Project 1 in Numerical Analysis

Introduction

The following diagram shows a pipe network through which water is flowing. The length of each pipe is listed. The positions of the junctions **B** and **E** can be freely adjusted. The pump on the left produces a constant outlet overpressure of Δp_0, that is, over atmospheric air pressure. The right end of pipe number 9 is at atmospheric pressure, while water exits the network at the point **C** at constant rate Q_C. The volume flow rates $q_1, q_2, ..., q_9$ (expressed in cubic meters/second) in each pipe are unknown and our goal is to compute them under various circumstances.

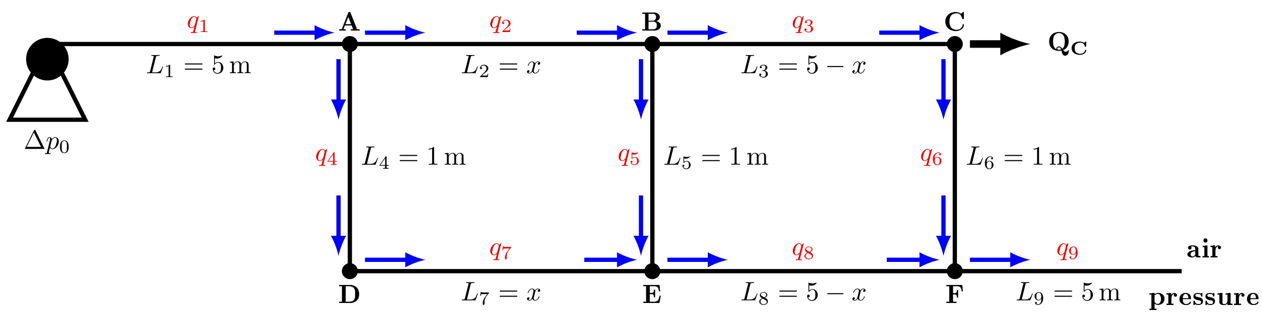

Figure 1: Schematic description of the pipe network. The blue arrows show the direction of water flow. The length of pipes 2 and 7 can be adjusted.

The analysis of such a pipe network has much in common with the analysis of an electric circuit. At the six junctions **A**, **B**, **C**, **D**, **E** and **F**, the rate at which water enters the junction must equal the rate at which it exits it. We obtain **Equations 1 to 6** by going through the junctions in alphabetical order.

$$
\begin{aligned}
q_1 - q_2 - q_4 &= 0 \\
q_2 - q_3 - q_5 &= 0 \\
q_3 - q_6 - Q_C &= 0 \\
q_4 - q_7 &= 0 \\
q_5 + q_7 - q_8 &= 0 \\
q_6 + q_8 - q_9 &= 0
\end{aligned}
$$

The signs are determined by the direction of the flow. The final three equations (we need nine equations to solve for our nine variables $q_1, ..., q_9$) can be obtained by studying the pressure drop along the two loops of the network, as well one open path, for instance through **A**, **D**, **E**, **F** and out. The precise nature of the pressure drop depends on whether the water flow is rather slow (so-called laminar flow) or faster (so called turbulent flow).

Laminar flow - Theory

Laminar flow occurs at low velocities. The water then tends to flow without turbulence or mixing. In the case of a cylindrical pipe we define the Reynolds number as

$$\mathrm{Re} = \frac{4\rho q}{\mu \pi D}$$

where q is the volume flow rate, ρ is the density of water, μ its dynamic viscosity and D is the diameter of the pipe. Then laminar flow occurs until the Reynolds number reaches 2000.

When the flow is laminar, the pressure drop Δp along a pipe is linearly related to the flow rate q via the Hagen-Poiseulle equation

$$\Delta p = \frac{128\mu L}{\pi D^4} q$$

where μ is the viscosity of water, L the length of the pipe and D its diameter. Going around the loops **ABED** and **BCFE**, and using the fact that the sum of the pressure drops around a loop must be zero, we obtain two additional equations:

$$\Delta p_2 - \Delta p_4 + \Delta p_5 - \Delta p_7 = \frac{128\mu}{\pi D^4}\Big(L_2 q_2 - L_4 q_4 + L_5 q_5 - L_7 q_7\Big) = 0$$

and

$$\Delta p_3 - \Delta p_5 + \Delta p_6 - \Delta p_8 = \frac{128\mu}{\pi D^4}\Big(L_3 q_3 - L_5 q_5 + L_6 q_6 - L_8 q_8\Big) = 0$$

Note that the pipes do not all have the same length, see Figure 1. In both equations we can cancel out the constant $128\mu/(\pi D^4)$ and obtain **Equations 7 and 8**.

$$L_2 q_2 - L_4 q_4 + L_5 q_5 - L_7 q_7 = 0$$

$$L_3 q_3 - L_5 q_5 + L_6 q_6 - L_8 q_8 = 0$$

For our final equation we consider the open path from the pump through **A**, **D**, **E**, **F** and out. The total pressure drop along the path is the overpressure Δp_0 so we obtain **Equation 9**:

$$\Delta p_0 = \frac{128\mu}{\pi D^4}\Big(L_1 q_1 + L_4 q_4 + L_7 q_7 + L_8 q_8 + L_9 q_9\Big)$$

We obtain a 9×9 linear system for the unknowns $q_1, ..., q_9$ which can be solved.

Laminar flow - Problems

1. Rewrite the linear system as a matrix equation $A\boldsymbol{q} = \boldsymbol{b}$ where A is a 9×9 matrix and $\boldsymbol{q} = [q_1\ q_2\ \ldots\ q_9]^T$. Solve it using the following parameters:

- Pipe lengths indicated on Figure 1, $x = 2.5\,\mathrm{m}$. Pipe diameter $D = 0.2\,\mathrm{m}$.
- Water viscosity $\mu = 1.002 \times 10^{-3}\,\mathrm{Pa}\cdot\mathrm{s}$ at $20°$ C.
- Pump over pressure $\Delta p_0 = 0.01\,\mathrm{Pa}$. From **C** we drain $Q_C = 10^{-5}\,\mathrm{m}^3/\mathrm{s}$.

2. Use the Reynolds number

$$\mathrm{Re} = \frac{4\rho q}{\mu \pi D}$$

to verify that the values of $q_1, ..., q_9$ obtained in Question 1 are compatible with laminar flow. Then use the bisection method to find with four correct significant digits the maximum value of Δp_0 such that we stay in laminar flow regime. $\rho = 998\,\mathrm{kg}/\mathrm{m}^3$ is the volumic mass of water at $20°$ C.

3. Plot the flow rates $q_1, ..., q_9$ as functions of Δp_0. Use the maximum value of Question 2 as an upper bound. Interpret the figure (e.g. why are the graph linear? why do they have different slopes?).

4. The viscosity of water μ is not an absolute constant. In particular it depends strongly on temperature and the experimental data in the table below suggests that $\mu(T)$ is a decreasing function.

Temperature T (Celsius)	20	40	60	80	100
Viscosity μ ($\times 10^{-3}\mathrm{N}\cdot\mathrm{s}/\mathrm{m}^2$)	1.003	0.657	0.467	0.355	0.283

A popular model for viscosity as a function of temperature is

$$\mu(T) = e^{a+b/(T+T_0)+c/(T+T_0)^2}$$

where a, b, c are constants, and $T_0 = 273.16°\mathrm{K}$ to allow for conversion from Kelvin to Celsius. We may rewrite it as a linear model

$$\ln(\mu) = a + \frac{b}{T+T_0} + \frac{c}{(T+T_0)^2}$$

Use (linear) least squares to find the values of a, b, c that fit best the data. Plot $\mu(T)$ as a function of T as well as the five given data points. Compute the root mean square error (RMSE).

5. We set again $\Delta p_0 = 0.01\,\mathrm{Pa}$. Using the least squares approximation for $\mu(T)$, solve the system for a range of temperature values between 20 and 100 Celsius. Plot and interpret the results. Does the whole network stay at laminar flow regime for all temperature values? Use the Reynolds number to justify your answer.

Turbulent flow - Theory

At higher velocities the flow ceases to increase linearly with pressure. Pressure drop Δp is now related to the flow rate q via the Darcy-Weissbach equation

$$\Delta p = \frac{8f\rho L}{\pi^2 D^5} q^2$$

where ρ is the density of water and f is the Darcy friction factor. **Equations 1 to 6** are the same as before, however **Equations 7 and 8** become (after simplifying the common factor $8f\rho/(\pi^2 D^5)$):

$$L_2 q_2^2 - L_4 q_4^2 + L_5 q_5^2 - L_7 q_7^2 = 0$$

$$L_3 q_3^2 - L_5 q_5^2 + L_6 q_6^2 - L_8 q_8^2 = 0$$

while **Equation 9** becomes

$$\Delta p_0 = \frac{8f\rho}{\pi^2 D^5}\left(L_1 q_1^2 + L_4 q_4^2 + L_7 q_7^2 + L_8 q_8^2 + L_9 q_9^2\right)$$

We now obtain a non-linear system of nine equations for the nine unknowns $q_1, ..., q_9$.

Turbulent flow - Problems

6. Rewrite the non-linear system as a vector equation

$$\boldsymbol{F}(\boldsymbol{q}) = \boldsymbol{0}$$

where $\boldsymbol{F} : \mathbb{R}^9 \to \mathbb{R}^9$ and $\boldsymbol{q} = [q_1\, q_2\, \dots\, q_9]^T$. Compute the Jacobi matrix $D\boldsymbol{F}$.

7. Write two programs for $\boldsymbol{F}$ or $D\boldsymbol{F}$ as functions of the vector $\boldsymbol{q}$. Then use multidimensional Newton's method to solve the 9×9 non-linear system using the parameters listed here below. Explain your choice of initial value.

- Pipe lengths indicated on Figure 1, $x = 2.5\,\mathrm{m}$. Pipe diameter $D = 0.2\,\mathrm{m}$.
- Water density at 20° C is $\rho = 998\,\mathrm{kg/m^3}$.
- Pump over pressure $\Delta p_0 = 0.5\,\mathrm{Pa}$. From **C** we drain $Q_C = 10^{-5}\,\mathrm{m^3/s}$.
- We assume a constant Darcy friction factor of $f = 0.02$.

8. We now let the variable length x be on the interval $[0.1, 4.9]$. Run Newton's method for equally spaced values of x on this interval, and plot $q_1, ..., q_9$ as functions of x. Interpret the graph.

9. The graph of $q_5(x)$ takes a minimum value somewhere on the interval $[0.1, 4.9]$. Use the golden section search algorithm to localize the minimum value of q_5 with four correct significant digits. Verify your solution either graphically or numerically.

10. In reality, the Darcy friction factor f is not constant. It depends on the Reynolds number

$$\mathrm{Re} = \frac{4\rho}{\mu\pi D}q$$

(and therefore on the flow rate q) in some complicated fashion. Note that the values of q differ greatly between the nine pipes. In the case of a perfectly smooth cylindrical pipe, experimental data suggests a power law of the form

$$f = \alpha Re^{\beta}$$

with $\alpha = 0.3164$ and $\beta = -0.25$. Correct the linear system $\boldsymbol{F}(\boldsymbol{q}) = \boldsymbol{0}$ to accommodate for this change. You also need to change the Jacobi matrix $D\boldsymbol{F}$. Compare the results to the case where f was held constant.

Independent work

Implement one or two experiments of your choice on this system. Here are a few suggestions but you are welcome to use your own ideas.

- The viscosity model of Question 4 could also be computed using non-linear least squares (Gauss-Newton method). Compare the results of both methods in terms of root square mean errors. More accurate viscosity data can be found online. Rerun Question 5 with Gauss-Newton data fit and compare the results.
- Study temperature-dependent turbulent flow. The Darcy friction factor f depends on temperature through μ (see Questions 4 and 10). The density of water ρ is also moving with temperature, although much less. Least squares analysis can be used on this density data.
- Study the effect of water intake Q_C, both at laminar and turbulent regime. Are there any unexpected effects for larger Q_C?
- The model for the Darcy friction factor f of Question 10 is not very precise and is only a valid approximation for a perfectly smooth pipe. This Wikipedia page contains various models. Implement one or some of these models and compare them. You will need to decide upon the surface roughness ε of the pipes and may want to consider a few cases. A few examples of surface roughness are listed here.
- Add some extra features to the pipe network, such as extra water intakes at other points than **C**, or a different shape altogether. Make some predictions and verify them numerically.

Grading for this part depends both on difficulty and the quality of the solution.