Ambrose Lo

Derivative Pricing

A Problem-Based Primer

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Derivative Pricing

A Problem-Based Primer

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Preface

Derivatives, which are financial instruments whose value depends on or is "derived" from (hence the name "derivatives") other more basic underlying variables, have become commonplace in financial markets all over the world. The proliferation of these relatively new financial innovations, options in particular, has underscored the ever-increasing importance of derivative literacy among a wide range of users that span students, practitioners, regulators, and researchers, all of whom are in need of a fundamental understanding of the mechanics, typical uses, and pricing theory of derivatives, though to different extents. Despite the diversity of such users, existing books on the subject have predominantly catered to only a very specific group of users and gone to two extremes. They either adopt a mostly descriptive approach to the intrinsically technical subject of derivatives, with occasional number crunching and slavish applications of pricing formulas taken without proof, or are preoccupied with sophisticated mathematical techniques from such areas as random processes and stochastic calculus, which can be inaccessible to students or practitioners lacking the necessary background and undesirably obscure the underlying conceptual ideas. Neither the "black box" approach nor the "purely mathematical" approach is of much pedagogical value.

Being an outgrowth of my lecture notes for a course entitled *ACTS:4380 Mathematics of Finance II* offered at the University of Iowa for advanced undergraduate and beginning graduate students in actuarial science, this book is a solid attempt to strike a balance between the two aforementioned methods to teach and learn derivatives, and to meet the needs of different types of readers. Adopting a mathematically rigorous yet widely accessible approach that will appeal to a wide variety of audience, the book is conceptually driven and strives to demystify the mechanics of typical derivatives and the fundamental mechanism of derivative pricing methodologies that should be part of the toolkit of every professional these days. This is accomplished by a combination of lucid explanations of the theory and assumptions behind common derivative pricing models, repeated emphasis on a small set of core ideas (e.g., no-arbitrage principle, replication, risk-neutral pricing), and a careful selection of fully worked-out illustrative examples and end-of-chapter problems. Readers of this book will leave with a firm understanding of "what" derivatives are, "how" and, more importantly, "why" derivatives are used and derivative pricing works.

Here is the skeleton of this book, divided into three parts.

- Part I (Chapters 1 to 3) lays the conceptual groundwork of the whole book by setting up the terminology of derivatives commonly encountered in the literature and introducing the definition, mechanics, typical use, and payoff structures of the two primary groups of derivatives, namely, forwards and options, which bestow upon their holders an obligation and a right to trade an underlying asset at a fixed price on a fixed date, respectively. Particular emphasis is placed on how and why a derivative works in a given scenario of interest. In due course, we also present the all-important *no-arbitrage assumption* and the method of *pricing by replication*. In loose terms, the former says that the prices of derivatives should be such that the market does not admit "free lunches," and the latter implements the former using the common-sense idea that if two derivatives possess the same payoff structure at expiration, they must enjoy the same initial price. Underlying the pricing and hedging of derivatives throughout this book, these two vehicles are applied in this part to determine the fair price of a forward, where "fair" is meant in the sense that the resulting price permits no free lunch opportunities.

- Whereas the pricing of forwards is model-independent in that it works for any asset price distribution, the pricing of options depends critically on the probabilistic behavior of the future asset price. In Part II (Chapters 4 to 8), which is the centerpiece of this book, we build upon the background material in Part I and tackle option pricing in two stages—first in the discrete-time binomial tree model (Chapter 4), which is simple, intuitive, and easy to implement, then in the technically more challenging continuous-time Black-Scholes model (Chapters 5 to 8). In this part, the no-arbitrage assumption and the method of replication continue to play a vital role in valuing options and lead to the celebrated *risk-neutral pricing formula*, which asserts that the price of a (European) derivative can be computed as its expected payoff at expiration in a risk-neutral sense, discounted at the risk-free interest rate. The implementation and far-reaching implications of the method of risk-neutral valuation for the pricing and hedging of derivatives are explored in Chapters 6 to 8.

- Finally, we end in Part III (Chapter 9) with a description of some general properties satisfied by option prices when no asset price model is prescribed. Even in this model-free framework setting, there is a rich theory describing the no-arbitrage properties universally satisfied by option prices. Although this part can be read prior to studying Part II, you will find that what you learn from Part II, especially the notion of an exchange option in Chapter 8, will provide you with surprisingly useful insights into the connections between different options.

It deserves mention that this book, as a primer, is indisputably not encyclopedic in scope. The choice of topics is geared towards the derivatives portion (Topics 6 to 10) of the Society of Actuaries' *Investment and Financial Markets* (IFM) Exam,[i] which is typically taken by advanced undergraduate students in actuarial science and allied disciplines. The theory of random processes and stochastic calculus, while conducive to understanding the pricing theory of derivatives in full but often an insurmountable barrier to first-time learners, is not covered in the book, neither are credit and interest rate derivatives (which, without doubt, are important in practice). By concentrating on the most essential conceptual ideas, we realize the huge "payoff" of being able to disseminate these core ideas to readers with minimal mathematical background; it is understandable that individuals interested in using and pricing derivatives nowadays come from a wide variety of background. To be precise, readers are only assumed to have taken a calculus-based probability and statistics course at the level of Hogg, Tanis and Zimmerman (2014) or Hogg, McKean and Craig (2013), where the basic notions of random variables, expectations, variances, are taught, and be able to perform simple discounted cash flow calculations as covered in a theory of interest or corporate finance course. With these modest prerequisites, this book is self-contained, with the necessary mathematical ideas presented progressively as the book unfolds. For readers interested in more advanced aspects of the use and pricing of derivatives, this book will provide them with a springboard for performing further studies in this burgeoning field.

It is widely acknowledged that the best way to learn a subject deeply is to test your understanding with a number of meaningful exercises. With this in mind, this primer lives up to its name and features an abundance of illustrative in-text examples and end-of-chapter problems (to be precise, 177 examples and 209 problems) on different aspects of derivatives. These problems are of a diverse nature and varying levels of difficulty (harder ones

[i]Section 4.5, Section 7.3 (the portion on the Black-Scholes equation), Subsection 8.1.2, and Section 9.1 are beyond the scope of the Exam IFM syllabus.

are labeled as [**HARDER!**]); while many emphasize calculating quantities such as payoffs, prices, profits, a primitive skill that most students in a derivatives course need to acquire, at least for exam purposes (in this respect, this book is an ideal exam preparation aid for students who will write Exam IFM), some concern more theoretical aspects of using and pricing derivatives, and consist of true-or-false items or derivations of formulas. All of these problems can be worked out in a pen-and-paper environment with the aid of a scientific calculator and a standard normal distribution function calculator (an example is `https://www.prometric.com/en-us/clients/soa/pages/mfe3f_calculator.aspx`, which is designed for students who will take Exam IFM.). If you do not have Internet access, you may use the less precise standard normal distribution table provided in Appendix A of this book. Readers who attempt these examples and problems seriously will benefit from a much more solid understanding of the relevant topics. To help you check your answers, full solutions to all odd-numbered end-of-chapter problems are provided in Appendix B. A solutions manual with solutions to all problems is available to qualified instructors.

It would be remiss of me not to thank my past ACTS:4380 students for personally class testing earlier versions of the book manuscript and many of the end-of-chapter problems, as well as my esteemed colleague, Professor Elias S.W. Shiu, at the University of Iowa, for sharing with me his old ACTS:4380 notes and questions, from which some of the examples and problems in this book were motivated. I am also grateful to the Society of Actuaries and Casualty Actuarial Society for kindly allowing me to reproduce their past and sample exam questions, of which they own the sole copyright, and which have proved instrumental in illustrating ideas in derivative pricing. Doctoral student Zhaofeng Tang at the University of Iowa merits a special mention for his professional assistance with some of the figures in this book and for meticulously proofreading part of the book manuscript. All errors that remain, typographical or otherwise, are solely mine. To help improve the content of the book, I would deeply appreciate it if you could bring any potential errors you have identified to my attention; my email address is `ambrose-lo@uiowa.edu`. For readers' benefits, an erratum and updates to the book will be maintained on my web page at `https://sites.google.com/site/ambroseloyp/publications/derivative-pricing`.

It is my sincere hope that this book will not only introduce you to the fascinating world of derivatives, but also to instill in you a little of the enthusiasm I have for this subject since my undergraduate studies. Welcome and may the fun begin!

Ambrose Lo, PhD, FSA, CERA
Iowa City, IA
May 2018

Symbols

Symbol Description

Symbol	Description
S or $S(0)$	time-0 price of an underlying asset
$S(t)$	time-t price of an underlying asset
$S^a(0)$	time-0 ask price of an underlying asset
$S^b(0)$	time-0 bid price of an underlying asset
$X(t)$	time-t exchange rate
K	strike price of an option
K_1	strike price of a gap option
K_2	payment trigger of a gap option
r	continuously compounded risk-free interest rate (per annum)
r^b	continuously compounded borrowing rate (per annum)
r^l	continuously compounded lending rate (per annum)
i	effective annual interest rate (per annum)
$\mathrm{PV}_{t,T}$	time-t (present) value of cash flows between time t and time$T$
$\mathrm{FV}_{t,T}$	time-T (future) value of cash flows between time t and time$T$
T	maturity time of a generic derivative
T_f	maturity time of a futures contract
T_1	maturity time of a compound option
T_2	maturity time of the underlying option of a compound option
$F^P_{t,T}$	time-t price of a prepaid forward maturing at time T
$F_{t,T}$	time-t price of a forward maturing at time T
$F^{\mathrm{obs}}_{t,T}$	time-t observed price of a forward maturing at time T
$F^{\mathrm{fair}}_{t,T}$	time-t fair price of a forward maturing at time T
V	time-0 price of a generic derivative
$V^{\max}$	time-0 price of a maximum contingent claim
$V^{\min}$	time-0 price of a minimum contingent claim
C	time-0 price of a generic call
C^E	time-0 price of a generic European call
C^A	time-0 price of a generic American call
$C(K,T)$	time-0 price of a K-strike T-year call
$C^{\mathrm{gap}}(K_1,K_2)$	time-0 price of a K_1- strike K_2-trigger generic gap call
$C(S(t),K,t,T)$	time-t price of a K-strike call maturing at time T when the time-t stock price is $S(t)$
P	time-0 price of a generic put
P^E	time-0 price of a generic European put
P^A	time-0 price of a generic American put
$P^{\mathrm{gap}}(K_1,K_2)$	time-0 price of a K_1-strike K_2-trigger generic gap put
$P(S(t),K,t,T)$	time-t price of a K-strike put maturing at time T when the time-t stock price is $S(t)$
BS	Black-Scholes pricing function
σ	volatility of an underlying asset
σ_{option}	volatility of an option
Δ	delta of a generic derivative

Δ_C	delta of a generic call
Δ_P	delta of a generic put
Γ	gamma of a generic derivative
Γ_C	gamma of a generic call
Γ_P	gamma of a generic put
Ω	elasticity of a generic derivative
Ω_C	elasticity of a generic call
Ω_P	elasticity of a generic put
$\mathbb{E}[X]$	expectation of random variable X
$\mathrm{Var}(X)$	variance of random variable X
$\mathrm{Cov}(X,Y)$	covariance between random variables X and Y
$\rho(X,Y)$	correlation coefficient between random variables X and Y
$N(\cdot)$	distribution function of the standard normal distribution
$N'(\cdot)$	density function of the standard normal distribution
$:=$	defined as
LHS	left-hand side
RHS	right-hand side
x_+	positive part of real number x
1_A	indicator function of event A
ZCB	zero-coupon bond

Part I

Conceptual Foundation on Derivatives

1

An Introduction to Forwards and Options

Chapter overview: This book centers on *derivatives*—not those you have studied in your calculus class, but those instruments whose values (or, in financial parlance, payoffs) depend on or are "derived" from other more basic underlying variables, such as the price of a stock or a commodity, an interest rate, a currency exchange rate, or even non-financial variables like the temperature of a city on a particular day or your semester GPA in college, so long as they can be quantified. These underlying driving variables are called the *underlying asset*, or in short, the *underlying*. To begin our systematic study of derivatives, this preparatory chapter provides a conceptual introduction to the two predominant types of derivatives, namely, forwards and options, and sets up the basic derivatives terminology that will be intensively used throughout this book. These primitive derivatives are basic building blocks of more sophisticated financial instruments that will be examined in later chapters. For each of forwards and options, we analyze its mechanics, typical use, and, most importantly, the structure and derivation of its payoff.

1.1 Forwards

Instead of spoon-feeding you directly with the terms and conditions of different kinds of derivatives, a much better way to get you to better understand their use and mechanics is through a daily-life example most familiar to actuarial students.

Motivating example.

It is now July 1, 3017 (assume that Doomsday has not yet come). You are planning to take Exam X, a notoriously difficult actuarial exam, in November 3017 and will be in severe need of a study manual in August for your exam preparation (assume that 3 months are enough for your study!). According to the Internet (assume that the Internet still exists in the next millennium), the current price of the study manual for Exam X is \$250 (per unit). You are worried, however, that the price of the study manual would rise dramatically over the next month, which would seriously jeopardize your financial well-being as a frugal student. Is there a way to protect yourself against adverse increases in the price of the study manual?

Now consider the following "contract" available in the market:

> You will be provided with one copy of the study manual on August 1, 3017 in exchange for \$275. The contract needs to be signed today and will bind you legally[i] to pay \$275 for the study manual on August 1.

Such a "contract" is the quintessence of a forward contract, the subject of the current section.

[i] We ignore counterparty risk, i.e., the risk that one or more parties fail to deliver on the obligations imposed by the derivative.

What is a forward?

In general, a *forward contract*, or simply a *forward*, is an agreement between two parties (also called counterparties) to buy or sell an asset at a certain time in the future at a certain price. The contract is agreed upon and signed today, with one party (the buyer) undertaking to pay the stipulated price in exchange for the asset and the other party (the seller) liable to deliver the asset and receive the same stipulated price at the particular future time specified in the contract. This way, a forward effectively removes the uncertainty that one needs to face with respect to the future price of the underlying asset. Here is a good mnemonic that helps you remember and make sense of a forward:

> A forward allows you to look *forward* in time and lock in the transaction price of an asset you wish to buy or sell in the future.

With respect to terminology, the essential elements in a forward are:

Term	**Description**	**Study Manual Example**
Underlying asset	The asset that the forward is based on	Study manual
Expiration date (or maturity date)	The time when the forward has to be settled	August 1, 3017
Forward price	The price that will be paid when the forward is settled	$275
Long or short	If you buy (resp. sell) the forward, you are said to be in a *long* (resp. *short*) forward position. The words long and short can serve as adjectives or verbs, so we may also say that you long (resp. short) the forward.	You are long the forward

The most important ingredient in a forward is the forward price. In this introductory chapter, we take the forward price as given. In Chapter 2, we will study in detail how the forward price can be determined in a way that is fair in a sense to be formalized. Notation-wise, we denote generically by $S(t)$ the time-t price of the underlying asset (the use of "S" is motivated by the fact that the most common underlying asset is a stock; in fact the terms "underlying asset" and "underlying stock" will be used interchangeably in the sequel when no confusion arises), and by $F_{0,T}$ the forward price that is determined at time 0 and will be paid at time T. Unless otherwise stated, in this book we will always measure time in years.

The payoff of a forward contract.

A very useful way to describe a derivative, a forward in particular, is to look at its *payoff*. This concept permeates virtually the entire book and is one of the most important dimensions of a derivative. Loosely speaking, the payoff of a derivative position at a particular time point at or before the expiration date is defined as the value of the position. An equivalent, but more informative and concrete definition that lends itself to practical computations is that the payoff of a derivative at a given point of time equals the amount of money that its holder would have had *if* he/she completely liquidated his/her position, i.e., to sell whatever he/she is holding, and buy back whatever he/she has owed.

Let us apply this line of reasoning to determine the payoff of a T-year long forward (i.e.,

Time	Transaction	Cash Flow
0	You buy a forward.	0
T	You settle the forward by paying the seller the forward price, $F_{0,T}$.	$-F_{0,T}$
	The seller delivers the asset to you.	0
	You sell the asset at the *spot price* (i.e., market price) of the asset, $S(T)$.	$+S(T)$
		Total: $S(T) - F_{0,T}$

TABLE 1.1
Cash flows associated with a long forward (physical settlement).

the forward expires in T years). Under a long forward, at expiration you pay $F_{0,T}$ for the underlying asset and can cash out by selling the asset at its market price for $S(T)$ (see Table 1.1). On a net basis, you receive $S(T) - F_{0,T}$. This is the payoff of a T-year long forward:

$$\boxed{\text{Payoff of long forward} = \underbrace{\text{Spot price}}_{\text{receive (income)}} - \underbrace{\text{forward price}}_{\text{pay (cost)}} = S(T) - F_{0,T}.} \tag{1.1.1}$$

The cash flows associated with a short forward are the exact opposite of those of a long forward. Upon receiving the forward price from the buyer, purchasing the underlying asset at the market price for $S(T)$, and delivering the asset thus purchased to the buyer, you as the short forward party receive a payoff of

$$\boxed{\text{Payoff of short forward} = \text{Forward price} - \text{spot price} = F_{0,T} - S(T),} \tag{1.1.2}$$

which is a mirror image of (1.1.1). You may notice that the sum of (1.1.1) and (1.1.2) is exactly zero. In other words, a forward is a zero-sum game, with what the buyer gains exactly offset by what the seller loses.

Example 1.1.1. (SOA Exam IFM Introductory Derivatives Sample Question 68: Payoff of long/short forward) For a nondividend-paying stock index, the current price is 1100 and the 6-month forward price is 1150. Assume the price of the stock index in 6 months will be 1210.

Which of the following is true regarding forward positions in the stock index?

(A) Long position gains 50

(B) Long position gains 60

(C) Long position gains 110

(D) Short position gains 60

(E) Short position gains 110

Solution. By (1.1.1) and (1.1.2), the long forward will gain $1210 - 1150 = \boxed{60}$ while the short forward will gain $1150 - 1210 = -60$, i.e., a loss of 60. **(Answer: (B))** □

It deserves mention that a forward imposes upon its counterparties an *obligation* to

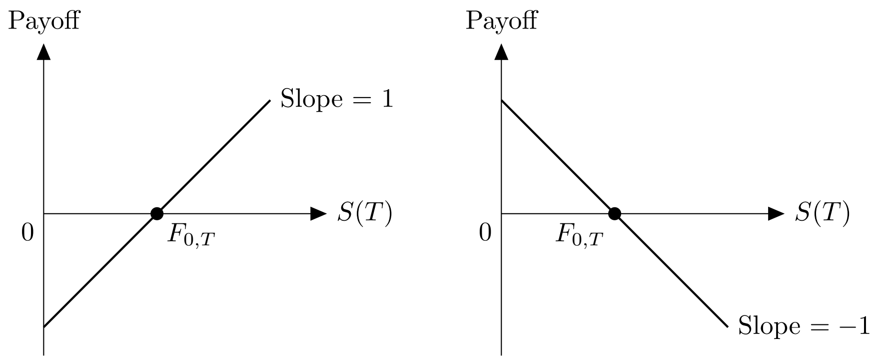


FIGURE 1.1.1
Payoff diagrams of a long forward (left) and a short forward (right).

execute the contract at expiration. Under a long forward position, you are obligated to pay the forward price at the expiration date, regardless of how high or low the then market price is. Even if the ending asset price is less than the forward price, in which case you suffer a loss, you still need to settle the forward. This is a characteristic of a forward that distinguishes it from other derivatives.

Payoff diagrams.

In Part I of this book, we will make intensive use of a visual device called a *payoff diagram*, which displays graphically the payoff of a derivative as a function of the underlying asset price at the point of interest. As we shall see soon, the payoff diagrams of many derivatives possess salient features that say a lot about the properties of the derivatives. In fact, one of the best ways to remember a derivative is arguably to associate it with the geometry of its payoff diagram.

Given (1.1.1) and (1.1.2), the payoff diagrams of a long forward and a short forward are sketched in Figure 1.1.1. They are straight lines cutting the horizontal axis at the forward price $F_{0,T}$ with a slope of 1 and -1, respectively. The linearity arises from a commitment to trade the underlying asset in the future. In contrast, we will see in Section 1.2 that options, the other primary category of derivatives, are characterized by nonlinear payoff functions because of their intrinsic optionality.

Physical vs cash settlement.

In the above discussions, we assumed that at the expiration of the forward, the seller indeed delivers the asset to the buyer for $F_{0,T}$, and the buyer indeed cashes out by selling the asset for $S(T)$, realizing a payoff of $S(T)-F_{0,T}$. This mode of settling a forward involving genuine delivery of the underlying asset is known as *physical* settlement of a forward. On second thought, however, it would make no difference if the seller directly gives the buyer an amount of $S(T)-F_{0,T}$ in cash (this amount can be positive or negative); see Table 1.2. The holder of the long forward will be in the same financial position as when there is genuine physical delivery of the underlying. Settling a forward this way by means of a direct exchange of cash without the corresponding delivery of the underlying asset is known as *cash* (or financial) settlement. Both physical settlement and cash settlement apply not only to forwards but also to other derivatives.

The two modes of settlement have their relative merits and demerits. Physical settlement aids our understanding of the mechanics of a derivative and the reason for using that

Time	Transaction	Cash Flow
0	You buy a forward.	0
T	The seller pays you $S(T) - F_{0,T}$.	$S(T) - F_{0,T}$

TABLE 1.2
Cash settlement of a long forward.

derivative in the first place. It is also critical to the fundamental derivation of its payoff formula (the long forward payoff formula $S(T) - F_{0,T}$, while taken by many for granted, can be derived from basics and need not be taken as a definition). However, a physical transaction in practice will incur possibly significant transaction costs and is applicable only to assets for which physical delivery is possible. This excludes more abstract assets such as temperature, your GPA, interest rates, market volatility, all of which cannot be physically traded. Cash settlement is widely applicable and simple to understand, but it does not lend itself to deriving the payoff of a derivative from first principles. Both modes of settlement will be used in the later part of this book, although cash settlement will play the dominant role.

Main motivation for using derivatives: Hedging and speculation.

With the payoff formula determined above, we are in a much better position to understand the typical use of a forward. Back to the study manual example, we again denote by $S(T)$ the unit price of the study manual on August 1. Suppose that the study manual is a necessity to you, so that you must buy the study manual however cheap or expensive it is. Under this assumption, your cash flow on August 1 will be $-S(T)$, which is inherently random. If you couple your position with a long forward on the study manual, then your payoff on August 1 will be constant at $-S(T) + (S(T) - F_{0,T}) = -F_{0,T}$. In effect, what you have to pay to own the study manual is transformed from a random amount of $S(T)$ to the constant forward price $F_{0,T}$. The long forward therefore allows you to *hedge* against the risk associated with the price of the study manual on August 1 by locking in the transaction price.

More generally, *hedging* aims to use the cash flows generated by a derivative to mitigate the (often random) cash flows from a given position. In the case of a forward, the payoff from the long forward counteracts the random cash outflow as a result of buying the underlying at its random future price. With a forward, you are shielded from the price risk of the underlying; the cash flow uncertainty arising from the future transaction has been completely eliminated. Long forwards are commonly employed by business companies to hedge against the future price risk associated with their necessary production inputs and outputs. They are also popular among importers or exporters concerned with the fluctuations of foreign exchange rates in the future.

In contrast to hedging, *speculation* refers to the attempt to profit from anticipated price movements of the underlying asset without an existing exposure in the asset. In the study manual example, if your mom, whom I assume does not need the study manual for her own use (unless she happens to be an actuarial science professor!), is convinced that the price of the study manual will skyrocket (resp. plummet) in August, she may take advantage of her belief by buying (resp. selling) a forward. Should the price of the study manual go up (resp. go down) as expected, she can reap a huge payoff from her long (resp. short) forward position. Of course, she may also suffer a huge loss from the forward if the price of the study manual does not move in the direction she predicts.

1.2 Options

1.2.1 Call Options

Motivation.

Recall from Section 1.1 that if you enter into a forward, you are required under any circumstances to settle the forward at the expiration date, even if doing so results in a negative payoff. In the study manual example, if the price of the study manual on August 1, 3017 is only \$260, it seems stupid (of course, with the benefit of hindsight!) to buy the manual at the forward price of \$275—you will pay \$15 more! You may wonder:

> Is there a derivative that entitles you the *option* to buy an asset if and only if doing so is to your interest, i.e., you have the right to walk away from the deal if you wish to?

Derivatives that give you an option to buy or sell an asset are naturally called *options* (pun intended!). There are two main types of options, namely, call options and put options. They are intended to provide one-sided protection against unfavorable movements in the price of the underlying asset.

Definition and terminology of a call option.

A *call* option (or a *call* in short) gives its holder the *right*, but not the obligation, to buy the underlying asset at a prespecified price. Here is a cheap mnemonic:

> By means of a call, you have the option to "call" the asset from someone and own it.

In addition to "underlying asset" and "long or short," which enjoy the same definition as a forward, the following terms are very useful in specifying an option in general and a call in particular.

1. *Exercise:* Exercising an option refers to the act of making use of the option to trade the underlying asset. If you exercise a call, you pay a certain price (see the next point) in return for the underlying asset.

2. *Strike price:* The *strike price*, also known as *exercise price* or simply the *strike*, is the price specified in the contract at which the option holder can exercise the option. In the case of a call option, the strike price is what the holder can choose to pay for the underlying asset. We shall denote the strike price of an option generically by K.

3. *Expiration:* The expiration date is the time when the option holder must decide whether to exercise the option. As in Section 1.1, the generic symbol for the time to expiration is T.

4. *Exercise style:* Exercise style is a characteristic unique to options but not forwards. It governs when an option can be exercised.

    - If the option can only be exercised at the expiration date, it is called a *European*-style or simply *European* option.
    - If the option can be exercised anytime prior to or on the expiration date, it is called an *American*-style or, in short, *American* option.

- A *Bermudan*-style option allows its holder to exercise it during only specified periods, but not throughout the entire life of the option.

Here is a mnemonic:

$$\begin{aligned} \boxed{\text{E}}\,\text{uropean} &= \boxed{\text{E}}\,\text{xpiration} \\ \boxed{\text{A}}\,\text{merican} &= \boxed{\text{A}}\,\text{nytime} \\ \boxed{\text{B}}\,\text{ermudan} &= \boxed{\text{B}}\,\text{between} \end{aligned}$$

Unless otherwise stated, options that are studied in the remainder of this book are European, whose analysis is mathematically more tractable. It is easy to conceive that American options are more valuable than otherwise identical European and Bermudan options because of the higher degree of freedom in relation to the time of exercise. We will see in Chapter 9 that in some cases, it turns out that the opportunity to early exercise an option is not exploited, so that European and otherwise identical American and Bermudan options do share the same price.

Payoff of a call option.

To determine the payoff of a long European call at the expiration time T, consider two cases:

Case 1. If $S(T) > K$, you will exercise the call to use the strike price K to buy the asset and sell it in the market immediately for the market price $S(T)$, earning $S(T) - K$. Here, we are assuming that the call is settled physically.

Case 2. If $S(T) \leq K$, it is irrational to exercise the call to buy the asset for K because you are better off buying the asset directly in the market at the cheaper price of $S(T)$. In this case, you will opt out of the call, which becomes worthless, and your payoff is zero.

To sum up, the payoff of holding a European[ii] call is

$$\begin{aligned} \text{Long call payoff} &= \begin{cases} S(T) - K, & \text{if } S(T) > K \\ 0, & \text{if } S(T) \leq K \end{cases} \\ &= \boxed{(S(T) - K)_+}, \end{aligned}$$

where $(\cdot)_+$ is the positive part function defined by $x_+ := \max(x, 0)$ for any $x \in \mathbb{R}$. For example, $6_+ = 6$, $11.3_+ = 11.3$, $(-4)_+ = 0$, and $(-9.21)_+ = 0$. The positive part function signifies the optionality inherent in options and is characteristic of option payoffs.

The payoff diagram of a long call is graphed in Figure 1.2.1. Unlike the payoff of a long forward, that of a long call is zero if the spot price is less than the strike price, and becomes a straight line emanating from the strike price and pointing to the right with a slope of 1 otherwise. The turning point at the strike price is an indication of the optionality of the call—the right is exercised when and only when $S(T) \geq K$. In essence, the payoff of a long call position is obtained from that of a long forward position by zeroing the negative portion of the latter and keeping the positive part intact.

[ii] The time-t payoff of a T-year K-strike American call is $(S(t) - K)_+$.

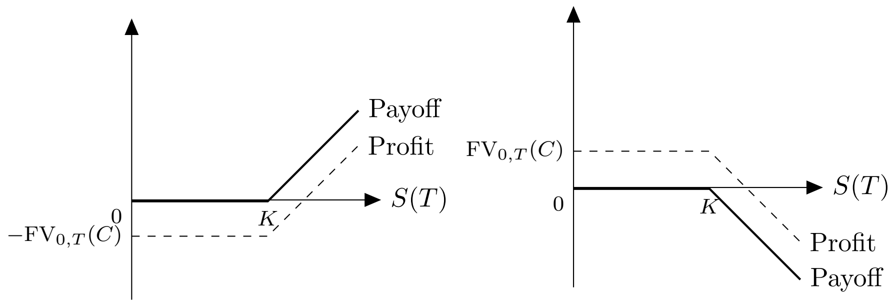


FIGURE 1.2.1
Payoff and profit diagrams of a long call (left) and a short call (right).

Time	Transaction	Cash Flow
0	You buy a call and pay the call premium C.	$-C$
T	You can decide whether to exercise the call, depending on the spot price.	$(S(T)-K)_+$

TABLE 1.3
Cash flows associated with a long European call position.

Option premium.

By entering into a call option, you have the option to benefit from a rise in the price of asset without the need for bearing any downside risk. In other words, your payoff at expiration must be non-negative regardless of the spot price. For the call to be fair to the seller, you must pay him/her an amount called the *option premium*, which we designate as C, at the inception of the option as a form of compensation—the upside protection offered by the call is not free! The premium is also called the *price* of the call option. Table 1.3 shows what and when you pay or receive if you buy a European call.

Parenthetically, the determination of the fair price of an option is a highly nontrivial task, much more technically complicated than that of the fair price of a forward, and is the subject of Part II of this book.

Profit of a call option.

The payoff of a derivative captures its value at a particular instant and ignores the cash flows at other times, particularly the initial cost to set up the derivative. A more global perspective on a derivative is through its *profit*, defined as the payoff of the derivative less the *future value* of cash flows at previous time points.

In the case of a long call, the profit is obtained by subtracting the future value of the option premium from the option payoff, i.e.,

$$\boxed{\text{Long call profit} = (S(T)-K)_+ - \mathrm{FV}_{0,T}(C),} \tag{1.2.1}$$

where $\mathrm{FV}_{0,T}(\cdot)$ denotes the future value from time 0 to time T. If r represents the continuously compounded risk-free interest rate,[iii] then $\mathrm{FV}_{0,T}(C) = Ce^{rT}$; if i is the effective annual interest rate, then $\mathrm{FV}_{0,T}(C) = C(1+i)^T$.

We note in passing that because a forward entails no initial investment by definition, its profit coincides with its payoff given in (1.1.1) and (1.1.2).

[iii] Throughout this book, interest rates are measured per annum.

Example 1.2.1. (Given the profit, find the spot price) You buy a 50-strike 6-month call option on a stock at a price of 5. The continuously compounded risk-free interest rate is 5%.

At the end of 6 months, the profit from the long call option is 4.

Calculate the price of the stock at the end of 6 months.

Ambrose's comments:

Problems directly asking for the payoff or profit of a derivative given other inputs via using formulas like (1.1.1) and (1.2.1) are somewhat too simple. A more realistic and non-trivial problem may turn things around and ask that you determine the unknown stock price at expiration to achieve a certain profit.

Solution. In terms of the 6-month stock price $S(0.5)$, the profit from the long call is $(S(0.5)-50)_+ - 5\mathrm{e}^{0.05(0.5)}$. Because the positive part function takes different expressions depending on the sign of the argument, we solve the equation $(S(0.5)-50)_+ - 5\mathrm{e}^{0.05(0.5)} = 4$ for $S(0.5)$ in two cases:

Case 1. If $S(0.5) < 50$, then the equation is $-5\mathrm{e}^{0.05(0.5)} = 4$, which has no solution.

Case 2. If $S(0.5) \geq 50$, then the equation becomes $(S(0.5)-50) - 5\mathrm{e}^{0.05(0.5)} = 4$, which gives $S(0.5) = 59.1266$.

The only possible solution is $S(0.5) = \boxed{59.1266}$. □

Example 1.2.2. (SOA Exam IFM Introductory Derivatives Sample Question 11: Comparing the profits of three calls) Stock XYZ has the following characteristics:

- The current price is 40.
- The price of a 35-strike 1-year European call option is 9.12.
- The price of a 40-strike 1-year European call option is 6.22.
- The price of a 45-strike 1-year European call option is 4.08.

The annual effective risk-free interest rate is 8%.

Let S be the price of the stock one year from now.

All call positions being compared are long.

Determine the range for S such that the 45-strike call produces a higher profit than the 40-strike call, but a lower profit than the 35-strike call.

(A) $S < 38.13$

(B) $38.13 < S < 40.44$

(C) $40.44 < S < 42.31$

(D) $S > 42.31$

(E) The range is empty

We present two solutions, one algebraic and one geometric.

Solution 1 (Algebraic). Denote by Pr_K the profit of a K-strike 1-year European call. We first express each Pr_K in terms of S:

$$\begin{aligned}\mathrm{Pr}_{35} &= (S-35)_+ - 9.12(1.08) = (S-35)_+ - 9.8496 &(1.2.2)\\ \mathrm{Pr}_{40} &= (S-40)_+ - 6.22(1.08) = (S-40)_+ - 6.7176 &(1.2.3)\\ \mathrm{Pr}_{45} &= (S-45)_+ - 4.08(1.08) = (S-45)_+ - 4.4064 &(1.2.4)\end{aligned}$$

To find the range for S such that $\mathrm{Pr}_{40} < \mathrm{Pr}_{45} < \mathrm{Pr}_{35}$, we consider each of the following cases:

Case 1. If $S < 35$, then all three positive part functions in (1.2.2), (1.2.3), and (1.2.4) vanish, so we have $\mathrm{Pr}_{35} < \mathrm{Pr}_{40} < \mathrm{Pr}_{45}$, which is not the order we seek.

Case 2. If $35 \le S < 40$, then $\mathrm{Pr}_{35} = S - 44.8496$, $\mathrm{Pr}_{40} = -6.7176$, and $\mathrm{Pr}_{45} = -4.4064$. To get the order of profits we want, we solve $-6.7176 < -4.4064 < S - 44.8496$, resulting in $S > 40.4432$, which contradicts the assumption $35 \le S < 40$.

Case 3. If $40 \le S < 45$, then $\mathrm{Pr}_{35} = S - 44.8496$, $\mathrm{Pr}_{40} = S - 46.7176$, and $\mathrm{Pr}_{45} = -4.4064$. To get the desired order of profits, we solve

$$S - 46.7176 < -4.4064 < S - 44.8496,$$

which gives $40.4432 < S < 42.3112$. Note that this range for S satisfies the hypothesis $40 \le S < 45$.

Case 4. If $S \ge 45$, then $\mathrm{Pr}_{35} = S - 44.8496$, $\mathrm{Pr}_{40} = S - 46.7176$, and $\mathrm{Pr}_{45} = S - 49.4064$. To get the order of profits we want, we solve

$$S - 46.7176 < S - 49.4064 < S - 44.8496,$$

in which the first inequality has no solution. Therefore, in this range of S, it is impossible that $\mathrm{Pr}_{40} < \mathrm{Pr}_{45} < \mathrm{Pr}_{35}$.

In conclusion, the required range for S is $S \in \boxed{(40.44, 42.31)}$. □

Solution 2 (Geometric). We first compute the future value of the 3 call premiums:

Strike	FV of Call Premium
35	$9.12 \times 1.08 = 9.8496$
40	$6.22 \times 1.08 = 6.7176$
45	$4.08 \times 1.08 = 4.4064$

Now we sketch the three call profit functions in Figure 1.2.2 (the diagram is not and need not be drawn to scale). Observe that the three profit functions are ordered as $\mathrm{Pr}_{40} < \mathrm{Pr}_{45} < \mathrm{Pr}_{35}$ when and only when S lies in the interval AB. To find the horizontal coordinates of the two points, A and B, recall that the slope of the profit function of a call must be 1 when the asset price is beyond the strike price. With this in mind, we can calculate the horizontal coordinates of A and B as $35 + [-4.41 - (-9.85)] = 40.44$ and $40 + [-4.41 - (-6.72)] = 42.31$. The required answer is $\boxed{S \in (40.44, 42.31)}$.
(Answer: (C)) □

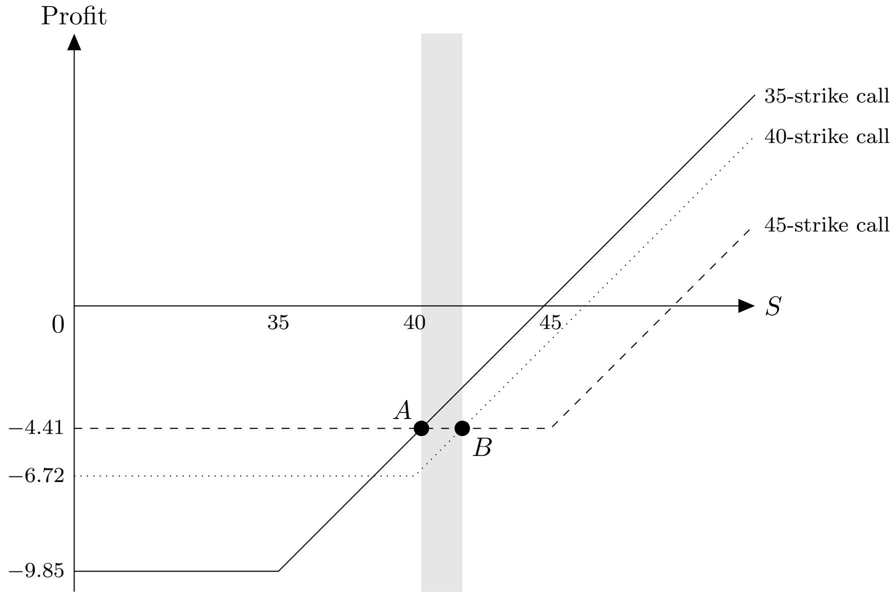


FIGURE 1.2.2
The profit functions of the three calls in Example 1.2.2.

> *Remark.* (i) The SOA provides the current stock price (= 40), which is not necessary for solving the problem.
>
> (ii) See whether you prefer a "geometric" or an "algebraic" solution.

Written call option.

What happens to the seller (a.k.a. *writer*) of a call option? He/she receives the option premium from the outset and has the *obligation*, not the right, to deliver the underlying asset if the buyer chooses to exercise the call. The payoff and profit of the writer are exactly the opposite of those of the buyer:

$$\begin{aligned} \text{Short call payoff} &= -(S(T)-K)_+, \\ \text{Short call profit} &= \mathrm{FV}_{0,T}(C) - (S(T)-K)_+. \end{aligned}$$

Note that the payoff of the call writer is always non-positive. This explains why he/she must insist on receiving compensation in the form of the call option premium at time 0.

> **Example 1.2.3. (Call vs forward)** Investor A wrote a 104-strike 1-year call option whose price is 2. Investor B entered into a 1-year long forward with a forward price of 105.
>
> The continuously compounded risk-free interest rate is 5%.
>
> It turns out that Investor A and Investor B earned the same profit.
>
> Calculate the 1-year stock price.

Solution. Equating the profits of the short call and long forward, we solve the equation

$$\underbrace{2e^{0.05} - (S(1) - 104)_+}_{\text{short call}} = \underbrace{S(1) - 105}_{\text{long forward}}$$

for $S(1)$. Depending on whether $S(1) < 104$ or $S(1) \geq 104$, we have:

Case 1. If $S(1) < 104$, then the equation above is $2e^{0.05} = S(1) - 105$, which implies $S(1) = 107.10$, a contradiction to the hypothesis $S(1) < 104$!

Case 2. If $S(1) \geq 104$, then the equation becomes $2e^{0.05} - (S(1) - 104) = S(1) - 105$, resulting in $S(1) = \boxed{105.5513}$, which complies with the hypothesis $S(1) \geq 104$. □

Example 1.2.4. (SOA Exam IFM Introductory Derivatives Sample Question 42: Profit of a written covered call) An investor purchases a nondividend-paying stock and writes a t-year, European call option for this stock, with call premium C. The stock price at time of purchase and strike price are both K.

Assume that there are no transaction costs.

The risk-free annual force of interest is a constant r. Let S represent the stock price at time t.

$S > K$.

Determine an algebraic expression for the investor's profit at expiration.

(A) Ce^{rt}

(B) $C(1 + rt) - S + K$

(C) $Ce^{rt} - S + K$

(D) $Ce^{rt} + K(1 - e^{rt})$

(E) $C(1 + r)^t + K[1 - (1 + r)^t]$

Solution. The time-0 investment is $S(0) - C = K - C$ and the time-t payoff is $S - (S - K)_+ = S - (S - K) = K$ because $S > K$. The profit at expiration is $K - (K - C)e^{rt} = \boxed{Ce^{rt} + K(1 - e^{rt})}$. **(Answer: (D))** □

1.2.2 Put Options

Definition of a put option.

It is now August 1, 3017, and you have got hold of the study manual of Exam X via the long forward or the long call, whichever way you prefer. You are not yet done with Exam X, but you are pretty sure that you will pass. Why still keep the abominable study manual and not sell it to the market after the exam, say in December 3017? A put option that expires in December 3017 can come to your rescue.

Whereas call options discussed in the preceding subsection allow their holders to buy a particular asset in the future, a *put* option (or a *put* in short) gives its holder the right to *sell* a certain asset for a certain price at a certain date or in a time period. Here is a cheap mnemonic:

> By means of a put option, you have the option to "put" the underlying asset to someone.

All contractual terms that apply to a call (e.g., strike price, expiration date, exercise style, premium, etc.) easily carry over to a put. In particular, the strike price K of a put is the price for which the put holder may elect to *sell* the underlying asset. At the other end of the transaction, the *seller* of a put is a potential *buyer* of the asset. Should the put holder decide to exercise the put, the seller is *obligated* to buy the asset against his/her own will and suffer a negative payoff.

Payoff and profit of a put option.

In parallel with the call payoff, we consider two cases to determine the payoff of a European put at the expiration time T:

Case 1. If $S(T) \leq K$, you will buy the asset for $S(T)$ and exercise the option to sell the asset for a higher price of K, earning $K - S(T)$.

Case 2. If $S(T) > K$, it is inadvisable to exercise the put because of selling the asset (K) at less than what it is worth in the market ($S(T)$). You will then forfeit the put, in which case your payoff is zero.

Combining both cases, we see that the payoff of holding a European put option is

$$\begin{aligned} \text{Long put payoff} &= \begin{cases} K - S(T), & \text{if } S(T) \leq K \\ 0, & \text{if } S(T) > K \end{cases} \\ &= \boxed{(K - S(T))_+}. \end{aligned}$$

The profit of the long put is simply the excess of its payoff over the future value of the put premium:

$$\text{Long put profit} = (K - S(T))_+ - \text{FV}_{0,T}(P).$$

The payoff and profit diagrams of a long put and short put are given in Figure 1.2.3.

Example 1.2.5. (SOA Exam IFM Introductory Derivatives Sample Question 48: Warm-up payoff manipulations) For a certain stock, Investor A purchases a 45-strike call option while Investor B purchases a 135-strike put option. Both options are European with the same expiration date. Assume that there are no transaction costs.

If the final stock price at expiration is S, Investor A's payoff will be 12.

Calculate Investor B's payoff at expiration, if the final stock price is S.

(A) 0

(B) 12

(C) 36

(D) 57

(E) 78

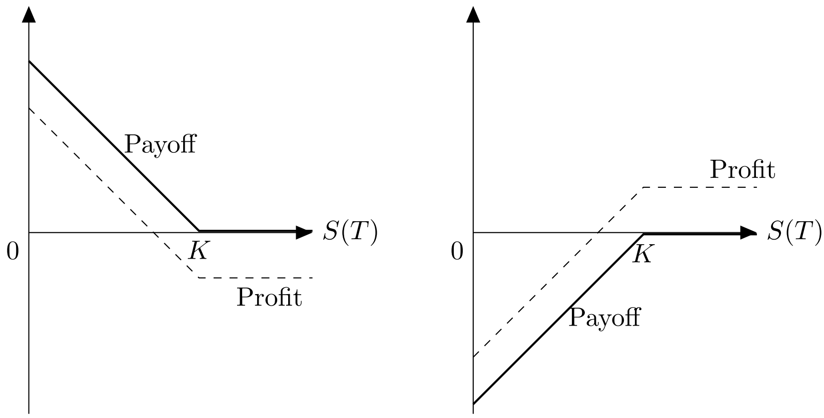


FIGURE 1.2.3
Payoff and profit diagrams of a long put (left) and a short put (right).

Solution. Considering Investor A, we have $(S - 45)_+ = 12$, or $S = 57$. For the same final stock price, the payoff of Investor B at expiration is $(135 - S)_+ = \boxed{78}$. **(Answer: (E))** □

Example 1.2.6. (SOA Exam IFM Introductory Derivatives Sample Question 33: Payoff of American, European and Bermudan options) Several years ago, John bought three separate 6-month options on the same stock.

- Option I was an American-style put with strike price 20.
- Option II was a Bermudan-style call with strike price 25, where exercise was allowed at any time following an initial 3-month period of call protection.
- Option III was a European-style put with strike price 30.

When the options were bought, the stock price was 20.

When the options expired, the stock price was 26.

The table below gives the maximum and minimum stock prices during the 6-month period:

Time Period	1^{st} 3 months of Option Term	2^{nd} 3 months of Option Term
Maximum Stock Price	24	28
Minimum Stock Price	18	22

John exercised each option at the optimal time.

Rank the three options, from highest to lowest payoff.

(A) I > II > III

(B) I > III > II

(C) II > I > III

(D) III > I > II

(E) III > II > I

Solution. The payoff of each option is determined and tabulated below:

Option	Remark	Payoff
I	• As an American option, it can be exercised at any time during the 6-month period. • As a put, its payoff is greatest when the stock price is smallest.	$20 - 18 = 2$
II	• As a Bermuda option, it can be exercised at any time during the second 3-month period. • As a call, its payoff is greatest when the stock price is greatest.	$28 - 25 = 3$
III	• As a European option, it can be exercised only at maturity. • We can directly apply the usual put payoff formula $(K - S(T))_+$.	$(30 - 26)_+ = 4$

It follows that the payoffs of the three options in descending order are $\boxed{\text{III} > \text{II} > \text{I}}$. **(Answer: (E))** □

Example 1.2.7. (SOA Exam IFM Introductory Derivatives Sample Question 35: Simple profit calculation for puts) A customer buys a 50-strike put on an index when the market price of the index is also 50. The premium for the put is 5. Assume that the option contract is for an underlying 100 units of the index.

Calculate the customer's profit if the index declines to 45 at expiration.

(A) −1000

(B) −500

(C) 0

(D) 500

(E) 1000

Ambrose's comments:

Be careful! The put option is on 100 units (not just one) of the index.

Solution. The customer's profit if the index declines to 45 at expiration is $100[(50-45)_+ - 5] = \boxed{0}$. **(Answer: (C))** □

Remark. This example makes no mention of the interest rate, so we have no choice but to assume it to be zero.

Example 1.2.8. (SOA Exam IFM Introductory Derivatives Sample Question 12: Long put, short put, same profit) Consider a European put option on a stock index without dividends, with 6 months to expiration and a strike price of 1,000. Suppose that the annual nominal risk-free rate is 4% convertible semiannually, and that the put costs 74.20 today.

Calculate the price that the index must be in 6 months so that being long in the put would produce the same profit as being short in the put.

(A) 922.83

(B) 924.32

(C) 1,000.00

(D) 1,075.68

(E) 1,077.17

Ambrose's comments:

This example shares a similar spirit as Example 1.2.1, with the quantity of interest being the asset price at expiration resulting in a certain profit.

Solution. The profit of the long put is $(1,000 - S(0.5))_+ - 74.2(1.02)$, while that of the short put is $74.2(1.02) - (1,000 - S(0.5))_+$. Equating these two gives $(1,000 - S(0.5))_+ = 75.684$, which in turn implies that $S(0.5) = \boxed{924.32}$. **(Answer: (B))** □

Remark. If being long and being short in the same derivative share the same profit, both positions must have a zero profit.

Example 1.2.9. (SOA Exam IFM Introductory Derivatives Sample Question 62: Expected profit of a put) The price of an asset will either rise by 25% or fall by 40% in 1 year, with equal probability. A European put option on this asset matures after 1 year.

Assume the following:

- Price of the asset today: 100
- Strike price of the put option: 130

- Put option premium: 7
- Annual effective risk free rate: 3%

Calculate the expected profit of the put option.

(A) 12.79

(B) 15.89

(C) 22.69

(D) 27.79

(E) 30.29

Solution. The 1-year payoff of the put option will be either $(130 - 100 \times 1.25)_+ = 5$ or $(130 - 100 \times 0.6)_+ = 70$ with equal probability. The expected profit is $(5 + 70)/2 - 7(1.03) = \boxed{30.29}$. **(Answer: (E))** □

1.3 Classification of Derivatives

Thus far, we have introduced altogether six different positions: forwards, calls, and puts, each of which can be long or short. In this section, we present several ways to compare and contrast different derivatives. These easy comparisons also provide opportunities for us to review the material covered in Sections 1.1 and 1.2.

Universal comparison.

Derivatives (not only forwards and options) can be classified according to the following general criteria:

Criterion 1. *Long or short with respect to the underlying asset:* By definition, a position is *long* (resp. *short*) with respect to the underlying asset if it benefits from increases (resp. decreases) in the price of the underlying asset, or, mathematically speaking, has a payoff function which is increasing (resp. decreasing) in the asset price. For example, a long forward is long with respect to the underlying asset, but a long put is short.

Note that the usage of the words "long" and "short" here is different from that in Sections 1.1 and 1.2, where "long" and "short" are synonyms for "buy" and "sell." You should be cautioned that your position with respect to the underlying asset can be different from your position in the derivative. For example, a short put is, by definition, short in the put, but actually long in the asset. Indeed, a way to distinguish whether a position is long or short with respect to the underlying asset is to see if it represents a right or an obligation to *buy* or *sell the asset*. A short put carries an obligation to buy the asset if the put holder exercises the option, so is fundamentally a long position.

Criterion 2. *Maximum profit and maximum loss:* Of particular interest is the maximum and minimum profit of a derivative. Unlimited maximum profit (e.g., long forward, long call) is certainly desirable, but unlimited loss (e.g., short forward, short call) is a great cause for concern—it is possible that you go bankrupt!

Example 1.3.1. (SOA Exam IFM Introductory Derivatives Sample Question 26: Unlimited loss) Determine which, if any, of the following positions has or have an unlimited loss potential from adverse price movements in the underlying asset, regardless of the initial premium received.

I. Short 1 forward contract
II. Short 1 call option
III. Short 1 put option

(A) None

(B) I and II only

(C) I and III only

(D) II and III only

(E) The correct answer is not given by (A), (B), (C), or (D)

Solution. Only a short forward and a short call have an unlimited loss potential because their payoff and profit functions exhibit an indefinite decreasing trend from a certain point onward. **(Answer: (B))** □

Example 1.3.2. (SOA Exam IFM Introductory Derivatives Sample Question 49: Maximum loss of a long put) The market price of Stock A is 50. A customer buys a 50-strike put contract on Stock A for 500. The put contract is for 100 shares of A.

Calculate the customer's maximum possible loss.

(A) 0

(B) 5

(C) 50

(D) 500

(E) 5000

Solution. Because the customer is long a put option, his/her maximum loss is attained when the price of Stock A at maturity is 50 or beyond. In that case, the payoff of the put is zero and the customer will have lost the entire initial investment of $\boxed{500}$. **(Answer: (D))** □

Remark. As in Example 1.2.7, we tacitly assume that the interest rate, which is not given, is zero.

Criterion 3. *Asset price contingency:* Derivatives can also be compared with respect to the conditions that trigger the settlement of the derivative. For forwards,

Position	Long/short w.r.t. Asset	Max. Profit	Min. Profit	Asset Price Contingency
Long forward	Long	$+\infty$	$-F_{0,T}$	Always
Short forward	Short	$F_{0,T}$	$-\infty$	Always
Long call	Long	$+\infty$	$-\mathrm{FV}_{0,T}(C)$	$S(T) > K$
Short call	Short	$\mathrm{FV}_{0,T}(C)$	$-\infty$	$S(T) > K$
Long put	Short	$K - \mathrm{FV}_{0,T}(P)$	$-\mathrm{FV}_{0,T}(P)$	$S(T) < K$
Short put	Long	$\mathrm{FV}_{0,T}(P)$	$\mathrm{FV}_{0,T}(P) - K$	$S(T) < K$

TABLE 1.5
Different criteria to compare derivatives.

settlement is an obligation and always takes place; for options, it is at the discretion of the buyer, depending on the relative magnitude of the spot price at expiration and the strike price. Note that whether a derivative is settled does not depend on whether it is long or short.

These three criteria are applied to compare the six positions involving forwards and options; see Table 1.5.

Comparison between options and forwards.

There are two critical differences between options and forwards.

1. Forward holders have the *obligation* to buy/sell the asset no matter how favorable/unfavorable the market is. In contrast, option holders have the *right* to buy/sell the underlying asset, depending on the market situation. In short, forwards carry commitment while options endow their holders with discretion.

2. Purchasing options require an upfront investment in the form of the option premium, while it is costless to enter into a forward.

Comparison across options: Moneyness.

Moneyness describes whether the payoff of an option would be positive or negative *had* the option been exercised immediately. This concept applies to both European, American, and Bermudan options even though European and Bermudan options may not be exercised before expiration.

- *In-the-money:* An *in-the-money* option is one with a strictly positive payoff if exercised immediately. A call is in-the-money if the current asset price is greater than the strike price, while a put is in-the-money if the current asset price is less than the strike price.

- *Out-of-the-money:* An *out-of-the-money* option is one with a strictly negative payoff if exercised immediately. The conditions for being out-of-the-money are the opposite of being in-the-money.

- *At-the-money:* An option is said to be *at-the-money* if it is both in-the-money and out-of-the-money. In other words, the payoff if exercised immediately is zero, which happens if and only if the current asset price is equal to the strike price.

The concept of moneyness will be useful when one wants to control the amount of insurance to purchase; see Chapter 3 for more details.

Example 1.3.3. (SOA Exam IFM Introductory Derivatives Sample Question 44: Simple deductions for moneyness – I) You are given the following information about two options, A and B:

(i) Option A is a one-year European put with exercise price 45.

(ii) Option B is a one-year American call with exercise price 55.

(iii) Both options are based on the same underlying asset, a stock that pays no dividends.

(iv) Both options go into effect at the same time and expire at $t = 1$.

You are also given the following information about the stock price:

(i) The initial stock price is 50.

(ii) The stock price at expiration is also 50.

(iii) The minimum stock price (from $t = 0$ to $t = 1$) is 46.

(iv) The maximum stock price (from $t = 0$ to $t = 1$) is 58.

Determine which of the following statements is true.

(A) Both options A and B are "at-the-money" at expiration.

(B) Both options A and B are "in-the-money" at expiration.

(C) Both options A and B are "out-of-the-money" throughout each option's term.

(D) Only option A is ever "in-the-money" at some time during its term.

(E) Only option B is ever "in-the-money" at some time during its term.

Solution. At expiration, the price is 50 and both options are out-of-the-money, eliminating Answers (A) and (B). With a strike price of 45 and a minimum stock price of 46, option A with payoff $(45 - S(t))_+ = 0$ for all $t \in [0, 1]$ is never in-the-money, eliminating Answer (D). With a strike price of 55, option B will be in-the-money at the time the stock price is 58 with a payoff of $58 - 55 = 3$, eliminating Answer (C) and verifying $\boxed{\textbf{Answer (E)}}$. □

Example 1.3.4. (SOA Exam IFM Introductory Derivatives Sample Question 61: Simple deductions for moneyness – II) An investor purchased Option A and Option B for a certain stock today, with strike prices 70 and 80, respectively. Both options are European one-year put options.

Determine which statement is true about the moneyness of these options, based on a particular stock price.

(A) If Option A is in-the-money, then Option B is in-the-money.

(B) If Option A is at-the-money, then Option B is out-of-the-money.

(C) If Option A is in-the-money, then Option B is out-of-the-money.

(D) If Option A is out-of-the-money, then Option B is in-the-money.

(E) If Option A is out-of-the-money, then Option B is out-of-the-money.

Solution. Fix any time $t \in [0, 1]$ and let Payoff_A and Payoff_B be the time-t payoffs of Option A and Option B, respectively. Notice the order $\text{Payoff}_B \geq \text{Payoff}_A$.

If Option A is in-the-money at time t, i.e., $\text{Payoff}_A > 0$, then necessarily $\text{Payoff}_B > 0$, i.e., Option B is also in-the-money. **(Answer: (A))** □

Example 1.3.5. (SOA Exam IFM Introductory Derivatives Sample Question 66: Simple deductions for moneyness – III) The current price of a stock is 80. Both call and put options on this stock are available for purchase at a strike price of 65.

Determine which of the following statements about these options is true.

(A) Both the call and put options are at-the-money.

(B) Both the call and put options are in-the-money.

(C) Both the call and put options are out-of-the-money.

(D) The call option is in-the-money, but the put option is out-of-the-money.

(E) The call option is out-of-the-money, but the put option is in-the-money.

Solution. The current stock price, 80, is higher than the strike price, 65. Since the call option would have a positive payoff of 15 if exercised immediately, it is in-the-money. On the other hand, the put option would have a negative payoff of -15 if exercised immediately, so it is out-of-the-money. **(Answer: (D))** □

1.4 Problems

Problem 1.4.1. (Payoff/profit of a forward for different ending stock prices) Aaron has purchased a forward contract on a stock. You are given:

(i) If the stock price at expiration is S, his payoff would be $-\$5$.

(ii) If the stock price at expiration is $1.1S$, his payoff would be $\$1$.

Calculate Aaron's profit on the long forward if the stock price at expiration is $1.2S$.

Problem 1.4.2. (Long put vs short forward) You are given:

(i) The current price of a 100-strike 9-month European put option is 12.

(ii) A 9-month forward has a forward price of 105.

(iii) The continuously compounded risk-free interest rate is 3%.

Calculate the stock price after 9 months such that the long put option and the *short* forward contract have the same profit.

Problem 1.4.3. (Call vs put) Jack buys a 50-strike 6-month European call option on stock ABC at a price of 8. Rose buys a 50-strike 6-month European put option on the same stock at a price of 6.

The continuously compounded risk-free interest rate is 4%.

6 months later, Jack suffers a loss while Rose realizes a profit, with Rose's profit being twice as large as Jack's loss.

Calculate the price of stock ABC at the end of 6 months.

Problem 1.4.4. (Maximum and minimum profits of a short put) Bob writes a two-year 100-strike European put with a premium of \$10. The continuously compounded risk-free interest rate is 4%.

Calculate the difference between Bob's maximum profit and his minimum profit.

Problem 1.4.5. (Put version of Example 1.2.1: Comparing the profits of three puts) You are given the following premiums of one-year European put options on stock ABC for various strike prices:

Strike	Put Premium
35	0.44
40	1.99
45	5.08

The effective annual risk-free interest rate is 8%.
Let $S(1)$ be the price of the stock one year from now.
Determine the range for $S(1)$ such that the 35-strike short put produces a higher profit than the 45-strike short put, but a lower profit than the 40-strike short put.

(Note: All put positions being compared are short.)

Problem 1.4.6. (European, American, and Bermuda options) Once upon a time, Leo entered into three separate positions involving 2-year options on the same stock.

- Option I was a *short* American-style call with strike price 30.
- Option II was a long Bermuda-style put with strike price 28, where exercise was allowed at any time following an initial 1-year period of put protection.
- Option III was a long European-style put with strike price 20.

At inception, the stock price was 27.

When the options expired, the stock price was 30.

The table below gives the maximum and minimum stock price during the 2-year period:

Time Period	1^{st} year of Option Term	2^{nd} year of Option Term
Maximum Stock Price	28	32
Minimum Stock Price	25	24

Each option was exercised by its holder at the optimal time.

Calculate the sum of the payoffs of the three options.

Problem 1.4.7. (Similarities between a long call and a short put) Determine which of the following statements about a long European call option and a short European put option on the same underlying asset is/are correct.

I. Both are long with respect to the underlying asset.

II. Both involve a possible purchase of the underlying asset in the future.

III. Both give you the right but not the obligation to buy the underlying asset at the strike price in the future.

(A) None

(B) I and II only

(C) I and III only

(D) II and III only

(E) The correct answer is not given by (A), (B), (C), or (D)

Problem 1.4.8. (Simple deduction for moneyness) An investor purchased Call X and Call Y for a certain stock today, with strike prices 50 and 60, respectively. Both options are European options with the same time to expiration.

Determine which of the following statements is true about the moneyness of these options, based on a particular stock price.

(A) If Call X is in-the-money, then Call Y is in-the-money.

(B) If Call X is at-the-money, then Call Y is in-the-money.

(C) If Call X is in-the-money, then Call Y is out-of-the-money.

(D) If Call X is out-of-the-money, then Call Y is in-the-money.

(E) If Call X is out-of-the-money, then Call Y is out-of-the-money.

2

Forwards and Futures

Chapter overview: Building upon the conceptual foundation laid in Chapter 1, this chapter explores in greater depth forward contracts on financial instruments, particularly stocks and stock indexes. Section 2.1 presents four different ways to own a stock at a fixed future time point, leading naturally to the introduction of the notions of prepaid forward and forward contracts. The discussion is expanded in Sections 2.2 and 2.3, where we study in detail how these contracts are priced and hedged for stocks that pay different modes of dividends under the all-important no-arbitrage assumption. In the course of our derivations, we present a synthetic construction of a prepaid forward and a forward which not only allows us to replicate the payoff of a forward by means of the underlying asset and the risk-free asset, but also furnishes a recipe to effect an arbitrage strategy if the observed price in the market deviates from the fair price. Finally, Section 2.4 concludes the chapter with a brief introduction of futures contracts, which are a variant of forwards, and their mechanics.

2.1 Alternative Ways to Buy a Stock

Four ways to own a stock.

There are intricately more ways to buy and own a stock than one might have imagined, if the payment and physical receipt of the stock are separated as two different activities. Table 2.1 lists altogether four different ways that result in the ownership of one unit of a stock at a fixed expiration time T.

1. *Outright purchase:* The simplest method is to pay for the stock at the outset and own it immediately.

2. *Fully leveraged purchase:* One can also receive the stock now and fund the required investment of $S(0)$ by making a loan to be repaid at time T. Given a continuously compounded risk-free interest rate of r, the required payment at time T is $S(0)\mathrm{e}^{rT}$.

3. *Prepaid forward contract:* If you pay for the stock now and receive it at time T, then you are said to have entered into a *prepaid forward* contract. We denote the payment made at time 0 under a prepaid forward contract, known as the *prepaid forward price*, by $F^P_{0,T}$, where the symbol F suggests "forward" and the superscript P signifies "prepaid."

   The difference between an outright purchase and a prepaid forward is that with the latter, you own the stock at time T, rather than at time 0.

4. *Forward contract:* As discussed in Chapter 1, we can also own the stock at time T via a forward contract by agreeing today to pay the prespecified forward price at time T and owning the stock. We denote, as in Chapter 1, the forward price by $F_{0,T}$.

Strategy	**Time of Payment**	**Time of Receipt of Asset**	**Amount of Payment**
Outright purchase	0	0	$S(0)$
Fully leveraged purchase	T	0	$S(0)e^{rT}$
Prepaid forward contract	0	T	$F^P_{0,T}$
Forward contract	T	T	$F_{0,T}$

TABLE 2.1
Four different ways to own one share of stock at time T.

The focus of this chapter is on the third and fourth arrangements. In fact, the principal objective of this chapter is to find the *fair* values of $F^P_{0,T}$ and $F_{0,T}$—"fair" is meant in the sense that the price will not give rise to "free lunch" opportunities (to be defined below) in the market. The technique used to derive the fair prices is known as the *no-arbitrage argument*, the mastery of which is equally important as being able to calculate $F^P_{0,T}$ and $F_{0,T}$ proficiently.

Example 2.1.1. (SOA Exam IFM Introductory Derivatives Sample Question 7: Name of arrangement) A nondividend-paying stock currently sells for 100. One year from now the stock sells for 110. The continuously compounded risk-free interest rate is 6%. A trader purchases the stock in the following manner:

- The trader pays 100 today
- The trader takes possession of the stock in one year

Determine which of the following describes this arrangement.

(A) Outright purchase

(B) Fully leveraged purchase

(C) Prepaid forward contract

(D) Forward contract

(E) This arrangement is not possible due to arbitrage opportunities

Solution. Because the trader pays now and possesses the stock in one year, such an arrangement is a $\boxed{\text{prepaid forward contract}}$. **(Answer: (C))** □

No-arbitrage principle.

The most important conceptual tool in this chapter is the *no-arbitrage principle*. At its core is the simple but reasonable idea that the prices of derivatives in a market should be set in a way that prohibits *arbitrage opportunities*, informally known as "free lunch," which allow investors to earn profits for sure and are something too good to be true in reality. Those prices are referred to as *fair prices* because they ensure that the market does not allow any completely "unfair" phenomena. Symbolically:

$$\boxed{\text{Market (Observed) price} = \text{Fair price} \quad \Leftrightarrow \quad \text{No arbitrage opportunity}}$$

The no-arbitrage assumption, which will be maintained throughout this book, is very natural but critical. It imposes the minimum form of regularity governing the prices of derivatives in the market and provides the theoretical basis for pricing and hedging all the derivatives in this book. If it does not hold, then it is possible to earn risk-free profits by buying underpriced derivatives and selling overpriced derivatives—the famous adage of "buy low, sell high."

Terminology-wise, a set of financial transactions designed to exploit an arbitrage opportunity is called an *arbitrage strategy*. It is a portfolio that carries only cash inflows, random or non-random, now or in the future, but not cash outflows. Holding such a portfolio will therefore always entitle us to a positive profit. Meanwhile, an individual engaging in an arbitrage strategy is termed an *arbitrageur*. In the later part of this chapter, we shall learn, when the market price deviates from the fair price, how to construct an arbitrage strategy, i.e., we will learn how to become a successful arbitrageur!

2.2 Prepaid Forwards

We separate the discussion of prepaid forwards according to whether the underlying stock pays dividends, and in what manner.

2.2.1 Nondividend-paying Stocks

We start by deriving the price of a prepaid forward on a nondividend-paying stock using two different methods. The more complicated case of dividend-paying stocks will be treated in the next subsection.

Method 1 (Pricing by "common sense").

Let's first discuss how to find the fair prepaid forward price using a common-sense approach. Intuitively, in the absence of dividends, it makes no difference between:

(a) Outright purchase (owning the stock starting from time 0 and holding it until time T)

(b) Prepaid forward (owning the stock starting from time T)

The cost of Method (a) is the current stock price $S(0)$, and that of Method (b) is the prepaid forward price $F_{0,T}^P$. These two methods both give rise to a payoff of $S(T)$ at time T[i] and entail no other cash flows between time 0 and time T other than the initial investment because of the absence of stock dividends. Due to the same financial nature of Methods (a) and (b), they must have the same time-0 cost, that is,

$$\boxed{F_{0,T}^P = S(0). \quad \text{(for nondividend-paying stocks)}}$$

[i]We ignore non-pecuniary benefits associated with owning a stock, such as voting and control rights. These benefits are hardly quantifiable.

Method 2 (Pricing by replication).

Pricing by replication is a more general and systematic method that can not only be easily extended to more complicated scenarios when common sense fails, but also be employed to take advantage of mispricing in the market. It is illustrated below for proving $F_{0,T}^{P} = S(0)$.

The following timeline diagram documents the cash flows associated with a long prepaid forward:

	$-F_{0,T}^{P}$ (target)		$+S(T)$
Time	0	$\cdots$	T
	You pay the prepaid forward price		The prepaid forward matures; you receive one unit of the asset

As its name suggests, the replication method entails trading other securities in the market, forming the *replicating portfolio*, to "replicate" the cash flows of the target derivative of interest, which is the prepaid forward in our case. What securities do we have at our disposal? In our simple market, we can trade the underlying stock and borrow or lend money. To reproduce the cash inflow of $S(T)$ at time T, the simplest way is to buy one unit of the stock at time 0 and sell it at time T. Doing so results in the following cash flows:

	$-S(0)$		$+S(T)$
Time	0	$\cdots$	T
	You pay the time-0 stock price	$\leftarrow$ Hold the stock $\rightarrow$	You sell the stock and receive the time-T price

Note that there are no cash flows between time 0 and time T because the stock is assumed to pay no dividends. This allows us to restrict our attention to only time 0 and time T.

As soon as the target derivative (the long prepaid forward in our case), has been replicated, the method of replication says that the fair price of our target is simply equal to the cost of creating the replicating portfolio (to buy one unit of the stock at time 0 in our case), the price of which is easily determined by market prices. After all, the long prepaid forward and the long stock have the same payoff at time T and should naturally cost the same at time 0. Our conclusion is that the fair value of the prepaid forward price should coincide with the time-0 stock price:

$$\boxed{F_{0,T}^{P} = S(0). \quad \text{(for nondividend-paying stocks)}}$$

In making this conclusion, we are using the following perfectly reasonable idea, which will be repeatedly used in this book:

> Two positions having the same *payoff* (at a reference future time) command the same *price* (at time 0).

What if this perfectly reasonable idea turns out to be wrong in the market? That is, what if we observe a prepaid forward price different from the current stock price? In this case, arbitrage opportunities exist and the method of replication has provided us with the recipe to take advantage of the mispricing. Consider:

Transaction	Cash Flows	
	Time 0	**Time T**
Buy 1 unit of stock @ $S(0)$	$-S(0)$	$+S(T)$
Sell prepaid forward @ $F^P_{0,T}$	$+F^P_{0,T}$	$-S(T)$
Total	$F^P_{0,T} - S(0)$	0

TABLE 2.2
Trading strategies to effect arbitrage when $F^P_{0,T} > S(0)$.

Case 1. If $F^P_{0,T} > S(0)$, we can follow the "buy low, sell high" strategy by buying one unit of the stock ("low") and selling the prepaid forward ("high"), immediately realizing a positive cash inflow of $F^P_{0,T} - S(0)$ (> 0). Buying the stock at time 0 guarantees that at time T we have at hand the stock, which we are obligated to deliver to the buyer of the prepaid forward. Note that the total cash flow at time T is exactly zero, but we already earn a positive cash inflow of $F^P_{0,T} - S(0)$ at time 0. This creates a risk-free profit and thus an arbitrage strategy (see Table 2.2 for the transactions and the associated cash flows).

Case 2. If $F^P_{0,T} < S(0)$, then reversing the transactions in Table 2.2 (i.e., buying the prepaid forward and short selling 1 unit of the stock) shows that an arbitrage is possible.

Combining the two cases, we infer that the only fair value of $F^P_{0,T}$ is $S(0)$.

In summary, the method of pricing by replication is a very useful tool to implement the no-arbitrage principle. It underlies the pricing of essentially all derivatives (not just forwards and options) and is one of the most fundamental ideas in this book. You should make every effort to master it fully!

2.2.2 Dividend-paying Stocks

General prepaid forward price formula.

In the presence of stock dividends, an outright purchase and a prepaid forward do make a difference. The reason is that if you own the stock starting from time 0 and hold it until time T, you receive the dividends payable from time 0 to time T as well, but the owner of the T-year prepaid forward does not. Because of the loss of dividends, intuition suggests and formal no-arbitrage arguments can be used to confirm that the holder of a long prepaid forward should be compensated (relative to an outright purchase) by having to pay *less* than the stock holder by the *price of the stock dividends*[ii] payable over the life of the prepaid forward, i.e.,

$$F^P_{0,T} = S(0) - \text{Price of dividends.}$$

We distinguish between two types of dividend payments.

Case 1: Discrete, non-random dividends.

Suppose it is expected that at each *known* time t_i, the stock will make a dividend payment with a *known* amount of $D(t_i)$, where $0 < t_1 < t_2 < \cdots < t_n \leq T$ and T is the maturity time of a T-year prepaid forward in question. If there is a dividend payable at time T, we assume that it is paid immediately before the prepaid forward matures. To replicate the long prepaid

[ii]We prefer not to use the term "present value of dividends" because stochastic dividends, discounted for time value for money, remain stochastic and thus cannot serve as a price.

forward by buying one unit of the stock, we will end up with the additional cash inflows of $D(t_i)$ at times $t_1, \ldots, t_n$ due to the discrete dividends that the holder of the stock is entitled to. To completely replicate the long prepaid forward, these interim cash inflows need to be eliminated. This can be accomplished by making a loan of $\mathrm{PV}_{0,t_i}(D(t_i)) = D(t_i)\mathrm{e}^{-rt_i}$ to be repaid at time t_i. Here shows the complete replicating portfolio and the associated cash flows:

Loan repaid at time t_1	$+\mathrm{PV}_{0,t_1}(D(t_1))$	$-D(t_1)$				
Loan repaid at time t_2	$+\mathrm{PV}_{0,t_2}(D(t_2))$		$-D(t_2)$			
$\vdots$	$\vdots$			$\ddots$		
Loan repaid at time t_n	$+\mathrm{PV}_{0,t_n}(D(t_n))$				$-D(t_n)$	
Long stock	$-S(0)$	$+D(t_1)$	$+D(t_2)$	$\cdots$	$+D(t_n)$	$+S(T)$
Time	0	t_1	t_2	$\cdots$	t_n	T

The fair value of the prepaid forward price then equals the cost of setting up the above replicating portfolio, which means that

$$\boxed{F^P_{0,T} = S(0) - \sum_{i=1}^{n} \mathrm{PV}_{0,t_i}(D(t_i)) = S(0) - \sum_{i=1}^{n} D(t_i)\mathrm{e}^{-rt_i}.} \tag{2.2.1}$$

Again, in the event that the observed value of $F^P_{0,T}$ differs from (2.2.1), the replicating portfolio will supply the necessary ingredients for effecting an arbitrage strategy, as we will see in the next subsection.

Example 2.2.1. (SOA Exam IFM Introductory Derivatives Sample Question 71: Calculation of $F^P_{0,T}$ with discrete dividends) A certain stock costs 40 today and will pay an annual dividend of 6 for the next 4 years. An investor wishes to purchase a 4-year prepaid forward contract for this stock. The first dividend will be paid one year from today and the last dividend will be paid just prior to delivery of the stock. Assume an annual effective interest rate of 5%.

Calculate the price of the prepaid forward contract.

(A) 12.85

(B) 13.16

(C) 17.29

(D) 18.72

(E) 21.28

Solution. By (2.2.1), the price of the 4-year prepaid forward contract is

$$\begin{aligned} F^P_{0,4} &= S(0) - \mathrm{PV}_{0,4}(\mathrm{Div}) \\ &= 40 - 6(1/1.05 + 1/1.05^2 + 1/1.05^3 + 1/1.05^4) \\ &= \boxed{18.72}. \qquad \textbf{(Answer: (D))} \end{aligned}$$

□

Case 2: Continuous proportional dividends.

For stock indexes comprising a number of stocks, it is common to model the dividends as being paid continuously at a rate proportional to the level of the stock index. This means that there is a non-negative constant δ, called the *dividend yield*, such that for each unit of the stock index, the amount of (stochastic) dividends paid between time t and $t+\mathrm{d}t$ for any infinitesimally small $\mathrm{d}t$ is $S(t)\delta\,\mathrm{d}t$, where $S(t)$ is the time-t stock index price. Throughout this book, we assume that the dividends received are not paid out in cash, but are reinvested immediately in the stock, resulting in more and more shares as time goes by.

To determine the increase in the number of shares, let $N(t)$ be the number of shares of the stock we hold at time t under the reinvestment policy. We give a calculus-based proof for the expression of $N(t)$. Between time t and time $t+\mathrm{d}t$, the amount of dividend payment is $S(t)\delta\,\mathrm{d}t$ per share, so the total amount of dividends we receive is $N(t)S(t)\delta\,\mathrm{d}t$. Reinvesting this amount in the stock allows us to buy $N(t)S(t)\delta\,\mathrm{d}t/S(t) = N(t)\delta\,\mathrm{d}t$ more shares. In other words, the change in the number of shares is given by

$$\mathrm{d}N(t) = N(t+\mathrm{d}t) - N(t) = N(t)\delta\,\mathrm{d}t,$$

which means that

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \delta N(t).$$

The solution to this separable ordinary differential equation in $N(t)$ with initial shares $N(0)$ is given by[iii]

$$N(t) = N(0)\mathrm{e}^{\delta t}.$$

In particular, 1 share (i.e., $N(0) = 1$) at time 0 will grow to $\mathrm{e}^{\delta T}$ shares at time T. By proportion, to obtain 1 share at time T and to replicate the payoff of $S(T)$, it suffices to buy $\mathrm{e}^{-\delta T}$ shares at time 0 and reinvest all dividends in the stock between time 0 and time T, which is what the replicating portfolio entails. It follows that the fair prepaid forward price in the presence of continuous dividends is the cost of buying $\mathrm{e}^{-\delta T}$ shares at time 0, or

$$\boxed{F^P_{0,T} = S(0)\mathrm{e}^{-\delta T}.} \tag{2.2.2}$$

Adjusting the initial position to offset the effect of dividend income so that exactly one unit of the underlying stock is received at expiration is called *tailing* the position.

You may draw an analogy between the continuous dividend yield acting on the number of shares and the force of interest acting on the amount of money in a risk-free account as you have seen in your theory of interest class. If the continuous dividend yield is δ, then 1 share at time 0 will accumulate to $\mathrm{e}^{\delta T}$ shares at time T. Likewise, if the force of interest is δ, then \$1 invested in the risk-free money account at time 0 will grow to \$$\mathrm{e}^{\delta T}$ at time T.

Let's review what we have learned from Cases 1 and 2 via a challenging example that combines discrete and continuous dividends.

Example 2.2.2. [HARDER!] (Discrete plus continuous dividends) For $t \geq 0$, let $S(t)$ be the time-t price of Stock ABC. You are given:

(i) $S(0) = 100$

[iii] If $\delta = \delta(t)$ is a deterministic function of time t, then it can be shown that the number of shares at time t is given by

$$N(t) = N(0)\exp\left(\int_0^t \delta(s)\,\mathrm{d}s\right).$$

(ii) At time 0.5, a cash dividend of \$10 per share will be paid.

(iii) From time 0.75 to time 1, dividends are paid continuously at a rate proportional to its price. The dividend yield is 10%.

(iv) The continuously compounded risk-free interest rate is 8%.

Calculate the price of a one-year prepaid forward contract on stock ABC.

Ambrose's comments:

This is a relatively hard problem combining discrete and continuous proportional dividends. It nicely illustrates the futility of slavishly memorizing prepaid forward price formulas without a solid understanding of the underlying dividend-paying mechanics. To help you answer this question, think about how you can end up with exactly 1 unit of stock at time 1.

Solution. To receive exactly one unit of stock ABC at time 1, we should start with only $e^{-\delta(1-0.75)} = e^{-0.025}$ shares of stock ABC because of the reinvestment of the continuous proportional dividends between time 0.75 and time 1. With $e^{-0.025}$ shares at time 0 and at time 0.5, we will receive a cash dividend of $\$10e^{-0.025}$ at time 0.5:

Method	Cash Flows		
	$t = 0$	$t = 0.5$	$t = 1$
Outright purchase	$-S(0)e^{-0.025}$	$10e^{-0.025}$	$S(T)$
Prepaid forward	$-F^P_{0,1}$	0	$S(T)$

To fully imitate the one-year prepaid forward, the replicating portfolio consists of:

1. Buying $e^{-0.025}$ number of shares of stock ABC at time 0
2. Making a loan of $\text{PV}_{0,0.5}(10e^{-0.025})$ at time 0 and repaying it at time 0.5.

The one-year prepaid forward price equals the cost of setting up the replicating portfolio, which in turn is

$$\begin{aligned} F^P_{0,1} &= S(0)e^{-0.025} - 10e^{-0.025} \times e^{-0.5r} \\ &= 100e^{-0.25(0.1)} - 10e^{-0.25(0.1)} \times e^{-0.08(0.5)} \\ &= \boxed{88.16}. \end{aligned}$$

□

Epilogue: Significance of prepaid forward prices.

The notion of prepaid forward prices is not a very popular and widely used one in the literature.[iv] However, it is a very useful concept which permeates the whole book. Its significance is (at least) twofold:

1. As we will see in the next section, prepaid forward prices and forward prices are inti-

[iv] As far as the author is aware, prepaid forward prices are introduced in McDonald (2013) (see Chapter 5 therein).

mately connected. We can understand the pricing of prepaid forwards if we understand the pricing of forwards, and the other way round.

2. Prepaid forward prices are also objects of independent interest. They provide a unifying treatment of the current price (or value) of a general cash inflow, *random or non-random*, in the future. As we have seen earlier, the T-year prepaid forward price of a stock, $F_{0,T}^P$, captures what you have to pay today in order to receive exactly one unit of the stock at a future time T, with a random payoff of $S(T)$. To receive a constant cash inflow of $\$K$ at time T, we need only pay $\$\mathrm{PV}_{0,T}(K) = \$K\mathrm{e}^{-rT}$ today. We can symbolize this with the prepaid forward price notation by writing

$$F_{0,T}^P(K) = K\mathrm{e}^{-rT}. \tag{2.2.3}$$

When confusion arises, we write $F_{0,T}^P(S)$ to signify the prepaid forward price on the stock to distinguish it from the present value of cash.

2.3 Forwards

2.3.1 Forward Prices

Most elementary textbooks on financial derivatives treat forward contracts directly, but now that we are armed with the formulas of the fair prices of prepaid forwards, the corresponding formulas of forwards can be obtained with considerable ease.

Going from the prepaid forward price to the forward price.

Think in this way:

What is the difference between a prepaid forward and a forward?

As we have seen in Section 2.1, whether you buy a prepaid forward or a forward, you will always end up receiving one unit of the stock at time T. You do not receive the interim dividends, if any, with both contracts. The only difference is the time of payment:

- With a prepaid forward, you pay the prepaid forward price $F_{0,T}^P$ at time 0.
- With a forward, you pay the forward price $F_{0,T}$ at time T.

Since both prices are constants, $F_{0,T}$ is simply the time-T future value of $F_{0,T}^P$ accumulated at the risk-free interest rate, i.e.,

$$\boxed{F_{0,T} = \mathrm{FV}_{0,T}(F_{0,T}^P) \quad \text{or} \quad F_{0,T}^P = \mathrm{PV}_{0,T}(F_{0,T}).} \tag{2.3.1}$$

Explicit forward price formulas.

Accumulating the time-0 prepaid forward price formulas at the continuously compounded risk-free interest rate r from time 0 to time T, we readily have the following forward price formulas:

$$\boxed{F_{0,T} = \begin{cases} S(0)\mathrm{e}^{rT} - \underbrace{\sum_{i=1}^{n} D(t_i)\mathrm{e}^{r(T-t_i)}}_{\mathrm{FV}_{0,T}(\mathrm{Div})}, & \text{for discrete dividends} \\ S(0)\mathrm{e}^{(r-\delta)T}, & \text{for continuous proportional dividends} \end{cases}.} \tag{2.3.2}$$

Example 2.3.1. (SOA Exam IFM Introductory Derivatives Sample Question 29: Ranking of different methods) The dividend yield on a stock and the interest rate used to discount the stock's cash flows are both continuously compounded. The dividend yield is less than the interest rate, but both are positive.

The following table shows four methods to buy the stock and the total payment needed for each method. The payment amounts are as of the time of payment and have not been discounted to the present date.

METHOD	TOTAL PAYMENT
Outright purchase	A
Fully leveraged purchase	B
Prepaid forward contract	C
Forward contract	D

Determine which of the following is the correct ranking, from smallest to largest, for the amount of payment needed to acquire the stock.

(A) C < A < D < B

(B) A < C < D < B

(C) D < C < A < B

(D) C < A < B < D

(E) A < C < B < D

Solution. The expressions for A, B, C, and D are

$$\begin{aligned} A &= S(0), \\ B &= S(0)e^{rT}, \\ C &= F_{0,T}^{P} = S(0)e^{-\delta T}, \\ D &= F_{0,T} = S(0)e^{(r-\delta)T}. \end{aligned}$$

As $e^{-\delta T} < 1 < e^{(r-\delta)T} < e^{rT}$, the correct ranking is $\boxed{\text{C} < \text{A} < \text{D} < \text{B}}$. **(Answer: (A))** □

Example 2.3.2. (SOA Exam IFM Introductory Derivatives Sample Question 37: Discrete dividends – I) A one-year forward contract on a stock has a price of \$75. The stock is expected to pay a dividend of \$1.50 at two future times, six months from now and one year from now, and the annual effective risk-free interest rate is 6%.

Calculate the current stock price.

(A) 70.75

(B) 73.63

(C) 75.81

(D) 77.87

(E) 78.04

Solution. By the first formula in (2.3.2), we solve the equation

$$S(0)(1.06) - \underbrace{1.5(1.06^{1/2} + 1)}_{\mathrm{FV}_{0,1}(Div)} = F_{0,1} = 75$$

yields $S(0) = \boxed{73.63}$. **(Answer: (B))** □

Example 2.3.3. (SOA Exam IFM Introductory Derivatives Sample Question 51: Discrete dividends – II) You are given the following information about Stock XYZ:

(i) The current price of the stock is 35 per share.

(ii) The expected continuously compounded rate of return is 8%.

(iii) The stock pays semi-annual dividends of 0.32 per share, with the next dividend to be paid two months from now.

The continuously compounded risk-free interest rate is 4%.

Calculate the current one-year forward price for stock XYZ.

(A) 34.37

(B) 35.77

(C) 36.43

(D) 37.23

(E) 37.92

Solution. By the first formula in (2.3.2) again, the one-year forward price is

$$F_{0,1} = 35\mathrm{e}^{0.04} - 0.32(\mathrm{e}^{0.04(10/12)} + \mathrm{e}^{0.04(4/12)}) = \boxed{35.77}. \quad \textbf{(Answer: (B))}$$

□

Remark. The continuously compounded rate of return on the stock is not needed.

Remark: Relationship between the forward price and expected future stock price.

You may be tempted to think that forward prices are a good estimator of the expected stock prices in the future. Such an impression may stem from the very definition that the forward price is what you agree to pay in the future in place of the random future stock price for owning the stock. Intuition suggests that the forward price, as a replacement of the expected future stock price, should estimate the latter reasonably well. It turns out, however, that the forward price is a systematic *underestimator* of the expected future stock price.

For concreteness, let α be the continuously compounded expected rate of return on the stock. By definition, α is the continuously compounded rate at which the initial stock price accumulates to the expected stock price, i.e., α satisfies $\mathbb{E}[S(T)] = S(0)e^{\alpha T}$. Because the stock is risky in the sense that its future price movements are random, investors (to be precise, risk-averse investors) naturally demand that $\alpha > r$, so that the stock is worse than the risk-free asset because of the presence of price variability but outperforms the risk-free asset with respect to the expected rate of return. The difference $\alpha - r$ is known as the *risk premium*. It follows that

$$\mathbb{E}[S(T)] = S(0)e^{\alpha T} > S(0)e^{rT} > S(0)e^{rT} - \mathrm{FV}_{0,T}(\mathrm{Div}) = F_{0,T}$$

for any positive T, meaning that the expected stock price at expiration is always *higher* than the forward price, no matter whether the stock pays dividends, and in what manner.

Example 2.3.4. (SOA Exam IFM Introductory Derivatives Sample Question 6: Bounds on forward price/expected stock price – I) The following relates to one share of XYZ stock:

- The current price is 100.
- The forward price for delivery in one year is 105.
- P is the expected price in one year.

Determine which of the following statements about P is TRUE.

(A) $P < 100$

(B) $P = 100$

(C) $100 < P < 105$

(D) $P = 105$

(E) $P > 105$

Solution. The expected 1-year stock price should be higher than the 1-year forward price. **(Answer: (E))** □

Example 2.3.5. (SOA Exam IFM Introductory Derivatives Sample Question 38: Bounds on forward price/expected stock price – II) The current price of a medical company's stock is 75. The expected value of the stock price in three years is 90 per share. The stock pays no dividends.

You are also given:

(i) The risk-free interest rate is positive.

(ii) There are no transaction costs.

(iii) Investors require compensation for risk.

The price of a three-year forward on a share of this stock is X, and at this price an investor is willing to enter into the forward.

Determine what can be concluded about X.

Transaction	Cash Flows	
	Time 0	Time T
Buy $\mathrm{e}^{-\delta T}$ shares of stock	$-S(0)\mathrm{e}^{-\delta T}$	$+S(T)$
Borrow $S(0)\mathrm{e}^{-\delta T}$	$+S(0)\mathrm{e}^{-\delta T}$	$-F_{0,T} = -S(0)\mathrm{e}^{(r-\delta)T}$
Total	0	$S(T) - F_{0,T}$

TABLE 2.3
Demonstration of (2.3.3) in the case of continuous proportional dividends.

(A) $X < 75$

(B) $X = 75$

(C) $75 < X < 90$

(D) $X = 90$

(E) $90 < X$

Solution. Because the stock pays no dividends, $X = F_{0,3} = S(0)\mathrm{e}^{3r} > S(0) = 75$. Moreover, $X \leq \mathbb{E}[S(3)] = 90$. **(Answer: (C))** □

2.3.2 Cash-and-Carry Arbitrage

It bears mention that the forward price formulas given in (2.3.2) represent the fair price of a forward. Should the price of the forward in the market be different from the fair price, it is possible to develop an arbitrage strategy to reap risk-free profits. The key vehicle is a synthetic construction of a forward by the underlying stock coupled with borrowing or lending. In what follows, we assume that the stock pays continuous proportional dividends with a dividend yield of δ.

Synthetic forwards.

Because a forward is identical to a prepaid forward except that the payment is made at expiration instead of at the outset, a forward can be replicated by the replicating portfolio of the prepaid forward together with a loan that erases the initial investment (recall that a genuine forward carries no initial investment) and defers the payment to the time of expiration. We know from Subsection 2.2.2 that it suffices to buy $\mathrm{e}^{-\delta T}$ shares of the stock to replicate the prepaid forward, which costs $S(0)\mathrm{e}^{-\delta T}$ at time 0, so at the same time we also take out a loan of $S(0)\mathrm{e}^{-\delta T}$ and repay it at time T with interest.

Table 2.3 confirms that these two actions successfully reproduce the cash flows of a long forward. The *synthetic construction* can be written symbolically as

$$\boxed{\text{(Genuine) Forward} = \underbrace{\text{Stock} + \text{Borrowing}}_{\text{Synthetic forward}},} \tag{2.3.3}$$

where "=" indicates having the same cash flows at every point of time.

Transaction	Cash Flows	
	Time 0	**Time T**
Buy $\mathrm{e}^{-\delta T}$ shares of stock	$-S(0)\mathrm{e}^{-\delta T}$	$+S(T)$
Borrow $S(0)\mathrm{e}^{-\delta T}$	$+S(0)\mathrm{e}^{-\delta T}$	$-S(0)\mathrm{e}^{(r-\delta)T}$
Short forward	0	$F_{0,T}^{\text{obs}} - S(T)$
Total	0	$F_{0,T}^{\text{obs}} - S(0)\mathrm{e}^{(r-\delta)T} > 0$

TABLE 2.4
Transactions and cash flows for a cash-and-carry arbitrage.

Example 2.3.6. (SOA Exam IFM Introductory Derivatives Sample Question 56: Rearranging the synthetic forward equation) Determine which of the following positions has the same cash flows as a short stock position.

(A) Long forward and long zero-coupon bond

(B) Long forward and short forward

(C) Long forward and short zero-coupon bond

(D) Long zero-coupon bond and short forward

(E) Short forward and short zero-coupon bond

Solution. Rearranging (2.3.3) yields

$$-\text{Stock} = -\text{Forward} + \text{Borrowing},$$

which corresponds to (E). **(Answer: (E))** □

(Reverse) Cash-and-carry arbitrage.

Given the synthetic forward construction, we now put ourselves in the shoes of an arbitrageur and demonstrate how risk-free profits can be earned when the observed forward price $F_{0,T}^{\text{obs}}$ is not equal to the fair price $F_{0,T}^{\text{fair}} = S(0)\mathrm{e}^{(r-\delta)T}$.

Case 1. Suppose that $F_{0,T}^{\text{obs}} > F_{0,T}^{\text{fair}} = S(0)\mathrm{e}^{(r-\delta)T}$ (i.e., the observed forward price is too high, or the forward is *overpriced*). We follow the "buy low, sell high" rule, which means selling the overpriced forward in the market and buying the synthetic forward in accordance with (2.3.3). Table 2.4 shows the transactions, called *cash-and-carry*, and the corresponding cash flows. Note that the cash inflow at time T involves quantities that are already known at time 0.

Here is a mnemonic about why we call the transactions in Table 2.4 "cash-and-carry":

Borrow the "cash" required and

"carry" the stock from time 0 to time T.

Example 2.3.7. (SOA Exam IFM Introductory Derivatives Sample Question 21: Cash-and-carry arbitrage in action!) A market maker in stock index forward contracts observes a 6-month forward price of 112 on the index. The index spot price is 110 and the continuously compounded dividend yield on the index is 2%.

The continuously compounded risk-free interest rate is 5%.

Describe actions the market maker could take to exploit an arbitrage opportunity and calculate the resulting profit (per index unit).

(A) Buy observed forward, sell synthetic forward, Profit = 0.34

(B) Buy observed forward, sell synthetic forward, Profit = 0.78

(C) Buy observed forward, sell synthetic forward, Profit = 1.35

(D) Sell observed forward, buy synthetic forward, Profit = 0.78

(E) Sell observed forward, buy synthetic forward, Profit = 0.34

Solution. The fair 6-month price of the forward contract is

$$F_{0,0.5}^{\text{fair}} = S(0)\mathrm{e}^{(r-\delta)T} = 110\mathrm{e}^{(0.05-0.02)(0.5)} = 111.6624,$$

which is 0.34 less than the observed price. Thus we can engage in cash-and-carry arbitrage by selling the observed forward at 112 and buying a synthetic forward (i.e., buying $\mathrm{e}^{-0.01}$ units of the index and borrowing $110\mathrm{e}^{-0.01}$) at 111.6624, realizing a profit of $112 - 111.6624 = \boxed{0.34}$. **(Answer: (E))** □

Remark. The arbitrage profit, realized at expiration, is equal to the absolute value of the difference between the observed forward price and the fair forward price.

Case 2. Suppose that $F_{0,T}^{\text{obs}} < F_{0,T}^{\text{fair}} = S(0)\mathrm{e}^{(r-\delta)T}$ (i.e., the observed forward price is too low, or the forward is *underpriced*). In this case, we simply take the opposite strategy of Case 1, namely:

1. Buy the observed forward in the market.
2. Sell the synthetic forward, i.e., short sell the stock and lend the proceeds.

These transactions constitute a *reverse cash-and-carry arbitrage* whose components and associated cash flows are shown in Table 2.5.

Example 2.3.8. (SOA Exam IFM Introductory Derivatives Sample Question 73) The current price of a nondividend-paying stock is 100. The annual effective risk-free interest rate is 4%, and there are no transaction costs.

The stock's two-year forward price is mispriced at 108, so to exploit this mispricing, an investor can short a share of the stock for 100 and simultaneously take a long position in a two-year forward contract. The investor can then invest the 100 at the risk-free rate, and finally buy back the share of stock at the forward price after two years.

Determine which term best describes this strategy.

(A) Hedging

Transaction	**Cash Flows**	
	Time 0	**Time T**
Short $e^{-\delta T}$ shares of stock	$+S(0)e^{-\delta T}$	$-S(T)$
Lend $S(0)e^{-\delta T}$	$-S(0)e^{-\delta T}$	$+S(0)e^{(r-\delta)T}$
Long forward	0	$S(T)-F_{0,T}^{\text{obs}}$
Total	0	$S(0)e^{(r-\delta)T}-F_{0,T}^{\text{obs}}>0$

TABLE 2.5
Transactions and cash flows for a reverse cash-and-carry arbitrage.

(B) Immunization

(C) Arbitrage

(D) Paylater

(E) Diversification

Solution. The fair 2-year forward price is $F_{0,2}^{\text{fair}}=100(1.04)^2=108.16$, which is higher than the observed forward price of 108. Thus the forward in the market is underpriced, and a reverse cash-and-carry $\boxed{\text{arbitrage}}$ should be undertaken. **(Answer: (C))** □

Example 2.3.9. (Reverse cash-and-carry arbitrage with discrete dividends) The current stock price is 80. A dividend of 2 will be paid 6 months from now. The continuously compounded risk-free interest rate is 6%.

If you observe a 1-year forward price of 82, describe actions you could take as an arbitrageur, and calculate the resulting arbitrage profit (per stock unit).

Ambrose's comments:

This example is a "reverse" and "discrete" counterpart of Example 2.3.7. To avoid repetition, the stock is changed to pay discrete dividends.

Solution. The fair 1-year forward price is $F_{0,1}^{\text{fair}}=80e^{0.06}-2e^{0.03}=82.8860$, which is higher than the observed price. To effect a reverse cash-and-carry arbitrage, we buy the observed forward, short sell the stock, and borrow and lend as follows:

Transaction	Cash Flows		
	Time 0	Time **1/2**	Time 1
Short 1 share of stock	$+80$	-2	$-S(1$
Lend $S(0)$ at time 0	-80	0	$+80e^{0.06}$
Borrow 2 at time 0.5	0	$+2$	$-2e^{0.03}$
Long forward	0	0	$S(1)-82$
Total	0	0	$\boxed{0.8860}$

Note the dividend of 2, which we have to *pay* in 6 months because of selling the stock. □

Sidebar: Short selling a stock.

In Example 2.3.9 above, we sell a share of the stock without actually owning it in the first place. This possibly strange act is known as *short selling*, which refers to selling an asset that you do not own. Although short selling is mathematically the opposite of an ordinary purchase, practically it is not as simple as it may seem. The precise procedure is as follows:

Step 1. Borrow the asset from a third party (e.g., your broker).

Step 2. Immediately sell the asset to the market, thereby creating a *short* position, and receive the proceeds.

Step 3. Buy back the same asset some time later and return it to the lender. The short position in the asset is said to be *closed out.*

Short selling is therefore essentially the same as borrowing money and paying back a random amount which is determined by the future asset price.

Some of the primary motivates for engaging in a short sale are:

1. *Speculation:* A short sale makes money only if the asset price goes down, or you sell high first and buy low later. Therefore, short-selling can be regarded as speculating that the asset price will drop in the future.

2. *Hedging:* A short sale can be undertaken to offset asset price risk. For example, a long forward coupled with a short sale will eliminate any uncertainty arising from future asset price movements.

3. *Arbitrage:* When you engage in a reverse cash-and-carry arbitrage, part of your transactions is to short sell the stock, as in Example 2.3.9.

Note that in the presence of dividends, a short seller must pay the broker any dividends payable by the asset that has been shorted. The broker will transfer such dividends to the lender from whom the asset has been borrowed.

Example 2.3.10. (Illustration of cash flows under a short sale) On January 1, the price of a stock is \$120 per share. An investor short sells 500 shares and closes out the position by buying them back on March 1 when the price per share is \$100. A dividend of \$1 *per share* is paid on February 1.

The cash flow of the investor is as follows:

Time	Transaction	Cash Flow
Jan 1	Borrow 500 shares and sell them at \$120	$\$60,000$
Feb 1	Pay dividend	$-\$500$
Mar 1	Buy 500 shares at \$100 and repay short sale	$-\$50,000$

Note that these cash flows are exactly the opposite of an investor who is long 500 shares.

2.3.3 Digression: Market Frictions

Thus far, we have been discussing the trading of derivatives in an idealistic, friction-free market. There are no taxes, transaction costs, bid/ask spreads, and no disparity between borrowing and lending interest rates. In this brief subsection, we take into account these practical factors and explore how they impact on fair forward prices.

Bid-ask spread	For any given asset, the price at which you can buy is called the *ask price*, $S^a(0)$, while the price at which you can sell is called the *bid price*, $S^b(0)$. At any moment of time, the ask price must be higher than or equal to the bid price, or else you can buy the asset at the ask price and immediately sell the asset at the bid price, realizing a risk-free profit. The difference between the two prices is called the *bid-ask spread*, which is a source of income to market makers.
Disparity between borrowing and lending rates	In general, the borrowing rate r^b and lending rate r^l need not be identical. To avoid arbitrage, it must hold that $r^b \geq r^l$ (why?).
Transaction costs	Buying or selling derivatives involves transaction costs, which can be a fixed amount or a variable amount proportional to the scale of transaction.

Example 2.3.11. (How much transaction cost do you incur?) You observe two prices \$65.1 and \$65.2 quoted in the market for the stock of Company ABC.

The brokerage commission includes:

(i) 0.3% of the transaction amount

(ii) A fixed cost of \$50 per transaction.

Calculate how much you gain/loss if you purchase 200 shares and then sell them all immediately.

Solution. Among the two observed prices, the bid price must be lower than or equal to the ask price, so the bid price is \$65.1 and the ask price is \$65.2.

To complete the desired transaction, you first spend $200 \times 65.2 \times (1 + 0.3\%) + 50 = \$13,129.12$ for purchasing 200 shares at the ask price, then receive $200 \times 65.1 \times (1 - 0.3\%) - 50 = \$12,930.94$ upon selling the 200 shares at the bid price. The amount lost is $13,129.12 - 12,930.94 = \boxed{\$198.18}$. □

The following example shows how market frictions affect no-arbitrage forward prices.

Example 2.3.12. [HARDER!] (SOA Exam IFM Introductory Derivatives Sample Question 52: No-arbitrage forward price) The ask price for a share of ABC company is 100.50 and the bid price is 100. Suppose an investor can borrow at an annual effective rate of 3.05% and lend (i.e., save) at an annual effective rate of 3%. Assume there are no transaction costs and no dividends.

Determine which of the following strategies does not create an arbitrage opportunity.

(A) Short sell one share, and enter into a long one-year forward contract on one share with a forward price of 102.50.

(B) Short sell one share, and enter into a long one-year forward contract on one share with a forward price of 102.75.

(C) Short sell one share, and enter into a long one-year forward contract on one share with a forward price of 103.00.

(D) Purchase one share with borrowed money, and enter into a short one-year forward contract on one share with a forward price of 103.60.

(E) Purchase one share with borrowed money, and enter into a short one-year forward contract on one share with a forward price of 103.75.

Solution. • *Prelude:* In the presence of market frictions, the fair forward price will no longer be a unique value. Instead, it will be relaxed to an *interval* whose length is governed by the extent to which the market frictions exist. It can be shown by no-arbitrage arguments that market frictions enter the no-arbitrage forward price interval in such a way to make it as wide as possible, e.g., if there exist a bid-ask spread and different interest rates for borrowing and lending, then the bid price will go with the lending rate to form the lower bound of the interval, and the ask price will go with the borrowing rate to form the upper bound:

$$F_{0,T}^{\text{fair}} \in [S^b(0)\mathrm{e}^{(r^l-\delta)T}, S^a(0)\mathrm{e}^{(r^b-\delta)T}].$$

• *Back to this example:* Given the above result, the no-arbitrage interval for the 1-year forward price can be (and will be) shown to be

$$[100(1.03), 100.50(1.0305)] = [103, 103.56525].$$

Observe that the forward prices in (A) and (B) are strictly less than the lower bound of 103 whereas those in (D) and (E) are strictly greater than the upper bound of 103.56525. For the purpose of illustration, we consider (A), (B), and (C), where $F_{0,1}^{\text{obs}} \leq 103$, and show that there is an arbitrage opportunity as long as $F_{0,1}^{\text{obs}} < 103$. Because the observed forward is underpriced, we pursue a reverse cash-and-carry arbitrage strategy by buying the forward and selling the synthetic forward by short selling one share (recall that there are no dividends) and lending the proceeds. The following table shows the resulting cash flows:

Transaction	Cash Flows	
	Time 0	Time 1
Short one share of stock	$S^b(0) = 100$	$-S(1)$
Lend $S^b(0)$	$-S^b(0) = -100$	$100(1.03) = 103$
Long forward	0	$S(1) - F_{0,1}^{\text{obs}}$
Total	0	$103 - F_{0,1}^{\text{obs}} \geq 0$

Note that in the "Time 0" column, it is the *bid* price that is used (we are selling the stock to the market) and in the "Time 1 column" the *lending* rate is used to accumulate the proceeds of 100. Now observe that the cash flow at time 1 is strictly positive so long as $F_{0,1}^{\text{obs}} < 103$ and an arbitrage strategy is created. If $F_{0,1}^{\text{obs}} = 103$, then the cash flows at both time 0 and time 1 are zero. Nothing is paid and received. **(Answer: (C))**

□

Remark. Using the above strategy, we can conclude that $F_{0,1}^{\text{obs}}$ should be at least 103 to preclude arbitrage opportunities. The upper bound of 103.56525 can be established by a cash-and-carry arbitrage strategy.

2.4 Futures

We conclude this chapter with a brief discussion on futures contracts and look at how they are implemented in practice.

2.4.1 Differences between Futures and Forwards

Futures vs forwards.

Futures contracts (make sure that you write 'futures," not 'future!") are similar to forward contracts in the sense that both of them involve a commitment to buy or sell the underlying asset at a prespecified price on a particular date. However, there are important contractual differences between them with respect to the time of settlement, liquidity, uniformity, the issue of credit risk, and the existence of price limits (see Table 2.7). These will become clearer as we learn the mechanics of futures in the next subsection.

Example 2.4.1. (SOA Exam IFM Introductory Derivatives Sample Question 30: Forward vs futures – I) Determine which of the following is NOT a distinguishing characteristic of futures contracts, relative to forward contracts.

(A) Contracts are settled daily, and marked-to-market.

(B) Contracts are more liquid, as one can offset an obligation by taking the opposite position.

(C) Contracts are more customized to suit the buyer's needs.

(D) Contracts are structured to minimize the effects of credit risk.

(E) Contracts have price limits, beyond which trading may be temporarily halted.

Solution. It is forwards that are more customized and futures that are more standardized. **(Answer: (C))** □

	Forwards	Futures
1.	Settled at expiration.	Settled at the end of every mark-to-market period (see Subsection 2.4.2).
2.	Traded over the counter, forwards are relatively illiquid.	Being exchange-traded, futures are liquid. It is easy to offset obligations by entering into opposite positions.
3.	Customized to suit the needs of buyer and seller.	Standardized with respect to expiration date, size, underlying asset, etc.
4.	Credit risk remains—the buyer/seller may fail to fulfill his/her obligations.	Credit risk is minimized as a result of marking to market.
5.	No price limits.	Price limits are imposed, triggering temporary halts in trading if the futures price moves dramatically.

TABLE 2.7
Differences between forwards and futures.

Example 2.4.2. (SOA Exam IFM Introductory Derivatives Sample Question 69: Forward vs futures – II) Determine which of the following statements about futures and forward contracts is false.

(A) Frequent marking-to-market and settlement of a futures contract can lead to pricing differences between a futures contract and an otherwise identical forward contract.

(B) Over-the-counter forward contracts can be customized to suit the buyer or seller, whereas futures contracts are standardized.

(C) Users of forward contracts are more able to minimize credit risk than are users of futures contracts.

(D) Forward contracts can be used to synthetically switch a portfolio invested in stocks into bonds.

(E) The holder of a long futures contract must place a fraction of the cost with an intermediary and provide assurances on the remaining purchase price.

Solution. Futures contracts are more useful than forwards for minimizing credit risk. This is due to the typically daily settlement of futures contracts. **(Answer: (C))** □

2.4.2 Marking to Market

How does marking to market work?

As an illustration of how a futures contract works, consider the S&P 500 futures contract, which has the following specifications:

Underlying	S&P 500 index
Notional value (size)	$250× S&P 500 futures price (i.e., multiplier = 250)
Months	March, June, September, December
Settlement	Cash-settled
Mark-to-market frequency	Daily

We also assume:

Margin	10% (of the notional value of the futures contract)
Risk-free rate	6%, compounded continuously per annum
Current futures price	1,100

With a slight abuse of notation, we denote the Day-t price of the futures maturing at Day n for $t \le n$ by $F_{t,n}$.

We now discuss various components that play a role in implementing a futures contract.

- *Margin:* Both buyers and sellers of a futures contract have to make a deposit, known as the *margin*, into an interest-bearing performance bond. This protects both parties against the failure to meet obligations. With the margin being 10% of the notional value of the contract, the *initial margin balance* is

$$\begin{aligned} B_0 &= \boxed{\text{Margin} \times \text{Multiplier} \times F_{0,n}} \qquad (2.4.1) \\ &= 10\% \times 250 \times 1,100 \\ &= 27,500. \end{aligned}$$

  Denote by B_t the balance in the margin account at the end of Day t, for $t = 1, 2, \ldots, n$.

- *Mark-to-market proceeds:* Under an S&P 500 futures contract, adjustments will be made daily to account for the changes in futures price. At the end of day t, the margin account receives a payment, known as a *mark-to-market proceed*, which reflects the change in value of the futures contract and is given mathematically, for a long futures position,[v] by

$$\boxed{D_t = \text{Multiplier} \times \underbrace{(F_{t,n} - F_{t-1,n})}_{\text{change in futures price}},} \qquad (2.4.2)$$

  which can be positive (the margin account gets credited) or negative (the margin account gets debited).

  To make sense of (2.4.2), we can think in this way:

  ▷ Consider $t = 1$ (i.e., Day 1). If the futures contracts *were* settled at the end of Day 1, you would be obligated to pay $F_{0,n}$. In contrast, a futures that is entered on Day 1 would require a payment of $F_{1,n}$. You therefore *save* $F_{1,n} - F_{0,n}$ (which can be positive or negative) per unit of the index by owning the futures purchased on Day 0. Multiplying this quantity by the multiplier specified in the futures contract yields the mark-to-market proceed on Day 1.

  ▷ On Day 2, the amount of "saving" is

$$F_{2,n} - F_{0,n} = (F_{2,n} - F_{1,n}) + \underbrace{(F_{1,n} - F_{0,n})}_{\text{Accounted for by } D_1}.$$

[v] The mark-to-market proceed for a *short* futures position is

$$D_t = \text{Multiplier} \times (F_{t-1,n} - F_{t,n}).$$

Day	Futures Price	Mark-to-Market Proceeds	Margin Account Balance
0	1,100	(NA)	27,500
1	1,050	−12,500	15,004.52
2	1,120	17,500	32,506.99
3	1,180	15,000	47,512.33
4	1,200	5000	52,520.14

TABLE 2.8
Mark-to-market proceeds and margin account balance over 4 days from a long position in S&P 500 futures contract.

Since the second term has been included in D_1, only the first term enters the mark-to-market proceed on Day 2.

▷ Inductively, we can regard each D_t as the *incremental* "saving" or as the *incremental value* brought by the futures on Day t.

- *Recursive calculations of margin account balances:* The balance in the margin account at the end of Day t can be determined recursively as

$$\boxed{B_t = B_{t-1} \underbrace{e^{r/365}}_{\substack{\text{accumulate previous} \\ \text{balance w/ interest}}} + D_t, \quad t = 1, 2, \ldots, n,}$$

with the starting value B_0 given in (2.4.1).

As an example, suppose that the futures price drops from 1,100 to 1,050 on Day 1. Then the mark-to-market proceed on Day 1 is

$$250 \times (1,050 - 1,100) = -12,500.$$

Since the initial balance has also earned interest for one day, the final balance at the end of Day 1 is

$$27,500e^{0.06/365} - 12,500 = 15,004.52.$$

- *Maintenance margin and margin call:* Since the mark-to-market proceeds can be negative, it can happen that the margin account balance will drop substantially below the initial margin. For this reason, there is usually a minimum level of the margin account balance, called the *maintenance margin* (often set at 70%-80% of the initial margin), below which the investor will receive a *margin call*, requiring him/her to make an additional deposit to bring the balance back to the initial margin level.

- *Overall profit:* The profit of the futures on Day t, should you choose to leave, is obtained by subtracting the future value of the original margin deposit from the final margin balance:

$$\text{Profit}_t = B_t - B_0 e^{r \times n/365}.$$

Table 2.8 displays the evolution of the margin account balance over 4 days with hypothetical futures prices. The profit of the futures position is

$$52,520.14 - 27,500e^{0.06 \times 4/365} = 25,002.05.$$

In contrast, the profit of a 4-day forward is

$$250(1,200 - 1,100) = 25,000,$$

which is slightly lower.

We end with two examples concerned with determining the mark-to-market proceeds and margin account balances for a small number of mark-to-market periods, e.g., one or two, which is amenable to pen-and-paper calculations.

Example 2.4.3. (SOA Exam IFM Introductory Derivatives Sample Question 45: Given the mark-to-market proceed) An investor enters a long position in a futures contract on an index (F) with a notional value of $200 \times F$, expiring in one year. The index pays a continuously compounded dividend yield of 4%, and the continuously compounded risk-free interest rate is 2%.

At the time of purchase, the index price is 1100. Three months later, the investor has sustained a loss of 100. Assume the margin account earns an interest rate of 0%.

Let S be the price of the index at the end of month three.

Calculate S.

(A) 1078

(B) 1085

(C) 1094

(D) 1105

(E) 1110

Solution. Since the margin account does not earn interest, the mark-to-market frequency plays no role. The time-0 and time-0.25 futures prices, assumed to be equal to forward prices, are, respectively,

$$F_{0,1} = 1,100\mathrm{e}^{(0.02-0.04)(1)} = 1,078.2185$$

and

$$F_{0.25,1} = S\mathrm{e}^{(0.02-0.04)(0.75)} = 0.985112S.$$

Solving $200(F_{0.25,1} - F_{0,1}) = -100$ results in $S = \boxed{1,094.01}$. **(Answer: (C))** □

Remark. It can be shown that when the risk-free interest rate is constant, futures prices coincide with forward prices (see, e.g., page 143 of McDonald (2013) for discussions).

Example 2.4.4. (SOA Exam IFM Introductory Derivatives Sample Question 32: Range of values of index price) Judy decides to take a short position in 20 contracts of S&P 500 futures. Each contract is for the delivery of 250 units of the index at a price of 1500 per unit, exactly one month from now. The initial margin is 5% of the notional value, and the maintenance margin is 90% of the initial margin. Judy earns a continuously compounded risk-free interest rate of 4% on her margin balance. The position is marked-to-market on a daily basis.

On the day of the first marking-to-market, the value of the index drops to 1498. On the day of the second marking-to-market, the value of the index is X and Judy is not required to add anything to the margin account.

Calculate the largest possible value of X.

(A) 1490.50

(B) 1492.50

(C) 1500.50

(D) 1505.50

(E) 1507.50

Solution. The initial margin is $5\% \times 20 \times 250 \times 1,500 = 375,000$. The maintenance margin is $0.9(375,000) = 337,500$.

On the day of the first marking-to-market, the margin account balance is

$$375,000e^{0.04/365} + 20(250)(1,500 - 1,498) = 385,041.0981.$$

On the day of the second marking-to-market, the margin account balance becomes

$$385,041.0981e^{0.04/365} + 20(250)(1,498 - X).$$

For this to be greater than $337,500$, we have $X < \boxed{1,507.52}$. **(Answer: (E))** □

Remark. Strictly speaking, the mark-to-market payments should be based on the changes in the *futures* price, not the index price after one day. However, the one-day futures price is not given in the question. Also refer to the (correct) solution of the previous example.

2.5 Problems

Prepaid forwards

Problem 2.5.1. (Direct application of prepaid forward price formula) The current price of a stock is 50. It pays a dividend of \$1 every 3 months, with the first dividend to be paid 3 months from today and the last dividend to be paid in 1 year.

The continuously compounded risk-free interest rate is 6%.

Calculate the price of a prepaid forward contract that expires 1 year from today, immediately after the last dividend.

Problem 2.5.2. (Given the forward price, deduce the interest rate – I) The current price of a stock is 60. A dividend of 2 will be paid 6 months from now.

The one-year forward price is 61.80.

Calculate the continuously compounded risk-free annual rate of interest.

Problem 2.5.3. (Given the forward price, deduce the interest rate – II) The current price of stock XYZ is \$80. A one-year forward contract on stock XYZ has a price of \$84. Stock XYZ is expected to pay a dividend of \$2 six months from now and a dividend of \$3 one year from now, immediately before the one-year forward contract expires.

Calculate the effective annual risk-free interest rate, assuming that it is non-negative.

Problem 2.5.4. (Price of continuous random dividends) The current price of a stock is 100. The stock pays dividends continuously at a rate proportional to its price. The dividend yields is 3%.

The continuously compounded risk-free interest rate is 7%.

Calculate the *price* of the stream of dividends to be paid in the next 5 years.

(Note: Because the dividends are stochastic, their present value is also stochastic and hence cannot be their price.)

Forwards

Problem 2.5.5. (SOA Exam IFM Introductory Derivatives Sample Question 20: Geometrically increasing discrete dividends) The current price of a stock is 200, and the continuously compounded risk-free interest rate is 4%. A dividend will be paid every quarter for the next 3 years, with the first dividend occurring 3 months from now. The amount of the first dividend is 1.50, but each subsequent dividend will be 1% higher than the one previously paid.

Calculate the fair price of a 3-year forward contract on this stock.

(A) 200

(B) 205

(C) 210

(D) 215

(E) 220

Problem 2.5.6. (Piecewise constant dividend yield) It is now January 1, 3018. You are given:

(i) The current price of the stock is 1,000.

(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield changes throughout the year. In March, June, September, and December, the dividend yield is 3%. In other months, the dividend yield is 2%.

(iii) The continuously compounded risk-free interest rate is 9%.

Calculate the price of a 1-year forward contract.

(Hint: How many shares should you buy at time 0 to end up with exactly one share in one year?)

Problem 2.5.7. (Arbitraging a mispriced forward) The current price of stock XYZ is 120. Stock XYZ pays dividends continuously at a rate proportional to its price. The dividend yield is 4%. The continuously compounded risk-free interest rate is 6%.

You observe a 1-year forward price of 121 on stock XYZ.

Describe, giving as many details as possible, actions you could take at time 0 to exploit an arbitrage opportunity and calculate the resulting profit (per unit of stock XYZ at expiration).

Problem 2.5.8. (Fair dividend yield) You are given:

(i) The current price of a stock is 1,000.

(ii) The stock pays dividends continuously at a rate proportional to its price.

(iii) The continuously compounded risk-free interest rate is 5%.

(iv) A 6-month forward price of 1,020 is observed in the market.

Describe actions you could take to exploit an arbitrage opportunity and calculate the resulting profit (per stock unit) in each of the following cases:

(a) The dividend yield of the stock is 0.5%

(b) The dividend yield of the stock is 2%

Problem 2.5.9. (Short selling stocks with continuous proportional dividends and transaction costs) You short sold 100 shares of stock X on November 1, 3016 and closed your position on November 1, 3018.

You are given:

(i) Stock X pays dividends continuously at a rate proportional to its price. The dividend yield is 4%.

(ii) The continuously compounded risk-free interest rate is 5%.

(iii) The commission rate (for purchase or sale) is 2% of the transaction amount.

(iv) The following bid and ask prices of stock X (per share) observed at various time points:

Date	Bid Price	Ask Price
November 1, 3016	95.0	95.5
November 1, 3017	98.0	98.5
November 1, 3018	100.0	100.5

Calculate your profit measured as of November 1, 3018.

Problem 2.5.10. (SOA Exam IFM Introductory Derivatives Sample Question 70: Bounds on forward price/expected stock price) Investors in a certain stock demand to be compensated for risk. The current stock price is 100.

The stock pays dividends at a rate proportional to its price. The dividend yield is 2%. The continuously compounded risk-free interest rate is 5%.

Assume there are no transaction costs.

Let X represent the expected value of the stock price 2 years from today. Assume it is known that X is a whole number.

Determine which of the following statements is true about X.

(A) The only possible value of X is 105.

(B) The largest possible value of X is 106.

(C) The smallest possible value of X is 107.

(D) The largest possible value of X is 110.

(E) The smallest possible value of X is 111.

Problem 2.5.11. (Comparison of various forward-related quantities) Consider a stock which pays dividends continuously at a rate proportional to its price. The dividend yield is less than the interest rate, but both are positive and continuously compounded.

Rank the following quantities in ascending order (i.e., from lowest to highest):

(A) = Current stock price
(B) = One-year forward price
(C) = Two-year forward price
(D) = Two-year prepaid forward price
(E) = Expected stock price at the end of two years

Problem 2.5.12. (No-arbitrage interval with discrete dividends) You are given the following information:

(i) The current bid price and ask price of stock X are 50 and 51, respectively.

(ii) A dividend of 3 will be paid 6 months from now.

(iii) The continuously compounded risk-free interest rate is 6%.

(iv) The one-year forward price on stock X is 55.

(v) The only transaction costs are:

- A \$1.5 transaction fee, paid at time 0, for buying or selling each unit of stock X.
- A \$1 transaction fee, paid at time 0, for buying or selling a forward contract on stock X.

Describe how arbitrage profits *in one year* can be locked in using actions taken *at time 0 only*. Calculate the resulting profit (per stock unit).

Problem 2.5.13. (No-arbitrage interval with continuous dividends) You are given the following information:

(i) The current bid price and ask price of stock Y are 40 and 41, respectively.

(ii) Stock Y pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(iii) The continuously compounded lending and borrowing rates are 6% and 7%, respectively.

(iv) The only transaction costs are:

- A \$1 transaction fee, paid at time 0, for buying or selling each unit of stock B.
- A \$2 transaction fee, paid at expiration, for settling a forward contract on stock B.

(v) The 3-year forward price on stock B is 38.

Describe actions you could take at time 0 to exploit an arbitrage opportunity (if any). Calculate the resulting profit (per stock unit).

Futures

Problem 2.5.14. (Margin account balance calculation given futures prices) You are given the following historical futures prices of the P&K 689 Index observed at different time points for various maturities:

	Futures Price		
Observation Date	1-month	2-month	3-month
March 1, 2018	624	626	629
April 1, 2018	712	715	718
May 1, 2018	640	643	645

On March 1, 2018, Cyrus decides to take a long position in ten 3-month P&K 689 index futures. Each contract permits the delivery of 250 units of the index. The initial margin is 10% of the notional value, and the maintenance margin is 75% of the initial margin. Cyrus earns a continuously compounded risk-free interest rate of 5% on his margin balance. The position is marked-to-market on a monthly basis.

Calculate the balance in Cyrus's margin account on the day of the second marking-to-market (i.e., on May 1, 2018).

(Hint: Only some of the futures prices are useful.)

Problem 2.5.15. (Margin call) The P&Q futures is currently trading at 1,629. The P&Q index pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

Today Peter enters into eight 3-month P&Q long futures contracts. Each contract permits the delivery of 250 units of the index. The initial margin is 10% of the notional value, and the maintenance margin is 70% of the initial margin. Peter earns the continuously compounded risk-free interest rate of 6% on his margin balance. The position is marked-to-market on a monthly basis.

Determine the range of values for the P&Q index price 1 month from now which results in a margin call.

3

Option Strategies

Chapter overview: In this chapter, we turn our attention to options and option strategies. We shall combine different option positions and obtain in this way a wide array of new derivatives which allow investors to better manage risk or speculate on asset price movements. In Section 3.1, we illustrate the use of a long put and a long call to hedge against the risks inherent in a long asset and a short asset, respectively. These results bear particular relevance to a business entity which sells its products or purchases its production inputs regularly at random future prices as part of its operating cycle. In Section 3.2, we demonstrate how calls and puts can be coupled to artificially create forwards. In the course of doing so, we obtain an all-important relationship known as put-call parity governing the prices of otherwise identical European call and put options. In Sections 3.3 and 3.4, we explore a number of new option strategies that involve the combinations of options of different types and/or strike prices and can be appropriately adopted to satisfy various risk management needs. The put-call parity developed in Section 3.2 allows us to understand the properties of and relations between the costs of some of these strategies.

Note that much of this conceptual chapter can be understood algebraically (i.e., by combining the payoff formulas of different positions), graphically (i.e., by looking at the payoff diagram of different positions), or by verbal reasoning (i.e., by thinking about how different positions work). Although all of these three approaches will be discussed, it is highly suggested that you strive to master the latter two approaches because of their pedagogical value— they are the key to understanding portfolios of options. As a key learning objective of this chapter, you should be acquainted with the shape and characteristics of the payoff diagram of a multitude of option strategies. You will find that the geometry of the payoff diagrams of these strategies can go a long way towards helping you remember their definitions.

3.1 Basic Insurance Strategies

This section introduces how options can be combined with a long or short position in the underlying asset to protect yourself against some adverse market scenarios. In each case, if we are long (resp. short) with respect to the underlying asset, then we take a counteracting position in an option which is short (resp. long) with respect to the asset.

3.1.1 Insuring a Long Position: Floors

Motivation.

At time 0, suppose that you invest in a certain asset, which for concreteness we assume is a share of a hypothetical stock called ABC, and plan to sell it in T years. If the price of stock ABC drops substantially after T years, then, against your original will, you can only sell

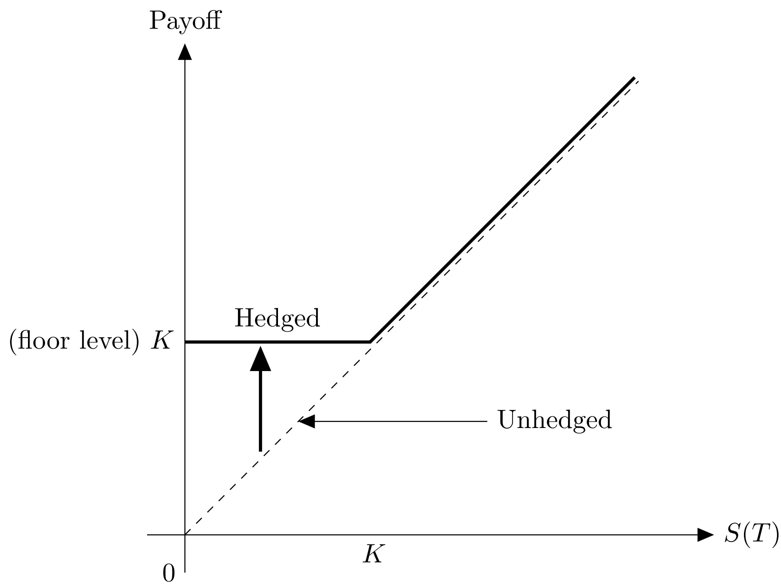


FIGURE 3.1.1
The payoff diagrams of a long asset (unhedged, dashed) and a long asset coupled with a long K-strike put (hedged, bold).

the stock for much less than the initial stock price. To hedge against this "downside risk," a put option on stock ABC can be of use.

Payoff and profit.

If you buy a K-strike T-year put option on one share of stock ABC, then your total time-T payoff is

$$\text{Payoff} = \underbrace{S(T)}_{\text{long asset}} + \underbrace{(K - S(T))_+}_{\text{long put}} = \begin{cases} K, & \text{if } S(T) < K, \\ S(T), & \text{if } S(T) \geq K, \end{cases}$$

or more compactly,

$$\text{Payoff} = \max(S(T), K),$$

which is bounded from below by the strike price K (see Figure 3.1.1 for the payoff diagram, where the payoff of the unhedged long asset between $S(T) = 0$ and $S(T) = K$ is pushed upward due to the long K-strike put). In other words, the downside risk is completely eliminated. For this reason, the put,[i] in the presence of a long stock position, is also called a *floor*—it places a "floor" on the sale price of the stock ABC (see the floor level of K in Figure 3.1.1).

A floor is a classical example of how options can serve as insurance. Intuitively, the put creates value in this context by allowing you to sell a share of stock ABC for at least K. If the time-T stock price of stock ABC is higher than K, then you can simply scrap the put and sell the stock at the higher time-T stock price. In the event that the time-T price of stock ABC falls below K, you can exercise the put option and sell the stock for K, which is the guaranteed minimum sale price. Of course, the downside protection that the floor

[i] A portfolio consisting of a long asset plus a long put is sometimes referred to as a *protective put*.

creates comes with a price: you need to pay the put option premium upfront (in addition to the time-0 price of stock ABC), which carries an interest cost.

Example 3.1.1. (SOA Exam IFM Introductory Derivatives Sample Question 75: Hedging an implicit long position) Determine which of the following risk management techniques can hedge the financial risk of an oil producer arising from the price of the oil that it sells.

I. Short forward position on the price of oil

II. Long put option on the price of oil

III. Long call option on the price of oil

(A) I only

(B) II only

(C) III only

(D) I, II, and III

(E) The correct answer is not given by (A), (B), (C), or (D)

Solution. Because the oil producer is to sell oil in the future, he/she will benefit from increases in the price of the oil. More precisely, his/her payoff of selling *each unit* of oil is the future (random) oil price, $S(T)$. As a result, he/she is *long* with respect to oil. The oil producer is therefore in need of positions which can help him hedge against the downside risk he/she faces arising from oil price. Here $\boxed{\text{I and II}}$ can serve this purpose.

For I, entering into a short forward position means that the oil producer agrees to sell its oil for a predetermined price in contrast to a random price at a fixed time in the future, which protects the producer from decreases in oil price.

For II, buying a put option allows the producer to sell oil for a minimum price, the strike price, which protects the producer from drops in oil price below the strike price. This sets up a floor. **(Answer: (E))** □

Remark. (i) Watch out! The producer is not short with respect to oil, although he/she is to *sell* oil.

(ii) Note that III protects the buyer of oil, who is vulnerable to increases in oil price, not the seller.

Example 3.1.2. (SOA Exam FM Derivatives Markets Sample Question 19: Expected profit of a floor) A producer of gold has expenses of 800 per ounce of gold produced. Assume that the cost of all other production-related expenses is negligible and that the producer will be able to sell all gold produced at the market price. In one year, the market price of gold will be one of three possible prices, corresponding to the following probability table:

Gold Price in 1-year	Probability
750 per ounce	0.2
850 per ounce	0.5
950 per ounce	0.3

The producer hedges the price of gold by buying a 1-year put option with an exercise price of 900 per ounce. The option costs 100 per ounce now, and the continuously compounded risk-free interest rate is 6%.

Calculate the expected 1-year profit per ounce of gold produced.

(A) 0.00

(B) 3.17

(C) 6.33

(D) 8.82

(E) 11.74

Ambrose's comments:

This FM sample question was deleted from the set of MFE/IFM Introductory Derivatives sample questions. However, it can be done using only the knowledge of a floor.

Solution. Let's determine the 1-year profit for each of the three 1-year gold prices:

1-year Gold Price	Profit per Ounce of Gold
750 per ounce	$750 - 800 + (900 - 750)_+ - 100e^{0.06} = -6.1837$
850 per ounce	$850 - 800 + (900 - 850)_+ - 100e^{0.06} = -6.1837$
950 per ounce	$950 - 800 + (900 - 950)_+ - 100e^{0.06} = 43.8163$

The expected 1-year profit is these three possible profits weighed by their respective probabilities, or $(0.2 + 0.5)(-6.1837) + (0.3)(43.8163) = \boxed{8.82}$. **(Answer: (D))**

□

Remark. (i) The profit when the gold price per ounce is 750 equals that when the gold price per ounce is 850 because of the floor of 900.

(ii) The expected 1-year profit without hedging is 60, which is much higher than 8.82, but in case the 1-year gold price turns out to be 750, your profit can be as low as −50.

An observation on Figure 3.1.1.

If you observe Figure 3.1.1 carefully, you will notice that the hedged position created by a long asset and a long put resembles a long call in shape. The payoff function is initially horizontal, then becomes upward sloping to the right of the strike price, just like a long call. The similarity between the hedged long asset and a long call (with the same strike) does make intuitive sense. When we own an asset and hedge it with a long put, we place a "floor" on the downside while still being able to benefit from the upside. Likewise, a long call, by definition, has a limited loss on the downside and allows its holder to profit from the upside. The difference between the payoff function of the hedged long position and that of a long call is that the former is an upward translation of the latter by K. Although the hedged long position and long call do not share the same payoff, towards the end of this section we will see that they indeed have the same profit.

3.1.2 Insuring a Short Position: Caps

Motivation.

A cap is used in the "mirror scenario" corresponding to a floor. Here you short sell one share of stock ABC at time 0 and plan to buy it back after T years in the hope that the stock price will plummet. In the adverse situation that the T-year stock price skyrockets, you will be exposed to a huge, indeed, infinite loss. Your short position can be insured by a long call on stock ABC to protect against repurchasing the stock at an exorbitant price.

Payoff and profit.

The payoff of a short asset position coupled with a long call is given by

$$\text{Payoff} = \underbrace{-S(T)}_{\text{short asset}} + \underbrace{(S(T)-K)_+}_{\text{long call}} = \begin{cases} -S(T), & \text{if } S(T) < K, \\ -K, & \text{if } S(T) \geq K, \end{cases}$$

or more compactly,

$$\text{Payoff} = -\min(S(T), K),$$

which is capped at $-K$ (see Figure 3.1.2 for the payoff diagram, where the payoff of the unhedged short asset is pushed upward to the right of the strike price K due to the long K-strike call). Note that the upside risk is eliminated: if the stock price after T years rises above K, we can still use $\$K$ instead of the higher price of $\$S(T)$ to close the short stock position. For this reason, the call option is also called a *cap* because it effectively "caps" the minimum payoff of the short asset position at $-K$. As in the case of a floor, the upside protection comes with a price: we need to pay the call option premium at time 0.

Example 3.1.3. (SOA Exam IFM Introductory Derivatives Sample Question 13: Profit on a cap) A trader shorts one share of a stock index for 50 and buys a 60-strike European call option on that stock that expires in 2 years for 10. Assume the annual effective risk-free interest rate is 3%.

The stock index increases to 75 after 2 years.

Calculate the profit on your combined position, and determine an alternative name for this combined position.

	Profit	**Name**
(A)	−22.64	Floor
(B)	−17.56	Floor
(C)	−22.64	Cap
(D)	−17.56	Cap
(E)	−22.64	"Written" Covered Call

Solution. A position consisting of a short asset position and a long call is, by definition, a cap. This eliminates Answers (A), (B), and (E). It remains to determine the profit on the cap, which is the payoff at time 2 less the future value of the initial investment. The payoff at time 2 is

$$\underbrace{-S(2)}_{\text{short stock}} \underbrace{+(S(2)-60)_+}_{\text{long call}} = -75 + (75-60)_+ = -60,$$

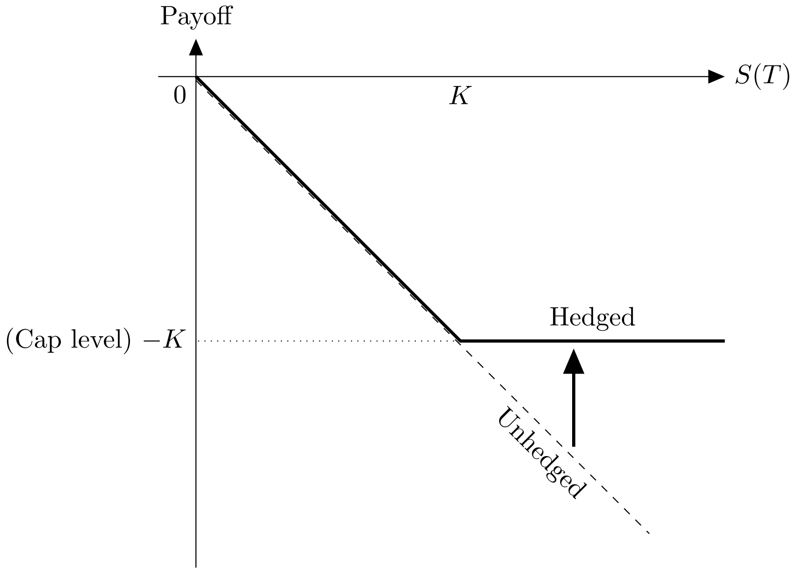


FIGURE 3.1.2
The payoff diagrams of a short asset (unhedged, dashed) and a short asset coupled with a long K-strike call (hedged, bold).

which is the cap level, and the investment made at time 0 is the call price less the initial stock price, or $10 - 50 = -40$, i.e., \$40 is received at time 0. The 2-year profit is $-60 - (-40)(1.03)^2 = \boxed{-17.564}$. **(Answer: (B))** □

Remark. The concept of a written covered call will be covered in the next subsection.

Example 3.1.4. (SOA Exam IFM Introductory Derivatives Sample Question 74: Hedging an implicit short position) Consider an airline company that faces risk concerning the price of jet fuel.

Select the hedging strategy that best protects the company against an increase in the price of jet fuel.

(A) Buying calls on jet fuel

(B) Buying collars on jet fuel

(C) Buying puts on jet fuel

(D) Selling puts on jet fuel

(E) Selling calls on jet fuel

Ambrose's comments:

This sample question is the opposite of Example 3.1.1 (IFM Introductory Derivatives Sample Question 75).

Solution. The airline company pays for jet fuel as an input and is therefore susceptible to increases in the price of jet fuel. It is short with respect to jet fuel. Answers (B), (C), and (E) are all short with respect to jet fuel, so that leaves only Answers (A) and (D). While both (A) and (D) are long with respect to jet fuel, (D) will leave the right tail risk associated with jet fuel unhedged, but (A), which gives rise to a cap, can make the airline invulnerable to extreme increases in jet fuel price. **(Answer: (A))** □

3.1.3 Selling Insurance

We now turn to how options can be *sold* for risk management purposes. The primary motive for selling options is to earn the option premium upfront, with the trade-off being that your payoff and profit will be lower than otherwise when the terminal asset price falls within a certain region.

Short covered calls.

We refer to writing a call in conjunction with a long position in the underlying asset as *writing a covered call*, where the word "covered" emanates from the fact that the short call, which alone can have an infinite loss, is covered by the corresponding long asset position. In contrast, a *naked call* is written when we simply sell a call without taking an offsetting long position in the underlying asset. This can be a highly dangerous activity because there is no limit to your maximum loss—as dangerous as being "naked!"

The payoff as a result of writing a covered call is

$$\begin{aligned}\underbrace{S(T)}_{\text{long asset}} \underbrace{-(S(T)-K)_+}_{\text{short call}} &= \begin{cases} S(T), & \text{if } S(T) \leq K \\ K, & \text{if } S(T) > K \end{cases} \\ &= \min(S(T), K),\end{aligned}$$

which is undesirably bounded from above by the strike price K. When $S(T)$ is higher than K, we have the obligation to sell the asset for K, meaning that we lose the potential upside gain on the underlying asset. To compensate for this loss of potential upside gain, we receive the call premium at time 0 and earn interest on it.

Example 3.1.5. (SOA Exam IFM Introductory Derivatives Sample Question 46: Simple true-or-false statements) Determine which of the following statements about options is true.

(A) Naked writing is the practice of buying options without taking an offsetting position in the underlying asset.

(B) A covered call involves taking a long position in an asset together with a written call on the same asset.

(C) An American style option can only be exercised during specified periods, but not for the entire life of the option.

(D) A Bermudan style option allows the buyer the right to exercise at any time during the life of the option.

(E) An in-the-money option is one which would have a positive profit if exercised immediately.

Solution. Statement (A) is false because naked "writing" involves selling, not buying, options. Statements (C) and (D) would be true if "American" and "Bermudan" were swapped. Statement (E) is also false because being in-the-money means that there is a positive payoff, not necessarily a positive profit. Only Statement (B) is correct. **(Answer: (B))** □

Example 3.1.6. (SOA Exam IFM Introductory Derivatives Sample Question 47: Composition of a written covered call) An investor has written a covered call. Determine which of the following represents the investor's position.

(A) Short the call and short the stock

(B) Short the call and long the stock

(C) Short the call and no position on the stock

(D) Long the call and short the stock

(E) Long the call and long the stock

Ambrose's comments:

When we speak of a written covered call, it implicitly means that a long asset position is in existence, or else the covered call is not well defined. If there is an answer choice saying "Short the call" only, don't choose it!

Solution. By definition, writing a covered call requires shorting the call option along with simultaneous ownership in the underlying asset. **(Answer: (B))** □

Short covered puts.

Analogous to writing a covered call, we call a short put that comes hand in hand with a short position in the underlying asset a *written covered put.* The payoff of writing a covered put is

$$\begin{aligned}\underbrace{-S(T)}_{\text{short asset}}\ \underbrace{-(K-S(T))_+}_{\text{short put}} &= \begin{cases}-K, & \text{if } S(T)\le K\\ -S(T), & \text{if } S(T)>K\end{cases}\\ &= -\max(S(T),K).\end{aligned}$$

Although the terminal payoff is always negative, you receive the put premium initially.

In practice, a written covered put is rarely used because it leaves the unlimited upside loss potential of the original short asset position unchanged. It can be as dangerous as writing a naked call!

3.1.4 A Simple but Useful Observation: Parallel Payoffs, Identical Profit

Before closing this section, we find it beneficial to bring forward a simple observation that will be immensely useful throughout this chapter. In simple terms, this observation says that two positions, say A and B, that enjoy parallel payoff functions (i.e., one payoff function is always higher than the other payoff function by a constant amount) must possess an identical profit function. Mathematically: (the subscripts A and B denote the payoff and profit functions of the respective positions)

Parallel payoffs, identical profit

If $\text{Payoff}_A = \text{Payoff}_B + c$ for some real constant c, then $\text{Profit}_A = \text{Profit}_B$.

The validity of this assertion can be argued as follows. Because the two payoff functions are parallel, their profit functions, being the payoff functions translated downward by the future value of the initial investment, are parallel as well. If $\text{Profit}_A \neq \text{Profit}_B$, then one profit function will be everywhere higher than the other profit function. Without loss of generality, we assume that $\text{Profit}_A > \text{Profit}_B$. Since Position A is always more attractive than Position B in the sense of having a higher profit, we take a long position in A and a short position in B. The overall profit is given by $\text{Profit}_A - \text{Profit}_B$, which is strictly positive by hypothesis. We have thus constructed an arbitrage strategy. By the no-arbitrage principle, $\text{Profit}_A \neq \text{Profit}_B$ cannot be true, and we must have the equality $\text{Profit}_A = \text{Profit}_B$.

A useful special case of the above result is that when the two payoff functions are identical (i.e., $c = 0$), so are the two profit functions.

Example 3.1.7. (Identical profit) Assume the same underlying asset, same time to expiration, and same strike price for all concerned options.

Which of the following must have the same profit as a written covered call?

(A) Short call

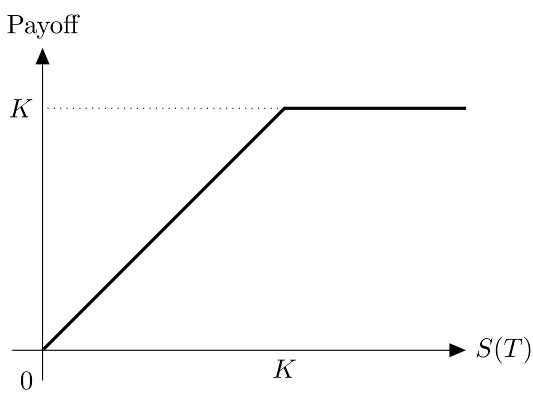
(B) Long call

(C) Short put

(D) Long put

(E) Written covered put

Solution. The payoff diagram of a written covered call is shown below.



From this figure, one observes that the payoff function of a written covered call has the same shape as that of a short put. It follows from the "parallel payoffs, identical profit" phenomenon that a written covered call has the same profit function as a short put. **(Answer: (C))** □

3.2 Put-call Parity

In Section 3.1, we saw that call and put options, when combined with positions in the underlying asset, are related in some ways. In this section, we make this relationship, called put-call parity, mathematically precise. We first introduce the concept of a synthetic forward constructed from options (thus distinct from that of Section 2.3 constructed using the underlying asset and borrowing or lending). This is crucial to deriving and understanding put-call parity.

3.2.1 Synthetic Forwards

What is a synthetic forward?

Perhaps to your astonishment, it is possible to combine options, which are meant to provide "options" in the first place, to create an *obligation* to buy or sell an asset, thereby imitating a forward. To see this, consider the following portfolio of options, which we call Portfolio $(*)$ for simplicity:

Portfolio $(*)$

Buy a European call option and sell a European put option, with both of them having the same underlying asset, strike price K, and time to expiration T.

What happens at the end of T years?

Case 1. If $S(T) \geq K$, you will exercise the call option to buy the asset at a price of K, while the holder of the put option you sold will not.

Case 2. If $S(T) < K$, then the holder of the put option you sold will exercise the put, forcing you to use K to *buy* the asset. Meanwhile, although holding a call option, you will be better off not to exercise it.

Combining both cases, we see that no matter what the asset price at expiration is, you always end up buying the underlying asset for K. The purchase price is thus guaranteed. With the asset at hand, the payoff of Portfolio $(*)$ at expiration is $S(T) - K$, which is the same as the payoff of a long forward contract with forward price K (see Figure 3.2.1 for a pictorial illustration of the payoff diagram of Portfolio $(*)$, where the turning points in the payoffs of the two options are smoothed into a straight line). In other words, we could use a long call plus a short put to mimic a long forward. We call Portfolio $(*)$ a *synthetic long forward*, where the word "synthetic" is meant in the sense that the long forward is created from other derivatives, to distinguish it from a *genuine* forward.

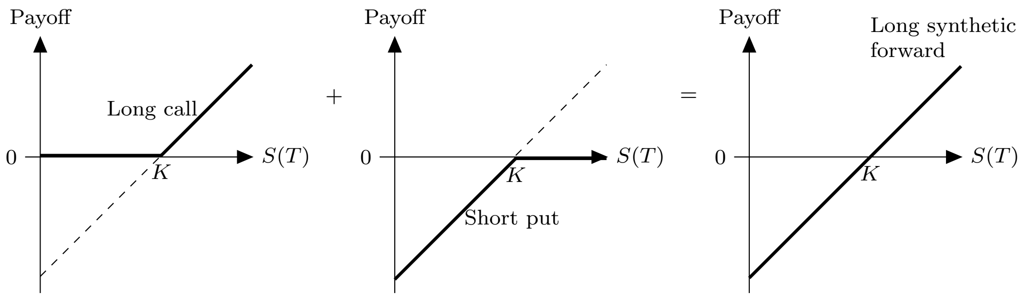


FIGURE 3.2.1
The payoff diagram of a long synthetic forward constructed by K-strike long call and short put options.

The equivalence between the synthetic long forward constructed by K-strike options and a long forward with a forward price of K can also been seen algebraically. The time-T payoff of Portfolio $(*)$ is

$$\text{Payoff of Portfolio } (*) = \underbrace{(S(T)-K)_+}_{\text{long call}} \underbrace{-(K-S(T))_+}_{\text{short put}}.$$

Due to the identity $x_+ - (-x)_+ = x$ for any real number x, the preceding payoff can be simplified into

$$\text{Payoff of Portfolio } (*) = S(T) - K,$$

without the use of the positive part function. This linear payoff function is exactly the payoff of a long forward with a forward price of K.

Example 3.2.1. (Profit on a synthetic forward) You enter into a long synthetic forward on a stock using 1-year European options. The price of a 50-strike call option is 6.8 and the price of a 50-strike put option is 4.2.

The continuously compounded risk-free interest rate is 3%.

The stock sells for 54 after 1 year.

Determine your profit from the long synthetic forward.

Solution. The long synthetic forward consists of buying the 50-strike call and selling the 50-strike put. The initial investment required is $C(50) - P(50) = 6.8 - 4.2 = 2.6$. After 1 year, the payoff of the synthetic forward is $S(1) - 50 = 54 - 50 = 4$. The 1-year profit is $4 - 2.6e^{0.03} = \boxed{1.3208}$. □

Differences between "synthetic" and "genuine" forwards.

There are two major differences between a synthetic forward and a genuine forward.

1. A genuine forward contract, by definition, has a zero premium; we neither pay nor receive money when we enter into a forward position at time 0. In contrast, the investment of a synthetic long (resp. short) forward at time 0 is $C(K,T) - P(K,T)$ (resp. $P(K,T) - C(K,T)$), which is generally non-zero.

   Here and in the remainder of this chapter, we denote the premium of a K-strike T-year call (resp. put) option by $C(K,T)$ (resp. $P(K,T)$). When no confusion arises, we may omit some or all of the arguments in $C(\cdot,\cdot)$ and $P(\cdot,\cdot)$.

2. With a synthetic forward, you pay the common strike price K for the asset, while you pay the fair forward price $F_{0,T}$ with a genuine forward. There can be many choices for K due to the availability of options with different strike prices in the market, whereas there is only one fair value for $F_{0,T}$.

3.2.2 The Put-call Parity Equation

Familiar idea: Same payoff, same price.

As we have seen in the previous subsection, a long synthetic forward, consisting of a long call and an otherwise identical short put, possesses the same payoff at expiration as a genuine forward with a forward price of K, which, in view of its payoff formula $S(T)-K$, is equivalent to a long asset coupled with a short zero-coupon bond with a face value of K. Symbolically and mathematically, we write

$$\begin{array}{ccccccc} \text{Long} & & \text{Short} & & \text{Long} & & \text{Short ZCB} \\ \text{call} & + & \text{put} & = & \text{asset} & + & \text{with face value } K \\ (S(T)-K)_+ & - & (K-S(T))_+ & = & S(T) & - & K \end{array} \quad \text{(in terms of }\underline{\text{payoff}}\text{)}$$

By the no-arbitrage principle, the two sides of the preceding equation should give rise to the same price (or cost) at time 0:

$$\begin{array}{ccccccc} \text{Long} & & \text{Short} & & \text{Long} & & \text{Short ZCB} \\ \text{call} & + & \text{put} & = & \text{asset} & + & \text{with face value } K \\ C(K,T) & - & P(K,T) & = & F_{0,T}^P & - & \mathrm{PV}_{0,T}(K) \end{array} \quad \text{(in terms of }\underline{\text{price}}\text{)}$$

Note that the price of receiving *exactly one unit* of the asset at time T is generally not the current stock price $S(0)$, but the T-year prepaid forward price $F_{0,T}^P$ as discussed in Section 2.2. The price equality above,

$$\boxed{C(K,T)-P(K,T)=F_{0,T}^P-\mathrm{PV}_{0,T}(K)=\mathrm{PV}_{0,T}(F_{0,T}-K),} \tag{3.2.1}$$

is known as *put-call parity*, which provides a mathematical equation governing the prices of otherwise identical European call and put options, the (prepaid) forward price for underlying asset, and the common strike price. It is one of the most fundamental results in the theory of options and will be used intensively in the remainder of this book.

How to better remember put-call parity?

If you have trouble writing the put-call parity equation correctly, struggling whether the right-hand side should be $F_{0,T}^P-\mathrm{PV}_{0,T}(K)$ or $\mathrm{PV}_{0,T}(K)-F_{0,T}^P$, you may keep the following in mind:

- The left-hand side of put-call parity gives the (time-0) price of the long synthetic forward when viewing it from the perspective of *options* (a long call plus a short put). This is the very definition of a synthetic forward.
- The right-hand side treats the long synthetic forward as a long *forward* with a forward price of K and expresses its (time-0) price as the fair price for receiving $S(T)-K$ at time T.

The put-call parity is therefore a mathematical manifestation of the dual identity of a synthetic forward—it originates from a combination of *options*, but can also serve as a *forward*.

Furthermore, (3.2.1) can be recast in terms of the prepaid forward notation as the symmetric and appealing form

$$\boxed{C(K,T) - P(K,T) = F^P_{0,T}(S) - F^P_{0,T}(K),} \tag{3.2.2}$$

which expresses the difference between the call price and the put price as the difference between the prepaid forward price of the stock (which you own by exercising the call) and the prepaid forward price of the strike (which you own by exercising the put).

Example 3.2.2. (SOA Exam IFM Introductory Derivatives Sample Question 65: Which one is put-call parity?) Assume that a single stock is the underlying asset for a forward contract, a K-strike call option, and a K-strike put option.

Assume also that all three derivatives are evaluated at the same point in time.

Which of the following formulas represents put-call parity?

(A) Call Premium – Put Premium = Present Value (Forward Price – K)

(B) Call Premium – Put Premium = Present Value (Forward Price)

(C) Put Premium – Call Premium = 0

(D) Put Premium – Call Premium = Present Value (Forward Price – K)

(E) Put Premium – Call Premium = Present Value (Forward Price)

Ambrose's comments:

It is surprising that a sample exam question can be as easy as recalling the form of put-call parity!

Solution. **(Answer: (A))** No need for explanations! □

Consequences of put-call parity.

Inspecting (3.2.1), we can make the following useful observations:

$$\underbrace{F_{0,T} > K}_{\substack{\text{pay less relative} \\ \text{to a genuine forward}}} \qquad \Leftrightarrow \qquad \underbrace{C(K,T) - P(K,T) > 0}_{\substack{\text{need to pay extra} \\ \text{at time 0}}}$$

and

$$\underbrace{F_{0,T} < K}_{\substack{\text{pay more relative} \\ \text{to a genuine forward}}} \qquad \Leftrightarrow \qquad \underbrace{C(K,T) - P(K,T) < 0}_{\substack{\text{get compensated} \\ \text{at time 0}}}.$$

These two equivalences shed light on the economic considerations underlying the use of a synthetic forward versus a genuine forward.

Case 1. If the forward price $F_{0,T}$ is higher than the strike price K, then it follows from put-call parity that $C(K,T)-P(K,T) > 0$. This makes sense because the holder of the long synthetic forward has the benefit of using a price lower than the fair forward price to buy the underlying asset at time T, and this benefit requires *paying* an extra amount equal to $C(K,T) - P(K,T)$ at time 0 when setting up the long synthetic forward.

Case 2. If the forward price $F_{0,T}$ is lower than the strike price K, then put-call parity implies that $C(K,T) - P(K,T) < 0$, which means that you actually *receive* a positive amount equal to $P(K,T)-C(K,T)$ when creating a synthetic forward at time 0. This amount serves as a way of compensation for buying the underlying asset at a price higher than the fair forward price at time T.

In particular, the cost required to set up the synthetic forward is zero if and only if the common strike price of the options equals the forward price:

$$F_{0,T} = K \quad \Leftrightarrow \quad C(K,T) = P(K,T).$$

In this case, the synthetic forward is nothing but a genuine forward, which requires zero investment.

Example 3.2.3. (SOA Exam IFM Introductory Derivatives Sample Question 5: Cost of a synthetic long forward) The PS index has the following characteristics:

- One share of the PS index currently sells for 1,000.
- The PS index does not pay dividends.

Sam wants to lock in the ability to buy this index in one year for a price of 1,025. He can do this by buying or selling European put and call options with a strike price of 1,025.

The annual effective risk-free interest rate is 5%.

Determine which of the following gives the hedging strategy that will achieve Sam's objective and also gives the cost today of establishing this position.

(A) Buy the put and sell the call, receive 23.81

(B) Buy the put and sell the call, spend 23.81

(C) Buy the put and sell the call, no cost

(D) Buy the call and sell the put, receive 23.81

(E) Buy the call and sell the put, spend 23.81

Solution. Because Sam wants to lock in the ability to *buy* the index for 1,025 in one year, he should enter into a long position in a synthetic forward. This means buying a 1,025-strike call and selling a 1,025-strike put, leaving only Answers (D) and (E).

The cost today of establishing Sam's position, by virtue of put-call parity, is $F_{0,1}^P(S) - F_{0,1}^P(K) = S(0) - F_{0,1}^P(K) = 1,000 - 1,025/1.05 = \boxed{23.81}$. **(Answer: (E))** □

Example 3.2.4. (SOA Exam IFM Introductory Derivatives Sample Question 53: Direct application of put-call parity) For each ton of a certain type of rice commodity, the four-year forward price is 300. A four-year 400-strike European call option costs 110.

The annual risk-free force of interest is a constant 6.5%.

Calculate the cost of a four-year 400-strike European put option for this rice commodity.

(A) 10.00

(B) 32.89

(C) 118.42

(D) 187.11

(E) 210.00

Solution. By put-call parity,

$$\underbrace{C(400,4)}_{110} - P(400,4) = \mathrm{PV}_{0,4}(F_{0,4} - 400) = \mathrm{e}^{-0.065(4)}(300 - 400),$$

which gives $P(400,4) = \boxed{187.11}$. **(Answer: (D))** □

Example 3.2.5. (SOA Exam IFM Introductory Derivatives Sample Question 41: Which is most costly?) XYZ stock pays no dividends and its current price is 100.

Assume the put, the call and the forward on XYZ stock are available and are priced so there are no arbitrage opportunities. Also, assume there are no transaction costs.

The annual effective risk-free interest rate is 1%.

Determine which of the following strategies currently has the highest net premium.

(A) Long a six-month 100-strike put and short a six-month 100-strike call

(B) Long a six-month forward on the stock

(C) Long a six-month 101-strike put and short a six-month 101-strike call

(D) Short a six-month forward on the stock

(E) Long a six-month 105-strike put and short a six-month 105-strike call

Solution. Note that buying or selling a (genuine) forward as in (B) and (D) entails no cost at time 0, while each of (A), (C), and (E) sets up a synthetic forward and requires a non-zero net investment (unless the strike price happens to be the fair forward price). By put-call parity, the cost of each of (A), (C), and (E) takes the form $P(K) - C(K) = \mathrm{PV}_{0,0.5}(K - F_{0,0.5}) = K/1.01^{0.5} - 100$, which increases with the strike price K. When $K = 105$ (i.e., the highest strike price), $\mathrm{PV}_{0,0.5}(K - F_{0,0.5}) = 105/1.01 - 100 > 0$. In other words, the net premium for (E) is the highest among the five choices. **(Answer: (E))** □

Example 3.2.6. (SOA Exam IFM Introductory Derivatives Sample Question 72: Synthetic short forward) CornGrower is going to sell corn in one year. In order to lock in a fixed selling price, CornGrower buys a put option and sells a call option on each bushel, each with the same strike price and the same one-year expiration date.

The current price of corn is 3.59 per bushel, and the net premium that CornGrower pays now to lock in the future price is 0.10 per bushel.

The continuously compounded risk-free interest rate is 4%.

Calculate the fixed selling price per bushel one year from now.

(A) 3.49

(B) 3.63

(C) 3.69

(D) 3.74

(E) 3.84

Solution. To lock in a fixed selling price, CornGrower enters into a *short* synthetic forward by buying a put option on oil and selling an otherwise identical call option. We are given that the cost of setting up this short synthetic forward is 0.10. This means that $P - C = 0.1$. By put-call parity (assuming that corn is nondividend-paying),

$$-0.1 = C - P = S(0) - K\mathrm{e}^{-rT} = 3.59 - K\mathrm{e}^{-0.04},$$

which gives $K = \boxed{3.8406}$. **(Answer: (E))** □

Remark. If you incorrectly take $C - P = +0.1$, you will end up with Answer (B). The answer choices for this question were very carefully set!

Example 3.2.7. (SOA Exam IFM Introductory Derivatives Sample Question 14: Given $C(K_1) - C(K_2)$, find $P(K_2) - P(K_1)$) The current price of a nondividend-paying stock is 40 and the continuously compounded risk-free interest rate is 8%. You are given that the price of a 35-strike call option is 3.35 higher than the price of a 40-strike call option, where both options expire in 3 months.

Calculate the amount by which the price of an otherwise equivalent 40-strike put option exceeds the price of an otherwise equivalent 35-strike put option.

(A) 1.55

(B) 1.65

(C) 1.75

(D) 3.25

(E) 3.35

Ambrose's comments:

A single application of put-call parity may seem too easy (do you agree?). For this reason, some problems may provide information about options with two different strike prices. In this case, we have to apply put-call parity twice, one for the first strike, and one for the second strike. Sometimes, solving a 2×2 linear system of equations is required.

Solution. Two applications of put-call parity show that

$$\begin{cases} C(35) - P(35) = \mathrm{PV}_{0,0.25}(F_{0,1/4} - 35) \\ C(40) - P(40) = \mathrm{PV}_{0,0.25}(F_{0,1/4} - 40) \end{cases}.$$

Subtracting the second equation from the first one, we get

$$\begin{aligned} \underbrace{[C(35) - C(40)]}_{3.35} + [P(40) - P(35)] &= \mathrm{PV}_{0,0.25}(5) \\ P(40) - P(35) &= \boxed{1.55}. \quad \textbf{(Answer: (A))} \end{aligned}$$

□

Example 3.2.8. (SOA Exam IFM Introductory Derivatives Sample Question 40: Put-call parity for two assets) An investor is analyzing the costs of two-year, European options for aluminum and zinc at a particular strike price.

For each ton of aluminum, the two-year forward price is 1400, a call option costs 700, and a put option costs 550.

For each ton of zinc, the two-year forward price is 1600 and a put option costs 550.

The annual effective risk-free interest rate is 6%.

Calculate the cost of a call option per ton of zinc.

(A) 522

(B) 800

(C) 878

(D) 900

(E) 1231

Solution. Applying put-call parity to options on aluminum and zinc yields

$$\begin{cases} 700 - 550 = \dfrac{1}{1.06^2}(1,400 - K) \\ C - 550 = \dfrac{1}{1.06^2}(1,600 - K) \end{cases}.$$

Subtracting the first equation from the second one gives $C = \boxed{878}$. **(Answer: (C))** □

Parity arbitrage.

Put-call parity is a mathematical equation that relates the *fair* prices of European call and put options having the same strike price, maturity date and underlying asset. If it is violated, then it is possible to design an arbitrage strategy, called a *parity arbitrage* in this context, to earn risk-free profits. The next example illustrates how this can be done.

Example 3.2.9. (Parity arbitrage) You are given:

(i) The price of a nondividend-paying stock is \$31.

(ii) The continuously compounded risk-free interest rate is 10%.

(iii) The price of a 3-month 30-strike European call option is \$3.

(iv) The price of a 3-month 30-strike European put option is \$2.25. Construct a trading strategy that will generate risk-free arbitrage profits at time 0.

Solution. It is easy to see that the call and put prices violate put-call parity:

- LHS:
$$C(30, 0.25) - P(30, 0.25) = 3 - 2.25 = 0.75,$$
- RHS:
$$F^P_{0,T}(S) - F^P_{0,T}(K) = S(0) - Ke^{-rT} = 31 - 30e^{-(0.1)(0.25)} = 1.7407.$$

To exploit arbitrage profits, we "buy the LHS" ("low") and "sell the RHS" ("high") by engaging in the following transactions:

	Cash Flows	
Transaction	**Time 0**	**Time 0.25**
Buy a 3-month 30-strike call	-3	$(S(0.25) - 30)_+$
Sell a 3-month 30-strike put	$+2.25$	$(30 - S(0.25))_+$
Short sell one share of the stock	$+31$	$-S(0.25)$
Lend $30e^{-(0.1)(0.25)} = 29.2593$	-29.2593	30
Total	0.9907	0

□

3.3 Spreads and Collars

This section and the next continue the spirit of Sections 3.1 and 3.2, and present several common strategies involving two or more options of possibly different types and strike prices. We shall examine the composition and payoff structure of and the motivation underlying each option strategy.