases} 10.4\alpha + 1.8\beta = 10 & \text{(time-0 price)} \\ 8\alpha + 0(\beta) = 2.5 & \text{(payoff in Outcome 2)} \end{cases}.$$

They imply that $\alpha = 0.3125$ and $\beta = 3.75$. Then the value of Security 2 in Outcome 1 must be $12\alpha + 3\beta = 15$.

Now that the price evolution of Security 2 is known, we can deduce that call option B pays $(15-11)_+ = 4$ in Outcome 1, and $(2.5-11)_+ = 0$ in Outcome 2. Call option B is therefore equivalent to 4/3 units of call option A and should cost

$$\frac{4}{3} \times \text{time-0 price of call option A} = \frac{4}{3} \times 1.8 = \boxed{2.4}.$$

□

Remark. (i) Notice carefully that having the same time-0 price does not mean having the same payoff at expiration. However, having the same time-0 price and the same payoff at one future outcome necessitates having the same payoff at the other outcome, or else arbitrage opportunities will exist (the question says that this is an "arbitrage-free binomial model").

(ii) You can replicate call option B by something else, but this is unnecessary.

Solution 2 (Risk-neutral pricing). To identify the risk-free rate implicit in this model, we replicate the risk-free bond using a portfolio of α' units of Security 1 and β' units of Call option A, so that the portfolio has the same payoff in Outcome 1 and in Outcome 2. Specifically, we require

$$12\alpha' + 3\beta' = 8\alpha' + 0(\beta') = 8\alpha',$$

which implies that α' and β' are related via

$$4\alpha' + 3\beta' = 0.$$

With this relation, the time-0 cost of the portfolio is

$$10.4\alpha' + 1.8\beta' = 10.4\alpha' + 1.8(-\frac{4}{3}\alpha') = 8\alpha'.$$

As $8\alpha'$ at time 0 grows to the same amount of $8\alpha'$ at the end of the binomial model, we deduce that the risk-free interest rate is $r = 0$.

Now we proceed to determine the risk-neutral probability of moving to Outcome 1. To this end, we apply the risk-neutral pricing formula to Security 1 and get

$$10.4 = \mathrm{e}^{-r}[12p^* + 8(1 - p^*)] = 12p^* + 8(1 - p^*),$$

meaning that $p^* = 0.6$. Then applying the risk-neutral pricing formula to Security 2 leads to

$$10 = ? \times p^* + 2.5(1 - p^*),$$

so that $? = 15$. A final application of the risk-neutral pricing formula shows that the current price of call option B is

$$p^*(15 - 11)_+ + (1 - p^*)(2.5 - 11)_+ = \boxed{2.4}.$$

□

7. *Solution.* Since $\mathrm{Var}[\ln[S(t)]] = \sigma^2 t$ for all $t > 0$, we have $\sigma = 0.5$. For a forward tree, the risk-neutral probability of an up move is

$$p^* = \frac{1}{1 + \mathrm{e}^{\sigma\sqrt{h}}} = \frac{1}{1 + \mathrm{e}^{0.5\sqrt{0.25}}} = \boxed{0.4378}.$$

□

9. *Solution.* In a forward tree,

$$\begin{aligned} u &= \exp[(r - \delta)h + \sigma\sqrt{h}] = \exp[(0.03 - 0.05)(1) + 0.3\sqrt{1}] = 1.323130, \\ d &= \exp[(r - \delta)h - \sigma\sqrt{h}] = \exp[(0.03 - 0.05)(1) - 0.3\sqrt{1}] = 0.726149. \end{aligned}$$

The risk-neutral probability of an up move is

$$p^* = \frac{\mathrm{e}^{(r-\delta)h} - d}{u - d} = \frac{\mathrm{e}^{(0.03-0.05)(1)} - 0.726149}{1.323130 - 0.726149} = 0.425557,$$

or, more simply,

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}} = \frac{1}{1+e^{0.3\sqrt{1}}} = 0.425557.$$

Then the four possible 1-year stock prices are

$$S_{uuu} = 440.1097, \quad S_{uud} = 241.5373, \quad S_{udd} = 132.5585, \quad S_{ddd} = 72.7496$$

and the corresponding payoffs of the 150-250 strangle (being $(S(3)-250)_+ + (150 - S(3))_+$) are

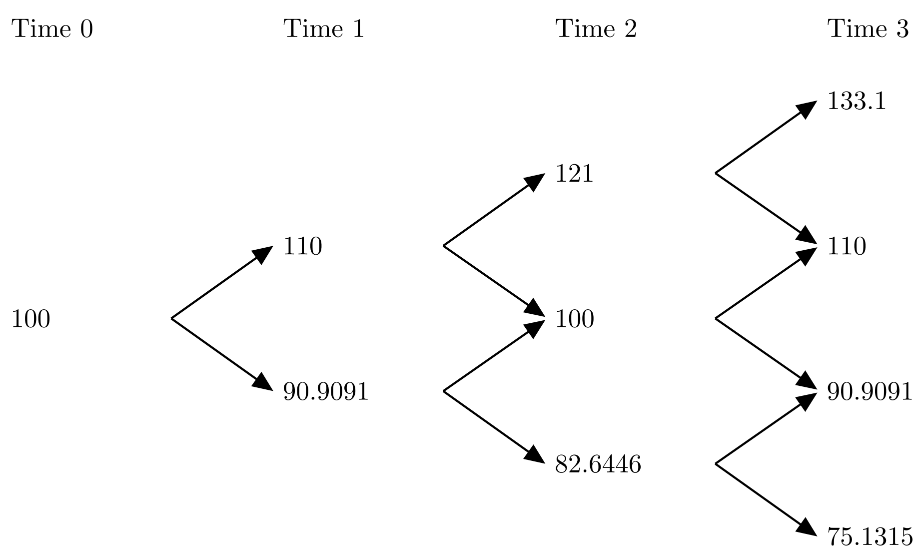
$$V_{uuu} = 190.1097, \quad V_{uud} = 0, \quad V_{udd} = 17.4415, \quad V_{ddd} = 77.2504.$$

By risk-neutral pricing, the current price of the strangle is

$$\begin{aligned} V_0 &= e^{-rT}[(p^*)^3 V_{uuu} + 3(p^*)^2(1-p^*)V_{uud} + 3p^*(1-p^*)^2 V_{udd} + (1-p^*)^3 V_{ddd}] \\ &= e^{-0.03(3)}[(0.425557)^3(190.1097) + 3(0.425557)(1-0.425557)^2(17.4415) \\ &\qquad +(1-0.425557)^3(77.2504)] \\ &= \boxed{33.4888}. \end{aligned}$$

□

11. *Solution.* The 3-period binomial tree is constructed as follows:



The risk-neutral probability of an up move is

$$p^* = \frac{e^{0.05} - 1/1.1}{1.1 - 1/1.1} = 0.744753.$$

The possible payoffs at expiration are

$$\begin{aligned} V_{uuu} &= 21 + 33.1 = 54.1, \\ V_{uud} &= 21 + 10 = 31, \\ V_{udu} = V_{duu} &= 0 + 10 = 10, \end{aligned}$$

and zero otherwise. By risk-neutral pricing, the price of the special derivative is

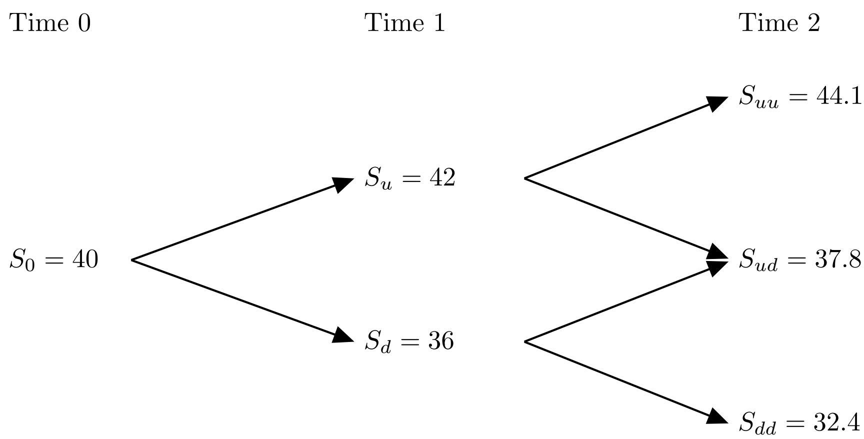
$$V_0 = \mathrm{e}^{-0.05(3)}[(p^*)^3 V_{uuu} + \underbrace{(p^*)^2(1-p^*)(V_{uud} + V_{udu} + V_{duu})}_{\text{not } 3(p^*)^2(1-p^*)V_{uud}!}] = \boxed{25.4495}.$$

□

13. This is a challenging question in which several financial instruments (stock, put, Derivative X, Derivative Y), some with different expiration dates, co-exist. To answer this question well, one needs to identify the nature and underlying asset of each derivative clearly.

Solution. (a) Note that Derivative X is a 1-year European *call* option on a European *put* option which matures two years from now. To value the call, we need the possible values of the put option (underlying asset) at the end of the first year.

We first start with the evolution of the stock price as shown below:



The risk-neutral probability of an up move is

$$p^* = \frac{\mathrm{e}^{0.03} - 0.9}{1.05 - 0.9} = 0.869697.$$

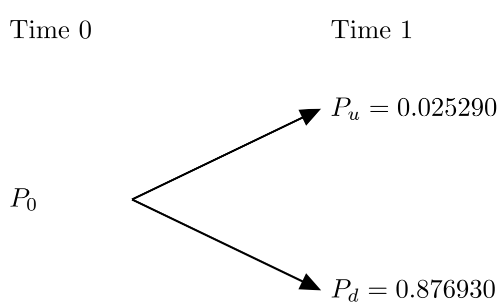
The possible time-2 payoffs of the 2-year put option are

$$P_{uu} = (38 - 44.1)_+ = 0, \quad P_{ud} = (38 - 37.8)_+ = 0.2, \quad P_{dd} = (38 - 32.4)_+ = 5.6.$$

By risk-neutral pricing, the possible time-1 values of this put option are

$$\begin{aligned} P_u &= \mathrm{e}^{-0.03}(1-p^*)(0.2) = 0.025290, \\ P_d &= \mathrm{e}^{-0.03}[p^*(0.2) + (1-p^*)(5.6)] = 0.876930. \end{aligned}$$

Here is the evolution of the put price:



As a call on the above put, Derivative X pays off only at the d node with a non-zero payoff of

$$V_d^X = \underbrace{(P_d - 0.5)_+}_{\text{like the usual } (S(T)-K)_+ \text{ formula}} = 0.376930.$$

By risk-neutral pricing again, the time-0 price of Derivative X is

$$V_0^X = \mathrm{e}^{-0.03}(1-p^*)V_d^X = \boxed{0.0477}.$$

(b) This time Derivative Y is a 1-year European *put* option on the *put* option which matures two years from now. Derivatives X and Y therefore form a call-put pair. To apply put-call parity, we need the time-0 price of the underlying asset, which is the 2-year put option:

$$P_0 = \mathrm{e}^{-0.03}[0.025290p^* + 0.876930(1-p^*)] = 0.13223.$$

By put-call parity for nondividend-paying underlying assets (the 2-year put in this case), we have

$$\begin{aligned} V_0^X - V_0^Y &= \underbrace{P_0}_{\text{not } S(0)} - \underbrace{0.5}_{\substack{\text{common}\\ \text{strike price}}} \times \underbrace{\mathrm{e}^{-r}}_{\substack{\text{discount for}\\ \text{1 year only}}} \\ 0.04766 - V_0^Y &= 0.13223 - 0.5\mathrm{e}^{-0.03} \\ V_0^Y &= \boxed{0.4007}. \end{aligned}$$

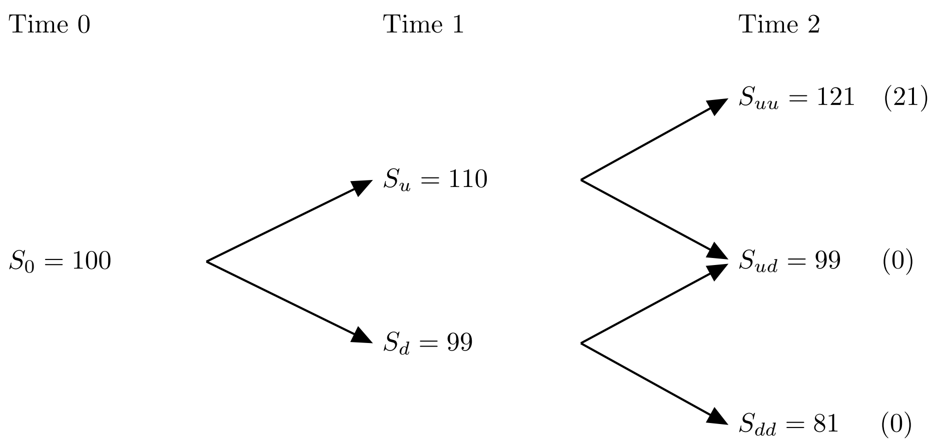
□

Remark. (i) Note that when applying the put-call parity, we discount for one year only because the time to maturity of Derivatives X and Y is one year, although their underlying asset, which is the 2-year put, matures in two years.

(ii) There are many good reasons for forcing you to use put-call parity to answer part (b). First, it challenges you with this compound option version of put-call parity. Second and more importantly, in the continuous-time Black-Scholes framework it is hard (at least beyond the scope of this book) to compute the price of a compound option directly. Given the price of Derivative X, one can first calculate the price of the underlying put by the Black-Scholes formula, then determine the price of Derivative Y by the compound option put-call parity.

15. This challenging problem illustrates the essence of *dynamic* replication in a multi-period binomial model and its applications to the synthetic construction of a derivative as well as to arbitraging a mispriced derivative. The most interesting feature is that your strategies vary with time and with the prevailing stock price. You cannot just sit on the sofa and watch TV forever!

Solution. Let's begin by determining the (one-period) replicating portfolios at the initial node, u node, and d node. The stock price evolution is described as follows:



- *Replicating portfolio:* With $C_{uu} = 21$ and $C_{ud} = C_{dd} = 0$, we have

$$\begin{aligned}\Delta_u &= \frac{C_{uu} - C_{ud}}{S_u - S_d} = \frac{21 - 0}{121 - 99} = 0.954545,\\ B_u &= \frac{uC_{ud} - dC_{uu}}{(1+i)(u-d)} = \frac{0 - 0.9(21)}{1.02(1.1 - 0.9)} = -92.647059,\\ C_u &= \Delta_u S_u + B_u = 12.352891\end{aligned}$$

and

$$\begin{aligned}\Delta_d &= \frac{C_{ud} - C_{dd}}{S_u - S_d} = 0,\\ B_d &= \frac{uC_{dd} - dC_{ud}}{(1+i)(u-d)} = 0.\\ C_d &= \Delta_d S_d + B_d = 0.\end{aligned}$$

(Note: If the stock price hits the d node, then the call will never pay off and becomes worthless.) At the initial node,

$$\begin{aligned}\Delta_0 &= \frac{C_u - C_d}{S_u - S_d} = \frac{12.352891 - 0}{110 - 90} = 0.617645,\\ B_0 &= \frac{uC_d - dC_u}{(1+i)(u-d)} = \frac{0 - 0.9(12.352891)}{1.02(1.1 - 0.9)} = -54.498049,\\ C_0 &= \Delta_0 S_0 + B_0 = 7.266451. \quad (< 7.5)\end{aligned}$$

In other words, the observed call is overpriced.

- *Arbitrage strategies:* To exploit the arbitrage opportunity, Peter should, *at time 0*, sell the observed call for \$7.5, buy $\Delta_0 = 0.617645$ shares and borrow \$54.498049 for a cost of $C_0 = \$7.266451$, thereby immediately realizing a positive cash inflow of \$0.233549.

  His actions at time 1 depend on the stock price that prevails at that time:

  Case 1. If $S(1) = S_u = 110$, then Peter should buy $\Delta_u - \Delta_0 = 0.3369$ *additional* shares and borrow an *additional* amount of $\$|B_u - B_0(1.02)| = \$37.0590$. No additional investment is required because the cost of $\$0.3369(110) = \$37.0590$ is the same (except for minor rounding errors) as the amount of borrowing.

(Alternatively, Peter can liquidate his position by selling $\Delta_0 = 0.617645$ shares for \$110 each, or for \$67.9410 in total and repaying the loan (with interest) of $\$54.4980(1.02) = \$55.5880$. Subsequently, he can use the \$12.353 he receives to buy $\Delta_u = 0.954545$ shares and borrow $\$|B_u| = \$92.6471$ for one year.)

Case 2. If $S(1) = S_d = 90$, then Peter should completely liquidate his position by selling $\Delta_0 = 0.617645$ shares for \$90 each, or for \$55.5881 in total, and repaying the loan (with interest) of $\$54.4980(1.02) = \$55.5880$. Again, no cash inflow and outflow arises.

In both cases, Peter will have identically zero payoff at time 2 because the payoff of his long synthetic call exactly offsets that of the short observed call. An arbitrage strategy is thus constructed due to the risk-less cash inflow of \$0.233549 at time 0. □

Remark. Peter cannot do nothing at time 0 and wait until time 1 for the exact actions to take, depending on whether the u node or d node is reached. The reason is that we are not sure of the observed call price, which can be greater than, smaller than or equal to the fair call price, at time 1.

17. *Solution.* In a forward tree,

$$\begin{aligned} u &= \exp[(r-\delta)h + \sigma\sqrt{h}] = \exp[(0.06)(0.5) + 0.2\sqrt{0.5}] = 1.186991, \\ d &= \exp[(r-\delta)h - \sigma\sqrt{h}] = \exp[(0.06)(0.5) - 0.2\sqrt{0.5}] = 0.894562. \end{aligned}$$

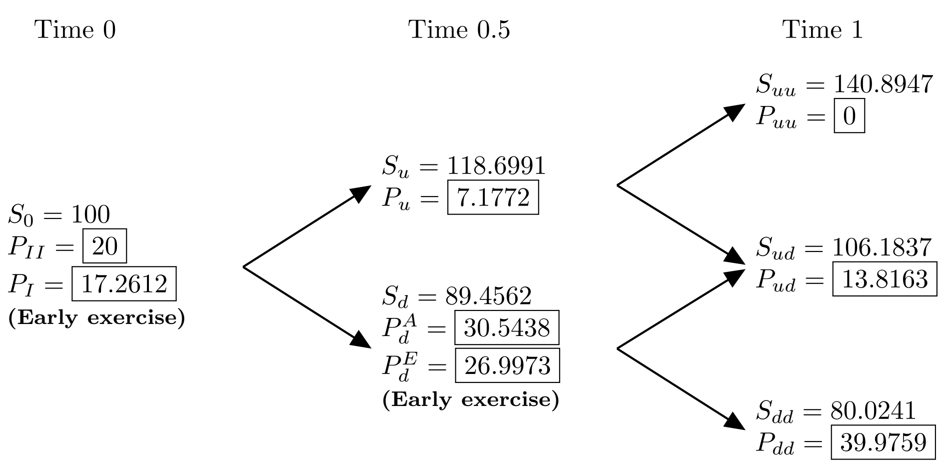
The risk-neutral probability of an up move is

$$p^* = \frac{\mathrm{e}^{(r-\delta)h} - d}{u - d} = \frac{\mathrm{e}^{(0.06)(0.5)} - 0.894562}{1.186991 - 0.894562} = 0.464703,$$

or

$$p^* = \frac{1}{1 + \mathrm{e}^{\sigma\sqrt{h}}} = \frac{1}{1 + \mathrm{e}^{0.2\sqrt{0.5}}} = 0.464703.$$

The stock prices are depicted below:



Note that early exercise is optimal at the d node because

$$\begin{aligned}\text{Holding value} &= \mathrm{e}^{-0.06(0.5)}[p^* P_{ud} + (1-p^*)P_{dd}] \\ &= 26.9973 \\ &< (120 - S_d)_+ \\ &= 30.5438 = \text{Exercise value,}\end{aligned}$$

so that $P_d^A = 30.5438$, as well as at the initial node, because

$$\begin{aligned}\text{Holding value} &= \mathrm{e}^{-0.06(0.5)}[p^* P_u + (1-p^*)P_d] \\ &= 19.1035 \\ &< (120 - S_0)_+ \\ &= 20 = \text{Exercise value.}\end{aligned}$$

Therefore, it pays to exercise the American put right away, with its price being $P_{II} = 20$. Meanwhile, the price of the corresponding European put is

$$P_I = \mathrm{e}^{-0.06(1)}[2p^*(1-p^*)(13.8163) + (1-p^*)^2(39.9759)] = 17.2612.$$

It follows that $P_{II} - P_I = \boxed{2.739}$. □

19. *Solution.* In a forward tree,

$$\begin{aligned}u &= \exp[(r-\delta)h + \sigma\sqrt{h}] = \exp[(0.05 - 0.035)/3 + 0.3\sqrt{1/3}] = 1.195070, \\ d &= \exp[(r-\delta)h - \sigma\sqrt{h}] = \exp[(0.05 - 0.035)/3 - 0.3\sqrt{1/3}] = 0.845180.\end{aligned}$$

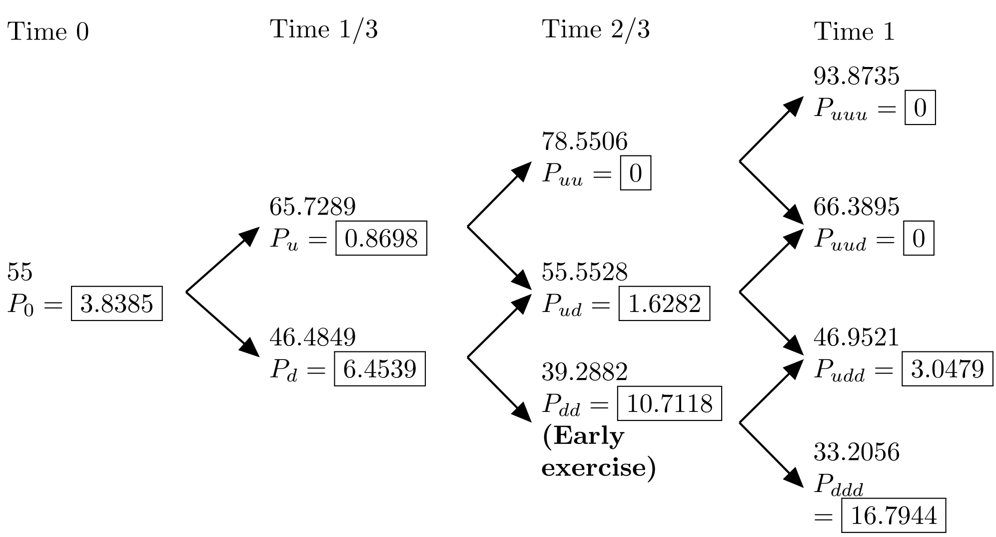
The risk-neutral probability of an up move is

$$p^* = \frac{\mathrm{e}^{(r-\delta)h} - d}{u - d} = \frac{\mathrm{e}^{(0.05-0.035)/3} - 0.845180}{1.195070 - 0.845180} = 0.456807,$$

or

$$p^* = \frac{1}{1 + \mathrm{e}^{\sigma\sqrt{h}}} = \frac{1}{1 + \mathrm{e}^{0.3/\sqrt{3}}} = 0.456807.$$

The stock prices and put prices are depicted below:



Early exercise is optimal at (and only at) the dd node because

$$\begin{aligned}\text{Holding value} &= \mathrm{e}^{-0.05}[p^* P_{udd} + (1-p^*)P_{ddd}] \\ &= 10.3411 \\ &< (50 - S_{dd})_+ \\ &= 10.7118 = \text{Exercise value.}\end{aligned}$$

The time-0 price of the American put is $P_0 = \boxed{3.8385}$. □

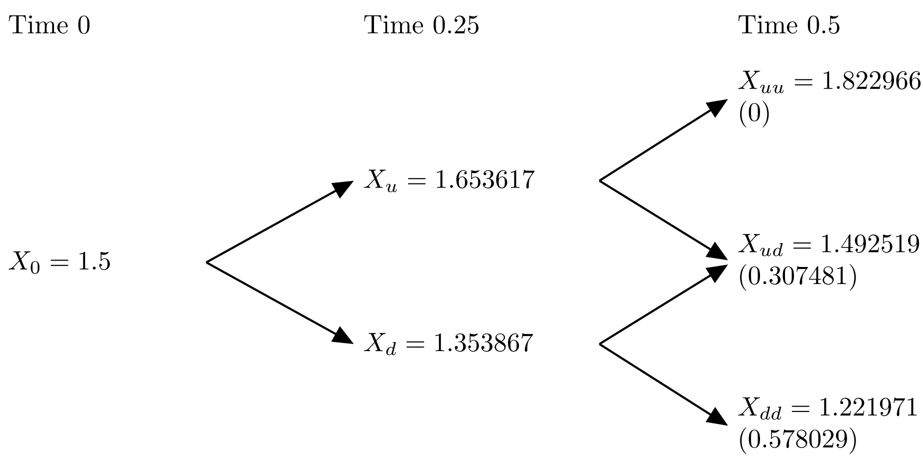
21. *Solution.* The forward tree parameters are

$$\begin{aligned} u &= \mathrm{e}^{(0.04-0.05)(0.25)+0.2\sqrt{0.25}} = 1.102411, \\ d &= \mathrm{e}^{(0.04-0.05)(0.25)-0.2\sqrt{0.25}} = 0.902578. \end{aligned}$$

The risk-neutral probability of an up move in the exchange rate is

$$p^* = \frac{1}{1+\mathrm{e}^{0.2\sqrt{0.25}}} = 0.475021.$$

The evolution of the dollar-euro exchange rate is shown below:



By risk-neutral pricing, the possible 3-month values of the put are

$$\begin{aligned} P_u &= \max\{\mathrm{e}^{-0.04(0.25)}[0.475021(0) + (1-0.475021)(0.307481)], (1.8-1.653617)_+\} \\ &= \max\{0.159815, 0.146383\} \\ &= 0.159815 \end{aligned}$$

and

$$\begin{aligned} P_d &= \max\{\mathrm{e}^{-0.04(0.25)}[0.475021(0.307481) + (1-0.475021)(0.578029)], (1.8-1.353867)_+\} \\ &= \max\{0.445040, 0.446133\} \\ &= 0.446133. \quad \textbf{(early exercise optimal here)} \end{aligned}$$

Finally, the current price of the put is

$$\begin{aligned} P_0 &= \max\{\mathrm{e}^{-0.04(0.25)}[0.475021(0.159815) + (1-0.475021)(0.446133)], (1.8-1.5)_+\} \\ &= \max\{0.307040, 0.3\} \\ &= \boxed{0.3070}. \end{aligned}$$

□

23. *Solution.* Given that $u = 1.2363$ and $d = 0.8089$, the risk-neutral probability of an up move is

$$p^* = \frac{1-d}{u-d} = 0.4471.$$

The evolution of the futures prices is shown below:

Time 0	Time 0.5	Time 1
		764.2324 $P_u = \boxed{0}$
	618.1555 $P_u = \boxed{0}$	
500		500 $P_u = \boxed{0}$
	404.4290 $P_u = \boxed{95.571}$	
		327.1256 $P_u = \boxed{172.8744}$

The possible values of the American put at time $t = 0.5$ are $P_u = 0$ at the u node, and

$$\begin{aligned} P_d &= \max\{e^{-0.06(0.5)}[p^*(0) + (1-p^*)(172.8744)], 500 - 404.4290\} \\ &= \max\{92.7465, 95.571\} \\ &= 95.571 \quad \textbf{(early exercise optimal)} \end{aligned}$$

at the d node. The replicating portfolio at $t = 0$ for the futures put is defined by

$$\Delta = \frac{P_u - P_d}{F_u - F_d} = \frac{0 - 95.571}{618.1555 - 404.4290} = \boxed{-0.4472}$$

and

$$B = e^{-0.06(0.5)}[p^*(0) + (1-p^*)(95.571)] = \boxed{51.27}.$$

□

25. *Solution.* The forward tree parameters are

$$u = e^{0.08/3+0.3/\sqrt{3}} = 1.221246 \quad \text{and} \quad d = e^{0.08/3-0.3/\sqrt{3}} = 0.863693.$$

The risk-neutral probability of an up move is

$$p^* = \frac{1}{1 + e^{0.3/\sqrt{3}}} = 0.456807.$$

As $C_{uu} = 34.5721$, $C_{ud} = 12.7391$ and $C_{dd} = 0$, we have

$$\begin{aligned} C_u &= e^{-0.08/3}[p^*(34.5721) + (1-p^*)(12.7391)] = 22.1149, \\ C_d &= e^{-0.08/3}[p^*(12.7391) + (1-p^*)(0)] = 5.6662. \end{aligned}$$

As the true probability of an up move is

$$p = \frac{e^{0.15/3} - 0.863693}{1.221246 - 0.863693} = 0.524616,$$

the continuously compounded true discount rate for the call option during the first time period, γ, satisfies

$$\mathrm{e}^{-\gamma/3}[pC_u + (1-p)C_d] = \mathrm{e}^{-r/3}[p^*C_u + (1-p^*)C_d],$$

or

$$\begin{aligned}&\mathrm{e}^{-\gamma/3}[0.524616(22.1149) + (1-0.524616)(5.6662)]\\ &\qquad = \mathrm{e}^{-0.08/3}[0.456807(22.1149) + (1-0.456807)(5.6662)],\end{aligned}$$

leading to $\gamma = \boxed{0.3237}$. □

B.5 Chapter 5

1. *Solution.* As
$$\hat{d}_2 = \frac{\ln(100/75) + (0.1 - 0.3^2/2)(0.75)}{0.3\sqrt{0.75}} = 1.26606,$$
the probability that a nine-month 75-strike European call option on the stock will be exercised is $N(1.26606) = \boxed{0.89725}$. □

   Remark. Because you are to find the actual probability, the risk-free interest rate is not used.

3. *Solution.* As $N^{-1}(0.975) = 1.95996$, a 95% lognormal prediction interval for the price of the stock in 3 months is
$$\begin{aligned}&S(0)\exp\left[\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)t \pm N^{-1}(0.975)\sigma\sqrt{t}\right]\\ =\ &100\exp\left[\left(0.15-\frac{1}{2}\times 0.35^2\right)(0.25) \pm 1.95996\times 0.35\sqrt{0.25}\right]\\ =\ &(100\mathrm{e}^{-0.32081}, 100\mathrm{e}^{0.36518})\\ =\ &\boxed{(72.56, 144.08)}.\end{aligned}$$
□

5. *Solution.* The 90% lognormal prediction interval for $S(2)$ is
$$\left(S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)(2)-\sigma\sqrt{2}\times 1.645}, S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)(2)+\sigma\sqrt{2}\times 1.645}\right).$$
The ratio of the upper end-point to the lower end-point gives
$$\mathrm{e}^{2\sigma\sqrt{2}\times 1.645} = \frac{41.9448}{13.1072},$$
which in turn implies that $\sigma = 0.25$. Moreover, the geometric averages of the two endpoints of the prediction intervals for $S(2)$ and $S(4)$ give
$$\begin{cases} S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)(2)} = \sqrt{(13.1072)(41.9448)} = 23.447364\\ S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)(4)} = \sqrt{(25.7923)(183.1083)} = 68.722516\end{cases}$$

resulting in $\alpha - \delta - \sigma^2/2 = 0.537659$ and $S(0) = 8$. As $N^{-1}(0.995) = 2.576$, it follows that the 99% lognormal prediction interval for $S(6)$ is

$$\begin{aligned}
&\left(S(0)e^{(\alpha-\delta-\sigma^2/2)(6)-\sigma\sqrt{6}\times 2.576}, S(0)e^{(\alpha-\delta-\sigma^2/2)(6)+\sigma\sqrt{6}\times 2.576}\right) \\
= &\left(8e^{0.537659(6)-0.25\sqrt{6}\times 2.576}, 8e^{0.537659(6)+0.25\sqrt{6}\times 2.576}\right) \\
= &(41.5927, 975.4188)
\end{aligned}$$

with a width of $975.4188 - 41.5927 = \boxed{934}$. □

7. *Solution.* We are given from (v) that $\mathbb{E}[S(1)] = S(0)e^{(\alpha-\delta)T} = 80e^{\alpha-\delta} = 84.2069$, or $\alpha - \delta = 0.051250$. Then

$$\begin{aligned}
\hat{d}_1 &= \frac{\ln[S(0)/K] + (\alpha - \delta + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(1) + (0.051250 + 0.25^2/2)(1)}{0.25\sqrt{1}} = 0.33, \\
\hat{d}_2 &= \hat{d}_1 - \sigma\sqrt{T} = 0.08 - 0.25\sqrt{1} = 0.08, \\
N(-\hat{d}_1) &= 0.37070, \\
N(-\hat{d}_2) &= 0.46812,
\end{aligned}$$

and the expected 1-year stock price, given that $S(1) < 80$, is

$$\begin{aligned}
\mathbb{E}[S(1)|S(1) < 80] &= S(0)e^{(\alpha-\delta)T}\frac{N(-\hat{d}_1)}{N(-\hat{d}_2)} \\
&= 84.2069 \times \frac{0.37070}{0.46812} \\
&= \boxed{66.6827}.
\end{aligned}$$

□

Remark. The risk-free interest rate is not used at all.

9. *Solution.* From (i), (ii), and (iii), we have:

(i) $N(-\hat{d}_2) = 0.512$

(ii) $S(0)e^{(\alpha-\delta)T} = 10.134$

(iii) $S(0)e^{(\alpha-\delta)T}\dfrac{N(-\hat{d}_1)}{N(-\hat{d}_2)} = 8.483$

Thus $N(-\hat{d}_1) = 8.483(0.512)/10.134 = 0.4286$. Then

$$\begin{aligned}
\hat{d}_1 &= -N^{-1}(0.4286) = -(-0.18) = 0.18, \\
\hat{d}_2 &= -N^{-1}(0.512) = -0.03, \\
\sigma &= \frac{\hat{d}_1 - \hat{d}_2}{\sqrt{T}} = \frac{0.18 - (-0.03)}{\sqrt{2/3}} = 0.257196.
\end{aligned}$$

The 98% lognormal prediction interval for the 8-month stock price $S(2/3)$ is

$$\begin{aligned}
S(0)e^{(\alpha-\delta-\sigma^2/2)T \pm \sigma\sqrt{T}N^{-1}(0.99)} &= 10.134e^{-(0.257196^2/2)(2/3)\pm 0.257196\sqrt{2/3}(2.326)} \\
&= \boxed{(6.08, 16.16)}.
\end{aligned}$$

□

B.6 Chapter 6

1. *Solution.* To insure against a reduction of more than 40% in the value of the equity index fund at the end of one year, the strike price of the put option should be set to $1,000(1-40\%) = 600$. As

$$\begin{aligned} d_1 &= \frac{\ln(1,000/600) + (0.025 - 0.02 + 0.2^2/2)(1)}{0.2\sqrt{1}} = 2.67913, \\ d_2 &= d_1 - 0.2\sqrt{1} = 2.47913, \\ N(-d_1) &= 0.00369, \\ N(-d_2) &= 0.00659, \end{aligned}$$

the price of the put option is given by

$$\begin{aligned} P &= Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1) \\ &= 600e^{-0.025(1)}(0.00659) - 1,000e^{-0.02}(0.00369) \\ &= \boxed{0.23944}. \end{aligned}$$

□

3. *Solution.* Note that the 3-year payoff of the contingent claim can be written as

$$\begin{aligned} \text{Payoff} &= \begin{cases} 0, & \text{if } S(3) < 45 \\ S(3) - 45, & \text{if } 45 \le S(3) < 75 \\ 2(S(3) - 60), & \text{if } 75 \le S(3) \end{cases} \\ &= (S(3) - 45)_+ + (S(3) - 75)_+, \end{aligned}$$

which is the sum of the payoff of a 45-strike long call and that of a 75-strike long call.

- At $K = 45$,

$$\begin{aligned} d_1 &= \frac{\ln(45/45) + (0.06 - 0.03 + 0.2^2/2)(3)}{0.2\sqrt{3}} = 0.43301, \\ d_2 &= d_1 - 0.2\sqrt{3} = 0.08660, \\ N(d_1) &= 0.66750, \\ N(d_2) &= 0.53451, \\ C &= 45e^{-0.03(3)}(0.66750) - 45e^{-0.06(3)}(0.53451) = 7.36150. \end{aligned}$$

- At $K = 75$,

$$\begin{aligned} d_1 &= \frac{\ln(45/75) + (0.06 - 0.03 + 0.2^2/2)(3)}{0.2\sqrt{3}} = -1.04161, \\ d_2 &= d_1 - 0.2\sqrt{3} = -1.38802, \\ N(d_1) &= 0.14880, \\ N(d_2) &= 0.08257, \\ C &= 45e^{-0.03(3)}(0.14880) - 75e^{-0.06(3)}(0.08257) = 0.94706. \end{aligned}$$

The price of the contingent claim is $V = 7.36150 + 0.94706 = \boxed{8.3086}$. □

5. *Solution.* The 9-month prepaid forward price of the stock is

$$F^P_{0,0.75}(S) = 50 - 2e^{-0.05(0.25)} = 48.0248444.$$

With

$$\begin{aligned} d_1 &= \frac{\ln(48.0248444/50e^{-0.05(0.75)}) + (0.3^2/2)(0.75)}{0.3\sqrt{0.75}} = 0.11911, \\ d_2 &= d_1 - 0.3\sqrt{0.75} = -0.14070, \\ N(-d_1) &= 0.45259, \\ N(-d_2) &= 0.55595, \end{aligned}$$

the price of the put option is

$$P = 50e^{-0.05(0.75)}(0.55595) - 48.0248444(0.45259) = \boxed{5.0388}.$$

To price this put option in the Black-Scholes framework, it is assumed that the prepaid forward prices of the stock are lognormally distributed, with σ being the volatility of the prepaid forward price, or the standard deviation per unit time of the natural logarithm of the prepaid forward price. Mathematically,

$$\sigma^2 t = \text{Var}\{\ln[F^P_{t,0.75}(S)]\} \quad \text{or} \quad \sigma = \sqrt{\frac{\text{Var}\{\ln[F^P_{t,0.75}(S)]\}}{t}}, \quad 0 < t \leq 0.75.$$

□

7. *Solution.* The 45-55 put bear spread is constructed by combining a short 45-strike put with a long 55-strike put. The 9-month prepaid forward price of the stock is

$$F^P_{0,0.75}(S) = 50 - 2.5(e^{-0.08(2/12)} + e^{-0.08(5/12)}) = 45.115072.$$

At the strike price of 45,

$$\begin{aligned} d_1 &= \frac{\ln(45.115072/45e^{-0.08(0.75)}) + (0.4^2/2)(0.75)}{0.4\sqrt{0.75}} = 0.35378, \\ d_2 &= d_1 - 0.4\sqrt{0.75} = 0.00737, \\ N(-d_1) &= 0.36317, \\ N(-d_2) &= 0.49601, \end{aligned}$$

and the price of the 45-strike put is

$$P^{\text{45-strike}} = 45e^{-0.08(0.75)}(0.49601) - 45.115072(0.36317) = 4.63617.$$

At the strike price of 55,

$$\begin{aligned} d_1 &= \frac{\ln(45.115072/55e^{-0.08(0.75)}) + (0.4^2/2)(0.75)}{0.4\sqrt{0.75}} = -0.22550, \\ d_2 &= d_1 - 0.4\sqrt{0.75} = -0.57191, \\ N(-d_1) &= 0.58920, \\ N(-d_2) &= 0.71631, \end{aligned}$$

and the price of the 55-strike put is

$$P^{\text{55-strike}} = 55e^{-0.08(0.75)}(0.71631) - 45.115072(0.58920) = 10.52094.$$

Finally, the price of the 45-55 put bear spread is $V = 10.52094 - 4.63617 = \boxed{5.8848}$. □

9. *Solution.* Observe that the 3-month payoff of the contingent claim, which is a cap, is parallel to that of a long 50-strike put. Therefore, the two derivatives must possess the same profit.

   The 3-month prepaid forward price of the stock is

$$F^P_{0,0.25}(S) = 50 - 1.5\mathrm{e}^{-0.1(2/12)} = 48.524793.$$

   With

$$\begin{aligned} d_1 &= \frac{\ln(48.524793/50) + (0.1 + 0.3^2/2)(0.25)}{0.3\sqrt{0.25}} = 0.04201, \\ d_2 &= d_1 - 0.3\sqrt{0.25} = -0.10799, \\ N(-d_1) &= 0.4840, \\ N(-d_2) &= 0.5438, \end{aligned}$$

   the price of the 3-month 50-strike put option is

$$P = 50\mathrm{e}^{-0.1(0.25)}(0.5438) - 48.524793(0.4840) = 3.0327.$$

   When $S(0.25) = 45$, the payoff of the put is $(50 - 45)_+ = 5$. Then the profit of the contingent claim, equal to that of the put, is $5 - 3.0327\mathrm{e}^{0.1(0.25)} = \boxed{1.8905}$. □

11. *Solution.* Let $X(t)$ be the time-t exchange rate of US dollars per British pound. In three months, you will pay 200,000 pounds, or $\$200,000X(0.25)$, but you will only have $\$320,000$. You are therefore worried about $X(0.25)$ being too high (to be precise, higher than 1.6). To cover the shortfall of

$$\$[200,000X(0.25) - 320,000]_+ = 200,000 \underbrace{\$[X(0.25) - 1.6]_+}_{\text{payoff of a currency call}},$$

   you can buy 200,000 \$1.6-strike dollar-denominated currency pound *call* options. As

$$\begin{aligned} d_1 &= \frac{\ln(1.6/1.6) + (0.01 - 0.02 + 0.2^2/2)(0.25)}{0.2\sqrt{0.25}} = 0.025, \\ d_2 &= d_1 - 0.2\sqrt{0.25} = -0.075, \\ N(d_1) &= 0.50997, \\ N(d_2) &= 0.47011, \end{aligned}$$

   the price of each currency call option is

$$C = 1.6\mathrm{e}^{-0.02(0.25)}(0.50997) - 1.6\mathrm{e}^{-0.01(0.25)}(0.47011) = 0.061585.$$

   The total cost of the currency options is $200,000(0.061585) = \boxed{12,316.90}$. □

   Remark. (i) Together with the long pound calls, you have set up a *cap*.

   (ii) This problem is dual to Example 6.2.4.

   - In Example 6.2.4, you are to *receive* an amount (therefore long) in the foreign currency, and worry about the exchange rate going *down*. A put option can hedge against the downside risk.
   - In the current problem, you are to *pay* an amount (therefore short) in the foreign currency, and worry about the exchange rate going *up*. This time a call option can hedge against the upside risk.

13. *Solution.* (a) Note that euro is the foreign (underlying) currency while dollar is the domestic (strike) currency. With $r = 0.04$, $\delta = 0.05$,

$$\begin{aligned} d_1 &= \frac{\ln(1.5/1.5) + (0.04 - 0.05 + 0.2^2/2)(0.5)}{0.2\sqrt{0.5}} = 0.03536, \\ d_2 &= d_1 - 0.2\sqrt{0.5} = -0.10607, \\ N(-d_1) &= 0.48590, \\ N(-d_2) &= 0.54224, \end{aligned}$$

the price of the euro put option is

$$P = 1.5\mathrm{e}^{-0.04(0.5)}(0.54224) - 1.5\mathrm{e}^{-0.05(0.5)}(0.48590) = \boxed{0.08640}.$$

(b) (1) No. Apple will need to pay an amount in euro in 6 months and is therefore short with respect to the 6-month dollar/euro exchange rate. He needs a long call on euros to hedge against his upside exchange rate risk.

(2) Yes. Ambrosio will receive a fixed amount in euro in 6 months and is thus long with respect to the 6-month dollar/euro exchange rate. The currency put in question allows him to hedge against the downside exchange rate risk he faces by setting up a floor.

(c) As

$$\hat{d}_2 = \frac{\ln(1.5/1.5) + (\overbrace{0.015}^{\text{exchange rate appreciation}} - 0.2^2/2)(0.5)}{0.2\sqrt{0.5}} = -0.01768,$$

the true probability that the put option will be exercised is $N(-\hat{d}_2) = \boxed{0.50705}$.

□

15. *Solution.* Note that the underlying futures contract matures three years from now. From the table in (iv), it has a current price of 563.75 and a volatility of 30%. With

$$\begin{aligned} d_1 &= \frac{\ln(563.75/550) + (0.3^2/2)(\overbrace{2}^{\text{lifespan of option}})}{0.3\sqrt{2}} = 0.27033, \\ d_2 &= d_1 - 0.3\sqrt{2} = -0.15393, \\ N(-d_1) &= 0.39345, \\ N(-d_2) &= 0.56117, \end{aligned}$$

the current price of the required put option is

$$P = 550\mathrm{e}^{-0.06(2)}(0.5596) - 563.75\mathrm{e}^{-0.06(2)}(0.3936) = \boxed{77.02}.$$

□

17. *Solution.* We are given in (ii) that $\Delta_P = -\mathrm{e}^{-0.03(0.5)}N(-d_1) = -0.3382$, or $N(-d_1) = 0.34331$. Then $d_1 = -N^{-1}(0.34331) = 0.40345$. Because

$$d_1 = \frac{\ln(95/90) + (\sigma^2/2)(0.5)}{\sigma\sqrt{0.5}} = 0.40345,$$

or

$$0.25\sigma^2 - 0.40345\sqrt{0.5}\sigma + \ln(95/90) = 0,$$

we have

$$\sigma = \frac{0.40345\sqrt{0.5} \pm \sqrt{0.40345^2(0.5) - 4(0.25)(\ln(95/90))}}{2(0.25)} = \underbrace{0.901132}_{\text{rejected}} \text{ or } 0.24.$$

Finally, the gamma of the futures put is

$$\begin{aligned} \Gamma &= e^{-rT} \times N'(d_1) \times \frac{1}{F_0\sigma\sqrt{T}} \\ &= e^{-0.03(0.5)} \times \frac{1}{\sqrt{2\pi}} e^{-0.40345^2/2} \times \frac{1}{95(0.24)\sqrt{0.5}} \\ &= \boxed{0.0225}. \end{aligned}$$

□

19. The correct graph is **(B)**. Delta of a (plain vanilla) European call is always bounded between 0 and 1. This eliminates (A), (C), and (E).

Solution 1 (Verbal explanations). A short-term deep out-the-money call option will almost surely end up out-of-the-money and is very insensitive to stock price movements, hence a delta of approximately 0. As the time to maturity increases, the call has a higher probability of ending up in-the-money. The increase in the sensitivity to stock price movements translates into an increase in delta (although it remains bounded from above by 1). □

Solution 2 (For math geeks). Mathematically, consider the partial derivative

$$\begin{aligned} \frac{\partial d_1}{\partial T} &= -\frac{\ln(S/K)}{2\sigma T^{3/2}} + \frac{r + \sigma^2/2}{2\sigma\sqrt{T}} \qquad \text{(B.6.1)} \\ &= \frac{\overbrace{-\ln(S/K)}^{>0} + (r + \sigma^2/2)T}{2\sigma T^{3/2}} \\ &> 0, \quad \text{if } S \ll K \text{ (i.e., deep out-of-the-money).} \end{aligned}$$

Then

$$\frac{\partial \Delta_C}{\partial T} = \frac{\partial}{\partial T} N(d_1) = N'(d_1)\frac{\partial d_1}{\partial T} > 0,$$

which means that Δ_C as a function of T is increasing. □

21. *Solution.* (a) The price of a European call option will decrease as K increases.
   (b) The payoff of a European call option is $(S(T) - K)_+$, which decreases with the strike price K (i.e., the call becomes less attractive). A lower payoff means a lower call price.
   (c) Mathematically, we examine the partial derivative of the European call price with respect to the strike price K (there is no Greek defined for this partial derivative):

$$\begin{aligned} \frac{\partial C}{\partial K} &= F_{0,T}^P(S)N'(d_1)\frac{\partial d_1}{\partial K} - e^{-rT}N(d_2) - Ke^{-rT}N'(d_2)\underbrace{\frac{\partial d_2}{\partial K}}_{=\partial d_1/\partial K} \\ &\overset{(6.3.2)}{=} -e^{-rT}N(d_2), \end{aligned}$$

which is negative. Therefore, the call price is a decreasing function of the strike price.

□

23. *Solution.* The gamma of the bull spread equals

$$\Gamma_{\text{bull spread}} = \Gamma^{\text{50-strike}} - \Gamma^{\text{60-strike}}.$$

Initially:

- For the 50-strike call,

$$\begin{aligned} d_1 &= \frac{\ln(50/50) + (0.05 + 0.2^2/2)(0.25)}{0.2\sqrt{0.25}} = 0.175, \\ \Gamma^{\text{50-strike}} &= \mathrm{e}^{-\delta T} \times N'(d_1) \times \frac{1}{S\sigma\sqrt{T}} \\ &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-0.175^2/2} \times \frac{1}{50(0.2)\sqrt{0.25}} = 0.07858. \end{aligned}$$

- For the 60-strike call,

$$\begin{aligned} d_1 &= \frac{\ln(50/60) + (0.05 + 0.2^2/2)(0.25)}{0.2\sqrt{0.25}} = -1.64822, \\ \Gamma^{\text{60-strike}} &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(-1.64822)^2/2} \times \frac{1}{50(0.2)\sqrt{0.25}} = 0.02051. \end{aligned}$$

The original gamma is thus $0.07858 - 0.02051 = 0.05807$.

After 1 month (the remaining time to expiration is 2 months):

- For the 50-strike call,

$$\begin{aligned} d_1 &= \frac{\ln(50/50) + (0.05 + 0.2^2/2)/6}{0.2\sqrt{1/6}} = 0.14289, \\ \Gamma^{\text{50-strike}} &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-0.14289^2/2} \times \frac{1}{50(0.2)\sqrt{1/6}} = 0.09673. \end{aligned}$$

- For the 60-strike call,

$$\begin{aligned} d_1 &= \frac{\ln(50/60) + (0.05 + 0.2^2/2)/6}{0.2\sqrt{1/6}} = -2.09009, \\ \Gamma^{\text{60-strike}} &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(-2.09009)^2/2} \times \frac{1}{50(0.2)\sqrt{1/6}} = 0.01100. \end{aligned}$$

The new gamma is $0.09673 - 0.01100 = 0.08573$. Therefore, the change in gamma is $0.08573 - 0.05807 = \boxed{0.02766}$. □

25. *Solution.* As

$$\begin{aligned} d_1 &= \frac{\ln(82/80) + (0.08 - 0.03 + 0.3^2/2)(0.25)}{0.3\sqrt{0.25}} = 0.32295, \\ d_2 &= d_1 - 0.3\sqrt{0.25} = 0.17295, \\ N(d_1) &= 0.62663, \\ N(d_2) &= 0.56865, \end{aligned}$$

the price of the call option is

$$C = 82e^{-0.03(0.25)}(0.62663) - 80e^{-0.08(0.25)}(0.56865) = 6.40853,$$

so its elasticity is

$$\Omega_C = \frac{S\Delta}{C} = \frac{82(e^{-0.03(0.25)})(0.62663)}{6.40853} = \boxed{7.95811}.$$

□

27. *Solution.* The straddle is constructed by a long 3-month 32-strike European call and a long otherwise identical European put. We first compute

$$\begin{aligned}
d_1 &= \frac{\ln(30/32) + (0.05 - 0.02 + 0.3^2/2)(0.25)}{0.3\sqrt{0.25}} = -0.30526, \\
d_2 &= d_1 - 0.3\sqrt{0.25} = -0.45526, \\
N(d_1) &= 0.38008, \\
N(-d_1) &= 0.61992, \\
N(d_2) &= 0.32446, \\
N(-d_2) &= 0.67554.
\end{aligned}$$

The current prices of the call and put are respectively

$$C = 30e^{-0.02(0.25)}(0.38008) - 32e^{-0.05(0.25)}(0.32446) = 1.09179$$

and

$$P = 32e^{-0.05(0.25)}(0.67554) - 30e^{-0.02(0.25)}(0.61992) = 2.84390.$$

Their current deltas are respectively

$$\Delta_C = e^{-0.02(0.25)}(0.38008) = 0.37818$$

and

$$\Delta_P = -e^{-0.02(0.25)}(0.61992) = -0.61683.$$

Therefore, the current elasticity of the straddle is

$$\Omega_{\text{straddle}} = \frac{S\Delta_{\text{straddle}}}{V_{\text{straddle}}} = \frac{S(\Delta_C + \Delta_P)}{C + P} = \frac{30[0.37818 + (-0.61683)]}{1.09179 + 2.84390} = \boxed{-1.81912}.$$

□

29. *Solution.* The contingent claim is composed of a long 9-month zero-coupon bond of face value 20 plus one 9-month at-the-money European call. With

$$\begin{aligned}
d_1 &= \frac{\ln(1) + (0.06 - 0.02 + 0.35^2/2)(0.75)}{0.35\sqrt{0.75}} = 0.25053, \\
N'(d_1) &= \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2} = 0.38662,
\end{aligned}$$

the current gamma of the contingent claim equals

$$\Gamma = e^{-\delta T}N'(d_1)\frac{1}{S\sigma\sqrt{T}} = e^{-0.02(0.75)}(0.38662)\frac{1}{S(0.35)\sqrt{0.75}} = 0.0314,$$

which gives $S = 40.0167$. With $N(d_1) = 0.59891$, $d_2 = d_1 - 0.35\sqrt{0.75} = -0.05258$, and $N(d_2) = 0.47903$, the time-0 price and time-0 delta of the contingent claim are, respectively,

$$\begin{aligned} V &= 20\mathrm{e}^{-rT} + C \\ &= 20\mathrm{e}^{-0.06(0.75)} + \underbrace{[40.0167\mathrm{e}^{-0.02(0.75)}(0.59891) - 40.0167\mathrm{e}^{-0.06(0.75)}(0.47903)]}_{5.28388} \\ &= 24.40383 \end{aligned}$$

and

$$\Delta_V = \Delta_C = \mathrm{e}^{-0.02(0.75)}(0.59891) = 0.58999.$$

It follows that the time-0 contingent-claim elasticity is

$$\Omega_V = \frac{S\Delta_V}{V} = \frac{40(0.58999)}{24.40383} = \boxed{0.9670}.$$

□

B.7 Chapter 7

1. *Solution.* For simplicity, we let four months ago be time 0 and the current time be time 1/3.

   (a) We use the current delta (the remaining time to expiration is 8 months) to deduce the volatility of the stock:

   $$d_1 = \frac{\ln(50/45) + (0.05 + \sigma^2/2)\overbrace{(2/3)}^{\text{not 1!}}}{\sigma\sqrt{2/3}} = N^{-1}(0.73507) = 0.62822.$$

   The solutions to this quadratic equation are $\sigma = 1.19$ and $\sigma = 0.35$. We take $\sigma = \boxed{0.35}$ as it is the only value less than 50%.

   (b) We also need the delta of the call 4 months ago and the current price of the call.

   - Four months ago,

     $$\begin{aligned} d_1(0) &= \frac{\ln(40/45) + (0.05 + 0.35^2/2)(1)}{0.35\sqrt{1}} = -0.01867, \\ \Delta_C(0) &= N(d_1) = 0.49255. \end{aligned}$$

     To delta-hedge his position, four months ago Eric should have *sold* $100(0.49255) = 49.255$ shares of the stock.
   - Currently, with

     $$\begin{aligned} d_2(1/3) &= d_1(1/3) - 0.35\sqrt{2/3} = 0.34245, \\ N(d_2(1/3)) &= 0.63399, \end{aligned}$$

     the call price is

$$C(1/3) \quad = \quad 50(0.73507) - 45e^{-0.05(2/3)}(0.63399) = 9.15926.$$

Then Eric's four-month holding profit is

$$\begin{aligned} & 100\{[C(1/3) - \Delta_C(0)S(1/3)] - e^{r/3}[C(0) - \Delta_C(0)S(0)]\} \\ = \; & 100\{[9.15926 - 0.49255(50)] - e^{0.05(1/3)}[4.45539 - 0.49255(40)]\} \\ = \; & \boxed{3.46}. \end{aligned}$$

□

3. *Solution.* We are told in (ii) that $\Delta_C = 750/1,000 = 0.75$. Since $\Delta_C = e^{-\delta T}N(d_1) = N(d_1)$, we have

$$d_1 = \frac{\ln(1) + (0.07 + \sigma^2/2)(1)}{\sigma\sqrt{1}} = N^{-1}(0.75) = 0.67449,$$

or

$$0.5\sigma^2 - 0.67449\sigma + 0.07 = 0.$$

Then

$$\sigma = \frac{0.67449 \pm \sqrt{(-0.67449)^2 - 4(0.5)(0.07)}}{2(0.5)} = 1.235682 \text{ or } 0.11330.$$

As $\sigma < 1$, we take $\sigma = 0.11330$. Then $d_2 = d_1 - \sigma\sqrt{T} = 0.67449 - 0.11330\sqrt{1} = 0.56119$ and $N(d_2) = 0.71267$, and the price of each call option is

$$C = 80(0.75) - 80e^{-0.07(1)}(0.71267) = \boxed{6.84}.$$

□

5. *Solution.* By put-call parity, the delta and theta of the put option are, respectively,

$$\begin{aligned} \Delta_P &= \Delta_C - 1 = -0.4579, \\ \theta_P &= \theta_C + rKe^{-rT} = -2.41519 + 0.02(20)e^{-0.02(0.25)} = -2.01719. \end{aligned}$$

Let x be the number of units of the put to *buy*. To maintain delta-neutrality, we have to solve the equation

$$-1000(0.54210) - 0.4579x = 0,$$

resulting in $x = -1,183.8829$. In other words, $1,183.8829$ units of the put should be *sold.*

Finally, the theta of the overall position is

$$\theta_V = -1000(-2.41519) - 1,183.8829(-2.01719) = \boxed{4,803}.$$

□

Remark. (i) To relate the call theta to the put theta, we differentiate both sides of put-call parity and *don't forget to negate*:

$$\theta_C - \theta_P = \boxed{-}\frac{\partial}{\partial T}(C - P) = \boxed{-}\frac{\partial}{\partial T}[S(0) - Ke^{-rT}] = -rKe^{-rT}.$$

(ii) That the interest rate is 2% can be deduced from the Black-Scholes equation.

7. *Solution.* For the 60-strike call,

$$\begin{aligned}
d_1 &= \frac{\ln(60/60) + (0.05 + 0.30^2/2)(1)}{0.30\sqrt{1}} = 0.31667, \\
\Delta_C^{\text{60-strike}} &= N(d_1) = 0.62425, \\
\Gamma_C^{\text{60-strike}} &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-d_1^2/2} \times \frac{1}{S\sigma\sqrt{T}} \\
&= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-0.31667^2/2} \times \frac{1}{60(0.30)\sqrt{1}} \\
&= 0.02108.
\end{aligned}$$

For the 65-strike put,

$$\begin{aligned}
d_1 &= \frac{\ln(60/65) + (0.05 + 0.30^2/2)(1)}{0.30\sqrt{1}} = 0.04986, \\
\Delta_P^{\text{65-strike}} &= -N(-d_1) = -0.48012, \\
\Gamma_P^{\text{65-strike}} &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-d_1^2/2} \times \frac{1}{S\sigma\sqrt{T}} \\
&= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-0.04986^2/2} \times \frac{1}{60(0.30)\sqrt{1}} \\
&= 0.02214.
\end{aligned}$$

After buying 200 units of the 60-strike call, our gamma is $200\Gamma_C^{\text{60-strike}} = 4.216$. To gamma-hedge our position, we should $\boxed{\text{sell}}$ $4.216/0.02214 = \boxed{190.42}$ 65-strike puts. The overall delta at this point is

$$200\Delta_C^{\text{60-strike}} - 190.42\Delta_P^{\text{65-strike}} = 216.52,$$

so we should also $\boxed{\text{sell } 216.52}$ shares of the stock to make our position both delta-neutral and gamma-neutral. □

Remark. There is no need to compute the two option prices.

9. *Solution.* (a) For the 50-strike call,

$$\begin{aligned}
d_1 &= \frac{\ln(50/50) + (0.05 + 0.25^2/2)(1)}{0.25\sqrt{1}} = 0.325, \\
\Delta^{\text{50-strike}} &= N(d_1) = 0.62741, \\
\Gamma^{\text{50-strike}} &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-d_1^2/2} \times \frac{1}{S\sigma\sqrt{T}} \\
&= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-0.325^2/2} \times \frac{1}{50(0.25)\sqrt{1}} \\
&= 0.030274.
\end{aligned}$$

For the 60-strike call,

$$\begin{aligned}
d_1 &= \frac{\ln(50/60) + (0.05 + 0.25^2/2)(1)}{0.25\sqrt{1}} = -0.40429, \\
\Delta^{\text{60-strike}} &= N(d_1) = 0.34300, \\
\Gamma^{\text{60-strike}} &= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-d_1^2/2} \times \frac{1}{S\sigma\sqrt{T}} \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(-0.40429)^2/2} \times \frac{1}{50(0.25)\sqrt{1}} \\
&= 0.02941.
\end{aligned}$$

After selling 1,000 50-strike calls, our gamma is $-1,000\Gamma^{\text{50-strike}} = -30.274$. To gamma-hedge our position, we should $\boxed{\text{buy}}$ $30.274/0.02941 = \boxed{1,029.38}$ 60-strike calls. The overall delta at this point is

$$-1,000\Delta^{\text{50-strike}} + 1,029.38\Delta^{\text{60-strike}} = -274.33,$$

so should also $\boxed{\text{buy } 274.33}$ shares for delta-hedging.

(b) Our initial investment as a result of the delta-gamma-hedged call position is

$$274.33(50) + 1,029.38(2.5127) - 1,000(6.1680) = 10,135.02313.$$

The payoff in one month is

$$274.33(50) + 1,029.38(2.2591) - 1,000(5.8611) = 10,180.87236.$$

Thus the one-month holding profit is

$$10,180.87236 - 10,135.02313\mathrm{e}^{0.05/12} = \boxed{3.53187}.$$

□

11. *Solution.* With $\epsilon = 10.50 - 10 = 0.5$, the estimated new call price is

$$\begin{aligned}
C + \Delta\epsilon + \frac{1}{2}\Gamma\epsilon^2 &= 2 + (0.6)(0.5) + \frac{1}{2}(0.2)(0.5)^2 \\
&= \boxed{2.325}.
\end{aligned}$$

□

13. *Solution.* In this problem, we deduce from the two approximations the values of the delta and gamma of the futures put, which in turn allow us to infer the values of its initial price and volatility. Exact Black-Scholes calculations can then be performed.

Using the delta approximation, we have $4.148 = \Delta_P\epsilon = -10\Delta_P$, or $\Delta_P = -0.4148$. Because $\Delta_P = -\mathrm{e}^{-\delta T}N(-d_1) = -\mathrm{e}^{-0.08(0.75)}N(-d_1)$, we get $N(-d_1) = 0.4404$. Then $d_1 = -N^{-1}(0.4404) = 0.15$. As the futures put is at-the-money,

$$d_1 = \frac{\ln(1) + (\sigma^2/2)T}{\sigma\sqrt{T}} = \frac{1}{2}\sigma\sqrt{0.75} = 0.15,$$

leading to $\sigma = 0.346410$.

Turning to the delta-gamma approximation, we now have

$$4.231 = \Delta_P \epsilon + \frac{1}{2}\Gamma_P \epsilon^2 = 4.148 + \frac{1}{2}\Gamma_P(-10)^2,$$

which gives $\Gamma_P = 0.00166$. Using the gamma formula,

$$\Gamma_P = \mathrm{e}^{-0.08(0.75)} \times \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-0.15^2/2} \times \frac{1}{F_0(0.346410)\sqrt{0.75}} = 0.00166,$$

so the *initial* futures price is $F = 745.9976$.

We are now ready for the exact calculation of the new price of the futures put. With

$$\begin{aligned}
d_1 &= \frac{\ln(\overbrace{735.9976}^{\text{drops by 10}}/745.9976) + (0.346410^2/2)(0.75)}{0.346410\sqrt{0.75}} = 0.10501,\\
d_2 &= d_1 - 0.346410\sqrt{0.75} = -0.19499,\\
N(-d_1) &= 0.45818,\\
N(-d_2) &= 0.57730,
\end{aligned}$$

the exact put price when the futures price decreases to 735.9976 is

$$\begin{aligned}
P &= \mathrm{e}^{-rT}[KN(-d_2) - F_0^{\text{new}}N(-d_1)]\\
&= \mathrm{e}^{-0.08(0.75)}[\underbrace{745.9976}_{\text{initial futures price}}(0.57730) - \underbrace{735.9976}_{\text{new futures price}}(0.45818)]\\
&= \boxed{88.00}.
\end{aligned}$$

□

15. *Solution.* We need the current delta of Put A. By the Black-Scholes equation,

$$0.05(70)(-0.2867) + \frac{1}{2}(0.25)^2(70)^2(0.0112) + \theta = 0.05(6.9389),$$

which gives $\theta = -0.364605$.

Let x and y be, respectively, the number of units of Put B and the stock to *buy*. To maintain delta-neutrality and theta-neutrality, we are prompted to solve the following system of two equations:

$$\begin{cases} -1000(-0.2867) - 0.3433x + y = 0 & \text{(delta-neutrality)}\\ -1000(-0.364605) - 0.2060x = 0 & \text{(theta-neutrality)} \end{cases}.$$

The second equation implies that $x = 1,769.9272$. Then it follows from the first equation that $y = 320.9077$. In other words, 1769.9029 units of Put B and 320.9160 shares of stock should be bought.

The net investment is

$$1,769.9272\underbrace{(9.0062)}_{\text{Put B's price}} + 320.9160\underbrace{(70)}_{\text{stock price}} - 1000\underbrace{(6.9389)}_{\text{Put A's price}} = \boxed{31,466}.$$

□

17. *Solution.* From the current elasticity, we have

$$5.4417 = \Omega_C(1) = \frac{S(1)\Delta_C(1)}{C(1)} = \frac{S(0.7296)}{5.7653} \quad \Rightarrow \quad S(1) = 43.$$

Furthermore, the Black-Scholes equation applied to the "Now" column says that

$$5.7653r = -2.1911 + r(43)(0.7296) + \frac{1}{2}(0.2)^2(43)^2(0.0385),$$

which gives $r = 0.03$.

Back to one year ago, another application of the Black-Scholes equation yields

$$5.6295(0.03) = -1.6569 + 0.03(40)\Delta + \frac{1}{2}(0.2)^2(40)^2(0.0331),$$

resulting in $\Delta = 0.6388$. By buying 1,000 units of the call, you incurred a delta of $1,000(0.6388) = 638.8$. To neutralize delta, we should have sold 638.8 shares of the stock one year ago.

To conclude, the 1-year holding profit is

$$\begin{aligned}&[-638.8(43) + 1,000(5.7653)] - \mathrm{e}^{0.03}[-638.8(40) + 1,000(5.6295)] \\ &\qquad = -21,703.1 - \mathrm{e}^{0.03}(-19,922.5) = \boxed{-1,174}.\end{aligned}$$

□

19. *Solution.* Because the delta three months ago is positive while the current delta is negative, the "Three months ago" column is for the otherwise identical call and the "Now" column is for the put. An application of the Black-Scholes equation to the values three months ago yields

$$5.2121r = -4.2164 + r(50)(0.5129) + \frac{1}{2}(0.3)^2(50)^2(0.0266),$$

which gives $r = 0.06$. By another application of the Black-Scholes equation to the current values, we have

$$0.06P = -2.1069 + 0.06(55)(-0.3809) + \frac{1}{2}(0.3)^2(55)^2(0.0267),$$

leading to $P = 4.511125$. Furthermore, by put-call parity, the price and delta of the put three months ago are respectively

$$P = C - S + K\mathrm{e}^{-rT} = 5.2121 - 50 + 55\mathrm{e}^{-0.06} = 7.009149$$

and

$$\Delta_P = \Delta_C - 1 = 0.5129 - 1 = -0.4871.$$

By selling 1,000 units of the put, you incurred a delta of $-1,000(-0.4871) = 487.1$. To neutralize delta, we should have sold 487.1 shares of the stock three months ago.

The 3-month holding profit is

$$\begin{aligned}&[-487.1(55) - 1,000(4.511125)] - \mathrm{e}^{0.06(0.25)}[-487.1(50) - 1,000(7.009149)] \\ &\qquad = -31,301.625 - \mathrm{e}^{0.06(0.25)}(-31,364.149) = \boxed{-563.53}.\end{aligned}$$

□

B.8 Chapter 8

1. *Solution.* Note that $\sigma = \sqrt{0.09} = 0.3$ (not 0.09). By (8.1.5),

$$d_1 = \frac{\ln[S(0)/1.1S(0)] + (0.06 - 0.02 + 0.09/2)(4)}{0.3\sqrt{4}} = 0.40782,$$

and

$$\pi = S(0)\mathrm{e}^{-\delta T}N(d_1) = S(0)\mathrm{e}^{-0.02(4)}N(0.40782),$$

so $\pi/S(0) = \mathrm{e}^{-0.08}\underbrace{N(0.40782)}_{=0.65830} = \boxed{0.60769}$. □

3. *Solution.* By risk-neutral pricing, the time-1 price of the derivative is given by

$$\begin{aligned} V(1) &= \overbrace{2S(1)\mathrm{e}^{-0.02(2)}N(d_1)}^{\text{2 units of } 1.5S(1)\text{-strike A/N call}} 1_{\{S(1)>120\}} \\ &\quad + \underbrace{S(1)\mathrm{e}^{-0.02(2)}}_{\text{1 unit of } 1.5S(1)\text{-strike A/N call}} N(d_1)1_{\{S(1)<120\}} \\ &= 2S(1)\mathrm{e}^{-0.04}N(d_1)1_{\{S(1)>120\}} + S(1)\mathrm{e}^{-0.04}N(d_1)1_{\{S(1)<120\}}, \end{aligned}$$

where

$$d_1 = \frac{\ln[S(1)/1.5S(1)] + (0.05 - 0.02 + 1/2 \times 0.25^2)(2)}{0.25\sqrt{2}} = -0.80035 \approx -0.80,$$

which is free of $S(1)$. Thus, the special 3-year partial asset-or-nothing option is equivalent to $2\mathrm{e}^{-0.04}N(d_1)$ units of a 1-year 120-strike asset-or-nothing *call*, plus $\mathrm{e}^{-0.04}N(d_1)$ units of a 1-year 120-strike asset-or-nothing *put*. With

$$d_1' = \frac{\ln(100/120) + (0.05 - 0.02 + 1/2 \times 0.25^2)(1)}{0.25\sqrt{1}} = -0.48429 \approx -0.48,$$

the time-0 price of the derivative, by risk-neutral pricing again, is

$$\begin{aligned} V(0) &= [2\mathrm{e}^{-0.04}N(d_1)][S(0)\mathrm{e}^{-0.02}N(d_1')] + [\mathrm{e}^{-0.04}N(d_1)][S(0)\mathrm{e}^{-0.02}]N(-d_1')] \\ &= 2\mathrm{e}^{-0.06}(100)\underbrace{N(-0.80)}_{0.2119}\underbrace{N(-0.48)}_{0.3156} + \mathrm{e}^{-0.06}(100)\underbrace{N(-0.80)}_{0.2119}\underbrace{N(0.48)}_{0.6844} \\ &= \boxed{26.2541}. \end{aligned}$$

□

5. *Solution.* With $K_1 = 100$, $K_2 = 110$,

$$\begin{aligned} d_1 &= \frac{\ln(95/110) + (0.05 - 0.1 + 0.1^2/2)(0.5)}{0.1\sqrt{0.5}} = -2.39148, \\ d_2 &= d_1 - 0.1\sqrt{0.5} = -2.46220, \end{aligned}$$

the price of the gap call option is

$$C^{\text{gap}} = 95\mathrm{e}^{-0.1(0.5)}\underbrace{N(-2.39148)}_{0.00839} - 100\mathrm{e}^{-0.05(0.5)}\underbrace{N(-2.46220)}_{0.00690} = \boxed{0.0852}.$$

□

7. *Solution.* Let $X(t)$ be the time-t exchange rate of US dollars per British pound. In three months, Michael will need to pay 200,000 pounds, or $\$200,000X(0.25)$, but he will receive $\$320,000$. The shortfall equals $(200,000X(0.25) - 320,000)_+ = 200,000\,(X(0.25) - 1.6)_+$, which can be compensated by 200,000 units of a 3-month \$1.6-strike European call option on pounds. To lower his cost, Michael can replace each plain vanilla call option by a 3-month \$1.6-strike European gap call option on pounds with a payment trigger of \$1.5, and with payoff equal to $[X(0.25) - 1.6]1_{\{X(0.25)>1.5\}}$. If the 3-month exchange rate is between 1.5 and 1.6, the payoff of such a gap option will be negative.

   As

$$\begin{aligned} d_1 &= \frac{\ln(1.6/\boxed{1.5}) + (0.01 - 0.02 + 0.2^2/2)(0.25)}{0.2\sqrt{0.25}} = 0.67039, \\ d_2 &= d_1 - 0.2\sqrt{0.25} = 0.57039, \\ N(d_1) &= 0.74870, \\ N(d_2) &= 0.71579, \end{aligned}$$

   the price of each currency gap call option is

$$C^{\text{gap}} = 1.6\text{e}^{-0.02(0.25)}(0.74870) - \boxed{1.6}\,\text{e}^{-0.01(0.25)}(0.71579) = 0.04954.$$

   The total cost of the currency gap options is $200,000(0.04954) = \boxed{9,908}$. □

   Remark. Notice that the total cost decreases quite remarkably from 12,316.90 in the plain vanilla case (see Problem 6.4.11) to 9,908 in the current gap option case.

9. *Solution.* (A) Yes
   (B) Yes
   (C) No
   (D) Yes
   (E) No

   We start with the most obvious representation *(is it obvious?)*

$$\begin{aligned} V(0) &= P(70,60) + C(70,80) + \text{PV}_{0,1}(10) \\ &= P(70,60) + C(70,80) + 10\text{e}^{-0.06}. \quad \textbf{(A)} \quad ☑ \end{aligned} \tag{B.8.1}$$

   By put-call parity for gap options,

$$\begin{aligned} C(70,60) - P(70,60) &= C(70,80) - P(70,80) \\ &= F_{0,1}(S) - \text{PV}_{0,1}(70) \\ &= 70\text{e}^{-0.03} - \boxed{70}\,\text{e}^{-0.06}. \end{aligned} \tag{B.8.2}$$

   Plugging (B.8.2) into (B.8.1) leads to

$$\begin{aligned} V(0) &= [C(70,60) - 70\text{e}^{-0.03} + \boxed{70}\,\text{e}^{-0.06}] + C(70,80) + 10\text{e}^{-0.06} \\ &= C(70,60) + C(70,80) - 70\text{e}^{-0.03} + 80\text{e}^{-0.06} \qquad \textbf{(B)} \quad ☑ \end{aligned}$$

   and

$$\begin{aligned} V(0) &= P(70,60) + [P(70,80) + 70\text{e}^{-0.03} - \boxed{70}\,\text{e}^{-0.06}] + 10\text{e}^{-0.06} \\ &= P(70,60) + P(70,80) + 70\text{e}^{-0.03} - 60\text{e}^{-0.06} \qquad \textbf{(D)} \quad ☑ \end{aligned}$$

□

Remark. Answers (C) and (E) would be obtained if you erroneously replaced the strike price 70 by the payment trigger (60 or 80) when writing the put-call parity for gap options.

11. *Solution.* The 90% lognormal prediction interval for $S(0.5)$ is given by

$$\left(S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)(0.5)-\sigma\sqrt{0.5}(1.645)}, S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)(0.5)+\sigma\sqrt{0.5}(1.645)}\right).$$

The ratio of the upper end-point to the lower end-point of the prediction interval gives

$$\mathrm{e}^{2\sigma\sqrt{0.5}(1.645)} = \frac{92.0014}{57.7734},$$

which in turn implies that $\sigma = 0.2$. Moreover, the geometric average of the two endpoints of the prediction interval for $S(0.5)$ gives

$$S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)(0.5)} = \sqrt{57.7734(92.0014)} \quad \Rightarrow \quad S(0)\mathrm{e}^{0.5(\alpha-\delta)} = 73.638363. \quad \text{(B.8.3)}$$

Now, we are also given in (i) that $\mathbb{P}(S(0.25) < 70) = N(-\hat{d}_2^{@K=70}) = 0.3669$ (not using $K = 75$, why?), or

$$\hat{d}_2 = \frac{\ln[S(0)/70] + (\alpha - \delta - 0.2^2/2)(0.25)}{0.2\sqrt{0.25}} = -N^{-1}(0.3669) = 0.34008,$$

which, together with (B.8.3), results in

$$\frac{\ln(73.638363\mathrm{e}^{-0.5(\alpha-\delta)}/70) + (\alpha - \delta - 0.2^2/2)(0.25)}{0.2\sqrt{0.25}} = 0.34.$$

This gives $\alpha - \delta = 0.046684$ and, by (B.8.3) again, $S(0) = 71.9394$. Finally, the first and second moments of $S(0.75)$, which is lognormal with parameters

$$\begin{cases} m = \ln S(0) + (\alpha - \delta - \sigma^2/2)(t) = \ln 71.9394 + (0.046684 - 0.2^2/2)(0.75) = 4.295837 \\ v^2 = \sigma^2 t = 0.2^2(0.75) = 0.03, \end{cases}$$

are, respectively,

$$\begin{aligned} \mathbb{E}[S(0.75)] &= \mathrm{e}^{m+v^2/2} = \mathrm{e}^{4.295837+0.03/2} = 74.5028 \\ \mathbb{E}[S(0.75)^2] &= \mathrm{e}^{2(m+v^2)} = \mathrm{e}^{2(4.295837+0.03)} = 5,719.7135, \end{aligned}$$

so $\mathrm{Var}(S(0.75)) = \mathbb{E}[S(0.75)^2] - \mathbb{E}[S(0.75)]^2 = \boxed{169.05}$. □

Remark. You can also take the first and second moments directly from the stock price equation

$$S(0.75) = S(0)\mathrm{e}^{(\alpha-\delta-\sigma^2/2)(0.75)+\sigma\sqrt{0.75}Z}$$

using the moment-generating function formula of a standard normal random variable.

13. *Solution.* Since

$$\begin{aligned} \text{Payoff} &= [S(1) - 60]1_{\{60 \le S(1) \le 80\}} \\ &= [S(1) - 60][1_{\{S(1) \ge 60\}} - 1_{\{S(1) > 80\}}] \\ &= [S(1) - 60]_+ - [S(1) - 60]1_{\{S(1) > 80\}}, \end{aligned}$$

the given "truncated" call option is equivalent to a long 60-strike plain vanilla call option plus a short 60-strike 80-trigger gap call option. Let $d_i^{@K}$ be the value of d_i evaluated at K for $i = 1, 2$. With

$$\begin{aligned}
d_1^{@60} &= \frac{\ln(65/60) + (0.06 - 0.03 + 0.25^2/2)(1)}{0.25\sqrt{1}} = 0.56517,\\
N(d_1^{@60}) &= 0.71402,\\
d_1^{@80} &= \frac{\ln(65/80) + (0.06 - 0.03 + 0.25^2/2)(1)}{0.25\sqrt{1}} = -0.58556,\\
d_2^{@80} &= d_1^{@60} - 0.25\sqrt{1} = -0.83556,\\
N(d_1^{@80}) &= 0.27909,\\
N(d_2^{@80}) &= 0.20170,
\end{aligned}$$

the current delta of the "truncated" call option is

$$\begin{aligned}
\Delta &= \underbrace{e^{-0.03}(0.7157)}_{\text{delta of 60-strike plain vanilla call}}\\
&\quad - \left[\underbrace{e^{-0.03}(0.27909)}_{\text{delta of 80-strike plain vanilla call}} + \underbrace{20e^{-0.06}N'(-0.83556) \times \frac{1}{65(0.25)\sqrt{1}}}_{\text{delta of 80-strike C/N call of \$20}}\right]\\
&= 0.69292 - (0.27084 + 0.32616)\\
&= \boxed{0.0959}.
\end{aligned}$$

□

15. *Solution.* By put-call parity for European gap options,

$$7.3528 - (-3.4343) = 100e^{-0.02(0.5)} - \underbrace{90}_{\text{not } 110} e^{-0.5r},$$

which gives $r = 4\%$. As the 90-strike and 110-payment-trigger gap call is equivalent to a 110-strike plain vanilla call plus a 110-strike cash-or-nothing call of 20, and

$$\begin{aligned}
d_1 &= \frac{\ln(100/110) + (0.04 - 0.02 + 0.2^2/2)(0.5)}{0.2\sqrt{0.5}} = -0.53252,\\
N(d_1) &= 0.29718,\\
d_2 &= d_1 - 0.2\sqrt{0.5} = -0.67394,
\end{aligned}$$

the delta of the gap call is

$$e^{-0.02(0.5)}(0.29718) + 20e^{-0.04(0.5)}\frac{1}{\sqrt{2\pi}}e^{-(-0.67394)^2/2} \times \frac{1}{100(0.2)\sqrt{0.5}} = 0.73489.$$

To delta-hedge his position, Ryan buys $10(0.73489) = 7.3489$ shares of the stock at time 0. After one month, his holding profit is

$$\underbrace{[7.3489e^{0.02/12}(95) - 10(4.2437)]}_{656.87305} - \underbrace{[7.3489(100) - 10(7.3528)]e^{0.04/12}}_{663.57027} = \boxed{-6.70}.$$

□

17. *Solution.* (a) The eight blanks should be filled in as follows:

- $\boxed{1} = 0$
- $\boxed{2} = F^P_{0,T}(K)$ or $K\mathrm{e}^{-rT}$
- $\boxed{3} = F^P_{0,T}(S)$
- $\boxed{4} = 0$
- $\boxed{5} = F_{0,T_f}$
- $\boxed{6} = T$ (not T_f)
- $\boxed{7} = F^P_{0,T}(S_2)$
- $\boxed{8} = 2F^P_{0,T}(S_1)$

(b)
- For option B, σ is the volatility of the prepaid forward on the stock.
- For option D, σ is the volatility of the ratio of the prepaid forward on Stock 1 to the prepaid forward on Stock 2.

□

19. *Solution.* The call price computed by Actuary A turns out to be identical to the put price computed by Actuary B.

To see this, note that the two options are related by swapping the price of the underlying stock and the strike price, and swapping the value of the dividend yield and value of the risk-free interest rate. From the perspective of an exchange option, the two options are the same in terms of the blended volatility ($\sigma = 0.3$), the current price and "dividend yield" of the asset acquired (190 and 5%, respectively), and the current price and "dividend yield" of the asset given up (200 and 3%, respectively). As a result, the two options share the same current price, which, by virtue of (8.2.2), is given by

$$\mathrm{BS}(190, 0.05; 200, 0.03; 0.3, 1) = 190\mathrm{e}^{-0.05(1)}N(d_1) - 200\mathrm{e}^{-0.03(1)}N(d_2),$$

with

$$d_1 = \frac{\ln(190/200) + (0.03 - 0.05 + 0.3^2/2)(1)}{0.3\sqrt{1}} \quad \text{and} \quad d_2 = d_1 - 0.3\sqrt{1}.$$

□

21. *Solution.* (A) Decrease.

(B) Increase.

(C) Decrease.

(D) Decrease.

(E) No change. When the underlying and strike assets both pay continuous proportional dividends, the exchange option pricing formula is free of the continuously compounded risk-free interest rate.

□

Remark. The conclusion for (E) may not be true for assets paying discrete, non-random dividends. The risk-free rate is used in determining the prepaid forward prices.

23. *Solution.* Note that the three options all share the same current price and "dividend yield" of the asset acquired, namely Stock 2 (200 and 5%, respectively), and the same current price and "dividend yield" of the asset given up (200 and 8%, respectively). Only their blended volatilities differ.

I. The volatility is simply the volatility of Stock 2, or σ_2.

II. The blended volatility equals

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \stackrel{(\sigma_1=\sigma_2)}{=} \sqrt{2\sigma_2^2 - 2(0.7)\sigma_2^2} = \sqrt{0.6}\sigma_2,$$

which is less than σ_2.

III. The blended volatility equals

$$\sigma = \sqrt{2\sigma_2^2 - 2(-0.25)\sigma_2^2} = \sqrt{2.5}\sigma_2,$$

which is higher than σ_2.

Recall that the price of a plain vanilla option increases with the volatility (equivalently, vega is positive; see Problem 6.4.18) and that the Black-Scholes price of an exchange option shares the same structure as that of a plain vanilla option. Because the blended volatilities of the three options are ordered as II < I < III, their prices are ordered as $\boxed{\text{II} < \text{I} < \text{III}}$ as well. □

25. *Solution.* (a) Since $\max(S_1(T), S_2(T)) = (S_1(T) - S_2(T))_+ + S_2(T)$, the time-0 price of the maximum option is

$$\begin{aligned}
V^{\max} &= [F_{0,T}^P(S_1)N(d_1) - F_{0,T}^P(S_2)N(d_2)] + F_{0,P}^P(S_2) \\
&= F_{0,T}^P(S_1)N(d_1) + F_{0,T}^P(S_2)N(-d_2) \\
&= F_{0,T}^P(S_1)N\left(\frac{\ln[F_{0,T}^P(S_1)/F_{0,T}^P(S_2)] + (\sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
&\quad + F_{0,T}^P(S_2)N\left(\frac{\ln[F_{0,T}^P(S_2)/F_{0,T}^P(S_1)] + (\sigma^2/2)T}{\sigma\sqrt{T}}\right),
\end{aligned}$$

where the first equality follows from $N(x) + N(-x) = 1$ for any $x \in \mathbb{R}$. Observe that the two terms

$$F_{0,T}^P(S_1)N\left(\frac{\ln[F_{0,T}^P(S_1)/F_{0,T}^P(S_2)] + (\sigma^2/2)T}{\sigma\sqrt{T}}\right)$$

and

$$F_{0,T}^P(S_2)N\left(\frac{\ln[F_{0,T}^P(S_2)/F_{0,T}^P(S_1)] + (\sigma^2/2)T}{\sigma\sqrt{T}}\right)$$

are symmetric with each other.

Remark. Note that

$$\max(S_1(T), S_2(T)) = S_1(T)1_{\{S_1(T)>S_2(T)\}} + S_2(T)1_{\{S_2(T)\geq S_1(T)\}}.$$

The two terms above correspond to the time-0 prices of the two asset-or-nothing options with time-T payoffs $S_1(T)1_{\{S_1(T)>S_2(T)\}}$ and $S_2(T)1_{\{S_2(T)\geq S_1(T)\}}$.

(b) As $\min(S_1(T), S_2(T)) = S_1(T) + S_2(T) - \max(S_1(T), S_2(T))$, the time-0 price of the minimum option is

$$\begin{aligned}
V^{\min} &= F^P_{0,T}(S_1) + F^P_{0,T}(S_2) - [F^P_{0,T}(S_1)N(d_1) + F^P_{0,T}(S_2)N(-d_2)] \\
&= F^P_{0,T}(S_1)N(-d_1) + F^P_{0,T}(S_2)N(d_2) \\
&= F^P_{0,T}(S_1)N\left(\frac{\ln[F^P_{0,T}(S_2)/F^P_{0,T}(S_1)] - (\sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
&\quad + F^P_{0,T}(S_2)N\left(\frac{\ln[F^P_{0,T}(S_1)/F^P_{0,T}(S_2)] - (\sigma^2/2)T}{\sigma\sqrt{T}}\right),
\end{aligned}$$

where the first line follows from the result of part (a). Observe again that the last two terms are symmetric with each other.

Remark. The two terms above correspond to the time-0 prices of the two asset-or-nothing options with time-T payoffs $S_2(T)1_{\{S_1(T)>S_2(T)\}}$ and $S_1(T)1_{\{S_2(T)\geq S_1(T)\}}$.

□

27. *Solution.* Rewrite the payoff of the contingent claim as

$$\begin{aligned}
\min(S_1(4), S_2(4)) &= S_1(4) + \min(S_2(4) - S_1(4), 0) \\
&= S_1(4) - (S_1(4) - S_2(4))_+ .
\end{aligned}$$

As

$$\begin{aligned}
\sigma &= \sqrt{0.12^2 + 0.15^2 - 2(0.12)(0.15)(-0.25)} = 0.214243, \\
F^P_{0,4}(S_1) &= 100e^{-0.03(4)} = 88.692044, \\
F^P_{0,4}(S_2) &= 120 - 5e^{-0.04(2)} = 115.384418, \\
d_1 &= \frac{\ln(88.692044/115.384418) + (0.214243^2/2)(4)}{0.214243\sqrt{4}} = -0.39978, \\
d_2 &= d_1 - 0.214243\sqrt{4} = -0.82826, \\
N(d_1) &= 0.34466, \\
N(d_2) &= 0.20376,
\end{aligned}$$

the price of the exchange option giving up one unit of Stock 2 in return for one unit of Stock 1 is

$$V = 88.692044(0.34466) - 115.384418(0.20376) = 7.05787.$$

Therefore, the price of the minimum option is

$$V^{\min} = 88.692044 - 7.05787 = \boxed{81.63417}.$$

□

29. *Solution.* Note that Derivatives A and B form a call-put pair with the underlying asset being the European call which matures at time 1. By compound option parity,

$$V_0^A - V_0^B = 6.4225 - 6 \times \underbrace{e^{-0.5r}}_{\text{discount for 6 months only}}.$$

To find the 6-month discount factor, we apply put-call parity to the 6-month call-put pair and the 1-year call-put pair, yielding the following two equations:

$$\begin{cases} 4.2581 - 2.7804 = S(0) - S(0)e^{-0.5r} \\ 6.4225 - 3.5107 = S(0) - S(0)e^{-r} \end{cases} \Rightarrow \begin{cases} 1.4777 = S(0)(1 - e^{-0.5r}) \\ 2.9118 = S(0)(1 - e^{-r}) \end{cases}.$$

Dividing the first equation from the second one (recall the identity $1-x^2 \equiv (1+x)(1-x)$) and noting that $\mathrm{e}^{-0.5r} \neq 1$, we have

$$\frac{1-\mathrm{e}^{-r}}{1-\mathrm{e}^{-0.5r}} = 1+\mathrm{e}^{-0.5r} = \frac{2.9118}{1.4777} \quad \Rightarrow \quad \mathrm{e}^{-0.5r} = 0.970495.$$

Hence the price of Derivative B is

$$V_0^B = V_0^A - 6.4225 + 6\mathrm{e}^{-0.5r} = 2.5944 - 6.4225 + 6(0.970495) = \boxed{1.9949}.$$

□

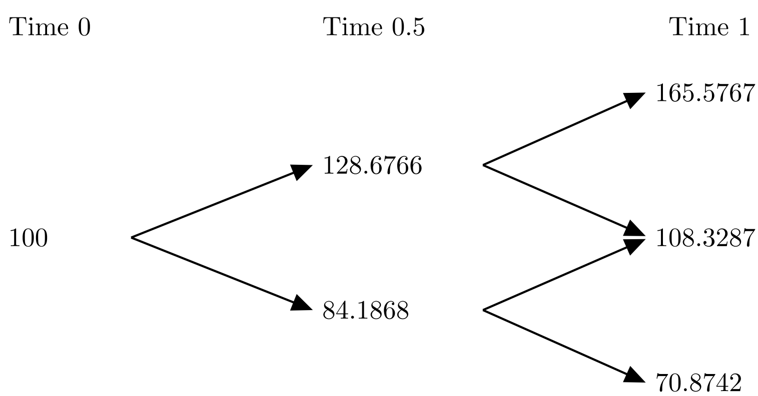
31. *Solution.* The forward tree is built by setting

$$\begin{aligned} u &= \exp[(r-\delta)h + \sigma\sqrt{h}] = \exp[(0.1-0.02)(0.5) + 0.3\sqrt{0.5}] = 1.286766, \\ d &= \exp[(r-\delta)h - \sigma\sqrt{h}] = \exp[(0.1-0.02)(0.5) - 0.3\sqrt{0.5}] = 0.841868, \end{aligned}$$

and the risk-neutral probability of an up move is

$$p^* = \frac{1}{1+\mathrm{e}^{\sigma\sqrt{h}}} = \frac{1}{1+\mathrm{e}^{0.3\sqrt{0.5}}} = 0.447165.$$

The resulting stock prices are depicted below:



(a) The arithmetic averages and the payoffs ($= (A - S(1))_+$) of the Asian put option corresponding to different stock price paths are:

Path	Arithmetic Average	Payoff
uu	$(128.6766 + 165.5767)/2 = 147.1266$	$(147.1266 - 165.5767)_+ = 0$
ud	$(128.6766 + 108.3287)/2 = 118.5026$	$(118.5026 - 108.3287)_+ = 10.1739$
du	$(84.1868 + 108.3287)/2 = 96.2578$	$(96.2578 - 108.3287)_+ = 0$
dd	$(84.1868 + 70.8742)/2 = 77.5305$	$(77.5305 - 70.8742)_+ = 6.6563$

Upon discounting the risk-neutral expected value of these payoffs back to time 0 at the risk-free interest rate, the current price of the arithmetic average strike Asian put option is

$$\begin{aligned} V_0 &= \mathrm{e}^{-rT}[p^*(1-p^*)V_{ud} + (1-p^*)^2 V_{dd}] \\ &= \mathrm{e}^{-0.1(1)}[(0.447165)(1-0.447165)(10.1739) + (1-0.447165)^2(6.6563)] \\ &= \boxed{4.1165}. \end{aligned}$$

(b) The geometric averages and the payoffs ($= (G - S(1))_+$) of the Asian put option corresponding to different stock price paths are:

Path	Geometric Average	Payoff
uu	$\sqrt{128.6766(165.5767)} = 145.9652$	$(145.9652 - 165.5767)_+ = 0$
ud	$\sqrt{128.6766(108.3287)} = 118.0651$	$(118.0651 - 108.3287)_+ = 9.7364$
du	$\sqrt{84.1868(108.3287)} = 95.4979$	$(95.4979 - 108.3287)_+ = 0$
dd	$\sqrt{84.1868(70.8742)} = 77.2442$	$(77.2442 - 70.8742)_+ = 6.3700$

By risk-neutral pricing again, the current price of the geometric average strike Asian put option is

$$\begin{aligned}V_0 &= \mathrm{e}^{-rT}[p^*(1-p^*)V_{ud} + (1-p^*)^2 V_{dd}] \\ &= \mathrm{e}^{-0.1(1)}[(0.447165)(1-0.447165)(9.7364) + (1-0.447165)^2(6.3700)] \\ &= \boxed{3.9394}.\end{aligned}$$

□

33. *Solution.* Put-call parity for otherwise identical European Asian options asserts that

$$\begin{aligned}C^{\text{Asian}} - P^{\text{Asian}} &= \text{Time-0 price of } \frac{S(1)+S(2)+S(3)}{3} \text{ payable at time 3} \\ &\quad - 15\mathrm{e}^{-rT}.\end{aligned}$$

To evaluate the time-0 price of the arithmetic average, we reason as follows:

- To obtain $S(1)$ payable at $t = 3$ (not $t = 1$!), we need $\mathrm{e}^{-2r}S(1)$ at $t = 1$ (this known amount will grow at the risk-free interest to exactly $S(1)$ at $t = 3$), or equivalently, $\mathrm{e}^{-2r}F_{0,1}^P(S) = \mathrm{e}^{-2r} \times S(0)\mathrm{e}^{-\delta} = \mathrm{e}^{-2(0.06)-0.03}(15) = 15\mathrm{e}^{-0.15}$ at $t = 0$.
- Similarly, the time-0 price of $S(2)$ payable at $t = 3$ is

$$\mathrm{e}^{-r}F_{0,2}^P(S) = \mathrm{e}^{-0.06-2(0.03)}(15) = 15\mathrm{e}^{-0.12}.$$

- The time-0 price of $S(3)$ payable at $t = 3$ is simply

$$F_{0,3}^P(S) = S(0)\mathrm{e}^{-3\delta} = 15\mathrm{e}^{-3(0.03)} = 15\mathrm{e}^{-0.09}.$$

Consequently, the time-0 price of the arithmetic average payable at $t = 3$ equals

$$\frac{15\mathrm{e}^{-0.15} + 15\mathrm{e}^{-0.12} + 15\mathrm{e}^{-0.09}}{3} = 13.3078,$$

and the required Asian put price is $1.5 - 13.3078 + 15\mathrm{e}^{-0.06(3)} = \boxed{0.7213}$.

□

Remark. Note that the time-0 price of the arithmetic average is *not*

$$\frac{F_{0,1}^P(S) + F_{0,2}^P(S) + F_{0,3}^P(S)}{3}.$$

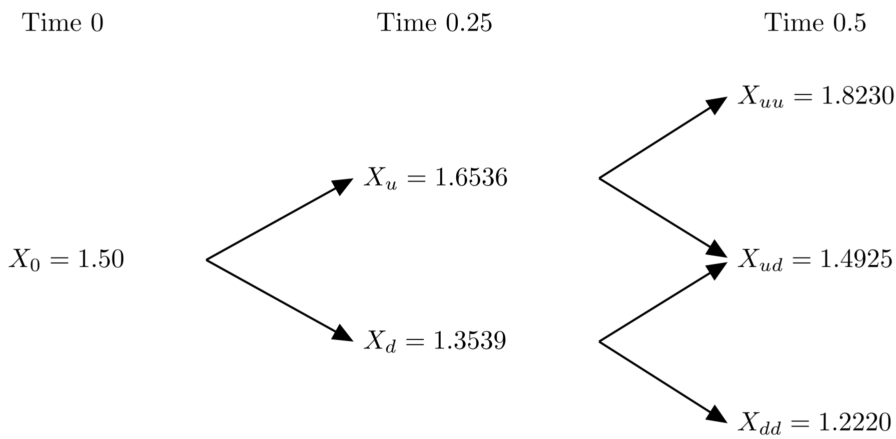
35. *Solution.* The forward tree is built by setting

$$\begin{aligned} u &= \exp[(r_{\$} - r_{€})h + \sigma\sqrt{h}] = \exp[(0.04 - 0.05)(0.25) + 0.2\sqrt{0.25}] = 1.102411, \\ d &= \exp[(r_{\$} - r_{€})h - \sigma\sqrt{h}] = \exp[(0.04 - 0.05)(0.25) - 0.2\sqrt{0.25}] = 0.902578, \end{aligned}$$

and the risk-neutral probability of an up move is

$$p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}} = \frac{1}{1 + e^{0.2\sqrt{0.25}}} = 0.475021.$$

The resulting dollar/euro exchange rates are depicted below:



From the binomial tree, we can calculate the 6-month payoffs of the standard American lookback call corresponding to different exchange rate paths:

Path	Terminal Exchange Rate (A)	Min. Exchange Rate (B)	Terminal Payoff = (A) – (B)
uu	1.8230	1.50	0.323
ud	1.4925	1.4925	0
du	1.4925	1.3539	0.1386
dd	1.2220	1.2220	0

Moving back to the end of three months, we have:

Path	Current Exchange Rate (A)	Min. Exchange Rate (B)	Holding Value	Exercise Value = (A) – (B)
u	1.6536	1.50	$e^{-0.04/4}[p^*(0.323)] = 0.1519$	0.1536
d	1.3539	1.3539	$e^{-0.04/4}[p^*(0.1386)] = 0.0652$	0

Early exercise is optimal at the u node, with $V_u = 0.1536$ and $V_d = 0.0652$. The replicating portfolio of the lookback call at the initial node is set up by

$$\Delta = e^{-r_{€}h}\left(\frac{V_u - V_d}{X_u - X_d}\right) = e^{-0.05(0.25)}\left(\frac{0.1536 - 0.0652}{1.6536 - 1.3539}\right) = 0.2913$$

and

$$\begin{aligned} B &= e^{-r_\$ h}\left(\frac{uV_d - dV_u}{u-d}\right) \\ &= e^{-0.04(0.25)}\left[\frac{1.102411(0.0652) - 0.902578(0.1536)}{1.102411 - 0.902578}\right] \\ &= -0.3307. \end{aligned}$$

It follows that the fair price of the lookback call is

$$V_0 = \Delta X + B = 0.2913(1.5) + (-0.3307) = 0.1062,$$

which is higher than the observed price. To pursue an arbitrage strategy, at the initial exchange rate node one should:

- Buy the lookback call from the market.
- Sell the replicating portfolio by selling 0.2913 units of euro and buying 0.3307 units of dollar.

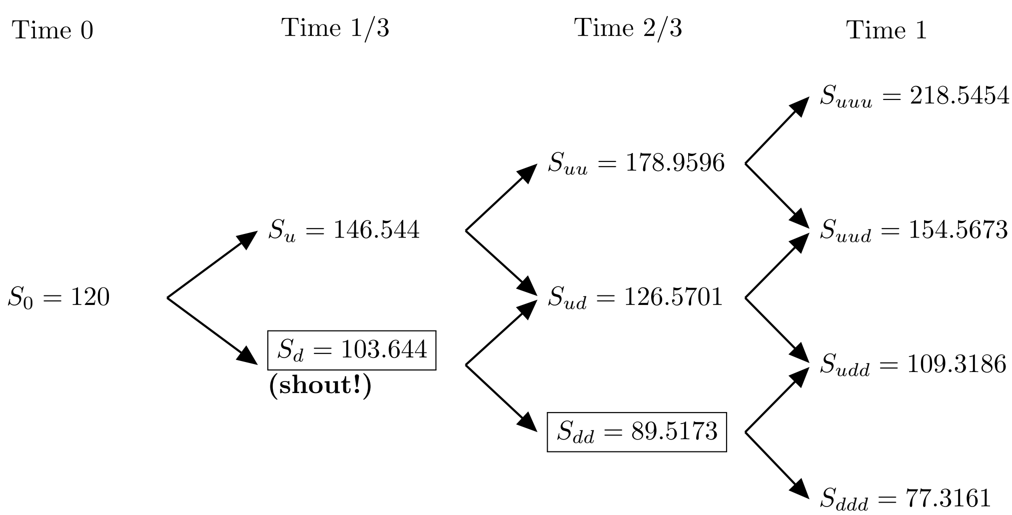
□

37. *Solution.* In terms of payoff, the four options are ordered as

$$\text{II} \le \text{I} \le \text{IV} \le \text{III}.$$

It follows that their (fair) prices are ordered in the same way. □

39. *Solution.* This problem uses an indirect way to request the value of the right to shout over the 1-year life of the shout option. Given that $u = 1.2212$ and $d = 0.8637$, we first construct the binomial tree below:



The risk-neutral probability of an up move is

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0)/3} - 0.8637}{1.2212 - 0.8637} = 0.456854.$$

Note that if we never shout during the life of the shout put, then its possible terminal payoffs are

$$P_{uuu} = P_{uud} = 0, \quad P_{udd} = 10.6814, \quad P_{ddd} = 42.6839.$$

We now investigate the optimal time to shout. Prior to maturity, the put option is in-the-money only at the d node and the dd node, so it suffices to focus on these two nodes.

- *dd node:* If we shout at the dd node, we can guarantee a minimum payoff of $120 - 89.5173 = 30.4827$ and the two possible terminal payoffs become

$$\begin{aligned} V_{udd} &= \max(10.6814, 30.4827) = 30.4827, \\ V_{ddd} &= \max(42.6839, 30.4827) = 42.6839. \end{aligned}$$

  By risk-neutral pricing, the value of the option at the dd node is

$$V_{dd} = \mathrm{e}^{-0.08/3}[0.456854(30.4827) + (1 - 0.456854)(42.6839)] = 36.1332.$$

- *d node:* We now further roll back one period to the d node. To determine whether to shout at the d node or to wait until the dd node (if the stock price indeed drops after 4 months), we have to evaluate the value of the shout put under these two mutually exclusive actions.

  Case 1. If we shout at the d node, then we will receive a payoff of at least $120 - 103.644 = 16.356$ at maturity, or more if the shout put expires deeper in the money. From the perspective of the d node, only the uud, udd, and ddd nodes are possible outcomes, with respective terminal payoffs being

$$\begin{aligned} V_{uud} &= \max(0, 16.356) = 16.356, \\ V_{udd} &= \max(10.6814, 16.356) = 16.356, \\ V_{ddd} &= \max(42.6839, 16.356) = 42.6839. \end{aligned}$$

  By risk-neutral pricing, the value of the shout put is

$$\begin{aligned} V_d &= \mathrm{e}^{-0.08(2/3)}\{16.356 + (1 - 0.456854)^2(42.6839 - 16.356)]\} \\ &= 22.8701. \end{aligned}$$

  Case 2. If we do not shout at the d node, but shout at the dd node, if we happen to reach that node, then the shout put will be worth $V_{ud} = \mathrm{e}^{-0.08/3}(1{-}0.456854)(10.6814) = 5.6489$ at the ud node or $V_{dd} = 36.1332$ at the dd node (where we shout). By risk-neutral pricing again, the value of the shout call is

$$V_d = \mathrm{e}^{-0.08/3}[0.456854(5.6489) + (1 - 0.456854)(36.1332)] = 21.6220.$$

Since the shout put is worth more in Case 1, it is advisable to shout at the d node, clinching the higher value of $V_d = 22.8701$.

For the corresponding plain vanilla put, its value at the d node is

$$\begin{aligned} P_d &= \mathrm{e}^{-0.08(2/3)}[2(0.456854)(1 - 0.456854)(10.6814) + (1 - 0.456854)^2(42.6839)] \\ &= 16.9637. \end{aligned}$$

The value of the right to shout is

$$P_{II} - P_I = \mathrm{e}^{-0.08/3}(1 - 0.456854)\underbrace{(22.8701 - 16.9637)}_{\text{change in } V_d} = \boxed{3.1236}.$$

□

Remark. It can be shown that $P_I = 10.3002$ and $P_{II} = 13.4238$, hence $P_{II} - P_I = 3.1236$, agreeing with the answer above.

41. *Solution.* For options I and II, $S(0) = 50 < B \le K = 60$, so their payoff formula can be simplified into

$$(S(T)-K)_+ \times 1_{\{M(T)>B\}} = (S(T)-K) \times 1_{\{S(T)>K, M(T)>B\}} = (S(T)-K) \times 1_{\{S(T)>K\}},$$

which is that of a 60-strike plain vanilla call. Hence options I and II have the same price.

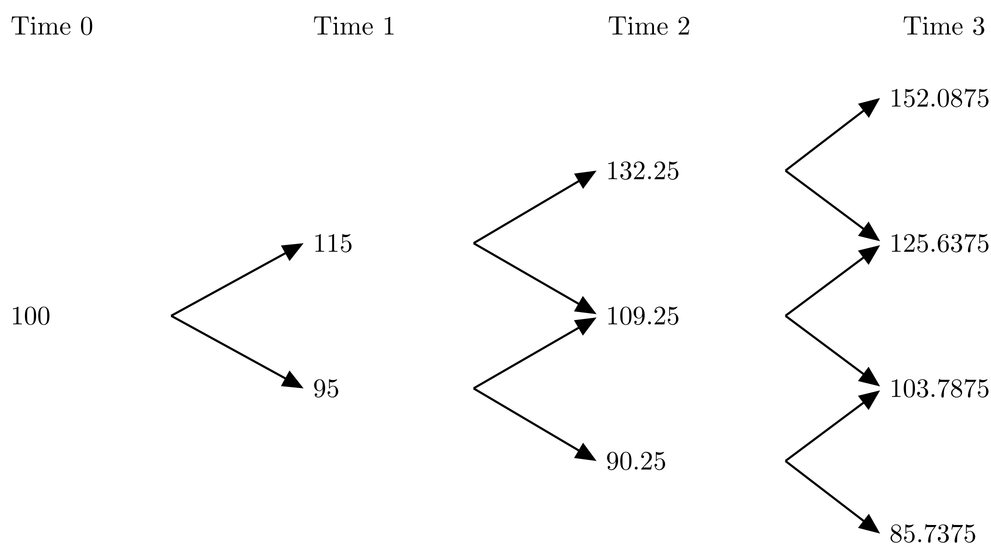
When $B > K = 60$, the payoff formula is

$$(S(T) - K)_+ \times 1_{\{M(T)>B\}},$$

which is non-increasing in B. Thus the prices of options III and IV are ordered as (III) $\ge$ (IV).

In conclusion, $\boxed{\text{(I)} = \text{(II)} \ge \text{(III)} \ge \text{(IV)}}$. **(Answer: (C))** □

43. *Solution.* Given that $u = 1.15$ and $d = 0.95$, the three-period binomial stock price tree is exhibited overleaf:



The risk-neutral probability of an up move is

$$p^* = \frac{e^{0.1(1)} - 0.95}{1.15 - 0.95} = 0.775855.$$

We tabulate below the possible payoffs of the up-and-in call along with the associated risk-neutral probabilities:

Node	Payoff	Number of Paths	Risk-neutral Probability
uuu	52.0875	1	$p^3 = 0.467027$
uud	25.6375	3	$3(p^*)^2(1-p^*) = 0.404773$
udd	3.7875	1	$p^*(1-p^*)^2 = 0.038980$

Note that the up-and-in call pays off at the *udd* node only via the *udd* path; both the *dud* and *ddu* paths do not cross the barrier of 110. By risk-neutral pricing, the price of the up-and-in call is

$$\begin{aligned} V &= \mathrm{e}^{-0.1(3)}[0.467027(52.0875) + 0.404773(25.6375) + 0.038980(3.7875)] \\ &= \boxed{25.82}. \end{aligned}$$

□

45. *Solution.* The forward tree parameters are

$$\begin{aligned} u &= \exp[(r-\delta)h + \sigma\sqrt{h}] = \exp[(0.05-0.03)(1) + 0.3\sqrt{1}] = 1.377128, \\ d &= \exp[(r-\delta)h - \sigma\sqrt{h}] = \exp[(0.05-0.03)(1) - 0.3\sqrt{1}] = 0.755784. \end{aligned}$$

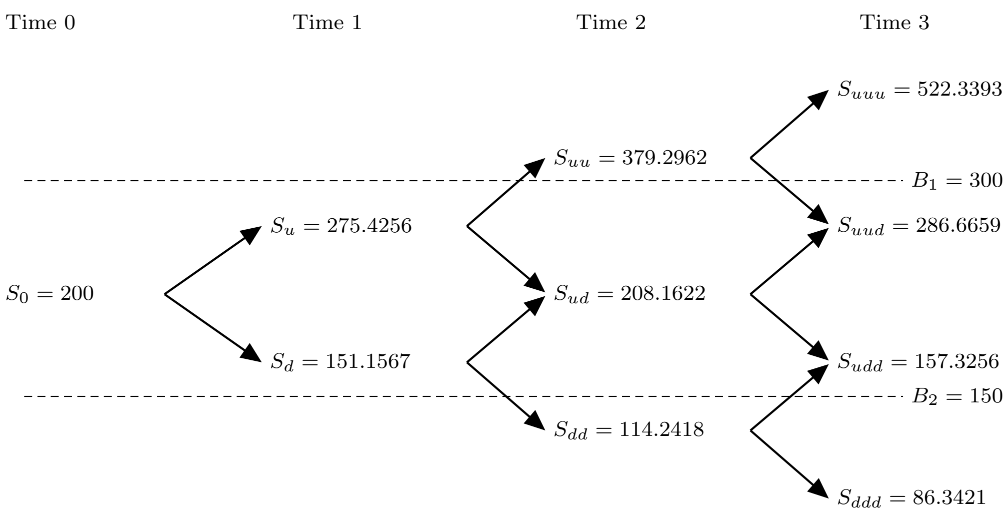
The risk-neutral probability of an up move is

$$p^* = \frac{\mathrm{e}^{(r-\delta)h} - d}{u-d} = \frac{\mathrm{e}^{(0.05-0.03)(1)} - 0.755784}{1.377128 - 0.755784} = 0.425557,$$

or

$$p^* = \frac{1}{1+\mathrm{e}^{\sigma\sqrt{h}}} = \frac{1}{1+\mathrm{e}^{0.3\sqrt{1}}} = 0.425557.$$

The stock price tree is depicted below:



Observe that the special collar may pay off only in the *uud* node attainable by 2 paths (the *udu* path and the *duu* path; the *uud* path crosses the upper barrier of 300), and the *udd* node attainable by 2 paths (the *udd* path and the *dud* path; the *ddu* path crosses the lower barrier of 150). The payoffs are summarized as follows:

Path	Geometric Average (G)	Payoff of Collar $= (180 - G)_+ - (G - 250)_+$
udu	$(275.4256 \times 208.1622 \times 286.6659)^{1/3} = 254.2499$	-4.2499
duu	$(151.1567 \times 208.1622 \times 286.6659)^{1/3} = 208.1622$	0
udd	$(275.4256 \times 208.1622 \times 157.3256)^{1/3} = 208.1622$	0
dud	$(151.1567 \times 208.1622 \times 157.3256)^{1/3} = 170.4288$	9.5712

By risk-neutral pricing, the price of the collar is

$$V_0 = \mathrm{e}^{-0.05(3)}[(p^*)^2(1-p^*)(-4.2499) + p^*(1-p^*)^2(9.5712)] = \boxed{0.7763}.$$

□

Remark. The presence of the double barriers actually simplifies your computations—you need not deal with $2^3 = 8$ stock price paths. Only four matter.

47. *Solution.* (a) The payoff at time $t = T = 6$ of the chooser option is

$$\begin{aligned}\max\{(S(6)-30)_+, (30-S(6))_+\} &= \begin{cases} 30 - S(6), & \text{if } S(6) < 30 \\ S(6) - 30, & \text{if } S(6) \geq 30 \end{cases} \\ &= |S(6) - 30|,\end{aligned}$$

which is the payoff of a $\boxed{\text{long 30-strike (6-year) straddle}}$. The price of the chooser option is therefore the sum of the price of a 30-strike 6-year call and that of a 30-strike 6-year put. To obtain the put price, we use put-call parity:

$$\begin{aligned}P(32,30,6) &= C(32,30,6) - S(0) + \mathrm{PV}_{0,6}(\mathrm{Div}) + K\mathrm{e}^{-rT} \\ &= 9.48 - 32 + 1.5(\mathrm{e}^{-0.05} + \mathrm{e}^{-0.05(3)} + \mathrm{e}^{-0.05(5)}) + 30\mathrm{e}^{-0.05(6)} \\ &= 3.590654.\end{aligned}$$

The price of the chooser option is then $C(32,30,6) + P(32,30,6) = 9.48 + 3.590654 = \boxed{13.07}$.

(b) The time-2 payoff of the chooser option is

$$P(S(2),30,4) + (C(S(2),30,4) - P(S(2),30,4))_+,$$

which, by put-call parity, equals

$$\begin{aligned}&P(S(2),30,4) + \left(S(2) - \mathrm{PV}_{2,6}(\mathrm{Div}) - 30\mathrm{e}^{-0.05(4)}\right)_+ \\ = \ &P(S(2),30,4) + \left(S(2) - 1.5(\mathrm{e}^{-0.05} + \mathrm{e}^{-0.05(3)}) - 30\mathrm{e}^{-0.05(4)}\right)_+ \\ = \ &P(S(2),30,4) + (S(2) - 27.279829)_+.\end{aligned}$$

It follows that the time-0 price of the chooser option is the sum of the price of a 6-year 30-strike put option and the price of a 2-year 27.28-strike call option, i.e.

$$P(32,30,6) + C(32,27.28,2) = \underbrace{3.590654}_{\text{from part (a)}} + 7.60 = \boxed{11.19}.$$

□

Remark. The performance in this question was catastrophic when it was offered in a real exam. As you can see from the above solutions, the question is not ridiculously hard. You can do it as long as you know how to manipulate put-call parity. Incidentally, this question also illustrates the futility of slavishly memorizing pricing formulas without understanding the assumptions governing their validity and the importance of being able to work out problems from first principles.

49. *Notation:* Let $V^{2B\to A}(t,T)$ (resp. $V^{A\to 2B}(t,T)$) be the time-t price of the European option maturing at time T and giving you the right to exchange two units of Stock B for one unit of Stock A (resp. exchange one unit of Stock A for two units of Stock B).

Solution 1. The time-1 payoff of the chooser option is

$$\begin{aligned}
& \max\{V^{2B\to A}(1,3), V^{A\to 2B}(1,3)\} \\
= \; & V^{A\to 2B}(1,3) + \left(V^{2B\to A}(1,3) - V^{A\to 2B}(1,3)\right)_+ \\
\overset{\text{(exchange option parity)}}{=} \; & V^{A\to 2B}(1,3) + \left(F^P_{1,3}(A) - 2F^P_{1,3}(B)\right)_+ \\
= \; & V^{A\to 2B}(1,3) + \Big(A(1)\mathrm{e}^{-0.02(2)} - 2B(1)\mathrm{e}^{-0.02(2)}\Big)_+ \\
= \; & \underbrace{V^{A\to 2B}(1,3)}_{\substack{\text{time-1 payoff of option to} \\ \text{exchange A for 2 units of B at time 3}}} + \mathrm{e}^{-0.02(2)} \underbrace{V^{2B\to A}(1,1)}_{\text{exchange 2 units of B for A at time 1}}.
\end{aligned}$$

Thus the time-0 price of the chooser is

$$V^{A\to 2B}(0,3) + \mathrm{e}^{-0.04} \underbrace{V^{2B\to A}(0,1)}_{=5 \text{ (from (iv))}} = V^{A\to 2B}(0,3) + 5\mathrm{e}^{-0.04}.$$

By the exchange option parity at time 0, we have

$$V^{A\to 2B}(0,3) - \underbrace{V^{2B\to A}(0,3)}_{=8 \text{ from (v)}} = 2F^P_{0,3}(B) - F^P_{0,3}(A) = \mathrm{e}^{-0.02(3)}[2(100) - 205],$$

so $V^{A\to 2B}(0,3) = 3.291177$. Hence the answer is $3.291177 + 5\mathrm{e}^{-0.04} = \boxed{8.0951}$. □

Solution 2. The time-1 payoff of the chooser option is

$$\begin{aligned}
& \max\{V^{2B\to A}(1,3), V^{A\to 2B}(1,3)\} \\
= \; & V^{2B\to A}(1,3) + \left(V^{A\to 2B}(1,3) - V^{2B\to A}(1,3)\right)_+ \\
\overset{\text{(exchange option parity)}}{=} \; & V^{2B\to A}(1,3) + \left(2F^P_{1,3}(B) - F^P_{1,3}(A)\right)_+ \\
= \; & V^{2B\to A}(1,3) + \Big(2B(1)\mathrm{e}^{-0.02(2)} - A(1)\mathrm{e}^{-0.02(2)}\Big)_+ \\
= \; & \underbrace{V^{2B\to A}(1,3)}_{\substack{\text{time-1 payoff of option to} \\ \text{exchange 2 units of B for A at time 3}}} + \mathrm{e}^{-0.02(2)} \underbrace{V^{A\to 2B}(1,1)}_{\text{exchange A for 2 units of B at time 1}}.
\end{aligned}$$

Thus the time-0 price of the chooser is

$$\underbrace{V^{2B\to A}(0,3)}_{=8 \text{ (from (v))}} + \mathrm{e}^{-0.04}V^{A\to 2B}(0,1) = 8 + \mathrm{e}^{-0.04}V^{A\to 2B}(0,1).$$

By the exchange option parity at time 0, we have

$$V^{A\to 2B}(0,1) - \underbrace{V^{2B\to A}(0,1)}_{=5 \text{ (from (iv))}} = 2F^P_{0,1}(B) - F^P_{0,1}(A) = \mathrm{e}^{-0.02(1)}[2(100) - 205],$$

so $V^{2B\to A}(0,1) = 0.099007$. Hence the answer is $8 + \mathrm{e}^{-0.04}(0.099007) = \boxed{8.0951}$. □

51. *Solution.* (a) From the perspective of time 1, the contingent claim maturing at time 3 is a 2-year $S(1)$-strike European cash-or-nothing call of $\$S(1)$ payable at time 3. Since

$$\begin{aligned} d_1 &= \frac{\ln[S(1)/S(1)] + (0.08 - 0.02 + 0.3^2/2)(2)}{0.3\sqrt{2}} = 0.49497, \\ d_2 &= d_1 - 0.3\sqrt{2} = 0.07071, \\ N(d_2) &= 0.52819, \end{aligned}$$

the time-1 price of the contingent claim is

$$S(1)\mathrm{e}^{-2r}N(d_2) = S(1)\mathrm{e}^{-0.08(2)}(0.52819).$$

Hence the time-0 price is

$$F^P_{0,1}(S)\mathrm{e}^{-0.08(2)}(0.52819) = 100\mathrm{e}^{-0.02-0.08(2)}(0.52819) = \boxed{44.12}.$$

(b) From the point of view of time 1, the contingent claim is a 2-year $S(1)$-strike European asset-or-nothing call. With $N(d_1) = 0.68969$, the time-1 price of the contingent claim is

$$S(1)\mathrm{e}^{-2\delta}N(d_1) = S(1)\mathrm{e}^{-0.02(2)}(0.68969).$$

Thus the time-0 price is

$$F^P_{0,1}(S)\mathrm{e}^{-0.02(2)}(0.68969) = 100\mathrm{e}^{-0.02(3)}(0.68969) = \boxed{64.95}.$$

□

53. *Solution.* The time-3 payoff of the forward start gap call option is

$$V(3) = [S(3) - 100]1_{\{S(3)>S(2)\}}.$$

At time 2, $S(2)$ is known and the forward start, by the Black-Scholes formula for gap options, is worth

$$V(2) = S(2)\mathrm{e}^{-\delta}N(d_1) - 100\mathrm{e}^{-r}N(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{\ln[S(2)/S(2)] + (0.06 - 0.025 + 0.24^2/2)(1)}{0.24\sqrt{1}} = 0.26583, \\ d_2 &= d_1 - 0.24\sqrt{1} = 0.02583, \end{aligned}$$

both of which do not depend on $S(2)$. It follows that the forward start is equivalent to $e^{-\delta}N(d_1)$ units of a 2-year prepaid forward on the stock and a short 2-year zero-coupon bond of face value $100e^{-r}N(d_2)$. As $N(d_1) = 0.6064$ (0.60481) and $N(d_2) = 0.5120$ (0.51030), the time-0 price of the forward start is

$$\begin{aligned}
V &= F_{0,2}^P(S)e^{-\delta}N(d_1) - 100e^{-3r}N(d_2) \\
&= S(0)e^{-3\delta}N(d_1) - 100e^{-3r}N(d_2) \\
&= 100e^{-3(0.025)}(0.60481) - 100e^{-3(0.06)}(0.51030) \\
&= \boxed{13.48701}.
\end{aligned}$$

□

Remark. If it is the strike price (instead of the payment trigger) of the gap call that equals $S(2)$, then d_1 and d_2 will depend on $S(2)$ and the current price of the forward start cannot be easily found.

55. *Solution.* Let's first derive a general expression for the current price of the T-year contingent claim which pays at time T $\min[S(t), S(T)]$ for some $t \leq T$, then evaluate the price at the numerical values given in the problem.

- *Method 1:* To begin with, we rewrite the time-T payoff of the contingent claim as

$$\min[S(t), S(T)] = S(T) + \min[S(t) - S(T), 0] = S(T) - (S(T) - S(t))_+,$$

which shows that the contingent claim is equivalent to a T-year prepaid forward on the stock plus a short T-year forward start option whose strike is determined by the time-t stock price. It follows from the results in Subsection 8.8.2 that the time-0 price of the contingent claim is

$$\begin{aligned}
V(0) &= S(0)e^{-\delta T} - S(0)[e^{-\delta T}N(d_1) - e^{-\delta t - r(T-t)}N(d_2)] \\
&= S(0)e^{-\delta T}N(-d_1) + S(0)e^{-\delta t - r(T-t)}N(d_2),
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \frac{\ln[S(t)/S(t)] + (r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \frac{(r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\
d_2 &= d_1 - \sigma\sqrt{T-t},
\end{aligned}$$

both of which are free of $S(t)$.

- *Method 2:* The time-T payoff of the contingent claim is

$$\begin{aligned}
\text{Payoff} &= \begin{cases} S(t), & \text{if } S(t) < S(T) \\ S(T), & \text{if } S(T) \leq S(t) \end{cases} \\
&= S(t)1_{\{S(T) > S(t)\}} + S(T)1_{\{S(T) \leq S(t)\}}.
\end{aligned}$$

From the perspective of time t, the contingent claim is equivalent to a $(T-t)$-year $S(t)$-strike *cash*-or-nothing *call* of $\$S(t)$, plus a $(T-t)$-year $S(t)$-strike *asset*-or-nothing *put*. Thus the time-t price is

$$\begin{aligned}
V(t) &= S(t)e^{-r(T-t)}N(d_2) + S(t)e^{-\delta(T-t)}N(-d_1) \\
&= S(t)[e^{-r(T-t)}N(d_2) + e^{-\delta(T-t)}N(-d_1)],
\end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln[S(t)/S(t)] + (r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \frac{(r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

both of which are free of $S(t)$. It follows that the time-0 price is

$$\begin{aligned} V(0) &= S(0)\mathrm{e}^{-\delta t}[\mathrm{e}^{-r(T-t)}N(d_2) + \mathrm{e}^{-\delta(T-t)}N(-d_1)] \\ &= S(0)[\mathrm{e}^{-\delta t - r(T-t)}N(d_2) + \mathrm{e}^{-\delta T}N(-d_1)]. \end{aligned}$$

With $S(0) = 20, r = 0.04, \delta = 0.015, \sigma = 0.25, t = 2, T = 3$, we have

$$\begin{aligned} d_1 &= \frac{(0.04 - 0.015 + 0.25^2/2)(3-2)}{0.25\sqrt{3-2}} = 0.225, \\ d_2 &= d_1 - 0.25\sqrt{1} = -0.025, \\ N(-d_1) &= 0.41099, \\ N(d_2) &= 0.49003, \end{aligned}$$

so the time-0 price of the contingent claim is

$$\begin{aligned} V &= 20\mathrm{e}^{-0.015(3)}(0.41099) + 20\mathrm{e}^{-0.015(2)-0.04(1)}(0.49003) \\ &= \boxed{16.9961}. \end{aligned}$$

□

B.9 Chapter 9

1. *Solution.* By put-call parity for nondividend-paying stocks, we have

$$0.15 = C - P = S(0) - 70\mathrm{e}^{-rT} = 60 - 70\mathrm{e}^{-4r},$$

which yields $r = \boxed{0.039}$. **(Answer: (A))** □

3. *Solution.* By currency option put-call duality, the prices of 3-year dollar-denominated put options on euros with strike prices of \$1.2 and \$1.5 are, respectively,

$$P_{\$}(1.25, 1.2) = 1.25 \times 1.2 \times \underbrace{C_{€}(1/1.25, 1/1.2)}_{0.03315 \text{ from (v)}} = 0.049725$$

and

$$P_{\$}(1.25, 1.5) = 1.25 \times 1.5 \times \underbrace{C_{€}(1/1.25, 1/1.5)}_{0.1641 \text{ from (iv)}} = 0.3076875.$$

Two applications of currency put-call parity, one at the strike of \$1.2, and one at the strike of \$1.5, yield

$$\begin{cases} C_{\$}(1.25, 1.2) - P_{\$}(1.25, 1.2) = X(0)\mathrm{e}^{-r_{€}T} - K_1\mathrm{e}^{-r_{\$}T} \\ C_{\$}(1.25, 1.5) - P_{\$}(1.25, 1.5) = X(0)\mathrm{e}^{-r_{€}T} - K_2\mathrm{e}^{-r_{\$}T} \end{cases}$$

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