

# Galois Theory notes

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## Abstract

Notes taken while studying Galois Theory, mostly from Ian Stewart's book "Galois Theory" [1].

Usually while reading books and papers I take handwritten notes in a notebook, this document contains some of them re-written to *LaTeX*.

The notes are not complete, don't include all the steps neither all the proofs.

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# 1 Galois Theory notes

## 1.1 Chapters 4-12

(Definitions, theorems, lemmas, corollaries and examples enumeration follows from Ian Stewart's book [1]).

**Definition 4.10.** A *simple extension* is  $L : K$  such that  $L = K(\alpha)$  for some  $\alpha \in L$ .

**Example 4.11.** Beware,  $L = \mathbb{Q}(i, -i, \sqrt{5}, -\sqrt{5}) = \mathbb{Q}(i, \sqrt{5}) = \mathbb{Q}(i + \sqrt{5})$ .

**Definition 5.5.** Let  $L : K$ , suppose  $\alpha \in L$  is algebraic over  $K$ . Then, the *minimal polynomial* of  $\alpha$  over  $K$  is the unique monic polynomial  $m$  over  $K$ ,  $m(t) \in K[t]$ , of smallest degree such that  $m(\alpha) = 0$ .

e.g.:  $i \in \mathbb{C}$  is algebraic over  $\mathbb{R}$ . The minimal polynomial of  $i$  over  $\mathbb{R}$  is  $m(t) = t^2 + 1$ , so that  $m(i) = 0$ .

**Lemma 5.9.** Every polynomial  $a \in K[t]$  is congruent modulo  $m$  to a unique polynomial of degree  $< \delta m$ .

*Proof.* Divide  $a/m$  with remainder,  $a = qm + r$ , with  $q, r \in K[t]$  and  $\delta r < \delta m$ . Then,  $a - r = qm$ , so  $a \equiv r \pmod{m}$ .

It remains to prove uniqueness.

Suppose  $\exists r \equiv s \pmod{m}$ , with  $\delta r, \delta s < \delta m$ . Then,  $r - s$  is divisible by  $m$ , but has smaller degree than  $m$ .

Therefore,  $r - s = 0$ , so  $r = s$ , proving uniqueness.  $\square$

**Theorem 5.10.**  $\forall 0 \neq f \in \frac{K[t]}{\langle m \rangle}$ ,  $\exists f^{-1}$  iff  $m$  is irreducible in  $K[t]$ .

Then  $\frac{K[t]}{\langle m \rangle}$  is a field.

**Theorem 5.12.** Let  $K(\alpha) : K$  simple algebraic extension, let  $m$  minimal polynomial of  $\alpha$  over  $K$ .

$K(\alpha) : K$  is isomorphic to  $\frac{K[t]}{\langle m \rangle}$ .

The isomorphism  $\frac{K[t]}{\langle m \rangle} \longrightarrow K(\alpha)$  can be chosen to map  $t$  to  $\alpha$ .

**Corollary 5.13.** Let  $K(\alpha) : K$  and  $K(\beta) : K$  be simple algebraic extensions. If  $\alpha, \beta$  have same minimal polynomial  $m$  over  $K$ , then the two extensions are isomorphic, and the isomorphism of the larger fields map  $\alpha$  to  $\beta$ .

*Proof.* By 5.12, both extensions are isomorphic to  $\frac{K[t]}{\langle m \rangle}$ .  $\square$

**Lemma 5.14.** Let  $K(\alpha) : K$  be a simple algebraic extension, let  $m$  be the minimal polynomial of  $\alpha$  over  $K$ , let  $\delta m = n$ .

Then  $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$  is a basis for  $K(\alpha)$  over  $K$ . In particular,  $[K(\alpha) : K] = n$ .

**Definition 6.2.** The degree  $[L : K]$  of a field extension  $L : K$  is the dimension of  $L$  considered as a vector space over  $K$ .

Equivalently, the dimension of  $L$  as a vector space over  $K$  is the number of terms in the expression for a general element of  $L$  using coefficients from  $K$ .

**Example 6.3.** 1.  $\mathbb{C}$  elements are 2-dimensional over  $\mathbb{R}$  ( $p + qi \in \mathbb{C}$ , with  $p, q \in \mathbb{R}$ ), because a basis is  $\{1, i\}$ , hence  $[\mathbb{C} : \mathbb{R}] = 2$ .

2.  $[\mathbb{Q}(i, \sqrt{5}) : \mathbb{Q}] = 4$ , since the elements  $\{1, \sqrt{5}, i, i\sqrt{5}\}$  form a basis for  $\mathbb{Q}(i, \sqrt{5})$  over  $\mathbb{Q}$ .

**Theorem 6.4. (Short Tower Law)** If  $K, L, M \subseteq \mathbb{C}$ , and  $K \subseteq L \subseteq M$ , then  $[M : K] = [M : L] \cdot [L : K]$ .

*Proof.* Let  $(x_i)_{i \in I}$  be a basis for  $L$  over  $K$ , let  $(y_j)_{j \in J}$  be a basis for  $M$  over  $L$ .  $\forall i \in I, j \in J$ , we have  $x_i \in L, y_j \in M$ .

Want to show that  $(x_i y_j)_{i \in I, j \in J}$  is a basis for  $M$  over  $K$ .

- i. prove linear independence:

Suppose that

$$\sum_{ij} k_{ij} x_i y_j = 0 \quad (k_{ij} \in K)$$

rearrange

$$\sum_j (\underbrace{\sum_i k_{ij} x_i}_{\in L}) y_j = 0 \quad (k_{ij} \in K)$$

Since  $\sum_i k_{ij} x_i \in L$ , and the  $y_j \in M$  are linearly independent over  $L$ , then  $\sum_i k_{ij} x_i = 0$ .

Repeating the argument inside  $L \rightarrow k_{ij} = 0 \quad \forall i \in I, j \in J$ .

So the elements  $x_i y_j$  are linearly independent over  $K$ .

- ii. prove that  $x_i y_j$  span  $M$  over  $K$ :

Any  $x \in M$  can be written

$$x = \sum_j \lambda_j y_j$$

for  $\lambda_j \in L$ , because  $y_j$  spans  $M$  over  $L$ . Similarly,

$$\forall j \in J, \lambda_j = \sum_i \lambda_{ij} x_i y_j$$

for  $\lambda_{ij} \in K$ .

Putting the pieces together,

$$x = \sum_{ij} \lambda_{ij} x_i y_j$$

as required. □

**Corollary 6.6. (Tower Law)**

If  $K_0 \subseteq K_1 \subseteq \dots \subseteq K_n$  are subfields of  $\mathbb{C}$ , then

$$[K_n : K_0] = [K_n : K_{n-1}] \cdot [K_{n-1} : K_{n-2}] \cdot \dots \cdot [K_1 : K_0]$$

*Proof.* From 6.4. □

**Theorem 6.7.** if  $K(\alpha) : K$

- transcendental  $\implies [K(\alpha) : K] = \inf$
- algebraic  $\implies [K(\alpha) : K] = \delta m$

(where  $m$  is the minimal polynomial of  $\alpha$  over  $K$ ).

**Definition 8.1.**  $L : K$ , a  $K$ -automorphism of  $L$  is an automorphism  $\alpha$  of  $L$  such that  $\alpha(k) = k \ \forall k \in K$ .  
ie.  $\alpha$  fixes  $k$ .

**Theorem 8.2, 8.3.** The set of all  $K$ -automorphisms of  $L$  forms a group,  $\Gamma(L : K)$ , the Galois group of  $L : K$ .

**Definition 8.12.** (Radical Extension)  $L : K$  is radical if  $L = K(\alpha_1, \dots, \alpha_m)$  where for each  $j = 1, \dots, m$ ,  $\exists n_j$  such that  $\alpha_j^{n_j} \in K(\alpha_1, \dots, \alpha_{j-1})$  ( $j \geq 1$ )

**Lemma 8.18.** Let  $q \in L$ . The minimal polynomial of  $q$  over  $K$  splits into linear factors over  $L$ .

**Exercise E.8.7.** TODO

**Definition 9.1.** For  $K \subseteq \mathbb{C}$ , and  $f \in K[t]$ ,  $f$  splits over  $K$  if it can be expressed as a product of linear factors

$$f(t) = k \cdot (t - \alpha_1) \cdot \dots \cdot (t - \alpha_n)$$

where  $k, \alpha_i \in K$ .

$\implies$  (Thm 9.3) if  $f$  splits over  $\Sigma$ ,  $\Sigma$  is the splitting field.  
If  $K \subseteq \Sigma' \subseteq \Sigma$  and  $f$  splits over  $\Sigma'$ , then  $\Sigma' = \Sigma$ .

**Theorem 9.6.** TODO

**Definition 9.8.**  $L : K$  is normal if every irreducible polynomial  $f \in K[t]$  that has at least one zero in  $L$ , splits in  $L$ .

**Theorem 9.9.** TODO

**Theorem 9.10.** An irreducible polynomial  $f \in K[t]$  ( $K \subseteq \mathbb{C}$ ) is separable over  $K$  if it has simple zeros in  $\mathbb{C}$ , or equivalently, simple zeros in its splitting field.

**Lemma 9.13.**  $f \in K[t]$  with splitting field  $\Sigma$ .  $f$  has multiple zeros (in  $\Sigma$  or  $\mathbb{C}$ ) iff  $f$  and  $Df$  have a common factor of degree  $\geq 1$  in  $\Sigma[t]$ .

More details at Rolle's theorem (2.7) section.

**Theorem 10.5.**  $|\Gamma(K : K_0)| = [K : K_0]$ , where  $K_0$  is the fixed field of  $\Gamma(K : K_0)$ .

**Definition 11.1.**  $K \subseteq L$ ,  $K \subseteq M \subseteq L$ . A  $K$ -monomorphism of  $M$  into  $L$  is a field monomorphism

$$\phi : M \longrightarrow L$$

such that  $\phi(k) = k \quad \forall k \in K$ .

**Theorem 11.3.**  $L : K$  normal,  $K \subseteq M \subseteq L$ . Let  $\tau$  any  $K$ -monomorphism  $\tau : M \longrightarrow L$ .

Then,  $\exists$  a  $K$ -automorphism  $\sigma$  of  $L$  such that  $\sigma \Big|_M = \tau$ .

*Proof.*  $L : K$  normal  $\implies$  by Thm 9.9,  $L$  splitting field for some poly  $f \in K[t]$ .

Hence,  $L$  is splitting field over  $M$  for  $f$  and over  $\tau(M)$  for  $\tau(f)$ .

Since  $\tau \Big|_K$  is the identity,  $\tau(f) = f$ .

We have

$$\begin{array}{ccc} M & \longrightarrow & L \\ \downarrow \tau & & \downarrow \\ \tau(M) & \longrightarrow & L \end{array}$$

with  $\sigma$  yet to be formed.

By Theorem 9.6,  $\exists$  isomorphism  $\sigma : L \longrightarrow L$  such that  $\sigma \Big|_M = \tau$ .

Therefore,  $\sigma$  is an automorphism of  $L$ , and since  $\sigma \Big|_K = \tau \Big|_K = id$ ,  $\sigma$  is a  $K$ -automorphism of  $L$ .  $\square$

**Proposition 11.4.**  $L : K$  finite normal,  $\alpha, \beta$  are zeros in  $L$  of the irreducible polynomial  $p \in K[t]$ .

Then,  $\exists$  a  $K$ -automorphism  $\sigma$  of  $L$  such that  $\sigma(\alpha) = \beta$ .

*Proof.* By Corollary 5.13,  $\exists$  isomorphism  $\tau : K(\alpha) \longrightarrow K(\beta)$  such that  $\tau \Big|_K$  is the identity, and  $\tau(\alpha) = \beta$ .

By Theorem 11.3,  $\tau$  extends to a  $K$ -automorphism  $\sigma$  of  $L$ .  $\square$

**Lemma 11.8.**  $K \subseteq L \subseteq N \subseteq M$ ,  $L : K$  finite,  $N$  normal closure of  $L : K$ .

Let  $\tau$  any  $K$ -monomorphism  $\tau : L \longrightarrow M$ .

Then  $\tau(L) \subseteq N$ .

*Proof.*  $\alpha \in L$ ,  $m$  minimal polynomial of  $\alpha$  over  $K$ .

$\implies m(\alpha) = 0$ , so  $\tau(m(\alpha)) = 0$

(since  $\tau$  is a  $K$ -automorphism, ie. maps the zeros of  $m(t)$ ).

Since  $\tau$  is a  $K$ -monomorphism,  $\tau(m(\alpha)) = m(\tau(\alpha)) = 0$

$\implies \tau(\alpha)$  is a zero of  $m$ .

Therefore,  $\tau(\alpha)$  lies in  $N$ , since  $N : K$  is normal.  
Henceforth,  $\tau(L) \subseteq N$ . □

**Theorem 11.9.** The following are equivalent:

1.  $L : K$  normal
2.  $\exists$  finite normal extension  $N$  of  $K$  containing  $L$ ,  
such that every  $K$ -monomorphism  $\tau : L \rightarrow N$  is a  $K$ -automorphism of  $L$ .
3. for every finite extension  $M$  of  $K$  containing  $L$ ,  
every  $K$ -monomorphism  $\tau : L \rightarrow M$  is a  $K$ -automorphism of  $L$ .

**Theorem 11.10.**  $[L : N] = 1$ ,  $N$  normal closure of  $L : K$ . Then,  
 $\exists n$   $K$ -monomorphisms  $L \rightarrow N$ .  
(the ones proven by Lemma 11.8).

**Corollary 11.11.**  $|\Gamma(L : K)| = [L : K]$  (if  $L : K$  is normal).  
ie. there are precisely  $[L : K]$  distinct  $K$ -automorphisms of  $L$ .

**Theorem 11.12.**  $\Gamma(L : K) = G$ . If  $L : K$  normal, then  $K$  is the fixed field of  $G$ .

*Proof.* let  $K_0$  be the fixed field of  $G$ . Let  $[L : K] = n$ .  
By 11.11,  $|G| = [L : K] = n$ .  
By 10.5,  $[L : K_0] = n$  ( $K_0$  fixed field).  
Since  $K \subseteq K_0$ , we must have  $K = K_0$ .  
 $\implies$  thus  $K$  is the fixed field of  $G$ . □

**Theorem 11.14.** if  $L$  any field,  $G$  any finite group of automorphisms of  $L$ , and  $K$  its fixed field,  
then  $L : K$  is *finite* and *normal*, with Galois group  $G$ .

**Theorem 12.2.** (Fundamental Theorem of Galois Theory) if  $L : K$  finite and normal inside  $\mathbb{C}$ , with  $\Gamma(L : K) = G$ , then:

1.  $|\Gamma(L : K)| = [L : K]$  (by Corollary 11.11)
2. the maps  $*$  and  $\dagger$  are mutual inverses, and setup an order-reversing one-to-one correspondence between  $\mathcal{F}$  and  $\mathcal{G}$ .
3. if  $M$  an intermediate field, then

$$[L : M] = |M^*| \quad [M : K] = \frac{|G|}{|M^*|}$$

4. for  $M$  an intermediate field,  $M : K$  normal iff

$$\underbrace{\Gamma(M : K)}_{=M^*} \triangleleft \underbrace{\Gamma(L : K)}_{=G}$$

5. for  $M$  intermediate, if  $M : K$  normal, then

$$\Gamma(M : K) \cong \frac{G}{M^*}$$

ie.

$$\Gamma(M : K) \cong \frac{\Gamma(L : K)}{\Gamma(L : M)}$$

*Proof.* TODO □

## 1.2 Chapter 13 - Full example

(Chapter 13 is basically a full example. More examples can be found at section 3.1)

## 1.3 Detour: Isomorphism Theorems

**Theorem i.1.** (*First Isomorphism Theorem*)

If  $\psi : G \rightarrow H$  a group homomorphism, then  $\ker(\psi) \triangleleft G$ .  
 Let  $\phi : G \rightarrow G/\ker(\psi)$  be the canonical homomorphism.  
 Then  $\exists$  unique isomorphism  $\eta : G/\ker(\psi) \rightarrow \psi(G)$  such  
 that  $\psi = \eta\phi$ .  
 $\iff$  ie.  $G/\ker(\psi) \cong \psi(G)$ .

$$\begin{array}{ccc} G & \xrightarrow{\psi} & H \\ \phi \searrow & & \nearrow \eta \\ & G/\ker(\psi) & \end{array}$$

*Proof.* (proof from Thomas W. Judson book "Abstract Algebra" [5])

Let  $K = \ker(\psi)$ . Since

$$\eta : G/K \rightarrow \psi(G)$$

let

$$\eta : gK \rightarrow \psi(g)$$

ie.  $\eta(gK) = \psi(g)$ .

i. show that  $\eta$  is a *well defined* map:

if we have two representatives of the same coset, ie.  $g_1K = g_2K$ , we want to show that  $\eta(g_1K) = \eta(g_2K)$ , so that  $\eta$  is a well-defined map.

By the coset properties for some  $k \in K$ ,  $g_1 = g_2k$ , so

$$\eta(g_1K) = \psi(g_1) = \psi(g_2k) = \eta(g_2kK) = \eta(g_2K)$$

Thus,  $\eta$  does not depend on the choice of coset representatives, and the map  $\eta : G/\ker(\psi) \rightarrow \psi(G)$  is uniquely defined since  $\psi = \eta\phi$ .

ii. show that  $\eta$  is a homomorphism:

Observe:

$$\eta(g_1Kg_2K) = \eta(g_1g_2K) = \psi(g_1g_2) = \psi(g_1)\psi(g_2) = \eta(g_1K)\eta(g_2K)$$

$\implies$  so  $\eta$  is a homomorphism.

iii. show that  $\eta$  is an isomorphism:

Since each element of  $H = \psi(G)$  has at least a preimage, then  $\eta$  is *surjective* (onto  $\psi(G)$ ).

Show that it is also *injective* (one-to-one):

Suppose 2 different preimages lead to the same image in  $\psi(G)$ , ie.  $\eta(g_1K) = \eta(g_2K)$

then,

$$\psi(g_1) = \psi(g_2)$$

which implies  $\psi(g_1^{-1}g_2) = e$ , ie.  $g_1^{-1}g_2 \in \ker(\psi)$ , hence

$$g_1^{-1}g_2K = K$$

$$g_1K = g_2K$$

so  $\eta$  is injective.

Since  $\eta$  is injective and surjective  $\implies \eta$  is a bijective homomorphism,  
ie.  $\eta$  is an *isomorphism*.  $\square$

**Theorem i.2.** (*Second Isomorphism Theorem*) Let  $H \subseteq G$ ,  $N \triangleleft G$ . Then

i.  $HN \subseteq G$

ii.  $H \cap N \triangleleft H$

iii.  $\frac{H}{H \cap N} \cong \frac{HN}{N}$

*Proof.* (proof from Thomas W. Judson book "Abstract Algebra" [5])

i. show  $HN \subseteq G$ :

Note that  $HN = \{hn : h \in H, n \in N\}$ . Let  $h_1n_1, h_2n_2 \in HN$ .

Since  $N$  normal  $\implies h_2^{-1}n_1h_2 \in N$ , so

$$(h_1n_1)(h_2n_2) = h_1h_2(h_2^{-1}n_1h_2) \cdot n_2 \in HN$$

[Recall: since  $N \triangleleft G$ ,  $gN = Ng \forall g \in G \implies gn = n'g$  for some  $n' \in N$ .]

To see that  $(hn)^{-1} \in HN$ :

since  $(hn)^{-1} = h^{-1}n^{-1} = h^{-1}(hn^{-1}h^{-1})$ , thus  $(hn)^{-1} \in HN$ .

Thus  $HN \subseteq G$ .

In fact,

$$HN = \bigcup_{h \in H} hN$$

(TODO: diagram)

- ii. show that  $H \cap N \triangleleft H$ :

Let  $h \in H$ ,  $n \in H \cap N$  (recall:  $H \cap N \subseteq H$ ).

Then  $h^{-1}nh \in H \longleftarrow$  since  $h^{-1}, n, h \in H$ .

Since  $N \triangleleft G$ ,  $h^{-1}nh \in N$ .

Therefore,  $h^{-1}nh \in H \cap N \implies H \cap N \triangleleft H$

- iii. show that  $\frac{H}{H \cap N} \cong \frac{HN}{N}$ :

Define a map

$$\begin{aligned} \phi : H &\longrightarrow \frac{HN}{N} \\ \text{by } \phi : h &\longmapsto hN \end{aligned}$$

$\phi$  is surjective (onto), since any coset  $hnN = hN$  is the image of  $h \in H$ , ie.  
 $\phi(h)$

$\phi$  is a homomorphism, since

$$\phi(hh') = hh'N = hNh'N = \phi(h)\phi(h')$$

By the First Isomorphism Theorem i.1,

$$\frac{HN}{N} \cong \frac{H}{\ker(\phi)}$$

and since

$$\begin{aligned} \ker(\phi) &= \{h \in H : h \in N\} \\ \text{then } \ker(\phi) &= H \cap N \end{aligned}$$

so then,

$$\frac{HN}{N} = \phi(H) \cong \frac{H}{\ker(\phi)} = \frac{H}{H \cap N}$$

thus

$$\frac{HN}{N} \cong \frac{H}{H \cap N}$$

□

**Theorem i.3.** (*Third Isomorphism Theorem*)

Let  $H \subseteq K$  and  $K \triangleleft G$ ,  $H \triangleleft G$ .

Then  $\frac{K}{H} \triangleleft \frac{G}{H}$  and

$$\frac{G/H}{K/H} \cong \frac{G}{K}$$

*Proof.* (proof from Dummit and Foote book “Abstract Algebra” [6])

Easy to see that  $\frac{K}{H} \triangleleft \frac{G}{H}$ .  
Define

$$\begin{aligned}\psi : \frac{G}{H} &\longrightarrow \frac{G}{K} \\ \text{by } \psi : gH &\longmapsto gK\end{aligned}$$

To show that  $\psi$  is *well defined*:

suppose  $g_1H = g_2H$ , then  $g_1 = g_2h$  for some  $h \in H$ .

Since  $H \subseteq K \implies h \in K$ , hence  $g_1K = g_2K$ ,

i.e.  $\psi(g_1H) = \psi(g_2H)$ , which shows that  $\psi$  is well defined.

Since  $g \in G$  may be chosen arbitrarily in  $G$ ,  $\psi$  is a surjective homomorphism.

Finally,

$$\begin{aligned}\ker(\psi) &= \{gH \in \frac{G}{H} \mid \psi(gH) = 1K\} \\ &= \{gH \in \frac{G}{H} \mid gK = 1K\} \\ &= \{gH \in \frac{G}{H} \mid g \in K\} \\ &= \frac{K}{H}\end{aligned}$$

By the First Isomorphism Theorem (i.1),

$$\begin{array}{ccc}\frac{G}{H} & \xrightarrow{\psi} & \frac{G}{K} \\ \phi \searrow & & \nearrow \eta \\ & \frac{G/H}{\ker(\psi)} = \frac{G/H}{K/H} & \end{array}$$

So, by

$$\eta : \frac{G/H}{K/H} \longrightarrow \frac{G}{K}$$

since  $\eta$  is bijective (we know it by the First Isomorphism Theorem),  $\eta$  is the isomorphism:

$$\frac{G/H}{K/H} \cong \frac{G}{K}$$

□

## 1.4 Chapter 14

**Definition 14.1.** a group  $G$  is soluble if it has a finite series of subgroups

$$1 = G_0 \subseteq G_1 \subseteq \dots \subseteq G_n = G$$

such that

- i.  $G_i \triangleleft G_{i+1}$  for  $i = 0, \dots, n - 1$
- ii.  $\frac{G_{i+1}}{G_i}$  is Abelian for  $i = 0, \dots, n - 1$

(Note:  $G_i \triangleleft G_{i+1} \triangleleft G_{i+2}$  does not imply  $G_i \triangleleft G_{i+2}$ )

**Theorem 14.4.**  $H \subseteq G$ ,  $N \triangleleft G$ , then

1. if  $G$  soluble  $\implies H$  soluble
2. if  $G$  soluble  $\implies G/N$  soluble
3. if  $N$  and  $G/N$  soluble  $\implies G$  soluble

*Proof.* 1. Since  $G$  soluble, by definition:  $\exists 1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_r = G$  with Abelian quotients  $\frac{G_{i+1}}{G_i}$ .

Let  $H_i = G_i \cap H$ , then  $H$  has a series  $1 = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_r = H$ , next we show that the quotients  $\frac{H_{i+1}}{H_i}$  are Abelian (so that  $H$  is soluble):

$$\frac{H_{i+1}}{H_i} = \frac{G_{i+1} \cap H}{G_i \cap H} \stackrel{(*)}{=} \frac{G_{i+1} \cap H}{G_i \cap (G_{i+1} \cap H)} \stackrel{(**)}{\cong} \frac{G_i(G_{i+1} \cap H)}{G_i} \subseteq \frac{G_{i+1}}{G_i}$$

(\*): to see why,  $H_i = G_i \cap H = G_i \cap H_i = G_i \cap H_{i+1} = G_i \cap (G_{i+1} \cap H)$ .

(\*\*): by the 2nd Isomorphism Theorem (i.3).

[TODO: diagram of subgroups]

Notice that  $\frac{G_{i+1}}{G_i}$  is Abelian, thus the left-hand-side of the congruence is also Abelian. Therefore,  $\frac{H_{i+1}}{H_i}$  is Abelian, thus  $H$  is soluble.

2. For  $G/N$  to be soluble, (by definition) it would have the series  $\frac{N}{N} = G_0 \frac{N}{N} \triangleleft G_1 \frac{N}{N} \triangleleft \dots \triangleleft G_r \frac{N}{N} = \frac{G}{N}$ , and any quotient being  $\frac{G_{i+1}N}{G_iN}$ .

The series clearly exists, so now we show that the quotients are Abelian, so that  $G/N$  is soluble:

$$\frac{G_{i+1}N}{G_iN} = \frac{G_{i+1}(G_iN)}{G_iN} \stackrel{(*)}{\cong} \frac{G_{i+1}}{G_{i+1} \cap (G_iN)} \cong \frac{G_{i+1}/G_i}{(G_{i+1} \cap (G_iN))/G_i}$$

(\*): by the 2nd Isomorphism Theorem (i.3).

The last quotient is a quotient of the Abelian group  $G_{i+1}/G_i$ , so it is Abelian.

Hence,  $\frac{G_{i+1}N}{G_iN}$  is also Abelian; so  $\frac{G}{N}$  is soluble.

3. By the definition of  $N$  and  $G/N$  being soluble,

$$\begin{aligned} N\text{soluble} &\implies 1 = N_0 \triangleleft N_1 \triangleleft \dots \triangleleft N_r = N \\ G/N\text{soluble} &\implies 1 = \frac{N}{N} = \frac{G_0}{N} \triangleleft \frac{G_1}{N} \triangleleft \dots \triangleleft \frac{G_r}{N} = \frac{G}{N} \end{aligned}$$

both with Abelian quotients.

Consider the series of  $G$  given by combining the two previous series:

$$1 = N_0 \triangleleft N_1 \triangleleft \dots \triangleleft N_r = N = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_r = G$$

the quotients are either

- $\frac{N_{i+1}}{N_i}$ , Abelian
- $\frac{G_{i+1}}{G_i}$ , isomorphic to  $\frac{G_{i+1}/N}{G_i/N}$ , which is Abelian.

Therefore, the quotients are always Abelian; hence  $G$  is soluble.  $\square$

**Definition 14.5.**  $G$  is *simple* if it's nontrivial and it's only normal subgroups are 1 and  $G$ .

**Theorem 14.6.** A *soluble* group is *simple* iff it is cyclic of prime order.

**Theorem 14.7.** if  $n \geq 5$ , then  $\mathbb{A}_n$  is simple.

**Corollary 14.8.**  $\mathbb{S}_n$  is not soluble if  $n \geq 5$ .

*Proof.* if  $\mathbb{S}_n$  were soluble, then  $\mathbb{A}_n$  would be soluble by Theorem 14.1(i), and simple by Theorem 14.7, hence of prime order by Theorem 14.6.

But observe:  $|\mathbb{A}_n| = \frac{1}{2}(n!)$  is not prime if  $n \geq 5$ .  
Thus  $\mathbb{S}_n$  is not soluble if  $n \geq 5$ .  $\square$

**Lemma 14.14.** if  $A$  finite and abelian group with  $p \mid |A|$  ( $p$  prime), then  $A$  has an element of order  $p$ .

*Proof.* i. if  $|A|$  prime and Abelian  $\implies$  then  $A$  is cyclic.

Since  $p \mid |A| \implies \exists! B \subseteq A$  such that  $|B| = p$ , where  $B = \langle b \rangle$  with  $\text{ord}(b) = p$ . So the lemma is proven.

ii. if  $|A|$  non-prime:

take  $M \subseteq A$  with  $|M| = m$ ,  $m$  maximal. Then

a. if  $p \mid m \implies \exists! B' = \langle b' \rangle$ ,  $b' \in A$  with  $|B'| = p$  and  $\text{ord}(b') = p$ .

b. if  $p \nmid m$ : Let  $b \in A \setminus M$  and  $B = \langle b \rangle$ .

Then  $MB \supseteq M$ , and by maximality must be  $MB = A$ .

By the 1st Isomorphism Theorem (i.1),

$$|A| = |MB| = \frac{|M| \cdot |B|}{|M \cap B|}$$

both  $|A|$  and  $|B|$  are divisible by  $p$  (but recall that  $p \nmid m = |M|$ ), since  $B$  is cyclic and  $p \mid |B|$

$\implies$  thus,  $B$  has an element of order  $p$ .

So, if  $|B| = r$ , and  $p \mid r \implies \text{ord}(b^{r/p}) = p$ .

Hence, in all cases i, ii.a, ii.b,  $A$  contains an element of order  $p$ .

□

**Theorem 14.15.** (Cauchy's Theorem) if  $p \mid |G|$  ( $p$  prime), then  $\exists x \in G$  such that  $\text{ord}(x) = p$ .

*Proof.* (induction on  $|G|$ )

For  $|G| = 1, 2, 3$ , trivial.

Induction step: class equation

$$|G| = 1 + |C_2| + \dots + |C_r|$$

since  $p \mid |G|$ , must have  $p \nmid |C_j|$  for some  $j \geq 2$ .

If  $x \in C_j \implies p \mid |C_G(x)|$  (since  $|C_j| = |G|/|C_G(x)|$ , recall  $p \mid |G|$ ).

i. if  $C_G(x) \neq G$ :

(by induction) since  $p \mid |C_G(x)|$ ,

$\exists a \in C_G(x)$  with  $\text{ord}(a) = p$ , and  $a \in G$  (since  $C_G(x) \subset G$ ).

ii. otherwise,  $C_G(x) = G$ :

implies  $x \in Z(G)$ , by choice  $x \neq 1$ , so  $Z(G) \neq 1$ .

Then either

I.  $p \mid |Z(G)| \implies$  Abelian case, Lemma 14.14.

II.  $p \nmid |Z(G)|$ : by induction,  $\exists x \in G$  such that  $\hat{x} \in G/Z(G)$ , with  $\text{ord}(\hat{x}) = p$ . (where  $\hat{x}$  is the image of  $x$ ).

$\implies x^p \in Z(G)$ , but  $x \notin Z(G)$ .

Let  $X = \langle x \rangle$ , cyclic.

$XZ(G)$  is Abelian, and  $p \mid |XZ(G)|$

$\implies$  by Lemma 14.14, it has an element of order  $p$ , and this element belongs to  $G$ .

□

**Definition 15.1.** (Soluble by radicals) let  $f \in K[t]$ ,  $K \subseteq \mathbb{C}$ , and  $\Sigma$  a splitting field of  $f$  over  $K$ .

$f$  is soluble by radicals if

$\exists$  a field  $M$  with  $\Sigma \subseteq M$  such that  $M : K$  is a radical extension (8.12).

Note: not required  $\Sigma : K$  to be radical.

**Lemma 15.3.**  $L : K$  radical extension  $\mathbb{C}$ , and  $M$  normal enclosure of  $L : K$ , then  $M : K$  is radical.

*Proof.* let  $L = K(\alpha_1, \dots, \alpha_r)$  with  $\alpha_i^{n_i} \in K(\alpha_1, \dots, \alpha_{j-1})$  (by definition of  $L : K$  being a radical extension).

Let  $f_i$  be the minimal polynomial of  $\alpha_i$  over  $K$ .

Then,  $M \supseteq L$  is splitting field of  $\prod_{i=1}^r f_i$ , since  $M$  is normal enclosure of  $L : K$ .

For every zero  $\beta_{ij}$  of  $f_i$  in  $M$ ,

$\exists$  an isomorphism  $\sigma : K(\alpha_i) \rightarrow K(\beta_{ij})$  by Corollary 5.13.

By Proposition 11.4, since  $K(\alpha_i), K(\beta_{ij}) \subset M$ ,  $\sigma$  extends to a  $K$ -automorphism

$$\tau : M \rightarrow M$$

since  $M$  is splitting field (ie. contains the zeros of  $f_i$ ).

Since  $\alpha$  is a member of radical sequence for a subfield of  $M$ , so it is  $\beta_{ij}$ .

By combining the sequences for  $M$ ,  $M : K$  is a radical extension.  $\square$

The next two lemmas show that certain Galois groups are Abelian.

**Lemma 15.4.**

*Proof.*

$\square$

## 2 Tools

This section contains tools that I found useful to solve Galois Theory related problems, and that don't appear in Stewart's book.

### 2.1 De Moivre's Theorem and Euler's formula

Useful for finding all the roots of a polynomial.

Euler's formula:

$$e^{i\psi} = \cos\psi + i \cdot \sin\psi$$

The n-th roots of a complex number  $z = x + iy = r(\cos\theta + i \cdot \sin\theta)$  are given by

$$z_k = \sqrt[n]{r} \cdot \left( \cos\left(\frac{\theta + 2k\pi}{n}\right) + i \cdot \sin\left(\frac{\theta + 2k\pi}{n}\right) \right)$$

for  $k = 0, \dots, n - 1$ .

So, by Euler's formula:

$$z_k = \sqrt[n]{r} \cdot e^{i(\frac{\theta+2k\pi}{n})}$$

Usually we will set  $\alpha = \sqrt[n]{r}$  and  $\zeta = e^{\frac{2\pi i}{n}}$ , and find the  $\mathbb{Q}$ -automorphisms from there (see 3.1 for examples).

### 2.2 Eisenstein's Criterion

*reference: Stewart's book*

Let  $f(t) = a_0 + a_1t + \dots + a_nt^n$ , suppose there is a prime  $q$  such that

1.  $q \nmid a_n$
2.  $q|a_i$  for  $i = 0, \dots, n - 1$
3.  $q^2 \nmid a_0$

Then,  $f$  is irreducible over  $\mathbb{Q}$ .

*TODO proof & Gauss lemma.*

### 2.3 Elementary symmetric polynomials

*TODO from orange notebook, page 36*

### 2.4 Cyclotomic polynomials

#### 2.4.1 From Elmyn Berlekamp's "Algebraic Coding Theory" book

The notes in this section are from the book "Algebraic Coding Theory" by Elmyn Berlekamp [3].

Some times we might find polynomials that have the shape of  $t^n - 1$ , those are *cyclotomic polynomials*, and have some properties that might be useful.

Observe that in a finite field of order  $q$ , factoring  $x^q - x$  gives

$$x^q - x = x(x^{q-1} - 1)$$

The factor  $x^{q-1} - 1$  is a special case of  $x^n - 1$ : if we assume that the field contains an element  $\alpha$  of order  $n$ , then the roots of  $x^n - 1 = 0$  are

$$1, \alpha, \alpha^2, \alpha^3, \dots, \alpha^{n-1}$$

and  $\deg(x^n - 1) = n$ , thus  $x^n - 1$  has at most  $n$  roots in any field, henceforth the powers of  $\alpha$  must include all the  $n$ -th roots of unity.

There fore, in any field which contains a primitive  $n$ -th root of unity we have:

**Theorem 4.31.**

$$x^n - 1 = \prod_{i=0}^{n-1} (x - \alpha^i) = \prod_{i=1}^n (x - \alpha^i)$$

If  $n = k \cdot d$ , then  $\alpha^k, \alpha^{2k}, \alpha^{3k}, \dots, \alpha^{dk}$  are all roots of  $x^d - 1 = 0$

Every element with order dividing  $n$ , must be a power of  $\alpha$ , since an element of order  $d$  is a  $d$ -th root of unity.

Every power of  $\alpha$  has order which divides  $n$ , and every field element whose order divides  $n$  is a power of  $\alpha$ . This suggests that we partition the powers of  $\alpha$  according to their orders:

$$x^n - 1 = \prod_{\substack{d \\ d|n}} \prod_{\beta} (x - \beta)$$

where at each iteration,  $\beta$  is a field element of order  $d$  for each  $d$ .

The polynomial whose roots are the field elements of order  $d$  is called the *cyclotomic polynomial*, denoted by  $Q^{(d)}(x)$ .

**Theorem 4.32.**

$$x^n - 1 = \prod_{\substack{d \\ d|n}} Q^{(d)}(x)$$

#### 2.4.2 From Ian Stewart's "Galois Theory" book

Notes from Ian Stewart's book [1].

Consider the case  $n = 12$ , let  $\zeta = e^{\pi i/6}$  be a primitive 12-th root of unity. Classify its powers  $(\zeta^j)$  according to their minimal power  $d$  such that  $(\zeta^j)^d = 1$  (ie. when they are primitive  $d$ -th roots of unity).

$$d = 1, \quad 1$$

$$d = 2, \quad \zeta^6$$

$$d = 3, \quad \zeta^4, \zeta^8$$

$$d = 4, \quad \zeta^3, \zeta^9$$

$$d = 6, \quad \zeta^2, \zeta^{10}$$

$$d = 12, \quad \zeta, \zeta^5, \zeta^7, \zeta^{11}$$

Observe that we can factorize  $t^{12} - 1$  by grouping the corresponding zeros:

$$\begin{aligned} t^{12} - 1 &= (t - 1) \times \\ &\quad (t - \zeta^6) \times \\ &\quad (t - \zeta^4)(t - \zeta^8) \times \\ &\quad (t - \zeta^3)(t - \zeta^9) \times \\ &\quad (t - \zeta^2)(t - \zeta^{10}) \times \\ &\quad (t - \zeta)(t - \zeta^5)(t - \zeta^7)(t - \zeta^{11}) \end{aligned}$$

which simplifies to

$$t^{12} - 1 = (t - 1)(t + 1)(t^2 + t + 1)(t^2 + 1)(t^2 - t + 1)F(t)$$

where  $F(t) = (t - \zeta)(t - \zeta^5)(t - \zeta^7)(t - \zeta^{11}) = t^4 - t^2 + 1$  (this last step can be obtained either by multiplying  $(t - \zeta)(t - \zeta^5)(t - \zeta^7)(t - \zeta^{11})$  together, or by dividing  $t^{12} - 1$  by all the other factors).

Let  $\Phi_d(t)$  be the factor corresponding to primitive  $d$ -th roots of unity, then we have proved that

$$t^{12} - 1 = \Phi_1 \Phi_2 \Phi_3 \Phi_4 \Phi_6 \Phi_{12}$$

**Definition 21.5.** The polynomial  $\Phi_d(t)$  defined by

$$\Phi_n(t) = \prod_{a \in \mathbb{Z}_n, (a, n) = 1} (t - \zeta^a)$$

is the  $n$ -th *cyclotomic polynomial* over  $\mathbb{C}$ .

**Corollary 21.6.**  $\forall n \in \mathbb{N}$ , the polynomial  $\Phi_n(t)$  lies in  $\mathbb{Z}[t]$  and is monic and irreducible.

**Theorem 21.9.** 1. The Galois group  $\Gamma(\mathbb{Q}(\zeta) : \mathbb{Q})$  consists of the  $\mathbb{Q}$ -automorphisms

$\psi_j$  defined by

$$\psi_j(\zeta) = \zeta^j$$

where  $0 \leq j \leq n - 1$  and  $j$  is prime to  $n$ .

2.  $\Gamma(\mathbb{Q}(\zeta) : \mathbb{Q}) \xrightarrow{\text{iso}} \mathbb{Z}_n^*$ , and is an abelian group.

3. its order is  $\phi(n)$

4. if  $n$  is prime,  $\mathbb{Z}_n^*$  is cyclic

### 2.4.3 Examples

Examples of cyclotomic polynomials:

$$\begin{aligned}\Phi_n(x) &= x^{n-1} + x^{n-2} + \dots + x^2 + x + 1 = \sum_{i=0}^{n-1} x^i \\ \Phi_{2p}(x) &= x^{p-1} + \dots + x^2 - x + 1 = \sum_{i=0}^{p-1} (-x)^i \\ \Phi_m(x) &= x^{m/2} + 1, \quad \text{when } m \text{ is a power of 2}\end{aligned}$$

## 2.5 Lemma 1.42 from J.S.Milne's book

Lemma from J.S.Milne's book [2].

Useful for when dealing with  $x^p - 1$  with  $p$  prime.

Observe that

$$x^p - 1 = (x - 1)(x^{p-1} + x^{p-2} + \dots + 1)$$

Notice that

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \dots + 1$$

is the  $p$ -th Cyclotomic polynomial.

**Lemma 1.42.** If  $p$  prime, then  $x^{p-1} + \dots + 1$  is irreducible; hence  $\mathbb{Q}[e^{2\pi i/p}]$  has degree  $p - 1$  over  $\mathbb{Q}$ .

*Proof.* Let  $f(x) = (x^p - 1)/(x - 1) = x^{p-1} + \dots + 1$  then

$$f(x+1) = \frac{(x+1)^p - 1}{x+1-1} = \frac{(x+1)^p - 1}{x} = x^{p-1} + \dots + a_i x^i + \dots + p$$

$$\text{with } a_i = \binom{p}{i+1}.$$

We know that  $p|a_i$  for  $i = 1, \dots, p-2$ , therefore  $f(x+1)$  is irreducible by Eisenstein's Criterion.

This implies that  $f(x)$  is irreducible.  $\square$

## 2.6 Dihedral groups - Groups of symmetries

Source: Wikipedia and [4].

Dihedral groups ( $\mathbb{D}_n$ ) represent the symmetries of a regular  $n$ -gon.

Properties:

- are non-abelian (for  $n > 2$ ), ie.  $rs \neq sr$
- order  $2n$
- generated by a rotation  $r$  and a reflection  $s$

- $r^n = s^2 = id$ ,  $(rs)^2 = id$

Subgroups of  $\mathbb{D}_n$ :

- rotations form a cyclic subgroup of order  $n$ , denoted as  $\langle r \rangle$
- for each  $d$  such that  $d|n$ ,  $\exists \mathbb{D}_d$  with order  $2d$
- normal subgroups
  - for  $n$  odd:  $\mathbb{D}_n$  and  $\langle r^d \rangle$  for every  $d|n$
  - for  $n$  even: 2 additional normal subgroups
- Klein four-groups:  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , of order 4

Total number of subgroups in  $\mathbb{D}_n$ :  $d(n) + s(n)$ , where  $d(n)$  is the number of positive divisors of  $n$ , and  $s(n)$  is the sum of those divisors.

**Example .** For  $\mathbb{D}_6$ , we have  $\{1, 2, 3, 6\}|6$ , so  $d(6) = d(6) = 4$ , and  $s(6) = 1+2+3+6 = 12$ ; henceforth, the total amount of subgroups is  $d(n)+s(n) = 4+12 = 16$ .

For  $n \geq 3$ ,  $\mathbb{D}_n \subseteq \mathbb{S}_n$  (subgroup of the Symmetry group).

## 2.7 Rolle's theorem

**Theorem .** (Rolle's Theorem) if a real-valued function  $f$  is

- *continuous* on a proper closed interval  $[a, b]$
- *differentiable* on the open interval  $(a, b)$
- $f(a) = f(b)$

then,  $\exists$  at least one  $c$  in  $(a, b)$  such that  $Df(c) = 0$ .

*Proof.* (proof source: cue math website) Notice that when  $Df(x_i) = 0$  occurs, is a maximum or minimum (extrema) value of  $f$ .  $\implies$  if a function is continuous, it is guaranteed to have both a maximum and a minimum point in the interval.

Two possibilities:

- i.  $f$  is constant  
 $\implies$  ie. a horizontal line ( $f(a) = f(b)$ ), ie. no slope  $\implies Df = 0$  everywhere in  $[a, b]$ .
- ii.  $f$  is not constant:  
since  $f$  not constant, must change directions in order to start and end at the same  $y$ -value ( $f(a) = f(b)$ ).  
Thus at some point between  $a$  and  $b$  it will either have a minimum, maximum or both.

a. does the maximum occur at a point where  $Df > 0$ ?

No, because if  $Df > 0$ , then  $f$  is increasing, but it can not increase since we're at its maximum point.

b. does the maximum occur at a point where  $Df < 0$ ?

No, because if  $Df < 0$ , then  $f$  is decreasing, which means that at our left it was larger, but we're at a maximum point, so a contradiction.

Same with minimus.

$\Rightarrow$  Hence, since  $Df \not< 0$  and  $Df \not> 0$ , and  $Df$  exists, then  $Df = 0$ .

Thus,  $f$  must have extrema (either max or min or both), and at that extrema  $Df$  must be zero.  $\square$

Consequence of Rolle's Theorem:

**Corollary .** (Zero separation property) Between any two distinct consecutive zeros of  $f$ , there lies at least one zero of  $Df$ .

**Example .** If  $f(t)$  has zeros  $t_1, t_2$ , with  $t_1 < t_2$ , and  $f$  is derivable, then by Rolle's theorem:

$\exists c \in (t_1, t_2)$  such that  $Df(c) = 0$ .

Hence, the zeros of  $Df$  separate the zeros of  $f$ .

### 3 Exercises

#### 3.1 Galois groups

##### 3.1.1 $t^6 - 7 \in \mathbb{Q}$

This exercise comes from a combination of exercises 12.4 and 13.7 from [1].

First let's find the roots. By De Moivre's Theorem (2.1),  $t_k = \sqrt[6]{7} \cdot e^{i\frac{2\pi k}{6}}$ .

From which we denote  $\alpha = \sqrt[6]{7}$ , and  $\zeta = e^{\frac{2\pi i}{6}}$ , so that the roots of the polynomial are  $\{\alpha, \alpha\zeta, \alpha\zeta^2, \alpha\zeta^3, \alpha\zeta^4, \alpha\zeta^5\}$ , ie.  $\{\alpha\zeta^k\}_0^5$ .

Hence the *splitting field* is  $\mathbb{Q}(\alpha, \zeta)$ .

*Degree of the extension*

In order to find  $[\mathbb{Q}(\alpha, \zeta) : \mathbb{Q}]$ , we're going to split it in tow parts. By the Tower Law (6.6),

$$[\mathbb{Q}(\alpha, \zeta) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \zeta) : \mathbb{Q}(\alpha)] \cdot [\mathbb{Q}(\alpha) : \mathbb{Q}]$$

To find each degree, we will find the minimal polynomial of the adjoined term over the base field of the extension:

- i. minimal polynomial of  $\alpha$  over  $\mathbb{Q}$

By Eisenstein's Criterion (2.2), with  $q = 7$  we have that  $q \nmid 1, 7| -7, 0, 0, \dots$ , and  $7^2 \nmid -7$ , hence  $f(t)$  is irreducible over  $\mathbb{Q}$ , thus is the minimal polynomial

$$m_i(t) = f(t) = t^6 - 7$$

which has roots  $\{\alpha\zeta^k\}_0^5$ .

- ii. minimal polynomial of  $\zeta$  over  $\mathbb{Q}(\alpha)$

Since  $\zeta$  is the primitive 6th root of unity, we know that the minimal polynomial will be the 6th cyclotomic polynomial (2.4):

$$m_{ii}(t) = \Phi_6(t) = t^2 - t + 1$$

which has roots  $\zeta, -\zeta$ .

Since  $\mathbb{Q}(\alpha) \subseteq \mathbb{R}$ , and the roots of  $\Phi_6(t) = t^2 - t + 1$  are in  $\mathbb{C}$ ,  $\Phi_6(t)$  remains irreducible over  $\mathbb{Q}(\alpha)$ .

Therefore, by the tower of law,

$$[\mathbb{Q}(\alpha, \zeta) : \mathbb{Q}] = \deg \Phi_6(t) \cdot \deg f(t) = 2 \cdot 6 = 12$$

and by the Fundamental Theorem of Galois Theory, we know that

$$|\Gamma(\mathbb{Q}(\alpha, \zeta) : \mathbb{Q})| = [\mathbb{Q}(\alpha, \zeta) : \mathbb{Q}] = 12$$

which tells us that there exist 12  $\mathbb{Q}$ -automorphisms of the Galois group.

Let's find the 12  $\mathbb{Q}$ -automorphisms. Start by defining  $\sigma$  which fixes  $\zeta$  and acts on  $\alpha$ , sending it to another of the roots of the minimal polynomial of  $\alpha$  over  $\mathbb{Q}$ ,  $f(t)$ , choose  $\alpha\zeta$ .

Now define  $\tau$  which fixes  $\alpha$  and acts on  $\zeta$ , sending it into another root of the minimal polynomial of  $\zeta$  over  $\mathbb{Q}(\alpha)$ , choose  $-\zeta$ .

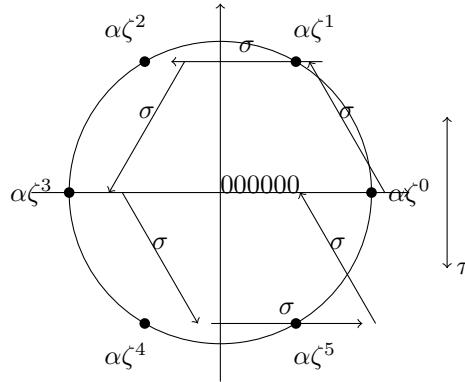
$$\sigma : \alpha \mapsto \alpha\zeta \quad \tau : \alpha \mapsto \alpha$$

$$\zeta \mapsto \zeta \quad \zeta \mapsto -\zeta = \zeta^{-1}$$

In other words, we have 12  $\mathbb{Q}$ -automorphisms, which are the combination of  $\sigma$  and  $\tau$ :

$$\begin{aligned}\sigma^k \tau^j : \alpha &\mapsto \alpha\zeta^k \\ \zeta &\mapsto \zeta^j\end{aligned}$$

for  $0 \leq k \leq 5$  and  $j = \pm 1$ .



NOTE: WIP diagram.

Observe, that  $\Gamma$  is generated by the combination of  $\sigma$  and  $\tau$ , and it is isomorphic to the group of symmetries of order 12, the dihedral group (2.6) of order 12,  $\mathbb{D}_6$ , ie.  $\Gamma \cong \mathbb{D}_6$ .

Let's find the subgroups of  $\Gamma$ , and the fixed fields of  $\mathbb{Q}(\alpha, \zeta)$ .

We know that  $\Gamma \cong \mathbb{D}_6$ , and we know from the properties of the dihedral group (2.6) that the number of subgroups of  $\mathbb{D}_6$  will be  $d(6) + s(6) = 4 + 12 = 16$  subgroups.

generators	order	group	fixed field	notes (check fixed field)
$\langle \rangle = \langle \sigma^6 \rangle = \langle \tau^2 \rangle$	1	id	$\mathbb{Q}(\alpha, \zeta)$	
$\langle \sigma \rangle = \langle \sigma^5 \rangle$	6	$\mathbb{Z}_6$	$\mathbb{Q}(\zeta)$	
$\langle \sigma^2 \rangle = \langle \sigma^4 \rangle$	3	$\mathbb{Z}_3$	$\mathbb{Q}(\alpha^3, \zeta)$	$\sigma^2(\alpha^3) = \alpha^3\zeta^{3 \cdot 2} = \alpha^3\zeta^6 = \alpha^3 \cdot 1 = \alpha^3$
$\langle \sigma^3 \rangle$	2	$\mathbb{Z}_2$	$\mathbb{Q}(\alpha^2, \zeta)$	$\sigma^3(\alpha^2) = (\alpha\zeta^3)^2 = \alpha^2\zeta^6 = \alpha^2$
$\langle \tau \rangle$	2	$\mathbb{Z}_2$	$\mathbb{Q}(\alpha)$	
$\langle \sigma\tau \rangle$	2	$\mathbb{Z}_2$	$\mathbb{Q}(\alpha + \alpha\zeta)$	$\sigma\zeta(\alpha + \alpha\zeta) = \sigma(\alpha + \alpha\zeta^{-1}) = \alpha\zeta + \alpha\zeta^{-1}\zeta = \alpha\zeta + \alpha$
$\langle \sigma^2\tau \rangle$	2	$\mathbb{Z}_2$	$\mathbb{Q}(\alpha + \alpha\zeta^2), \mathbb{Q}(\alpha\zeta)$	$\sigma^2\tau(\alpha + \alpha\zeta^2) = \sigma(\alpha + \alpha\zeta^{-2}) = \alpha\zeta^2 + \alpha\zeta^{-2}\zeta^2 = \alpha\zeta^2 + \alpha$
$\langle \sigma^3\tau \rangle$	2	$\mathbb{Z}_2$	$\mathbb{Q}(\alpha + \alpha\zeta^3)$	$\sigma^3\tau(\alpha + \alpha\zeta^3) = \sigma(\alpha + \alpha\zeta^{-3}) = \alpha\zeta^3 + \alpha\zeta^{-3}\zeta^3 = \alpha\zeta^3 + \alpha$
$\langle \sigma^4\tau \rangle$	2	$\mathbb{Z}_2$	$\mathbb{Q}(\alpha + \alpha\zeta^4), \mathbb{Q}(\alpha\zeta^2)$	$\sigma^4\tau(\alpha + \alpha\zeta^4) = \sigma(\alpha + \alpha\zeta^{-4}) = \alpha\zeta^4 + \alpha\zeta^{-4}\zeta^4 = \alpha\zeta^4 + \alpha$
$\langle \sigma^5\tau \rangle$	2	$\mathbb{Z}_2$	$\mathbb{Q}(\alpha + \alpha\zeta^5)$	$\sigma^5\tau(\alpha + \alpha\zeta^5) = \sigma(\alpha + \alpha\zeta^{-5}) = \alpha\zeta^5 + \alpha\zeta^{-5}\zeta^5 = \alpha\zeta^5 + \alpha$
$\langle \sigma, \tau \rangle = \langle \sigma^5, \tau \rangle$	$6 \cdot 2 = 12$	$\mathbb{D}_6$	$\mathbb{Q}$	
$\langle \sigma^2, \tau \rangle = \langle \sigma^4, \tau \rangle$	$3 \cdot 2 = 6$	$\mathbb{D}_3$	$\mathbb{Q}(\alpha^3)$	$\sigma^2(\alpha^3) = \alpha^3\zeta^{3 \cdot 2} = \alpha^3$ and $\tau(\alpha^3) = \alpha^3$
$\langle \sigma^3, \tau \rangle$	$2 \cdot 2 = 4$	$\mathbb{D}_2$	$\mathbb{Q}(\alpha^2)$	$\sigma^3(\alpha^2) = \alpha^2\zeta^{2 \cdot 2} = \alpha^2$ and $\tau(\alpha^2) = \alpha^2$
$\langle \sigma^2, \sigma\tau \rangle$	$3 \cdot 2 = 6$	$\mathbb{D}_3$	$\mathbb{Q}(\alpha^3 + \alpha^3\zeta^3)$	$\sigma^2(\alpha^3 + \alpha^3\zeta^3) = \alpha^3\zeta^3 + \alpha^3\zeta^3\zeta^3 = \alpha^3\zeta^3 + \alpha^3\zeta^6 = \alpha^3\zeta^3 + \alpha^3$
$\langle \sigma^3, \sigma\tau \rangle$	$2 \cdot 2 = 4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Q}(\alpha^2\zeta^2), \mathbb{Q}(\alpha^2 + \alpha^2\zeta^2)$	$\sigma^3(\alpha^2 + \alpha^2\zeta^2) = \alpha^2\zeta^{2 \cdot 3} + \alpha^2\zeta^{2 \cdot 3}\zeta^2 = \alpha^2 + \alpha^2\zeta^2$ and $\sigma\tau(\alpha^2 + \alpha^2\zeta^2) = \alpha^2\zeta^2 + \alpha^2\zeta^{-2}\zeta^2 = \alpha^2\zeta^2 + \alpha^2$
$\langle \sigma^3, \sigma^2\tau \rangle$	$2 \cdot 2 = 4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Q}(\alpha^2\zeta^4), \mathbb{Q}(\alpha^2 + \alpha^2\zeta^4)$	$\sigma^2\zeta(\alpha^2\zeta^4) = \alpha^2\zeta^2\zeta^{-4} = \alpha^2\zeta^{-2} = \alpha^2\zeta^4$ and $\sigma^3(\alpha^2\zeta^4) = \alpha^2\zeta^{2 \cdot 3}\zeta^4 = \alpha^2\zeta^4$

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