4 - Forwards with Market Frictions

2.3.3 Market Frictions

THIS SECTION EXTENDS THE TEXTBOOK.
We consider now the effect of market frictions on the fair forward price.
We consider the following types of market frictions:

Bid-ask spread
For any given asset, the price at which you can buy is called the ask price, $S^a(0)$, while the price at which you can sell is called the bid price, $S^b(0)$. At any moment of time, the ask price must be higher than or equal to the bid price, or else you can buy the asset at the ask price and immediately sell the asset at the bid price, realizing a risk-free profit. The difference between the two prices is called the bid-ask spread, which is a source of income to market makers.

Disparity between borrowing and lending rates
In general, the borrowing rate r^b and lending rate r^l need not be borrowing and identical. To avoid arbitrage, it must hold that $r^b \geq r^l$ (why?).

Transaction costs
Buying or selling derivatives involves transaction costs, which can be a fixed amount or a variable amount proportional to the scale of transaction. Transaction costs can occur at purchase or settlement (or both) of a contract and may be a cost or an inflow depending on your position.

Keystone Example: Forwards with Market Frictions I

Discrete dividends, symmetric transactions costs at 0.
This example expands on Problem 2.5.12.

- The current bid and ask price of a stock are $S^a(0) \geq S^b(0)$.
- The continuously compounded risk-free rates for borrowing and lending are $r^b \geq r^a$.
- The relevant transactions costs are:
  - C^S per unit of stock bought or sold, paid at time 0.
  - C^F per unit of forward contract bought or sold, paid at time 0.
    - You are trading with an exchange, hence symmetrical fees.
- The stock pays discrete dividends, d_i for $i = 1, \ldots, N$, at times $0 < t_1 < t_2 < \cdots < t_n \leq T$.
- Derive the no-arbitrage bounds for the forward price, $F_{0,T}$.

Solution:
Part 1:
Consider the cashflows of a long forward position: $(-C^F, S(T) - F_{0,T})$

- Transactions to cancel the terminal cashflows:
  - Lend / invest cash: $(-F_{0,T}e^{-r_l T}, F_{0,T})$
  - Short stock: $(S^b(0) - C^S, -d_1, -d_2, \ldots, -d_n, -S(T))$
  - Lend / invest cash: $(-\sum_{i=1}^N d_i e^{-r_l t_i}, d_1, d_2, \ldots, d_n)$
- Final cashflow stream:

$$(S^b(0) - C^S - C^F - F_{0,T}e^{-r_l T} - \sum_{i=1}^N d_i e^{-r_l t_i}, 0)$$

  Therefore:

$$S^b(0) - C^S - C^F - F_{0,T}e^{-r_l T} - \sum_{i=1}^N d_i e^{-r_l t_i} \leq 0$$

$$F_{0,T} \geq (S^b(0) - C^S - C^F)e^{r_l T} - \sum_{i=1}^N d_i e^{r_l (T - t_i)}$$

Part 2:
Consider the cashflows of a short forward position: $(-C^F, F_{0,T} - S(T))$

- Transactions to cancel the terminal cashflows:
  - Borrow cash: $(F_{0,T}e^{-r_bT}, -F_{0,T})$
  - Long stock: $(-S^a(0) - C^S, d_1, d_2, \ldots, d_n, S(T))$
  - Borrow cash: $(\sum_{i=1}^N d_i e^{-r_b t_i}, -d_1, -d_2, \ldots, -d_n)$
- Final cashflow stream:

$$(-S^a(0) - C^S - C^F + F_{0,T}e^{-r_bT} + \sum_{i=1}^N d_i e^{-r_b t_i}, 0)$$

Therefore:

$$\begin{aligned} &-S^a(0) - C^S - C^F + F_{0,T}e^{-r_bT} + \sum_{i=1}^N d_i e^{-r_b t_i} \le 0 \\ &F_{0,T} \le (S^a(0) + C^S + C^F)e^{r_bT} - \sum_{i=1}^N d_i e^{r_b(T-t_i)} \end{aligned}$$

The bounds above can be placed into the context of our previously derived prices.
Without market frictions:

$$F_{0,T} = \begin{cases} S(0)\mathrm{e}^{rT} - \underbrace{\sum_{i=1}^n D(t_i)\mathrm{e}^{r(T-t_i)}}_{\mathrm{FV}_{0,T}(\mathrm{Div})}, & \text{for discrete dividends} \\ S(0)\mathrm{e}^{(r-\delta)T}, & \text{for continuous proportional dividends} \end{cases}$$

With market frictions - discrete dividend case:

$$\begin{aligned} F_{0,T} \in \Bigg[&(S^b(0) - C^S)e^{r_lT} - \sum_{i=1}^N d_i e^{r_l(T-t_i)} - C^F e^{r_lT}, \\ &(S^a(0) + C^S)e^{r_bT} - \sum_{i=1}^N d_i e^{r_b(T-t_i)} + C^F e^{r_bT}\Bigg] \end{aligned}$$

With market frictions - continous, proportional dividend case:

$$\begin{aligned} F_{0,T} \in \Big[&(S^b(0) - C^S)e^{(r_l-\delta)T} - C^F e^{r_lT}, \\ &(S^a(0) + C^S)e^{(r_b-\delta)T} + C^F e^{r_bT}\Big] \end{aligned}$$

IMPORTANT: Problem 2.5.13 is specifically set up so that you cannot use this formula directly. The transaction costs are not symmetrical!

Problem 2.5.12 (No-arbitrage Interval with Discrete Dividends EXTENDED)

You are given the following information:
(i) The current bid price and ask price of stock ABC are $50 and $51, respectively.
(ii) A dividend of $3 will be paid 6 months from now.
(iii) The continuously compounded risk-free interest rate is 6%.
(iv) The only transaction costs are:

- A $\$1.5$ transaction fee, paid at time 0, for buying or selling each unit of stock ABC.
- A $\$1$ transaction fee, paid at time 0, for buying or selling a forward contract on stock ABC.

a) Derive the lower bound of the no-arbitrage interval for $F_{0,1}$, a 1-year forward contract written on stock ABC. Clearly indicate the financial transactions taken.

b) Derive the upper bound of the no-arbitrage interval for $F_{0,1}$. Clearly indicate the financial transactions taken.

c) If the no-arbitrage bounds for $F_{0,1}$ are given by

$$F_{0,1} \in [47.34587, 53.71689],$$

but the current one-year forward price on stock ABC is $55, what amount of arbitrage profit could be generated at time 0?

Problem 2.5.13 (No-arbitrage Interval with Continuous Dividends - EXTENDED)

You are given the following information:
(i) The current bid price and ask price of stock Y are $40 and $41, respectively.
(ii) Stock Y pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.
(iii) The continuously compounded lending and borrowing rates are 6% and 7%, respectively.
(iv) The only transaction costs are:

- A $1 transaction fee, paid at time 0, for buying or selling each unit of stock Y.
- A $2 transaction fee, paid at expiration, for settling a forward contract on stock Y.
    - If you are long the forward, you pay this amount to settle the contract. If you are short the forward, you earn this amount when the contract is settled.

a) Derive the lower bound of the no-arbitrage interval for $F_{0,3}$, a 3-year forward contract written on stock Y. Clearly indicate the financial transactions taken.

b) Derive the upper bound of the no-arbitrage interval for $F_{0,3}$. Clearly indicate the financial transactions taken.

c) If the no-arbitrage bounds for $F_{0,3}$ are given by

$$F_{0,3} \in [40.6728, 45.3549],$$

but the current one-year forward price on stock Y is $38, what amount of arbitrage profit could be generated at time 0?