T-503-AFLE: Class Test 1 - Solution

- **Duration:** 1.5 hours
- **Total Marks:** 20 marks

1 - Example 1.2.9 (Expected Profit of a Put) (2 marks)

The price of an asset will either rise by 25% or fall by 40% in 1 year, with equal probability. A European put option on this asset matures after 1 year.
Assume the following:

- Price of the asset today: 100
- Strike price of the put option: 130
- Put option premium: 7
- Annual effective risk free rate: 3%

Calculate the expected profit of the put option.

Solution:
The 1-year payoff of the put option will be either $(130 - 100 \times 1.25)_+ = 5$ or $(130 - 100 \times 0.6)_+ = 70$ with equal probability. The expected profit is $(5 + 70)/2 - 7(1.03) = 30.29$.

2 - Problem 1.4.5 (Comparing the Profits of Three Puts) (3 marks)

You are given the following premiums of one-year European put options on stock ABC for various strike prices:

Strike	Put Premium
35	0.44
40	1.99
45	5.08

The effective annual risk-free interest rate is 8%. Let $S(1)$ be the price of the stock one year from now.

Determine the range for $S(1)$ such that the 35-strike short put produces a higher profit than the 45-strike short put, but a lower profit than the 40-strike short put. (Note: All put positions being compared are short.)

Solution:
We begin by computing the future value of the put premiums:

Strike	FV of Put Premium
35	$0.44 \times 1.08 = 0.48$
40	$1.99 \times 1.08 = 2.15$
45	$5.08 \times 1.08 = 5.49$

The profit functions of the three short puts are sketched in Figure B.1.1. The 35-strike line crosses the 40-strike and 45-strike lines, respectively, at $40-(2.15-0.48)=38.33$ and $45-(5.49-0.48)=39.99$. Visually inspecting the profit diagram, we conclude that the 35 -strike put produces a higher profit than the 45-strike put, but a lower profit than the 40 -strike put, when $38.33<S(1)<39.99$.

Algebraically:

Problem 1.4.5 (3)

- $P(1,35) = 0.44$ · All puts are short.
- $P(1,40) = 1.99$
- $P(1,45) = 5.08$ $\quad Pr_{45} < Pr_{35} < Pr_{40}$
- Effective annual RFR = 8%

Q: Values for $S(1)$ s.t. $Pr_{35} > Pr_{45}$ and $Pr_{35} < Pr_{40}$?

Solution:

$Pr_{35} = 0.44(1.08) - (35-S)^+ = 0.48 - (35-S)^+$

$Pr_{40} = 1.99(1.08) - (40-S)^+ = 2.15 - (40-S)^+$

$Pr_{45} = 5.08(1.08) - (45-S)^+ = 5.49 - (45-S)^+$

(1) Case 1: $S < 35$ ✗ ✓

$Pr_{35} = S - 34.52$ $\quad S - 39.51 < S - 34.52 < S - 37.85$?

$Pr_{40} = S - 37.85$

$Pr_{45} = S - 39.51$

(2) Case 2: $35 < S < 40$ $\quad S - 39.51 < 0.48 < S - 37.85$?

$Pr_{35} = 0.48$ $\quad -39.99 < -S < -38.33$

$38.33 < S < 39.99$ ✓ ✓

(3) Case 3: $40 < S < 45$

$Pr_{35} = 0.48$ $\quad S - 39.51 < 0.48 < 2.15$?

$Pr_{40} = 2.15$ $\quad S < 39.99$ ✗ ✓

(4) Case 4: $45 < S$

$Pr_{35} = 0.48$ $\quad 5.49 < 0.48 < 2.15$?

$Pr_{40} = 2.15$ ✗

$Pr_{45} = 5.49$

Thus: $38.33 < S < 39.99$

3 - Problem 1.4.6 (European, American and Bermudan Options) (3 marks)

Once upon a time, Leo entered into three separate positions involving 2-year options on the same stock.

- Option I was a short American-style call with strike price 30.

- Option II was a long Bermuda-style put with strike price 28, where exercise was allowed at any time following an initial 1-year period of put protection.
- Option III was a long European-style put with strike price 20.
  At inception, the stock price was 27. When the options expired, the stock price was 30.
  The table below gives the maximum and minimum stock price during the 2-year period:

Time Period	1^{st} year of Option Term	2^{nd} year of Option Term
Maximum Stock Price	28	32
Minimum Stock Price	25	24

Calculate the payoffs of each of the three options.

Solution:

Option	Payoff	Remarks
I	$-(32-30)=-2$	The holder of the American call (not Leo!) exercised it when the stock price was the highest over the 2-year period, i.e., when $S=32$
II	$28-24=4$	Leo exercised the Bermuda put when the stock price in the 2nd year was the lowest, i.e., when $S=24$
III	$(20-30)_{+}=0$	Usual payoff formula for a European put at expiration

The sum of the payoffs is 2.
Remark. Note that Leo, being short the American call, was not the one to decide when to exercise the option.

4 - Example 2.2.1 (Calculation of $F_{0,T}^{P}$ with Discrete Dividends) (2 marks)

A certain stock costs 40 today and will pay an annual dividend of 6 for the next 4 years. An investor wishes to purchase a 4-year prepaid forward contract for this stock. The first dividend will be paid one year from today and the last dividend will be paid just prior to delivery of the stock. Assume an annual effective interest rate of 5%.

Calculate the price of the prepaid forward contract.

Solution:
The price of the 4-year prepaid forward contract is

$$\begin{aligned} F_{0,4}^{P} &= S(0)-\mathrm{PV}_{0,4}(\text{ Div}) \\ &= 40-6\left(1/1.05+1/1.05^{2}+1/1.05^{3}+1/1.05^{4}\right) \\ &= 18.72. \end{aligned}$$

5 - Example 2.2.2 (Discrete Plus Continuous Dividends) (4 marks)

For $t \geq 0$ let $S(t)$ be the time-t price of Stock ABC. You are given:

- $S(0)=100$
- At time 0.5 , a cash dividend of $\$10$ per share will be paid.
- From time 0.75 to time 1 , dividends are paid continuously at a rate proportional to its price. The dividend yield is 10%.
- The continuously compounded risk-free interest rate is 8%.

Calculate the price of a one-year prepaid forward contract on Stock ABC.

Solution:
To receive exactly one unit of stock ABC at time 1, we should start with only $\mathrm{e}^{-\delta(1-0.75)}=\mathrm{e}^{-0.025}$ shares of stock ABC because of the reinvestment of the continuous proportional dividends between time 0.75 and time 1 . With $\mathrm{e}^{-0.025}$ shares at time 0 and at time 0.5, we will receive a cash dividend of $\$10\mathrm{e}^{-0.025}$ at time 0.5 :

Method	Cash Flows		
	$t=0$	$t=0.5$	$t=1$
Outright purchase	$-S(0)\mathrm{e}^{-0.025}$	$10\mathrm{e}^{-0.025}$	$S(T)$
Prepaid forward	$-F_{0,1}^{P}$	0	$S(T)$

To fully imitate the one-year prepaid forward, the replicating portfolio consists of:

1. Buying $\mathrm{e}^{-0.025}$ number of shares of stock ABC at time 0
2. Making a loan of $P_{0,0.5}\left(10\mathrm{e}^{-0.025}\right)$ at time 0 and repaying it at time 0.5 .
   The one-year prepaid forward price equals the cost of setting up the replicating portfolio, which in turn is

$$\begin{aligned} F_{0,1}^{P} &= S(0)\mathrm{e}^{-0.025}-10\mathrm{e}^{-0.025}\times\mathrm{e}^{-0.5r} \\ &= 100\mathrm{e}^{-0.25(0.1)}-10\mathrm{e}^{-0.25(0.1)}\times\mathrm{e}^{-0.08(0.5)} \\ &= 88.16. \end{aligned}$$

6 - Problem 2.5.8 (Fair Dividend Yield) (6 marks)

You are given:
(i) The current price of a stock is 1,000 .
(ii) The stock pays dividends continuously at a rate proportional to its price.
(iii) The continuously compounded risk-free interest rate is 5%.
(iv) A 6-month forward price of 1,020 is observed in the market.

Describe actions you could take to exploit an arbitrage opportunity and calculate the resulting profit (per stock unit) in each of the following cases:
(a) The dividend yield of the stock is 0.5% (3 marks)
(b) The dividend yield of the stock is 2% (3 marks)

Solution. (a)
If the dividend yield over the next 6 months will be only 0.5%, then the fair forward price is $F_{0,0.5}^{\text{fair}}=S(0)\mathrm{e}^{(r-\delta)T}=1,000\mathrm{e}^{(5\%-0.5\%)(0.5)}=1,022.7550$, which is higher than the observed forward price of 1,020 , i.e., the forward in the market is underpriced. Then we engage in a reverse cash-and-carry arbitrage, buying the forward in the market, selling $\mathrm{e}^{-0.5\%(0.5)}$ units of the stock and lending $S(0)\mathrm{e}^{-0.5\%(0.5)}$. The associated cash flows are:

Transaction	Cash Flows	
	Time 0	**Time 0.5**
Buy the observed forward	0	$S(0.5) - 1,020$
Sell $e^{-0.5\%(0.5)}$ units of stock	$+1,000e^{-0.5\%(0.5)}$	$-S(0.5)$
Lend $1,000e^{-0.5\%(0.5)}$	$-1,000e^{-0.5\%(0.5)}$	$+1,000e^{(5\%-0.5\%)(0.5)}$
Total	0	$\boxed{2.7550}$

Solution. (b)

If the dividend yield over the next 6 months will be 2%, then the fair forward price is $F_{0,0.5}^{\text{fair}} = S(0)e^{(r-\delta)T} = 1,000e^{(5\%-2\%)(0.5)} = 1,015.1131$, which is lower than the observed forward price of 1,020 , i.e., the forward in the market is overpriced. Then we engage in a cash-and-carry arbitrage, selling the forward in the market, buying $e^{-2\%(0.5)}$ units of the stock and borrowing $S(0)e^{-2\%(0.5)}$. The associated cash flows are:

Transaction	Cash Flows	
	Time 0	**Time 0.5**
Sell the observed forward	0	$1,020 - S(0.5)$
Buy $e^{-2\%(0.5)}$ units of stock	$-1,000e^{-2\%(0.5)}$	$+S(0.5)$
Borrow $1,000e^{-2\%(0.5)}$	$+1,000e^{-2\%(0.5)}$	$-1,000e^{(5\%-2\%)(0.5)}$
Total	0	$\boxed{4.8869}$