Derivatives

T-503-AFLE

Teacher: Dr. Maxime Segal

Assistant Teacher: Aqib Ahmed

August 18, 2025 – November 7, 2025

Introduction

Few words about your teacher

Dr. Maxime Segal

Currently:

- Senior Risk Modelling, **Íslandsbanki**.
- Adjunct Prof., **Reykjavik University**.

In Reykjavik University:

- 5 times **Teacher**, 5 times **Assistant Teacher**

2019-2023:

- PhD in Financial Engineering, **Reykjavik University**.

2020-2022:

- Head of Markets, **Technology Metals Market**. (2020-2022)
- Strucutred Products Sales, **Leonteq,** Monaco (2019)
- Financial Engineer, **Crédit Agricole,** Paris (2018

Class Methodology

Timetable

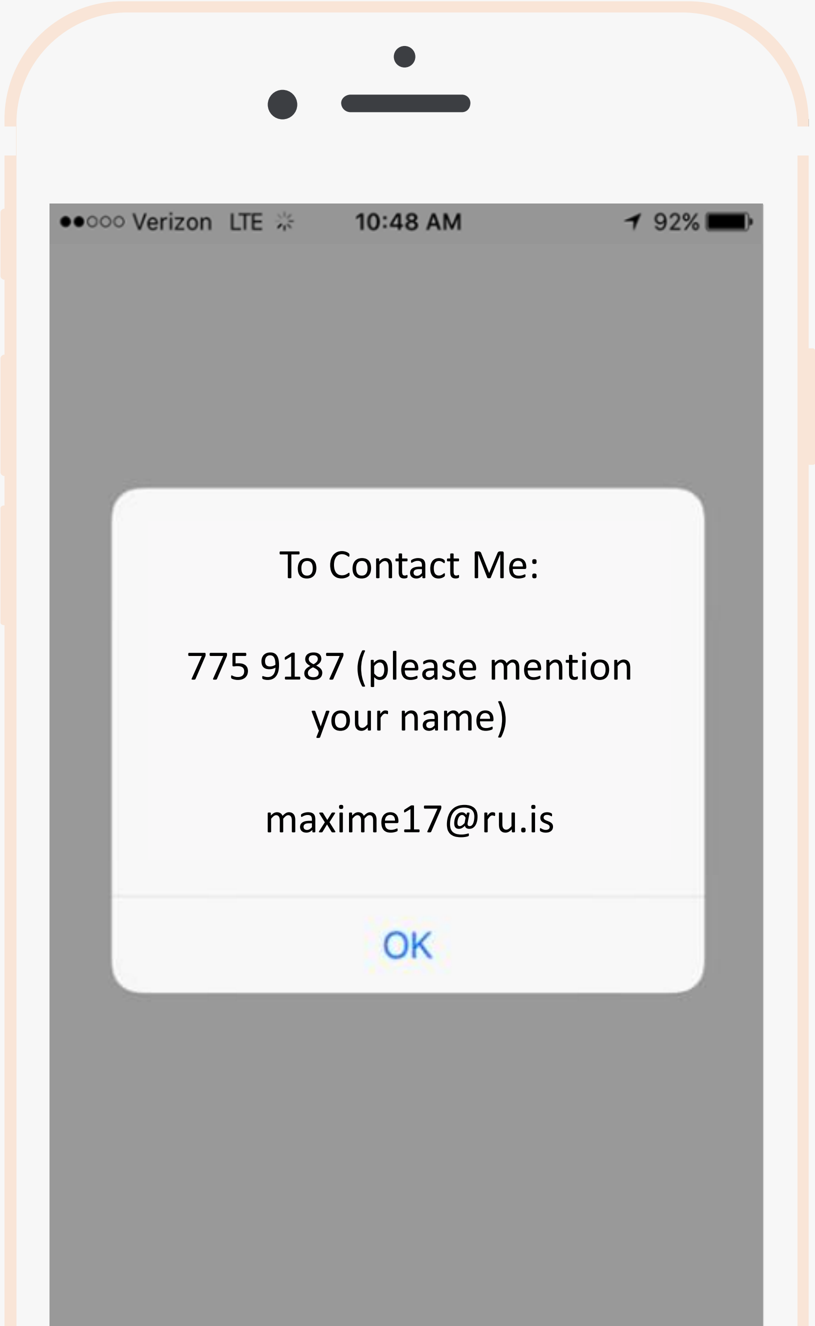
- Mondays, 12.40 – 14.20, session with TA
- Tuesdays, 12.40 – 14.20
- Fridays, 08:30 – 10.10
- 12 weeks

Teaching

- Active teaching + Class discussions with 2-3 speakers to give an insight from the trading floor.
- Ambrose Lo, Derivative Pricing: A Problem-Based Primer.
- Options, Futures, and Other Derivatives, J. Hull.

Exams

- Final Exam (40%)
- Mid-Term Exam (30%)
- Class quizz, 15', bi-weekly (30%)
  - The worst grade will be dropped



Introduction

I/ Futures & Forward

A first approach

The first module will aim to appreciate the difference between spot, futures, forward and swap rates. We will see how they are traded and priced as well as which payoff to expect.

II/ Options

A deeper insight in derivatives

An option is a contract that gives the owner some flexibility in its execution. It gives the right, by opposition with the obligation, to buy or sell an underlying at a prefixed strike price on a given date. The pricing becomes more challenging, but you will love them!

III/ Hedging

Risk mitigation

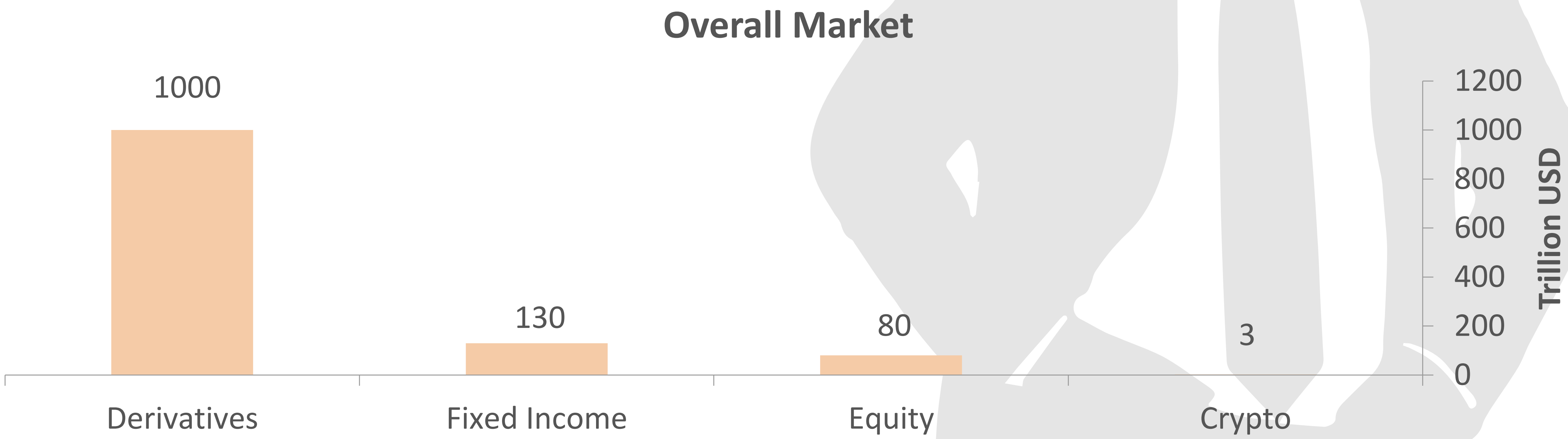
Derivatives are not only aiming to get leverage and allows to speculate. They can be used as an insurance tool to hedge an inverstor on potential losses in case of advserse market movements.
This module will introduce a few hedging techniques.

IV/ Structured Products

Unleash the full potential of derivatives

Mixing asset classes or combining derivatives together are creating products with complex payoffs that are called Structured Products.
In addition to introduce some, we will learn how to simulate their prices.

Financial World

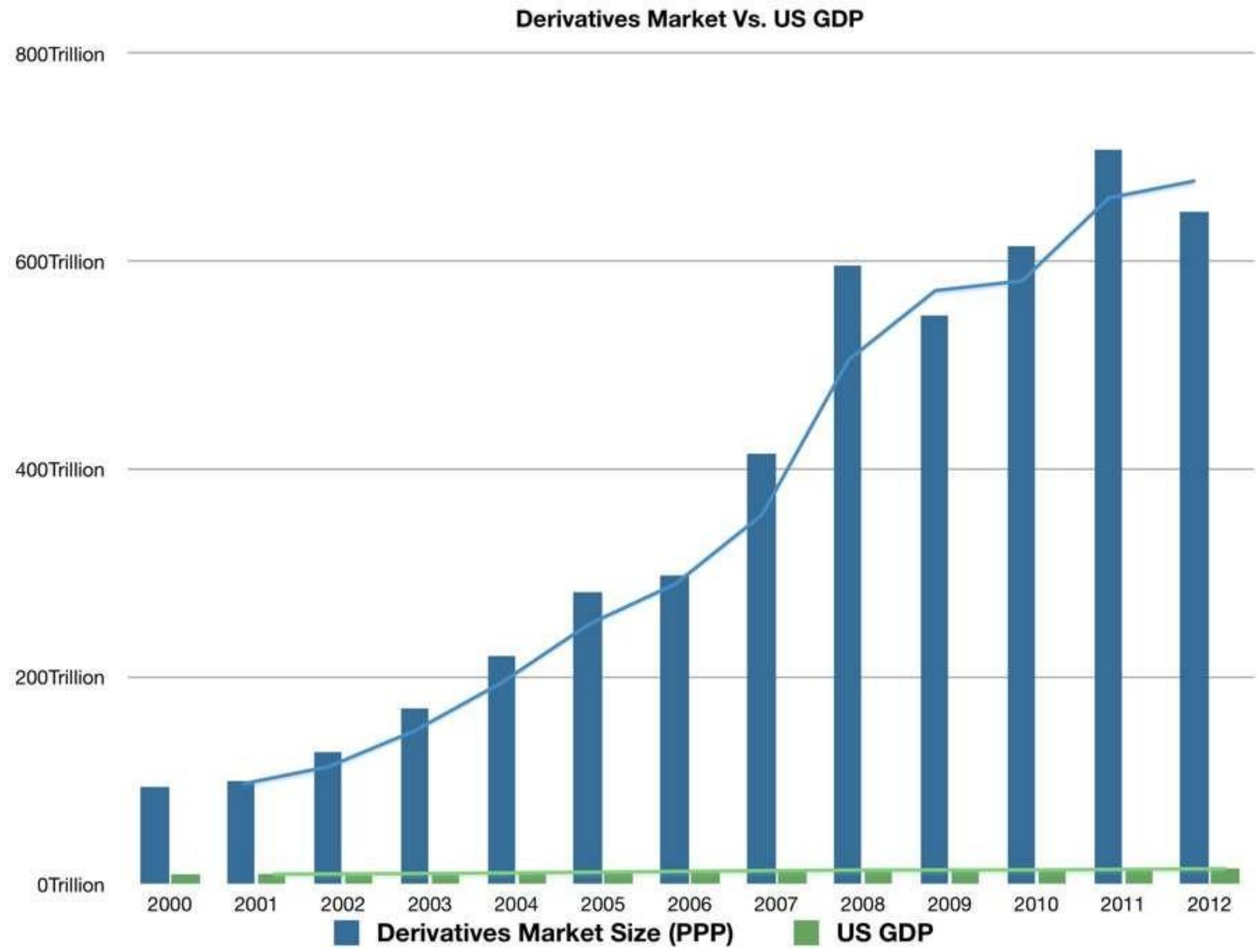


By far the largest market

Every mature market naturally develops its own derivatives segment that can end up being way larger than its underlying market.

Derivatives market size

Size of derivatives markets in relation to the US economy



What are Derivatives?

Why using them? Who use them? When to use them?

A derivative is a financial instrument that derives its value from the performance of an underlying entity.

The most traded are the following one

FORWARDS

FUTURES

OPTIONS

SWAPS

Two investment strategies:

- LONG position
- SHORT position

Futures / Forwards

Agreement to deliver or receive assets at a future date for a price set today.
Some intrinsic differences in their design, that will be covered later.

Options

Contract that offers the buyer the right (but not the obligation) to buy or sell an asset at a given time horizon for a pre-fixed price.

Swaps

Agreement through which two parties exchange a fixed against a variable cashflow (usually from different instruments).

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FUTURES

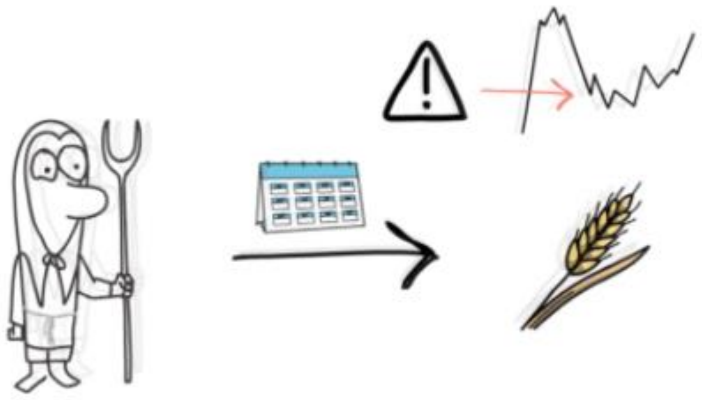
OPTIONS

SWAPS

The two main uses are:

- **Hedging (risk mitigation)**
- Speculation

Example 1: The Wheat farmer



- Price Fluctuating
- How to mitigate the naked position risk?

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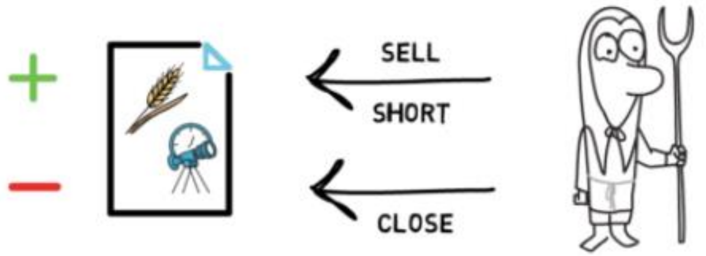

- Short sell wheat future contract for the amount predicted to harvest

If spot wheat goes **down** -> the short position makes **profit**
If spot wheat goes **up** -> the short position incurs a **loss**.

The total revenue becomes **predictable**!

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FORWARDS

FUTURES

OPTIONS

SWAPS

The two main uses are:

- Hedging (risk mitigation)
- **Speculation**

Example 2: The Reykjavik Univ Student

- Think the price of the Bitcoin will go up.
- Cannot afford to buy 1 BTC for 105,000 USD.
- How to take leverage of this belief?

What are Derivatives?

Why using them? Who use them? When to use them?

A derivative is a financial instrument that derives its value from the performance of an underlying entity.

The most traded are the following one

FORWARDS

FUTURES

OPTIONS

SWAPS

The two main uses are:

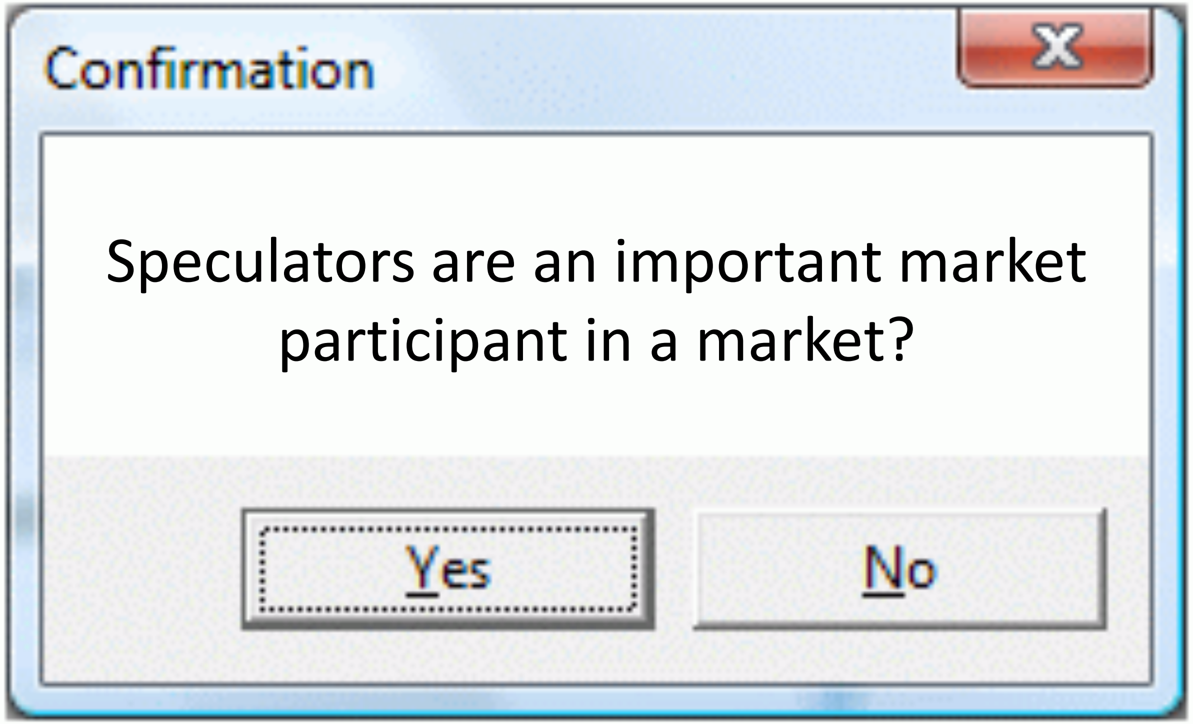
- Hedging (risk mitigation)
- **Speculation**

Example 2: The Reykjavik Univ Student

- Take a LONG position on Call option. Each call option costs 400 USD and allows in a 6 month time to buy 1 BTC @ 107,000 USD.

-> If the price goes up to 110,000 USD, the student do 2,600 USD of profit.
-> If the price goes down to 70,000 USD, the student “just” lose 400 USD.

Open Question



YES.

Speculators are **liquidity providers.**

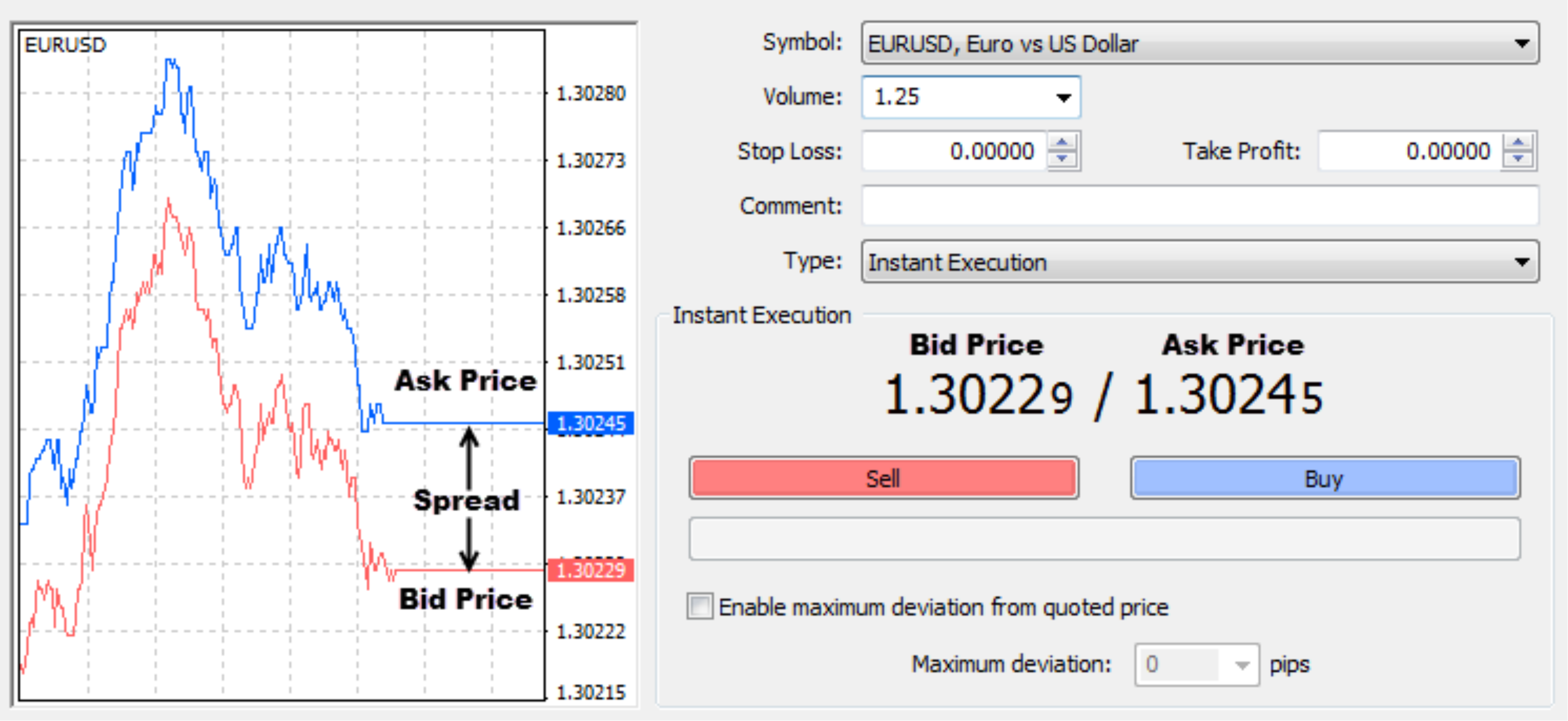
By supplying the market with liquidity, it allows investors, who want to buy a specific instrument to hedge their risk, in addition to easily open and close a position.

BID
The price a buyer is willing to pay for an asset

SPREAD
The difference between the lowest ask price and the highest bid price

ASK - BID = SPREAD

ASK
The price a seller is willing to accept for an asset



Course
LVMH

488,5000 (c) EUR
+4.15%

FR0000121014 MC
EURONEXT PARIS REAL-TIME DATA
Enforcement Policy
Quotation on other markets

SECTOR
Clothing and accessories

BENCHMARK INDEX
CAC 40

ORDER BOOK

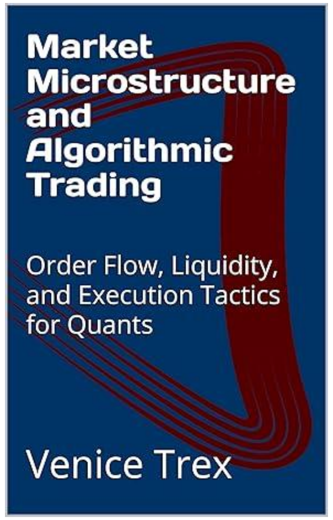
ORDERS	QTY	PURCHASE	SALE	QTY	ORDERS
1	**15**	**488,0000**	**492,0000**	**8**	**2**
1	2	487,5000	492,2000	1	1
1	100	486,6000	492,8000	3	1
1	10	486,0000	493,0000	48	3
1	1	485,8500	493,4500	8	2
2	7	485,0000	494,0000	26	3
4	58	482,0000	494,2500	4	1
1	22	481,9500	494,5000	1	1
1	29	481,1000	494,9000	1	1
10	93	480,0000	495,0000	449	20
23	**337**	**TOTAL**	**TOTAL**	**549**	**35**

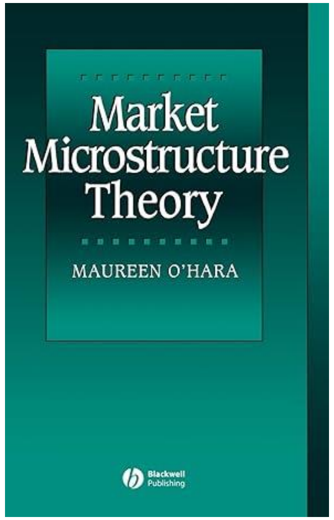
Bid-Ask Spread

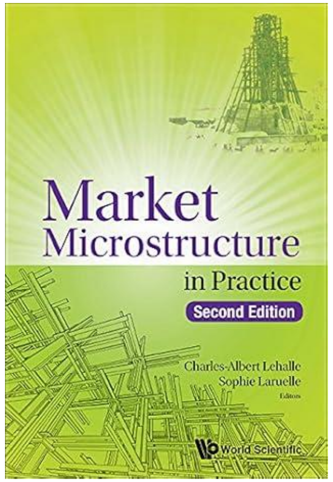
- Reflects **liquidity, transaction costs, market efficiency**.
- Narrow spreads: highly liquid & competitive market
- Wide spreads: information asymmetry, low liquidity and inefficiencies

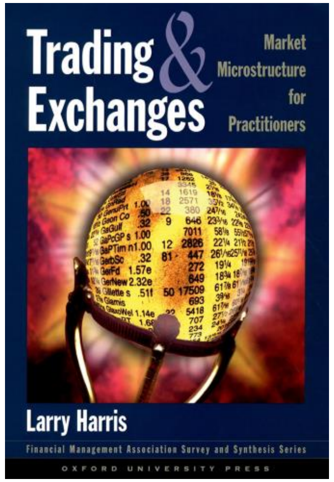
Order Book

- Insight into **market depth, order flow, and supply/demand dynamics**
- Reveals:
    - Liquidity concentration around price levels
    - Presence of spoofing/manipulation
    - Latency arbitrage and HFT behaviour

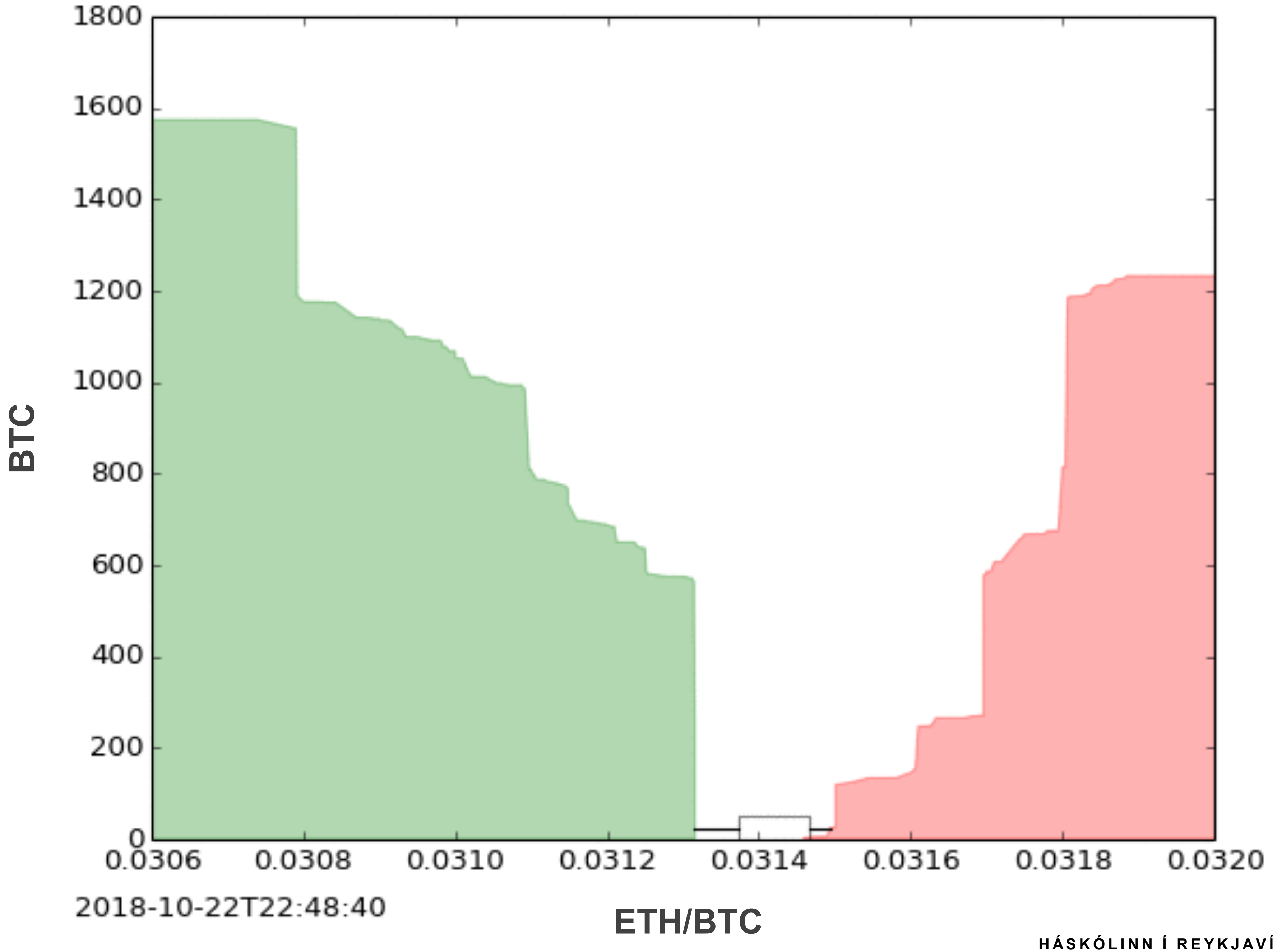








ETH/BTC order book over 10 hours, starting Oct 22, 2018



Where to trade derivatives

- The Chicago Board of Trade (CBOT) was established in 1848 to bring farmers and merchants together where standardized quantities of agricultural products could be traded.
- The Chicago Mercantile Exchange (CME) was established in 1919
- The Chicago Board Options Exchange (CBOE) started trading call options on 16 stocks in 1973
- Put options started trading on CBOE in 1977

- **There is also a very active over-the-counter (OTC) market in derivatives.**

OTC vs. Exchange trades.

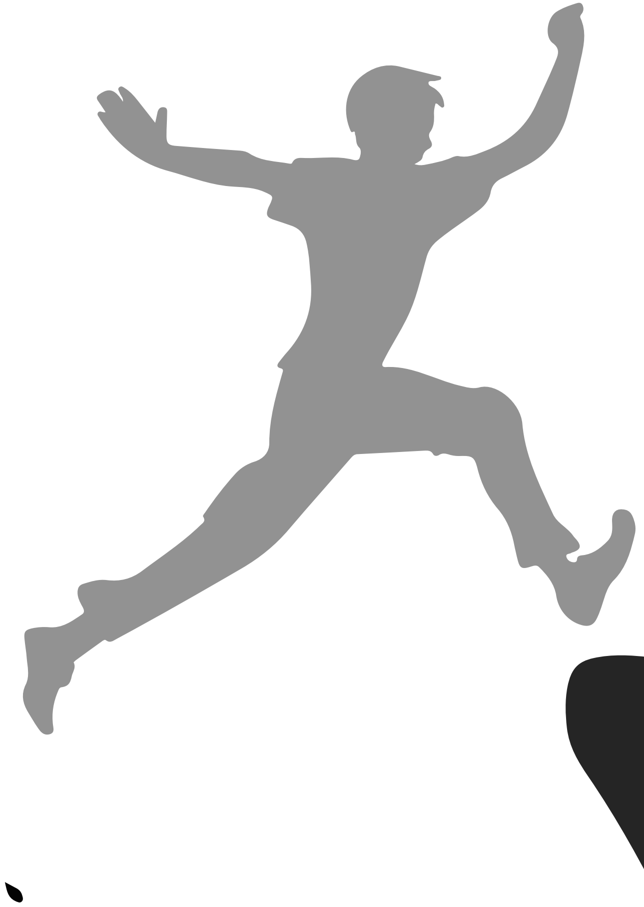
Downsides

- Lower liquidity on a contract OTC traded
- Credit Risk if the seller does not honor its contract in the OTC market
- Organized exchanges trade in standardized contracts

Upsides

- OTC traded products allow to design more flexible instruments (customized on the client requirements)
- Clearing house for derivatives traded on the market
- Low prices can mean the potential for big percent moves (OTC).

The concept of **Arbitrage**



In 1 example…

On the NYSE, $MSFT is trading
@ USD 244.5 / 246.1

OFFER QTY	LAST PRICE	OPEN	HIGH	LOW	TRADES	VOLUM

Order Book By Order ×

BID			OFFER	
QUANTITY	PRICE		PRICE	QUANTITY
2	244.5		246.1	2
2	243.8		247.0	4
2	241.0		249.0	4

In Frankfurt, $MSFT is trading
@ USD 246.3 / 246.5

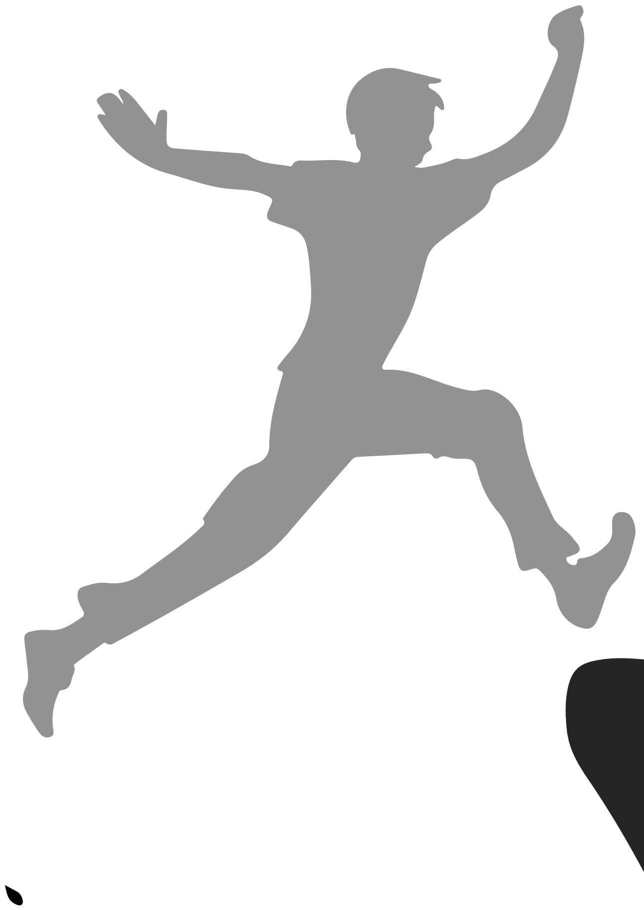
Order Book By Order ×

BID			OFFER	
QUANTITY	PRICE		PRICE	QUANTITY
2	246.3		246.5	2
2	246.7		246.8	4
2	246.9		247.0	4

WHAT WOULD YOU DO?

As a delta one (equity) trader on the Reykjavik University trading floor

The concept of
Arbitrage



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Order Book By Order ×

BID		OFFER	
QUANTITY	PRICE	PRICE	QUANTITY
2	246.3	246.5	2
2	246.7	246.8	4
2	246.9	247.0	4

BUY 2 shares @ 246.1 USD on Nasdaq

and instantaneously

SELL 2 shares @ 246.3 USD in Frankfurt

=

$ 0.2 * 2 = $ 0.4 risk-free profit

The concept of **Arbitrage**

In 1 example…

Arbitrage is the practice of taking advantage of price discrepancies without taking any risk.

If 2 equally risky portfolios P_1 and P_2 are expected to have the same value at the future time T, i.e.

$$E_t P_1(T) = E_t P_2(T)$$

Then, they should have the same price at all previous times, $t < T$ i.e.

$$P_1(t) = P_2(t)$$

In the previous example, we do say that an arbitrage opportunity was existing. As we take profit of this opportunity, the prices are said to converge following our actions.

Be careful, if you had to borrow money to engage the initial purchase of 2 MSFT shares, the resulting profit still need to be above your borrowing costs.

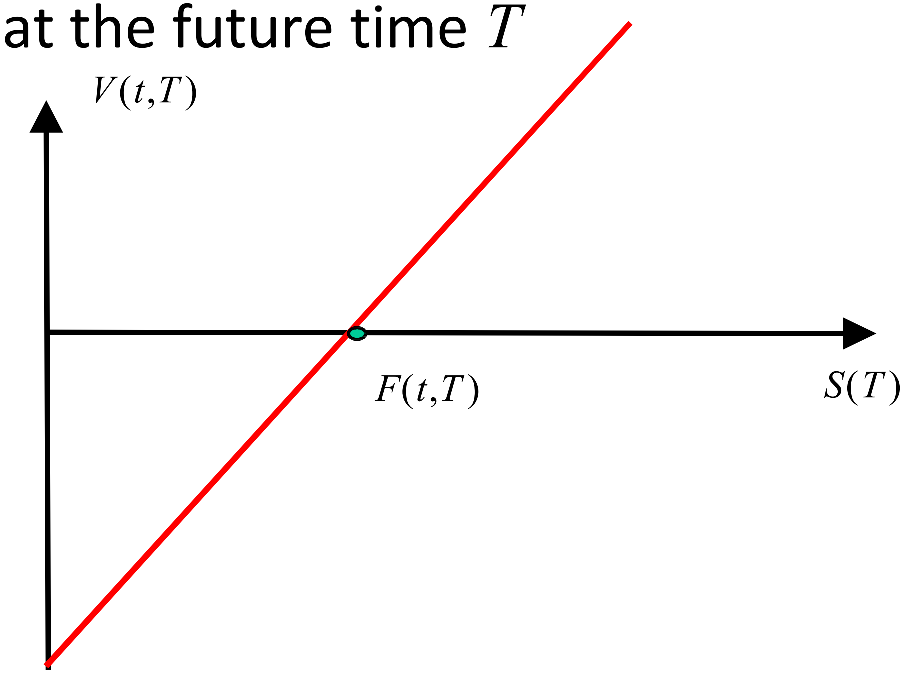
Module 1

Futures & Forward

1.1. Forwards

Example – Long Forward Contract

- Let's now consider the payment and the physical receipt of a stock as two different activities. This allows to introduce the concept of forwards
- A **long** position in a **forward contract** on a market asset **obliges** its holder **to buy** this asset at the future time T at a price $F(t,T)$ which is fixed today at time t.
- The present price of the asset is $S(t)$ but its price $S(T)$ at the future time T is unknown today.



- At T, the value of the long position is:

$$V(t,T) = S(T) - F(t,T) = \begin{cases} > 0 \; if \; S(T) > F(t,T) \\ < 0 \; if \; S(T) < F(t,T) \end{cases}$$

Forward contracts and Futures

- The essential difference between forward contracts and futures is that:
  - Forward contracts are settled only at the maturity of the contract and they are traded as over the counter (OTC).
  - Futures are settled daily (in a process called Mark-to-Market) and they are traded on organized exchanges

Spot vs. Forward

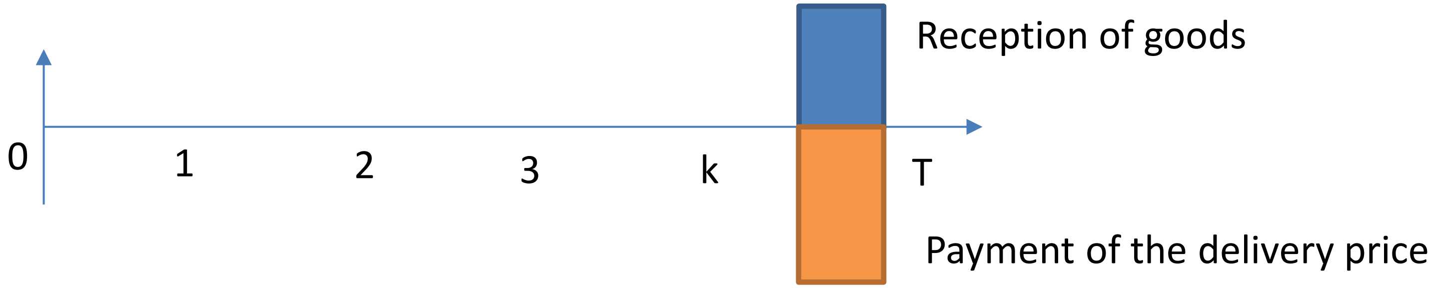
- Do not mix with **spot** contract, which is an agreement to buy or sell an asset today – or instantly. E.g. Buying a Viking in Vinbudin is a spot transaction.

- In a forward contract one of the parties takes a long position which agrees to buy the asset at specified time in the future (maturity) but prefixed price (delivery price). The party with the short position agrees to sell the asset on the same date and for the same price.

- At maturity, forward can be **settled in cash** (exchange of cashflows only) or **physically** (with delivery of the goods).

Forward contracts

- At the time a forward contract is entered into, the value of each position is **<u>zero</u>** - i.e. it costs nothing to enter either long or short position – in other words no exchange of cash takes place when the contract is entered into
- Forward contracts are settled at maturity – the short position delivers the asset to the long position in exchange for cash - i.e. the delivery price.



- However, because the value of a forward contract is determined by the market price of the underlying asset, any movement of the underlying asset in the market affects the value of the forward contract. If the underlying goes UP, the contract value becomes > 0 (as the beneficiary will receive the same quantity for the same pre-fixed price).

Use of forward contracts

- Forward contracts allow the parties to the contracts to **remove the future price uncertainty** by fixing today the price of that will be paid or received in the future.
- The long position knows today the cost of acquiring the asset in the future.
- The short position knows today how much the asset can be sold for in the future.
- However, the forward price may **deviate significantly** from the future spot prices leading to gain/losses for either party.
- <u>Therefore, the risk has been removed but opportunity cost remains i.e. to secure a fixed price one has to give up the benefits from possible positive price movements</u>

Use of forward contracts

- Companies trading in the international markets have a strong cost exposure to price fluctuations in a whole range of market variables including:
  - Commodity prices
  - Interest rates
  - FX exchange rates
- Forward contracts can be used to stabilize the cost exposure to these fluctuations
- Forward contracts therefore present an important tool to manage the exposure to a whole range of price risks

Use of forward contracts

- A US company planning to buy goods from the UK in the near future expects the GBP to strengthen against the USD - making the cost of the goods higher in terms of the USD.

GBP/USD pair
The GBP is strenghtening, meaning that more USD are required to buy 1 £
Google Finance. July 2, 2025.

- Instead of waiting and buy the £ when needed, the US company can enter a forward currency contract for the GBP/USD rate (USD per 1 GBP)
- By entering this contract the uncertainty (risk) caused by possible currency fluctuations can be removed (to the price of not benefiting from the potential weakening British Pounds).

An example

- Spot and forward quotes for the GBP/USD exchange rate.

Spot	**1.3645**
1-month forward (30 days)	1.3712
3-month forward (90 days)	1.3798
6-month forward (180 days)	1.3830

How many $ you need to buy 1 £

- You are the FX trader in the Treasury division of this US company. You decide to enter a forward contract to buy 1 million GBP in 90 days for the fixed exchange rate 1.3798.

- Was it a wise move? Let's compute two different scenarios:
  - ❖ FX rate rises to 1.4000
  - ❖ FX rate falls to 1.3450

Spot	1.3645
1-month forward	1.3712
3-month forward	1.3798
6-month forward	1.3830

How many $ you need to buy 1 £

❖ FX rate rises to 1.4000 (stronger GBP, weaker USD)

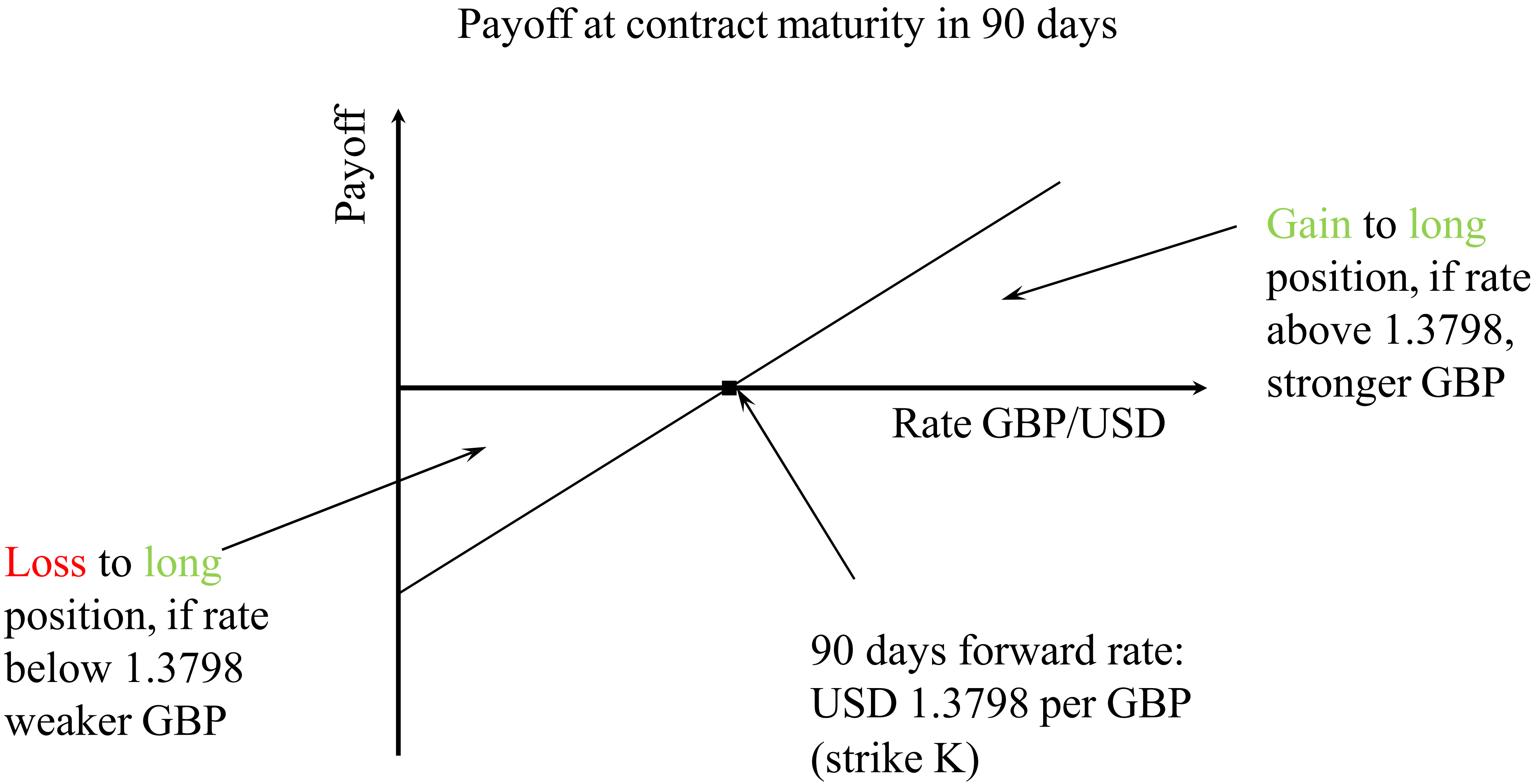
The company would gain (1.4-1.3798)*1M = USD 20,200

❖ FX rate falls to 1.3450

The company would loose (1.345-1,3798)*1M = - USD 34,800

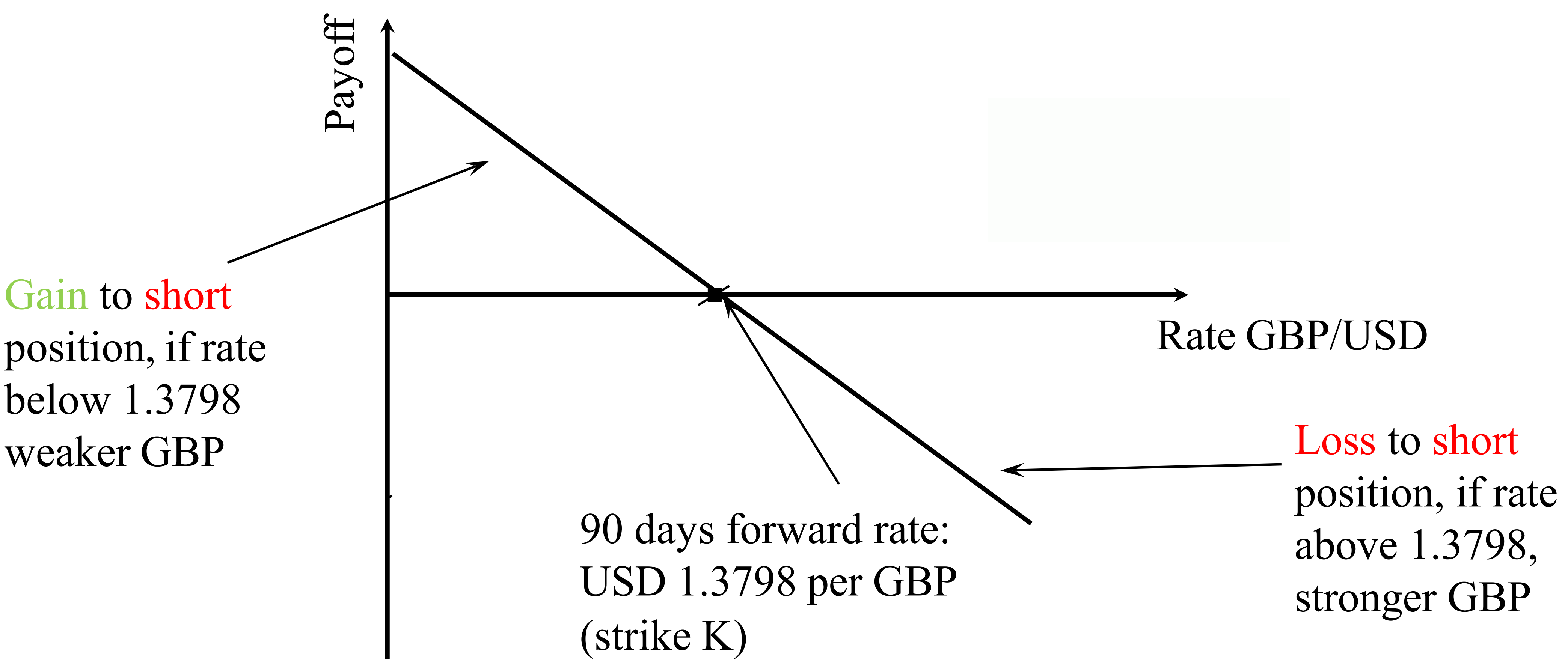
❖ **The forward contract can lead to gains or losses, but it fixes the company's GBP/USD exposure.**

Payoff to the US company



Payoff to the forward seller

Payoff at contract maturity in 90 days



Fixing the forward price

- Generally the purchasing of an asset consists of three action points:
  1) Determining the mutually agreeable price
  2) Making the payment – the buyer (long position) pays the seller (short position)
  3) The asset is transferred from seller to the new owner
- The fact that the three action points can take place at different times offers an interesting pricing, investment and financing alternatives
- In our consideration it is assumed that all the action points can take place at two different times:
  - Today at time t
  - Or in the future at time T

Fixing the forward price

- There are four different pay – delivery alternatives:

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)\mathrm{e}^{rT}$
Prepaid forward contract	0	T	$F_{0,T}^{P}$
Forward contract	T	T	$F_{0,T}$

What is the fair price for a forward?

It is possible to derive the fair price for a forward contract, given a few simplyfing assumptions that needs to always hold. We won't repeat them, but they are considered valid throughout this class. Real market conditions can differ.

01

Liquidity

The market is considered liquid, in the sense that entering or closing a position can be done at any time.

02

Transaction Costs

Assumed to be 0.

03

Taxation

All the market participants pay the same taxes on profits.

04

Interest Rate

Market participants are able to borrow and lend money at the same rate r.

05

Arbitrage

Market participants can take advantage of arbitrage opportunities. Let $S(t)$ be the spot price. Any arbitrage activity has been already accounted in the reflected price.

Basic case – fair price for a forward

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)e^{rT}$
Prepaid forward contract	0	T	$F_{0,T}^{P}$
Forward contract	T	T	$F_{0,T}$

- A market traded asset is presently priced @ $S(t)$
- An investor wants to pay for it today at time t and then take delivery at the future time T

For the time being we assume that the asset pays **no dividends** between t and T, that it incurs **no storage or other costs**, and that it provides its holder with **no rights that can be valued**. Therefore it is irrelevant whether the investor has a physical possession of the stock or not in the time from t to T.

In that case the pre-paid fair price (AOA) is simply

$$\boxed{F(t,T) = S(t)}$$

'Less' Basic case – fair price for a forward

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)e^{rT}$
Prepaid forward contract	0	T	$F^P_{0,T}$
Forward contract	T	T	$F_{0,T}$

Fully leveraged purchase.

The asset is priced at $S(t)$ today and pays no dividends in the time from t to T.

The investor borrows money to purchase the asset at time t $(i.e.\ \$\ S(t))$ and at the same time takes a short position in a forward contract on the asset with maturity at T.

- ✓ The forward price is set at $F(t,T)$.
- ✓ The interest rate is set at $R(t,T)$.

At maturity T

The investor delivers the asset at the agreed forward price $F(t,T)$. The investor also pays back the loan with interest:

$$S(t) * \exp(R(t,T) * (T-t))$$

To exclude any arbitrage opportunity

The forward price must have been set to:

$$F(t,T) = S(t) * \exp(R(t,T) * (T-t))$$

Otherwise

It could have been possible at inception $(t = 0)$ to take profit from the existing arbitrage opportunity.

‘Less’ Basic case – fair price for a forward

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
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Prepaid forward contract	0	T	$F^P_{0,T}$
Forward contract	T	T	$F_{0,T}$

Fully leveraged purchase.

Cashflows

Transaction	**Time t**	**Time T**
Borrow	S(t)	
Buy	-S(t)	-S(t)exp(R(t,T)(T-t))
Enter a short forward	0	F(t,T)

- At time t the net transaction value is zero, $S(t) - S(t) = 0$
- For there not to be a risk free arbitrage opportunity the forward price needs to be set at

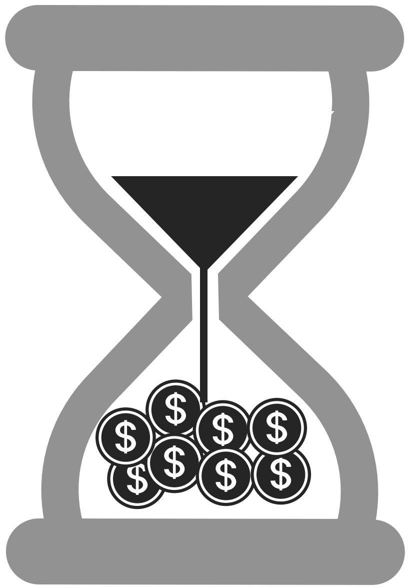
$$F(t,T) = S(t) * \exp(R(t,T) * (T - t))$$

Fair price for a forward contract

- **Statement**. The arbitrage free forward price for an asset, which pays no dividends and is presently priced at $S(t)$, and to be received at the future time T is

$$F(t,T) = S(t) * \exp\big(R(T-t) * (T-t)\big) = \frac{S(t)}{D(t,T)}$$

Where $R(t,T)$ is the rate for maturity T at which market makers can lend and borrow money and $D(t,T)$ the discount rate (the ‘time value’).



Demonstration

We draw the proof by showing that any other situation would lead to an arbitrage opportunity by constituting a specific portfolio

Demo- Fair price for a forward contract

- Let's assume that:

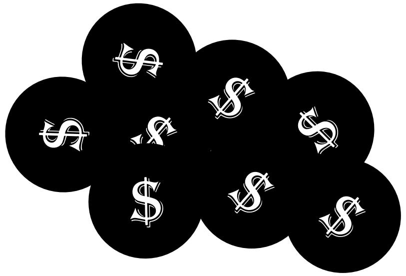
$$F(t,T) > S(t) * \exp\big(R(T-t) * (T-t)\big)$$

If a trader observes this price inequality (s)he can undertake the following transactions:

- Take a short position in a forward contract – i.e. sell a forward contract
- Borrow $S(t)$ and buy the asset at time t
- At time T deliver the asset according to the forward contract, receive $F(t,T)$ and pay back the loan

- Total risk-free profit made from this portfolio:

$$P(T) = F(t,T) - S(t) * \exp\big(R(T-t) * (T-t)\big) > 0$$



Demo- Fair price for a forward contract

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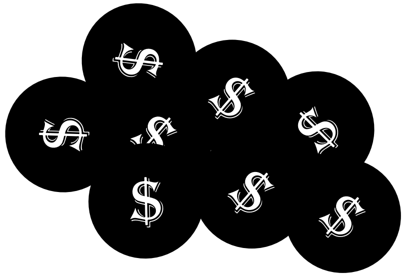
$$F(t,T) < S(t) * \exp\big(R(T-t) * (T-t)\big)$$

If a trader observes this price inequality (s)he can undertake the following transactions:

  - Take a long position in a forward contract and short sell the stock to receive S(t) – i.e. buy a forward contract and sell the stock.
  - Lend $S(t)$ at the interest rate $R(t,T)$ at time t
  - Receive the loan with interest, take delivery of the asset and pay $F(t,T)$

- Total risk-free profit made from this portfolio:

$$P(T) = S(t) * \exp\big(R(T-t) * (T-t)\big) - F(t,T) > 0$$



Demo- Fair price for a forward contract

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)e^{rT}$
Prepaid forward contract	0	T	$F^P_{0,T}$
Forward contract	T	T	$F_{0,T}$

- For there not to be an arbitrage opportunity:

$$F(t,T) = S(t) * \exp\big(R(T-t) * (T-t)\big)$$

- It follows that the price of a pre-paid forward contract would simply be the spot price of the asset:

$$F_P(t,T) = PV\big(F(t,T)\big) = F(t,T) * D(t,T) = F(t,T) * \exp\big(-R(t,T)(T-t)\big)$$
$$= \mathrm{S(t)} * \exp\big(R(t,T)(T-t)\big) * \exp\big(-R(t,T)(T-t)\big) = \mathrm{S(t)}$$

Strategy	Time of Payment	Time of Receipt of Asset	Amount of Payment
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)e^{rT}$
Prepaid forward contract	0	T	$F^P_{0,T}$
Forward contract	T	T	$F_{0,T}$

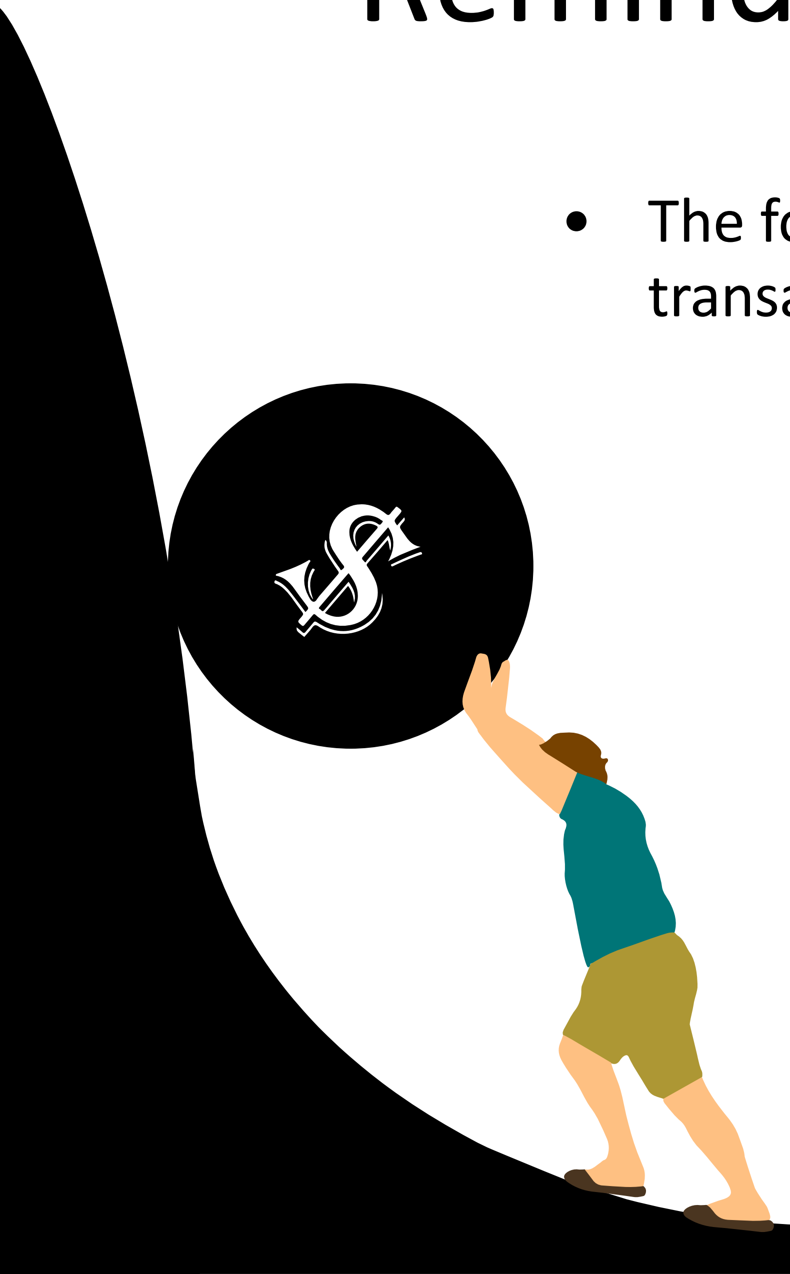
Fair price for a forward contract

- So far we have only considered forward contracts on assets that are only valued in terms of their **capital appreciation,** i.e. no attention has been paid to the fact that some assets pay the holder an income in terms of dividends or interest payments

- Income from an asset **impacts** the forward price of the asset

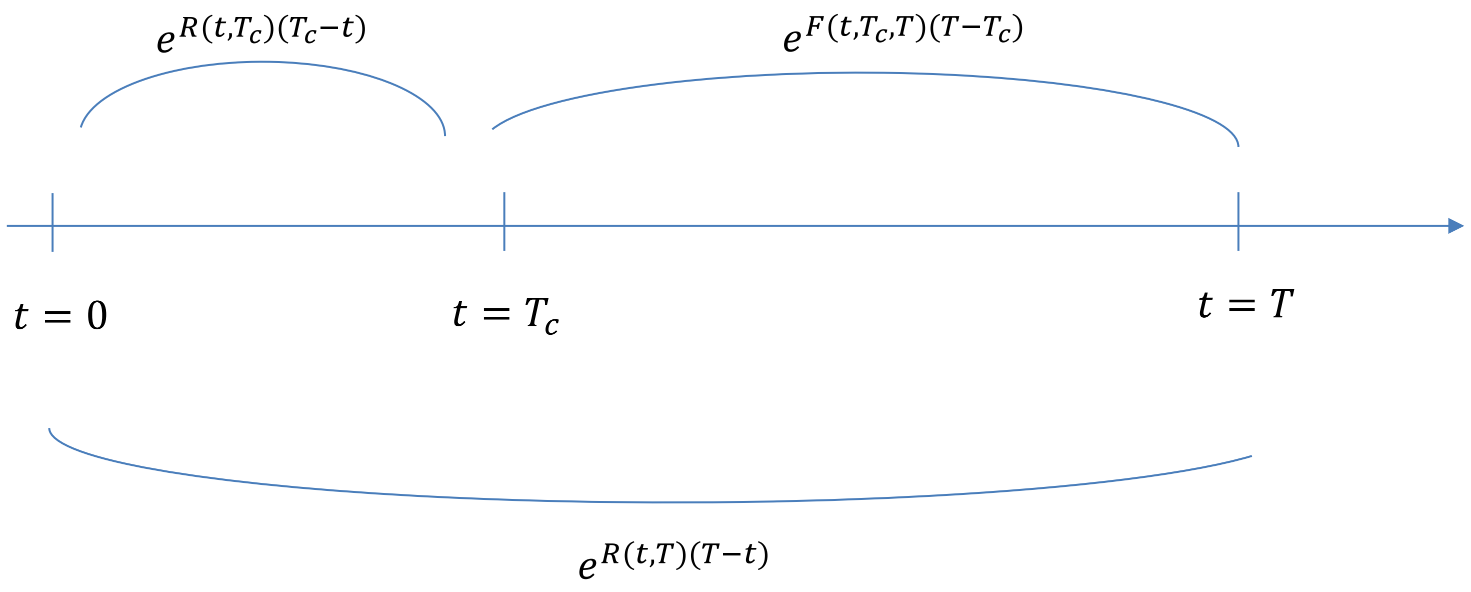
- As in the case of assets that pay no income we will proceed to calculate the forward price of income-paying assets by also using absence of arbitrage arguments

Reminder



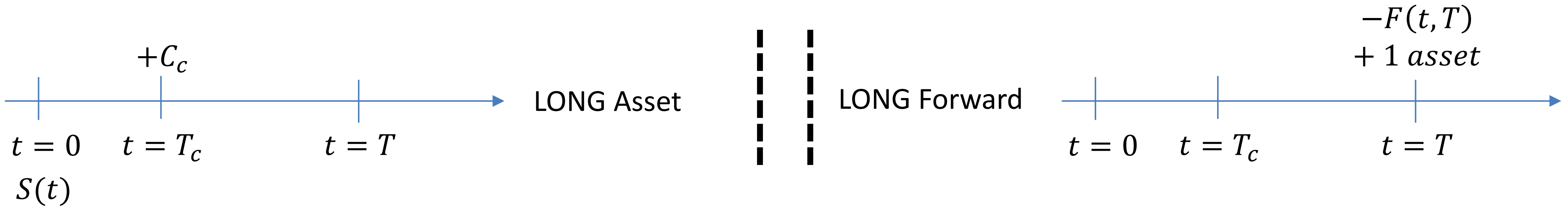
- The forward rate $F(t, T_c, T)$ is an interest rate applicable to a financial transaction that will take place in the future:

$$e^{R(t,T_c)(T_c-t)}e^{F(t,T_c,T)(T-T_c)} = e^{R(t,T)(T-t)}$$



Fair price for a forward contract

- Now consider a forward contract on an asset that pays predictable income at discrete times in the future. The present price of the asset is $S(t)$.
- What is the fair price of a forward contract that matures at T if there is one cash income payment $C_c = C(T_c)$ at time $T_c < T$?



- **Statement**. For there to be no arbitrage the following relationship has to hold:

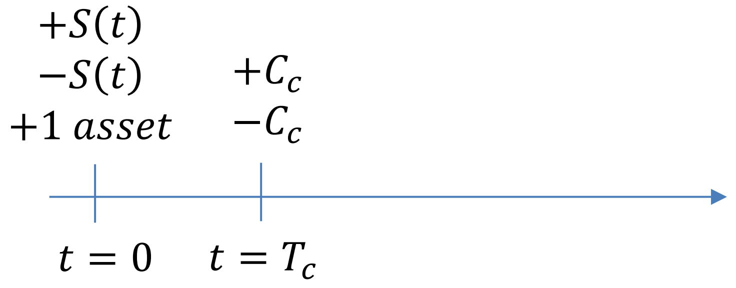
$$F(t,T) = \left(S(t) - C(T_c)\exp\left(-R(t,T_c)(T_c - t)\right)\right) * \exp\left(R(t,T) * (T - t)\right)$$
$$= \left(S(t) - PV(C_c)\right) * \exp\left(R(t,T) * (T - t)\right)$$

Where $PV(C_c) = C(T_c) * \exp\left(-R(t,T_c)(T_c - t)\right)$.

Fair price for a forward contract

- **Proof of Statement**
- At time t, we borrow $S(t)$, buy the asset and take a short position in a forward contract with delivery price set at $F(t,T)$
- At time T_c the cash payment C_c is received and used to reduce the accruing loan to

$$L(t,T_c) = S(t) * \exp\big(R(t,T_c) * (T_c - t)\big) - C_c$$



- The loan then accrues for the remaining time from T_c to T at the forward rate $F(t,T_c,T)$ set at t

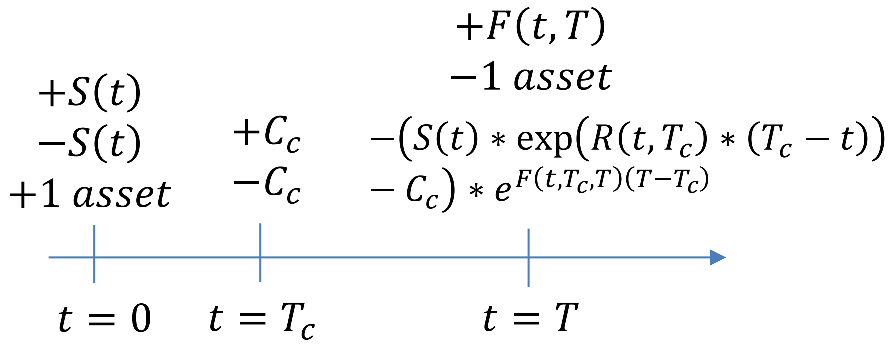
$$L(t,T_c,T) = L(t,T_c) * e^{F(t,T_c,T)(T-T_c)} = \big(S(t) * \exp\big(R(t,T_c) * (T_c - t)\big) - C_c\big) * e^{F(t,T_c,T)(T-T_c)}$$

Careful: $F(t,T)$ is the **forward price** of the asset presently priced at $S(t)$ but $F(t,T_c,T)$ is the **forward interest rate** set at time t for the future interval $[T_c,T]$! These are two things different !

To be continued..

Fair price for a forward contract

- At time T the short position receives the agreed forward price $F(t,T)$ and pays the accrued debt.
- For there not to be an arbitrage opportunity the following must hold:



$$F(t,T)=\left(S(t)e^{R(t,T_c)(T_c-t)}-C_c\right)*e^{F(t,T_c,T)(T-T_c)}$$

- Using the relationship $C_c = PV(C_c)*e^{R(t,T_c)(T_c-t)}$

We have:

$$\begin{aligned} F(t,T) &= \left(S(t)e^{R(t,T_c)(T_c-t)}-PV(C_c)e^{R(t,T_c)(T_c-t)}\right)e^{F(t,T_c,T)(T-T_c)} \\ &= \left(S(t)-PV(C_c)\right)e^{R(t,T_c)(T_c-t)}e^{F(t,T_c,T)(T-T_c)} \\ &= \left(S(t)-PV(C_c)\right)e^{R(t,T)(T-t)} \end{aligned}$$

Fair price for a forward contract

- **Comment**. If we assume a flat term-structure, i.e. the same interest rate r for all maturities then the previous relationship takes the simpler form: $F(t,T) = \left(S(t) - PV(C_c)\right)e^{r(T-t)}$
- The proof is now similar but **simpler**
- At time t, we borrow $S(t)$, buy the asset and take a short position in a forward contract for $F(t,T)$
- At time T_c, we receive the payment C_c and use it to pay part of the outstanding debt. Remaining debt at T_c: $S(t)\exp\left(r(T_c - t)\right) - C_c$
- At time T, we pay the outstanding debt which has accrued to $\left(S(t)e^{r(T_c-t)} - C_c\right)e^{r(T-T_c)}$ and receive the forward price $F(t,T)$

To be continued..

Fair price for a forward contract

- For there not to be an arbitrage opportunity the following has to hold:

$$F(t,T) = \left(S(t)e^{r(T_c-t)} - C_c\right)e^{r(T-T_c)}$$

- We multiply out the previous equation

$$F(t,T) = S(t)e^{r(T_c-t)}e^{r(T-T_c)} - C_c e^{r(T-T_c)} = S(t)e^{r\underline{(T-t)}} - C_c e^{r\underline{(T-T_c)}}$$

- By inserting $C_c = PV(C_c)e^{r(T_c-t)}$, it follows that:

$$F(t,T) = S(t)e^{r(T-t)} - PV(C_c)e^{r(T_c-t)}e^{r(T-T_c)} = S(t)e^{r(T-t)} - PV(C_c)e^{r(T-t)}$$

$$\boxed{F(t,T) = \left(S(t) - PV(C_c)\right)e^{r(T-t)}}$$

Numerical example

- In 9 months, Arion Banki is expected to pay ISK 5.0 dividend.

  (a) What is the fair price for a one year forward contract on this asset assuming that the 9 month interest rate is 7.8% and the one year rate 8%?

  (b) What is the forward rate $F(0, 9\ months, 1\ year)$?

Shares Closed on the stock exchange

XICE First North All stocks ETFs Selected stocks

IDENTIFICATION	PRICE	CHANGE	LAST 3 MONTHS	PURCHASE	SALES	TURNOVER
OMX OMXI15	2,601.98	0.72%		-	-	-
ALVO	1,105.00	-0.90%		1,100.00	1,105.00	122m
AMRQ	131.00	0.77%		130.00	131.00	21m
ARION	169.50	0.89%		168.50	169.50	462m
BRIEF	59.00	0.85%		58.50	59.00	2m
EIK OAK	13.90	0.00%		13.60	13.90	4m
EIM	370.00	-0.54%		370.00	376.00	287th
FAST	286.00	0.70%		284.00	286.00	200m
HAGE	106.00	1.19%		105.00	107.00	163m
HEMP	117.00	-0.43%		117.00	118.00	3m
HOME	37.00	0.54%		36.80	37.20	45m
ICEAIR	1.10	-1.79%		1.09	1.10	31m

Simplification (realistic though)

- Rather than assuming that an asset pays predictable income (dividends or coupons) at fixed times, it is common -and quite convenient- to assume that income is paid at a continuous rate δ

- This assumption can be computationally quite useful, particularly if we consider a portfolio of assets: all paying dividends at different times. In aggregation, the effect of that can be viewed as an asset (portfolio) paying continuous dividend.

- That allows us to talk about **dividend yield $\boldsymbol{\delta}$**

Fair price for a forward contract

- Let $S(t)$ be the price of an asset that pays continuous and constant dividend yield δ
- **Statement**. For there not to be an arbitrage opportunity the forward price on this asset, with maturity T, needs to be:

$$F(t,T) = S(t)e^{(R(t,T)-\delta)(T-t)}$$

- **How to think the proof**. We assume that an investor takes a loan for $S(t)$ to buy the asset. The dividends the asset pays over the time from t to T are reinvested continuously in the asset.

Therefore, at time T the investor holds a position of $S(t)e^{\delta(T-t)}$ in the asset. As the dividend payments are certain the investor can, at time t, enter a short position in a forward contract

cont.

Fair price for a forward contract

- The forward price on **one unit** of the asset is set at $F(t,T)$ and the price on $\mathbf{1} * \exp\big(\delta(T-t)\big)$ **units** is therefore, $\mathrm{F}(t,T) * \exp\big(\delta(T-t)\big)$

- The cash flow balance equation for the absence of risk free arbitrage is,
$$-S(t)\exp\big(R(t,T)(T-t)\big) + F(t,T)\exp\big(\delta(T-t)\big) = 0$$

- Therefore,
$$F(t,T) = S(t)e^{(R(t,T)-\delta)(T-t)}$$

Which is the AOA forward price on **one unit** of asset presently priced at $S(t)$

Forward contracts on dividend paying stocks

- We look at a different, calculus-based proof for the pricing of a forward contract on an asset that pays continuous dividend.
- Assuming again a non-negative constant δ, such that for each unit of the stock, the amount of stochastic dividends paid between time t and $t + dt$ for any infinitesimally small dt is $S(t)\delta dt$, where $S(t)$ is the time-t stock price
- We assume again that the dividends received are not paid out in cash, but immediately reinvested in the stock, resulting in more and more shares as time goes by.
- To determine the increase in the number of shares, we define $N(t)$ the number of shares of the stock we hold at time t under the reinvestment policy.

Forward contracts on dividend paying stocks

- Between time t and $t + dt$, the amount of dividend payment is $S(t)\delta dt$ per share, so the total amount of dividends we receive is $N(t)S(t)\delta dt$.
- Reinvesting this amount in the stock allows us to buy $\frac{N(t)S(t)\delta dt}{S(t)} = N(t)\delta dt$ more shares. In other words, the change in the number of shares is given by:

$$dN(t) = N(t + dt) - N(t) = N(t)\delta \mathrm{dt}$$

which means that:

$$\frac{dN(t)}{dt} = \delta \mathrm{N}(t)$$

- The solution to this ordinary differential equation in $N(t)$ with initial shares $N(0)$ is given by $N(t) = N(0)e^{\delta t}$

Forward contracts on dividend paying stocks

- Thus, 1 share (i.e. $N(0) = 1$) at time $t = 0$ will grow to $e^{\delta t}$ shares at time T. By proportion, to obtain 1 share at time T and to replicate the payoff of $S(T)$, it requires to buy $e^{-\delta \mathrm{T}}$ shares at time 0 and reinvest all dividends in the stock between 0 and T.
- It follows that the fair prepaid forward price in the presence of continuous dividends is the cost of buying $e^{-\delta T}$ shares at time 0, or:

$$F(0, T) = S(0)e^{-\delta T}$$

To go further

And bridge with Section 1.2 Futures.

Forward Premium

+ Contango / Backwardation concept

Definition

The ratio between the forward and the spot price gives the forward premium at maturity T

CONTANGO

Name of the market when the forward future price is **above** the expected spot price

Always larger than 1.

This premium will always be larger than one (look at the equation), except if the interest rates are negatives!

BACKWARDATION

Name of the market when the forward future price is **below** the expected spot price

$$FP(t,T) = \frac{F(t,T)}{S(t)} = e^{R(t,T)(T-t)}$$

In the case of non-dividend paying asset

Forward Premium

+ Contango / Backwardation concept

$$FP(t,T) = \frac{F(t,T)}{S(t)} = e^{(R(t,T)-\delta)*(T-t)}$$

In the case of dividend paying asset

Definition

The ratio between the forward and the spot price gives the forward premium at maturity T

CONTANGO

Name of the market when the forward future price is **above** the expected spot price

NOT always larger than 1.

With dividend paying asset, the forward premium is affected and might easily end up smaller than 1.

BACKWARDATION

Name of the market when the forward future price is **below** the expected spot price

More on the contango / backwardation

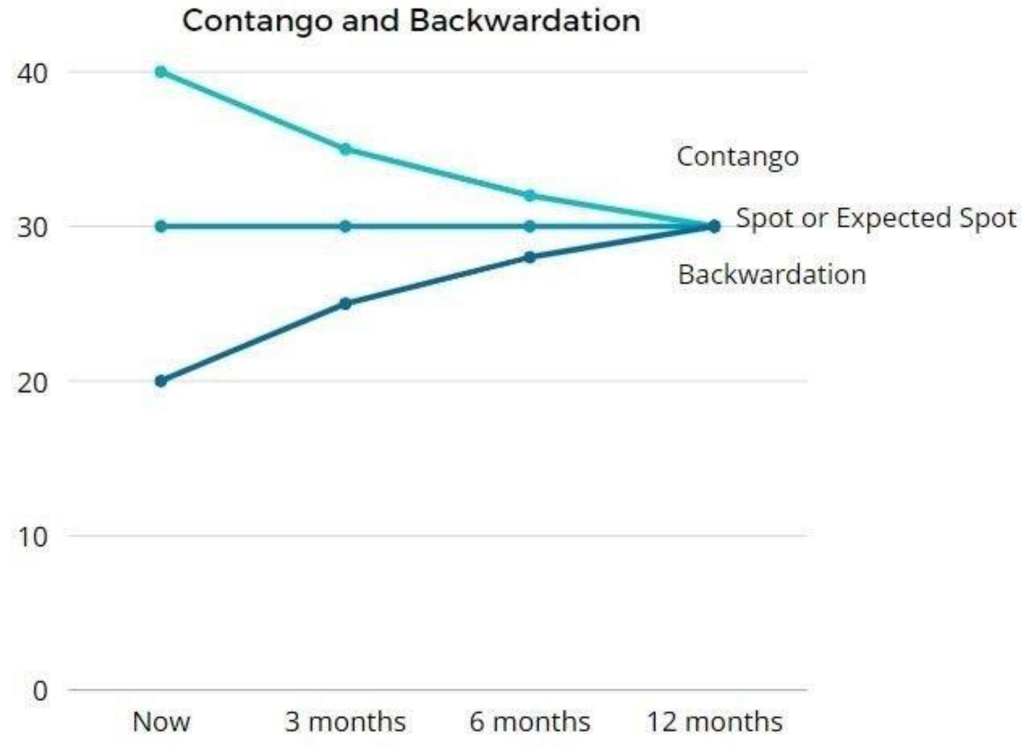
Figure

Let's assume the figure is about Brent Crude Oil. The price today is $30. You buy a future contract that expires in 3 months:

- If the price is $35 (> S(t)) -> contango
- If the price is $25 (< S(t)) -> backwardation

Observation

Over time, as the contract nears expiration, it will move closer to the spot price.
Therefore, unless the price rises above the price paid, the value of a contango contract will drop to meet the spot price at expiry.



Market Opportunity

The market can flip from contango to backwardation, or vice versa. Building a trading strategy based on a specific condition could become **unprofitable** if the conditions flip rapidly. NOT A TRADING ADVICE.

BACKWARDATION TRADES

A trader could buy a futures contract in the hope that it moves higher to meet the spot price. This can be profitable if the price of the commodity is trending higher.

If futures prices are below the expected price, though, this could also mean that traders are anticipating less demand for the commodity.

Profiting from backwardation is not as simple as it sounds and it does not solely depend on whether you should or buy or sell the asset.

More on the contango / backwardation

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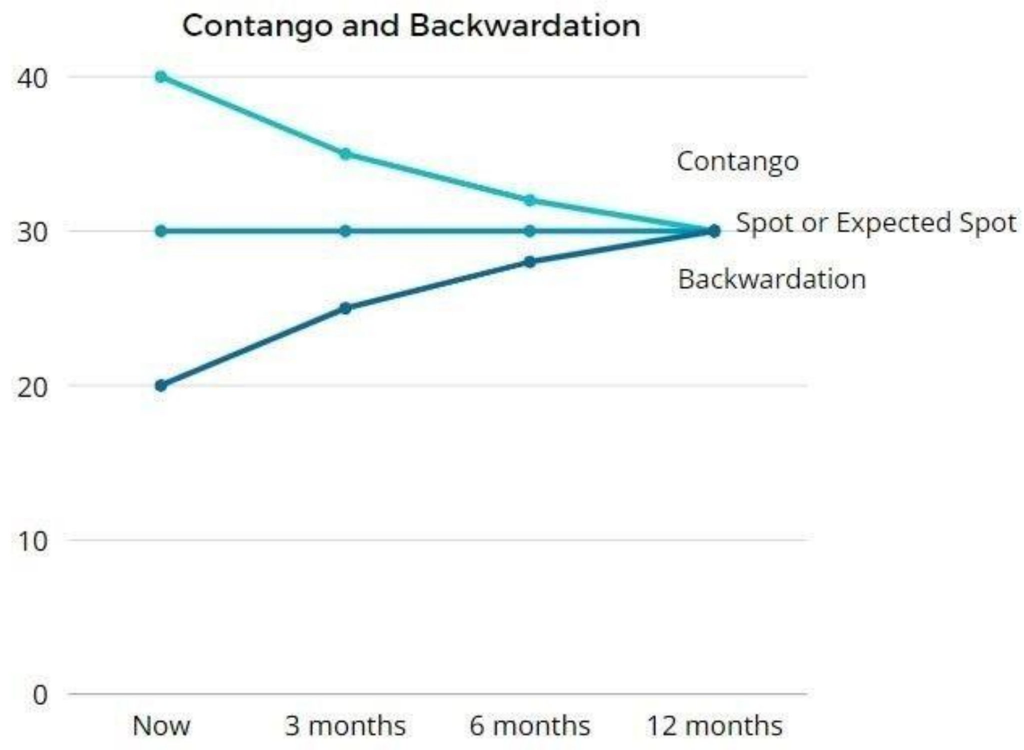

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CONTANGO TRADES

A trader believes that the spot price of oil will go even lower versus the future month's contract. A trader would short the spot month contract and buy the further out month. This trade would profit if the market increases its contango structure.

The trade would lose money if the market reverts to a normal backwardation structure.

Silver

Why is it usually in contango?

Crude Oil

Why is it usually in contango?

Gasoline

Why –in that case- could it be in backwardation?

The WTI 'super-contango' structure

FUTURES CONTRACTS ROLL OVER

Traders will roll over futures contracts that are about to expire to a longer-dated contract in order to maintain the same position following expiry. The roll involves selling the front-month contract already held to buy a similar contract but with longer time to maturity.

WHAT HAPPENED?

On April, 20th 2020, at the peak of the COVID-19 crisis, the WTI oil prices plunge to
- $37.63.

The panic was apparent in the futures market as the May contract expiry approached and traders wondered how they would take delivery of physical barrels of oil when storage sites are reaching full capacity.

This large front-month spread made traders not want to roll, nor did they want to hold and take delivery, hence they dumped instead.

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Module 1

Futures & Forward

1.2. Futures

Main differences with Futures contracts

Futures contracts are traded on open exchanges

Changes in the contract price are settled daily - this is called Mark-to-Market (MtM).

A contract is referred to by its delivery month

Existence of Forwards/Futures

For your own understanding.

Enough of the underlying in a standardized form

Sufficient price variability in the asset to create a demand for risk sharing between hedgers and speculators

Conditions.

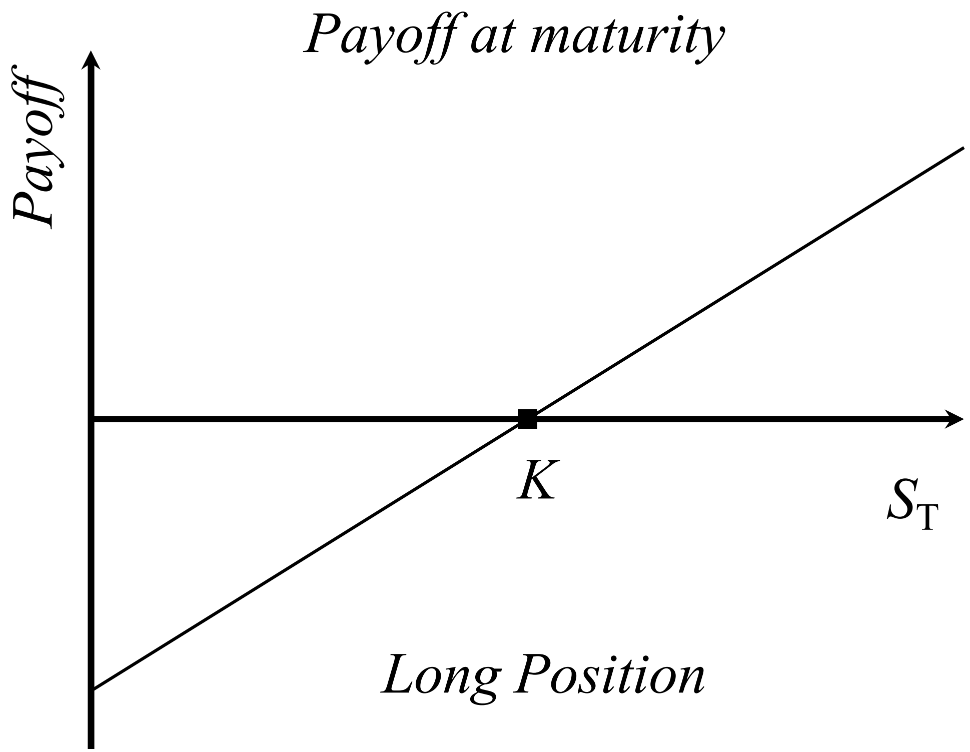
Traded commodities include: pork bellies, live cattle, sugar, wool, lumber, copper, aluminium, gold, tin, cocoa, coffee

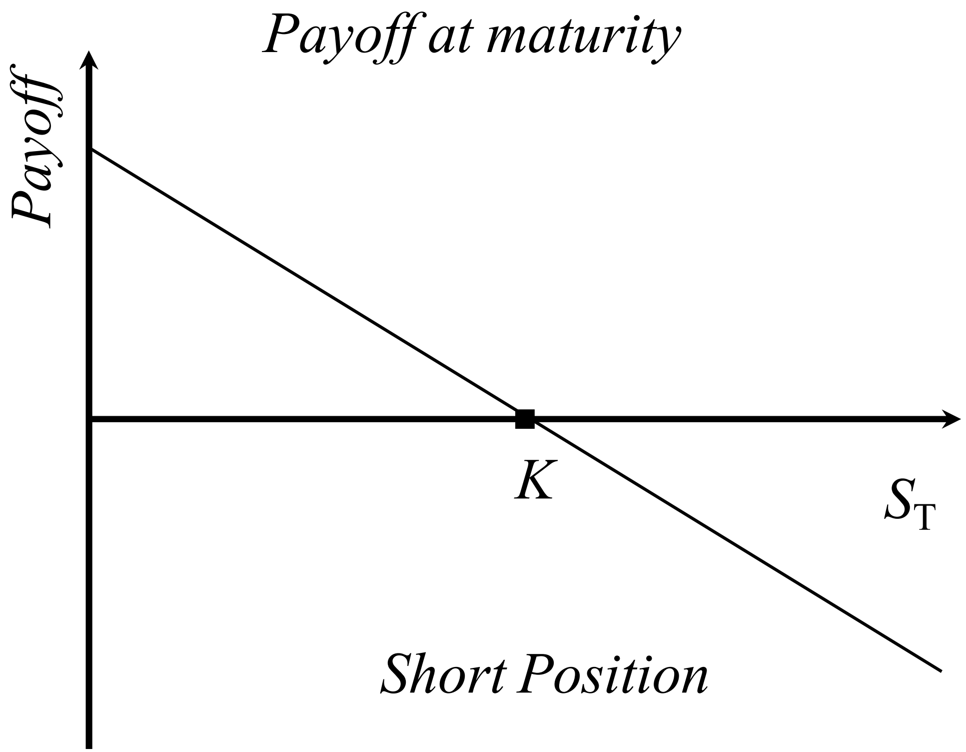
Financial assets include: stocks, stock indices, currencies, interests rates and bonds

Live Markets.

Payoffs for futures contracts at maturity

It should not come at a surprise after our discussion on forwards.





$Payoff_{LONG}(T) = S(T) - K$

$Payoff_{SHORT}(T) = K - S(T)$

K = Delivery Price ; $S(T)$ = Price at maturity

Zero-Sum Game

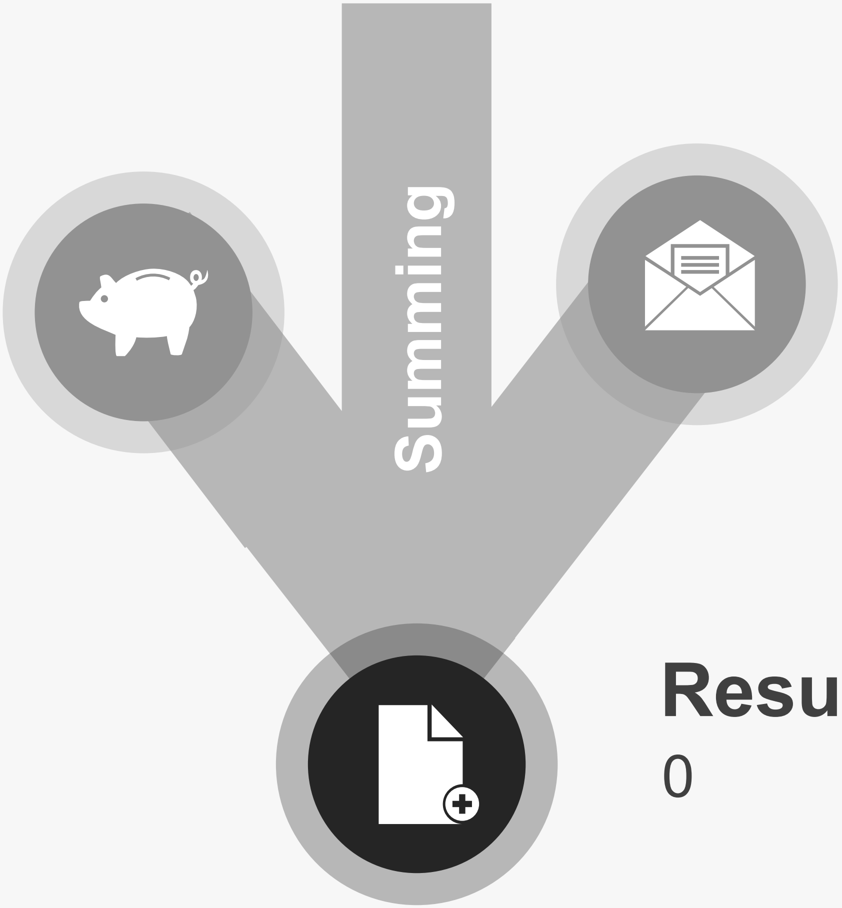
Futures contracts present a zero-sum game – i.e. what is lost by one side of the contract is gained by the other side and vice versa.

Payoff Long (T)

$S(T) - K$

Payoff Short (T)

$K - S(T)$



Result

0

Who trades in the futures market?

- Hedgers
  - Those wanting to reduce their risk exposures
- Speculators
  - Those who take positions in the market
- Arbitrageurs
  - Try to lock in riskless profit by simultaneously taking a position in two or more markets

About the notation

- To simplify the notation we will frequently use the lighter notation:
$$F(t) = F(t, T)$$

- Sometimes we may write:
$$F_T(t) = F(t, T)$$

- When a futures or forward contract is entered into at time t the forward price of an asset is fixed at $K = F(t, T)$but no cash is exchanged – in other words **the net value of the contract is zero.**

- At a later time t the contract may have a positive of negative value,
$$f(\tau, T) = F(\tau, T) - K$$

Margin Accounts

- OTC contracts carry some credit risk as one of the parties can back out of the deal or may simply not have the resources to honour the commitments
- It is one of the main roles of exchanges to organize trading in such a manner that contract defaults are avoided

---

- This is done by the operation of margins through clearing houses

Margin accounts

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- It is one of the main roles of exchanges to organize trading in such a manner that contract defaults are avoided
- This is done by the operation of margins through clearing houses

Margin cash flows

- Futures contracts are settled at the end of each trading day
- We consider a futures contract entered into today at time t with maturity T and use the notion $F(t) = \mathrm{F}(t,T)$,
- We set

$$t = T_0 < T_1 < \cdots < T_{n-1} < T_n = T \quad \& \quad F(t) = F(t,T)$$

Where T_k indicates the end of trading on day k

Day cash flows for a futures contract: Mark-To-Market (MtM)						
Day	$t = T_0$	T_1	T_2		T_{n-1}	$T_n = T$
Cash flow	0	$F(T_1) - F(T_0)$	$F(T_2) - F(T_1)$	. . .	$F(T_{n-1}) - F(T_{n-2})$	$F(T_n) - F(T_{n-1})$

Remember that $F(T_n) = S(T)$, i.e. the forward price converges towards the spot price as time approaches the maturity of the contract

Margin account

- A margin account is required to guarantee that both parties to a contract can honour their requirements: We have seen that if ‘one leg’ is winning, the other is loosing. Initially the margin account is set to generally around 10% of the initial contract value.
- Generally, payment to margin account is required whenever the **account balance** $\boldsymbol{AB(t)}$ is less that 10% of the **contract value** $\boldsymbol{F(t,T)}$
- Therefore, the **margin payment** $\boldsymbol{MP(t)}$ is given by:

$$MP(t) = \max(10\% * F(t,T) - AB(t)\,; 0)$$

On the basis of the said we can now determine the account balance at time t as:

$$AB(t) = \big(AB(t-1) + MP(t-1)\big)e^{r*1} + N_{contracts} * (F(t,T) - F(t-1,T))$$

Example 1 – Margin Account

The margin account is updated on a daily basis according to the equation on previous page

Risk Free rate	5%
Req Margin	10%

Day	No Contracts	Futures price (p/c)	Total price	Pr change per contract	Pr change per pos	New balance	Min Req Marg	Pay to acc
0	5000	$ 1 000,00	$ 5 000 000	0	0	$ 500 000	$ 500 000	0
1	5000	$ 987,90	$ 4 939 500					
2	5000	$ 987,97	$ 4 939 850					
3	5000	$ 990,53	$ 4 952 650					
4	5000	$ 988,37	$ 4 941 850					
5	5000	$ 973,89	$ 4 869 450					
6	5000	$ 968,70	$ 4 843 500					
7	5000	$ 980,82	$ 4 904 100					

Solution 1 – Margin Account

The margin account is updated on a daily basis according to the equation on previous page

Risk Free rate	5%
Req Margin	10%

Day	No Contracts	Futures price (p/c)	Total price	Pr change per contract	Pr change per pos	New balance	Min Req Marg	Pay to acc
0	5000	$ 1 000,00	$ 5 000 000	0	0	$ 500 000	$ 500 000	0
1	5000	$ 987,90	$ 4 939 500	$ -12,10	$ -60 500,00	$ 439 568,50	$ 493 950	$ 54 381,50
2	5000	$ 987,97	$ 4 939 850	$ 0,07	$ 350,00	$ 494 367,67	$ 493 985	$ -
3	5000	$ 990,53	$ 4 952 650	$ 2,56	$ 12 800,00	$ 507 235,40	$ 495 265	$ -
4	5000	$ 988,37	$ 4 941 850	$ -2,16	$ -10 800,00	$ 496 504,88	$ 494 185	$ -
5	5000	$ 973,89	$ 4 869 450	$ -14,48	$ -72 400,00	$ 424 172,90	$ 486 945	$ 62 772,10
6	5000	$ 968,70	$ 4 843 500	$ -5,19	$ -25 950,00	$ 461 061,71	$ 484 350	$ 23 288,29
7	5000	$ 980,82	$ 4 904 100	$ 12,12	$ 60 600,00	$ 545 016,35	$ 490 410	$ -

Example 2 – Fwd vs Fut

Case study

Consider a WTI contract on NYMEX. Suppose short positions in two contracts are entered into on 21 October 2010, a forward contract and a DEC10 futures contract, each to be closed out on 19 November 2010 when the futures contract matures. Suppose that the US interest rate is 0.25% and that strike K = 85. With initial and variation margins of $9,788 and $7,250 respectively and a contract size of 1,000 barrels, i.e. $9.788 and $7.25 respectively per barrel, we can tabulate the cashflows on a per-barrel basis for each of these two contracts, given NYMEX oil prices over that period of interest.

Contract entered	21/10/2010	Strike	85	Contract Size N	1000	Base	365
Maturity	19/11/2010	US interest rate	0,25%	Initial Margin	9788	Variation margin (maintenance margin)	7250

			Margin a/c balance			Cashflow on		
t	f(t,T)	P&L	c/o	+P&L	+P&L + Margin	SHORT Forward	SHORT Futures	PV(CF Fut)
21/10/2010	80,36	4,64	9,788	14,428	14,428	0,000	-9,788	-9,788000
22/10/2010	81,63	-1,27	14,428	13,158	13,158	0,000	0,000	0,000000
25/10/2010	82,52	-0,89	13,158	12,268	12,268	0,000	0,000	0,000000
26/10/2010	82,57	-0,05	12,268	12,218	12,218	0,000	0,000	0,000000
27/10/2010	81,91	0,66	12,218	12,878	12,878	0,000	0,000	0,000000
28/10/2010	82,18	-0,27	12,878	12,608	12,608	0,000	0,000	0,000000
29/10/2010	81,45	0,73	12,609	13,339	13,339	0,000	0,000	0,000000
01/11/2010	82,98	-1,53	13,339	11,809	11,809	0,000	0,000	0,000000
02/11/2010	83,91	-0,93	11,809	10,879	10,879	0,000	0,000	0,000000
03/11/2010	84,67	-0,76	10,879	10,119	10,119	0,000	0,000	0,000000
04/11/2010	86,47	-1,8	10,119	8,319	8,319	0,000	0,000	0,000000
05/11/2010	86,85	-0,38	8,319	7,939	7,939	0,000	0,000	0,000000
08/11/2010	87,02	-0,17	7,939	7,769	7,769	0,000	0,000	0,000000
09/11/2010	86,73	0,29	7,769	8,059	8,059	0,000	0,000	0,000000
10/11/2010	87,8	-1,07	8,059	6,989	7,250	0,000	-0,261	-0,260623
11/11/2010	87,81	-0,01	7,250	7,240	7,250	0,000	-0,010	-0,009930
12/11/2010	84,91	2,9	7,250	10,150	10,150	0,000	0,000	0,000000
15/11/2010	84,84	0,07	10,150	10,220	10,220	0,000	0,000	0,000000
16/11/2010	82,34	2,5	10,220	12,720	12,720	0,000	0,000	0,000000
17/11/2010	80,46	1,88	12,720	14,600	14,600	0,000	0,000	0,000000
18/11/2010	81,87	-1,41	14,600	13,190	13,190	0,000	0,000	0,000000
19/11/2010	81,51	0,36	13,191	13,551	0,000	3,490	13,551	13,547860

Source: Commodity Option Pricing. Wiley Finance

HEDGING STRATEGIES

WHEN

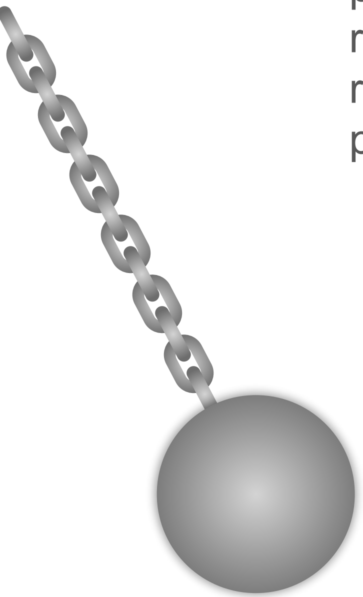
When hedging by the use of futures, the aim is to take a position that reduces the risk as much as possible.

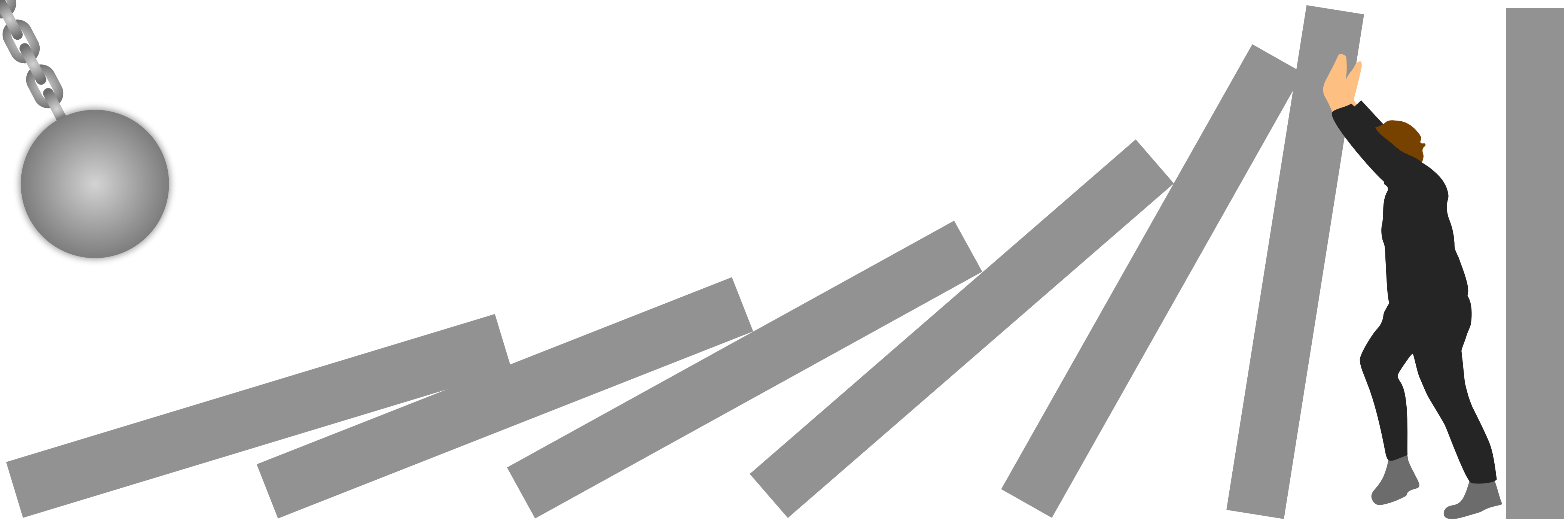
WHY

Generally, the hedger has an exposure to the price of an asset.

HOW

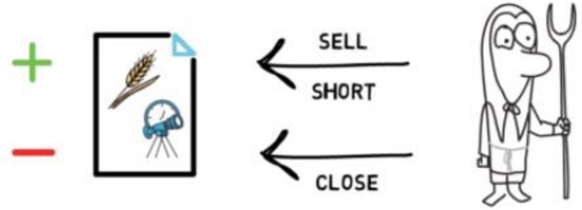
By taking an offsetting position in the futures market that risk is reduced.





Short hedge

- Short hedge involves taking a short position in a futures contract
- This is used when the hedger already owns an asset and expects to sell it sometimes in the future
- For example
  - A producer of grain or oil that will be sold in the market in few months time
  - A US based exporter who expects to receive a foreign currency, for example GBP in the near future. Gains will be realized if the GBP strengthens relative to the GBP and losses if the GBP weakens. This situation can be hedged with a short position in the rate GBP/USD



Arbitrage opportunity

- If the futures price is not equal the spot price during the delivery period there is an arbitrage opportunity which will force

$$F(T,T) = S(T)$$

or equivalently in the abbreviated notation:

$$F_T = S_T$$

Incoming proof

Arbitrage opportunity

We will now show that a simultaneous position in a forward contract $F(t,T)$ and cash, equal in value to the price of an asset $S(t)$, is at contract maturity T equal to a long position in the asset at contract maturity, $S(T)$

- Value of position at time t

$$(F(t,T) - K) + S(t)$$

- Remember that $F(T,T) = S(T)$ and $K = S(t) * e^{R(t,T)(T-t)}$

- Value of position at time T

$$(F(T,T) - K) + S(t)e^{R(t,T)(T-t)} = S(T)$$

Futures and spot markets

- Futures contracts can be used to replicate the risks of the spot market which amounts to the creation of a synthetic security
- This fact is fundamental for the understanding of how forwards and futures contracts can be used for hedging purposes
- As shown on the previous page, the two sides have the same risk and return profile

$$Futures\ \ Contract\ \ +\ \ Cash\ Reserve\ \ \ <=>\ Underlying\ Security$$

- At expiry,
  - Left hand side gives the value of futures contract plus cash reserve
  - Right hand side gives the value if the security had been bought

Futures and spot markets

- The value of the synthetic security at maturity is (left hand side of equation previous page – stated here again)

$$Futures\ \ Contract\ \ +\ \ Cash\ Reserve\ \ \ <=>\ Underlying\ Security$$

$$F_T - F_0 + S_0 e^{rT} = S_T - S_0 e^{rT} + S_0 e^{rT} = S_T$$

- This is precisely the value of the security itself at maturity
- Result
  - The forward contract provides the same risk - return profile as the underlying security

Futures contracts for hedging market risk

- Rearranging the relationship

$$Futures\ \ Contract\ \ +\ \ Cash\ Reserve\ \ <=>\ Underlying\ Security$$

to

$$Underlying\ Security - Futures\ \ Contract\ \ \ <=>\ Cash\ Reserve$$

Shows that holding the underlying security long and simultaneously taking a short position in a futures contract results in a cash reserve i.e. fixed exposure

The risk in the underlying security is offset by the short position in the futures contract … and this is KEY.

Futures contracts for hedging market risk

- The arbitrage relationship for the pricing of the forward contract is such that the returns earned on the hedged (risk-less) position are consistent with the risk-less rate

$$Underlying\,Security \; - \; Futures\;\; Contract <=> Cash\,Reserve$$

$$S_T - (F_T - F_0) = S_T - (S_T - S_0 e^{rT}) = S_0 e^{rT}$$

- The value of the hedged position does not depend on the value of the underlying security at expiry

Futures contracts for hedging market risk

- We show by simple arguments how futures/forward contracts can be used to hedge various risks firms may be exposed to
- Consider a firm that needs to purchase or sell an asset at some future time T. The future price of the asset is unknown today and therefore the firm is exposed to "price risk" i.e. it faces unknown future expense or income
- By entering a future contract for this asset the firm can reduce this risk significantely.
- **Strategy:**
  - If the firm needs to buy an asset (because short the asset) it buys a forward contract (and becomes long the forward contract)
  - If the firm needs to sell an asset (because long the asset) it sells a forward contract (and becomes short the forward contract)

Hedged and non-hedged positions

- Consider a firm that needs to buy an asset at a future time T . The assets present price is S_0

=> If the firm does not hedge it pays the unknown future price S_T

=> If the firm hedges it pays the known futures/forward price $S_0 e^{rT}$

- Relative to today's price the expense/gain is:

$$Non\ Hedge: C_{nh} = -(S_T - S_0)$$
$$Hedge: C_h = -(F(0,T) - S_0) = -(S_0 e^{rT} - S_0) = -S_0(e^{rT} - 1)$$

- C_{nh} is a random variable (stochastic) – i.e. its value at time T is unknown today
- C_h is deterministic – i.e. its value at time T is known today

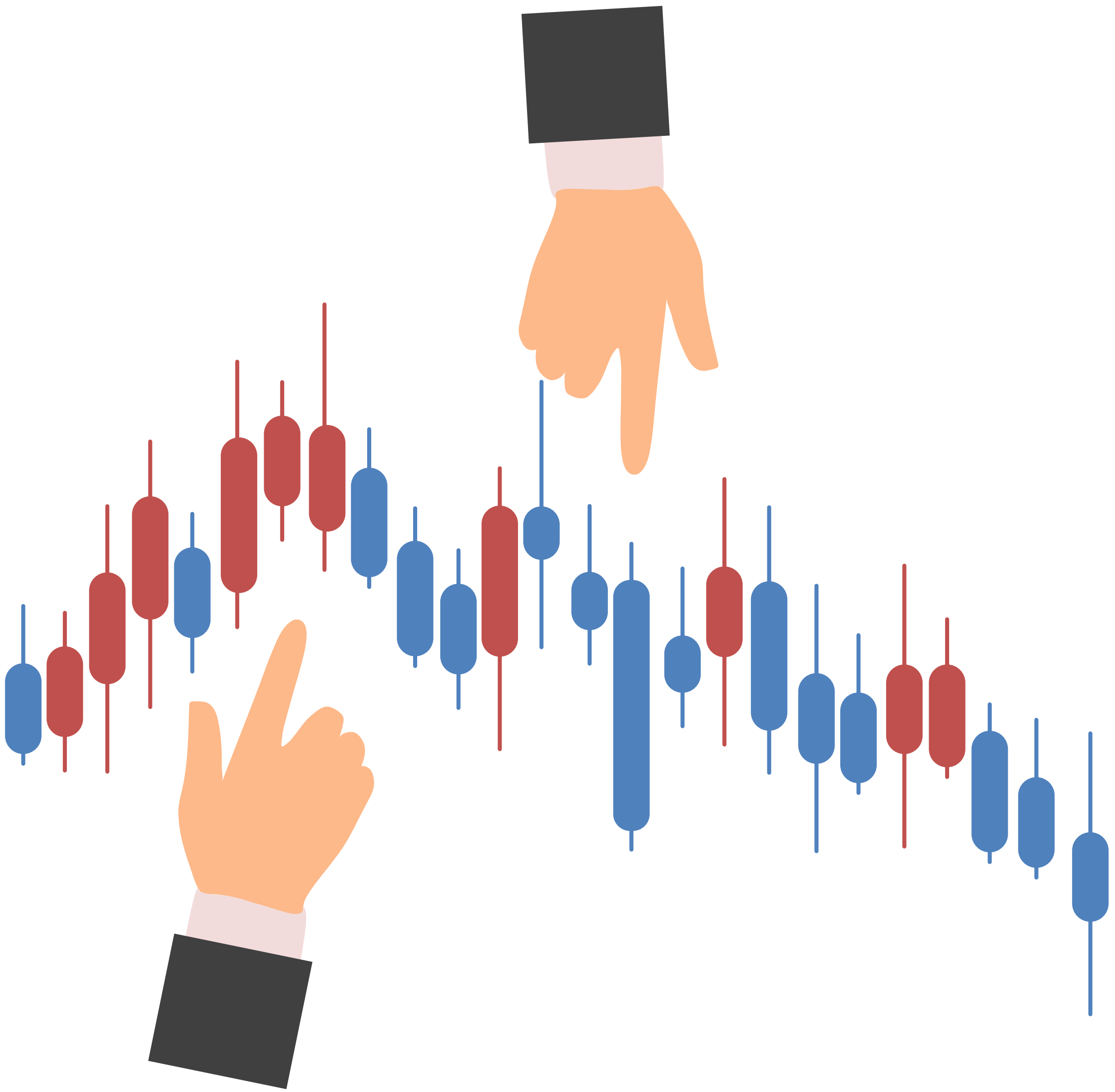
Example 1 - Oil

- Oil producer
  - Long in asset + short in futures contract

$$P(t) = S(t) - F(t)$$

- Oil purchaser
  - Short in asset + long in futures contract

$$P(t) = F(t) - S(t)$$



MONTH	OPTIONS	CHART	LAST	CHANGE	PRIOR SETTLE	OPEN	HIGH	LOW	VOLUME	UPDATED
AUG 2025 CLQ5	OPT		66.50	-0.50 (-0.75%)	-	67.13	67.18	66.04	56,217	11:59:59 CT 04 Jul 2025
SEP 2025 CLU5	OPT		65.15	-0.47 (-0.72%)	-	65.64	65.73	64.68	28,287	11:59:59 CT 04 Jul 2025
OCT 2025 CLV5	OPT		63.95	-0.45 (-0.70%)	-	64.41	64.47	63.55	20,165	11:59:59 CT 04 Jul 2025
NOV 2025 CLX5	OPT		63.11	-0.45 (-0.71%)	-	63.59	63.62	62.76	9,820	11:59:56 CT 04 Jul 2025
DEC 2025 CLZ5	OPT		62.65	-0.35 (-0.56%)	-	63.00	63.06	62.27	15,963	11:59:59 CT 04 Jul 2025
JAN 2026 CLF6	OPT		62.25	-0.43 (-0.69%)	-	62.66	62.66	62.05	3,718	11:59:55 CT 04 Jul 2025

Example 1 - OIL

TODAY is July 5, 2025

An oil producer has agreed to sell 1m barrels of crude oil for delivery in Sept 2025 (the expiry date of the closest futures contract is 16 Sept 2025)

AGREED PRICE

The agreed selling price is the spot price observed on September 16, 2025

PRESENT SPOT

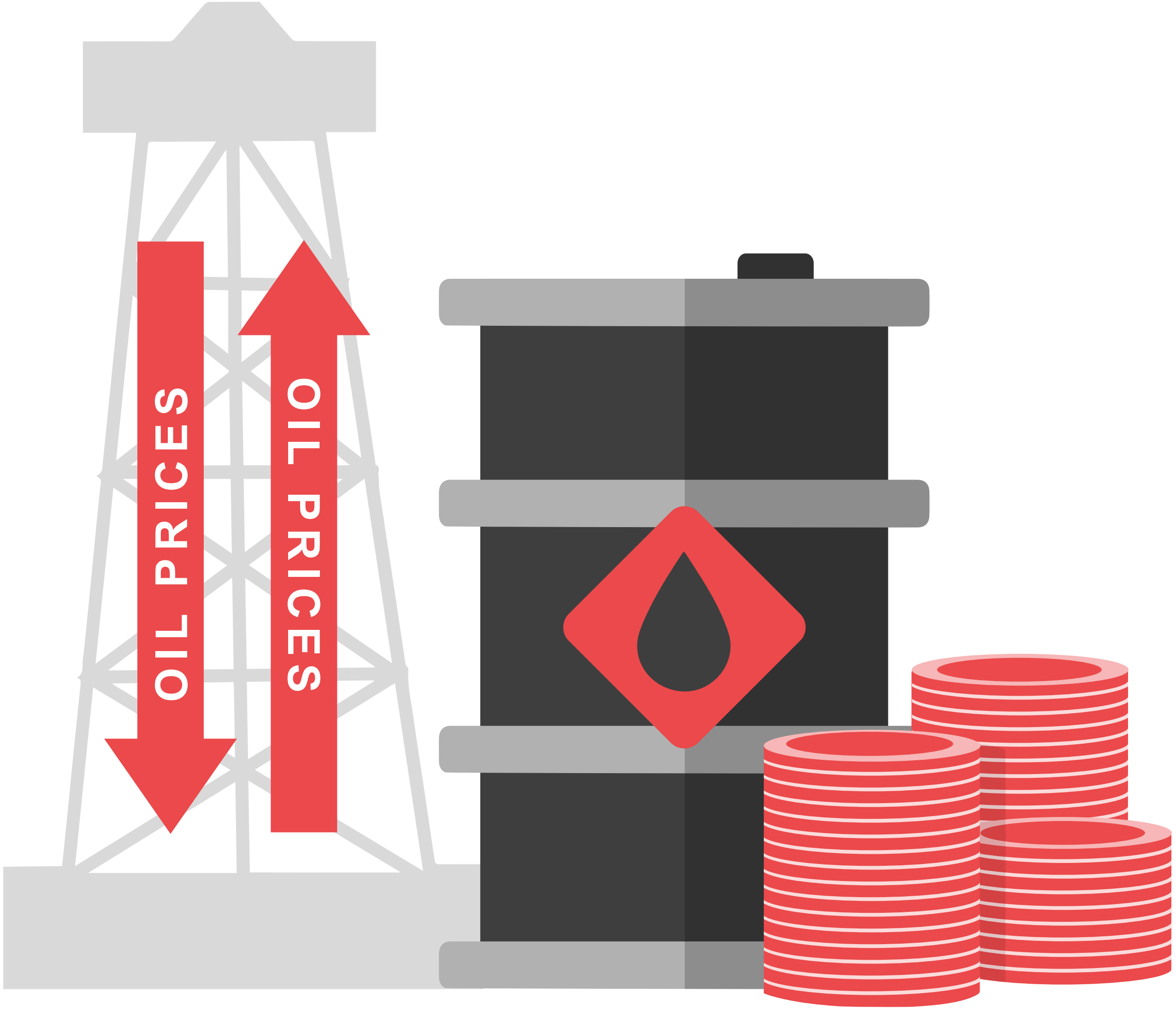
Present spot price for crude oil is USD 66.455 and the today's price of Sep-25 contract is USD 65.15

WITHOUT HEDGE

Producer is long crude oil and would on delivery earn:

$$Q_T = 1m * S_T$$

Example 1 - Oil



SHORT HEDGE

1 | Assumption

The oil producer takes a short position in the Sept-25 futures contract, presently priced at $F(t,T) = K = 65.15$

2 | On 16 Sept 2025

We do assume 2 scenarios (A & B)

A | SPOT PRICE USD 60

Profit from selling the crude:

$$1m * 60 = \$\, 60m$$

Gains on selling the futures contract:

$$1m * (65.15 - 60) = 1m * 5.15 = \$5.15m$$

B | SPOT PRICE USD 70

Profit from selling the crude:

$$1m * 70 = \$\, 70m$$

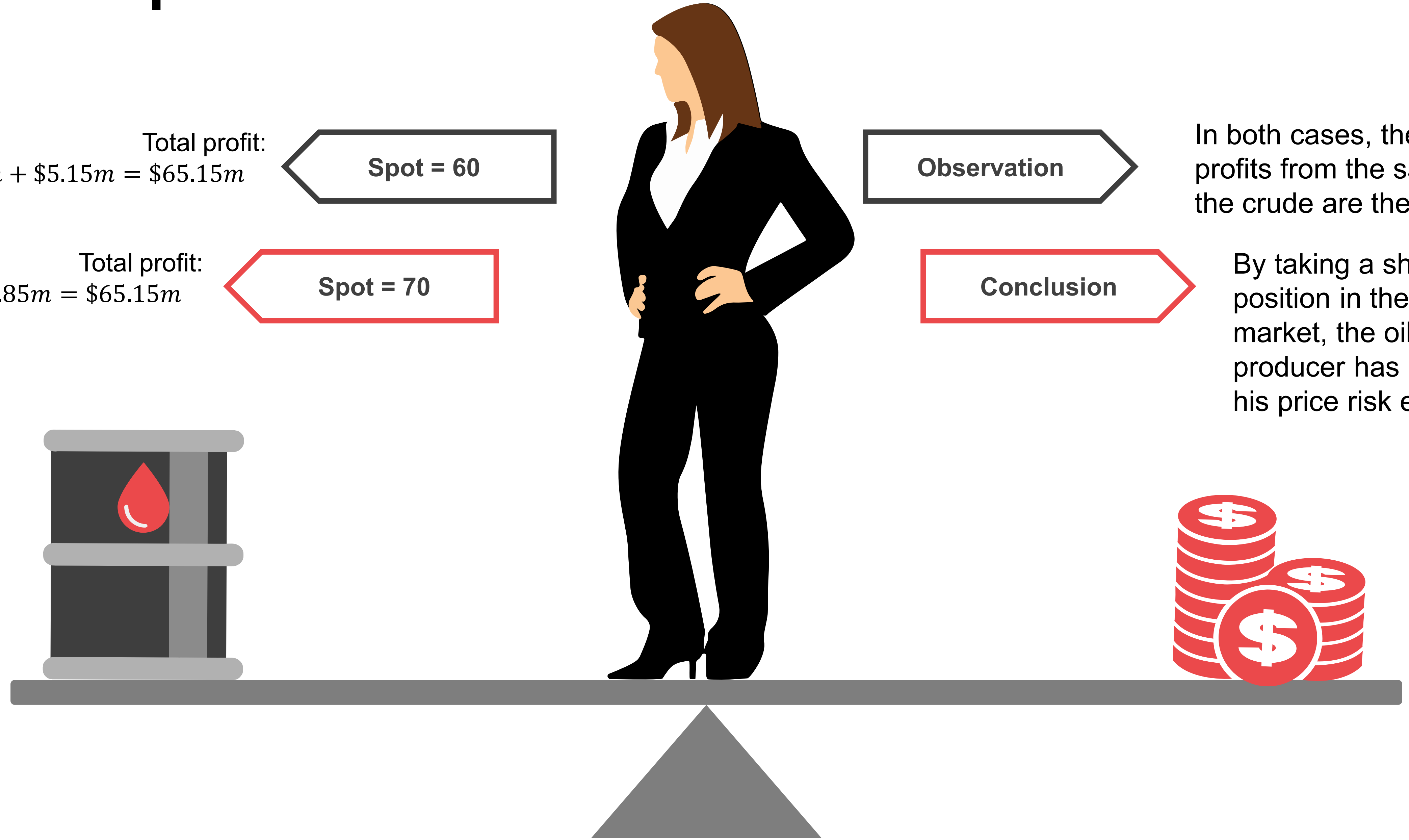
Loss on selling the futures contract:

$$1m * (65.15 - 70) = 1m * 4.85 = \$4.85m$$

Example 1 - Oil

Total profit:
$\$60m + \$5.15m = \$65.15m$

Total profit:
$\$70m - \$4.85m = \$65.15m$



Observation

In both cases, the profits from the sale of the crude are the same!

Conclusion

By taking a short position in the futures market, the oil producer has removed his price risk exposure

Example 2 – Coffee

Today is 05/07/2025

A company wants to purchase 375,000 pounds of coffee for delivery in July 2026.

Contracts

Trading unit on NYMEX is 37,500 pounds (lbs)

Present Spot Price

USD 3.1944 per pound (lb)

Jul-26 Contract Price

USD 2.6870 per pound (lb)

As we already said

The spot price at delivery S_T is unknown – ‘price risk’

Two possible alternatives for the company:
- Accept the risk and pay spot on delivery
- Creating a long hedge to reduce or even remove the risk.

Example 2 - Coffee

LONG HEDGE

1 | **Assumption**

The company takes a long position in the Jul-26 futures contract, presently priced at $F(t,T) = K = 2.6870$

2 | **On 11 May 2022**

We do assume 2 scenarios (A & B)

A | **SPOT PRICE USD 2.1**

Cost of purchasing the coffee:

$$375k \ * 2.1 = \$\ 787{,}500$$

Loss on buying the futures contract:

$$375k * (2.6870 - 2.1) = 375k * 0.587 = \$\ 220{,}125$$

B | **SPOT PRICE USD 3.05**

Cost of purchasing the coffee:

$$375k \ * 3.05 = \$\ 1{,}143{,}750$$

Gain on buying the futures contract:

$$375k * (3.05 - 2.687) = 375k * 0{,}363 = \$\ 136{,}125$$

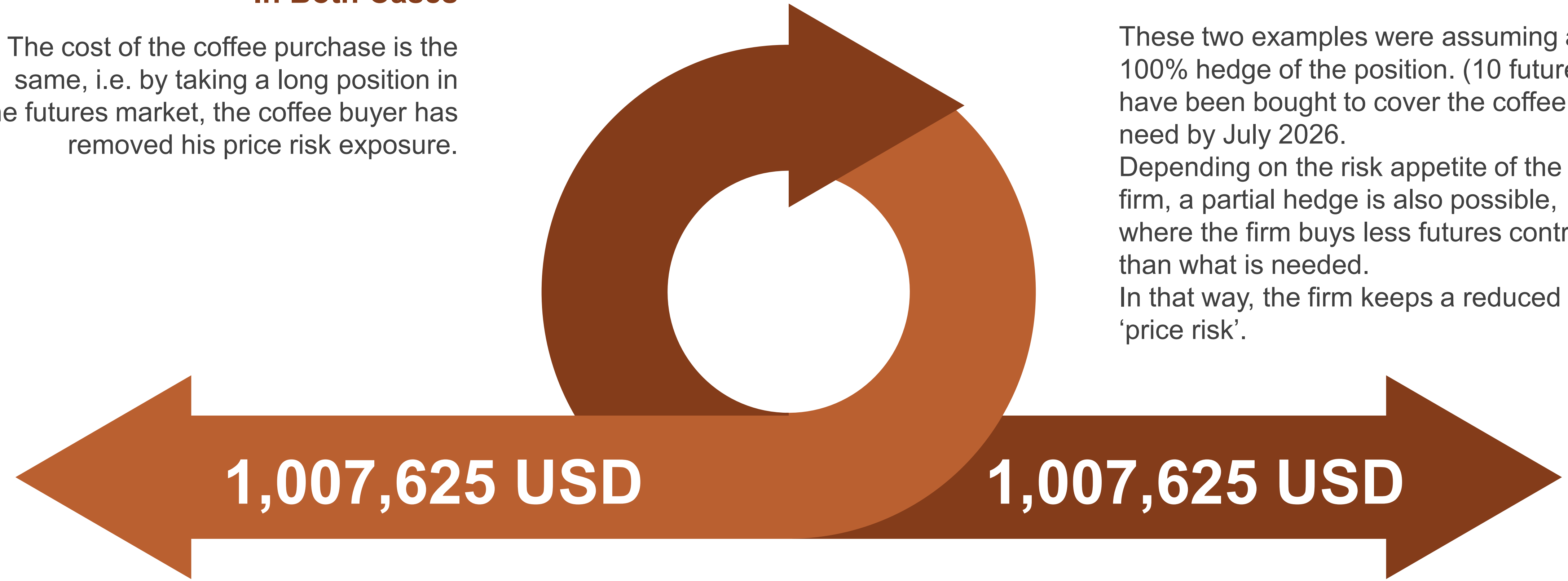
Example 2 - Coffee

In Both Cases

The cost of the coffee purchase is the same, i.e. by taking a long position in the futures market, the coffee buyer has removed his price risk exposure.

Full Hedge

These two examples were assuming a 100% hedge of the position. (10 futures have been bought to cover the coffee need by July 2026.
Depending on the risk appetite of the firm, a partial hedge is also possible, where the firm buys less futures contract than what is needed.
In that way, the firm keeps a reduced 'price risk'.



Hedging with a futures contract

- A company which plans to purchase an asset at a future date can take a long position in a futures contract
- At time $t = 0$ the value of the portfolio is (long futures + short underlying position)

$$P_0 = F_0 - S_0$$

- At time $t = T$ the value of the portfolio is

$$P_T = F_T - S_T \quad (if\ T = expiry, then\ P_T = 0)$$

- The total gain/loss is therefore

$$\Delta P = P_T - P_0 = F_T - S_T - (F_0 - S_0) = F_T - F_0 - (S_T - S_0) = \Delta F - \Delta S$$

Hedging with a futures contract

- The relationship

$$\Delta P = \Delta F - \Delta S$$

tells us that loss on the asset purchase is compensated for by gains on the futures contract and vice versa

- If the contract is held to expiry then $(S_T = F_T)$

$$\Delta P = P_T - P_0 = F_T - S_T - (F_0 - S_0) = S_0 - F_0$$

- If the futures contract is on non dividend paying asset and fairly priced, then

$$\Delta P = S_0 - F_0 = S_0 - S_0 e^{rT} = S_0(1 - e^{rT})$$

which is the difference between the spot and the futures price at $t = 0$

Hedging and Basis Risk

For your own understanding

- If the futures contract is not held to expiry $(t < T)$
  $$\Delta P = P_t - P_0 = F_t - S_t - (F_0 - S_0) = S_0 - F_0 - (S_t - F_t)$$
- The only risk associated with the futures contract is in the basis
  $$b_t = -(S_t - F_t)$$
- As the purchasing time gets closer to its expiry the basis risk gets smaller and disappears completely in the delivery time
  $$\lim_{t \to T} b_t \to 0$$

Comments on basis risk

For your own understanding

- A firm needs to buy an asset at a future time T_d – time today t
- Hedge with a short futures contract that expires at time $T_c > \ T_d$
- Futures price $F(t,T_c)$
- At time T_d the firm purchases the asset at the spot price $S(T_d)$ and closes the futures contract
- The net price paid for the asset is

$$Cost(T_d) = S(T_d) - \big(F(T_d,T_c) - F(t,T_c)\big) = F(t,T_c) + S(T_d) - F(T_d,T_c)$$

- At time t, $F(t,T_c)$ is known – but the hedging risk is the uncertainty associated with the basis, the basis – risk

$$b(T_d) = S(T_d) - F(T_d,T_c)$$

- Of course

$$\lim_{T_d \to T_c} b(T_d) \to 0$$

Closing contract before delivery

- Need to buy an asset at time T_d
- Enter a futures contract at $F(t,T_c)$ with expiry $T_c < T_d$
- Close contract at time T_c
- Gains from contract are
$$g = F(T_c,T_c) - F(t,T_c) = S(T_c) - F(t,T_c)$$
- The net price paid for asset at time T_d is
$$S(T_d) - \big(S(T_c) - F(t,T_c)\big) = F(t,T_c) - \big(S(T_c) - S(T_d)\big)$$
- $F(t,T_c)$ is known at time t. The risk is associated with the unknown quantity
$$\big(S(T_c) - S(T_d)\big)$$
- This is an exposure to the spot market but over shorter period of time than in the case of no hedge

Minimum variance hedge

- Need to sell N_A units of an asset at time T_d
- There are no futures contracts in this asset
- Use similar asset for hedging – how many units N_F of the hedging asset should I use?
- Take a short position in the T_c - futures contract and close it at T_d
- Net profit from selling the asset and closing the futures contract is

$$\begin{aligned} P(t,T_d) &= N_A S(T_d) - N_F\big(F(T_d,T_c) - F(t,T_c)\big) \\ &= N_A S(t) + N_A\big(S(T_d) - S(t)\big) - N_F\big(F(T_d,T_c) - F(t,T_c)\big) \\ &= N_A S(t) + N_A \Delta S(t,T_d) - N_F \Delta F(t,T_d) \end{aligned}$$

- The first term on the right hand side is known – I want to minimise the standard deviation of the remaining part on the right hand side

Minimum variance hedge

- I want to minimize the variance of

$$W(t,T_d) = N_A \Delta S(t,T_d) - N_F \Delta F(t,T_d)$$

- Which is the same as minimizing the variance of

$$\Omega(t,T_d) = \Delta S(t,T_d) - \frac{N_F}{N_A} \Delta F(t,T_d) = \Delta S(t,T_d) - h * \Delta F(t,T_d)$$

- The variance is $\sigma_\Omega^2 = \sigma_S^2 + h^2 * \sigma_F^2 - 2h * \sigma_S * \rho_{S,F} * \sigma_F$
- Minimum variance is found from

$$\frac{\partial \sigma_\Omega^2}{\partial h} = 0 \Rightarrow h\sigma_F^2 - \sigma_S \rho_{S,F} \sigma_F = 0 \Rightarrow h = \rho_{F,S} * \frac{\sigma_S}{\sigma_F} = \frac{cov(F,S)}{\sigma_F^2}$$

- h gives the ratio N_F/N_A which minimizes the variance of W

Minimum variance hedge

- From the minimum variance hedge ratio we find the optimum number of contracts
  - N_A size of position being hedged
  - Q_F size of one futures contract
  - N_{op} optimal number of futures contracts

- Then

$$N_F = hN_A = Q_F N_{op} \Rightarrow N_{op} = h * \frac{N_A}{Q_F}$$

Futures on stock indexes

- Use stock index futures to hedge a well diversified equity portfolio
- Define
  - P, current value of portfolio
  - V_S , current value of the stocks underlying one futures contract
- Beta of the portfolio is

$$\beta = \rho_{I,P} * \frac{\sigma_P}{\sigma_I} = \frac{cov(I,P)}{\sigma_I^2}$$

- The number of futures contracts required for minimum variance hedging

$$N_{op} = \beta * \frac{P}{V_S}$$

$$N_F = \beta N_A = V_S N_{op}$$

$$\Rightarrow N_{op} = \beta * \frac{P}{V_S}$$

$$(N_A = P)$$

Minimum Variance Hedging – Example 1

- I want to use a futures contract in a stock index to hedge a portfolio of stocks
  - Value of index, $V_I = 1000$
  - Value of portfolio, $V_P = \$5\ 000\ 000 = \$5\text{m}$
  - Portfolio beta, $\beta = 1.5$
  - One futures contract is $\$250$ times the index
- The number of futures contracts required for minimum variance hedging:

$$N_{op} = \beta \frac{V_P}{250 * V_I} = 1.5 * \frac{5m}{250 * 1000} = 30$$

Minimum Variance Hedging – Example 1

Consequence

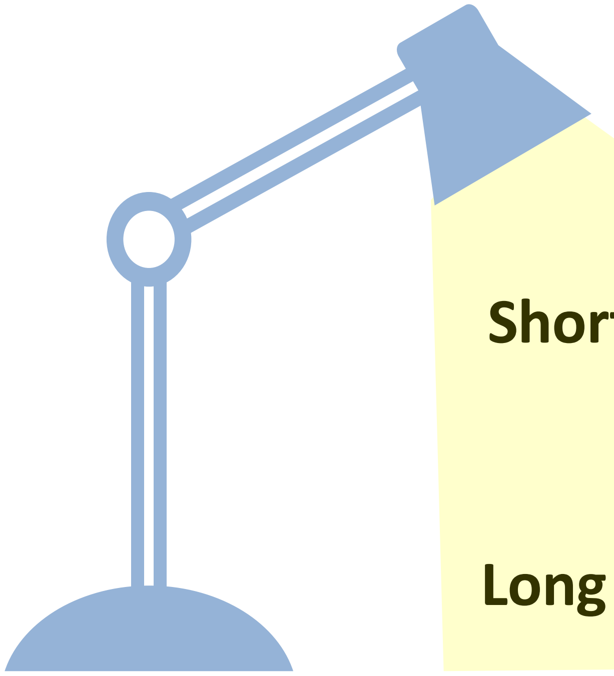
- **Assume the index goes down, from $\boldsymbol{V_I = 1000}$ to $\boldsymbol{V_I = 900}$**

- The gain on a short futures contract is
$$30 * 250 * (1000 - 900) = 750\,000\ \$$$

- When the index goes down by 10% the portfolio goes down by
$$1.5 * 10\% = 15\%$$

- Loss on portfolio is
$$0{,}15 * 5m = 750\,000\ \$$$

Long and short hedge



Short hedge: Long asset and short futures. Used when asset has to be sold in the future

$$P = S - aF \quad ; \quad \Delta P = \Delta S - a\Delta F$$

Long hedge: Short asset and long futures. Used when asset needs to be bought in the future

$$P = aF - S \quad ; \quad \Delta P = a\Delta F - \Delta S$$

Long hedge Modelling

- Change in portfolio at time t, $0 < t < T$

$$\Delta P(0,t) = P(t) - P(0) = aF(t) - S(t) - \big(aF(0) - S(0)\big)$$
$$= -\big(S(t) - S(0)\big) + a\big(F(t) - F(0)\big)$$

- Change in portfolio at expiry

$$\Delta P(0,T) = -\big(S(T) - S(0)\big) + a\big(F(T) - F(0)\big) = -S(T) + aS(T) + S(0) - aF(0)$$
$$= -S(T) + aS(T) + S(0) - aS(0)e^{rT} = -S(T)(1-a) + S(0)(1 - ae^{rT})$$

$$(We\ remind\ that\ F(T) = S(T)\ and\ that\ F(0) = S(0)e^{rT})$$

Long hedge Modelling

- The change in the value of the long hedged portfolio is a line where the variable is the final spot price and the steepness (gradient) is given by $(1\text{-}a)$

$$\Delta P(0,T) = -S(T)(1-a) + S(0)(1-ae^{rT})$$

- Special cases:

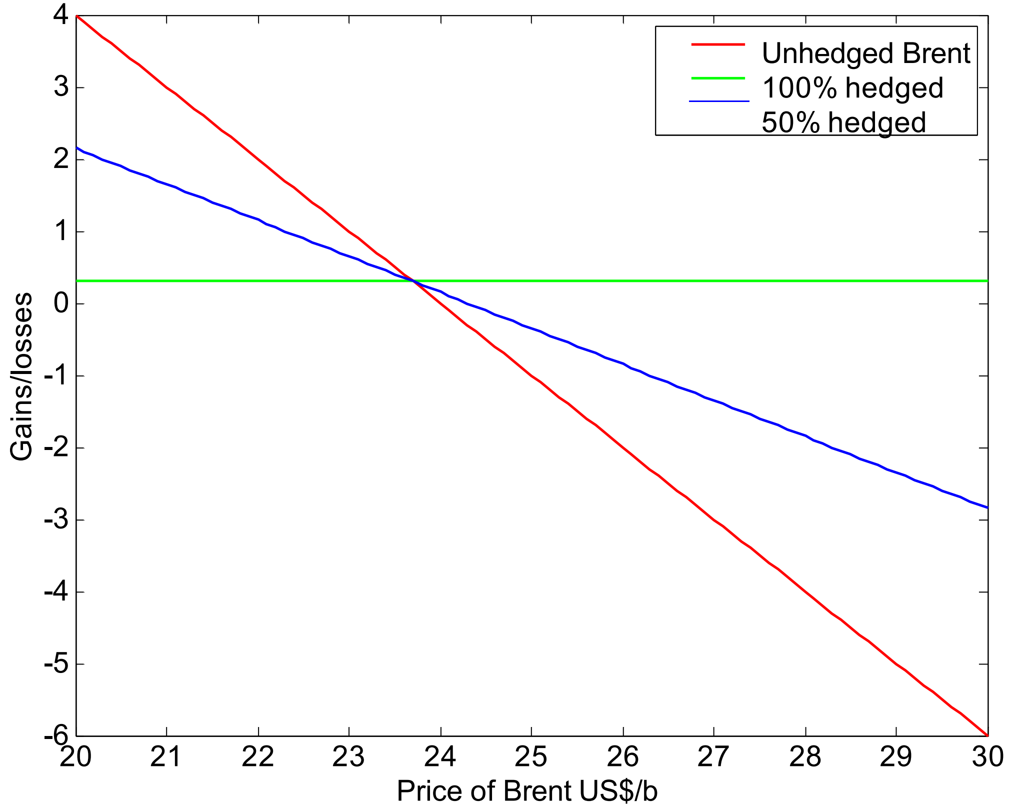
  - $a = 0$, no hedge:
  $$\Delta P(0,T) = -\big(S(T) - S(0)\big)$$

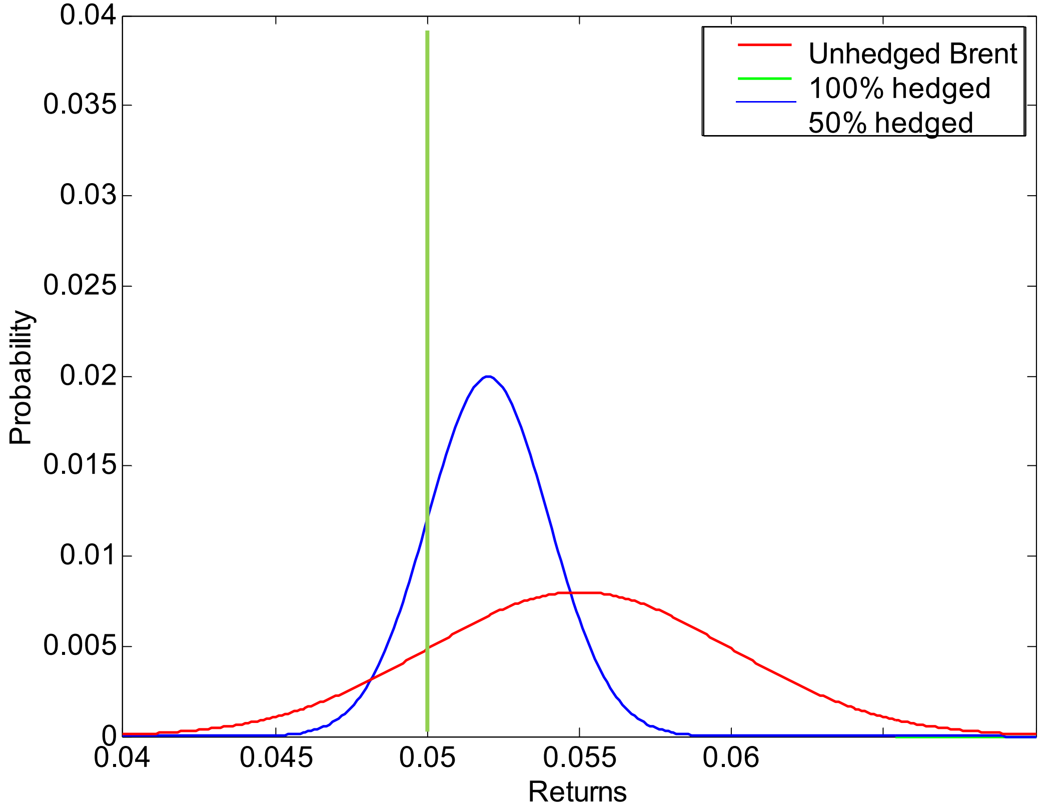
  - $a = 1$, full (100%) hedge:
  $$\Delta P(0,T) = S(0)(1-e^{rT})$$

Payoffs and returns for hedged portfolios

- Short asset and long futures contract – **long hedge**

Spot S_0 = \$24
3 months forward price $F_{0,1/4}$ = \$23.7





Futures prices–future spot prices

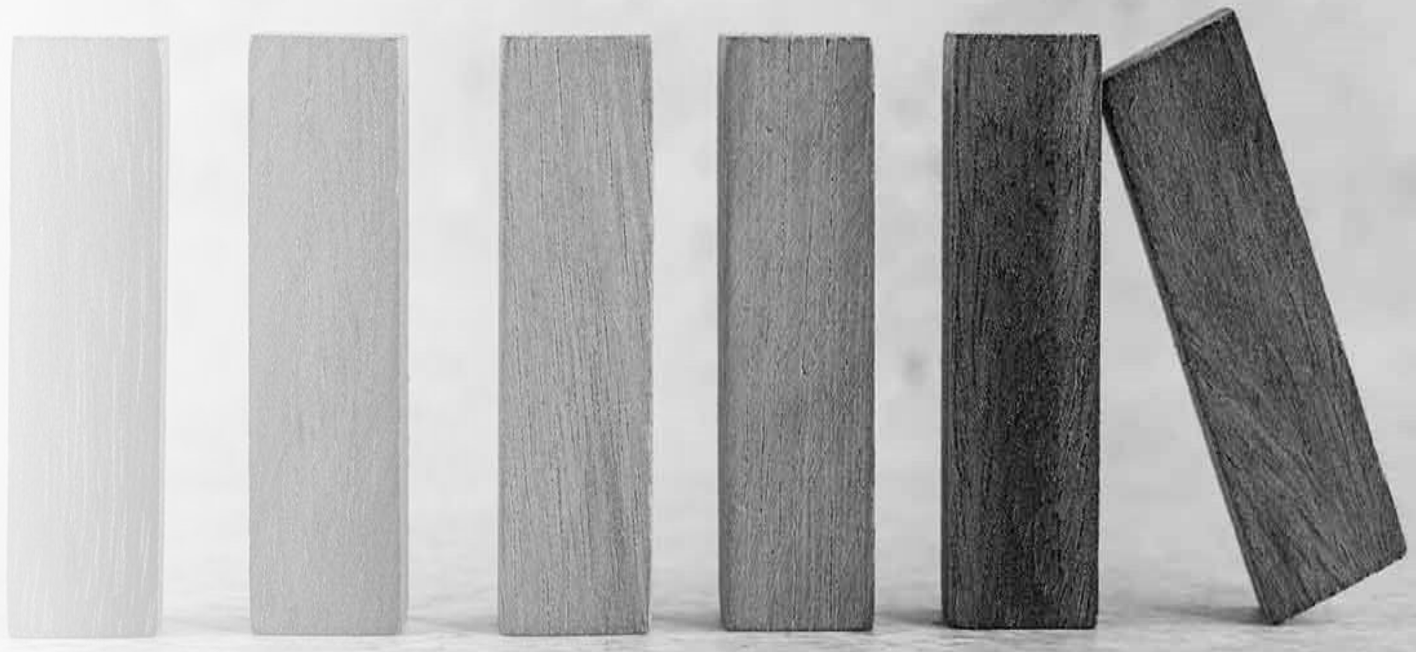
- At time t_0 an investor chooses to enter a commitment (long futures contract) to buy a security at $T > t_0$. How much should he be willing to pay for receiving the security at $t = T$?
- If he has reliable expectations on the future spot price $E_{t_0}\{S(T)\}$ then a possible forward price would be:

$$F(t_0, T) = E_{t_0}\{S(T)\}$$

- The problem is, no one knows what the future spot price will be - if one did futures contracts would not be necessary (and we would not be in this room).
- **In view of that forward prices need to be set by using absence of arbitrage arguments**

To go further

And bridge concepts in your mind.



Example 1 of forward contracts on assets that pay income

- Long 9 month forward contract on coupon paying bond.
  - The forward price is $910, the current bond price is $900.
  - The bond pays $40 coupon in four months time.
  - Rates: *R*(4m) = 3.0 %, *R*(9m) = 4.0 %.
- Strategy:
  - Borrow $900 and buy the bond and short the forward contract
  - The present value of the coupon is $PV(C) = 40e^{-0,03*\left(\frac{1}{3}\right)} = \$39.6$
  - Of the $900, $39.60 is borrowed at 3% and is repaid with the coupon
  - The remaining $900 – $39.60 = $860.40 is borrowed at 4% over nine months
  - The amount owed at the end of nine months is $860.4e^{0.04*\left(\frac{9}{12}\right)} = \$886.6$
  - $910 is received for the forward contract and a profit made:

    $910 \;- 886.6 = 23.4\ USD$

Example 2 of forward contracts on assets that pay income

- Long 9 month forward contract on coupon paying bond.
  - The forward price is \$870, the current bond price is \$900.
  - The bond pays \$40 coupon in four months time.
  - Rates: R(4m) = 3.0 %, R(9m) = 4.0 %.
- Strategy:
  - Short the bond and enter a long forward contract
  - Of the \$900 realized from shorting \$39.60 is invested at 3% for four months and used to pay the bond coupon
  - The remaining \$860.40 is invested at 4% for nine months and grows to \$886.60
  - Under the terms of the forward contract the bond is bought for \$870 and the following gain made: 886.60 – 870 = \$16.60

Forward contracts on assets that pay income

For your own understanding

Forward price = $910

Action now:

- Borrow $900: $39.60 for 4 months and $860.40 for 9 months
- Buy 1 unit of asset
- Enter a short position in a 9 month forward contract for $910
- *Action in 4 months*:
- Receive $40 coupon – use to pay first loan
- *Action in 9 months*:
- Sell asset for $910
- Use $886.60 to repay second loan with interest
- Profit realized: $23.40

Forward price = $870

Action now:

- Short one unit of asset to realize $900
- Invest $39.40 for 4 months and $860.40 for 9 months
- Enter long position in a 9 month forward contract for $870
- *Action in 4 months*:
- Receive $40 from 4 month investment
- Pay income of $40 on asset
- *Action in 9 months*:
- Receive $886.60 from 9 month investment
- Buy asset for $870 and close short position
- Profit realized: $16.60

Generalization

- If an asset provides an income with **present value of I** during the life of the contract then

$$F(t,T) = \big(S(t) - PV(I)\big)\exp\big(r(t,T)(T-t)\big)$$

where $r(t,T)$ is the risk free interest rate for the term $T-t$

- **Previous example**: $F(t,T) = (900 - 39{,}6)\exp(0{,}04 * 0{,}75) = \$886{,}6$

Generalization

- Consider an asset presently priced at $S(t)$ which pays dividends $D_1, D_2, \dots, D_T$ at the times $T_1, T_2, \dots, T_N$
  Then, the forward price for the asset is:

$$F(t,T) = \left(S(t) - \sum_{k=1}^{N} PV(D_k) \right) e^{R(t,T)(T-t)} \quad ; \quad T > T_N$$

- And, the present value of the forward price:

$$F^P(t,T) = e^{-R(t,T)(T-t)} F(t,T) = S(t) - \sum_{k=1}^{N} PV(D_k) \quad ; \quad T > T_N$$

Forward contracts on currencies

- Companies wanting to fix their risk-exposure to foreign currency fluctuations can enter a forward contract on the FX rates
- Let $X_{d,f}(t)$ be the FX spot rate between two currencies - meaning the **amount of domestic currency per unit of foreign currency**
- The higher value $X_{d,f}(t)$ takes the stronger is the foreign currency
- A domestic firm wants to fix its exposure to a foreign currency by entering a forward contract on the FX rate
- It can be shown that for there to be no risk-free arbitrage opportunity the forward rate, set today at t for maturity at T needs to be set as,

$$FX_{d,f}(t,T) = X_{d,f}(t)\exp\left(\left(R_d(t,T) - R_f(t,T)\right)(T-t)\right)$$

- Where $R_d(t,T)$ and $R_f(t,T)$ are respectively the domestic and foreign spot rates at t for the term T respectively.

Module 2

Options

2.1 Introduction

Options

- Options are examples of derivative securities that derive their value from the price evolution of an underlying security.
- There are essentially two different types of options – **call** options and **put** options

- *Call option* gives the right or obligation to *buy* specified quantity of some underlying security (asset) on or before specified date for a certain specified price
- *Put option* gives the right or obligation to *sell* specified quantity of some underlying security (asset) on or before specified date for a certain specified price

Options

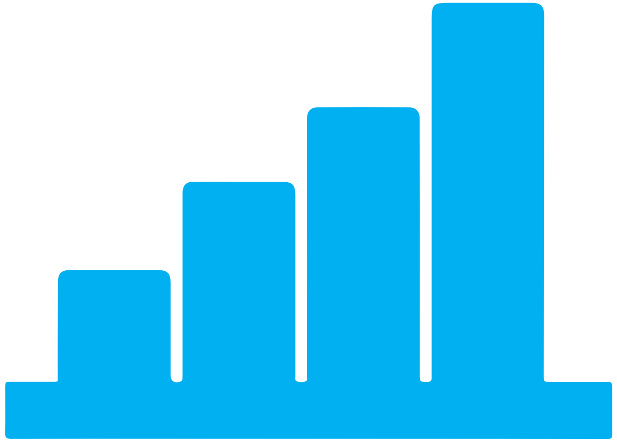
- ***Buying*** an option (the contract) gives the holder ***right*** to *buy* or *sell* some asset at a fixed price, on or before a fixed date.
- This ***right*** is priced as a derivative security and can be traded on the open market.

- ***Selling*** an option (the contract) means taking on an ***obligation*** to *buy* or *sell* an asset at fixed price before a fixed date
- This ***obligation*** is tradable and can be sold on the open market.

Parties to an option contract

- To each option contract there are two parties
  - *option holder or the long position* (has rights) - buys a right provided by an option - **pays a premium**
  - *option writer (issuer) or the short position* (has obligations) - sells an option right - **receives a premium**

- **Four different combinations are possible:**
  - long position in a call option – right to buy
  - long position in a put option – right to sell
  - short position in a call option – obligation to sell
  - short position in a put option – obligation to buy



The benefits of options

- Why trade in options rather than directly in the underlying?
  - Options require less initial capital – **you only pay a premium**
  - The downside risk can be **quantified** and **limited**
  - Better prospects for **high return on capital**
  - Decisions to invest **can be delayed**

Why buy a call option?

- To gain from rise in asset prices without investing in the assets themselves

- To bet on price rises with only limited possible loss

- For hedging purposes,
  - If short in an asset - hedge against upwards moves

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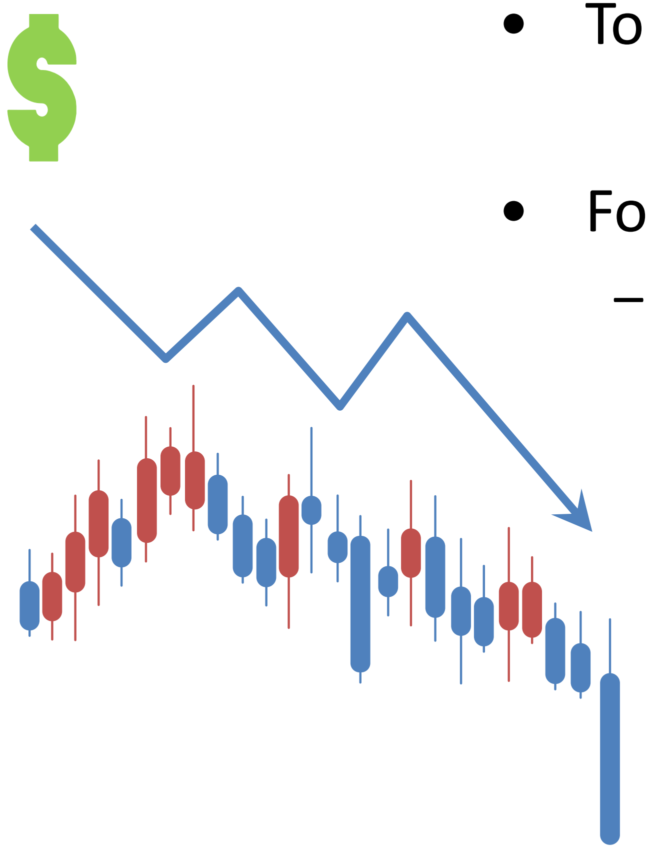
  - If short in an asset - hedge against upwards mo

Why buy a put option?

- To gain from decline in asset prices without investing in the assets themselves through a short position

- To bet on price losses with only limited possible loss

- For hedging purposes,
  - If long in an asset - hedge against downwards moves



Anti-Correlation Call/Put

Call Option

The call option will logically be positively correlated with the price of the underlying

Put Option

The put option will logically be negatively correlated with the price of the underlying

Call and Put options will be anti-correlated.

Some basic option concepts and notations

- *Exercise* or *strike* price – notation : K or X
  - the price at which the option can be exercised

- *Expiration* or *exercise* date or *maturity* – notation : T
  - the date on or before which the option has to be exercised

- *Volatility* – notation : σ
  - the standard deviation in the returns of the underlying asset

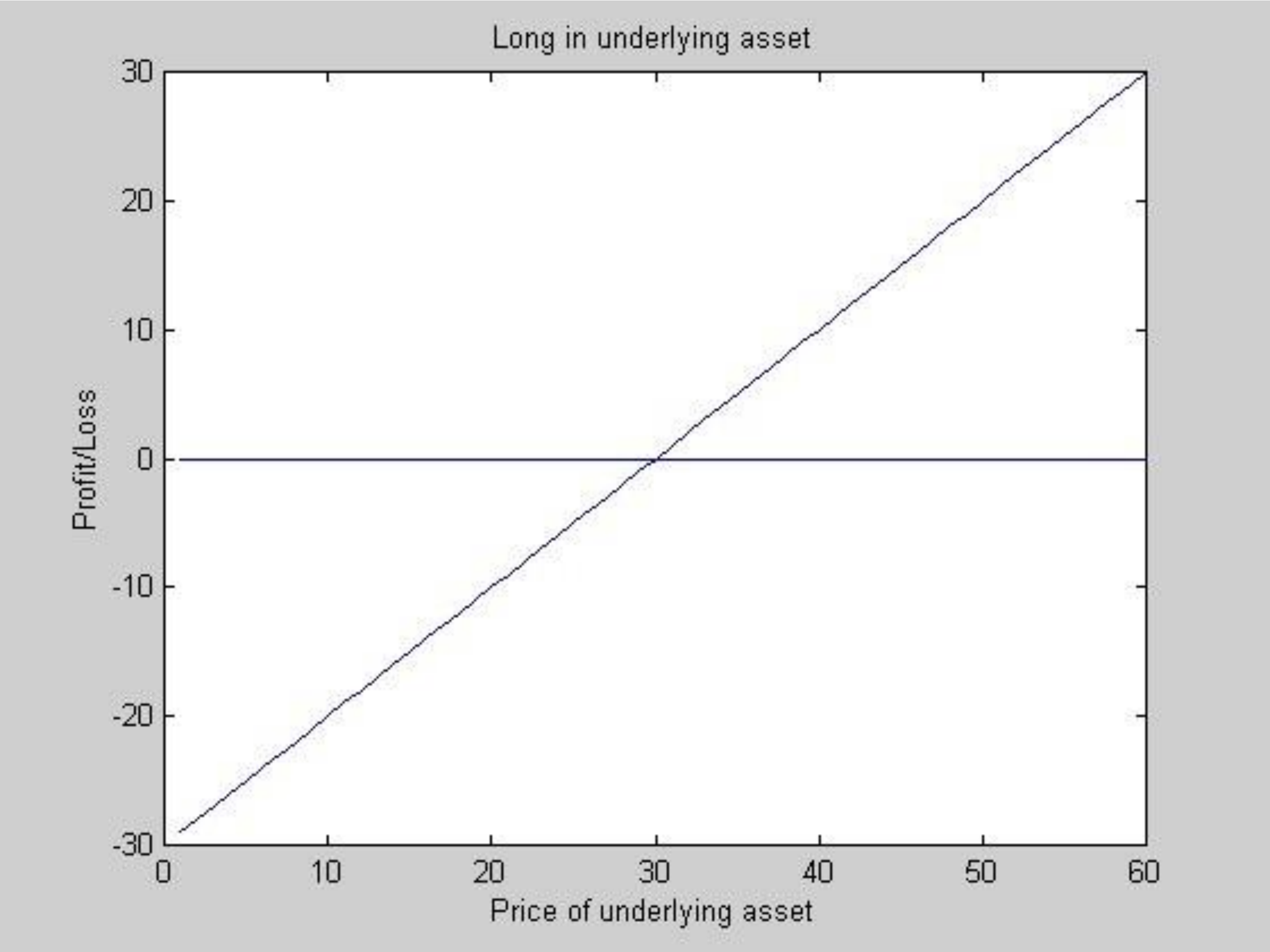
The Vanilla Option

Our focus now is only on the simplest options possible: A call and a put are named « plain vanilla options ».

It is named in analogy with the common vanilla ice cream flavour.

Vanilla ice cream became widely and cheaply available thanks to the artificial development of the « vanillin » flavour **<- It won't be in any exam.**

Profit on long position in the underlying



Features

Initial cost: $S_0 = 30$

Max profit: Unlimited

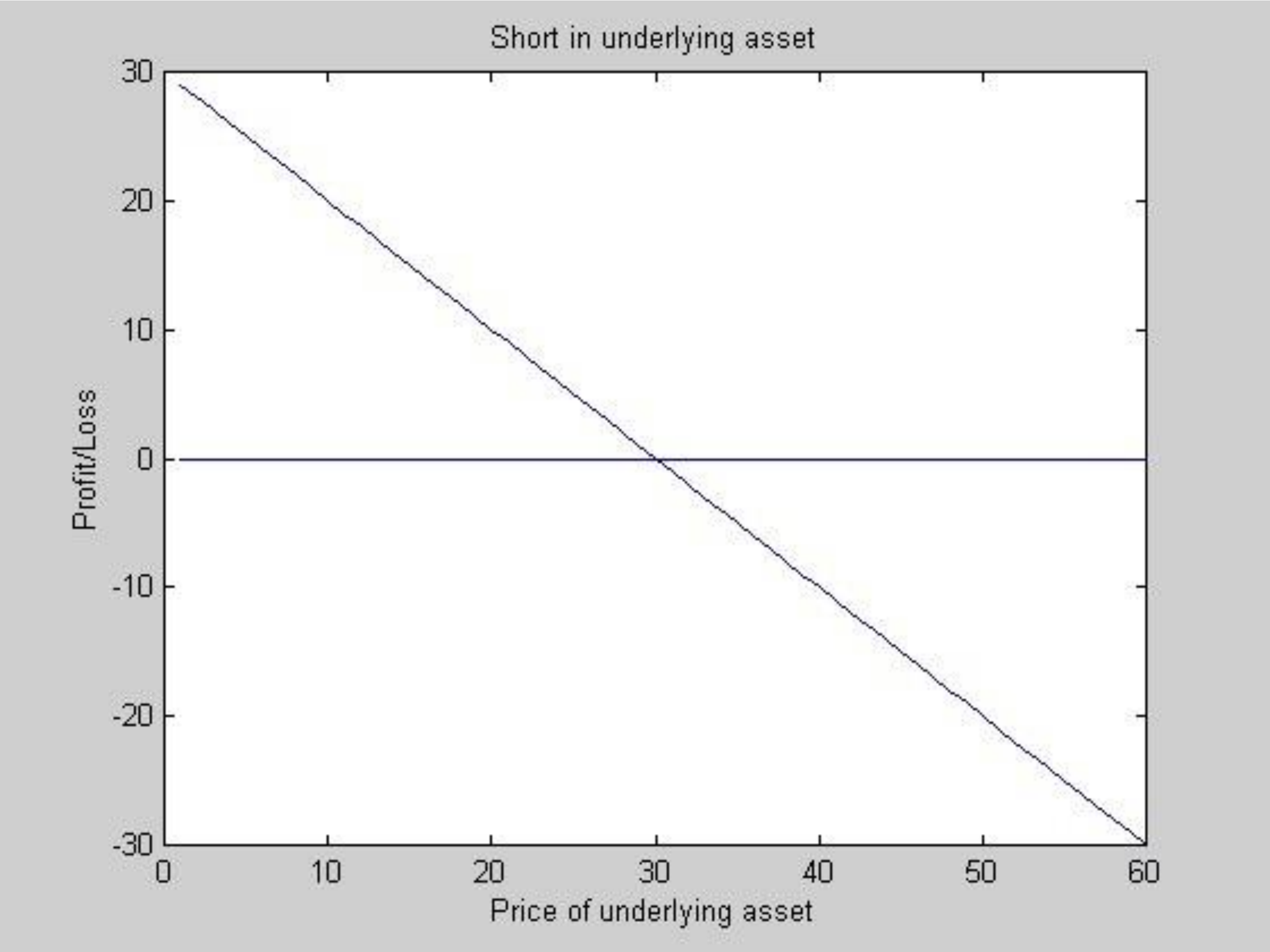
Max Loss: $L_M = -30$

Breakeven Price: $S_0 = 30$

Market outlook: Bullish

Risk Posture: Aggressive

Profit on short position in the underlying



Features

Initial credit: $S_0 = 30$

Max profit: $P_M = 30$

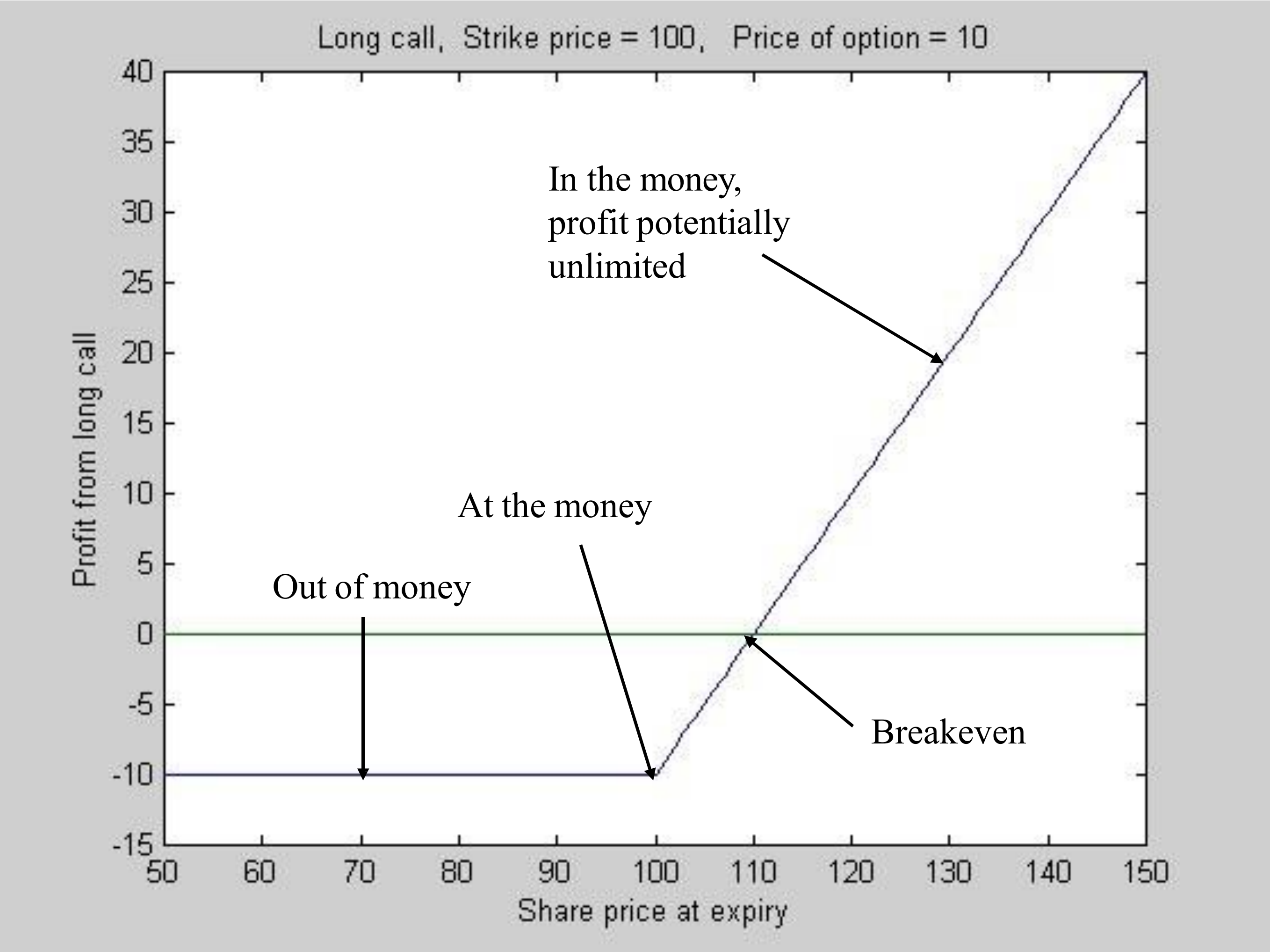
Max Loss: Unlimited

Breakeven Price: $S_0 = 30$

Market outlook: Bearish

Risk Posture: Aggressive

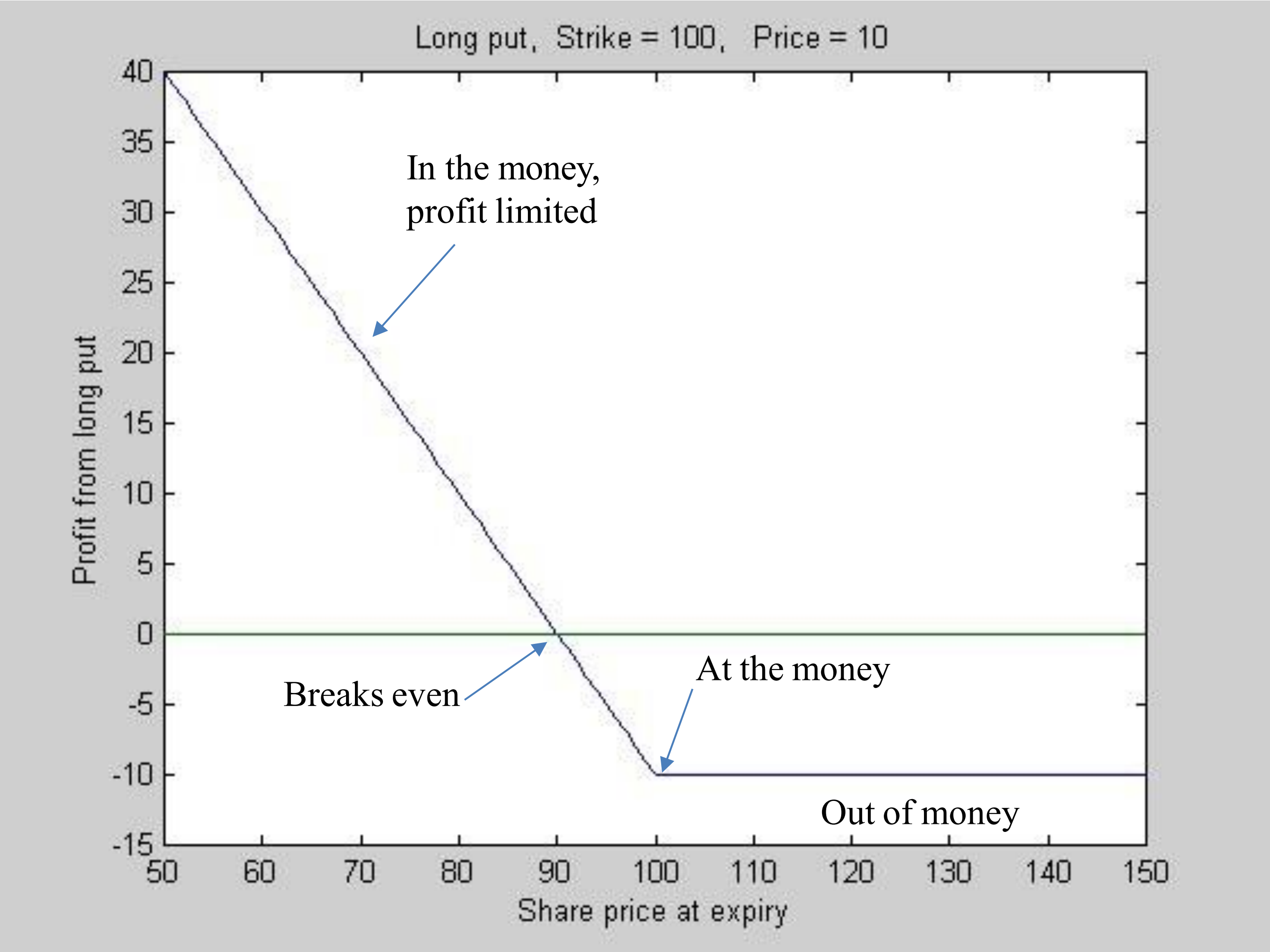
Option profit profile - long call



Features

Initial cost:	$C_0 = 10$
Max profit:	*Unlimited*
Max Loss:	$L_M = -10$
Breakeven Price:	$K + C_0 = S$
Market outlook:	Bullish
Risk Posture:	Defensive

Option profit profile - long put



Features

Initial cost: P=10

Max profit: K-P

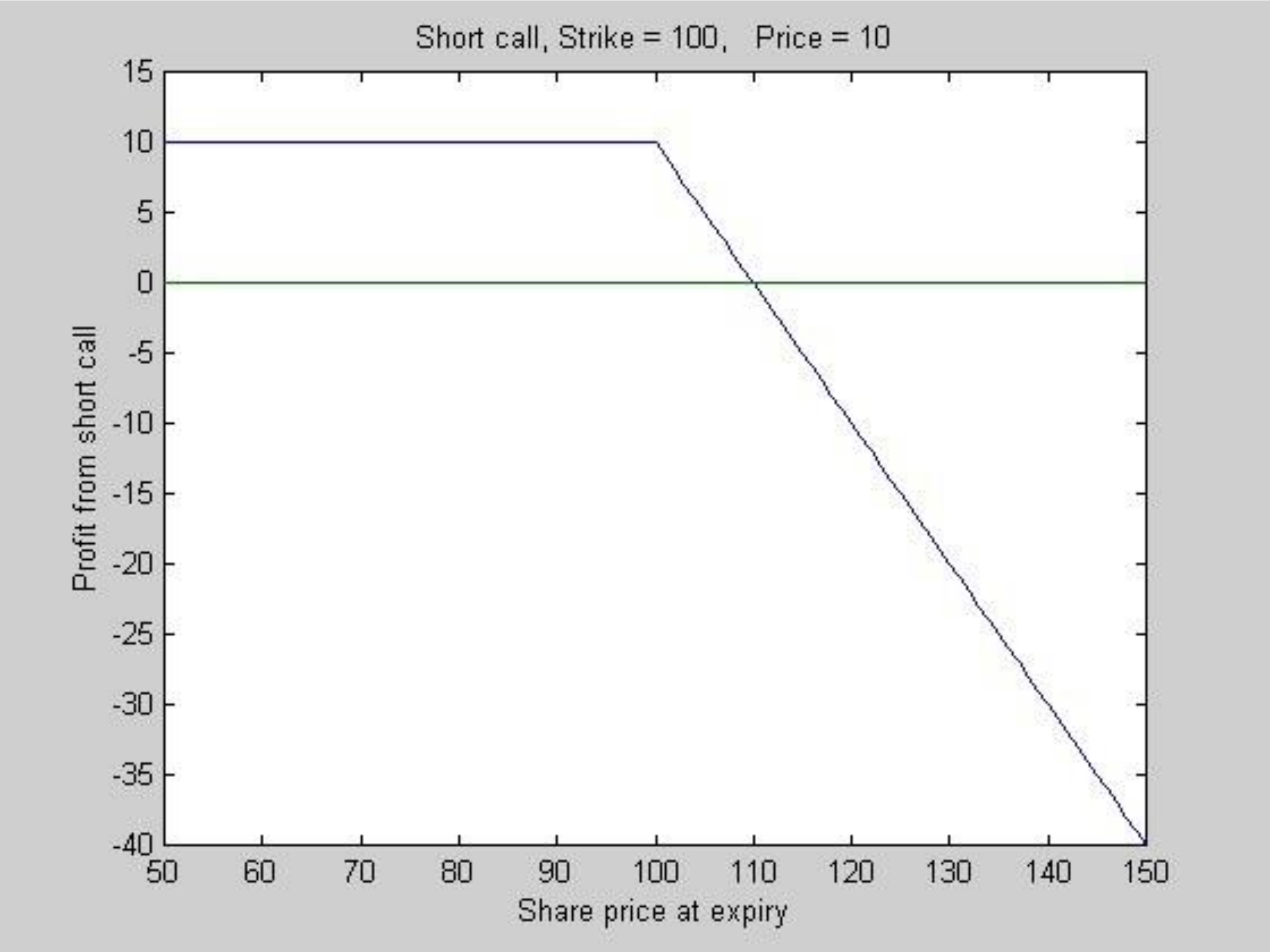
Max Loss: $L_M = P$

Breakeven Price: S=K-P

Market outlook: Bearish

Risk Posture: Defensive

Option profit profile - short call



Features

Initial profit: $C = 10$

Max profit: $C = 10$

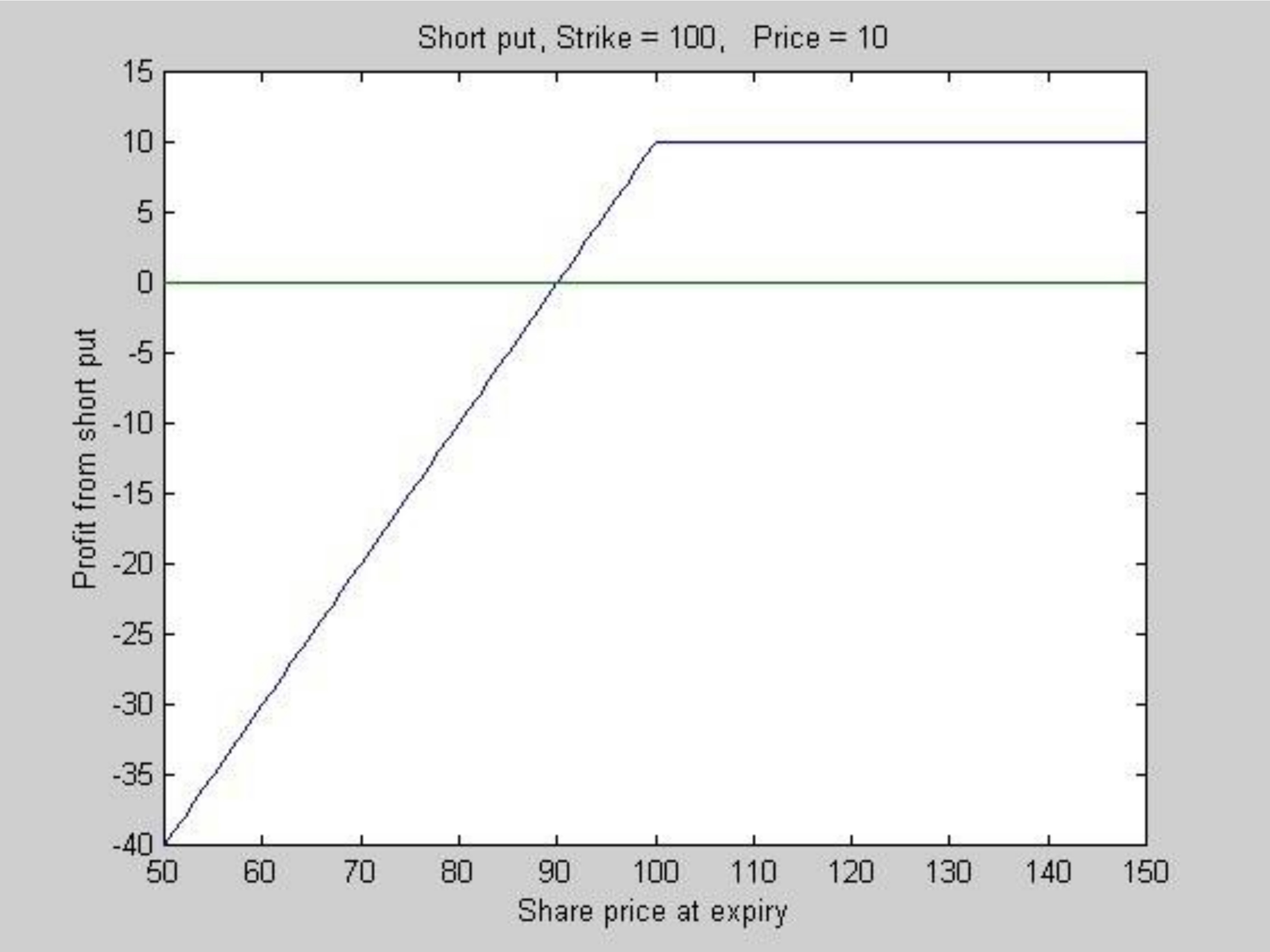
Max Loss: *Unlimited*

Breakeven Price: $S = X + C$

Market outlook: Bearish

Risk Posture: Aggressive

Option profit profile - short put



Features

Initial profit: $P = 10$

Max profit: $P = 10$

Max Loss: $L_M = X - P$

Breakeven Price: $S = X - P$

Market outlook: Bullish

Risk Posture: Aggressive

Intrinsic option values

- The mathematical expressions for the intrinsic values i.e. option values at expiry:

$$c_L(T) + c_S(T) = 0 \quad as \quad \begin{cases} c_L(T) = \max(S_T - K, 0) \\ c_S(T) = \min(X - S_T, 0) \end{cases}$$

$$p_L(T) + p_S(T) = 0 \quad as \quad \begin{cases} p_L(T) = \max(K - S_T, 0) \\ p_S(T) = \min(S_T - K, 0) \end{cases}$$

- Options are therefore a **zero-sum game**

Price Relationships

- We need to make distinction between American and European options and then between assets that pay dividends and assets that don’t
- Initially our approach will be formal based on arbitrage arguments followed by numerical examples
- Apart from trivial situations we will use portfolio approach, i.e. we construct two different portfolios which at expiry need to satisfy certain price relationships. Therefore, these price relationships must also be satisfied at all previous times

European options

We start by stating two trivial pricing relationships that must hold for there not to be an arbitrage opportunity.

- **For European calls**: $c(t,T) \leq S_t$

If this inequality is not satisfied investors could simply buy the underlying and sell calls on it – a strategy that would lead to risk free profits

- **For European puts**: $p(t,T) \leq K * \exp\big(-r(T-t)\big)$

If this inequality is not satisfies investors could sell European options and invest the premium at the risk free rate

Basic inequalities – calls

For your own understanding

- Next we establish by arbitrage arguments that the following price inequality needs to be satisfied for there not to be arbitrage opportunities:

$$c(t,T) \geq S(t) - K * exp\big(-r(T-t)\big)$$

- To establish this we construct the two portfolios,
  a) European call priced at $c(t,T)$ plus the amount of cash $K\ \exp(-r(T-t))$
  b) Long position in one unit of the underlying
- At maturity of the call the two portfolios have the following values

$$a)\ c(T,T,) + K = \max(S(T) - K, 0) + K = \max(S(T), K)$$
$$b)\ S(T)$$

Therefore, portfolio a) is worth more at expiry of the call.

Basic inequalities – calls

For your own understanding

- To avoid arbitrage, portfolio a) must be more valuable at all previous times:

$$c(t,T) + K * \exp\big(-r(T-t)\big) \geq S(t)$$

- Which implies:

$$c(t,T) \geq S(t,T) - K * \exp\big(-r(T-t)\big)$$

Basic inequalities – puts

For your own understanding

- The price of European puts needs to satisfy,

$$p(t,T) \geq K * \exp\big(-r(T-t)\big) - S(t)$$

- To establish this we construct the two portfolios,
  a) European put priced at $p(t,T)$ plus one unit of the underlying $S(t)$
  b) Cash position $K\exp(-r(T-t))$
- At expiry of the option:

$$a)\ p(T,T) + S(T) = \max(K - S(T), 0) + S(T) = \max\big(K, S(T)\big)$$

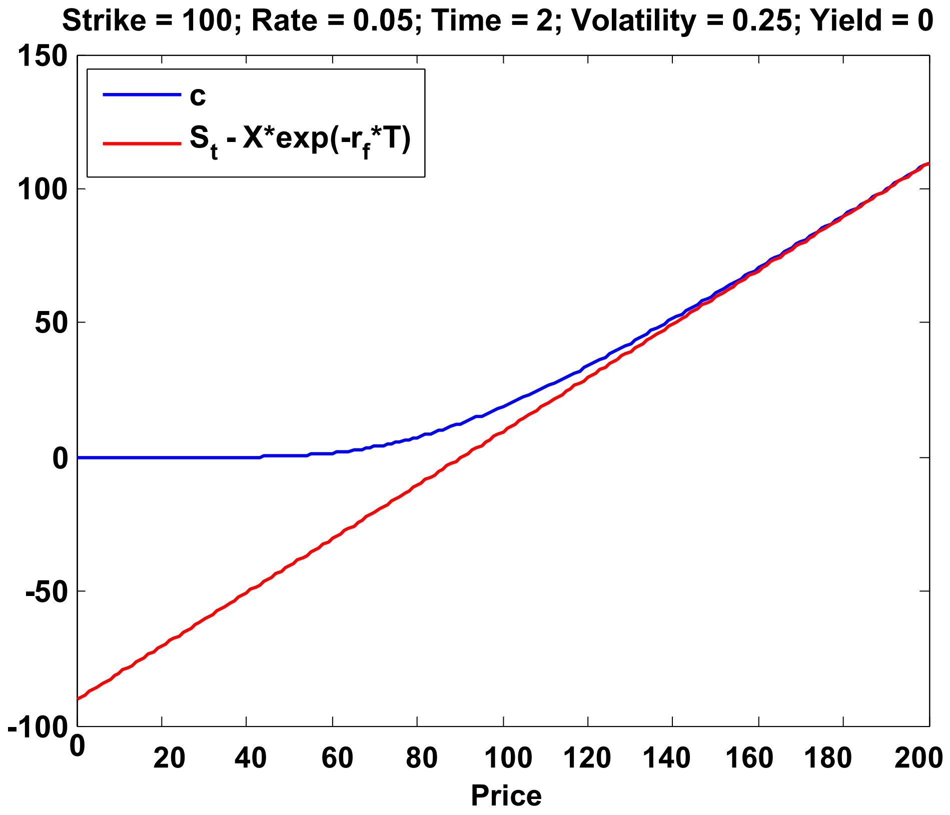
$$b)\ K$$

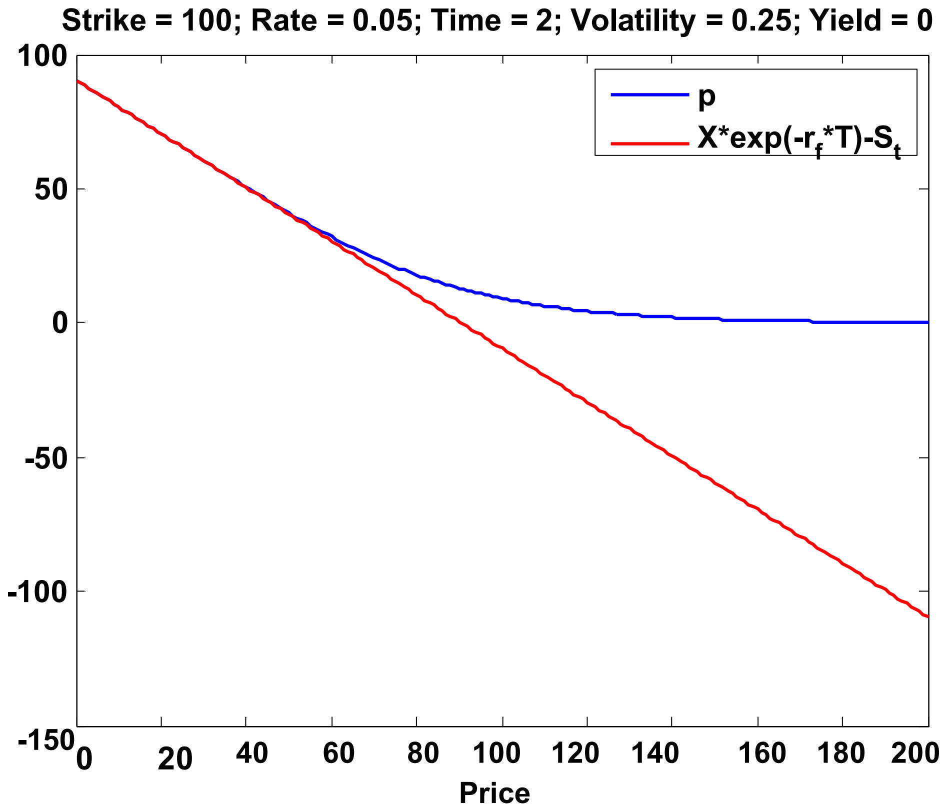
Therefore, portfolio a) is more valuable at expiry and at all previous times.

Graphical display of inequalities

For your own understanding

The figures below describe the two previous inequalities.





Put call parity

- We create two portfolios:

$a)\ c(t,T) + K * \exp\big(-r(T-t)\big)$

$b)\ p(t,T) + S(t)$

- In the limit $t \rightarrow T$,

$a)\ \max(S(T) - K, 0) + K = \max(S(T), K)$

$b)\ \max(K - S(T), 0) + S(T) = \max(S(T), K)$

- As the two portfolios have the same value at maturity they must have the same value at all previous times for there not to be an arbitrage opportunity. Therefore:

$$c(t,T) + K * \exp\big(-r(T-t)\big) = p(t,T) + S(t)$$

  Which is called the put call parity

Put call parity

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$a)\ c(t,T) + K * \exp\left(-r(T-t)\right)$

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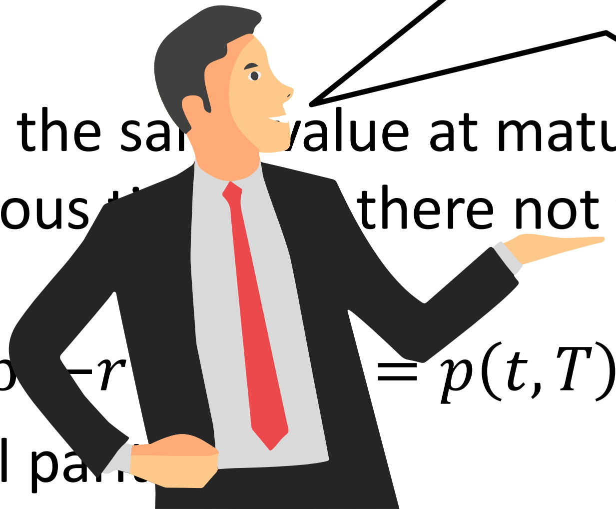
- As the two portfolios have the sa value at maturity they must have the same value at all previous there not to be an arbitrage opportunity. Therefore:

$$c(t,T) + K * \exp -r \quad = p(t,T) + S(t)$$

Which is called the put call pa



Option profile versus Black-Scholes price

- Prior to expiry the option value is larger than the intrinsic value of the option
- This is due to the fact that we don’t know where the value of the underlying will end up at maturity.

- The value, in excess of the intrinsic value, is therefore “opportunity value” – it depends amongst other things on:
  - **Time left to maturity**
  - **Volatility of the underlying asset**

$$\textit{Option Value} = \textit{Intrinsic Value} + \textit{Time Value}$$

Put call parity

- We have a call option and a put option on an underlying asset S_t. Both expire at the same date T and both have the same strike K
- Then, at any time $t < T$ the following relationship must hold:
$$C_t - P_t = S_t - PV(K)$$
- **Proof**: We show that at maturity the left hand side and the right hand side are portfolios with the same value. Consequently, for there not to be an arbitrage opportunity, they must have the same value at all previous times
- The proof goes essentially as follows:
  - If at maturity $S_T \leq K$ then $C_T = 0$ and $P_T = K - S_T$
  - If at maturity $S_T > K$ then $C_T = S_T - K$ and $P_T = 0$

This is summarized in the Table below (next page)

Put call parity

Table

	ST ≤ X	ST > X
c_T	0	ST - X
p_T	X - ST	0
$c_T - p_T$	ST - X	ST - X

- The first row shows the two possible relationships between the price of the underlying and the strike at maturity;
- The second row gives the associated values for the price of the call at maturity;
- The third row gives the associated values for the price of the put at maturity;
- The forth row is the second row minus the third row;

<u>Both outcomes in first row lead to the same result in the fourth row</u>

Put call parity

- The put-call parity demonstrates that buying a call (long call position) and selling a put (short put option) produces the same payoff as buying the underlying asset (long in the underlying) by taking a loan (short bond position) – we write this as:

$$Long\,Call + Short\,Put = Long\,Asset + Short\,Bond$$

- The nominal value of the bond -the amount of borrowing- is equal to the present value of the strike value K

- **The equation above shows how to create a leveraged long position in the long asset** – i.e. how to buy stock on margin**.**

Put call parity

- The previous equation can be arranged in various ways
  Different long positions as portfolios:

$$Long\,Asset = Long\,Call + Short\,Put + Long\,Bond$$

$$Long\,Call = Long\,Asset + Long\,Put + Short\,Bond$$

$$Long\,Put = Long\,Call + Short\,Asset + Long\,Bond$$

Put call parity

- The put call parity also allows us to write a Long Bond position i.e. riskless lending as a portfolio,

$$Long\ Bond = Long\ Asset + Short\ \ Call + Long\ Put$$

- This shows that a riskless hedged position can be created by: buying the underlying asset; sell a call on this asset and buy a put on the same asset, with same maturity and the same strike.

  This position is riskless until the expiration date of the options

- **Example**: A trader who has sold a call (short call) can create a riskless position (conversion) by buying the underlying asset and a put on it with same strike and maturity as the call he sold.

Put call parity for dividend paying stock

- Today is time t. Stock is priced at S_t and pays dividends of D at time t_d. If we have a call option priced at c_t and a put option priced at p_t, both with the same strike K and maturity T, $t < t_d < T$, then the following equality must hold:

$$c_t - p_t = S_t - De^{-r(t_d - t)} - Ke^{-r(T-t)}$$

- **Proof**. At maturity the left hand side will be:

$$c_t - p_t = \max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K$$

- At maturity the right hand side will be assuming that the dividend payment is reinvested at the risk free rate:

$$S_T - De^{-r(t_d - T)} - K + De^{r(T - t_d)} = S_T - K$$

Put call parity for dividend paying stock

- Today is time t. Stock is priced at S_t and pays dividends of D_i at time $t_i, i=1,2,\ldots,n$. If we have a call option priced at c_t and a put option priced at p_t on the stock, both with the same strike K and maturity T, where $t<t_1<t_2 \ldots t_n<T$, then the following equality must hold:

$$c_t - p_t = S_t - \sum_{i=1}^{n} D_i e^{-r(t_i - t)} - K e^{-r(T-t)}$$

- **Proof**. At maturity the left hand side will be:

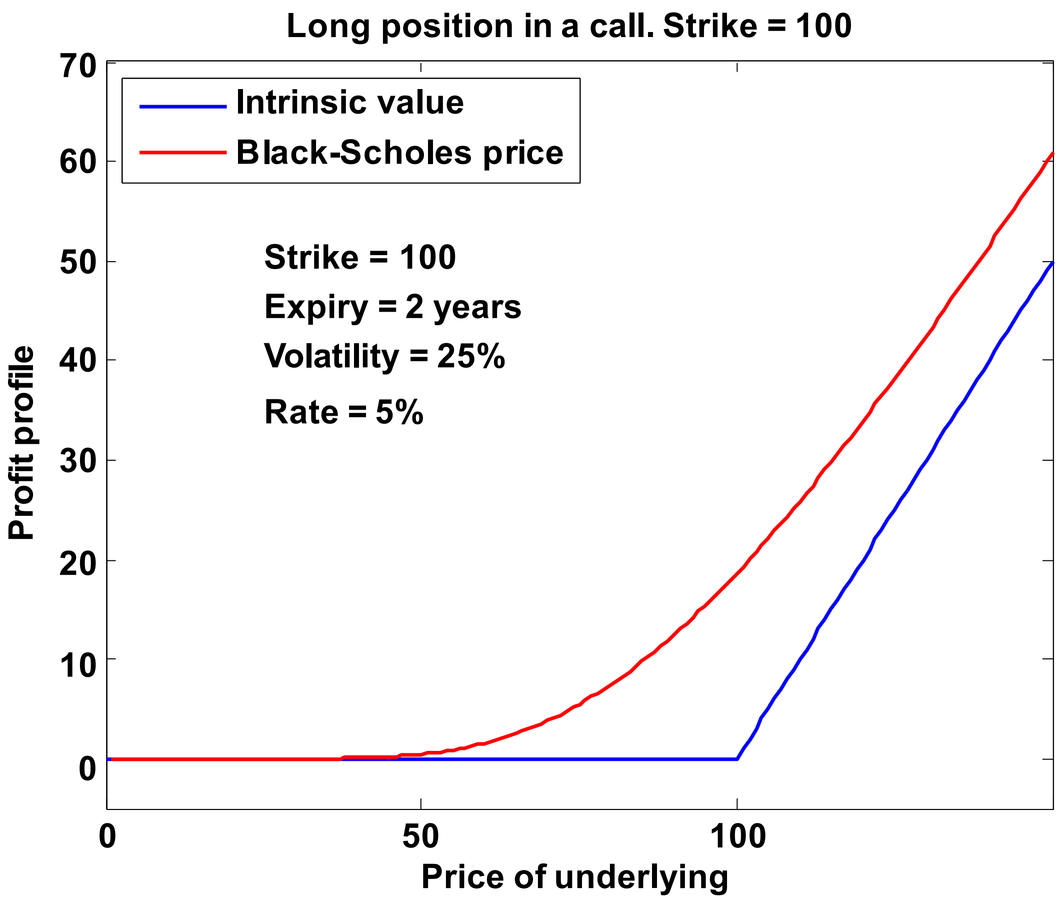
$$c_T - p_T = \max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K$$

- At maturity the right hand side will be assuming that the dividend payments are reinvested at the risk free rate

$$S_T - \sum_{i=1}^{n} D_i e^{-r(t_i - T)} - K + \sum_{i=1}^{n} D_i e^{r(T - t_i)} = S_T - K$$

Option profile versus Black-Scholes price

Intrinsic value at expiry and the Black-Scholes price



Different investment scenarios (1/5)

- **Selling a naked call** – sell a call on an asset I don't own
- At time t sell the call at strike K and receive the premium C_t
- At time T the value of the call option can be:

$$c_T = \max(S_T - K, 0)$$

- The value of the naked call position at time T (the profit) is:

$$V_T = c_t - c_T = c_t - \max(S_T - K, 0)$$
$$= \begin{cases} C_t - (S_T - K) & if\ S_T \geq K \\ C_t & if\ S_T < K \end{cases}$$

- The break-even condition is:

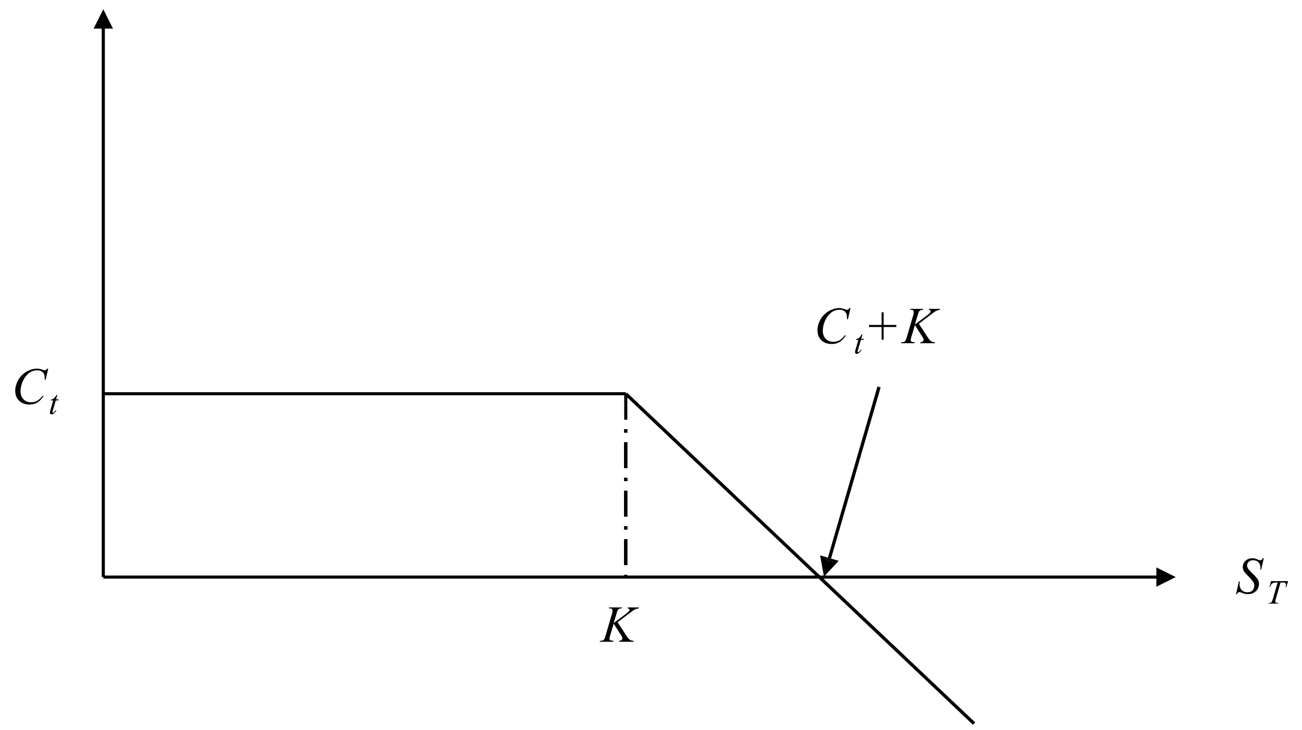
$$V_T = 0 \Rightarrow S_T = K + C_t$$

- There is loss in this investment if:

$$S_T > K + C_t$$

Different investment scenarios (1/5)

- **Profit from selling a naked call.**



Different investment scenarios (2/5)

- **Selling a covered call.** The investor buys the asset for S_t (long asset) and simultaneously sells a call option for C_t (short call) on the asset for strike $K > S_t$ (i.e. the option is out-of-the money at time t)
- At time t buy the asset for S_t and sell a call for C_t , value of portfolio,
$$V_t = S_t - c_t$$
- The cost of putting it together:
$$Cost_t = -V_t = -(S_t - c_t)$$
- At time T (the time the option expires) the value of this portfolio is:

$$V_T = S_T - C_T = S_T - \max(S_T - K, 0) = \begin{cases} K \quad if \;\; S_T \geq K \\ S_T \;\; if S_T < K \end{cases}$$

Different investment scenarios (2/5)

- **Selling a covered call – cont.** The change in the value of the portfolio (profit) is:

$$\Delta V[t,T] = V_T - V_t = S_T - c_T - (S_t - c_t)$$

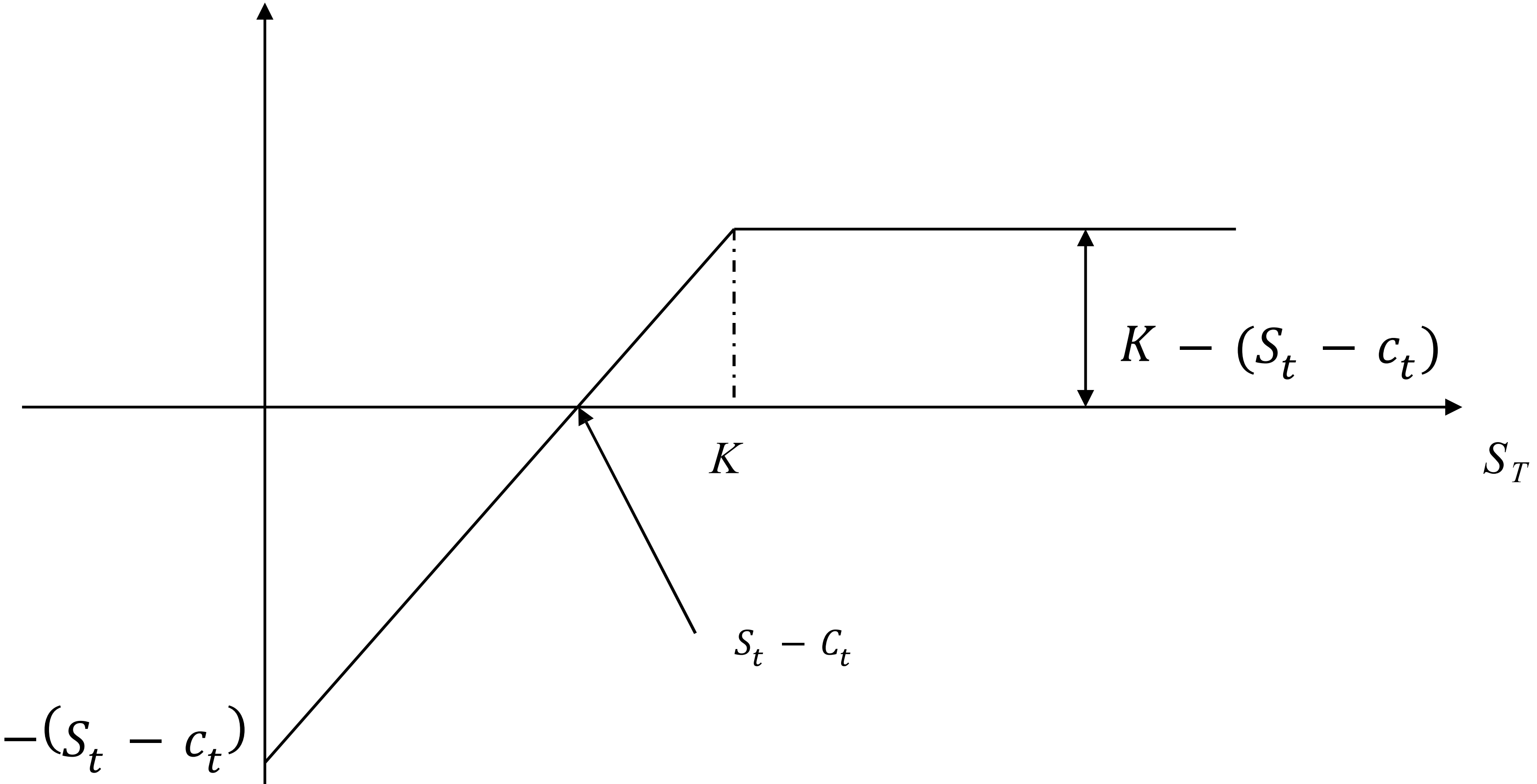
- Or explicitly:

$$\Delta V[t,T] = S_T - \max(S_T - K, 0) - (S_t - c_t) = \begin{cases} K - (S_t - c_t) & if \quad S_T \geq K \\ S_T - (S_t - c_t) & if \quad S_T < K \end{cases}$$

- Some observations:
  - Maximum profit of $K - (S_t - C_t)$ if $S_T \geq K$
  - Break even profit condition if $\Delta V[t,T] = 0 \rightarrow S_T = S_t - c_t$
  - Loss if $S_T < S_t - C_t$
  - Maximum loss of - $(S_t - C_t)$ when $S_T = 0$

Different investment scenarios (2/5)

- **Selling a covered call – cont.** The results from previous page give rise to the following profit-profile,



Different investment scenarios (3/5)

- **Uncovered put** - sell a put on an asset I don't have a short position in
- At time t sell the put at strike K and receive the premium p_t
- At time T the value of the put option can be,

$$p_t = \max(K - S_T, 0)$$

- The value of the naked put position at time T (the profit) is:

$$V_T = P_t - P_T = P_t - \max(K - S_T, 0) = \begin{cases} P_t & if\ S_T \geq K \\ P_t - (K - S_T) & if\ S_T < K \end{cases}$$

- The break-even condition is:

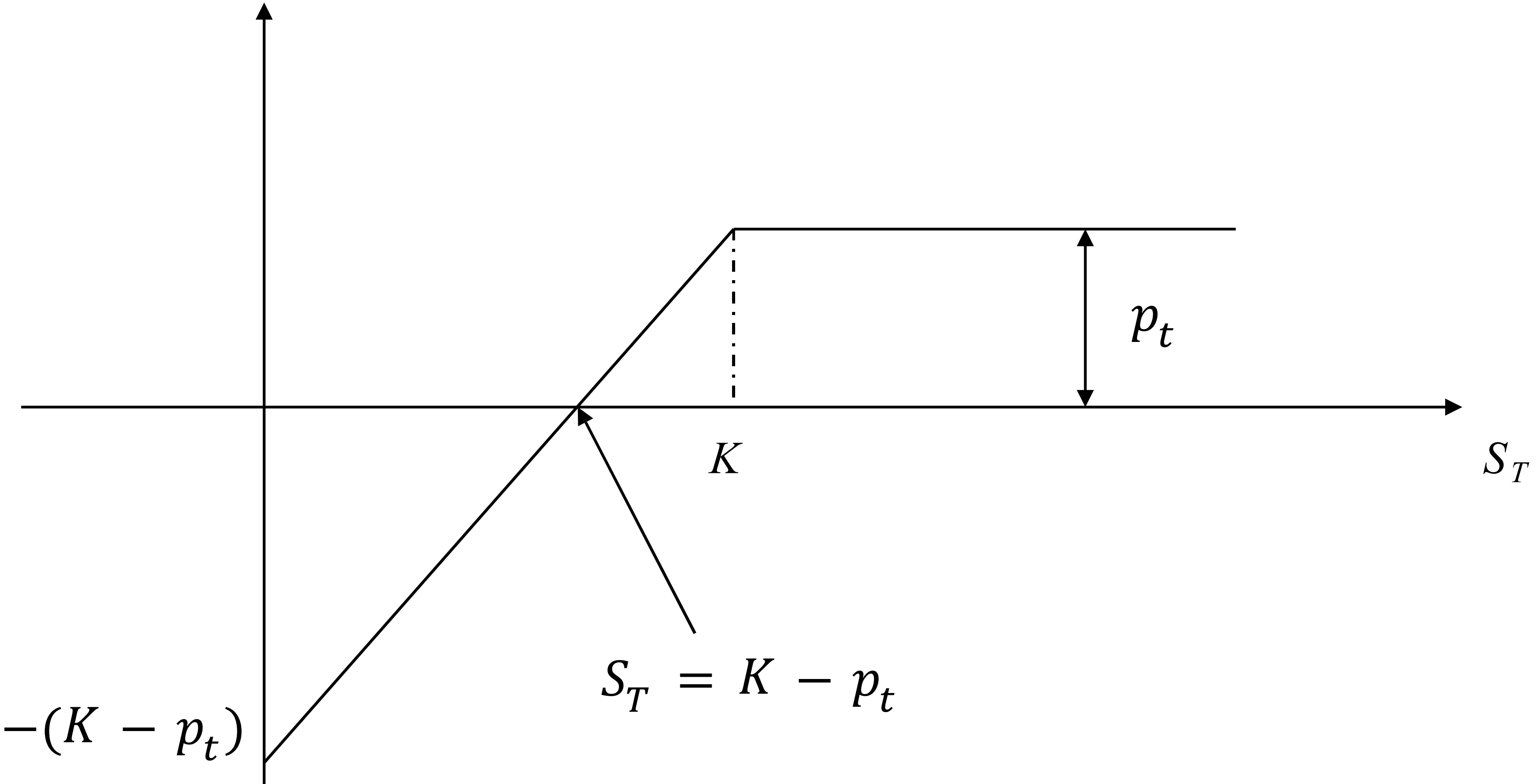
$$V_T = 0 \Rightarrow S_T = K - p_t$$

- There is loss in this investment if:

$$S_T < K - p_t$$

Different investment scenarios (3/5)

- **Uncovered put –** cont. previous results lead to the following profit profile



Different investment scenarios (4/5)

- **Covered put** – sell a put with a short position in the underlying asset
- At time t take a short position in the underlying asset priced at S_t and sell a strike K put for p_t. Value of portfolio (two short positions):

$$V_t = -p_t - S_t$$

- At time T – the maturity of the put option the value of the portfolio is

$$V_T = -p_T - S_T$$

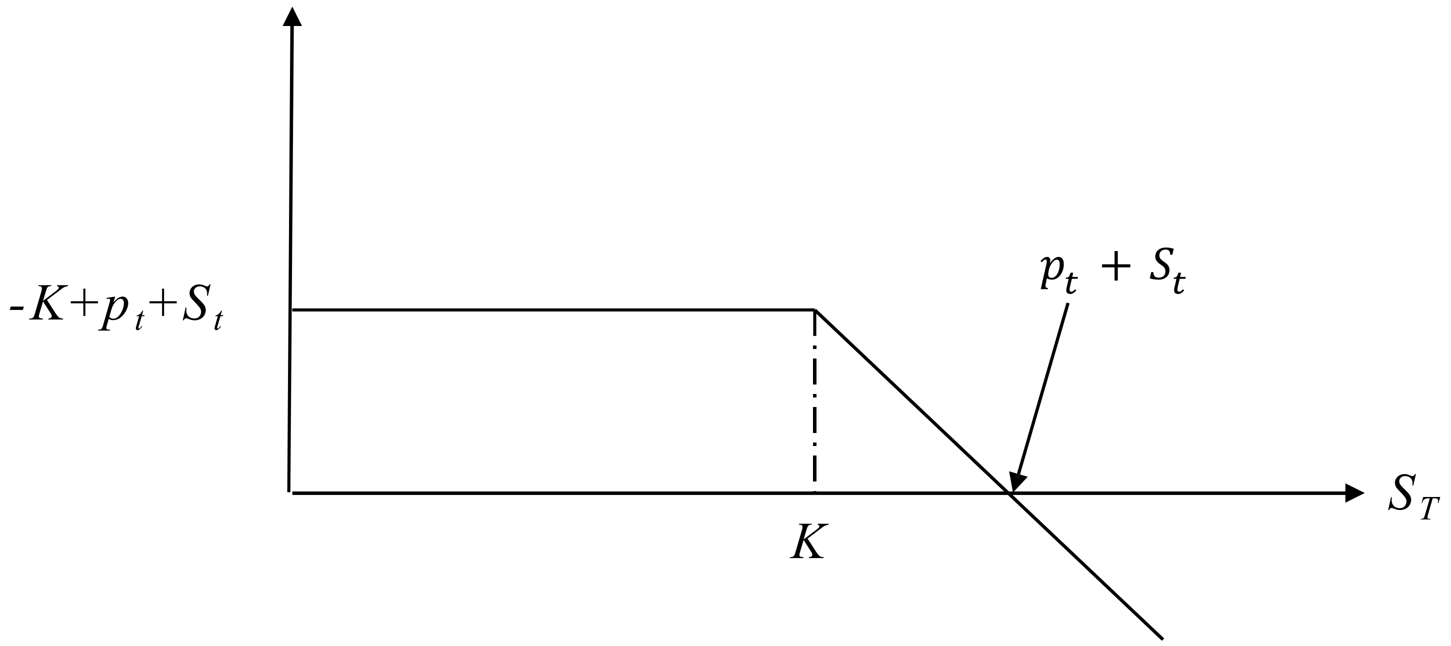
- The profit made:

$$\Delta \mathrm{V}[t,T] = V_T - V_t = -(P_T + S_T) + (P_t + S_t)$$

$$= -\max(K - S_T) - S_T + P_t + S_t = \begin{cases} -K + p_t + S_t & if\ S_T \leq K \\ -S_T + p_t + S_t & if\ S_T > K \end{cases}$$

Different investment scenarios (4/5)

- **Covered put – cont**.



- Here losses occur if the price of the underlying increases beyond the value $P_t + S_t$ as then it becomes expensive to close the short position in the underlying

Different investment scenarios (5/5)

- **Protective put** – At time t take a long position in the underlying asset priced at S_t and buy a strike K put for P_t.

- Value of portfolio (two long positions):

$$V_t = S_t + P_t$$

- At time T the value of the portfolio is,

$$V_T = S_T + P_T$$

- Profit made:

$$\Delta V[t,T] = V_T - V_t = (P_T + S_T) - (P_t + S_t)$$

$$= \max(K - S_T, 0) + S_T - (P_t + S_t) = \begin{cases} K - P_t - S_t & if\ \ S_T \leq K \\ S_T - P_t - S_t & if\ S_T > K \end{cases}$$

Different investment scenarios

- **Protective put**