Class Test 3 - Solution

T-503-AFLE: Class Test 3 - Solution

- **Duration:** 1.5 hours
- **Total Marks:** 20 marks

1: Problem 2.5.6 (Piecewise Constant Dividend Yield) (2 marks)

It is now January 1, 3018. You are given:

- The current price of the stock is 1,000 .
- The stock pays dividends continuously at a rate proportional to its price. The dividend yield changes throughout the year. In March, June, September, and December, the dividend yield is 3%. In other months, the dividend yield is 2%.
- The continuously compounded risk-free interest rate is 9%.

Calculate the 1-year fair forward price.
(Hint: How many shares should you buy at time 0 to end up with exactly one share in one year?)

Solution:
The forward price is

$$
\begin{aligned}
F_{0,1} &= S(0)\mathrm{e}^{rT} \times \mathrm{e}^{-\int_0^T \delta_s \,\mathrm{d}s} \\
&= 1,000\mathrm{e}^{0.09(1)} \times \underbrace{\mathrm{e}^{-0.03(4/12)-0.02(8/12)}}_{\substack{\text{only need this no. of shares at time 0} \\ \text{to have one share at time 1}}} \\
&= 1,068.94.
\end{aligned}
$$

Remark. (i) When the dividend yield is a (continuous) function of time, say δ_t, the differential equation for the number of shares is $\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \delta_t N(t)$, whose solution is $N(t) = N(0)\mathrm{e}^{\int_0^t \delta_s \,\mathrm{d}s}$

2: Problem 3.5.2. (Simple profit calculation for a floor) (2 marks)

You buy a non-dividend-paying stock at $\$300$ and buy an at-the-money 9-month European put option on the stock at a price of $\$15$.
The continuously compounded risk-free interest rate is 5%.
Calculate your 9-month profit if the 9-month stock price is $\$280$.

Solution. The initial investment is $300+15=315$. The 9 -month payoff is $(300-280)_{+}+280=300$, which is the floor level. The 9 -month profit is $300-315\mathrm{e}^{0.05(0.75)}=$ -27.0368 .

3: Example 4.1.11 (Estimating σ) (3 marks)

You are to estimate a nondividend-paying stock's annualized volatility using its prices in the past nine months.

Month	Stock Price ($/share)
1	80
2	64
3	80
4	64
5	80
6	100
7	80
8	64
9	80

Calculate the volatility for this stock over the period.

Solution:
Let r_i be the continuously compounded monthly returns for the i^{th} month. Then:

Month i	$r_i = \ln\{S(ih)/S[(i-1)h]\}$
1	$\cos =$
2	$\ln(64/80) = \ln 0.8$
3	$\ln(80/64) = \ln 1.25$
4	$\ln(64/80) = \ln 0.8$
5	$\ln(80/64) = \ln 1.25$
6	$\ln(100/80) = \ln 1.25$
7	$\ln(80/100) = \ln 0.8$
8	$\ln(64/80) = \ln 0.8$
9	$\ln(80/64) = \ln 1.25$

Note that four of the r_i 's are $\ln 1.25$ and the other four are $\ln 0.8 = -\ln 1.25$. In particular, their mean $\bar{r}$ is zero.

The (unbiased) sample variance of the non-annualized monthly returns is

$$\hat{\sigma}^2_{1/12} = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2 = \frac{1}{7}\sum_{i=1}^{8} r_i^2 = \frac{8}{7}(\ln 1.25)^2$$

and the estimated annualized volatility is

$$\hat{\sigma} = \frac{\hat{\sigma}_{1/12}}{\sqrt{1/12}} = \sqrt{12} \times \sqrt{\frac{8}{7}(\ln 1.25)^2} = 82.64\%. \text{ (Answer: (A))}$$

Mark Allocation:

- 1 - Computing returns
- 1 - Computing montly volatility
- 1 - Computing annual volatility

4: Problem 4.6.6. (A market consisting only of risky securities - II) (4 marks)

In an arbitrage-free securities market, there are two non-dividend-paying stocks, A and B, both with current price $\$90$. There are two possible outcomes for the prices of A and B one year from now:

Outcome	A	B
1	\$100	\$80
2	\$60	$\$x$

The current price of a one-year 100-strike European put option on B is $\$15$. Determine all possible values of x.

Solution:
Solution:
Replicating the one-year 100-strike European put by α units of Stock A and β units of Stock B requires

$$\begin{cases} 90\alpha + 90\beta = 15 & \text{(time-0 price)} \\ 100\alpha + 80\beta = (100-80)_+ = 20 & \text{(Outcome 1 payoff)} \end{cases}$$

resulting in $\alpha = 1/3$ and $\beta = -1/6$. The replicating portfolio and the put option must have the same payoff at Outcome 2, so

$$(100 - x)_+ = 60\alpha + x\beta = 20 - \frac{x}{6}.$$

To solve this equation in x, we distinguish two cases:
Case 1. If $x < 100$, then equation (4.1) becomes

$$100 - x = 20 - \frac{x}{6}$$

which gives $x = 96$, consistent with $x < 100$.
Case 2. If $x \geq 100$, then equation (4.1) becomes

$$0 = 20 - \frac{x}{6},$$

so that $x = 120$, consistent with $x \geq 100$.
In conclusion, the possible values of x are 96 and 120.

Problem 4.6.6.

(a) 90 → 100, 60

(B) 90 → 80, 20; $S_d^{(B)}$

$P^B(100) = 15$

· What is $S_d^{(B)}$? (All values)

Replication:

$\Delta^{(1)} S_0^{(1)} + \Delta^{(2)} S_0^{(2)} = P^B(100)$

$90 \cdot \Delta^{(1)} + 90 \cdot \Delta^{(2)} = 15$

$100 \cdot \Delta^{(1)} + 80 \cdot \Delta^{(2)} = 20$

$\Delta^{(1)} = \frac{20 - 80\Delta^{(2)}}{100} = 0.2 - 0.8\Delta^{(2)}$

$\frac{2}{10} + \frac{1}{6} \cdot \frac{8}{10} = \frac{12+8}{60} = \frac{1}{3}$

$90 \cdot (0.2 - 0.8\Delta^{(2)}) + 90 \cdot \Delta^{(2)} = 15$

$18 - 72\Delta^{(2)} + 90\Delta^{(2)} = 15$

$18\Delta^{(2)} = -3$

$\Delta^{(2)} = -\frac{1}{6}$

$\Delta^{(1)} = \frac{1}{3}$

$\frac{1}{3} \cdot 60 - \frac{1}{6} \cdot S_d^{(B)} = (100 - S_d^{(B)})^+$

Case 1: $S_d^{(B)} > 100$

$S_d^{(B)} = 120$

Case 2: $S_d^{(B)} < 100$

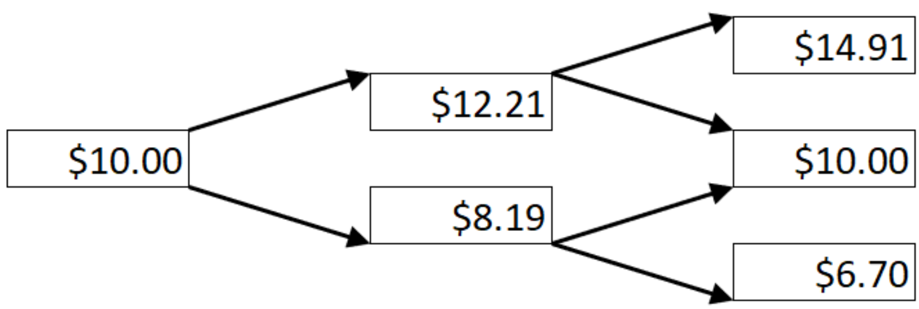
$-80 + \frac{5}{6} S_d^{(B)} = 0$

$S_d^{(B)} = 96$

5: Problem 4.6.20. (Pricing a warrant as an American call with varying strike prices) (5 marks)

The Ash Company needs to raise capital to support its rapidly growing business. One proposal is to publicly issue a certain number of warrants.

Assume that the price of the underlying asset follows a binomial tree with 1-year time steps as follows:



The warrant provides the right to purchase one share of the stock for $\$9$ at the first anniversary or, if not exercised, for $\$10$ at the second anniversary.

Assume further that:

- The stock pays no dividend.
- The risk-free interest rate is 4% per annum (CCRFR)
  Calculate the value of the warrant using the binomial tree.

Solution:

Solution. With $u=1.221$ and $d=0.819$, the risk-neutral probability of an move is

$$p^*=\frac{\mathrm{e}^{0.04(1)}-0.819}{1.221-0.819}=0.551768$$

The warrant is essentially an American call on the stock with a strike price of $\$9$ at the first anniversary and $\$10$ at the second anniversary. The possible terminal payoffs of the warrant are $C_{uu}=(14.91-10)_+=4.91, C_{ud}=(10-10)_+=0$, and $C_{dd}=(6.70-10)_+=0$. Going from $t=2$ to $t=1$, we have

$$\begin{aligned}
C_u&=\max\{\mathrm{e}^{-0.04(1)}\left[p^*(4.91)+(1-p^*)(0)\right], \quad \underbrace{12.21-9}_{\text{early exercise optimal}} \quad\}=3.21,\\
C_d&=0
\end{aligned}$$

It follows that the time-0 value of the warrant is

$$C_0=\mathrm{e}^{-0.04(1)}\left[p^*(3.21)+(1-p^*)(0)\right]=1.70.$$

Remark.
(i) The values of u and d in the second period are close to, but not exactly the same as 1.221 and 0.819 , respectively, but we ignore such small discrepancies.
(ii) We assume that exercising the warrant at time 0 is not possible.

Mark allocation:
Risk neutral probability 1
V(2) 1
V(1) 2

V(0) 1

PROBLEM 4.6.20				
2-PERIOD BINOMIAL TREE				
Equity		**FX**	**Futures**	
S(0)=	10	X(0)	F(0)	
r=	0.04	r_d		
delta=	0	r_f	=r	
u=	1.221			
d=	0.819			
T=	2			
TREE PARAMETERS				
h=	1			
p*=	0.55177			
1-p*	0.44823		DF=	0.96079
UNDERLYING				
	Value	**P(S|S(0))**	*# Successes*	*# Trials*
S_uu=	14.9084	0.30445	*2*	*2*
S_ud=	9.99999	0.49464	*1*	*2*
S_dd=	6.70761	0.20091	*0*	*2*
S_u=	12.21	0.55177	*1*	*1*
S_d=	8.19	0.44823	*0*	*1*
S_0=	10	1	*0*	*0*
CUSTOM WARRANT				
K(2)=	10		K(1)=	9
Call?	TRUE	FALSE=PUT		
American?	TRUE	FALSE=European		
	Value	*Exercise Val.*	*Cont. Val.*	*Max(Ex; Cont)*
V_uu=	4.90841	4.90841	4.90841	4.90841
V_ud=	0	0	0	0
V_dd=	0	0	0	0
V_u=	3.21	3.21	2.6021099	3.21
V_d=	0	0	0	0
V_0=	1.70173	0	1.7017268	1.7017268

6: Problem 5.3.2. (v1 - Calculations of miscellaneous probabilistic quantities) (4 marks)

Assume the Black-Scholes framework. You are given:
(i) The current stock price is 100 .
(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 2.5%.
(iii) The continuously compounded expected rate of return on the stock is 6%.
(iv) The stock's volatility is 30%.

Calculate:
(a) The probability that a 3-year at-the-money European put option on the stock is exercised.
(b) The expected 3-year stock price.
(c) The expected 3-year stock price, given that the put option in (a) pays off at maturity.
(d) The variance of the 3-year stock price.

Solution:
(a) With

$$\hat{d}_2=\frac{\ln(100/100)+\left(0.06-0.025-0.3^2/2\right)(3)}{0.3\sqrt{3}}=-0.05774,$$

the exercise probability of the put option is $N\left(-\hat{d}_2\right)=0.52302$.

Z-table:
N(0.06) = 0.5239

(b) The expected 3-year stock price is

$$\mathbb{E}[S(3)]=S(0)\mathrm{e}^{(\alpha-\delta)T}=100\mathrm{e}^{(0.06-0.025)(3)}=111.0711.$$

(c) As $\hat{d}_1=\hat{d}_2+\sigma\sqrt{T}=-0.05774+0.3\sqrt{3}=0.46188$ and $N\left(-\hat{d}_1\right)=0.32208$, the expected 3-year stock price, given that $S(3)<S(0)$, is

$$\mathbb{E}[S(3)\mid S(3)<S(0)]=\mathbb{E}[S(3)]\times\frac{N\left(-\hat{d}_1\right)}{N\left(-\hat{d}_2\right)}=\underbrace{111.0711}_{\text{from part (c)}}\times\frac{0.32208}{0.52302}=68.40.$$

Z-table:
N(-d1)= 1-N(d1) = 1 - N(0.46) = 1 - 0,6772 = 0.3228
N(-d2) = 0.5239
Answer: 68.43624943691544

(d) Since $S(3)$ is lognormally distributed with parameter $v^2=\sigma^2T=0.3^2(3)=0.27$, the variance of $S(3)$ is

$$\operatorname{Var}[S(3)]=(\underbrace{111.0711}_{\text{from (c)}})^2\left(\mathrm{e}^{v^2}-1\right)=(111.0711)^2\left(\mathrm{e}^{0.27}-1\right)=3,823.96.$$

Remark. You can also obtain the second moment of $S(3)$ by squaring and taking expectation of the stock price equation

$$S(T)=S(0)\exp\left[\left(\alpha-\delta-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z\right].$$