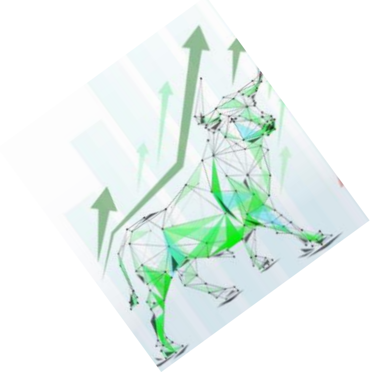
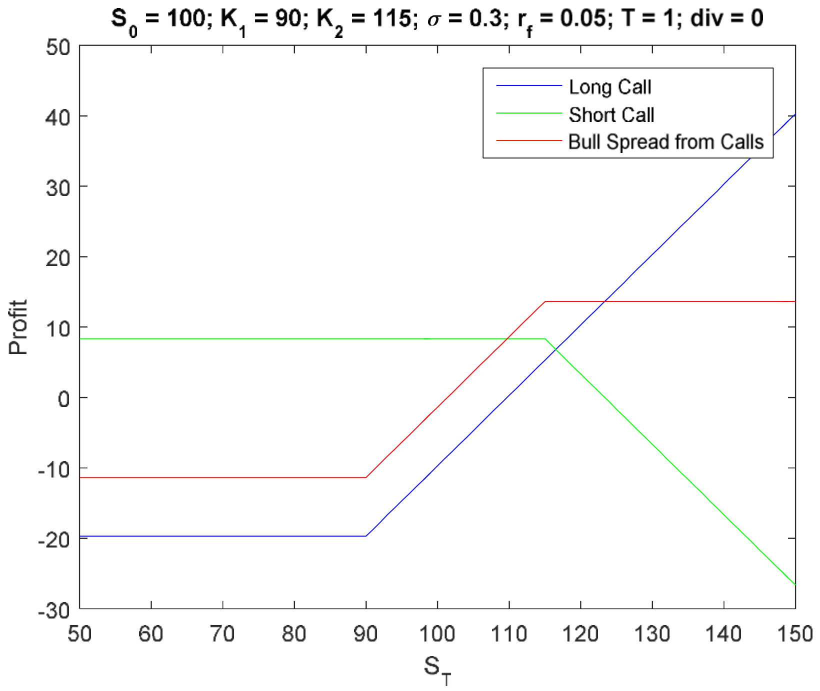
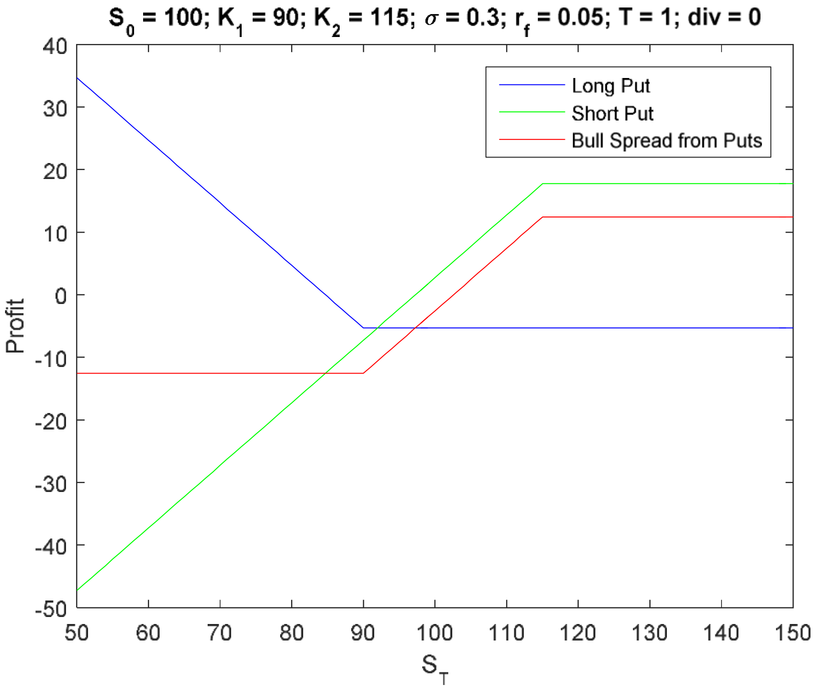


Bull spread

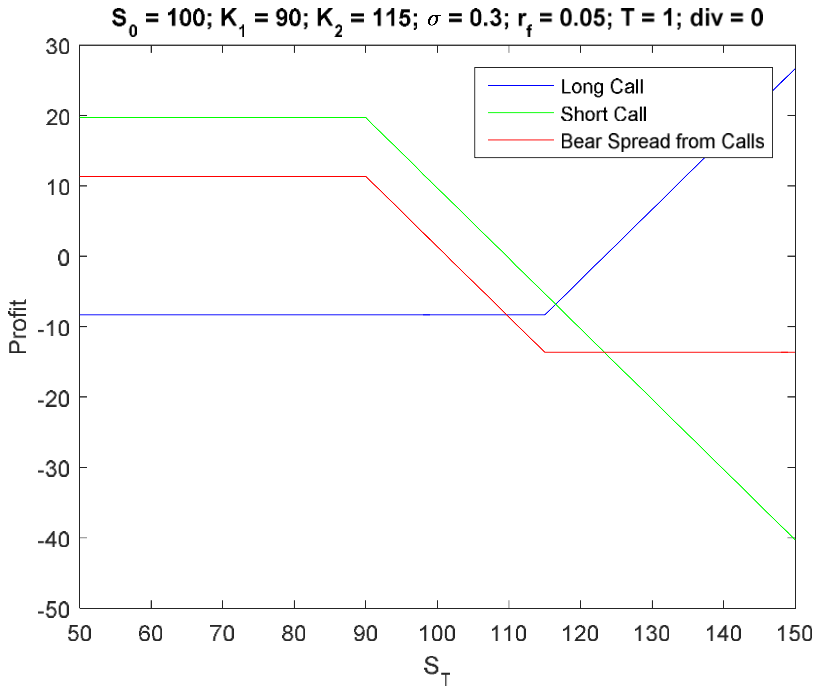
- Bull spread can either be constructed from **two calls** or from **two puts**
- Calls: Buy a call at K_1 and sell a call at K_2, $K_1 < K_2$
- Puts: Buy a put at K_1 and sell a put at K_2, $K_1 < K_2$

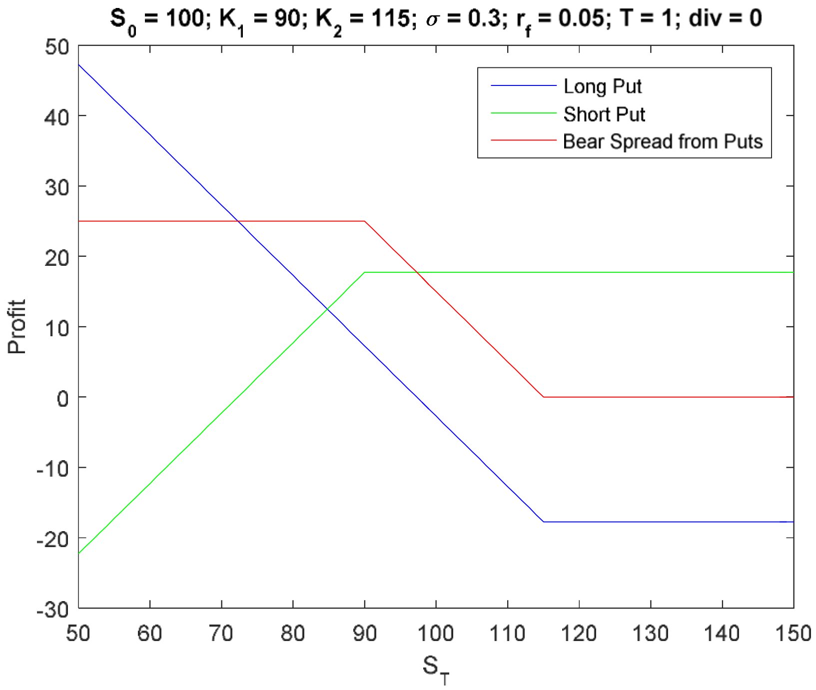




Bear spread

- Bear spread can either be constructed from two calls or from two puts
- Calls: Buy a call at K_2 and sell a call at K_1, $K_1 < K_2$
- Puts: Buy a put at K_2 and sell a put at K_1, $K_1 < K_2$





Bull and Bears

Under which conditions Bull and Bears spreads are beneficial ?

Different investment scenarios

- **Long asset**. The investor buys the asset
- At time t by the stock for S_t
- At the later time T sell the asset for S_T
- Capital gain made is:

$$G([t,T]) = S_T - S_t$$

- Return made on this investment:

$$R([t,T]) = \frac{S_T - S_t}{S_t} = \frac{G([t,T])}{S_t}$$

- **Comment.** If the asset price follows geometric Brownian motion:

$$E_t\big[G[t,T]\big] = \mu S_t (T-t) \ ; \quad var\big[G[t,T]\big] = \sigma^2 S_t^2 var[W(T) - W(t)] = \sigma^2 {S_t}^2 (T-t)$$
$$E_t\big[R[t,T]\big] = \mu (T-t) \ ; \quad var\big[R[t,T]\big] = \sigma^2 var[W(T) - W(t)] = \sigma^2 \ (T-t)$$

American options

- An American option is an option that can be exercised before the maturity T.
- Open discussion: Where would you see an interest?

Module 2

Options

2.2 Binomial Pricing Methods

Binomial trees and options

- How much should I pay for the option to buy or sell the stock at the strike K at the future time T ?
- We use binomial trees to address this question
- Here the basic assumption is that asset prices follow a **random walk:** At each time step prices move up or down with a certain probabilities

- Combining this approach with the construction of "risk-less" portfolios leads to the important principle of "risk neutral valuation"

Binomial Tree Method

- A binomial tree presents all the possible paths the price of an asset (the underlying) can take during the lifetime of an option
- Each node in the tree has associated with it certain value for the underlying and for the option to buy it or sell it

- The binomial tree method provides a relationship with the risk neutral valuation method

Risk less Portfolio and Pricing

- **An option can be priced by assuming there is no arbitrage opportunity for the investor**

- One constructs a **risk less** portfolio of assets and options on these assets
- The portfolio weights are constructed in such a manner that the value of the portfolio is independent of what price path the underlying asset takes – i.e. The portfolio is (temporarily) risk less
- As the portfolio is risk less it can be discounted backwards at the risk free rate to find its value today
- Knowing the portfolio value allows us to price the option

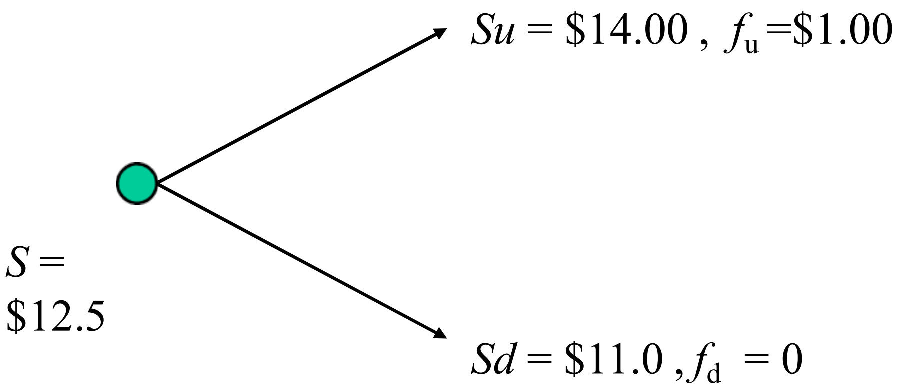
Easy Approach

- Shares in company A are today trading at \$12.50
- Assume that in three months time they will be either
  - down to \$11.00   or   up to \$14.00
- How much should you pay today for a *K* = \$13.00 call that expires at time *T* ?

- The option value at expiry is

$$f(T) = \max(0, S(T) - K)$$

- But, what is the value (price) of the option when the contract is entered into ?



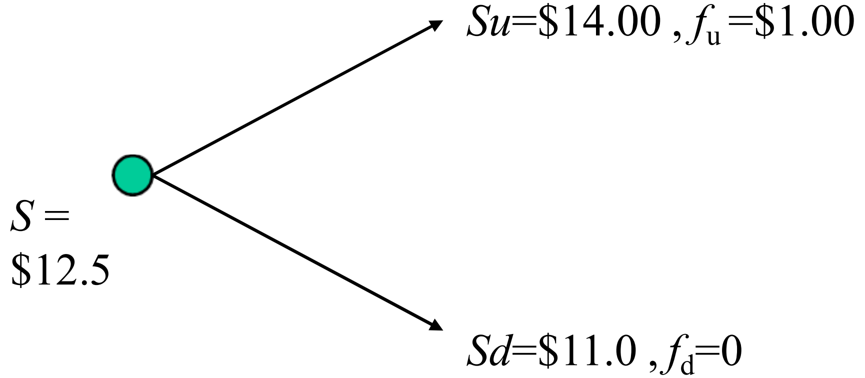
Easy Approach

- To price the call we construct a portfolio of the stock and the option - choose the “mixing” such that the value of the portfolio is independent of the stock value going up or going down

- ***Portfolio***: For every short position in a call hold Δ shares long. Then

$$V(0) = \Delta S - f \quad ; \quad V(P_u) = \Delta * 14 - 1 \quad ; \quad V(P_d) = \Delta * 11$$

- From $V(P_u) = V(P_d)$ we find Δ = 1/3
- The portfolio therefore consists of
  - Long: 1/3 share
  - Short: 1 option



Easy Approach

- The value of the portfolio at the end of three months:

$$V_u(T) = \frac{14}{3} - 1 = \$3.67 = V_{d(T)} = \frac{11}{3}$$

- As the portfolio is risk less its value at the beginning (three months earlier) must be (assuming 7% risk free rate):

$$V(0) = 3.67 * e^{-0,07*0,25} = \$3.61$$

- Putting this back into the portfolio expression $V(0) = S\Delta - f$

$$f = S * \Delta - V(0) = \frac{12.5}{3} - 3.61 = \$0,56$$

which gives the call premium at $\boldsymbol{t = 0}$

Easy Approach

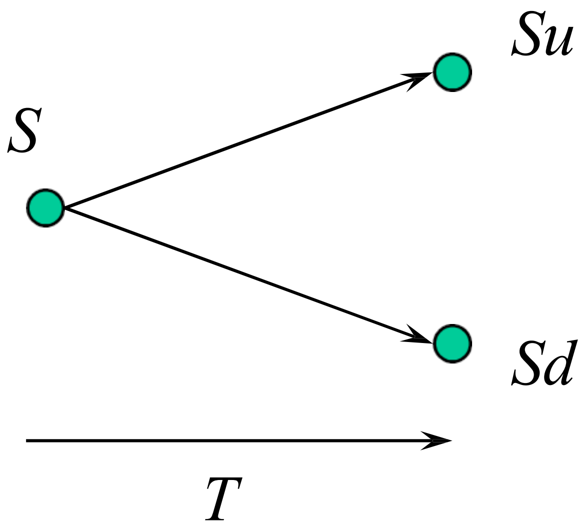
Assuming that there are no arbitrage opportunities one finds for the current value of the option, $f(0) = \$0.56$

- If $f(0) < \ \$\ 0.56$ shorting the portfolio would make it possible to borrow money at a rate lower than the risk free rate.
- If $f(0) > \ \$\ 0.56$ the portfolio would cost less that $\$\ 3.61$ to set up and would therefore earn more than the risk free rate.

Generalisation of example

- Start by making some assumptions about the future value of the stock
- Assume, it either goes up or it goes down
- The simplest possible assumption is:

  The stock value goes up to Su or it goes down to Sd

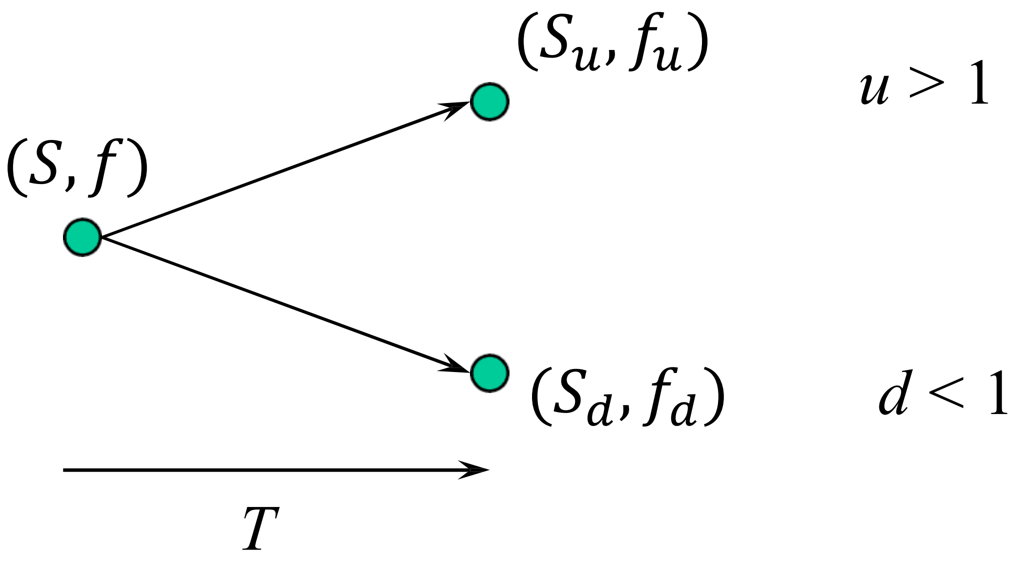


$u > 1$

$d < 1$

Immunising a portfolio

- At time $t = 0$ the price of stock and *call* option is $(S\ ,f)$
- At later time the prices of these two assets can be either (Su, f_u) or (Sd, f_d)
- Put together a portfolio whose value does not depend on the value of the stock at $t = T$



Risk neutral portfolio

- Sell a call at $t = 0$. Hold Δ shares for each short position in a call. Value of position at t = 0:

$$V(0) = \Delta S - f$$

- At the end of period, at $t = T$ the value can be

$$V_{u(T)} = \Delta Su - f_u \qquad or \qquad V_{d(T)} = \Delta \mathrm{Sd} - f_d$$

- The purpose of the exercise was to choose Δ such that

$$V_u(T) = V_d(T)$$

Choice of delta

- By demanding

$$\Delta S u - f_u = \Delta S d - f_d$$

- Solve for Δ:

$$\Delta = \frac{f_u - f_d}{S_u - S_d} \equiv \frac{\delta f}{\delta S}$$

- As $V_u(T) = \Delta S u - f_u$ is the end of period value, the present value of the portfolio must be

$$\Delta S - f = \exp(-rT)\,\{\Delta S u - f_u\}$$

Because the portfolio is risk less it only earns the risk free rate !

Another point of view

- Inserting Δ into previous eq. and solve for f:

$$f = \exp(-rT) * \{p_u f_u + p_d f_d\}$$

With $\quad p_u = \frac{\exp(rT) - d}{u-d} \quad ; \quad p_d = \frac{u - \exp(rT)}{u-d}$

- It is natural to look at p_u as a probability that the stock value increases to Su
- Therefore, the value of the derivative today is given by its expected future value discounted at the risk free rate r

Proof

For your own understanding

Insert $\Delta = \frac{f_u - f_d}{S_u - S_d}$ into $\Delta S - f = \exp(-rT)\{\Delta S u - f_u\}$ then

$$\frac{f_u - f_d}{u - d} - f = \exp(-rT)\left[\left(\frac{f_u - f_d}{u - d}\right)u - f_u\right] \Rightarrow$$

$$\frac{f_u - f_d - f(u - d)}{u - d} = \exp(-rT)\left[\frac{(f_u - f_d)u - f_u(u - d)}{u - d}\right] \Rightarrow$$

$$f = \frac{1}{u - d} * [f_u(1 - \exp(-rT)\, d) - f_d(\, 1 - \exp(-rT)\, u\,)] \Rightarrow$$

$$f = \exp(-rT) * \left[f_u * \frac{(\exp(rT) - d)}{u - d} + f_d * \frac{(u - \exp(rT))}{u - d}\right] = \exp(-rT)\,[p_u f_u + p_d f_d]$$

With $p_u = \frac{(\exp(rT) - d)}{u - d}$; $p_d = \frac{(u - \exp(rT))}{u - d}$

Back to example

- Apply this result to previous example

$$r = 0.07 \ , \ u = 14/\ 12.5 = 1.12 \ , d = 11/\ 12.5 = 0.88$$

- Then

$$p_u = \frac{\exp(0.07 * 0.25) - 0.88}{1.12 - 0.88} = 0.57$$

And

$$f(0) = \exp(-0.07 * 0.25)\,(0.57 * 1) = 0.56$$

Stock prices in risk neutral world

- If we assumed that p_u (resp. p_d) is the probability of an up (resp. down) movement in the stock price then the expected stock price at T is

$$E\{S(T)\} = p_u S_u + p_d S_d$$

or, after substituting for p_u and p_d

$$E\{S(T)\} = S * \exp(rT)$$

- Therefore, the stock grows on average at the risk free rate

Proof

For your own understanding

- Straight forward calculation – inserting the expressions for p_u and p_d :

$$E\{S(t)\} = \left(\frac{\exp(rT) - d}{u - d}\right) S_u + \left(\frac{u - \exp(rT)}{u - d}\right) S_d$$

$$= \frac{S_u * \exp(rT) - S_u d + S_d u - S_d \exp(rT)}{u - d}$$

$$= \frac{(u - d) S * \exp(rT)}{u - d} = S * \exp(rT)$$

Comments

- Setting the probability of up and down movement for stock price equal to

$$p_u = \frac{\exp(rT) - d}{u - d} \quad ; \quad p_d = \frac{u - \exp(rT)}{u - d}$$

  is the same as assuming it is growing at risk free rate

- **In the risk neutral world, investors require no compensation for risk and the expected return on all securities is the risk free interest rate**

Risk neutral valuation applied to previous example

- p_u probability that stock increases, $(1\text{-}p_u)$ that it decreases.
- Use this and the assumption that the stock grows (on average) at the risk free interest rate:

$$14 * p_u + 11 * (1 - p_u) = 12.50 * \exp(0.07 * 0.25)$$

i.e. $p_u = \frac{1.72}{3} = 0.57$

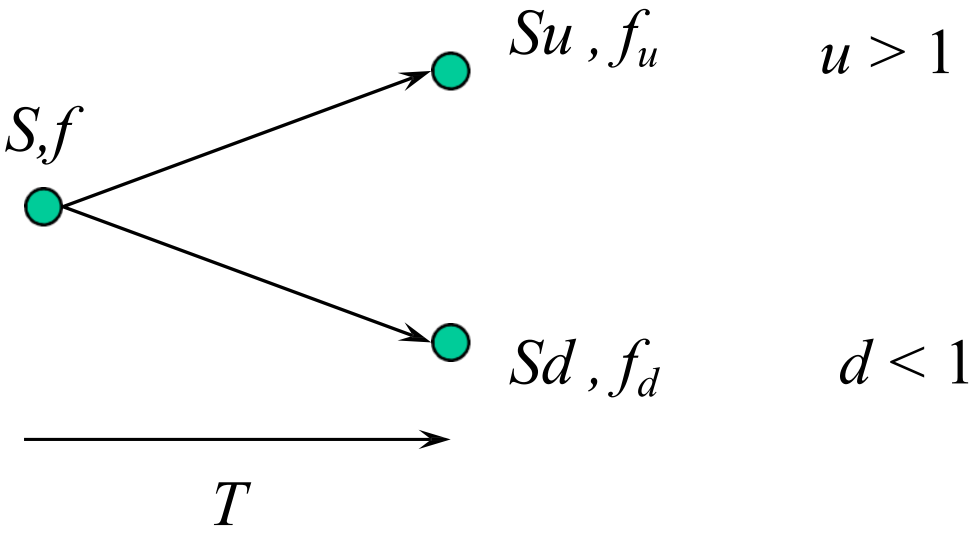
i.e. after 3 months the option has the probability $p_u = 0.57$ to have the value 1, and $p_d = 0.43$ to have the value zero. The expected value of option is therefore

$0.57 * 1 + 0.43 * 0 = 0.57$ and after discounting at risk free rate

$0.57 * \exp(-0.07 * 0.25) = 0.56$ which is the same value found by arbitrage arguments.

Single step binomial tree for pricing

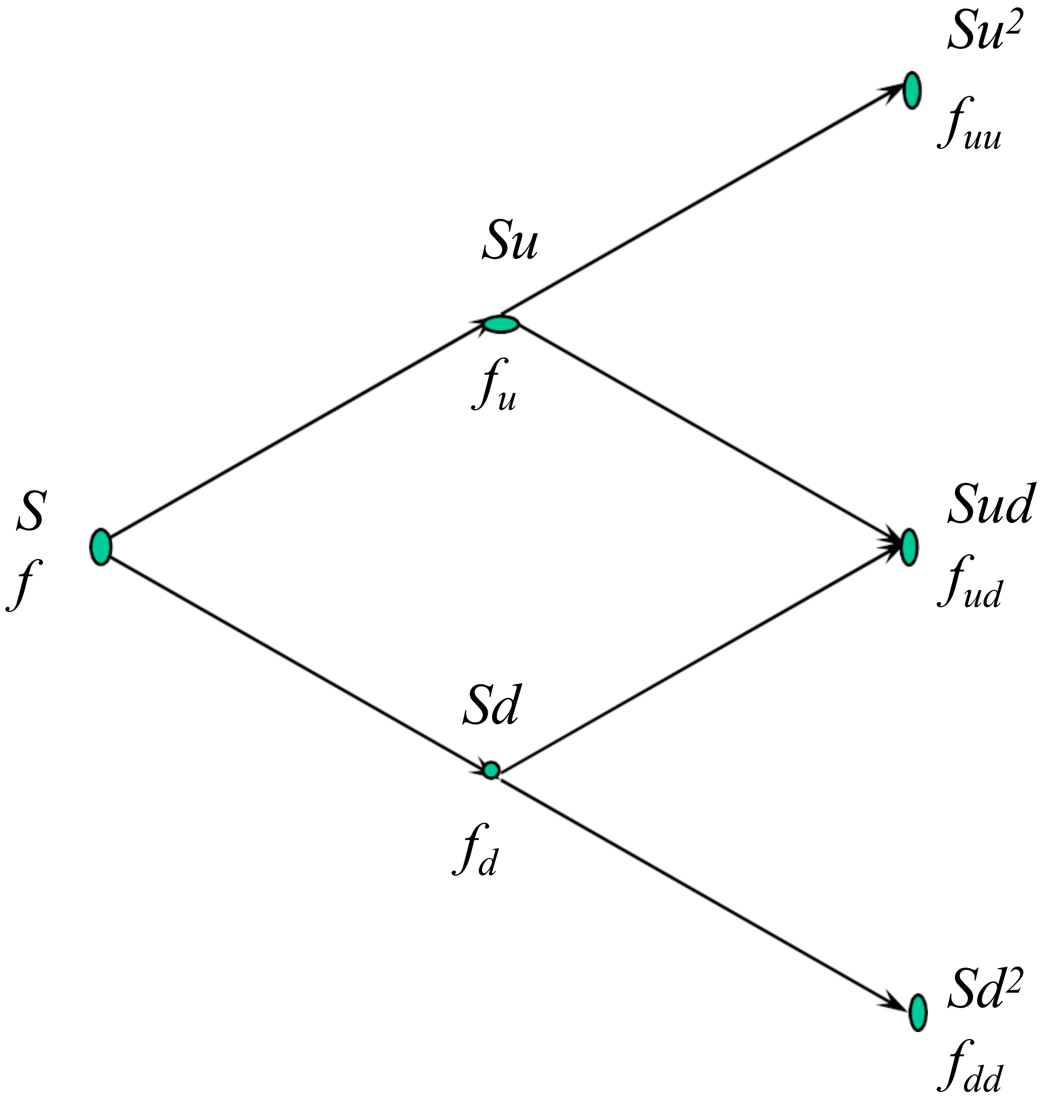
- Single step binomial tree for option valuation



$$\boxed{f = e^{-rT}(p_u f_u + p_d f_d)}$$

- This can be generalised to *two* and *higher* step binomial trees

Multi step binomial pricing



The stock and option price start at (*S* , *f*). From there at each time step they can go one up or one down.

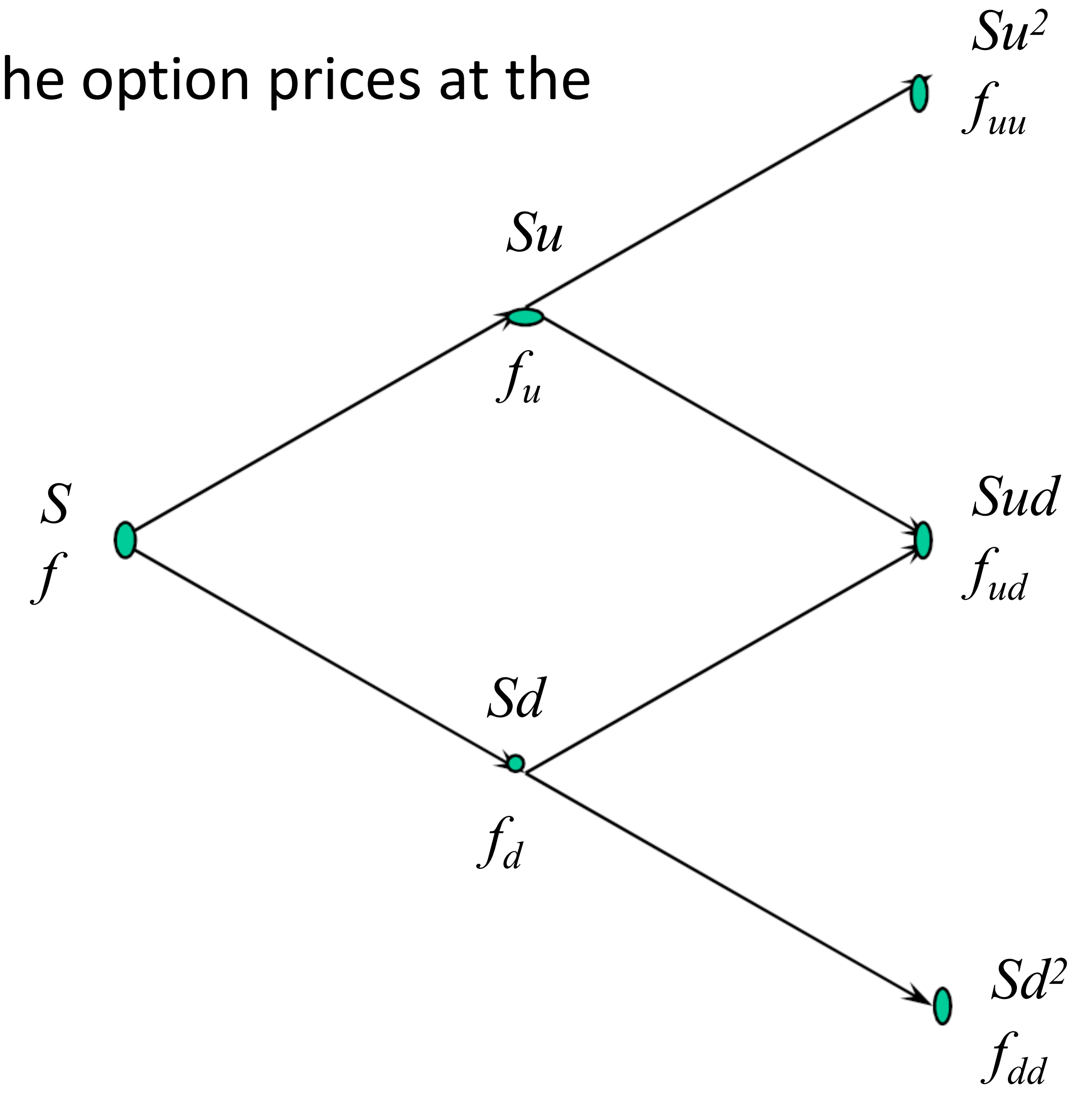
Each node is characterised by a stock price and an associated price of an option - call or put.

The pricing procedure is to assume knowledge of the final prices and then find the previous prices discounting at the risk free interest rate.

Call prices at final nodes

- Consider a call with exercise price K. Then the option prices at the final nodes are found to be:

$$f_{uu} = \max(Su^2 - K, 0)$$
$$f_{ud} = \max(Sud - K, 0)$$
$$f_{dd} = \max(Sd^2 - K, 0)$$



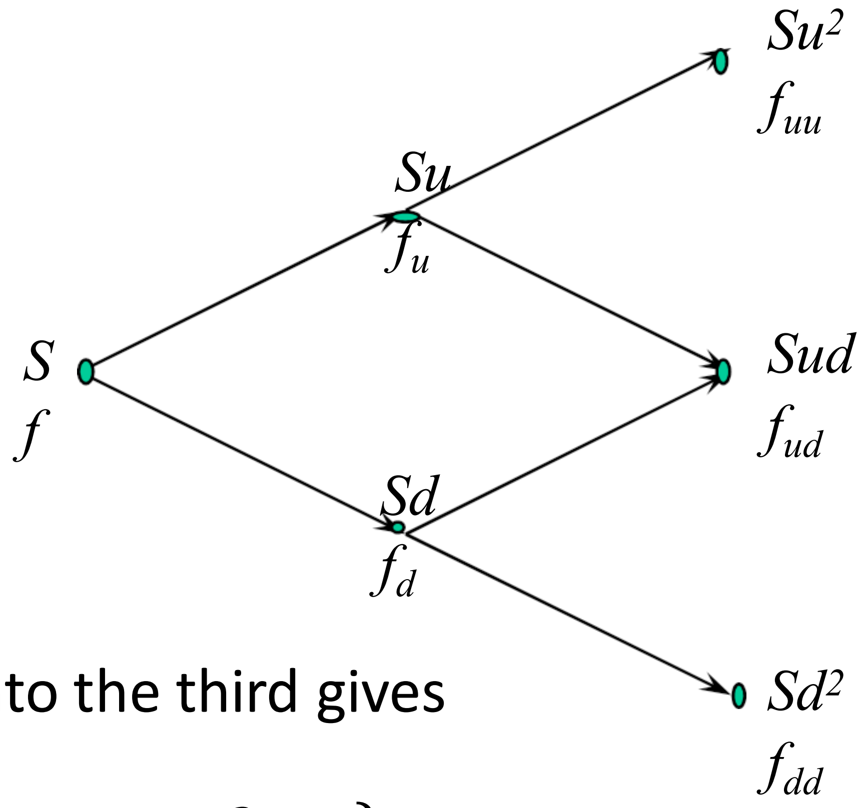
Binomial pricing - how it works

- Discount from last level to second - and then from the second level to the first

$$f_u = e^{-r\Delta t}\{pf_{uu} + (1-p)f_{ud}\}$$

$$f_d = e^{-r\Delta t}\{pf_{ud} + (1-p)f_{dd}\}$$

$$f = e^{-r\Delta t}\{pf_u + (1-p)f_d\}$$

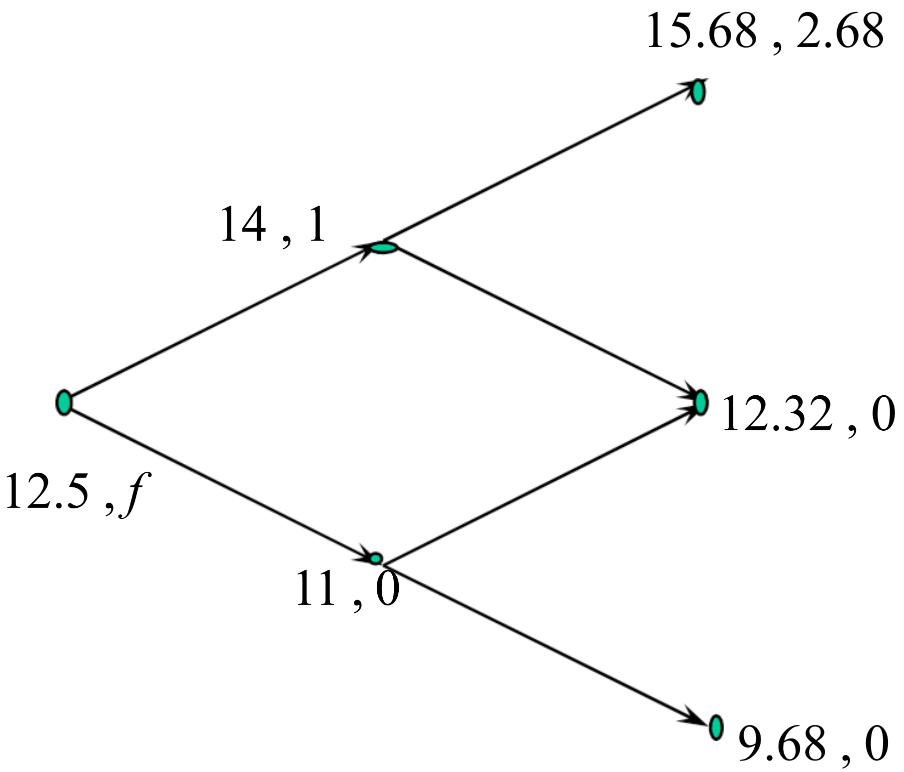


- Substituting the first two equations into the third gives

$$f = e^{-2r\Delta t}\{p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}\}$$

Examples & applications

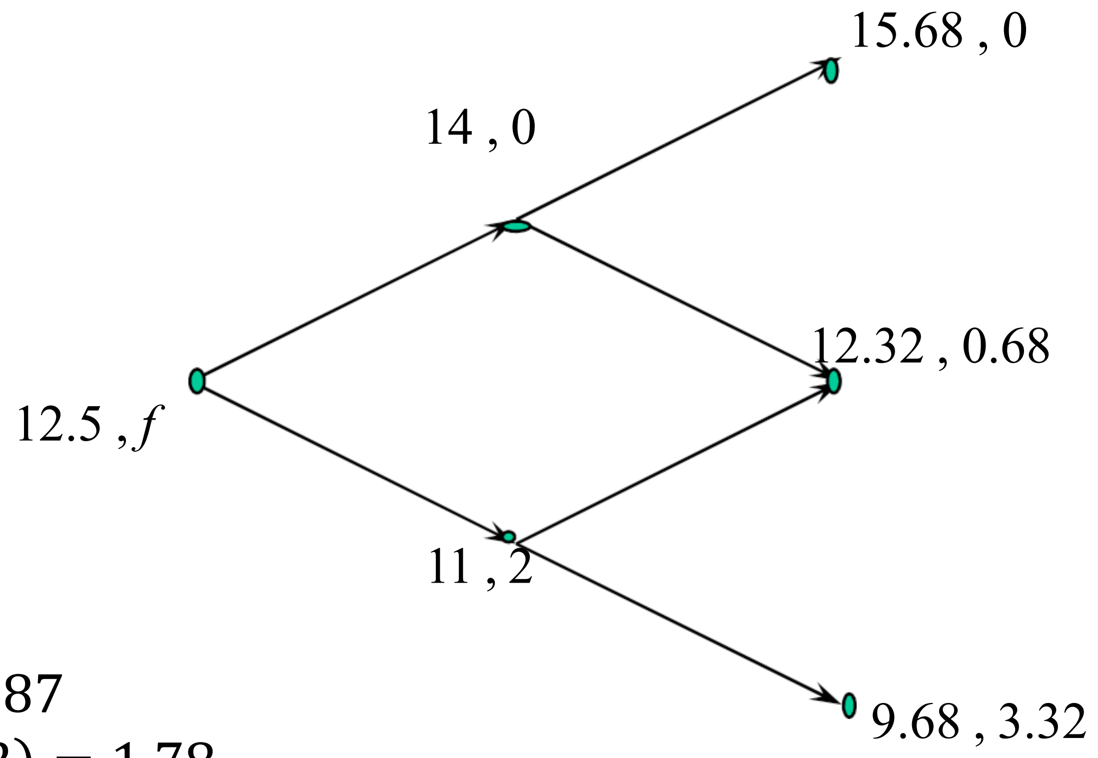
- **Call option** - Numerical Example. Generalise Example to two steps:
- $u = 1.12$; $d = 0.88$; $p = 0.57$; $r = 0.07$; $\Delta t = 0.25$; $K = \$13$



$$f_u = \exp(-0.07 * 0.25)\,(0.57 * 2.68) = 1.5$$
$$f_d = 0$$
$$f = \exp(-0.07 * 0.25) * 0.57 * 1.5 = 0.84$$

Examples & applications

- **Put option** - Numerical Example. Generalise Example to two steps:
- $u = 1.12$; $d = 0.88$; $p = 0.57$; $r = 0.07$; $\Delta t = 0.25$; $K = \$13$



$$f_u = \exp(-0.07 * 0.25)\,(0.43 * 0.68) = 0.287$$
$$f_d = \exp(-0.07 * 0.25)\,(0.57 * 0.68 + 0.43 * 3.32) = 1.78$$
$$f = \exp(-0.07 * 0.25)\,(0.57 * 0.287 + 0.43 * 1.78) = 0.9$$

Put-call parity – verification

- Inserting previous numerical results into the put-call parity

$$c(t) + K * \exp\big(-r(T - t)\big) = p(t) + S(t)$$

- Left side:

$$0.84 + 13 * \exp(-0.07 * 2 * 0.25) = 13.393$$

- Right side:

$$0.9 + 12.5 = 13.4$$

Application to American options

- The binomial tree approach can be applied to American options with slight modification
- The value of the option at the final nodes is the same as in the case of European options
- At earlier nodes the value is modified as

$$\boldsymbol{f = \max(\exp(-r\Delta t)\,(p_u f_u + (1-p_u) f_d), S_u - K)}$$

- In other words, calculate the price using binomial pricing as in the case of European options. **Compare this value with the options intrinsic value**. **Take the larger of the two**. The result gives the value of the option at the considered node

Numerical example

- American **put** option
  $u = 1.12$; $d = 0.88$; $p = 0.57$; $r = 0.07$; $\Delta t = 0.25$; $K = \$13$

- From previous evaluation, $f_u = 0.287, f_d = 1.78.$ f_d has to be modified as:
  $$f_d = \max(1.78, 2) = 2$$

- Therefore
  $$f = \exp(-0.07 * 0.25)\,(0.57 * 0.287 + 0.43 * 2.0) = 1.002$$

- The American option is $\$1.002$ as opposed to $\$0.9$ for the equivalent European option

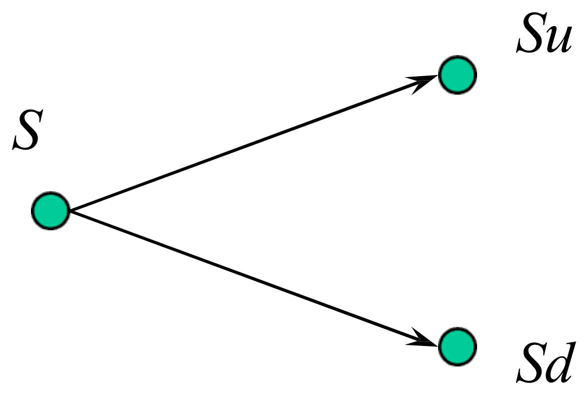
Module 3

Hedging

3.1 Beyond the simple concept

Delta hedging

- In the single-step binomial model we create a risk less portfolio as follows,$V = \Delta S - f$, where Δ was fixed such as:



$$\Delta S_u - f_u = \Delta S_d - f_d$$

$$\Delta = \frac{f_u - f_d}{S_u - S_d}$$

- This procedure is called *delta hedging*. It tells an investor who sells a call, how many shares to hold for each call sold **to create a risk less hedge.**

Delta hedging

- In the single-step binomial model we create a risk less portfolio as follows,$V = \Delta S - f$, where Δ was fixed such as:

In addition to be a very typical question during interviewes. The Delta Hedge is widely used on all the Derivatives trading floor across the world. Understanding that it 'kills' the risk associated to the underlying price movements is key here.

$$\Delta S_u - f_u = \Delta S_d - f_d$$

$$\Delta = \frac{f_u - f_d}{S_u - S_d}$$

This procedure is called *delta hedging*. It tells an investor who sells a call how many shares to hold for each call sold **to create a risk less hedge.**

Dynamic delta hedging

- In a multi step model D changes over time. Therefore, to maintain a risk less hedge an investor must at each time step adjust his holding in the stock he has sold a call on
- Delta at $t = 0$

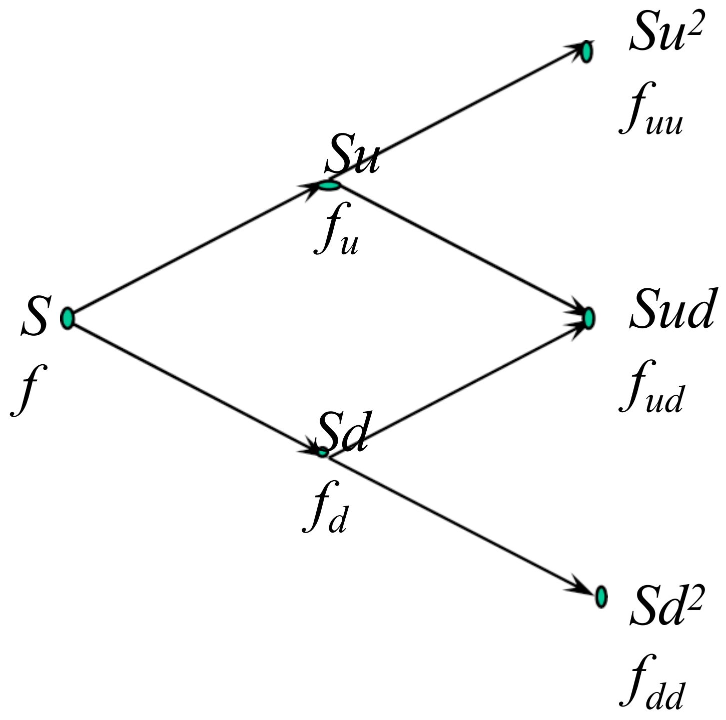
$$\Delta = \frac{f_u - f_d}{S_u - S_d}$$

- Delta at $t = dt$ if '*up*'

$$\Delta = \frac{f_{uu} - f_{ud}}{S_{u^2} - S_{ud}}$$

- Delta at $t = dt$ if '*down*'

$$\Delta = \frac{f_{ud} - f_{dd}}{S_{ud} - S_{d^2}}$$



Fixing the values for *u* and *d*

- We suspect that the values of *u* and *d* depend on the volatility and the length of considered time intervals
- To derive the expressions for *u* and *d* we use the fact that the standard deviation of asset returns is given by:

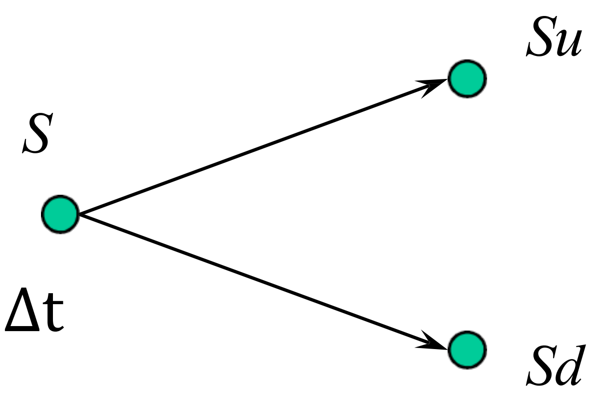
$$std(r) = \sigma\sqrt{dt} \quad ; \quad var(r) = \sigma^2 * dt$$

- The variance of a stochastic variable *r* can be written in terms of the expectation value as follows:

$$var(r) = E[r^2] - \left[E[r]\right]^2$$

- Now we have for the variance on the tree

$$var(r) = p_u u^2 + p_d d^2 - [p_u u + p_d d]^2 = \sigma^2 \Delta\text{t}$$



Fixing the values for *u* and *d*

- We have one equation and two unknown

$$var(r) = p_u u^2 + p_d d^2 - [p_u u + p_d d]^2 = \sigma^2 \Delta\mathrm{t}$$

- Insert the expressions for p_u and p_d

$$var(r) = \exp(r\Delta t)\,(u + d) - ud - \exp(2r\Delta t) = \sigma^2 \Delta\mathrm{t}$$

- Introduce the constraint $u = 1/d$ - then after inserting and arranging

$$\exp(r\Delta t)\,u^2 - (1 + \exp(2r\Delta t) + \sigma^2\Delta\mathrm{t})\mathrm{u} + \exp(\mathrm{r}\Delta t) = 0$$

- This is a quadratic equation with a solution

$$u = \frac{2 + 2r\Delta t + \sigma^2 \Delta\mathrm{t} \pm 2\sigma\sqrt{\Delta t}}{2 + 2r\Delta t} \xrightarrow[\lim_{\Delta t \to 0}]{} 1 \pm \sigma\sqrt{\Delta t}$$

- Accepting only the "+ part"

$$u = \exp\left(\sigma\sqrt{\Delta t}\right) \quad ; \quad d = \exp\left(-\sigma\sqrt{\Delta t}\right)$$

Fixing *u* and *d*

- The values for *u* and *d* are related to the stock price volatility in the following way:

$$\boxed{u = \exp\left(\sigma\sqrt{\Delta t}\right)} \quad ; \quad \boxed{d = \exp\left(-\sigma\sqrt{\Delta t}\right)}$$

and therefore

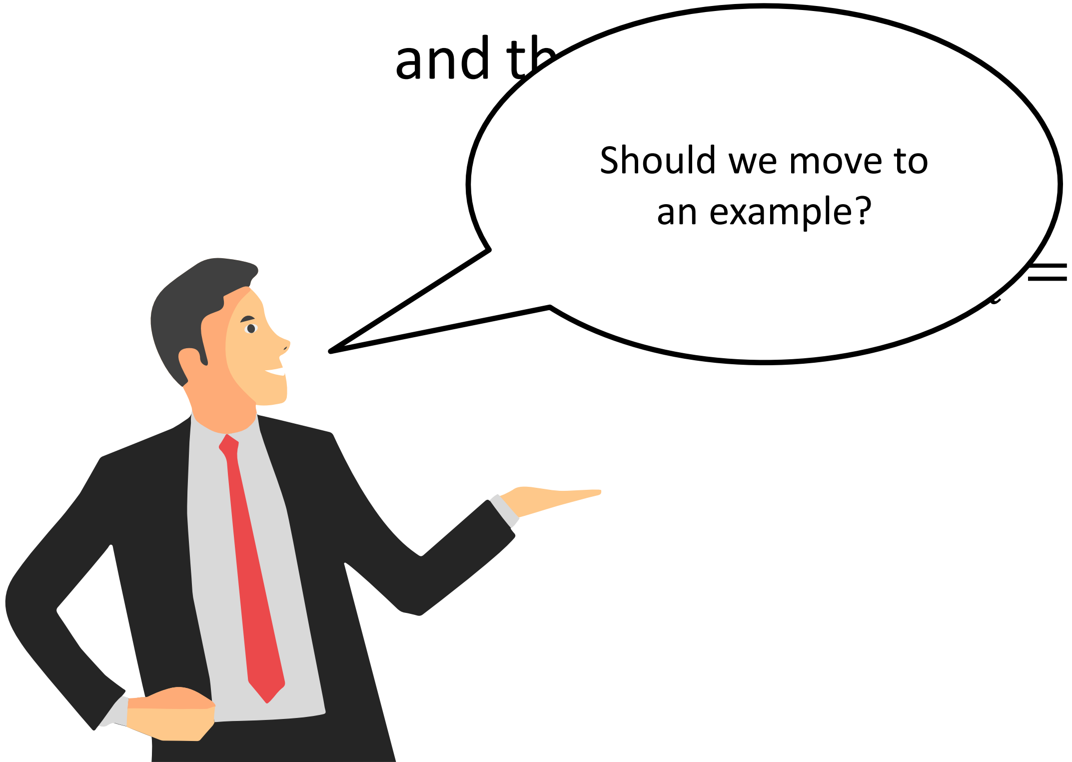
$$\boxed{p_u = \frac{\exp(r\Delta t) - d}{u - d}}$$

Fixing u and d

- The values for u and d are related to the stock price volatility in the following way:

$$u = \exp\left(\sigma\sqrt{\Delta t}\right) \quad ; \quad d = \exp\left(-\sigma\sqrt{\Delta t}\right)$$

and th



$$= \frac{\exp(r\Delta t) - d}{u - d}$$

Example 1

- Consider a call option on an asset. The parameters are:

$$S_0 = 50, K = 51, r = 5\%, \sigma = 20\%, \Delta t = 1$$

- Calculate the price of a call option using 1-step binomial tree
- First calculate

$$u, \qquad d, \qquad \exp(\mathrm{r}\Delta t)$$

- Then,

$$f_u, \qquad f_d$$

- And finally,

$$f$$

Example 1

- Consider a call option on an asset. The parameters are:

$$S_0 = 50, K = 51, r = 5\%, \sigma = 20\%, \Delta t = 1$$

- Calculate the price of a call option using 1-step binomial tree
- First calculate

$$u = \exp\left(0.2\sqrt{1}\right) = 1.22, \qquad d = \exp\left(-0.2\sqrt{1}\right) = 0.82, \qquad \exp(0.05 * 1) = 1.05$$

- Then,

$$f_u = \max(S_u - K, 0) = \max\left(50\exp\left(0.2\sqrt{1}\right) - 51{,}0\right) = 10.07$$

$$f_d = \max(S_d - K, 0) = \max\left(50\exp\left(-0.2\sqrt{1}\right) - 51{,}0\right) = 0$$

- And finally,

$$f = \exp(-0.05)\left(\frac{1.05 - 0.82}{1.22 - 0.82}\right) * 10.07 = 5.51$$

Example-2

- Consider a call option on an asset. The parameters are:

$$S_0 = 50, K = 51, r = 5\%, \sigma = 20\%, \Delta t = 1$$

- Calculate the price of a call option using 2-step binomial tree – now we need to set $\Delta t = 1/2$

- Your turn…

Example-2

- Consider a call option on an asset. The parameters are:

$$S_0 = 50, K = 51, r = 5\%, \sigma = 20\%, \Delta t = 1$$

- Calculate the price of a call option using 2-step binomial tree – now we need to set $\Delta t = 1/2$

$$f_{uu} = \max(S_{u^2} - K, 0) = \max\left(50\left(0.2\sqrt{\frac{1}{2}}\right)\exp\left(0.2\sqrt{\frac{1}{2}}\right) - 51{,}0\right) = 15.34$$

$$f_{ud} = \max(S_{ud} - K, 0) = 0 \quad ; \quad f_{dd} = \max(S_{d^2} - K, 0) = 0$$

$$f_u = \exp\left(-0.05 * \frac{1}{2}\right) * \left(\frac{\exp\left(0.05 * \frac{1}{2}\right) - \exp\left(-0.2 * \sqrt{\frac{1}{2}}\right)}{\exp\left(0.2 * \sqrt{\frac{1}{2}}\right) - \exp\left(-0.2 * \sqrt{\frac{1}{2}}\right)}\right) * f_{uu} = 0.98 * \left(\frac{1.03 - 0.87}{1.15 - 0.87}\right) * 15.34$$

$$\Rightarrow f = \exp\left(-0.05 * \frac{1}{2}\right)\left(\frac{1.03 - 0.87}{1.15 - 0.87}\right) * 8.59 = 4.81$$

Example-3

- Consider a call option on an asset. The parameters are:
$$S_0 = 50, K = 51, r = 5\%, \sigma = 20\%, \Delta t = 1$$
- Calculate the price of a call option using Black – Scholes equation

$$call\ c = S_0 * N(d_1) - Kexp(-rT)N(d_2)$$

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = 0.25 \quad ; \quad d_2 = d_1 - \sigma\sqrt{T} = 0.05$$

$$\Rightarrow c = 4.71$$

Stochastic behaviour of assets

- Assume that the price changes of an underlying asset follow geometric Brownian motion

$$dS = S\mu dt + S\sigma dW \Rightarrow S(t+dt) = S(t) + dS$$
$$dW \sim \eta\sqrt{dt} \quad ; \; \eta \in N(0,1)$$

- The μ-term is called the drift term as it controls the drift for the process
- The σ-term is called 'the noise' or the volatility terms as it controls the variability of the process

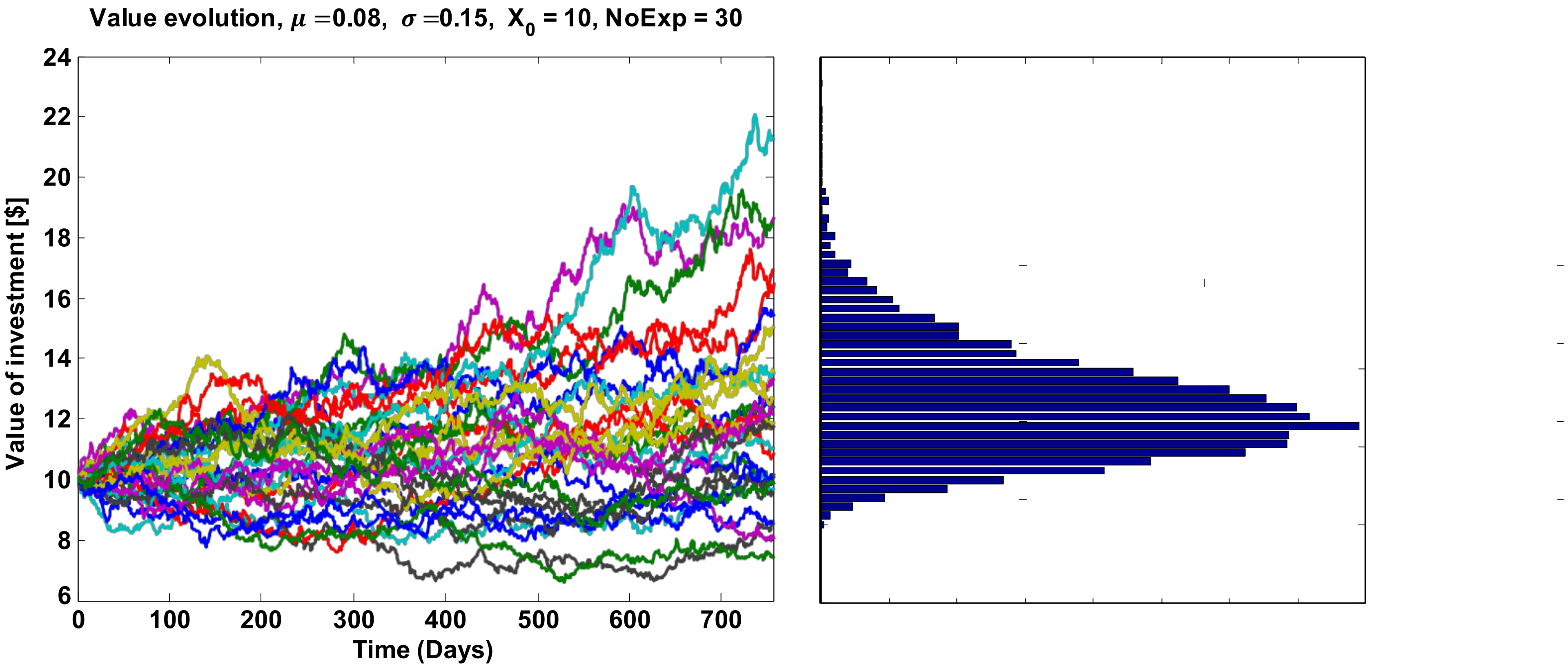
-----

- The μ component is deterministic
- The σ component is stochastic

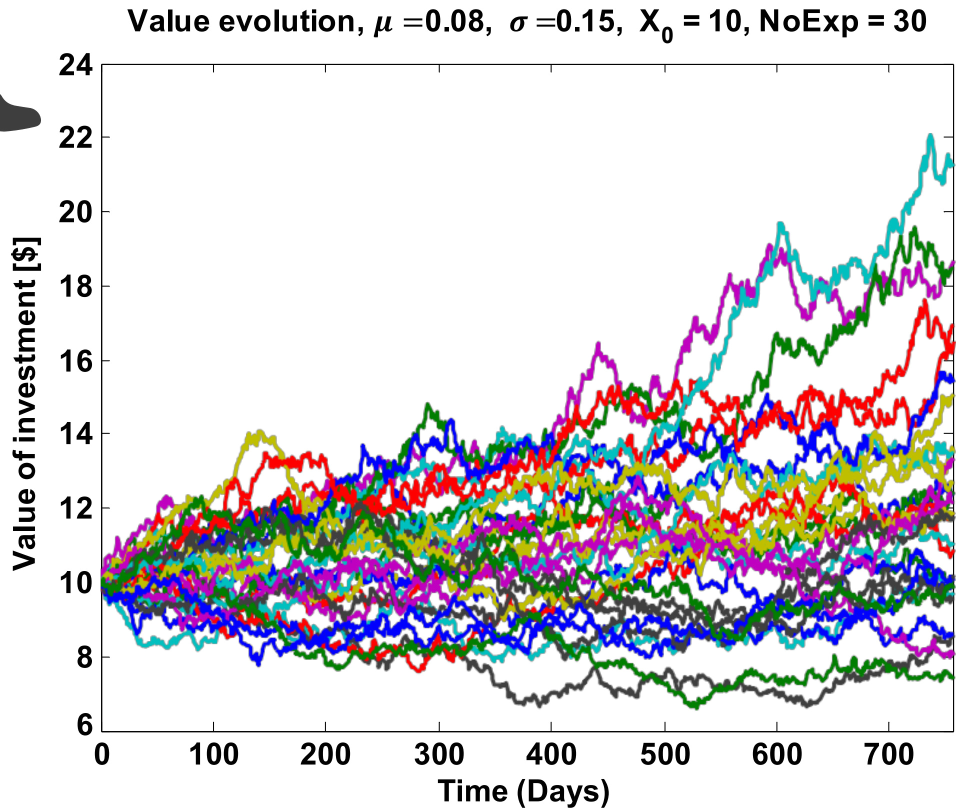
Stochastic behaviour of assets

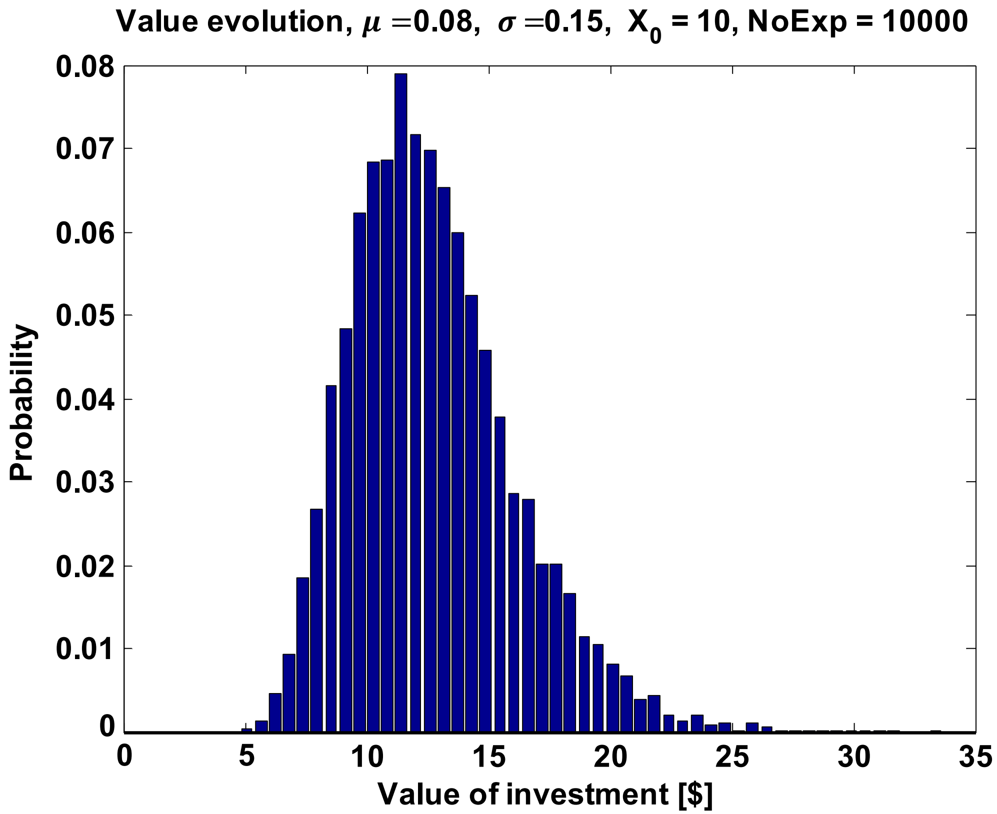
- Asset price evolution (left) and end distribution (right)

$$dS = S\mu dt + S\sigma dW \Rightarrow S(t+dt) = S(t) + dS$$



Can we find an order in this chaos?

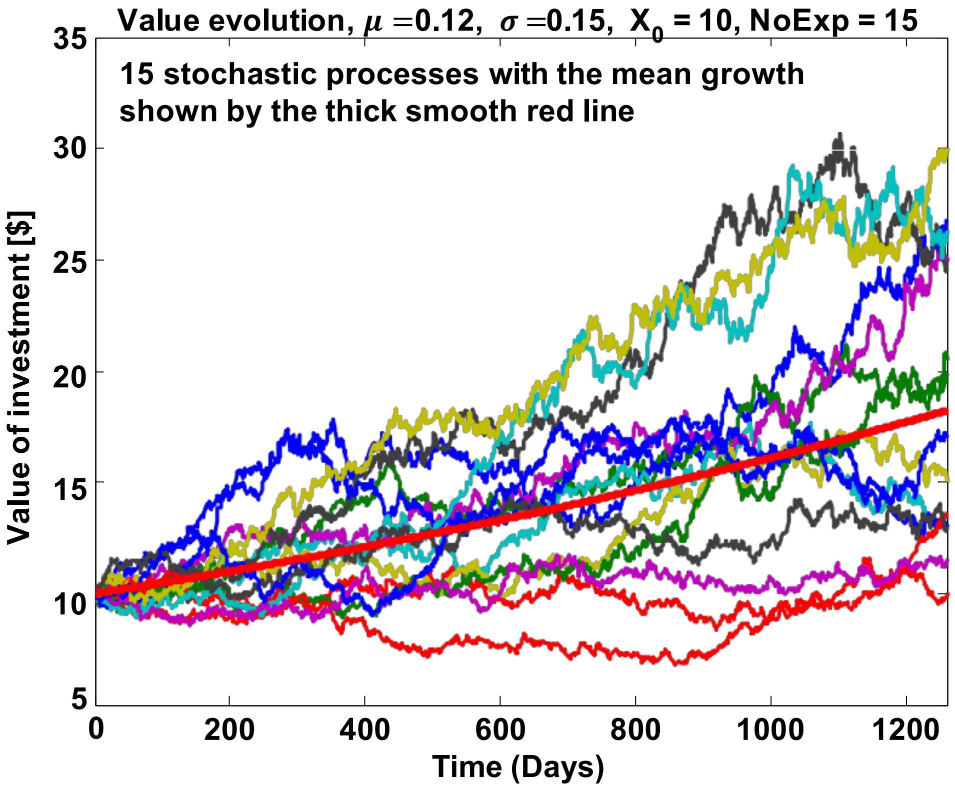




Stochastic behaviour of assets

- Geometric Brownian motion with the mean controlled by the drift term given by

$$S(t) = S(0)\mathrm{e}^{\mathrm{rT}}$$



Stochastic behaviour of options

- The price of the derivative security F derives it's value from an underlying asset S
- This underlying asset is generally assumed to follow some stochastic process – for example geometric Brownian motion – in other words,

$$F = F\left(S, \sigma, \ldots\right)$$

- What is the expression for the price changes dF of the derivative security ?

▶ The answer is given by Ito's Lemma ☢

Ito process

- The price of underlying asset changes according to a geometric Brownian motion process

$$dS = S\mu dt + S\sigma dW$$

- The price of an option on the primary security also follows a stochastic process:

$$dF(S,t) = \left(\mu S \frac{\partial F}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial t}\right) dt + \sigma S \frac{\partial F}{\partial S} dW$$

- The first process describes the underlying asset the second the derivative security. The two can be “combined” to reduce/remove the stochastic component presented by the Wiener component dW

Choosing a risk free portfolio

- If the price of the underlying follows GBM-process then so does the price of the derivative security
- In both cases the price risk is modelled by a Wiener process
- By choosing an appropriate portfolio combination of underlying and derivative securities the risk can be removed temporarily. The change in this portfolio can be expressed as

$$dP = A * dF + B * dS$$

- By fixing the weights

$$B = \frac{\partial F}{\partial S} \quad ; \quad A = -1 \;\Rightarrow\; dP = \frac{\partial F}{\partial S} dS - dF$$

i.e. for each short position in the derivative security hold $\partial F / \partial S$ underlying securities

Black-Scholes equation

- By inserting the expressions for dS and dF we find:

$$dP = \frac{\partial F}{\partial S} dS - dF = -\left(\frac{\sigma^2 S^2}{2} \frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial t}\right) dt$$

  which we rewrite as

$$dP = d\left(\frac{\partial F}{\partial S} S - F\right) = -\left(\frac{\sigma^2 S^2}{2} \frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial t}\right) dt$$

- We have constructed a portfolio:

$$P = \frac{\partial F}{\partial S} S - F$$

  which is temporarily is risk less – i.e. it contains no dW term

Black-Scholes equation

- If the portfolio P is risk less one should demand it grows at the risk free interest rate *r*, i.e

$$dP = rPdt$$

- Inserting all the previous expressions results in the Black-Scholes equation

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} = rF$$

- This differential equation can only be **solved after fixing boundary conditions.**
- The price of any derivative dependent on non-dividend paying stock has to satisfy the Black-Scholes equation

Black-Scholes Equation

To derive at the Black Scholes equation various assumptions have been made:

01

Liquidity

The market is considered liquid, in the sense that entering or closing a position can be done at any time.

Transaction Costs **02**

Assumed to be 0.

Taxation **03**

All the market participants pay the same taxes on profits.

Constant parameters **04**

The gBm (geometric Brownian motion) is assumed to have constant parameters r, μ, σ

05

Arbitrage

Market participants can take advantage of arbitrage opportunities. Let $S(t)$ be the spot price. Any arbitrage activity has been already accounted in the reflected price.

Comments on Black-Scholes

- The Black-Scholes equation:

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} = rF$$

  may have many different solutions depending on the type of derivative defined with S as the underlying

- Each particular derivative is found from specific boundary conditions - for example
  - $f = \max(S - X, 0)$ when $t = T$ for European call
  - $f = \max(X - S, 0)$ when $t = T$ for European put

Comments on Black-Scholes

- The solutions to the BS-equation with these boundary conditions are:

$$c(t) = S(t)N(d_1) - Xexp\big(-r(T-t)\big)N(d_2)$$
$$p(t) = Xexp\big(-r(T-t)\big)N(-d_2) - S(t)N(-d_1)$$

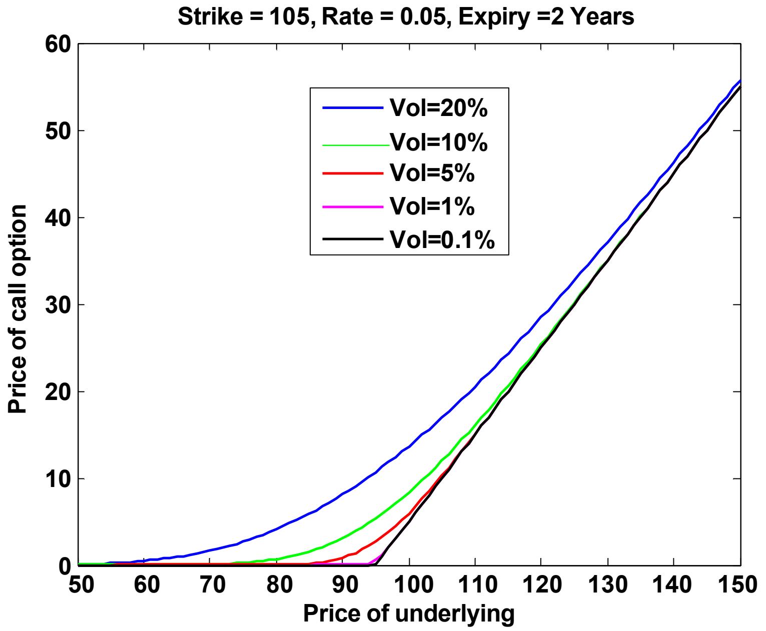
With:

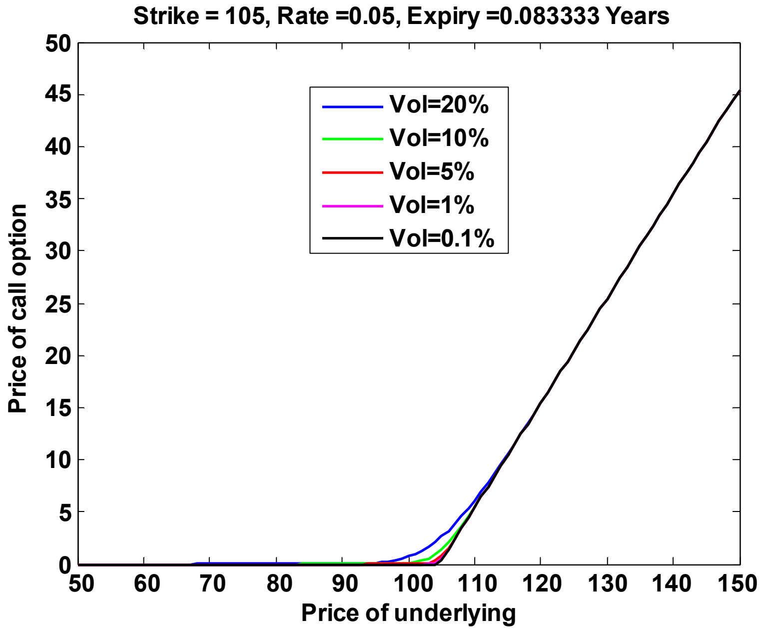
$$d_1 = \frac{\log\left(\frac{S(t)}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \; ; \; d_2 = \frac{\log\left(\frac{S(t)}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

$N(x)$ is the cumulative normal distribution function, i.e. $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy$

The BS solutions

- BS call price for different volatilities and expiry in two years (left)
- BS call price for different volatilities and expiry in one month (right)





- We do see the time value ‘footprint’ of the option.

Comments on Black-Scholes

With Dividends

$$c(t) = S(t)\exp\big(-\delta(T-t)\big)\,N(d_1) - Xexp\big(-r(T-t)\big)N(d_2)$$
$$p(t) = Xexp\big(-r(T-t)\big)N(-d_2) - S(t)\exp\big(-\delta(T-t)\big)\,N(-d_1)$$

With:

$$d_1 = \frac{\log\left(\frac{S(t)}{X}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\ ;\ d_2 = \frac{\log\left(\frac{S(t)}{X}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

$N(x)$ is the cumulative normal distribution function, i.e. $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy$

Black-Scholes - Example

- BS prices on assets with no dividend payments:

Simple Black-Scholes				
Inputs				
Price	120		d1 =	0.266207
Strike	135		d2 =	-0.0802
Rate	0.05		N(d1) =	0.60496
Time	3		N(d2) =	0.468038
Volatility	0.2		N(-d1) =	0.39504
Dividend	0		N(-d2) =	0.531962
Outputs				
Call =	**18.21129**			
Put =	**14.40686**			

- BS prices on assets with known dividend payments: [3% per/y]

Simple Black-Scholes				
Inputs				
Price	100		d1 =	0.289914
Strike	100		d2 =	-0.06364
Rate	0.05		N(d1) =	0.614059
Time	2		N(d2) =	0.474629
Volatility	0.25		N(-d1) =	0.385941
Dividend	0.03		N(-d2) =	0.525371
Outputs				
Call =	**14.88372**			
Put =	**11.19101**			

Integration of the geometric Brownian motion

- Integrating a stochastic process requires techniques from stochastic integration – different from conventional integration
- For this we need to introduce Ito’s lemma
- This will be done in a non-rigorous way – with focus on its practical consequences

Comments on option elasticity

- Assume we have an asset priced at $S(t)$ and an option on this asset priced at $f(S,K,r,\sigma,T)$ (this option can either be a call or a put)
- We are interested in evaluating **the relative price change for the option i.e. df/f in response to the relative price change in the underlying i.e. dS/S**
- In particular we are interested in the so called option elasticity Ω defined as the ratio: $$\Omega_f = \frac{df/f}{dS/S} = \frac{S}{f}\frac{df}{dS} = \frac{S}{f}\Delta_f$$
- In the case of a call or a put we have respectively:

$$\boxed{\begin{aligned} \Omega_c &= \frac{S}{c}N(d_1) \\ \Omega_p &= -\frac{S}{p}N(-d_1) \end{aligned}}$$

Risk neutral portfolio

- Consider an asset whose price follows the geometric Brownian motion process:
$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

- With $V(t)$ the price of an option we now construct the following portfolio:
$$\Pi(t) = V(t) - \Delta S$$
- Applying Ito, we find:
$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS$$

- Putting $\Delta = \frac{\partial V}{\partial S}$

We find for the following expression:

Risk neutral portfolio

$$d\Pi - \mathrm{r}\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \ \frac{\partial^2 V}{\partial S^2}\right) dt - r(V - \Delta S)dt$$
$$= \left(\frac{\partial V}{\partial t} + r\frac{\partial V}{\partial S}S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV\right) = 0$$

Therefore:

$$d\Pi = r\Pi dt$$

Which tells us that the portfolio Π grows at the risk free rate r.

The Ito process

- A stochastic process is said to be an Ito process if
$$dX = \mu(t, X)dt + \sigma(t, X)dW$$
where $\mu(t,X)$ is the drift function and $\sigma(t,X)$the volatility function

- Special cases
  - Brownian motion: $\mu(t, X) = \mu$ and $\sigma(t, X) = \sigma$
  - Geometric Brownian motion $\mu(t, X) = \mu X$ and $\sigma(t, X) = \sigma X$

- We know how to integrate the Brownian motion process
- How do we integrate the **geometric** Brownian motion (**g**Bm) process is were the Ito Lemma comes in.

Ito’s lemma

- Let $f\,(t,X)$ be a function which is at least twice differentiable with respect to X and at least once with respect to t
- If X follows an Ito process

$$dX = \mu(t,X)dt + \sigma(t,X)dW$$

then f follows the following process:

$$df = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma\frac{\partial f}{\partial X}dW$$

- **Remark**. At this stage think of X as being the price of an “underlying” security and f the price of a “derivative” security
- The price evolution of f depends on the price evolution of X – and perhaps other parameters such as σ

Ito's lemma – sketch of a proof

For your own understanding

- A Taylor expansion of f gives:

$$df = \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \cdots + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2} + \cdots$$

- Ignoring terms of higher order than dt we have

$$dX = \mu(t,X)dt + \sigma(t,X)dW$$
$$dX^2 = \sigma(t,X)^2 dt$$

and therefore

$$df = \left(\frac{\partial f}{\partial t} + \mu(t,X)\frac{\partial f}{\partial X} + \frac{\sigma(t,X)^2}{2}\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma(t,X)\frac{\partial f}{\partial X}dW$$

Ito's lemma – special cases

- Brownian motion

$$dX = \mu dt + \sigma dW$$

$$df = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial X} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma \frac{\partial f}{\partial X} dW$$

- Geometric Brownian motion

$$dX = \mu X dt + \sigma X dW$$

$$df = \left(\frac{\partial f}{\partial t} + \mu X \frac{\partial f}{\partial X} + \frac{\sigma^2 X^2}{2}\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma X \frac{\partial f}{\partial X} dW$$

Geometric Brownian motion

- To integrate the Geometric Brownian Motion we use Ito's Lemma
- Introduce

$$f = \log X \Rightarrow \frac{\partial f}{\partial t} = 0 \quad ; \quad \frac{\partial f}{\partial X} = \frac{1}{X} \quad ; \quad \frac{\partial^2 f}{\partial X^2} = -\frac{1}{X^2}$$

- Insert back into Ito's expression for the gBm:

$$df = d\log X = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW$$

- Therefore $f = \log X$ follows a generalised Brownian process with drift rate $\mu - \sigma^2/2$ and variance rate σ^2

$$f(t_2) - f(t_1) \in N\left(\left(\mu - \frac{\sigma^2}{2}\right)(t_2 - t_1), \sigma\sqrt{t_2 - t_1}\right)$$

Geometric Brownian motion

- We found that $d\ \log X$ follows a generalized Brownian process

$$d\log X = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW$$

Which we can now integrate:

$$\int_{X(0)}^{X(t)} d\log X = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right)dt + \int_{W(0)}^{W(t)} \sigma dW$$

To find:

$$\log X(t) - \log X(0) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)$$

Normal & log-normal distributions

- From the solution of the GBM we find:

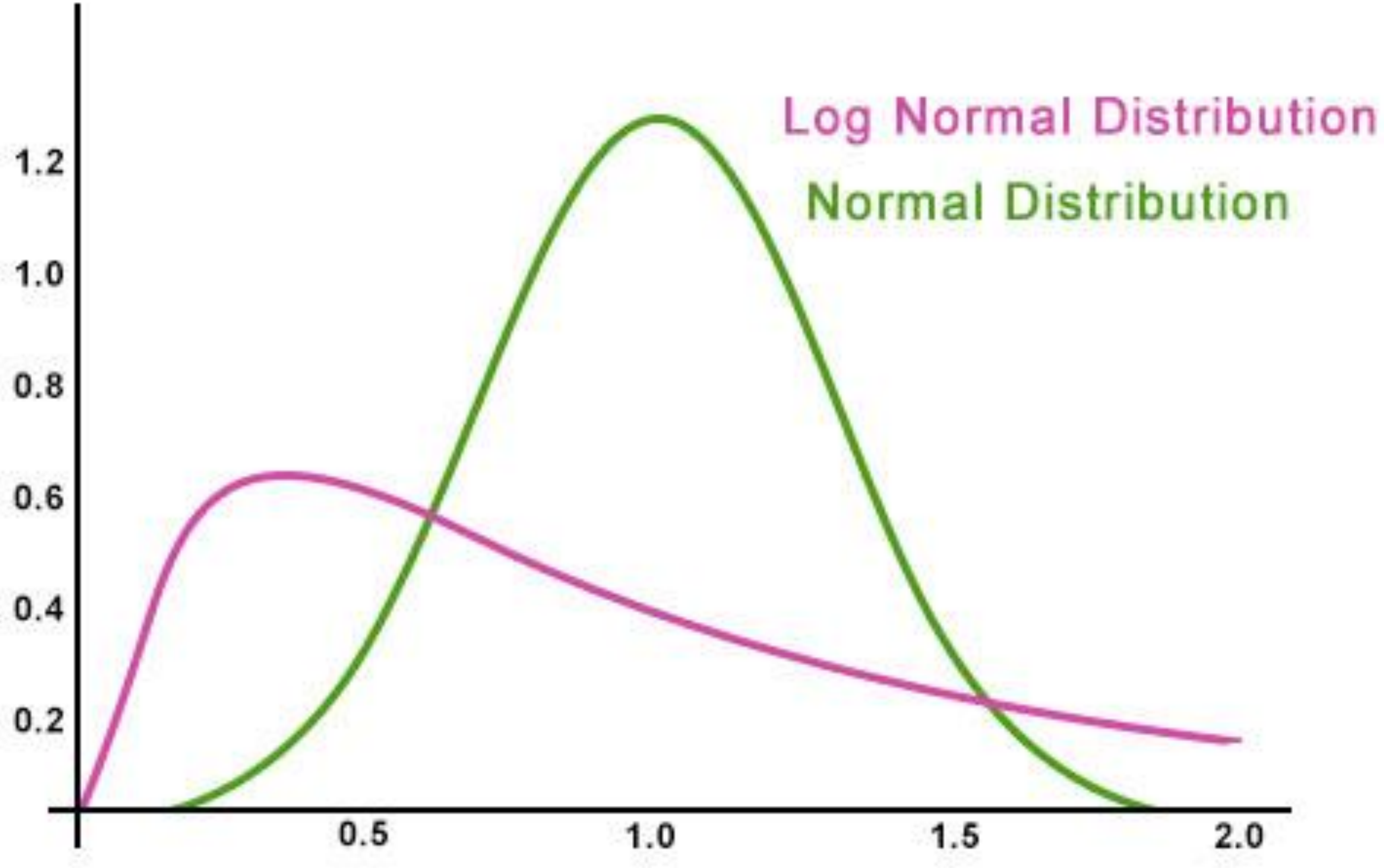
$$\log\left(\frac{X(t)}{X(0)}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)$$

- This is normally distributed i.e.

$$\log\left(\frac{X(t)}{X(0)}\right) = (\log\big(\mathrm{X(t)}\big) - \log\big(\mathrm{X(0)}\big) \in \mathrm{N}\left(\left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma\sqrt{t}\right)$$

$$\log\big(\mathrm{X(t)}\big) \in N\left(\log\big(X(0)\big) + \left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma\sqrt{t}\right)$$

- X itself has therefore log-normal distribution



Could you guess which problem the use of log-normal distribution over normal distribution can solve?

(specifically when used in the analysis of stock prices)

Integration of the geometric Brownian motion

- **Summary**. If X follows a GBM process

$$dX = \mu X dt + \sigma X dW$$

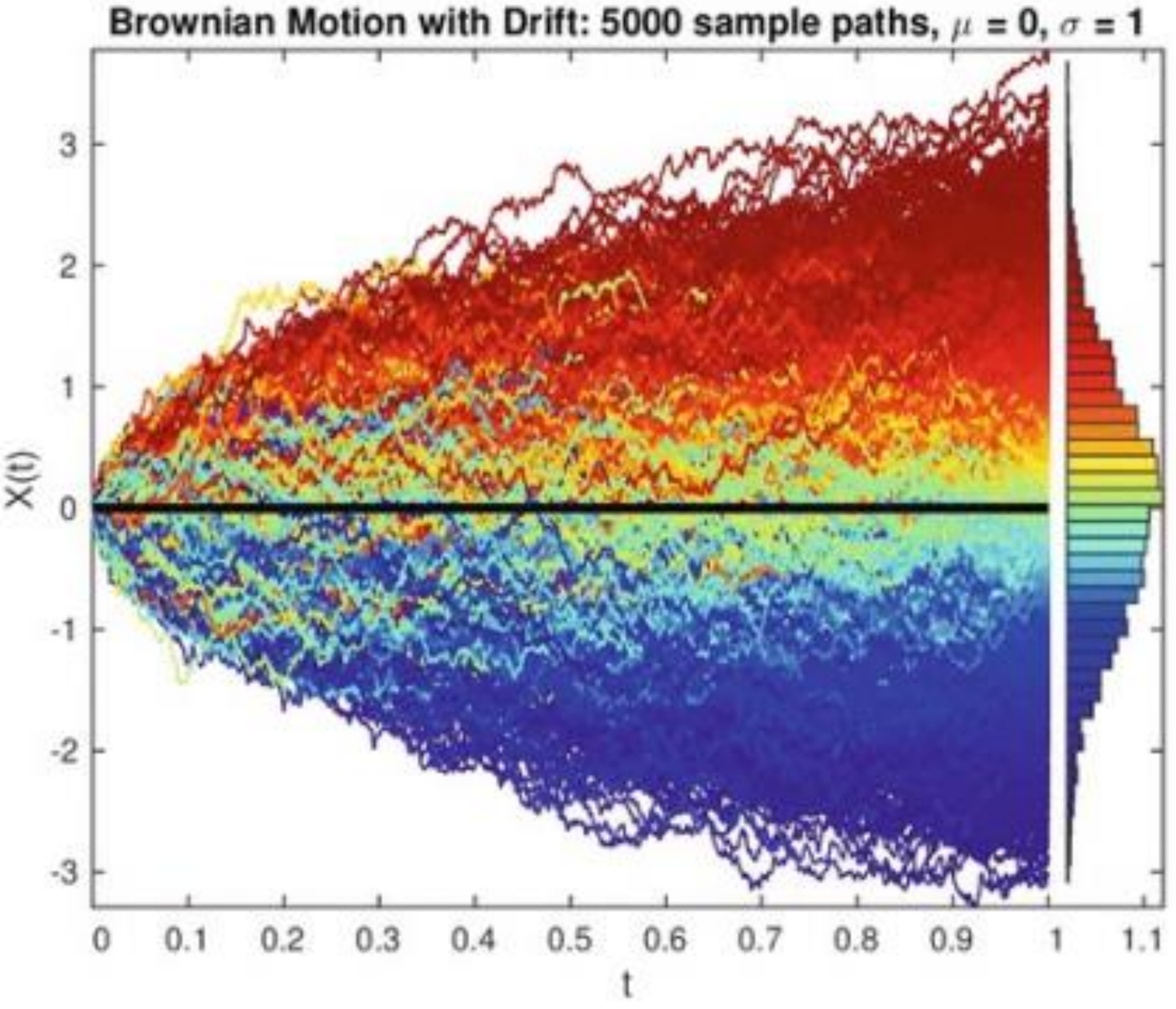
- Then, by using Ito´s lemma we found

$$\log\left(\frac{X(t)}{X(0)}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)$$

- And therefore by applying the $\exp$ – function on both sides:

$$X(t) = X(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right)$$

Estimating future asset values



Estimating future asset values

- Consider an asset which is assumed to follow a geometric Brownian motion:

$$dX = \mu \mathrm{Xdt} + \sigma X dW$$

- Then, we know from previous slides that the logarithm of the asset's values, at some future time T, is normally distributed:

$$\log X(T) \in N(\log\left(X(t) + \left(\mu - \frac{1}{2}\sigma^2\right)(T - t), \sigma\sqrt{T - t}\right) = N\big(M(t,T), \Sigma(t,T)\big)$$

- From this knowledge we can ask questions like: What is the probability that the asset's future value are in the interval:

$$I = [M(t,T) - \alpha\Sigma(t,T), M(t,T) + \alpha\Sigma(t,T)]$$

Some properties of the normal distribution

- Consider a random variable X distributed according to the normal distribution: $X \in N(\mu, \sigma)$

- The probability that this variable takes values in the interval $[\mu - \alpha\sigma, \mu + \alpha\sigma]$ is given by:

$$P_\alpha = \Pr(\mu - \alpha\sigma \leq x \leq \mu + \alpha\sigma) = \mathrm{erf}\left(\frac{\alpha}{\sqrt{2}}\right) \quad ; \quad \alpha \geq 0$$

- Therefore: $\sqrt{2} * erfinv(P_\alpha) = \alpha$

- The table below gives the connection between α and the associated probabilities P_α

Pa	0.6	0.7	0.8	0.9	0.95
a	0.8416	1.0364	1.2816	1.6449	1.96

Example

- By applying previous results:

$$\log S(T) \in N(\log\left(S(t) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t), \sigma\sqrt{T-t}\right)$$

- We find:

$$\log S(t) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t) - \alpha\sigma\sqrt{T-t} \leq \log S(T) \leq \log S(t) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \alpha\sigma\sqrt{T-t}$$

with probability P_{a}

- **Exercise**. There is 95% probability that a normally distributed variable has values within 1.96 standard deviations of its mean. From this find the value range within which the stock price of Icelandair will be with 95% probability in two years time.
Assume $\mathrm{S}(0) = 1.15\, ISK,\ \sigma = 25\%\,\&\, \mu = 4\%$

Geometric Brownian motion

- gBm:

$$dX = \mu X dt + \sigma X dW$$

- Because of $E\{dW\} = 0$ we find

$$E\{dX\} = \mu X dt$$

  and therefore

$$E\{X(t)\} = X_0 \exp(\mu t)$$

- For the variance one finds

$$var\{X(t)\} = X_0^2 e^{2\mu t}(\exp(\sigma^2 t) - 1)$$

The log-normal distribution

- A random variable Y is said to be a lognormal random variable with parameters μ and σ if log(Y) is a normal random variable with mean μ and standard deviation σ.
- Therefore, Y is lognormal if it can be expressed as $Y = \exp(X)$ where X is a normal random variable
- The probability density function (PDF) is:

$$f(y;\mu,\sigma) = \frac{1}{y\,\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(y)-\mu)^2}{2\sigma^2}\right)$$

The cumulative distribution (CDF) is:

$$F(y;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^y \frac{\exp\left(-\frac{(\ln(w)-\mu)^2}{2\sigma^2}\right)}{w} dw = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left[\frac{\ln(y)-\mu}{\sigma\sqrt{2}}\right] = \Phi\left(\frac{\ln(y)-\mu}{\sigma}\right)$$

The log-normal distribution

- Now, let S be the price of an asset assumed to be log-normally distributed – then:

$$f(S_T;\mu,\sigma) = \frac{1}{S_T\ \sigma_{\ln S_T}\sqrt{2\pi}} \exp\left(-\frac{(\ln(S_T) - E_t \ln S_T)^2}{2\sigma_{\ln S_T}^2}\right)$$

- We know:

$$E_t lnS_T = \ln S_t + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)$$

$$\sigma_{\ln S_T}^2 = \sigma^2(T-t)$$

- Therefore:

$$f(S_T;\mu,\sigma) = \frac{1}{S_T\ \sigma\sqrt{T-t}\sqrt{2\pi}} \exp\left(-\frac{\left(\ln\left(\frac{S_T}{S_t}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma(T-t)}\right)$$

Price at risk

- The price of an asset is assumed to be log-normally distributed
- Then, it can be shown that the probability that the asset price is below some value at the future time T is given by:

$$\Pr(S(T) \leq X)$$
$$= \frac{1}{\sqrt{2\pi}} * \int_{-\infty}^{\frac{\ln\left(\frac{X}{S(t)}\right)-\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}} \exp\left(-\frac{u^2}{2}\right) du = N(d)$$
$$= N(-d_2)$$

With

$d = \frac{\ln\left(\frac{X}{S(t)}\right)-\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$ or $d_2 = \frac{\ln\left(\frac{S(t)}{X}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$

Example

- An asset is log-normally distributed
- The details are

$$S_0 = 105 \quad ; \quad \mu = 0.08 \quad ; \quad \sigma = 0.2$$

- Calculate the probability that the assets price is lower than $X = 80$ in two years time

- Answer:

$$\Pr(S_2 \leq 80) = 8.292\%$$

Module 3

Hedging

3.2 The Greeks

The 'Greeks'

is a term used in the options market to describe the different dimensions of risk involved in taking an options position.

The Delta $\Delta = \frac{\partial f}{\partial S}$

Delta is the amount an option price is expected to move based on a $1 change in the underlying stock. Interestingly, this is also the % of probability to end in the moneyness.

The Gamma $\Gamma = \frac{\partial f}{\partial \Delta} = \frac{\partial^2 f}{\partial S^2}$

Gamma is used to determine how stable an option's delta is: higher gamma values indicate that delta could change dramatically in response to even small movements in the underlying's price. Gamma is higher for options that are at-the-money and lower for options that are in- and out-of-the-money, and accelerates in magnitude as expiration approaches

The Theta $\Theta = \frac{\partial f}{\partial T}$

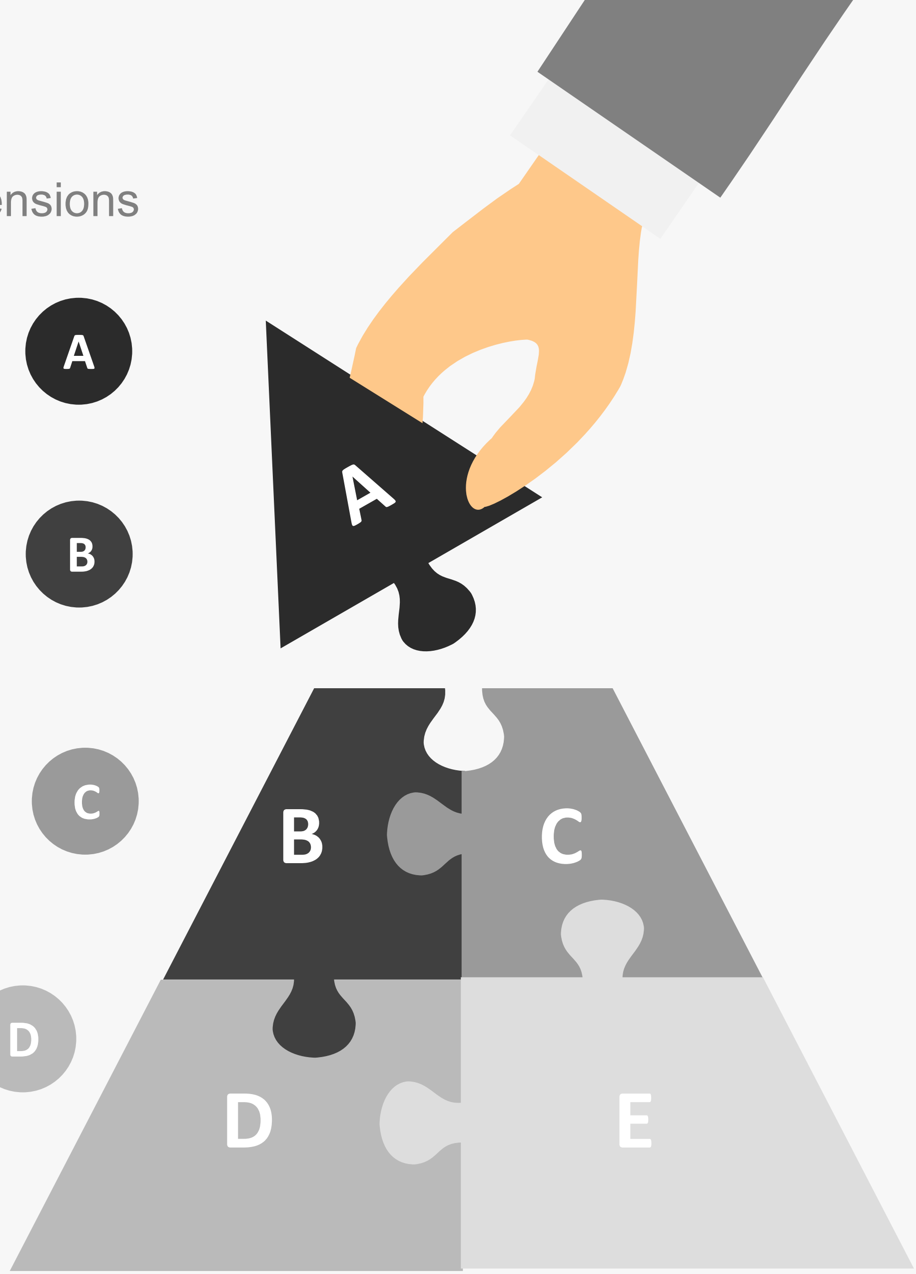
represents the rate of change between the option price and time, or time sensitivity - sometimes known as an option's time decay. Theta indicates the amount an option's price would decrease as the time to expiration decreases, all else equal. For example, assume an investor is long an option with a theta of -0.50. The option's price would decrease by 50 cents every day that passes, all else being equal.

Vega $V = \frac{\partial f}{\partial \sigma}$

This is the option's sensitivity to volatility. Vega indicates the amount an option's price changes given a 1% change in implied volatility. For example, an option with a Vega of 0.10 indicates the option's value is expected to change by 10 cents if the implied volatility changes by 1%

Rho $\rho = \frac{\partial f}{\partial r}$

This measures sensitivity to the interest rate. For example, assume a call option has a rho of 0.05 and a price of $1.25. If interest rates rise by 1%, the value of the call option would increase to $1.30, all else being equal.



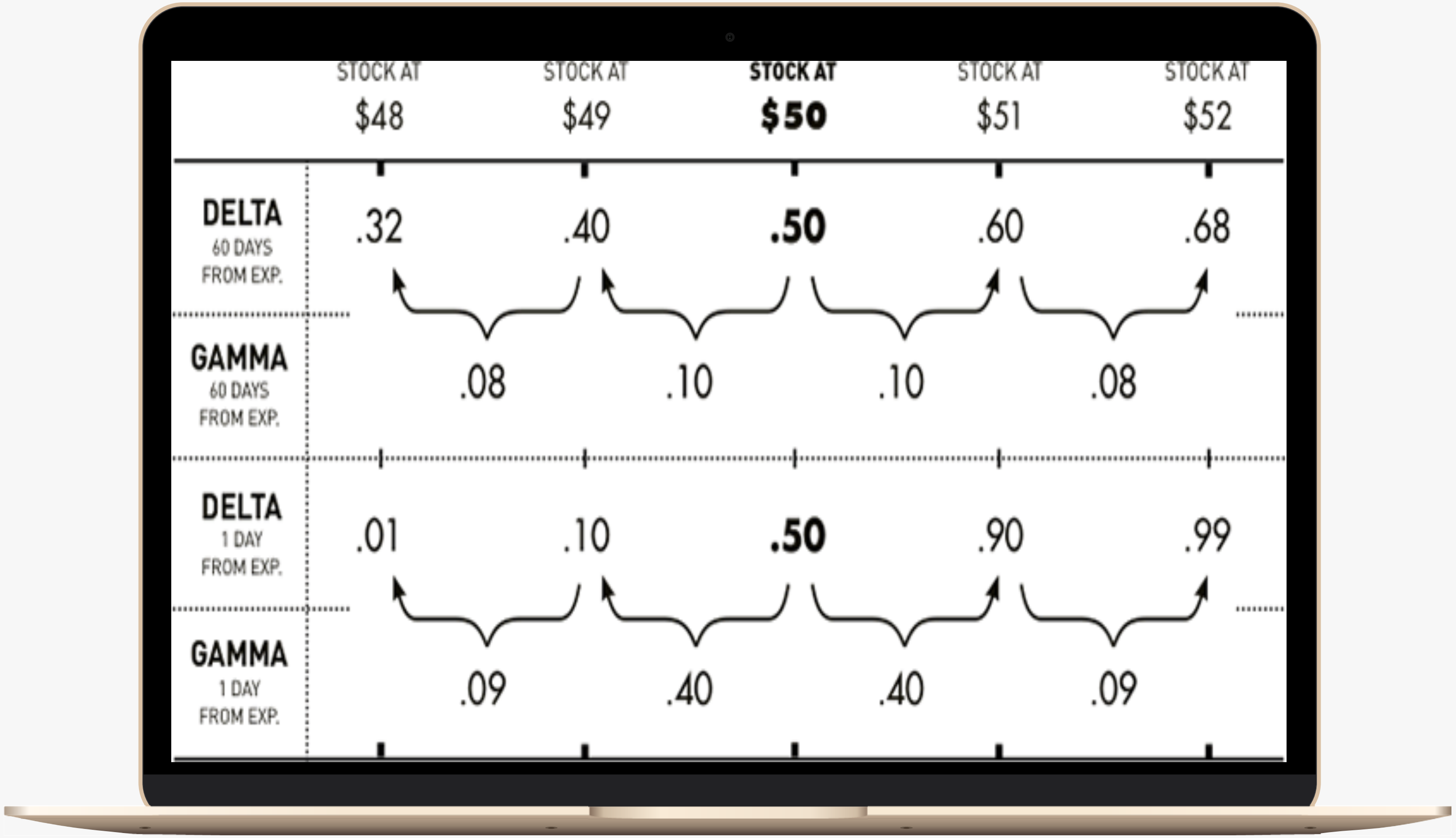
Greeks are very important in risk management. Studying them deeply is outside of the scope of this class, but it is very important to understand their role.

Note also, that they can be extended beyond the first or second partial derivative, but considered having a negligible effect at these levels.

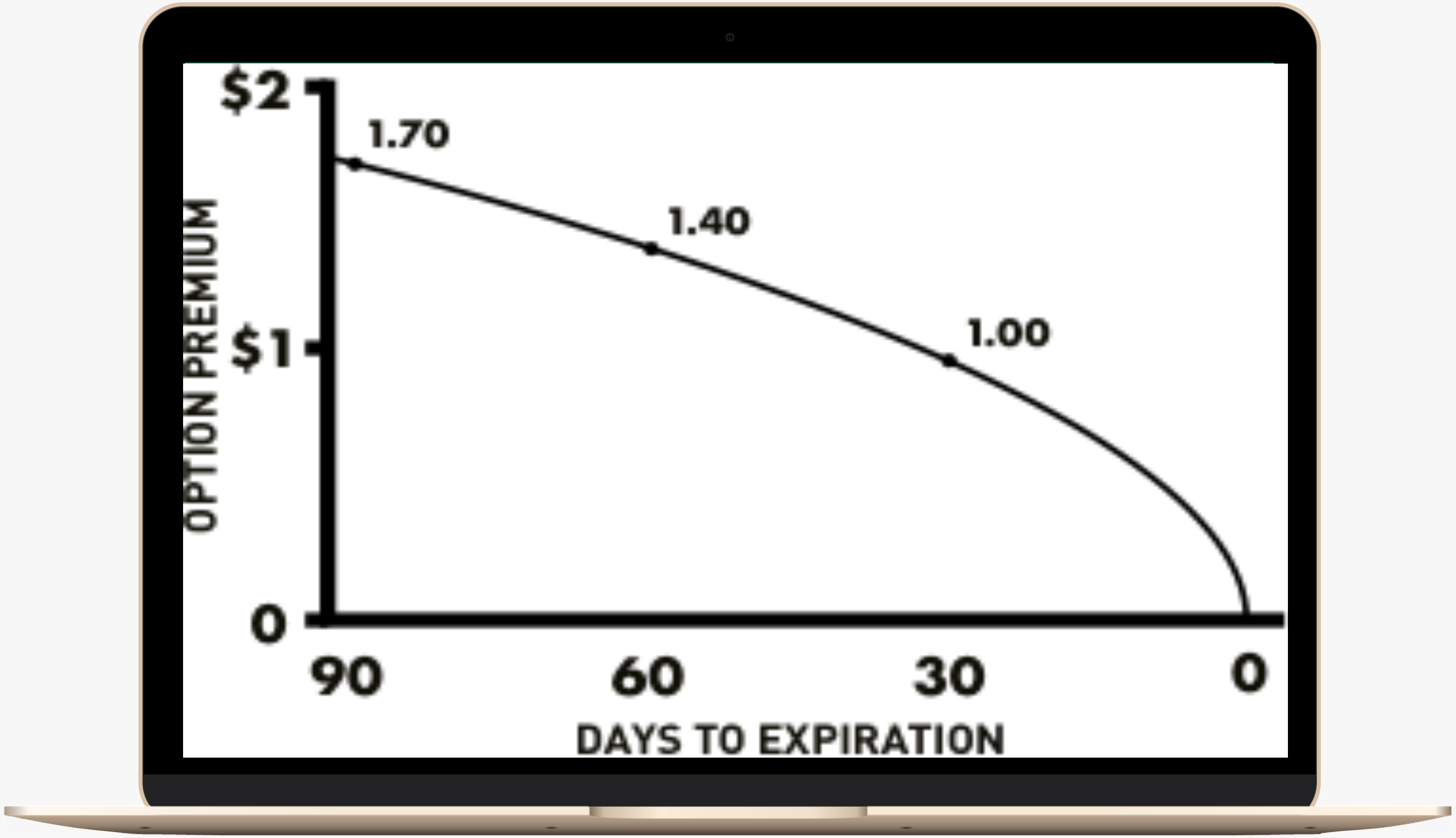
The aim for us is more to have a discussion about it than entering into a lot of calculations.

Delta & Gamma

$$\Delta_{call} = e^{-qT} * N(d_1)$$
$$\Delta_{put} = -e^{-qT} * N(-d_1)$$



Theta



Vega

(XYZ = $50, STRIKE PRICE = $50)		
TIME UNTIL EXP.	30 days	365 days
OPTION COST	$1.50	$5.36
VEGA	**.03**	**.20**

Option Greeks

Long/Short \ Greeks	Delta	Vega	Theta	Gamma	Rho
Long call	+	+	-	+	+
Long put	-	+	-	+	-
Short call	-	-	+	-	-
Short put	+	-	+	-	+

Module 4

Structured Products

Barrier Options

Knock-In or Knock-Out

Down or Up

Barrier Options - Example

Down or Up

Knock-In or Knock-Out

Payoff CALL: 0

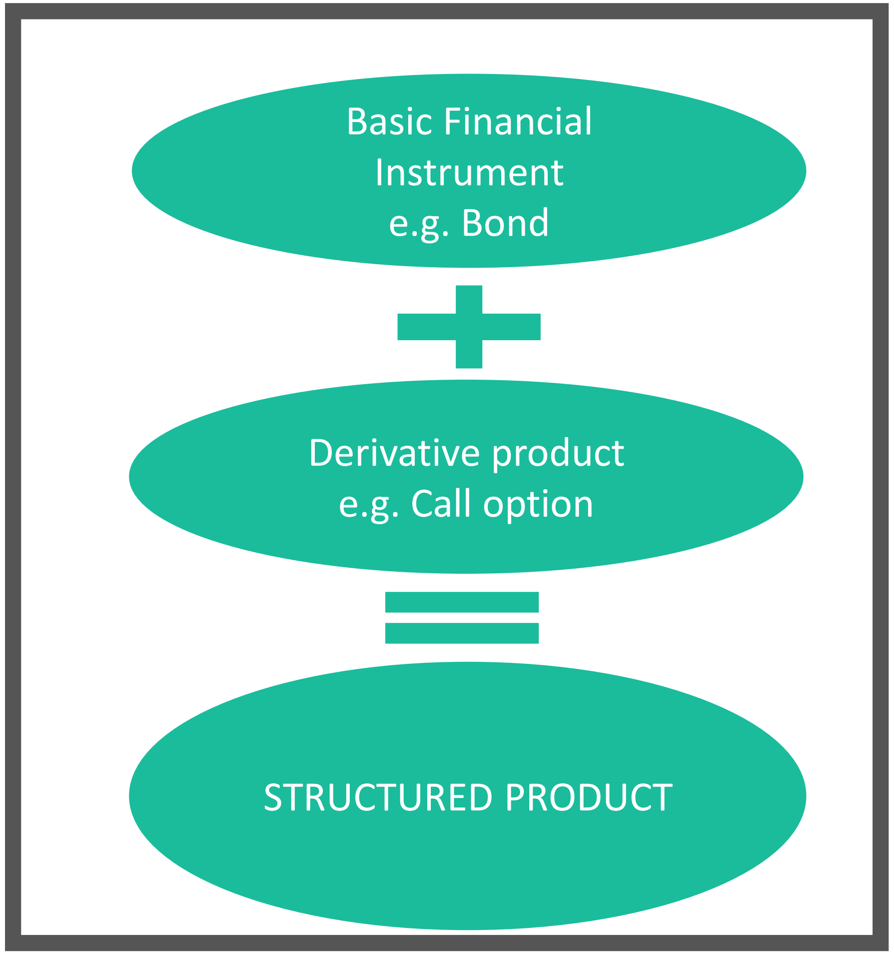
Payoff PUT: 0

Payoff CALL: 0

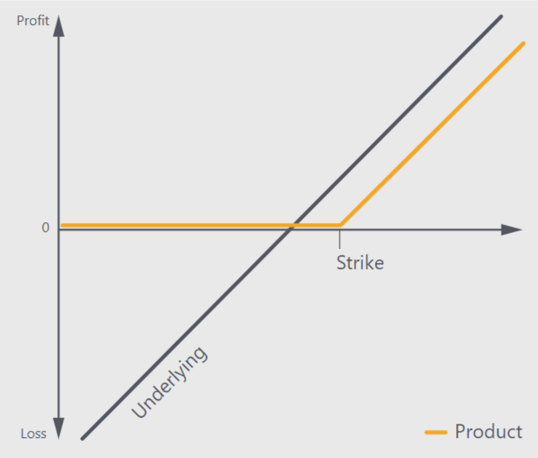
Payoff PUT: $K - S_T$

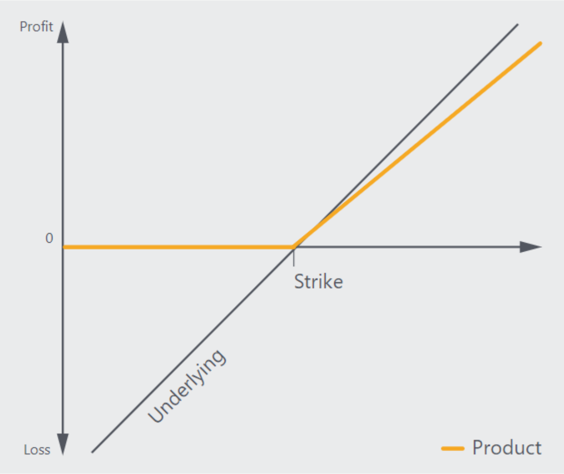
Structured Products

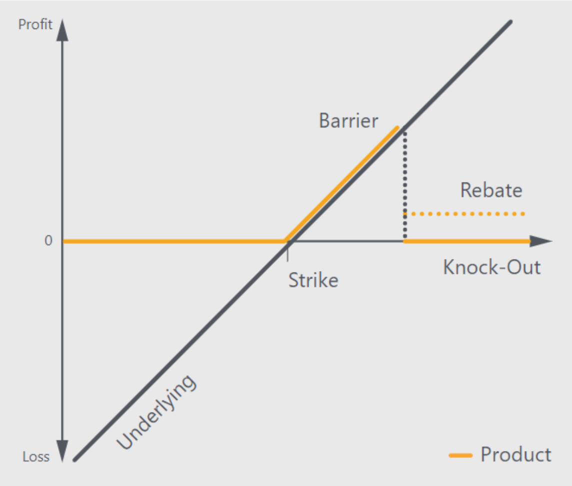
4 categories providing various risk/reward payoff.

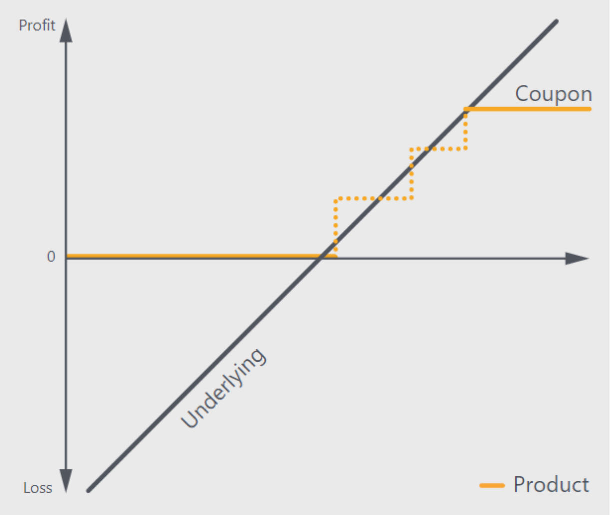


I/ Capital protection certificate



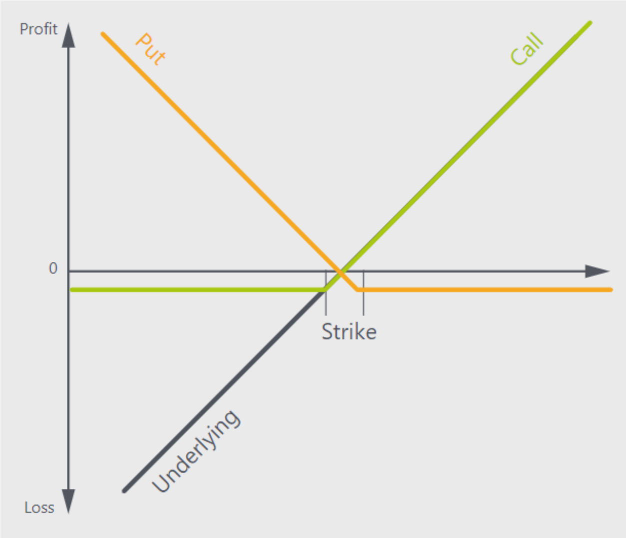


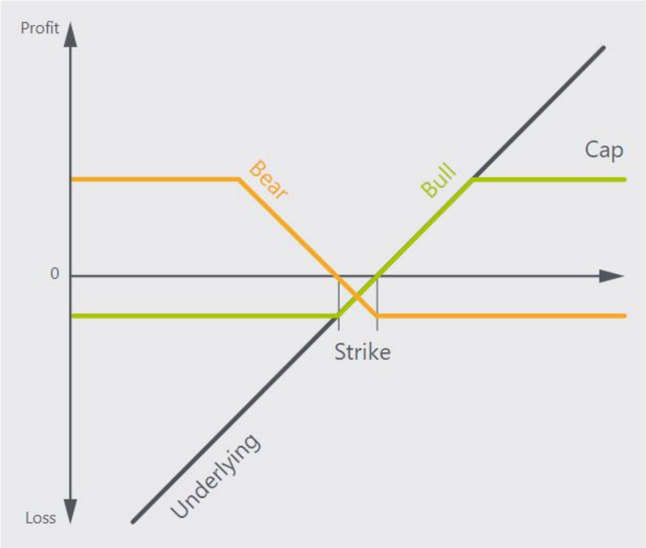


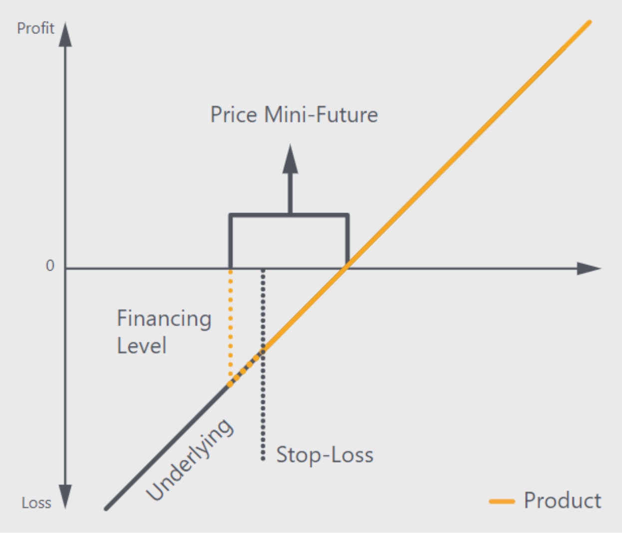


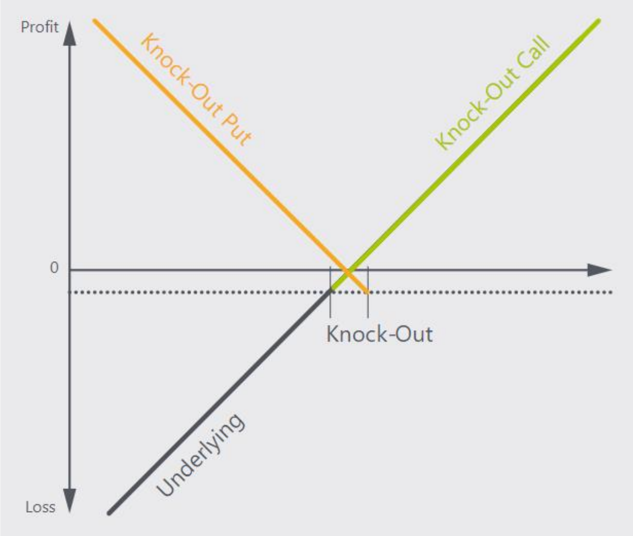
Source:

II/ Leverage



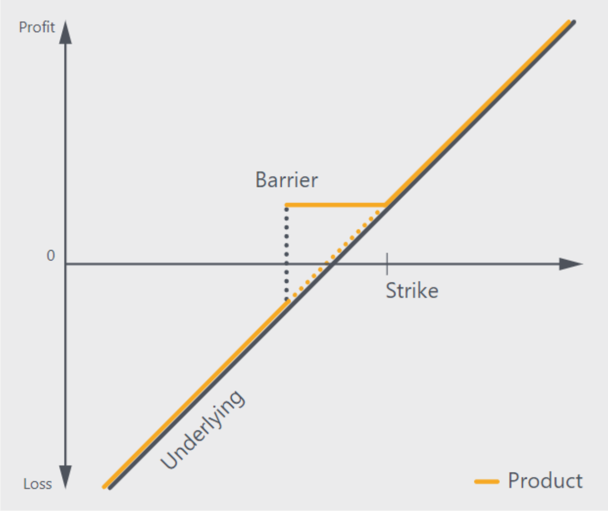




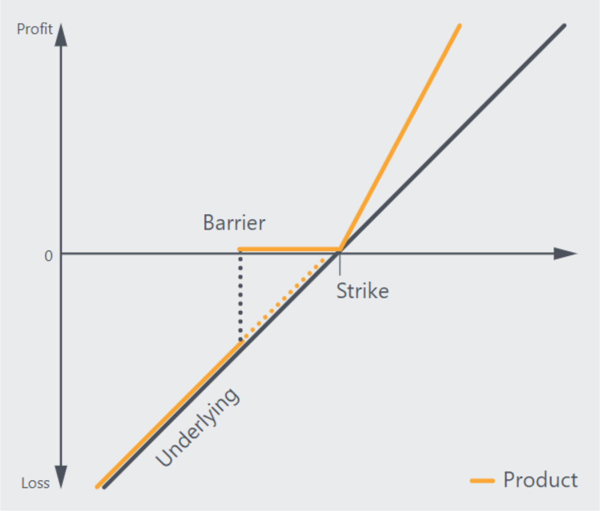


Source:

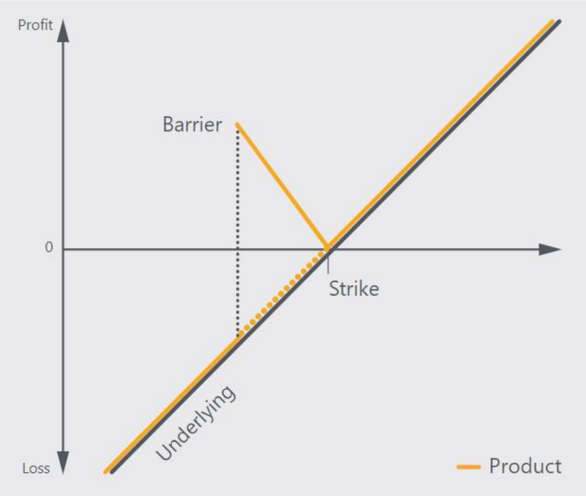
III/ Participation



Bonus Certificate



Outperformance Certificate

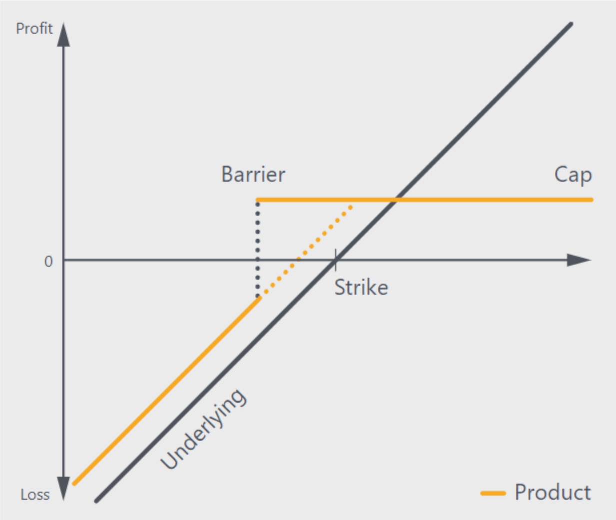


Twin-Win

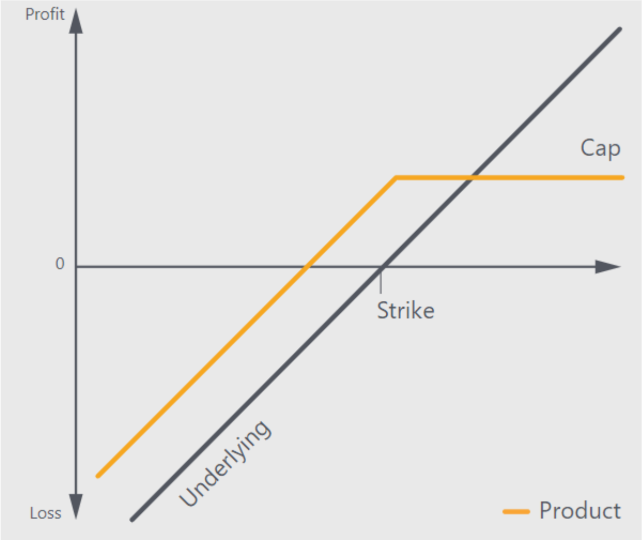
Source:

LEONTEQ
SWISS INVESTMENT ENGINE

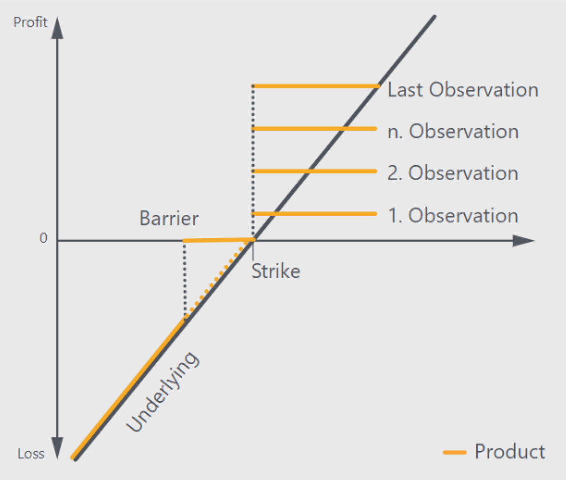
III/ Yield Enhancement



Barrier Reverse Convertible (BRC)
(Bond + Short Down-and-in Put)



Discount Certificate
(Bond + Short Put)



Autocallable BRC

Source:

LEONTEQ
SWISS INVESTMENT ENGINE

Everything is modulable/customizable:

Single underlying or Basket of underlyings

Barrier observable at maturity (EU), continuously (US), at some specific time (Bermudean), on average (Asian)

Autocallable feature

Barrier on coupon or not, with memory effect or not.

Any change modifying the payoff can be impacted on either the strike, barrier, or product price.

Brain gymnastic that requires a good understanding of derivatives and greeks.

References

Some literature

- Introduction
  - Bodie, Kane and Marcus, *Investments*, McGraw-Hill 2017
- Intermediate
  - Hull, *Options, Futures and other Derivatives*, Pearson-Prentice Hall, 2017
- Advanced
  - Björk, *Arbitrage Theory in Continuous Time*, Oxford University Press, 2009
  - Chin, Dian and Ólafsson, *Problems and Solutions in Mathematical Finance Volume* 1: *Stochastic Calculus*, John Wiley 2014
  - Chin, Dian and Ólafsson, *Problems and Solutions in Mathematical Finance Volume* 2: *Equity Derivatives*, John Wiley 2017