T-503-AFLE Derivatives

Forwards

The fair forward price, $F_{0,T}$, for a forward contract on the underlying $S(t)$, struck at time 0 with expiry T, is given by

$$F_{0,T} = S(0)e^{rT} - \underbrace{\sum_{i=1}^{n} d_i e^{r(T-t_i)}}_{\mathrm{FV}_{0,T}(\mathrm{Div})}$$

in the case the underlying pays discrete deterministic dividends of size d_i at times t_i, with $i = 1, \ldots, n$, and $0 \le t_1 < t_2 < \cdots < t_n \le T$. In the case the underlying pays a continuous proportional dividend yield of δ that is fully re-invested, the fair forward price is

$$F_{0,T} = S(0)e^{(r-\delta)T}.$$

The relationship between the prepaid forward price and the forward price is given by

$$F_{0,T} = \mathrm{FV}_{0,T}\left(F_{0,T}^P\right) \text{ or } F_{0,T}^P = \mathrm{PV}_{0,T}\left(F_{0,T}\right).$$

Put-call Parity

The prices of a maturity-T European call with strike K and an otherwise identical European put are related by

$$\begin{aligned} C(K,T) - P(K,T) &= F_{0,T}^P - \mathrm{PV}_{0,T}(K) \\ &= \mathrm{PV}_{0,T}\left(F_{0,T} - K\right) \end{aligned}$$

where

- $F_{0,T}^P$ the price of a maturity-T prepaid forward on the same underlying stock, and
- $\mathrm{PV}_{0,T}(x)$ represents the generalized present-value of time-T cash-flow x.

Option Strategies

The table contains fundamental option trading strategies. All options are written on the same underlying, S, for the same maturity, T. Furthermore, $K_1 < K_2 < K_3$, and $m \ne n$. Note that the initial cashflow is the negative of the cost.

Option Strategy	**Cost**
Floor	$S + P(K)$
Cap	$-S + C(K)$
Short Covered Call	$S - C(K)$
Short Covered Put	$-S - P(K)$
Synthetic Forward	$C(K) - P(K)$
Call Bull Spread	$C(K_1) - C(K_2)$
Put Bull Spread	$P(K_1) - P(K_2)$
Call Bear Spread	$-C(K_1) + C(K_2)$
Put Bear Spread	$-P(K_1) + P(K_2)$
Call Ratio Spread	$\pm mC(K_1) \pm nC(K_2)$
Put Ratio Spread	$\pm mP(K_1) \pm nP(K_2)$
Collar	$P(K_1) - C(K_2)$
Straddle	$P(K) + C(K)$
Strangle	$P(K_1) + C(K_2)$
Butterfly Spread	$P(K_1) + C(K_3) - (P(K_2) + C(K_2))$
Box Spread	$C(K_1) - P(K_1) - (C(K_2) - P(K_2))$

Binomial Option Pricing Models

Consider a one-period binomial model. The underlying, S, has a continuously reinvested proportional dividend yield of δ. It begins at S_0 and can move to either $S_u = uS_0$ or $S_d = dS_0$, in one time period of length h. The continuously compounded risk-free rate is r.
The replicating portfolio for a derivative, V, written on S, is given by

$$\Delta = e^{-\delta h}\left(\frac{V_u - V_d}{S_u - S_d}\right)$$

shares of the stock and

$$B = e^{-rh}\left(\frac{uV_d - dV_u}{u - d}\right)$$

dollars in the bank account, where V_u is the payoff of the derivative in the event S attains S_u and V_d is the payoff of the derivative in the event S attains S_d. Thus,

$$V_0 = \Delta S_0 + B.$$

Parameterization: The Forward Tree
A *forward* binomial tree is parameterized by

$$u = e^{(r-\delta)h + \sigma\sqrt{h}} \quad \text{and} \quad d = e^{(r-\delta)h - \sigma\sqrt{h}}$$

where

$$\sigma^2 = \mathbb{V}\mathrm{ar}\left[\ln \frac{S(t+1)e^{\delta}}{S(t)}\right].$$

Thus, σ is the standard deviation of the annual log-returns of the stock, also known as the *volatility* of the stock. In this parameterization,

$$p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}}.$$

Risk-neutral Pricing

A binomial tree is free of arbitrage if and only if

$$d \le e^{(r-\delta)h} \le u$$

which is equivalent to

$$0 \le p^* := \frac{e^{(r-\delta)h} - d}{u - d} \le 1.$$

Thus a binomial tree is free of arbitrage if and only if the risk-neutral binomial measure defined by p^* exists. Then, for any traded asset or derivative, V,

$$V_0 = e^{-rh}\mathbb{E}^*\left[V(h)\right].$$

The Black-Scholes Framework

Let

$$S(t) = S(0)\exp\left[\left(r - \delta - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z\right].$$

model the the time-t price of a stock paying continuous proportional dividends at the rate δ, **under the risk-neutral probability measure.** Here, $r, \delta, t \ge 0$ and $Z \sim \mathcal{N}(0,1)$. Then

$$\mathbb{E}^*[S(t)] = S(0)e^{(r-\delta)t}$$

and

$$\mathbb{V}\mathrm{ar}^*[S(t)] = S(0)^2 e^{2(r-\delta)t}\left(e^{\sigma^2 t} - 1\right).$$

Exercise Probabilities
The European call and put exercises probabilities, under the risk-neutral measure, are given by

$$\mathbb{P}^*(S(T) > K) = N(d_2) \text{ and } \mathbb{P}^*(S(T) < K) = N(-d_2),$$

for

$$d_2 = \frac{\ln[S(0)/K] + \left(r - \delta - \sigma^2/2\right)T}{\sigma\sqrt{T}}.$$

Prediction Intervals
The $100(1-p)\%$ equal-tailed prediction interval for $S(t)$, under the risk-neutral measure, is given by

$$\mathbb{P}^*\left(S^L < S(t) < S^U\right) = 1 - p$$

with

$$S^L = S(0)e^{\left(r-\delta-\sigma^2/2\right)t + \sigma\sqrt{t}N^{-1}(p/2)}$$

and

$$S^U = S(0)e^{\left(r-\delta-\sigma^2/2\right)t + \sigma\sqrt{t}N^{-1}(1-p/2)}.$$

Conditional Expected Stock Price
The expected stock price, under the risk-neutral measure, conditional on being in the exercise region for a European call struck at K is

$$\mathbb{E}^*[S(T) \mid S(T) > K] = S(0)e^{(r-\delta)T}\frac{N(d_1)}{N(d_2)}.$$

where

$$d_1 = d_2 + \sigma\sqrt{T} = \frac{\ln[S(0)/K] + \left(r - \delta + \sigma^2/2\right)T}{\sigma\sqrt{T}}.$$

Correspondingly,

$$\mathbb{E}^*[S(T) \mid S(T) < K] = S(0)e^{(r-\delta)T}\frac{N(-d_1)}{N(-d_2)}.$$

Pricing Formulae

The Black-Scholes pricing formulae for European calls and puts are given by

$$C = S_0 e^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$
$$P = Ke^{-rT}\Phi(-d_2) - S_0 e^{-\delta T}\Phi(-d_1)$$

where

- S_0 is the current price of the underlying,
- δ is the (fully-reinvested) continuous proportional dividend rate,
- K is the strike price,
- r is the continuously compounded risk-free interest rate,
- σ is the volatility of the annual log-returns of the underlying,
- T is the maturity of the option

and

$$d_1 = \frac{\ln[S_0/K] + \left(r - \delta + \sigma^2/2\right)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln[S_0/K] + \left(r - \delta - \sigma^2/2\right)T}{\sigma\sqrt{T}}.$$

Discrete Deterministic Dividends

In the presence of discrete, deterministic dividends, the Black-Scholes pricing formulae for a European call and put are given by

$$C = F^P_{0,T}(S)\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$
$$P = Ke^{-rT}\Phi(-d_2) - F^P_{0,T}(S)\Phi(-d_1)$$

where $F^P_{0,T}(S)$ is the price of a T-maturity prepaid forward on the underlying stock and

$$d_1 = \frac{\ln\left[F^P_{0,T}(S)/K\right] + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

Currency Options

The Garman-Kohlhagen pricing formulae for European calls and puts on a foreign currency are given by

$$C = X_0 e^{-r^f T}\Phi(d_1) - Ke^{-r^d T}\Phi(d_2),$$
$$P = Ke^{-r^d T}\Phi(-d_2) - X_0 e^{-r^f T}\Phi(-d_1),$$

where

- X_0 is the current exchange rate expressed as units of domestic currency per single unit of foreign currency,
- r^f is the continuously compounded foreign risk-free rate,
- r^d is the continuously compounded domestic risk-free rate,

and

$$d_1 = \frac{\ln[X/K] + \left(r^d - r^f + \sigma^2/2\right)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

Futures Options

The Black-Scholes pricing formulae for European calls and puts on futures are

$$C = F_{0,T_f} e^{-rT}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$
$$P = Ke^{-rT}\Phi(-d_2) - F_{0,T_f} e^{-rT}\Phi(-d_1)$$

where

- T is the time to maturity of the futures call and put,
- T_f is the time to maturity of the underlying futures with $T \leq T_f$,
- F_{0,T_f} is the current price of the T_f-year futures

and

$$d_1 = \frac{\ln\left(F_{0,T_f}/K\right) + \sigma^2 T/2}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

Greek		Calls	Puts
fair value (V)	V	$S_0 e^{-\delta T}\Phi(d_1) - e^{-rT}K\Phi(d_2)$	$e^{-rT}K\Phi(-d_2) - S_0 e^{-\delta T}\Phi(-d_1)$
delta (Δ)	$\frac{\partial V}{\partial S_0}$	$e^{-\delta T}\Phi(d_1)$	$-e^{-\delta T}\Phi(-d_1)$
vega ($\mathcal{V}$)	$\frac{\partial V}{\partial \sigma}$	$S_0 e^{-\delta T}\varphi(d_1)\sqrt{T} = Ke^{-rT}\varphi(d_2)\sqrt{T}$	
theta (Θ)	$\frac{\partial V}{\partial t}$	$-e^{-\delta T}\frac{S_0\varphi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2) + \delta S_0 e^{-\delta T}\Phi(d_1)$	$-e^{-\delta T}\frac{S_0\varphi(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}\Phi(-d_2) - \delta S_0 e^{-\delta T}\Phi(-d_1)$
rho (ρ)	$\frac{\partial V}{\partial r}$	$KTe^{-rT}\Phi(d_2)$	$-KTe^{-rT}\Phi(-d_2)$
epsilon (ϵ)	$\frac{\partial V}{\partial \delta}$	$-S_0 T e^{-\delta T}\Phi(d_1)$	$S_0 T e^{-\delta T}\Phi(-d_1)$
gamma (Γ)	$\frac{\partial^2 V}{\partial S_0^2}$	$e^{-\delta T}\frac{\varphi(d_1)}{S_0\sigma\sqrt{T}} = Ke^{-rT}\frac{\varphi(d_2)}{S_0^2\sigma\sqrt{T}}$	
omega (Ω)	Ω	$\Delta\frac{S_0}{V}$	

Table 1: The first-order Greeks, as well as the second-order Greek gamma and the elasticity, denoted omega, for European call and put options in the Black-Scholes framework.

Table 2: Entries in the table represent the area under the standard normal distribution function from $-\infty$ to $z, \Phi(z) = \mathbb{P}(Z \le z)$. The value of z to the first decimal is given in the left column. The second decimal is given in the top row. Areas for negative values of z can be obtained by symmetry, i.e., $\Phi(z) = 1 - \Phi(-z)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

Table 3: Selected quantiles for the standard normal distribution. Note that $\Phi^{-1}(1-y) = z \Leftrightarrow \Phi^{-1}(y) = -z$.

Values of z for selected values of $\mathbb{P}(Z \le z)$							
z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
$\mathbb{P}(Z \le z)$	0.800	0.850	0.900	0.950	0.975	0.990	0.995