T-503-AFLE Derivatives

Forwards

The fair forward price, $F_{0,T}$, for a forward contract on the underlying $S(t)$, struck at time 0 with expiry T, is given by

$$F_{0,T} = S(0)\mathrm{e}^{rT} - \underbrace{\sum_{i=1}^{n} d_i \mathrm{e}^{r(T-t_i)}}_{\mathrm{FV}_{0,T}(\mathrm{Div})}$$

in the case the underlying pays discrete deterministic dividends of size d_i at times t_i, with $i = 1, \ldots, n$, and $0 \le t_1 < t_2 < \cdots < t_n \le T$. In the case the underlying pays a continuous proportional dividend yield of δ that is fully re-invested, the fair forward price is

$$F_{0,T} = S(0)\mathrm{e}^{(r-\delta)T}.$$

Put-call Parity

The prices of a maturity-T European call with strike K and an otherwise identical European put are related by

$$C(K,T) - P(K,T) = F^P_{0,T} - \mathrm{PV}_{0,T}(K) = \mathrm{PV}_{0,T}\left(F_{0,T} - K\right)$$

where

- $F^P_{0,T}$ the price of a maturity-T prepaid forward on the same underlying stock, and
- $\mathrm{PV}_{0,T}(x)$ represents the generalized present-value of time-T cash-flow x.

Option Strategies

The table contains fundamental option trading strategies. All options are written on the same underlying, S, for the same maturity, T. Furthermore, $K_1 < K_2 < K_3$, and $m \neq n$. Note that the initial cashflow is the negative of the cost.

Option Strategy	**Cost**
Floor	$S + P(K)$
Cap	$-S + C(K)$
Short Covered Call	$S - C(K)$
Short Covered Put	$-S - P(K)$
Synthetic Forward	$C(K) - P(K)$
Call Bull Spread	$C(K_1) - C(K_2)$
Put Bull Spread	$P(K_1) - P(K_2)$
Call Bear Spread	$-C(K_1) + C(K_2)$
Put Bear Spread	$-P(K_1) + P(K_2)$
Call Ratio Spread	$\pm mC(K_1) \pm nC(K_2)$
Put Ratio Spread	$\pm mP(K_1) \pm nP(K_2)$
Collar	$P(K_1) - C(K_2)$
Straddle	$P(K) + C(K)$
Strangle	$P(K_1) + C(K_2)$
Butterfly Spread	$P(K_1) + C(K_3) - (P(K_2) + C(K_2))$
Box Spread	$C(K_1) - P(K_1) - (C(K_2) - P(K_2))$

Binomial Option Pricing Models

Consider a one-period binomial model. The underlying, S, has a continuously reinvested proportional dividend yield of δ. It begins at S_0 and can move to either $S_u = uS_0$ or $S_d = dS_0$, in one time period of length h. The continuously compounded risk-free rate is r.

The replicating portfolio for a derivative, V, written on S, is given by

$$\Delta = \mathrm{e}^{-\delta h}\left(\frac{V_u - V_d}{S_u - S_d}\right)$$

shares of the stock and

$$B = \mathrm{e}^{-rh}\left(\frac{uV_d - dV_u}{u - d}\right)$$

dollars in the bank account, where V_u is the payoff of the derivative in the event S attains S_u and V_d is the payoff of the derivative in the event S attains S_d. Thus,

$$V_0 = \Delta S_0 + B.$$

Parameterization: The Forward Tree

A *forward* binomial tree is parameterized by

$$u = \mathrm{e}^{(r-\delta)h + \sigma\sqrt{h}} \quad \text{and} \quad d = \mathrm{e}^{(r-\delta)h - \sigma\sqrt{h}}$$

where

$$\sigma^2 = \mathbb{V}\mathrm{ar}\left[\ln \frac{S(t+1)e^{\delta}}{S(t)}\right].$$

Thus, σ is the standard deviation of the annual log-returns of the stock, also known as the *volatility* of the stock. In this parameterization,

$$p^* = \frac{1}{1 + \mathrm{e}^{\sigma\sqrt{h}}}.$$

Risk-neutral Pricing

A binomial tree is free of arbitrage if and only if

$$d \le \mathrm{e}^{(r-\delta)h} \le u$$

which is equivalent to

$$0 \le p^* := \frac{\mathrm{e}^{(r-\delta)h} - d}{u - d} \le 1.$$

Thus a binomial tree is free of arbitrage if and only if the risk-neutral binomial measure defined by p^* exists. Then, for any traded asset or derivative, V,

$$V_0 = e^{-rh}\mathbb{E}^*[V(h)].$$

A Log-normal Stock Price Model

Let

$$S(t) = S(0)\exp\left[\left(\alpha - \delta - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z\right].$$

model the the time-t price of a stock paying continuous proportional dividends at the rate δ. Here, $\alpha, \delta, t \ge 0$ and $Z \sim \mathcal{N}(0,1)$. Then

$$\mathbb{E}[S(t)] = S(0)\mathrm{e}^{(\alpha-\delta)t}$$

and

$$\mathbb{V}\mathrm{ar}[S(t)] = S(0)^2 e^{2(\alpha-\delta)t}\left(e^{\sigma^2 t} - 1\right).$$

Exercise Probabilities

The European call and put exercises probabilities are given by

$$\mathbb{P}(S(T) > K) = N\left(\hat{d}_2\right) \quad \text{and} \quad \mathbb{P}(S(T) < K) = N\left(-\hat{d}_2\right),$$

for

$$\hat{d}_2 = \frac{\ln[S(0)/K] + \left(\alpha - \delta - \sigma^2/2\right)T}{\sigma\sqrt{T}}.$$

Prediction Intervals

The $100(1-p)\%$ equal-tailed prediction interval for $S(t)$ is given by

$$\mathbb{P}\left(S^L < S(t) < S^U\right) = 1 - p$$

with

$$S^L = S(0)\mathrm{e}^{\left(\alpha-\delta-\sigma^2/2\right)t + \sigma\sqrt{t}N^{-1}(p/2)}$$

and

$$S^U = S(0)\mathrm{e}^{\left(\alpha-\delta-\sigma^2/2\right)t + \sigma\sqrt{t}N^{-1}(1-p/2)}.$$

Conditional Expected Stock Price

The expected stock price conditional on being in the exercise region for a European call struck at K is

$$\mathbb{E}[S(T) \mid S(T) > K] = S(0)\mathrm{e}^{(\alpha-\delta)T}\frac{N\left(\hat{d}_1\right)}{N\left(\hat{d}_2\right)}.$$

where

$$\hat{d}_1 = \hat{d}_2 + \sigma\sqrt{T} = \frac{\ln[S(0)/K] + \Big(\alpha - \delta\Big[+\sigma^2/2\Big)T}{\sigma\sqrt{T}}.$$

Correspondingly,

$$\mathbb{E}[S(T) \mid S(T) < K] = S(0)\mathrm{e}^{(\alpha-\delta)T}\frac{N\left(-\hat{d}_1\right)}{N\left(-\hat{d}_2\right)}.$$