

# Commutative Algebra notes

arnaucube

## Abstract

Notes taken while studying Commutative Algebra, mostly from Atiyah & MacDonald book [1] and Reid's book [2].

Usually while reading books and papers I take handwritten notes in a notebook, this document contains some of them re-written to *LaTeX*.

The proofs may slightly differ from the ones from the books, since I try to extend them for a deeper understanding.

## Contents

<b>1</b>	<b>Ideals</b>	<b>1</b>
1.1	Definitions . . . . .	1
1.2	Lemmas, propositions and corollaries . . . . .	3
<b>2</b>	<b>Modules</b>	<b>3</b>
2.1	Modules . . . . .	3
2.2	Cayley-Hamilton theorem, Nakayama lemma, and corollaries . .	4
<b>3</b>	<b>Noetherian rings</b>	<b>9</b>
<b>4</b>	<b>Exercises</b>	<b>11</b>
4.1	Exercises Chapter 1 . . . . .	11
4.2	Exercises Chapter 2 . . . . .	13

## 1 Ideals

### 1.1 Definitions

**Definition ideal.**  $I \subset R$  ( $R$  ring) such that  $0 \in I$  and  $\forall x \in I, r \in R, xr, rx \in I$ .

ie.  $I$  absorbs products in  $R$ .

**Definition prime ideal.** if  $a, b \in R$  with  $ab \in P$  and  $P \neq R$  ( $P$  a prime ideal), implies  $a \in P$  or  $b \in P$ .

**Definition principal ideal.** generated by a single element,  $(a)$ .

$(a)$ : principal ideal, the set of all multiples  $xa$  with  $x \in R$ .

**Definition maximal ideal.**  $\mathfrak{m} \subset A$  ( $A$  ring) with  $\mathfrak{m} \neq A$  and there is no ideal  $I$  strictly between  $\mathfrak{m}$  and  $A$ . ie. if  $\mathfrak{m}$  maximal and  $\mathfrak{m} \subseteq I \subseteq A$ , either  $\mathfrak{m} = I$  or  $I = A$ .

**Definition unit.**  $x \in A$  such that  $xy = 1$  for some  $y \in A$ . ie. element *which divides 1*.

**Definition zerodivisor.**  $x \in A$  such that  $\exists 0 \neq y \in A$  such that  $xy = 0 \in A$ . ie.  $x$  *divides 0*.

If a ring does not have zerodivisors is an integral domain.

**Definition prime spectrum -  $\text{Spec}(A)$ .** set of prime ideals of  $A$ . ie.

$$\text{Spec}(A) = \{P \mid P \subset A \text{ is a prime ideal}\}$$

**Definition integral domain.** Ring in which the product of any two nonzero elements is nonzero.

ie. no zerodivisors.

ie.  $\forall 0 \neq a, 0 \neq b \in A, ab \neq 0 \in A$ .

Every field is an integral domain, not the converse.

**Definition principal ideal domain - PID.** integral domain in which every ideal is principal. ie. ie.  $\forall I \subset R, \exists a \in I$  such that  $I = (a) = \{ra \mid r \in R\}$ .

**Definition nilpotent.**  $a \in A$  such that  $a^n = 0$  for some  $n > 0$ .

**Definition nilrad  $A$ .** set of all nilpotent elements of  $A$ ; is an ideal of  $A$ .  
if  $\text{nilrad}A = 0 \implies A$  has no nonzero nilpotents.

$$\text{nilrad}A = \bigcap_{P \in \text{Spec}(A)} P$$

**Definition idempotent.**  $e \in A$  such that  $e^2 = e$ .

**Definition radical of an ideal.**

$$\text{rad}I = \{f \in A \mid f^n \in I \text{ for some } n\}$$

$\text{rad}I$  is an ideal.

$$\text{nilrad}A = \text{rad}0$$

$$\text{rad}I = \bigcap_{\substack{P \in \text{Spec}(A) \\ P \supset I}} P$$

**Definition local ring.** A *local ring* has a unique maximal ideal.

Notation: local ring  $A$ , its maximal ideal  $\mathfrak{m}$ , residue field  $K = A/\mathfrak{m}$ :

$$A \supset \mathfrak{m} \text{ or } (A, \mathfrak{m}) \text{ or } (A, \mathfrak{m}, K)$$

## 1.2 Lemmas, propositions and corollaries

**Theorem AM.1.X.** Zorn's lemma TODO

**Theorem AM.1.3.** Every ring  $A \neq 0$  has at least one maximal ideal.

*Proof.* By Zorn's lemma AM.1.X. □

**Corollary AM.1.4.** if  $I \neq (1)$  an ideal of  $A$ ,  $\exists$  a maximal ideal of  $A$  containing  $I$ .

**Corollary AM.1.5.** Every non-unit of  $A$  is contained in a maximal ideal.

**Definition Jacobson radical.** The *Jacobson radical* of a ring  $A$  is the intersection of all the maximal ideals of  $A$ .

Denoted  $Jac(A)$ .

$Jac(A)$  is an ideal of  $A$ .

**Proposition AM.1.9.**  $x \in Jac(A)$  iff  $(1 - xy)$  is a unit in  $A$ ,  $\forall y \in A$ .

*Proof.* Suppose  $1 - xy$  not a unit.

By AM.1.5,  $1 - xy \in \mathfrak{m}$  for  $\mathfrak{m}$  some maximal ideal.

But  $x \in Jac(A) \subseteq \mathfrak{m}$ , since  $Jac(A)$  is the intersection of all maximal ideals of  $A$ .

Hence  $xy \in \mathfrak{m}$ , and therefore  $1 \in \mathfrak{m}$ , which is absurd, thus  $1 - xy$  is a unit.

Conversely:

Suppose  $x \notin \mathfrak{m}$  for some maximal ideal  $\mathfrak{m}$ .

Then  $\mathfrak{m}$  and  $x$  generate the unit ideal  $(1)$ , so that we have  $u + xy = 1$  for some  $u \in \mathfrak{m}$  and some  $y \in A$ .

Hence  $1 - xy \in \mathfrak{m}$ , and is therefore not a unit. □

## 2 Modules

### 2.1 Modules

Let  $A$  be a ring. An  $A$ -module is an Abelian group  $M$  with a multiplication map

$$\begin{aligned} A \times M &\longrightarrow M \\ (f, m) &\longmapsto fm \end{aligned}$$

satisfying  $\forall f, g \in A, m, n \in M$ .

i.  $f(m \pm n) = fm \pm fn$

ii.  $(f \pm g)m = fm \pm gm$

iii.  $(fg)m = f(gm)$

iv.  $1_A m = m$

Let  $\psi : M \longrightarrow M$  an  $A$ -linear endomorphism of  $M$ .  
 $A[\psi] \subset \text{End}M$  is the subring generated by  $A$  and the action of  $\psi$ .

- since  $\psi$  is  $A$ -linear,  $A[\psi]$  is a commutative ring.
- $M$  is a module over  $A[\psi]$ , so  $\psi$  becomes multiplication by a ring element.

## 2.2 Cayley-Hamilton theorem, Nakayama lemma, and corollaries

**Proposition AM.2.4.** (Cayley-Hamilton Theorem) Let  $M$  a finitely generated  $A$ -module. Let  $\mathfrak{a}$  an ideal of  $A$ , let  $\psi$  an  $A$ -module endomorphism of  $M$  such that  $\psi(M) \subseteq \mathfrak{a}M$ .

Then  $\psi$  satisfies

$$\psi^n + a_1\psi^{n-1} + \dots + a_{n-1}\psi + a_n = 0$$

with  $a_i \in \mathfrak{a}$ .

*Proof.* Since  $M$  finitely generated, let  $\{x_1, \dots, x_n\}$  be generators of  $M$ .

By hypothesis,  $\psi(M) \subseteq \mathfrak{a}M$ ; so for any generator  $x_i$ , its image  $\psi(x_i) \in \mathfrak{a}M$ .

Any element in  $\mathfrak{a}M$  is a linear combination of the generators with coefficients in the ideal  $\mathfrak{a}$ , thus

$$\psi(x_i) = \sum_{j=1}^n a_{ij}x_j$$

with  $a_{ij} \in \mathfrak{a}$ .

Thus, for a module with  $n$  generators, we have  $n$  different  $\psi(x_i)$  equations:

$$\left. \begin{array}{l} \psi(x_1) = a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n \\ \psi(x_2) = a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n \\ \dots \\ \psi(x_n) = a_{n,1}x_1 + a_{n,2}x_2 + \dots + a_{n,n}x_n \end{array} \right\} \begin{array}{l} n \text{ elements } \psi(x_i) \in \mathfrak{a}M \text{ which} \\ \text{are linear combinations of the} \\ n \text{ generators of } M \end{array}$$

Next step: rearrange in order to use matrix algebra.

Observe that each row equals 0, and rearranging the elements at each row we get

$$\begin{array}{l} \psi(x_1) - (a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n) = 0 \\ \psi(x_2) - (a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n) = 0 \\ \dots \\ \psi(x_n) - (a_{n,1}x_1 + a_{n,2}x_2 + \dots + a_{n,n}x_n) = 0 \end{array}$$

Then, group the  $x_i$  terms together; as example, take the row  $i = 1$ :

$$(\psi - a_{1,1})x_1 - a_{1,2}x_2 - \dots - a_{1,n}x_n = 0$$

for  $i = 2$ :

$$-a_{2,1}x_1 + (\psi - a_{2,2})x_2 - \dots - a_{2,n}x_n = 0$$

So,  $\forall i \in [n]$ , as a matrix:

$$\begin{pmatrix} \psi - a_{1,1} & -a_{1,2} & \dots & -a_{1,n} \\ -a_{2,1} & \psi - a_{2,2} & \dots & -a_{2,n} \\ \vdots & & & \\ -a_{n,1} & -a_{n,2} & \dots & \psi - a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Kronecker delta:  $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$

With the Kronecker delta,  $\psi(x_i)$  can be expressed as

$$\psi(x_i) = \sum_{j=1}^n \delta_{ij} \psi(x_j)$$

so the previous matrix can be characterized as

$$\sum_{j=1}^n (\delta_{ij} \psi - a_{ij}) x_j = 0$$

The entries of the matrix are *endomorphisms* (elements of the ring  $A[\psi]$ )

- the term  $(\psi - a_{11})$  is an operator that acts on  $x_1$ ; as  $(\psi(x_1) - a_{11} \cdot x_1)$
- the term  $(-a_{12})$  is an operator that acts on  $x_2$ ; as multiplication by it, ie.  $(-a_{12} \cdot x_2)$

Since  $A$  is a commutative ring, and  $\psi$  commutes with any  $a \in A$ , the ring of operators  $A[\psi]$  is a commutative ring.

$\implies$  so we can treat the matrix as a matrix of real numbers and calculate its determinant.

We're interested in the determinant because it is the only way to turn a system of multiple equations in a single scalar-like equation that describes the endomorphism  $\psi$ .

$\rightarrow$  Because in module theory, we lack of "division", so can not "solve for  $\psi$ " the system of equations.

$\rightarrow$  The determinant provides a way to find a polynomial that *annihilates* the module; the *characteristic polynomial*, which related  $\psi$  to the ideal  $\mathfrak{a}$

$$\det(M) \cdot x_i = 0 \quad \forall i$$

where  $x_i$  are the generators of  $M$ .

Since  $\det(M)$  kills every generator, it must kill every element in  $M$

$\implies \det(M)$  is the zero map.

Leibniz formula of the determinant of an  $n \times n$  matrix:

$$\det(M) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n M_{i, \sigma(i)}$$

so,

$$(\psi - a_{11})(\psi - a_{22}) \dots (\psi - a_{nn})$$

expanding it,

- highest power is  $1 \cdot \psi^n$
- coefficient of  $\psi^{n-1}$  is  $-(\underbrace{a_{11} + a_{22} + \dots + a_{nn}}_{a_1})$ ,  
where, since each  $a_{ii} \in \mathfrak{a}$ ,  $a_1 \in \mathfrak{a}$

- the rest of coefficients of  $\psi^k$  are also elements in  $\mathfrak{a}$

So we have

$$p(\psi) = \psi^n + a_1 \psi^{n-1} + a_2 \psi^{n-2} + \dots + a_{n-1} \psi + a_n$$

with  $a_i \in \mathfrak{a}$ .

Since this determinant annihilates the generators (ie.  $\det(M)x_i = 0$ ), the resulting endomorphism  $p(\psi)$  is the zero map on the entire module  $M$ , so:

$$\psi^n + a_1 \psi^{n-1} + a_2 \psi^{n-2} + \dots + a_{n-1} \psi + a_n = 0$$

with  $a_i \in \mathfrak{a}$ , as stated in the Cayley-Hamilton theorem.  $\square$

**Corollary AM.2.5.** Let  $M$  a fingen  $A$ -module, let  $\mathfrak{a}$  an ideal of  $A$  such that  $\mathfrak{a}M = M$ .

Then,  $\exists x \equiv 1 \pmod{\mathfrak{a}}$  such that  $xM = 0$ .

*Proof.* take  $\psi = \text{identity}$ . Then in Cayley-Hamilton (AM.2.4):

$$\begin{aligned} \psi^n + a_1 \psi^{n-1} + a_2 \psi^{n-2} + \dots + a_{n-1} \psi + a_n &= 0 \\ \implies id_M + a_1 id_M + a_2 id_M + \dots + a_{n-1} id_M + a_n &= 0 \\ \implies (1 + a_1 + \dots + a_n) id_M &= 0 \end{aligned}$$

apply it to  $m \in M$ , where since  $id_M(m) = m$  (by definition of the identity), we then have

$$(1 + a_1 + \dots + a_n) \cdot m = 0$$

with  $a_i \in \mathfrak{a}$ .

part i.  $xM = 0$ :

Thus the scalar  $x = (1 + a_1 + \dots + a_n)$  annihilates every  $m \in M$ , ie. the entire module  $M$ .

part ii.  $x \equiv 1 \pmod{\mathfrak{a}}$ :

$$x \equiv 1 \pmod{\mathfrak{a}} \iff (x - 1) \in \mathfrak{a}$$

then from  $x = (1 + \underbrace{a_1 + \dots + a_n}_b) \in \mathfrak{a}$ , set  $b = a_1 + \dots + a_n$ ,

so that  $x = (1 + b) \in \mathfrak{a}$ .

Then  $x - 1 = (1 + b) - 1 = b \in \mathfrak{a}$

so  $x - 1 \in \mathfrak{a}$ , thus  $x \equiv 1 \pmod{\mathfrak{a}}$  as stated. □

**Proposition AM.2.6.** Nakayama's lemma Let  $M$  a fingen  $A$ -module, let  $\mathfrak{a}$  an ideal of  $A$  such that  $\mathfrak{a} \subseteq \text{Jac}(A)$ .

Then  $\mathfrak{a}M = M$  implies  $M = 0$ .

*Proof.* By AM.2.5: since  $\mathfrak{a}M = M$ , we have  $xM = 0$  for some  $x \equiv 1 \pmod{\text{Jac}(A)}$ . (notice that at AM.2.5 is  $\pmod{\mathfrak{a}}$  but here we use  $\pmod{\text{Jac}(A)}$ ), since we have  $\mathfrak{a} \subseteq \text{Jac}(A)$ ).

(recall AM.1.9:  $x \in \text{Jac}(A)$  iff  $(1 - xy)$  is a unit in  $A$ ,  $\forall y \in A$ ).

By AM.1.9,  $x$  is a unit in  $A$  (thus  $x^{-1} \cdot x = 1$ ).

$$\text{Hence } M = x^{-1} \cdot \underbrace{x \cdot M}_{=0 \text{ (by AM.2.5)}} = 0.$$

Thus, if  $\mathfrak{a}M = M$  then  $M = 0$ . □

**Corollary AM.2.7.** Let  $M$  a fingen  $A$ -module, let  $N \subseteq M$  a submodule of  $M$ , let  $\mathfrak{a} \subseteq \text{Jac}(A)$  an ideal.

Then  $M = \mathfrak{a}M + N \xRightarrow{\text{implies}} M = N$ .

*Proof.* The idea is to apply Nakayama (AM.2.6) to  $M/N$ .

Since  $M$  fingen  $\implies M/N$  is fingen and an  $A$ -module.

Since  $\mathfrak{a} \subseteq \text{Jac}(A) \implies$  Nakayama applies to  $M/N$  too.

By definition,

$$\mathfrak{a}M = \left\{ \sum a_i \cdot m_i \mid a_i \in \mathfrak{a}, m_i \in M \right\}$$

where  $m_i$  are the generators of  $M$ .

Then, for  $M/N$ ,

$$\mathfrak{a}\left(\frac{M}{N}\right) = \left\{ \sum a_i \cdot (m_i + N) \mid a_i \in \mathfrak{a}, m_i \in M \right\}$$

observe that  $a_i(m_i + N) = a_i m_i + N$ , thus

$$\sum_i a_i \cdot (m_i + N) = \underbrace{\left( \sum_i a_i \cdot m_i \right)}_{\in \mathfrak{a}M} + N \in \mathfrak{a}M + N$$

Hence,

$$\mathfrak{a}\left(\frac{M}{N}\right) = \{x + N \mid x \in \mathfrak{a}M\} = \mathfrak{a}M + N \quad (1)$$

By definition, if we take  $\frac{\mathfrak{a}M+N}{N}$ , then

$$\frac{\mathfrak{a}M + N}{N} = \{y + N \mid y \in \mathfrak{a}M + N\} = \mathfrak{a}M + N$$

thus every  $y \in \mathfrak{a}M + N$  can be written as

$$y = x + n, \text{ with } x \in \mathfrak{a}M, n \in N$$

which comes from (1).

Thus,  $y + N = (x + n) + N = x + N$ , since  $n \in N$  is zero in the quotient.

Hence, every element of  $\frac{\mathfrak{a}M+N}{N}$  has the form

$$\frac{\mathfrak{a}M + N}{N} = \{x + N \mid x \in \mathfrak{a}M\}$$

as in (1).

Thus

$$\mathfrak{a}\left(\frac{M}{N}\right) = \mathfrak{a}M + N = \frac{\mathfrak{a}M + N}{N} \quad (2)$$

By the Collorary assumption,  $M = \mathfrak{a}M + N$ ; quotient it by  $N$ :

$$\frac{M}{N} = \frac{\mathfrak{a}M + N}{N} \quad (3)$$

So, from (2) and (3):

$$\mathfrak{a}\left(\frac{M}{N}\right) = \mathfrak{a}M + N = \frac{\mathfrak{a}M + N}{N} = \frac{M}{N}$$

thus,  $\mathfrak{a}\left(\frac{M}{N}\right) = \frac{M}{N}$ .

By Nakayama's lemma AM.2.6, if  $\mathfrak{a}\left(\frac{M}{N}\right) = \frac{M}{N} \implies \frac{M}{N} = 0$

Note that

$$\frac{M}{N} = \{m + N \mid m \in M\}$$

(the zero element in  $\frac{M}{N}$  is the coset  $N = 0 + N$ )

Then,  $\frac{M}{N} = 0$  means that the quotient has exactly one element, the zero coset  $N$ .

Thus, every coset  $m + N$  equals the zero coset  $N$ , so  $m - 0 \in N \implies m \in N$ .

Hence every  $m \in M$  lies in  $N$ , ie.  $\forall m \in M, m \in N$ .

So  $M \subseteq N$ . But notice that by the Corollary, we had  $N \subseteq M$ , therefore  $M = N$ .

Thus, if  $M = \mathfrak{a}M + N \implies M = N$ . □

**Proposition AM.2.8.** Let  $x_i \forall i \in [n]$  be elements of  $M$  whose images  $\frac{M}{mM}$  from a basis of this vecctor space. Then the  $x_i$  generate  $M$ .



*Proof.* Let  $N$  submodule  $M$ , generated by the  $x_i$ .

Then the composite map  $N \longrightarrow M \longrightarrow \frac{M}{mM}$  maps  $N$  onto  $\frac{M}{mM}$ , hence  $N + \mathfrak{a}M = M$ , which by AM.2.7 implies  $N = M$ .  $\square$

**Proposition AM.2.10.** Split exact sequence. TODO

### 3 Noetherian rings

**Definition .** Ascending Chain Condition A partially ordered set  $\Sigma$  has the *ascending chain condition* (a.c.c.) if every chain

$$s_1 \leq s_2 \leq \dots \leq s_k \leq \dots$$

eventually breaks off, that is,  $s_k = s_{k+1} = \dots$  for some  $k$ .

$\implies$   $\Sigma$  has the a.c.c. iff every non-empty subset  $S \subset \Sigma$  has a maximal element.

if  $\neq S \subset \Sigma$  does not have a maximal element, choose  $s_1 \in S$ , and for each  $s_k$ , an element  $s_{k+1}$  with  $s_k < s_{k+1}$ , thus contradicting the a.c.c.

**Definition R.3.2.** Noetherian ring Let  $A$  a ring; 3 equivalent conditions:

- i. the set  $\Sigma$  of ideals of  $A$  has the a.c.c.; in other words, every increasing chain of ideals

$$I_1 \subset I_2 \subset \dots \subset I_k \subset \dots$$

eventually stops, that is  $I_k = I_{k+1} = \dots$  for some  $k$ .

- ii. every nonempty set  $S$  of ideals has a maximal element
- iii. every ideal  $I \subset A$  is finitely generated

If these conditions hold, then  $A$  is *Noetherian*.

*Proof.* TODO  $\square$

**Definition R.3.4.D.** Noetherian modules An  $A$ -module  $M$  is Noetherian if the submodules of  $M$  have the a.c.c., that is, any increasing chain

$$M_1 \subset M_2 \subset \dots \subset M_k \subset \dots$$

of submodules eventually stops.

As in with rings, it is equivalent to say that

- i. any nonempty set of modules of  $M$  has a maximal element
- ii. every submodule of  $M$  is finite

**Proposition R.3.4.P.** Let  $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$  be a s.e.s. (split exact sequence, AM.2.10).

Then,  $M$  is Noetherian  $\iff L$  and  $N$  are Noetherian.

*Proof.*  $\implies$ : trivial, since ascending chains of submodules in  $L$  and  $N$  correspond one-to-one to certain chains in  $M$ .

$\impliedby$ : suppose  $M_1 \subset M_2 \subset \dots \subset M_k \subset \dots$  is an increasing chain of submodules of  $M$ .

Then identifying  $\alpha(L)$  with  $L$  and taking intersection gives a chain

$$L \cap M_1 \subset L \cap M_2 \subset \dots \subset L \cap M_k \subset \dots$$

of submodules of  $L$ , and applying  $\beta$  gives a chain

$$\beta(M_1) \subset \beta(M_2) \subset \dots \beta(M_k) \subset \dots$$

of submodules of  $N$ .

Each of these two chains eventually stop, by the assumption on  $L$  and  $N$ , so that we only need to prove the following lemma which completes the proof.  $\square$

**Lemma R.3.4.L.** for submodules  $M_1 \subset M_2 \subset M$ ,

$$L \cap M_1 = L \cap M_2 \text{ and } \beta(M_1) = \beta(M_2) \implies M_1 = M_2$$

*Proof.* if  $m \in M_2$ , then  $\beta(m) \in \beta(M_1) = \beta(M_2)$ , so that there is an  $n \in M_1$  such that  $\beta(m) = \beta(n)$ .

Then  $\beta(m - n) = 0$ , so that

$$m - n \in M_2 \cap \ker(\beta) = M_1 \cap \ker(\beta)$$

Hence  $m \in M_1$ , thus  $M_1 = M_2$ .  $\square$

## 4 Exercises

For the exercises, I follow the assignments listed at [3].

The exercises that start with **R** are the ones from the book [2], and the ones starting with **AM** are the ones from the book [1].

### 4.1 Exercises Chapter 1

**Exercise R.1.1.** Ring  $A$  and ideals  $I, J$  such that  $I \cup J$  is not an ideal. What's the smallest ideal containing  $I$  and  $J$ ?

*Proof.* Take ring  $A = \mathbb{Z}$ . Set  $I = 2\mathbb{Z}$ ,  $J = 3\mathbb{Z}$ .

$I, J$  are ideals of  $A (= \mathbb{Z})$ . And  $I \cup J = 2\mathbb{Z} \cup 3\mathbb{Z}$ .

Observe that for  $2 \in I$ ,  $3 \in J \implies 2, 3 \in I \cup J$ , but  $2 + 3 = 5 \notin I \cup J$ .

Thus  $I \cup J$  is not closed under addition; thus is not an ideal.

Smallest ideal of  $\mathbb{Z} (= A)$  containing  $I$  and  $J$  is their sum:

$$I + J = \{a + b \mid a \in I, b \in J\}$$

$\gcd(2, 3) = 1$ , so  $I + J = \mathbb{Z}$ .

Therefore, smallest ideal containing  $I$  and  $J$  is the whole ring  $\mathbb{Z}$ .  $\square$

**Exercise R.1.5.** let  $\psi : A \longrightarrow B$  a ring homomorphism. Prove that  $\psi^{-1}$  takes prime ideals of  $B$  to prime ideals of  $A$ .

In particular if  $A \subset B$  and  $P$  a prime ideal of  $B$ , then  $A \cap P$  is a prime ideal of  $A$ .

*Proof.* (Recall: prime ideal is if  $a, b \in R$  and  $a \cdot b \in P$  (with  $R \neq P$ ), implies  $a \in P$  or  $b \in P$ ).

Let

$$\psi^{-1}(P) = \{a \in A \mid \psi(a) \in P\} = A \cap P$$

The claim is that  $\psi^{-1}(P)$  is prime ideal of  $A$ .

i. show that  $\psi^{-1}(P)$  is an ideal of  $A$ :

$0_A \in \psi^{-1}(P)$ , since  $\psi(0_A) = 0_B \in P$  (since every ideal contains 0).

If  $a, b \in \psi^{-1}(P)$ , then  $\psi(a), \psi(b) \in P$ , so

$$\psi(a - b) = \psi(a) - \psi(b) \in P$$

hence  $a - b \in \psi^{-1}(P)$ .

If  $a \in \psi^{-1}(P)$  and  $r \in A$ , then  $\psi(ra) = \psi(r)\psi(a) \in P$ , since  $P$  is an ideal.

Thus  $ra \in \psi^{-1}(P)$ .

$\implies$  so  $\psi^{-1}$  is an ideal of  $A$ .

ii. show that  $\psi^{-1}(P)$  is prime:

$\psi^{-1}(P) \neq A$ , since if  $\psi^{-1}(P) = A$ , then  $1_A \in \psi^{-1}(P)$ , so  $\psi(1_A) = 1_B \in P$ , which would mean that  $P = B$ , a contradiction since  $P$  is prime ideal of  $B$ .

Take  $a, b \in A$  with  $ab \in \psi^{-1}(P)$ ; then  $\psi(ab) \in P$ , and since  $\psi$  is a ring homomorphism,  $\psi(ab) = \psi(a)\psi(b)$ .

Since  $P$  prime ideal, then  $\psi(a)\psi(b) \in P$  implies either  $\psi(a) \in P$  or  $\psi(b) \in P$ . Thus  $a \in \psi^{-1}(P)$  or  $b \in \psi^{-1}(P)$ .

Hence  $\psi^{-1}(P)$  ( $= A \cap P$ ) is a prime ideal of  $A$ .

□

**Exercise R.1.6.** prove or give a counter example:

- a. the intersection of two prime ideals is prime
- b. the ideal  $P_1 + P_2$  generated by 2 prime ideals  $P_1, P_2$  is prime
- c. if  $\psi : A \rightarrow B$  ring homomorphism, then  $\psi^{-1}$  takes maximal ideals of  $B$  to maximal ideals of  $A$

*Proof.* a. let  $I = 2\mathbb{Z} = (2)$ ,  $J = 3\mathbb{Z} = (3)$  be ideals of  $\mathbb{Z}$ , both prime.

Then  $I \cap J = (2) \cap (3) = (6)$ .

The ideal  $(6)$  is not prime in  $\mathbb{Z}$ , since  $2 \cdot 3 \in (6)$ , but  $2 \notin (6)$  and  $3 \notin (6)$ .

Thus the intersection of two primes can not be prime.

- b.  $P_1 = (2)$ ,  $P_2 = (3)$ , both prime.

Then,

$$P_1 + P_2 = (2) + (3) = \{a + b \mid a \in P_1, b \in P_2\}$$

$\rightarrow$  in a principal ideal domain (like  $\mathbb{Z}$ ), the sum of two principal ideals is again principal, and given by  $(m) + (n) = (gcd(m, n))$ .

(recall: principal = generated by a single element)

So,  $P_1 + P_2 = (2) + (3) = (gcd(2, 3)) = (1) = \mathbb{Z}$ .

The whole ring is not a prime ideal (by the definition of the prime ideal), so  $P_1 + P_2$  is not a prime ideal.

Henceforth, the sum of two prime ideals is not necessarily prime.

- c. let  $A = \mathbb{Z}$ ,  $B = \mathbb{Q}$ ,  $\psi : A \rightarrow B$ .

Since  $\mathbb{Q}$  is a field, its only maximal ideal is  $(0)$ .

Then

$$\begin{aligned} \psi^{-1}((0)) &= (0) \subset \mathbb{Z} \\ \text{ie. } \psi^{-1}(m_B) &= (m_B) \subset A \end{aligned}$$

But  $(0)$  is not maximal in  $\mathbb{Z}$ , because  $\mathbb{Z}/(0) \cong \mathbb{Z}$  is not a field.

Thus the preimages of maximal ideals under arbitrary ring homomorphisms need not be maximal.

□

## 4.2 Exercises Chapter 2

### References

- [1] M. F. Atiyah and I. G. MacDonald. Introduction to Commutative Algebra, 1969.
- [2] Miles Reid. Undergraduate Commutative Algebra, 1995.
- [3] Steven Kleiman. Commutative Algebra - MIT OpenCourseWare, 2008. <https://ocw.mit.edu/courses/18-705-commutative-algebra-fall-2008/>.