2 - The No-arbitrage Assumption

NB: Here we deviate from the textbook and formalize no-arbitrage pricing.

The No-arbitrage Assumption

Trading Strategy: A set of financial transactions, .e.g, going long or short a financial instrument, investing/borrowing cash, etc.

Assumption: (No Arbitrage)
There are no trading strategies in the market model that create a risk-free (riskless) profit.

We will consider financial instruments as cash flow streams of the following form:
cash flow amounts $(P, a_1, a_2, \ldots, a_n)$, occurring at times $(0, t_1, t_2, \ldots, t_n)$, with P the price of the instrument, $a \in \mathbb{R}$ the cash flows that the instrument creates, and $0 < t_1 < t_2 < \cdots < t_n = T < \infty$ with T the expiry or maturity of the instrument.

Notes:

- *Arbitrage:* a risk-free profit.
- True arbitrages are rare and for the purposes of analysis and modelling we often assume no arbitrage opportunities exist. The existence of well developed markets justifies this no arbitrage assumption: if obvious arbitrage opportunities existed, that is all the market participants would do.
- The no-arbitrage assumption has profound consequences and **forms the foundation of modern mathematical finance**. With it, we will be able to show that **asset prices must satisfy certain constraints** and **price complex derivative securities**.
- Arbitrage Strategy: A set of financial transactions (trading strategy) designed to exploit an arbitrage opportunity.
- Later in this chapter we will learn how to construct arbitrage strategies, when prices violate the no-arbitrage criterion.

The Law of One Price

Now consider two financial instruments with the same random future cash flows (at the same future times), but different initial prices. The corresponding cash flow streams are $(P_1, a_1, a_2, \ldots, a_n)$ for the first instrument and $(P_2, a_1, a_2, \ldots, a_n)$, and $P_2 > P_1$.

In a frictionless financial market, we purchase the first instrument and sell the second, we can generate the final cash flows

$$(-P_1 + P_2, a_1 - a_1, a_2 - a_2, \ldots, a_n - a_n) = (P_2 - P_1, 0, 0, \ldots, 0).$$

This creates an immediate risk-free profit of $P_2 - P_1$, and requires no initial investment since the short sale of the second instrument generates enough money to purchase the first. This is an arbitrage strategy and is excluded by our no-arbitrage assumption. This result is formalized into the *Law of One Price*.

Result: (The Law of One Price)
Two financial instruments that generate the same future cash flows at the same times, command the same initial price.

Pricing by Replication

It should be clear that any trading strategy (any series of financial transactions in the market model), can be considered as a *synthetic financial instrument*. If there is a instrument in our model with an unknown price, the idea of *pricing by replication* is to create a trading strategy with the same future cash flows. Then, by the Law of One Price, the initial cost of the trading strategy dictates the cost of the financial instrument.

Special Case: Deterministic Cashflows

In the presence of an ideal bank, the no-arbitrage assumption requires that the cost of any trading strategy that generates a series of deterministic cash flow be the net present value of those future cash flows.

This is because an ideal bank can be used to *generate any deterministic cash flow stream at the cost of its net present value*. For any deterministic cash flow stream, the replicating trading strategy consists of borrowing and lending from the ideal bank.

Corollary: (Deterministic Cash Flow Streams)
The net present value of any trading strategy that generates purely deterministic cash flows, when including all outlays, must be zero.

In our previous example, the final cashflow stream, $(P_2 - P_1, 0, 0, \ldots, 0)$ is deterministic, so its NPV must be zero, which provides $P_2 = P_1$.

You can also interpret pricing via replication as transforming a stochastic cash flow into a deterministic cash flow: by removing any uncertainty about the future cash flows, we can then use the NPV criterion to price the instrument.

Market Frictions

In the absence of market frictions a trading strategy can be executed long and short at the same price. This enforces the strict equality in the Law of One Price: if the replicating strategy cost less than the instrument, you could short the replicating strategy and long the instrument to make a risk-less profit.

However, in the presence of market frictions, it is only required that any strategy, when accounting for costs, generate a strictly non-positive total net present value. This is because the strategy cannot be reversed (at the same price) to generate a strictly positive arbitrage.