Derivatives

T-503-AFLE

Teacher: Dr. Maxime Segal

Assistant Teacher: Aqib Ahmed

August 18, 2025 – November 7, 2025

Introduction

Few words about your teacher

Dr. Maxime Segal

Currently:

- Senior Risk Modelling, **Íslandsbanki**.
- Adjunct Prof., **Reykjavik University**.

In Reykjavik University:

- 5 times **Teacher**, 5 times **Assistant Teacher**

2019-2023:

- PhD in Financial Engineering, **Reykjavik University**.

2020-2022:

- Head of Markets, **Technology Metals Market**. (2020-2022)
- Strucutred Products Sales, **Leonteq,** Monaco (2019)
- Financial Engineer, **Crédit Agricole,** Paris (2018

Class Methodology

Timetable

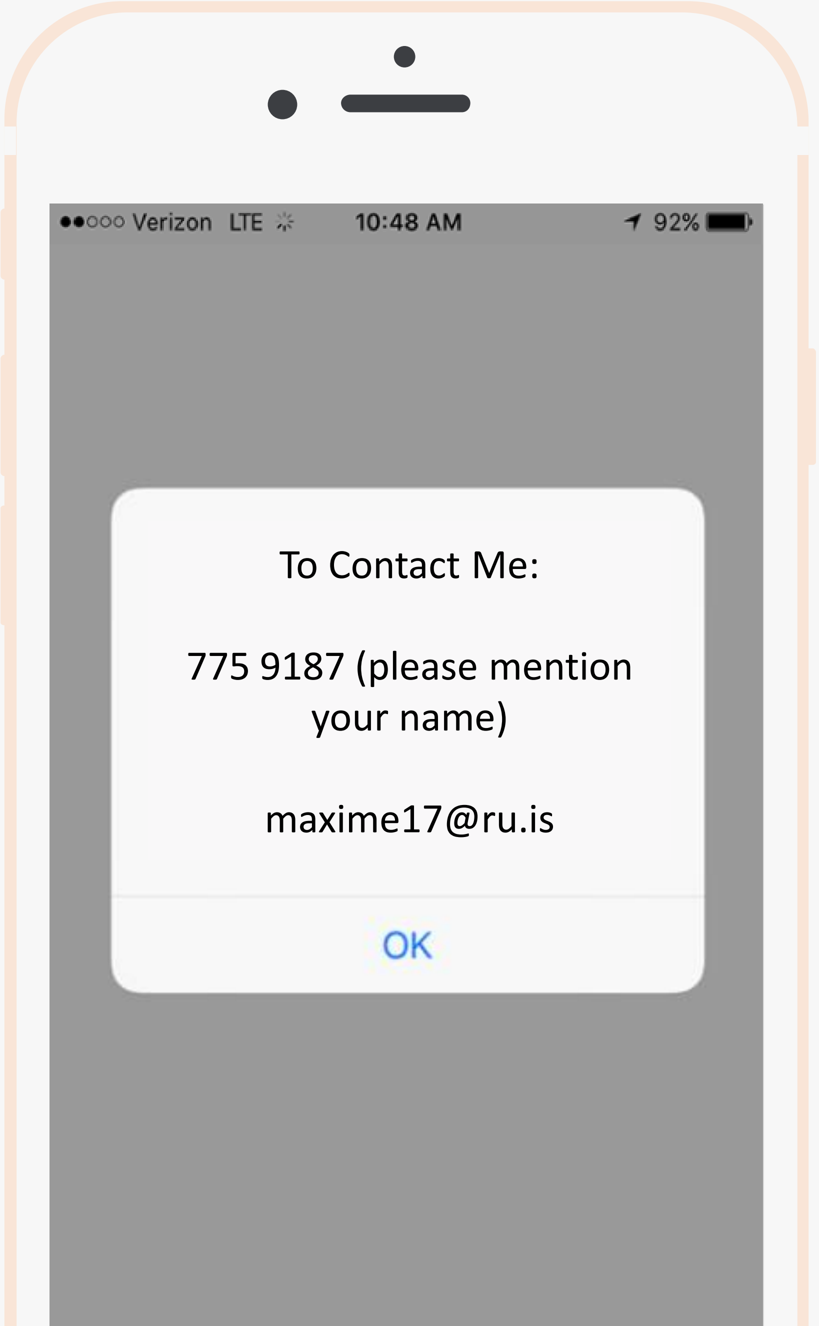
- Mondays, 12.40 – 14.20, session with TA
- Tuesdays, 12.40 – 14.20
- Fridays, 08:30 – 10.10
- 12 weeks

Teaching

- Active teaching + Class discussions with 2-3 speakers to give an insight from the trading floor.
- Ambrose Lo, Derivative Pricing: A Problem-Based Primer.
- Options, Futures, and Other Derivatives, J. Hull.

Exams

- Final Exam (40%)
- Mid-Term Exam (30%)
- Class quizz, 15’, bi-weekly (30%)
  - The worst grade will be dropped



Introduction

I/ Futures & Forward

A first approach

The first module will aim to appreciate the difference between spot, futures, forward and swap rates. We will see how they are traded and priced as well as which payoff to expect.

II/ Options

A deeper insight in derivatives

An option is a contract that gives the owner some flexibility in its execution. It gives the right, by opposition with the obligation, to buy or sell an underlying at a prefixed strike price on a given date. The pricing becomes more challenging, but you will love them!

III/ Hedging

Risk mitigation

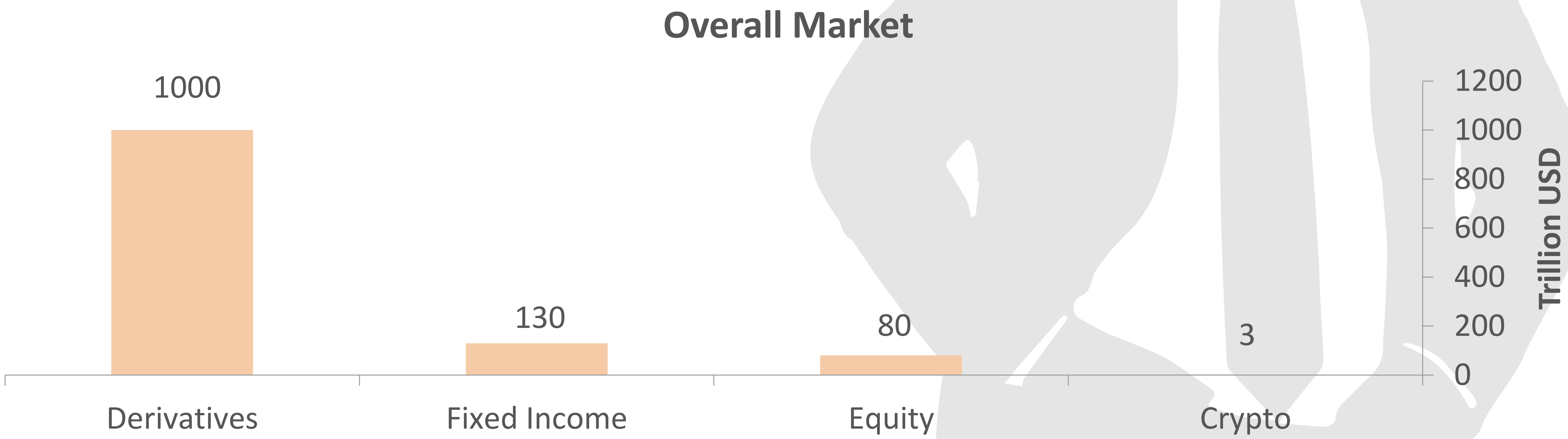
Derivatives are not only aiming to get leverage and allows to speculate. They can be used as an insurance tool to hedge an inverstor on potential losses in case of advserse market movements.
This module will introduce a few hedging techniques.

IV/ Structured Products

Unleash the full potential of derivatives

Mixing asset classes or combining derivatives together are creating products with complex payoffs that are called Structured Products.
In addition to introduce some, we will learn how to simulate their prices.

Financial World

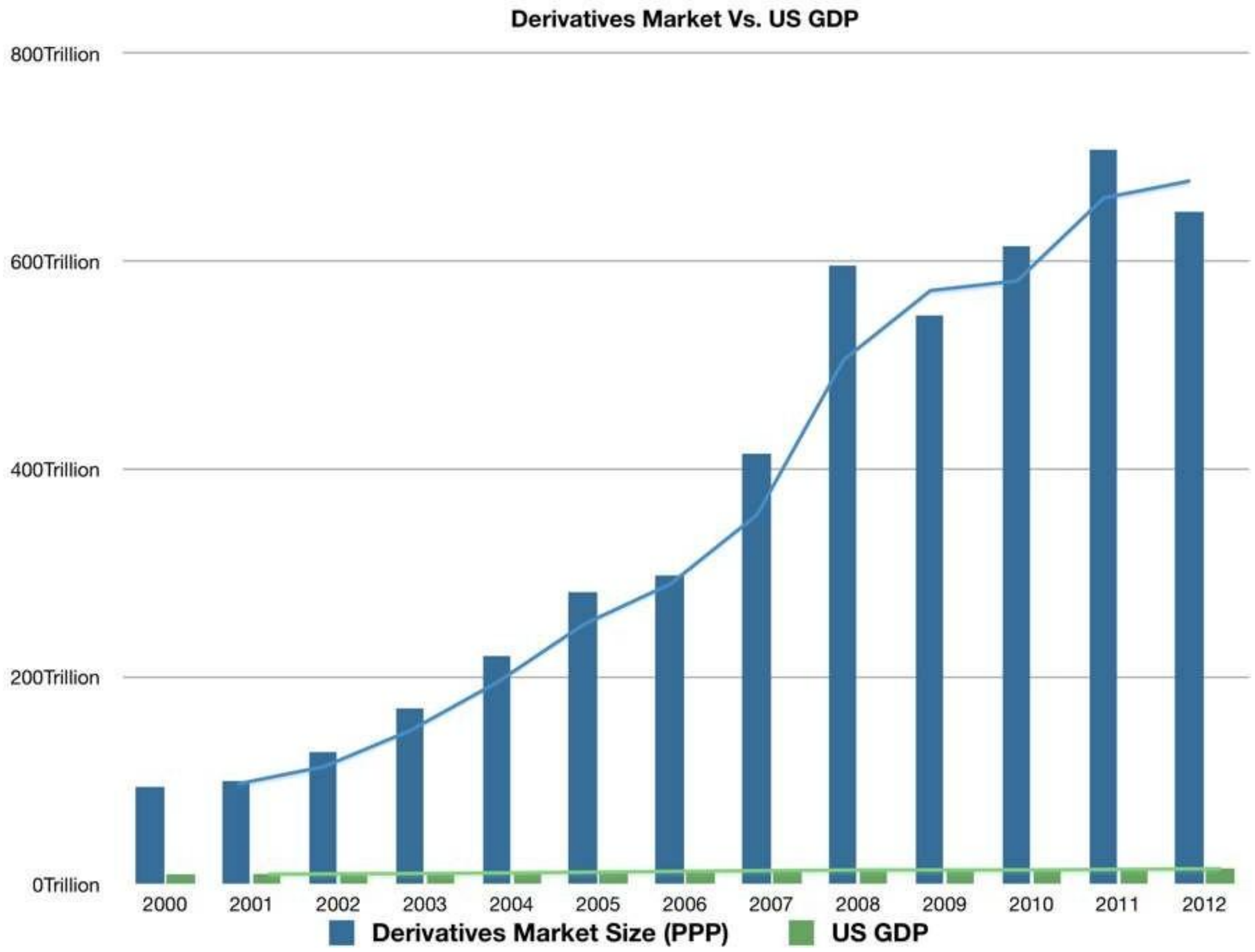


By far the largest market

Every mature market naturally develops its own derivatives segment that can end up being way larger than its underlying market.

Derivatives market size

Size of derivatives markets in relation to the US economy



What are Derivatives?

Why using them? Who use them? When to use them?

A derivative is a financial instrument that derives its value from the performance of an underlying entity.

The most traded are the following one

FORWARDS

FUTURES

OPTIONS

SWAPS

<u>Two investment strategies:</u>

- LONG position
- SHORT position

<u>Futures / Forwards</u>

Agreement to deliver or receive assets at a future date for a price set today.
Some intrinsic differences in their design, that will be covered later.

<u>Options</u>

Contract that offers the buyer the right (but not the obligation) to buy or sell an asset at a given time horizon for a pre-fixed price.

<u>Swaps</u>

Agreement through which two parties exchange a fixed against a variable cashflow (usually from different instruments).

What are Derivatives?

Why using them? Who use them? When to use them?

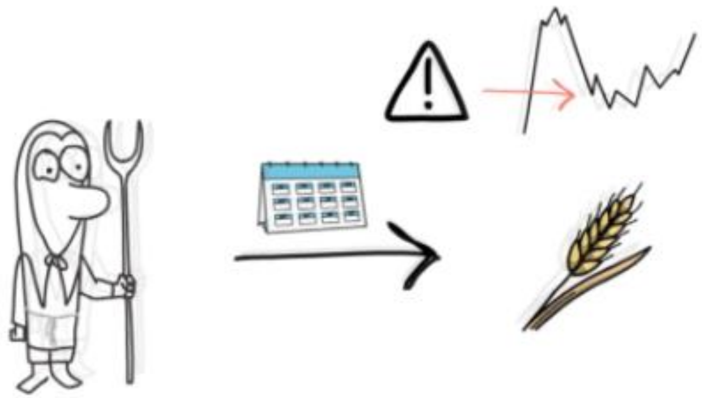
A derivative is a financial instrument that derives its value from the performance of an underlying entity.

The most traded are the following one

The two main uses are:

- **Hedging (risk mitigation)**
- Speculation

Example 1: The Wheat farmer



- Price Fluctuating
- How to mitigate the naked position risk?

What are Derivatives?

Why using them? Who use them? When to use them?

A derivative is a financial instrument that derives its value from the performance of an underlying entity.

The most traded are the following one

FORWARDS

FUTURES

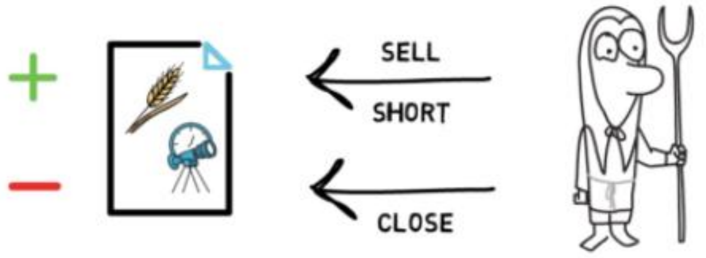
OPTIONS

SWAPS

The two main uses are:

- **Hedging (risk mitigation)**
- Speculation

Example 1: The Wheat farmer



- Short sell wheat future contract for the amount predicted to harvest

If spot wheat goes **down** -> the short position makes **profit**
If spot wheat goes **up** -> the short position incurs a **loss**.

The total revenue becomes **predictable**!

What are Derivatives?

Why using them? Who use them? When to use them?

A derivative is a financial instrument that derives its value from the performance of an underlying entity.

The most traded are the following one

FORWARDS

FUTURES

OPTIONS

SWAPS

The two main uses are:

- Hedging (risk mitigation)
- **Speculation**

Example 2: The Reykjavik Univ Student

- Think the price of the Bitcoin will go up.
- Cannot afford to buy 1 BTC for 105,000 USD.
- How to take leverage of this belief?

What are Derivatives?

Why using them? Who use them? When to use them?

A derivative is a financial instrument that derives its value from the performance of an underlying entity.

The most traded are the following one

FORWARDS

FUTURES

OPTIONS

SWAPS

The two main uses are:

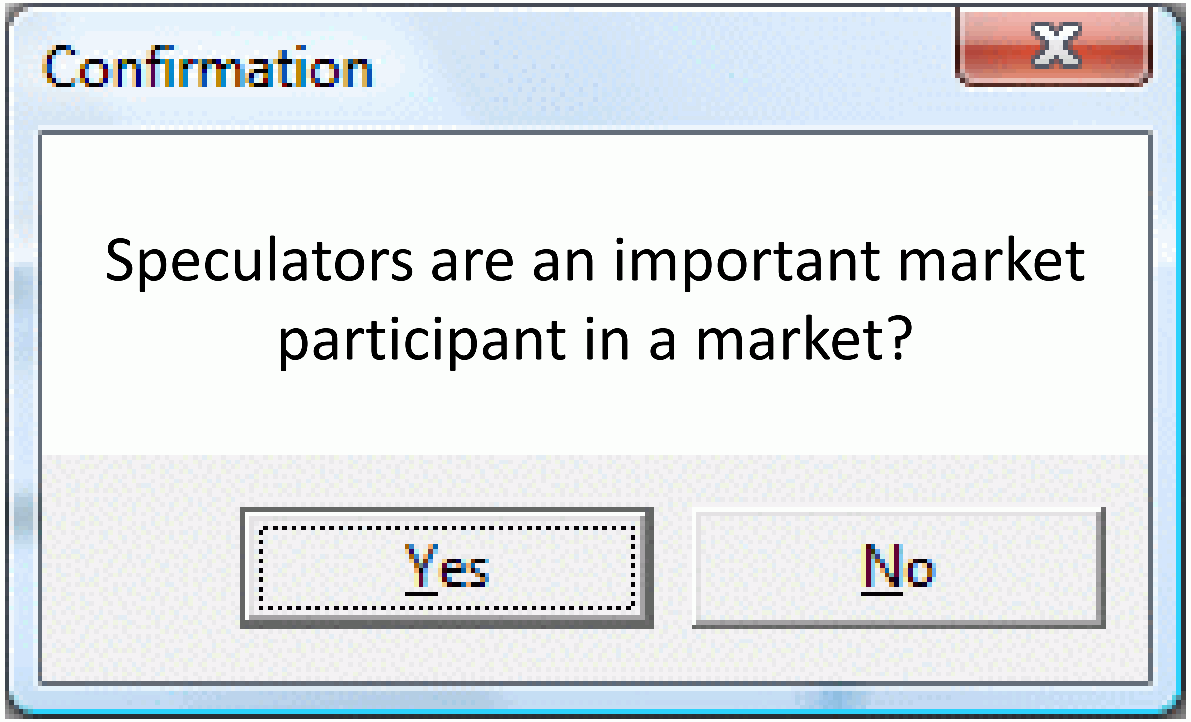
- Hedging (risk mitigation)
- **Speculation**

Example 2: The Reykjavik Univ Student

- Take a LONG position on Call option. Each call option costs 400 USD and allows in a 6 month time to buy 1 BTC @ 107,000 USD.

-> If the price goes up to 110,000 USD, the student do 2,600 USD of profit.
-> If the price goes down to 70,000 USD, the student “just” lose 400 USD.

Open Question



YES.

Speculators are **liquidity providers.**

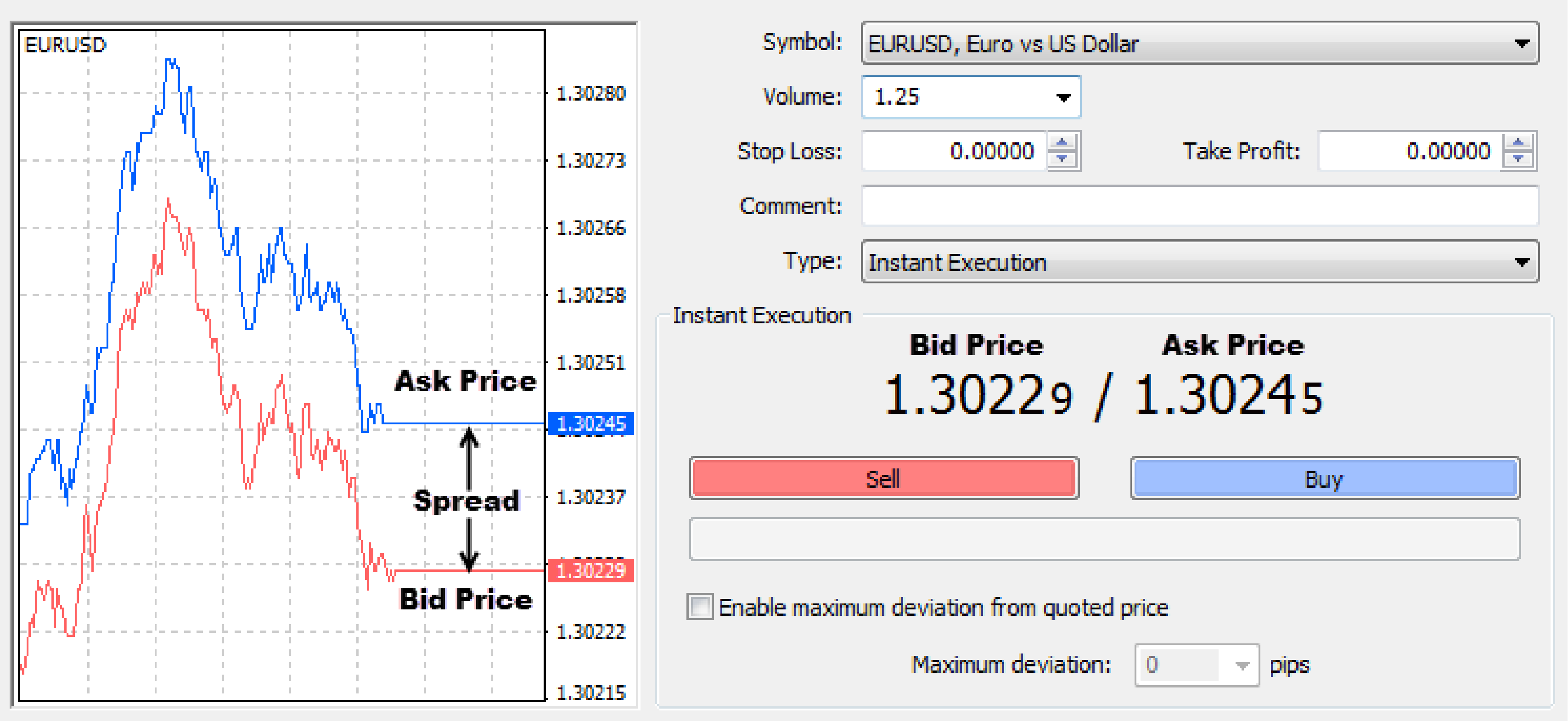
By supplying the market with liquidity, it allows investors, who want to buy a specific instrument to hedge their risk, in addition to easily open and close a position.

BID
The price a buyer is willing to pay for an asset

SPREAD
The difference between the lowest ask price and the highest bid price

ASK - BID = SPREAD

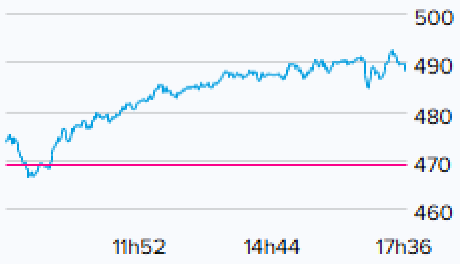
ASK
The price a seller is willing to accept for an asset



Course
LVMH

488,5000 (c) EUR
+4.15%

FR0000121014 MC
EURONEXT PARIS REAL-TIME DATA
Enforcement Policy
Quotation on other markets



SECTOR
Clothing and accessories

BENCHMARK INDEX
CAC 40

ORDER BOOK

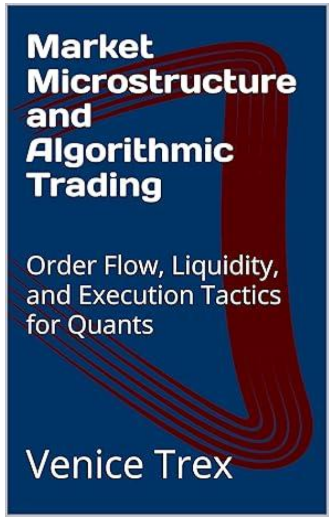
ORDERS	QTY	PURCHASE	SALE	QTY	ORDERS
1	**15**	**488,0000**	**492,0000**	**8**	**2**
1	2	487,5000	492,2000	1	1
1	100	486,6000	492,8000	3	1
1	10	486,0000	493,0000	48	3
1	1	485,8500	493,4500	8	2
2	7	485,0000	494,0000	26	3
4	58	482,0000	494,2500	4	1
1	22	481,9500	494,5000	1	1
1	29	481,1000	494,9000	1	1
10	93	480,0000	495,0000	449	20
23	**337**	**TOTAL**	**TOTAL**	**549**	**35**

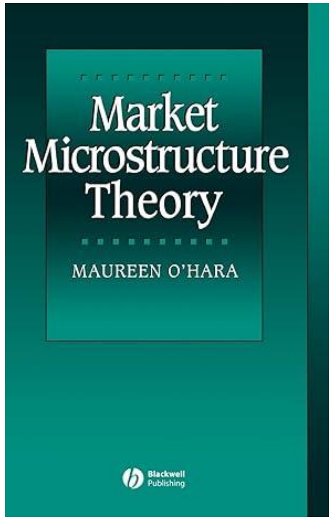
Bid-Ask Spread

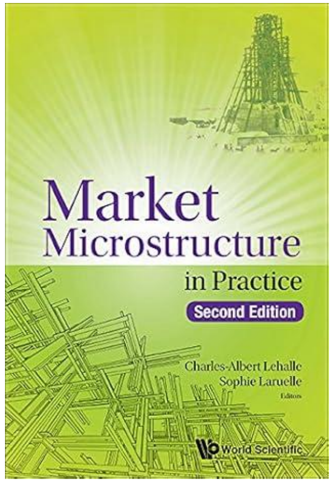
- Reflects **liquidity, transaction costs, market efficiency**.
- Narrow spreads: highly liquid & competitive market
- Wide spreads: information asymmetry, low liquidity and inefficiencies

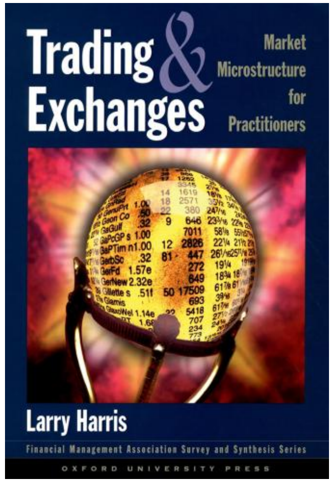
Order Book

- Insight into **market depth, order flow, and supply/demand dynamics**
- Reveals:
    - Liquidity concentration around price levels
    - Presence of spoofing/manipulation
    - Latency arbitrage and HFT behaviour

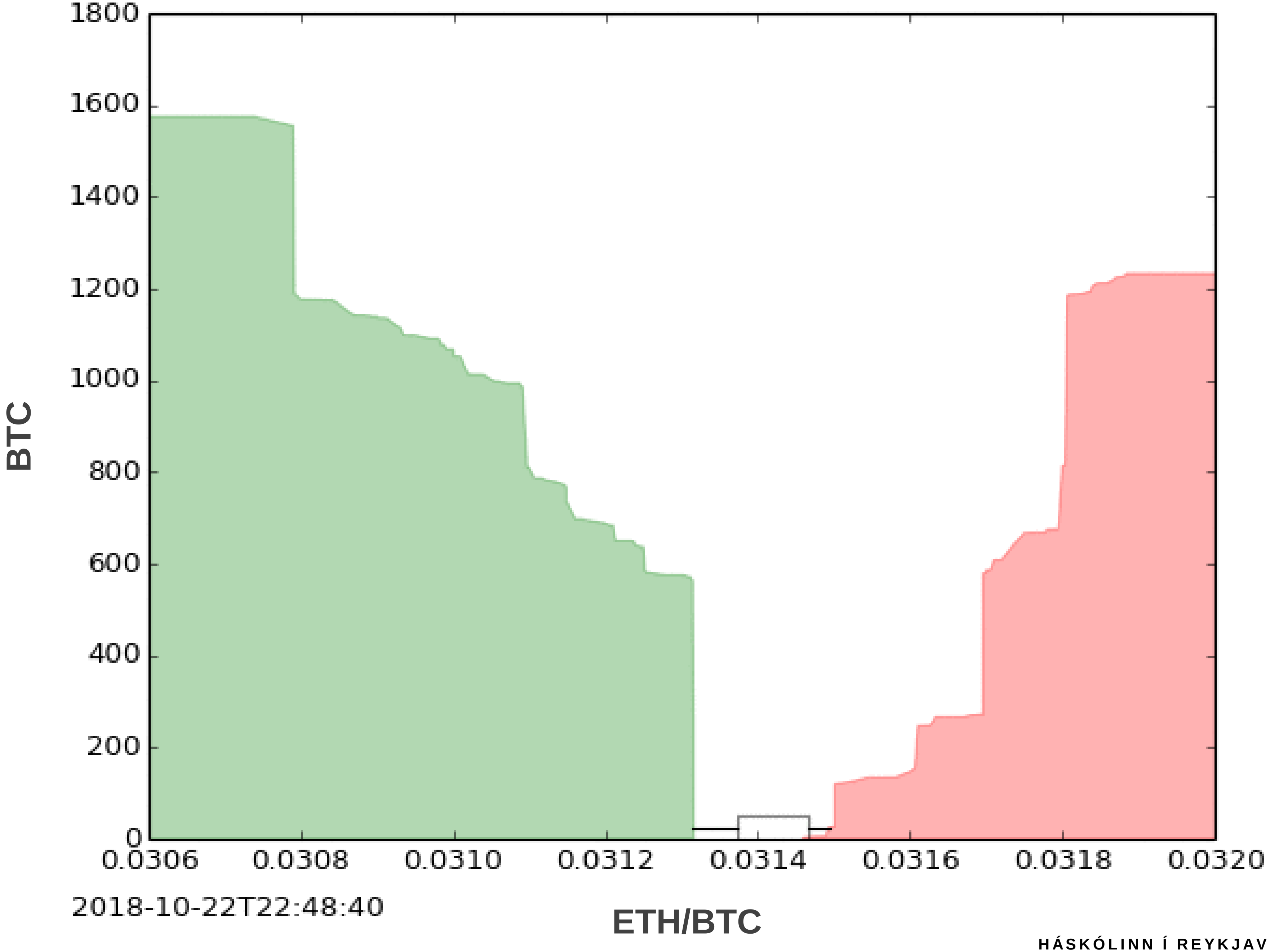








ETH/BTC order book over 10 hours, starting Oct 22, 2018



Where to trade derivatives

- The Chicago Board of Trade (CBOT) was established in 1848 to bring farmers and merchants together where standardized quantities of agricultural products could be traded.
- The Chicago Mercantile Exchange (CME) was established in 1919
- The Chicago Board Options Exchange (CBOE) started trading call options on 16 stocks in 1973
- Put options started trading on CBOE in 1977

- **There is also a very active over-the-counter (OTC) market in derivatives.**

OTC vs. Exchange trades.

Upsides

OTC traded products allow to design more flexible instruments (customized on the client requirements)

Clearing house for derivatives traded on the market

Low prices can mean the potential for big percent moves (OTC).

Downsides

Lower liquidity on a contract OTC traded

Credit Risk if the seller does not honor its contract in the OTC market

Organized exchanges trade in standardized contracts

The concept of **Arbitrage**

In 1 example...

On the NYSE, $MSFT is trading
@ USD 244.5 / 246.1

OFFER QTY	LAST PRICE	OPEN	HIGH	LOW	TRADES	VOLUM

Order Book By Order ×

BID		OFFER	
QUANTITY	PRICE	PRICE	QUANTITY
2	244.5	246.1	2
2	243.8	247.0	4
2	241.0	249.0	4

WHAT WOULD YOU DO?

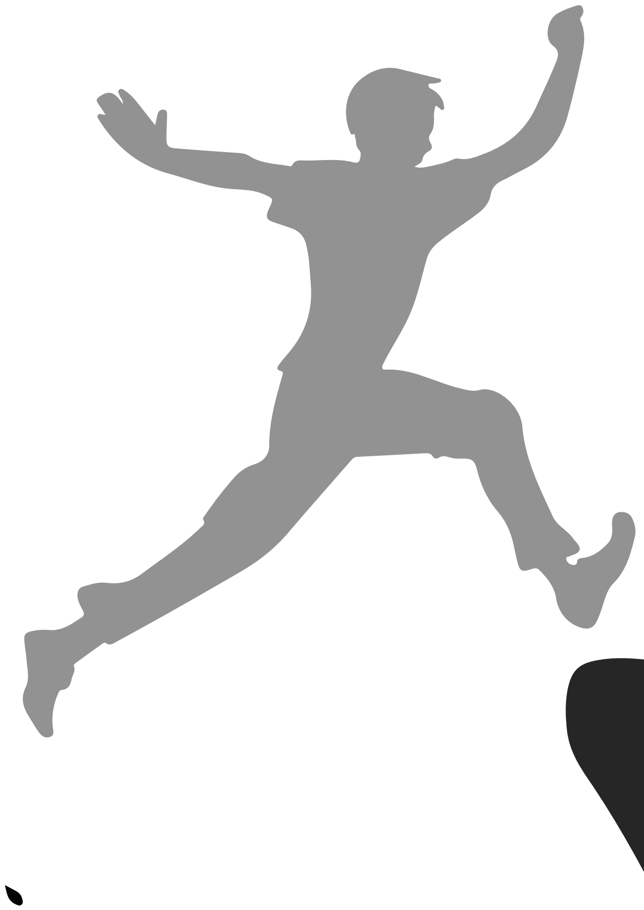
As a delta one (equity) trader on the Reykjavik University trading floor

In Frankfurt, $MSFT is trading
@ USD 246.3 / 246.5

Order Book By Order ×

BID		OFFER	
QUANTITY	PRICE	PRICE	QUANTITY
2	246.3	246.5	2
2	246.7	246.8	4
2	246.9	247.0	4

The concept of **Arbitrage**



In 1 example...

On the NYSE, $MSFT is trading
@ USD 244.5 / 246.1

OFFER QTY	LAST PRICE	OPEN	HIGH	LOW	TRADES	VOLUM

Order Book By Order ×

BID	
QUANTITY	PRICE
2	244.5
2	243.8
2	241.0

OFFER	
PRICE	QUANTITY
246.1	2
247.0	4
249.0	4

In Frankfurt, $MSFT is trading
@ USD 246.3 / 246.5

Order Book By Order ×

BID	
QUANTITY	PRICE
2	246.3
2	246.7
2	246.9

OFFER	
PRICE	QUANTITY
246.5	2
246.8	4
247.0	4

BUY 2 shares @ 246.1 USD on Nasdaq

and instantaneously

SELL 2 shares @ 246.3 USD in Frankfurt

=

$ 0.2 * 2 = $ 0.4 risk-free profit

The concept of **Arbitrage**

In 1 example…

Arbitrage is the practice of taking advantage of price discrepancies without taking any risk.

If 2 equally risky portfolios P_1 and P_2 are expected to have the same value at the future time T, i.e.

$$E_t P_1(T) = E_t P_2(T)$$

Then, they should have the same price at all previous times, $t < T$ i.e.

$$P_1(t) = P_2(t)$$

In the previous example, we do say that an arbitrage opportunity was existing. As we take profit of this opportunity, the prices are said to converge following our actions.

<u>**Be careful**</u>, if you had to borrow money to engage the initial purchase of 2 MSFT shares, the resulting profit still need to be above your borrowing costs.

Module 1

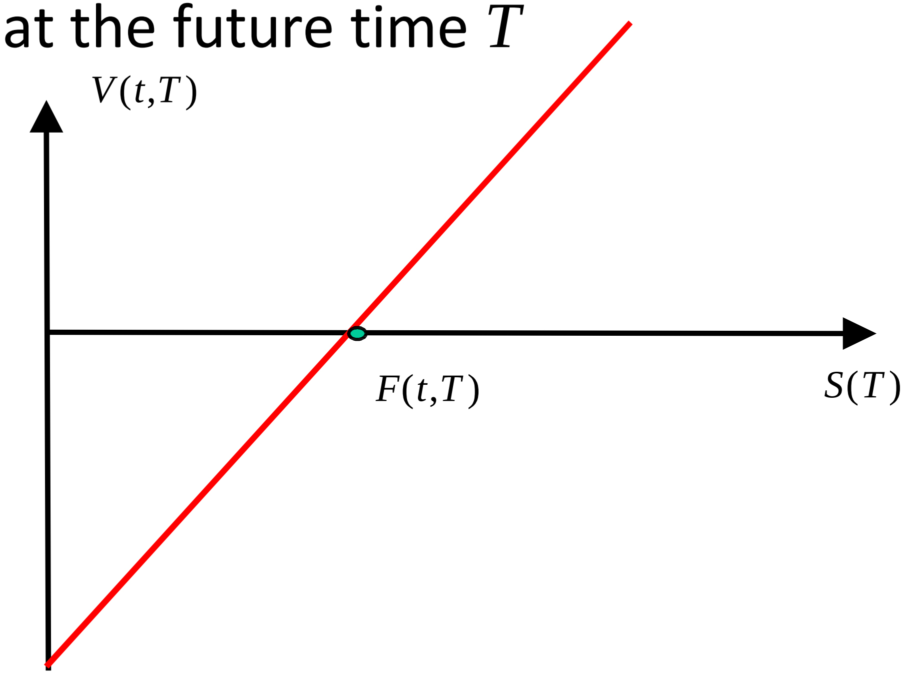
Futures & Forward

1.1. Forwards

Example – Long Forward Contract

- Let’s now consider the payment and the physical receipt of a stock as two different activities. This allows to introduce the concept of forwards
- A **long** position in a **forward contract** on a market asset <u>**obliges**</u> its holder <u>**to buy**</u> this asset at the future time T at a price $F(t,T)$ which is fixed today at time t.
- The present price of the asset is $S(t)$ but its price $S(T)$ at the future time T is unknown today.
- At T, the value of the long position is:

$$V(t,T) = S(T) - F(t,T) = \begin{cases} > 0 \; if \; S(T) > F(t,T) \\ < 0 \; if \; S(T) < F(t,T) \end{cases}$$



Forward contracts and Futures

- The essential difference between forward contracts and futures is that:
  - Forward contracts are settled only at the maturity of the contract and they are traded as over the counter (OTC).
  - Futures are settled daily (in a process called Mark-to-Market) and they are traded on organized exchanges

Spot vs. Forward

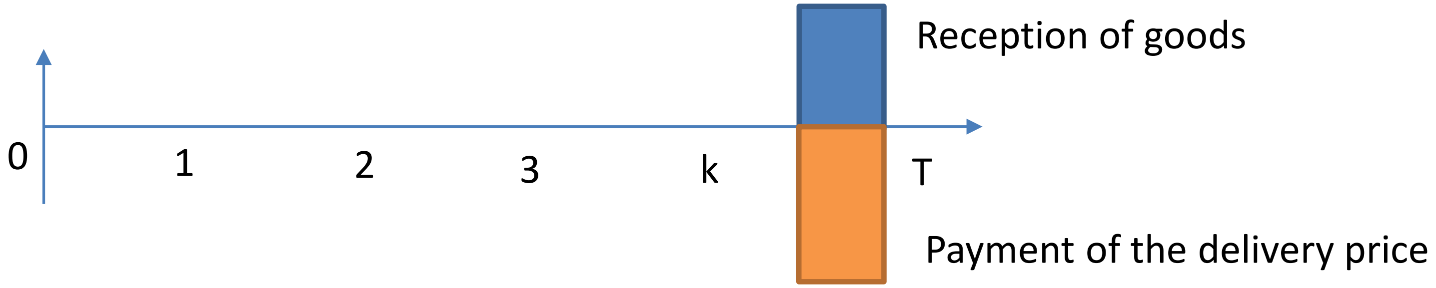
- Do not mix with **spot** contract, which is an agreement to buy or sell an asset today – or instantly. E.g. Buying a Viking in Vinbudin is a spot transaction.

- In a forward contract one of the parties takes a long position which agrees to buy the asset at specified time in the future (maturity) but prefixed price (delivery price). The party with the short position agrees to sell the asset on the same date and for the same price.

- At maturity, forward can be **settled in cash** (exchange of cashflows only) or **physically** (with delivery of the goods).

Forward contracts

- At the time a forward contract is entered into, the value of each position is **zero** - i.e. it costs nothing to enter either long or short position – in other words no exchange of cash takes place when the contract is entered into
- Forward contracts are settled at maturity – the short position delivers the asset to the long position in exchange for cash - i.e. the delivery price.



- However, because the value of a forward contract is determined by the market price of the underlying asset, any movement of the underlying asset in the market affects the value of the forward contract. If the underlying goes UP, the contract value becomes > 0 (as the beneficiary will receive the same quantity for the same pre-fixed price).

Use of forward contracts

- Forward contracts allow the parties to the contracts to **remove the future price uncertainty** by fixing today the price of that will be paid or received in the future.
- The long position knows today the cost of acquiring the asset in the future.
- The short position knows today how much the asset can be sold for in the future.
- However, the forward price may **deviate significantly** from the future spot prices leading to gain/losses for either party.
- <u>Therefore, the risk has been removed but opportunity cost remains i.e. to secure a fixed price one has to give up the benefits from possible positive price movements</u>

Use of forward contracts

- Companies trading in the international markets have a strong cost exposure to price fluctuations in a whole range of market variables including:
  - Commodity prices
  - Interest rates
  - FX exchange rates
- Forward contracts can be used to stabilize the cost exposure to these fluctuations
- Forward contracts therefore present an important tool to manage the exposure to a whole range of price risks

Use of forward contracts

- A US company planning to buy goods from the UK in the near future expects the GBP to strengthen against the USD - making the cost of the goods higher in terms of the USD.

GBP/USD pair
The GBP is strenghtening, meaning that more USD are required to buy 1 £
Google Finance. July 2, 2025.

- Instead of waiting and buy the £ when needed, the US company can enter a forward currency contract for the GBP/USD rate (USD per 1 GBP)
- By entering this contract the uncertainty (risk) caused by possible currency fluctuations can be removed (to the price of not benefiting from the potential weakening British Pounds).

An example

- Spot and forward quotes for the GBP/USD exchange rate.

Spot	1.3645
1-month forward (30 days)	1.3712
3-month forward (90 days)	1.3798
6-month forward (180 days)	1.3830

How many $ you need to buy 1 £

- You are the FX trader in the Treasury division of this US company. You decide to enter a forward contract to buy 1 million GBP in 90 days for the fixed exchange rate 1.3798.

- Was it a wise move? Let's compute two different scenarios:
  - ❖ FX rate rises to 1.4000
  - ❖ FX rate falls to 1.3450

Spot	1.3645
1-month forward	1.3712
3-month forward	1.3798
6-month forward	1.3830

How many $ you need to buy 1 £

- FX rate rises to 1.4000 (stronger GBP, weaker USD)

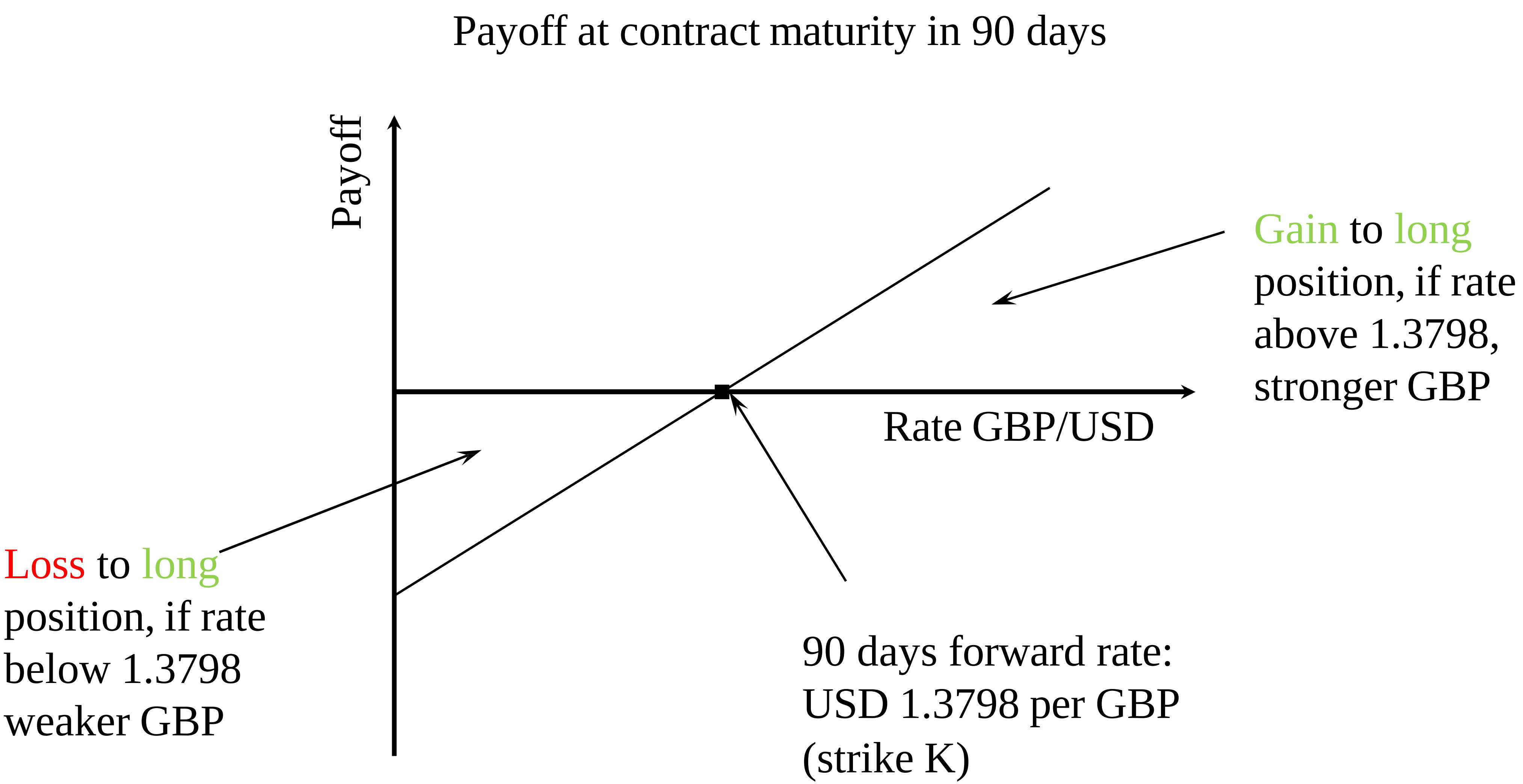
The company would gain (1.4-1.3798)*1M = USD 20,200

- FX rate falls to 1.3450

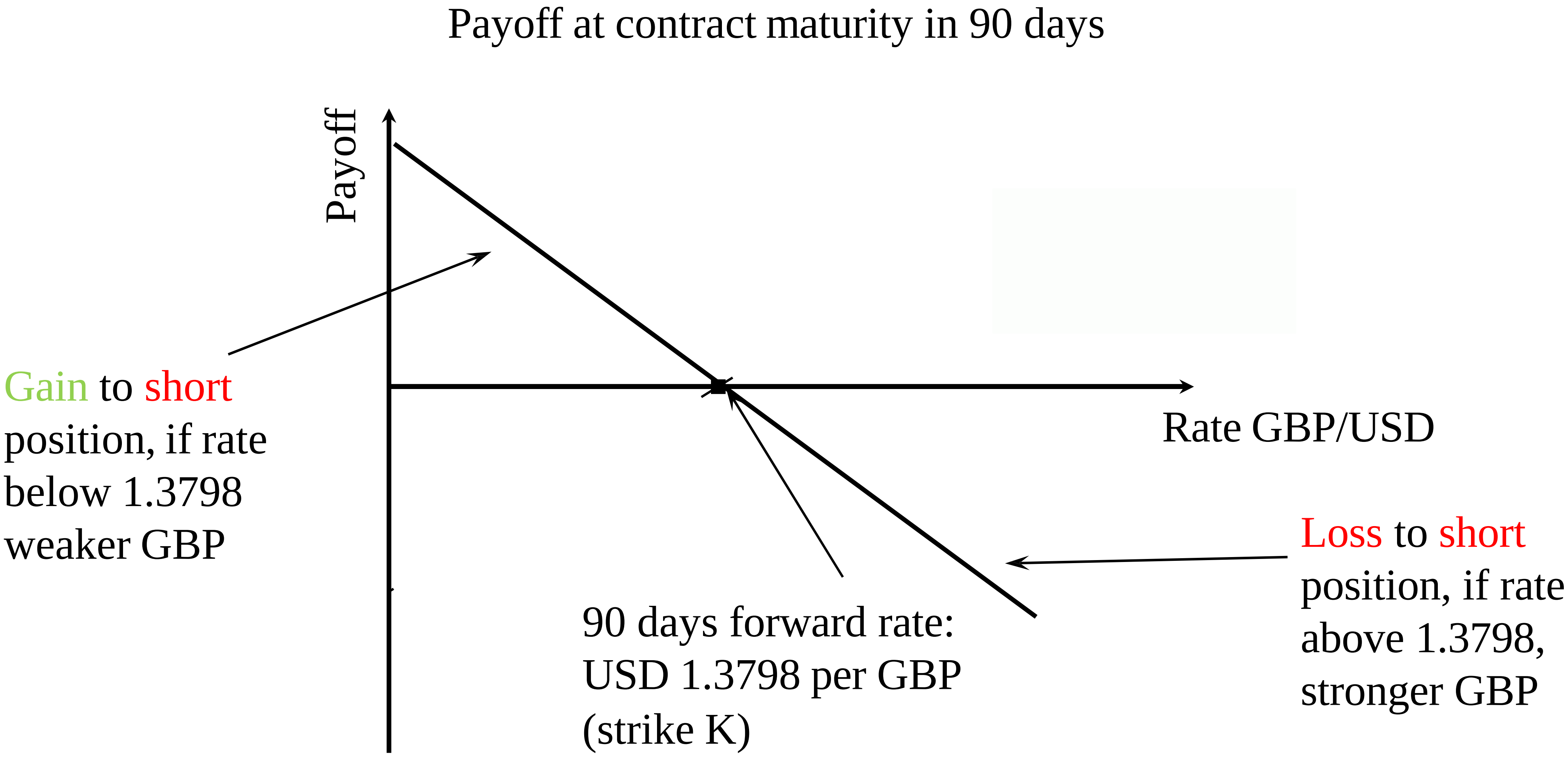
The company would loose (1.345-1,3798)*1M = - USD 34,800

- **The forward contract can lead to gains or losses, but it fixes the company's GBP/USD exposure.**

Payoff to the US company



Payoff to the forward seller



Fixing the forward price

- Generally the purchasing of an asset consists of three action points:
    1) Determining the mutually agreeable price
    2) Making the payment – the buyer (long position) pays the seller (short position)
    3) The asset is transferred from seller to the new owner
- The fact that the three action points can take place at different times offers an interesting pricing, investment and financing alternatives
- In our consideration it is assumed that all the action points can take place at two different times:
    - Today at time t
    - Or in the future at time T

Fixing the forward price

- There are four different pay – delivery alternatives:

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)\mathrm{e}^{rT}$
Prepaid forward contract	0	T	$F_{0,T}^{P}$
Forward contract	T	T	$F_{0,T}$

What is the fair price for a forward?

It is possible to derive the fair price for a forward contract, given a few simplyfing assumptions that needs to always hold. We won’t repeat them, but they are considered valid throughout this class. Real market conditions can differ.

01

Liquidity

The market is considered liquid, in the sense that entering or closing a position can be done at any time.

Transaction Costs 02

Assumed to be 0.

Taxation 03

All the market participants pay the same taxes on profits.

Interest Rate 04

Market participants are able to borrow and lend money at the same rate r.

05 Arbitrage

Market participants can take advantage of arbitrage opportunities. Let $S(t)$ be the spot price. Any arbitrage activity has been already accounted in the reflected price.

Basic case – fair price for a forward

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)e^{rT}$
Prepaid forward contract	0	T	$F^P_{0,T}$
Forward contract	T	T	$F_{0,T}$

- A market traded asset is presently priced @ $S(t)$
- An investor wants to pay for it today at time t and then take delivery at the future time T

For the time being we assume that the asset pays **no dividends** between t and T, that it incurs **no storage or other costs**, and that it provides its holder with **no rights that can be valued**. Therefore it is irrelevant whether the investor has a physical possession of the stock or not in the time from t to T.

In that case the pre-paid fair price (AOA) is simply

$$\boxed{F(t,T) = S(t)}$$

‘Less’ Basic case – fair price for a forward

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)e^{rT}$
Prepaid forward contract	0	T	$F^P_{0,T}$
Forward contract	T	T	$F_{0,T}$

Fully leveraged purchase.

The asset is priced at $S(t)$ today and pays no dividends in the time from t to T.

The investor borrows money to purchase the asset at time t $(i.e.\ \$\ S(t))$ and at the same time takes a short position in a forward contract on the asset with maturity at T.

- ✓ The forward price is set at $F(t,T)$.
- ✓ The interest rate is set at $R(t,T)$.

At maturity T

The investor delivers the asset at the agreed forward price $F(t,T)$. The investor also pays back the loan with interest:

$$S(t) * \exp(R(t,T) * (T-t))$$

To exclude any arbitrage opportunity

The forward price must have been set to:

$$F(t,T) = S(t) * \exp(R(t,T) * (T-t))$$

Otherwise

It could have been possible at inception $(t=0)$ to take profit from the existing arbitrage opportunity.

'Less' Basic case – fair price for a forward

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)e^{rT}$
Prepaid forward contract	0	T	$F^P_{0,T}$
Forward contract	T	T	$F_{0,T}$

Fully leveraged purchase.

Cashflows

Transaction	**Time t**	**Time T**
Borrow	S(t)	
Buy	-S(t)	-S(t)exp(R(t,T)(T-t))
Enter a short forward	0	F(t,T)

- At time t the net transaction value is zero, $S(t) - S(t) = 0$
- For there not to be a risk free arbitrage opportunity the forward price needs to be set at

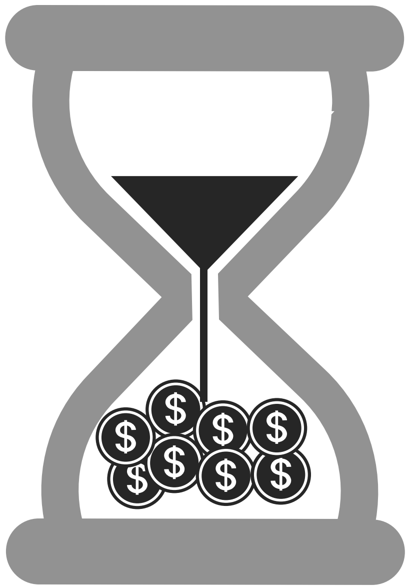
$$F(t,T) = S(t) * \exp(R(t,T) * (T-t))$$

Fair price for a forward contract

- **Statement**. The arbitrage free forward price for an asset, which pays no dividends and is presently priced at $S(t)$, and to be received at the future time T is

$$F(t,T) = S(t) * \exp\big(R(T-t) * (T-t)\big) = \frac{S(t)}{D(t,T)}$$

Where $R(t,T)$ is the rate for maturity T at which market makers can lend and borrow money and $D(t,T)$ the discount rate (the ‘time value’).



Demonstration

We draw the proof by showing that any other situation would lead to an arbitrage opportunity by constituting a specific portfolio

Demo- Fair price for a forward contract

- Let's assume that:

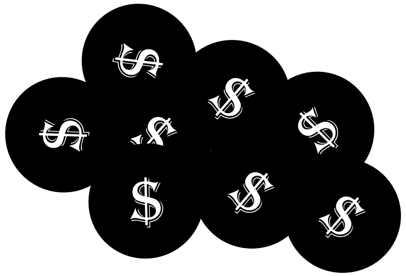
$$F(t,T) > S(t) * \exp\big(R(T-t) * (T-t)\big)$$

If a trader observes this price inequality (s)he can undertake the following transactions:

- Take a short position in a forward contract – i.e. sell a forward contract
- Borrow $S(t)$ and buy the asset at time t
- At time T deliver the asset according to the forward contract, receive $F(t,T)$ and pay back the loan

- Total risk-free profit made from this portfolio:

$$P(T) = F(t,T) - S(t) * \exp\big(R(T-t) * (T-t)\big) > 0$$



Demo- Fair price for a forward contract

- Let's assume that:

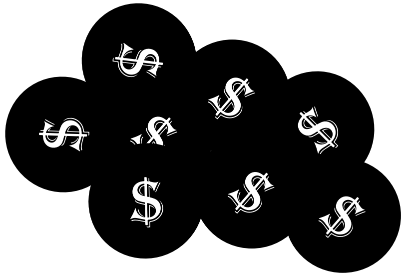
$$F(t,T) < S(t) * \exp\big(R(T-t) * (T-t)\big)$$

If a trader observes this price inequality (s)he can undertake the following transactions:

- Take a long position in a forward contract and short sell the stock to receive S(t) – i.e. buy a forward contract and sell the stock.
- Lend $S(t)$ at the interest rate $R(t,T)$ at time t
- Receive the loan with interest, take delivery of the asset and pay $F(t,T)$

- Total risk-free profit made from this portfolio:

$$P(T) = S(t) * \exp\big(R(T-t) * (T-t)\big) - F(t,T) > 0$$



Demo- Fair price for a forward contract

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)e^{rT}$
Prepaid forward contract	0	T	$F_{0,T}^{P}$
Forward contract	T	T	$F_{0,T}$

- For there not to be an arbitrage opportunity:

$$F(t,T) = S(t) * \exp\big(R(T-t) * (T-t)\big)$$

- It follows that the price of a pre-paid forward contract would simply be the spot price of the asset:

$$F_P(t,T) = PV\big(F(t,T)\big) = F(t,T) * D(t,T) = F(t,T) * \exp\big(-R(t,T)(T-t)\big)$$
$$= \mathrm{S(t)} * \exp\big(R(t,T)(T-t)\big) * \exp\big(-R(t,T)(T-t)\big) = \mathrm{S(t)}$$

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)e^{rT}$
Prepaid forward contract	0	T	$F_{0,T}^{P}$
Forward contract	T	T	$F_{0,T}$

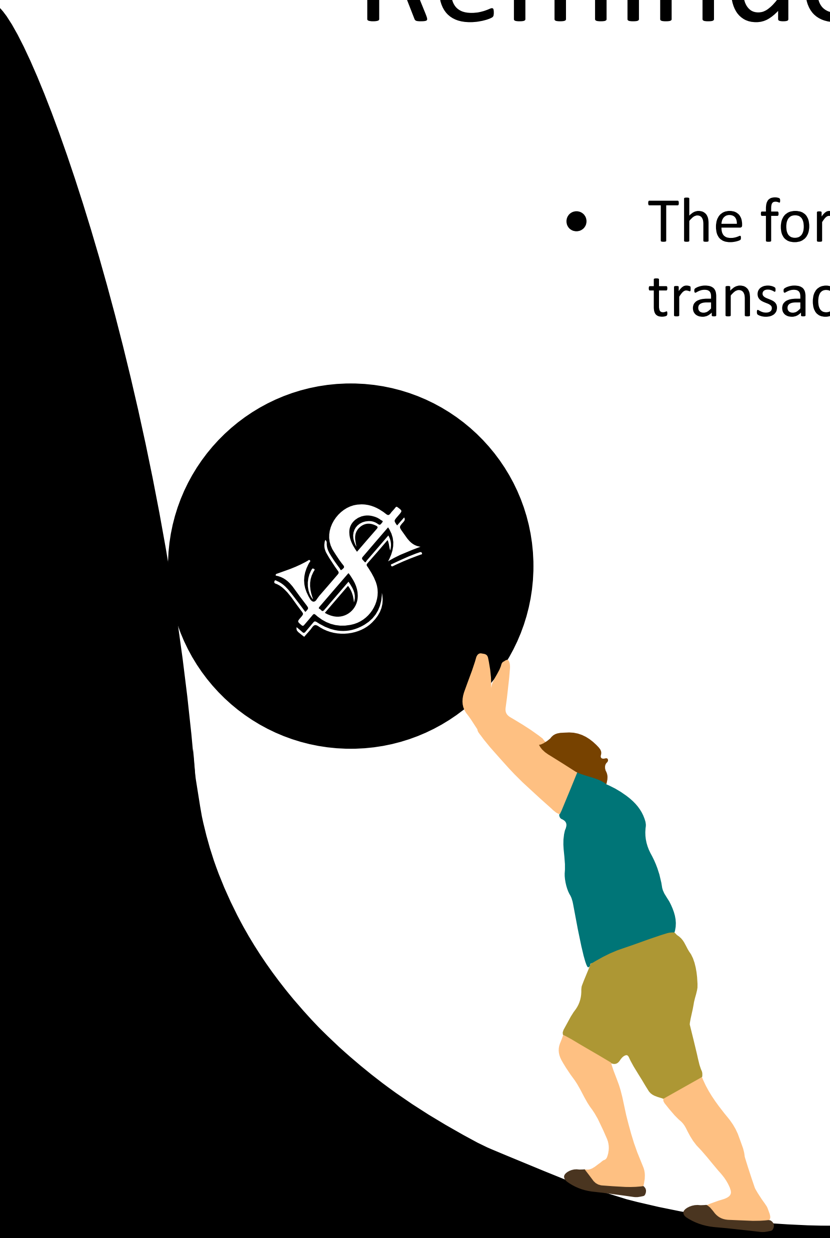
Fair price for a forward contract

- So far we have only considered forward contracts on assets that are only valued in terms of their **capital appreciation,** i.e. no attention has been paid to the fact that some assets pay the holder an income in terms of dividends or interest payments

- Income from an asset **impacts** the forward price of the asset

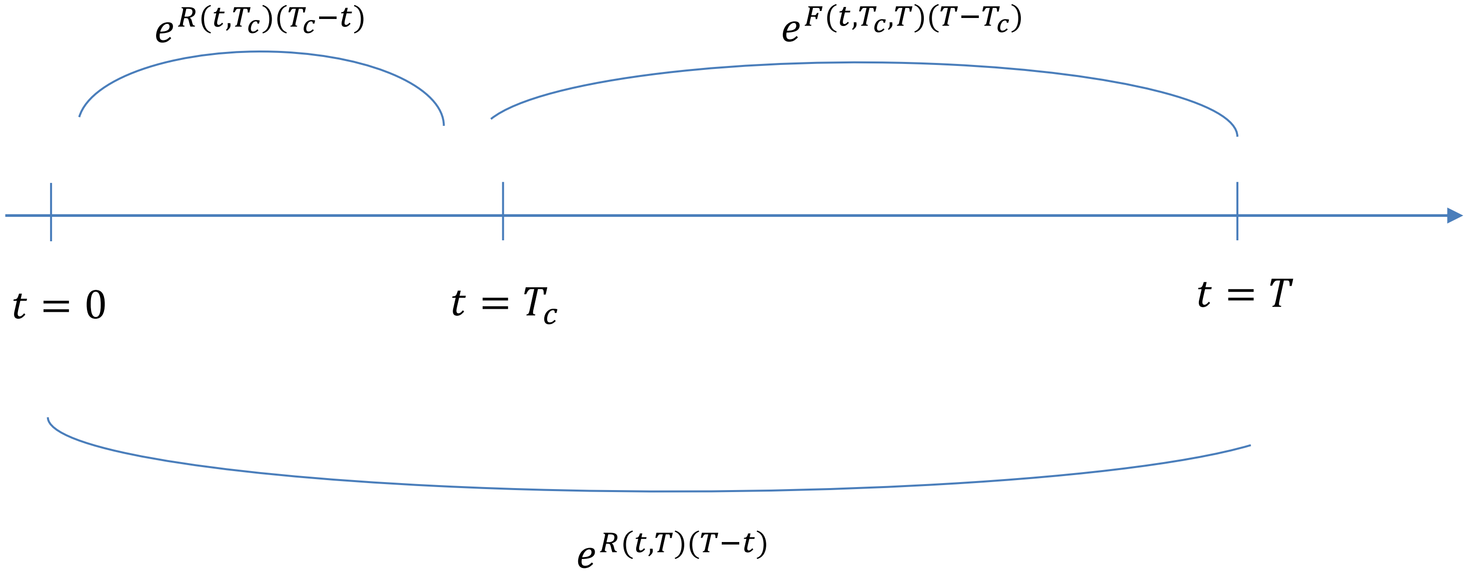
- As in the case of assets that pay no income we will proceed to calculate the forward price of income-paying assets by also using absence of arbitrage arguments

Reminder



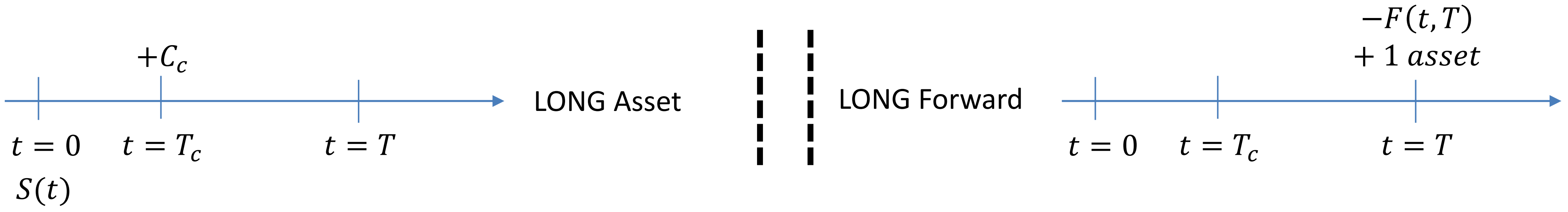
- The forward rate $F(t, T_c, T)$ is an interest rate applicable to a financial transaction that will take place in the future:

$$e^{R(t,T_c)(T_c-t)}e^{F(t,T_c,T)(T-T_c)} = e^{R(t,T)(T-t)}$$



Fair price for a forward contract

- Now consider a forward contract on an asset that pays predictable income at discrete times in the future. The present price of the asset is $S(t)$.
- What is the fair price of a forward contract that matures at T if there is one cash income payment $C_c = C(T_c)$ at time $T_c < T$?



- **Statement**. For there to be no arbitrage the following relationship has to hold:

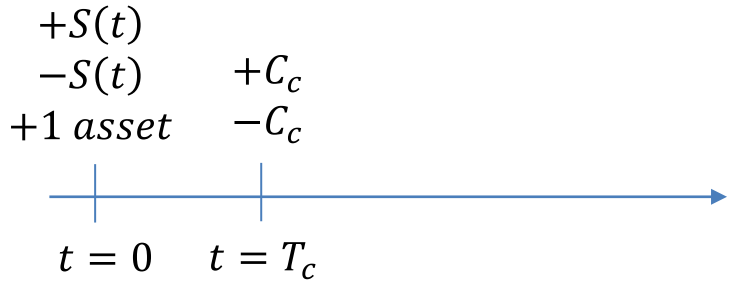
$$F(t,T) = \big(S(t) - C(T_c)\exp\big(-R(t,T_c)(T_c - t)\big)\big) * \exp\big(R(t,T) * (T - t)\big)$$
$$= \big(S(t) - PV(C_c)\big) * \exp\big(R(t,T) * (T - t)\big)$$

Where $PV(C_c) = C(T_c) * \exp\big(-R(t,T_c)(T_c - t)\big)$.

Fair price for a forward contract

- **Proof of Statement**
- At time t, we borrow $S(t)$, buy the asset and take a short position in a forward contract with delivery price set at $F(t,T)$
- At time T_c the cash payment C_c is received and used to reduce the accruing loan to

$$L(t,T_c) = S(t) * \exp\big(R(t,T_c) * (T_c - t)\big) - C_c$$



- The loan then accrues for the remaining time from T_c to T at the forward rate $F(t,T_c,T)$ set at t

$$L(t,T_c,T) = L(t,T_c) * e^{F(t,T_c,T)(T-T_c)} = \big(S(t) * \exp\big(R(t,T_c) * (T_c - t)\big) - C_c\big) * e^{F(t,T_c,T)(T-T_c)}$$

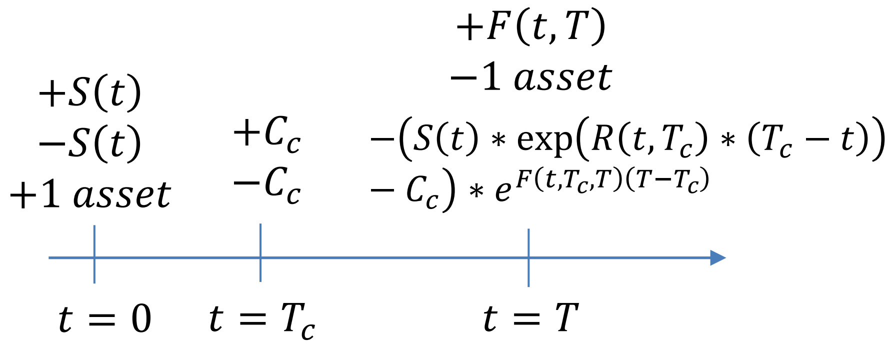
Careful: $F(t,T)$ is the **forward price** of the asset presently priced at $S(t)$ but $F(t,T_c,T)$ is the **forward interest rate** set at time t for the future interval $[T_c,T]$! These are two things different !

To be continued..

Fair price for a forward contract

- At time T the short position receives the agreed forward price $F(t,T)$ and pays the accrued debt.
- For there not to be an arbitrage opportunity the following must hold:

$$F(t,T) = \left(S(t)e^{R(t,T_c)(T_c-t)} - C_c\right) * e^{F(t,T_c,T)(T-T_c)}$$



- Using the relationship $C_c = PV(C_c) * e^{R(t,T_c)(T_c-t)}$

We have:

$$\begin{aligned} F(t,T) &= \left(S(t)e^{R(t,T_c)(T_c-t)} - PV(C_c)e^{R(t,T_c)(T_c-t)}\right)e^{F(t,T_c,T)(T-T_c)} \\ &= \left(S(t) - PV(C_c)\right)e^{R(t,T_c)(T_c-t)}e^{F(t,T_c,T)(T-T_c)} \\ &= \left(S(t) - PV(C_c)\right)e^{R(t,T)(T-t)} \end{aligned}$$

Fair price for a forward contract

- **Comment**. If we assume a flat term-structure, i.e. the same interest rate r for all maturities then the previous relationship takes the simpler form: $F(t,T) = \big(S(t) - PV(C_c)\big)e^{r(T-t)}$
- The proof is now similar but **simpler**
- At time t, we borrow $S(t)$, buy the asset and take a short position in a forward contract for $F(t,T)$
- At time T_c, we receive the payment C_c and use it to pay part of the outstanding debt. Remaining debt at T_c: $S(t)\exp\big(r(T_c - t)\big) - C_c$
- At time T, we pay the outstanding debt which has accrued to $\big(S(t)e^{r(T_c-t)} - C_c\big)e^{r(T-T_c)}$ and receive the forward price $F(t,T)$

To be continued..

Fair price for a forward contract

- For there not to be an arbitrage opportunity the following has to hold:

$$F(t,T) = \left(S(t)e^{r(T_c-t)} - C_c\right)e^{r(T-T_c)}$$

- We multiply out the previous equation

$$F(t,T) = S(t)e^{r(T_c-t)}e^{r(T-T_c)} - C_c e^{r(T-T_c)} = S(t)e^{r(T-t)} - C_c e^{r(T-T_c)}$$

- By inserting $C_c = PV(C_c)e^{r(T_c-t)}$, it follows that:

$$F(t,T) = S(t)e^{r(T-t)} - PV(C_c)e^{r(T_c-t)}e^{r(T-T_c)} = S(t)e^{r(T-t)} - PV(C_c)e^{r(T-t)}$$

$$\boxed{F(t,T) = \left(S(t) - PV(C_c)\right)e^{r(T-t)}}$$

Numerical example

- In 9 months, Arion Banki is expected to pay ISK 5.0 dividend.

  (a) What is the fair price for a one year forward contract on this asset assuming that the 9 month interest rate is 7.8% and the one year rate 8%?

  (b) What is the forward rate $F(0, 9\ months, 1\ year)$?

Shares Closed on the stock exchange

XICE First North All stocks ETFs Selected stocks

IDENTIFICATION	PRICE	CHANGE	LAST 3 MONTHS	PURCHASE	SALES	TURNOVER
OMXI15	2,601.98	0.72%		-	-	-
ALVO	1,105.00	-0.90%		1,100.00	1,105.00	122m
AMRQ	131.00	0.77%		130.00	131.00	21m
ARION	169.50	0.89%		168.50	169.50	462m
BRIEF	59.00	0.85%		58.50	59.00	2m
OAK	13.90	0.00%		13.60	13.90	4m
EIM	370.00	-0.54%		370.00	376.00	287th
FAST	286.00	0.70%		284.00	286.00	200m
HAGE	106.00	1.19%		105.00	107.00	163m
HEMP	117.00	-0.43%		117.00	118.00	3m
HOME	37.00	0.54%		36.80	37.20	45m
ICEAIR	1.10	-1.79%		1.09	1.10	31m

Simplification (realistic though)

- Rather than assuming that an asset pays predictable income (dividends or coupons) at fixed times, it is common -and quite convenient- to assume that income is paid at a continuous rate δ

- This assumption can be computationally quite useful, particularly if we consider a portfolio of assets: all paying dividends at different times. In aggregation, the effect of that can be viewed as an asset (portfolio) paying continuous dividend.

- That allows us to talk about **dividend yield** $\boldsymbol{\delta}$

Fair price for a forward contract

- Let $S(t)$ be the price of an asset that pays continuous and constant dividend yield δ
- **Statement**. For there not to be an arbitrage opportunity the forward price on this asset, with maturity T, needs to be:

$$F(t,T) = S(t)e^{(R(t,T)-\delta)(T-t)}$$

- **How to think the proof**. We assume that an investor takes a loan for $S(t)$ to buy the asset. The dividends the asset pays over the time from t to T are reinvested continuously in the asset.

Therefore, at time T the investor holds a position of $S(t)e^{\delta(T-t)}$ in the asset. As the dividend payments are certain the investor can, at time t, enter a short position in a forward contract

cont.

Fair price for a forward contract

- The forward price on **one unit** of the asset is set at $F(t,T)$ and the price on $\mathbf{1} * \exp\big(\delta(T-t)\big)$ **units** is therefore, $\mathrm{F}(t,T) * \exp\big(\delta(T-t)\big)$

- The cash flow balance equation for the absence of risk free arbitrage is,
$$-S(t)\exp\big(R(t,T)(T-t)\big) + F(t,T)\exp\big(\delta(T-t)\big) = 0$$

- Therefore,
$$F(t,T) = S(t)e^{(R(t,T)-\delta)(T-t)}$$

Which is the AOA forward price on **one unit** of asset presently priced at $S(t)$

Forward contracts on dividend paying stocks

- We look at a different, calculus-based proof for the pricing of a forward contract on an asset that pays continuous dividend.
- Assuming again a non-negative constant δ, such that for each unit of the stock, the amount of stochastic dividends paid between time t and $t + dt$ for any infinitesimally small dt is $S(t)\delta dt$, where $S(t)$ is the time-t stock price
- We assume again that the dividends received are not paid out in cash, but immediately reinvested in the stock, resulting in more and more shares as time goes by.
- To determine the increase in the number of shares, we define $N(t)$ the number of shares of the stock we hold at time t under the reinvestment policy.

Forward contracts on dividend paying stocks

- Between time t and $t + dt$, the amount of dividend payment is $S(t)\delta dt$ per share, so the total amount of dividends we receive is $N(t)S(t)\delta dt$.
- Reinvesting this amount in the stock allows us to buy $\frac{N(t)S(t)\delta dt}{S(t)}$ $= N(t)\delta dt$ more shares. In other words, the change in the number of shares is given by:

$$dN(t) = N(t + dt) - N(t) = N(t)\delta \mathrm{dt}$$

which means that:

$$\frac{dN(t)}{dt} = \delta \mathrm{N}(t)$$

- The solution to this ordinary differential equation in $N(t)$ with initial shares $N(0)$ is given by $N(t) = N(0)e^{\delta t}$

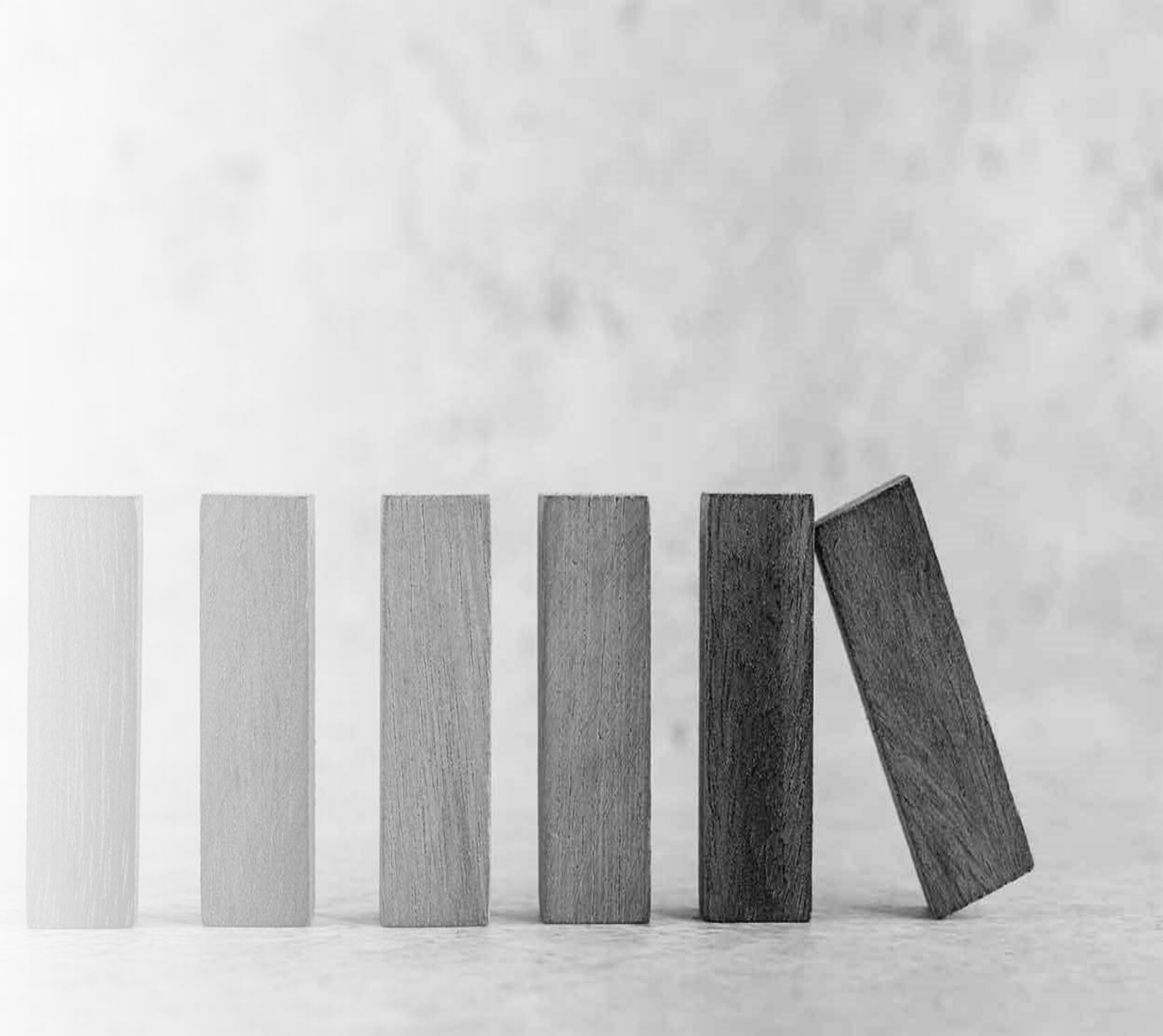
Forward contracts on dividend paying stocks

- Thus, 1 share (i.e. $N(0) = 1$) at time $t = 0$ will grow to $e^{\delta t}$ shares at time T. By proportion, to obtain 1 share at time T and to replicate the payoff of $S(T)$, it requires to buy $e^{-\delta \mathrm{T}}$ shares at time 0 and reinvest all dividends in the stock between 0 and T.
- It follows that the fair prepaid forward price in the presence of continuous dividends is the cost of buying $e^{-\delta T}$ shares at time 0, or:

$$F(0, T) = S(0)e^{-\delta T}$$

To go further

And bridge with Section 1.2 Futures.



Forward Premium

+ Contango / Backwardation concept

Definition

The ratio between the forward and the spot price gives the forward premium at maturity T

CONTANGO

Name of the market when the forward future price is **above** the expected spot price

Always larger than 1.

This premium will always be larger than one (look at the equation), except if the interest rates are negatives!

BACKWARDATION

Name of the market when the forward future price is **below** the expected spot price

$$FP(t,T) = \frac{F(t,T)}{S(t)} = e^{R(t,T)(T-t)}$$

In the case of non-dividend paying asset

Forward Premium

+ Contango / Backwardation concept

Definition

The ratio between the forward and the spot price gives the forward premium at maturity T

CONTANGO

Name of the market when the forward future price is **above** the expected spot price

NOT always larger than 1.

With dividend paying asset, the forward premium is affected and might easily end up smaller than 1.

BACKWARDATION

Name of the market when the forward future price is **below** the expected spot price

$$FP(t,T) = \frac{F(t,T)}{S(t)} = e^{(R(t,T)-\delta)*(T-t)}$$

In the case of dividend paying asset

More on the contango / backwardation

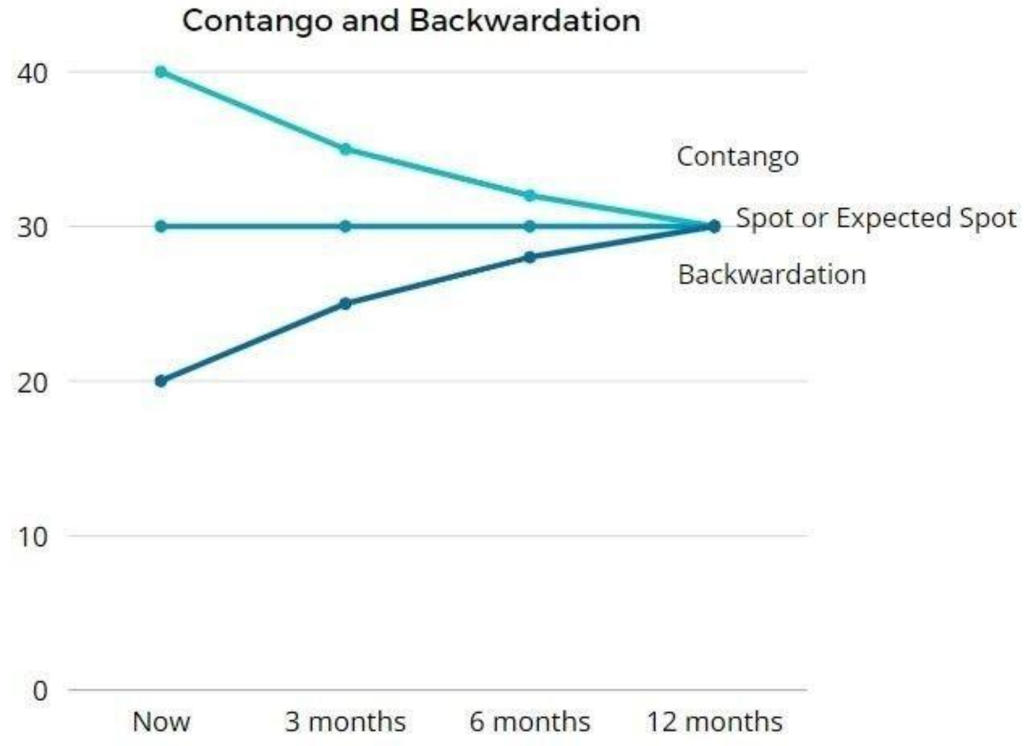
Figure

Let's assume the figure is about Brent Crude Oil. The price today is $30. You buy a future contract that expires in 3 months:

- If the price is $35 (> S(t)) -> contango
- If the price is $25 (< S(t)) -> backwardation

Observation

Over time, as the contract nears expiration, it will move closer to the spot price.
Therefore, unless the price rises above the price paid, the value of a contango contract will drop to meet the spot price at expiry.



Market Opportunity

The market can flip from contango to backwardation, or vice versa. Building a trading strategy based on a specific condition could become **unprofitable** if the conditions flip rapidly. NOT A TRADING ADVICE.

BACKWARDATION TRADES

A trader could buy a futures contract in the hope that it moves higher to meet the spot price. This can be profitable if the price of the commodity is trending higher.

If futures prices are below the expected price, though, this could also mean that traders are anticipating less demand for the commodity.

Profiting from backwardation is not as simple as it sounds and it does not solely depend on whether you should or buy or sell the asset.

More on the contango / backwardation

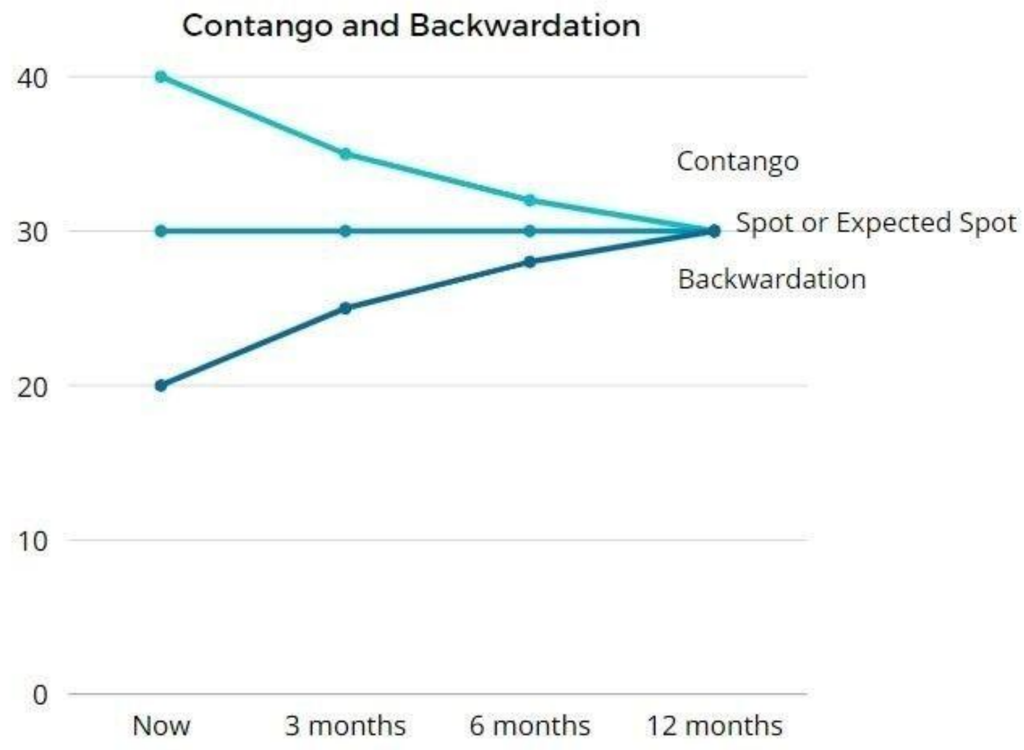
Figure

Let's assume the figure is about Brent Crude Oil. The price today is $30. You buy a future contract that expires in 3 months:

- If the price is $35 (> S(t)) -> contango
- If the price is $25 (< S(t)) -> backwardation

Observation

Over time, as the contract nears expiration, it will move closer to the spot price.
Therefore, unless the price rises above the price paid, the value of a contango contract will drop to meet the spot price at expiry.



Market Opportunity

The market can flip from contango to backwardation, or vice versa. Building a trading strategy based on a specific condition could become **unprofitable** if the conditions flip rapidly. NOT A TRADING ADVICE.

CONTANGO TRADES

A trader believes that the spot price of oil will go even lower versus the future month's contract. A trader would short the spot month contract and buy the further out month. This trade would profit if the market increases its contango structure.

The trade would lose money if the market reverts to a normal backwardation structure.

Silver

Why is it usually in contango?

Crude Oil

Why is it usually in contango?

Gasoline

Why –in that case- could it be in backwardation?

The WTI 'super-contango' structure

FUTURES CONTRACTS ROLL OVER

Traders will roll over futures contracts that are about to expire to a longer-dated contract in order to maintain the same position following expiry. The roll involves selling the front-month contract already held to buy a similar contract but with longer time to maturity.

WHAT HAPPENED?

On April, 20th 2020, at the peak of the COVID-19 crisis, the WTI oil prices plunge to
- $37.63.

The panic was apparent in the futures market as the May contract expiry approached and traders wondered how they would take delivery of physical barrels of oil when storage sites are reaching full capacity.

This large front-month spread made traders not want to roll, nor did they want to hold and take delivery, hence they dumped instead.

The WTI 'super-contango' structure

FUTURES CONTRACTS ROLL OVER

Traders will roll over futures contracts that are about to expire to a longer-dated contract in order to maintain the same position following expiry. The roll involves selling the front-month contract already held to buy a similar contract but with longer time to maturity.

WHAT HAPPENED?

On April, 20th 2020, at the peak of the Coronavirus crisis, the WTI oil prices plunge to **- $37.63 EOD**.

The panic was apparent in the futures market as the May contract expiry approached and traders wondered how they would take delivery of physical barrels of oil when storage sites were reaching full capacity.

This large front-month spread made traders not want to roll, nor did they want to hold and take delivery, hence they dumped instead.

The WTI 'super-contango' structure

FUTURES CONTRACTS ROLL OVER

Traders will roll over futures contracts that are about to expire to a longer-dated contract in order to maintain the same position following expiry. The roll involves selling the front-month contract already held to buy a similar contract but with longer time to maturity.

WHAT HAPPENED?

On April, 20th 2020, at the peak of the Coronavirus crisis, the WTI oil prices plunge to **- $37.63 EOD**.

The panic was apparent in the futures market as the May contract expiry approached and traders wondered how they would take delivery of physical barrels of oil when storage sites were reaching full capacity.

This large front-month spread made traders not want to roll, nor did they want to hold and take delivery, hence they dumped instead.