Online Appendix to "Endogenous Production Networks and the Business Cycle"

A Computational Algorithms

A.1 Exogenous network market equilibrium solution algorithm

Given the matching function m and the associated quantity of labor L_f used for relationship costs, the exogenous network market equilibrium specified in Definition 1 can be solved for using the following algorithm.

1. Make initial guesses $\hat{\Phi}$ and $\hat{\Delta}$ for the network productivity and quality functions, and iterate on equations (2.15) and (2.16) until convergence.

2. Solve for Δ_H using equation (2.28).

3. Compute the allocation $\left\{l\left(\chi\right), X\left(\chi\right), x\left(\chi, \chi'\right), x_H\left(\chi\right)\right\}_{\chi \in S_\chi}$ using (2.24), (2.25), (2.27), and (2.31) respectively.

Since the functional equations (2.15) and (2.16) constitute contraction mappings with Lipschitz constants $\left(\frac{\alpha}{\mu}\right)^{\sigma-1}$ and $\frac{\alpha^{\sigma-1}}{\mu^{\sigma}}$ respectively, the iteration procedure in step 1 of the algorithm is guaranteed to converge at those rates. In practice, numerical solution of the model requires discretization of the state space S_χ into a mesh grid, of say $N_{grid} \times N_{grid}$ points. One can then solve for the functions $\Phi\left(\cdot\right)$ and $\Delta\left(\cdot\right)$ in step 1 at each point in the mesh grid, and then use bilinear interpolation to obtain numerical approximations of these functions for any desired value of $\chi \in S_\chi$.

A.2 Endogenous network market equilibrium solution algorithm

Given the distribution of fundamental firm characteristics G_χ and relationship costs G_ξ at any date, the endogenous network market equilibrium specified in Definition 2 can be solved for using the following algorithm.

1. Make initial guesses $\hat{\Phi}$ and $\Delta_{\hat{H}}\Delta$ for the network productivity function and the network quality function scaled by the household demand shifter.

2. Compute the implied profit function $\tilde{\pi}$ from equation (3.6).

3. Compute the implied matching functions, $\tilde{m}$, from equation (3.7).

4. Compute the implied network productivity and quality functions, $\tilde{\Phi}$ and $\tilde{\Delta}$, from equations (2.15) and (2.16).

5. Compute the implied household demand shifter $\tilde{\Delta}_H$ from equations (2.28), (3.8), (3.9), and (3.10), and obtain the implied guess for the scaled network quality function, $\widetilde{\Delta_H\Delta} = \tilde{\Delta}_H\tilde{\Delta}$.

6. Compute the residual $\mathcal{R} \equiv \max\{\mathcal{R}_\Phi, \mathcal{R}_\Delta\}$, where:

$$\mathcal{R}_\Phi \equiv \max_{\chi\in S_\chi}\left|\hat{\Phi}(\chi) - \tilde{\Phi}(\chi)\right|$$
$$\mathcal{R}_\Delta \equiv \max_{\chi\in S_\chi}\left|\widehat{\Delta_H\Delta}(\chi) - \widetilde{\Delta_H\Delta}(\chi)\right|$$

   If $\mathcal{R} > \epsilon$ for some tolerance level ϵ, update the guesses for the network productivity and scaled quality functions according to $\hat{\Phi}' = \tilde{\Phi}$ and $\widehat{\Delta_H\Delta}' = \widetilde{\Delta_H\Delta}$, and repeat from step 1 until $\mathcal{R} \leq \epsilon$.

Note that once the matching function is endogeneized through equation (3.7), the functional equations (2.15) and (2.16) no longer satisfy Blackwell's sufficient conditions for a contraction mapping. Nonetheless, the iterations described above always converge in practice.

B Efficiency of the market equilibrium

B.1 Exogenous network market equilibrium efficiency

To characterize the efficiency of the static market equilibrium, I compare the resulting allocation with the allocation that would be chosen by a social planner whose goal is to maximize household welfare subject to the production technology and market clearing constraints. Given the matching function m, the social planner chooses the allocation $\mathcal{A} \equiv \left\{l(\chi), X(\chi), \{x(\chi,\chi')\}_{\chi'\in S_\chi}, x_H(\chi)\right\}_{\chi\in S_\chi}$ according to :

$$U=\max_{\mathcal{A}}\left[\int_{S_\chi}\left[\delta x_H(\chi)\right]^{\frac{\sigma-1}{\sigma}}dG_\chi(\chi)\right]^{\frac{\sigma}{\sigma-1}}$$

subject to the following constraints:

$$X(\chi)=\left[\left[\phi l(\chi)\right]^{\frac{\sigma-1}{\sigma}}+\int_{S_\chi}m\left(\chi,\chi^{'}\right)\left[\alpha x\left(\chi,\chi^{'}\right)\right]^{\frac{\sigma-1}{\sigma}}dG_\chi\left(\chi^{'}\right)\right]^{\frac{\sigma}{\sigma-1}} \tag{B.1}$$

$$X(\chi)=x_H(\chi)+\int_{S_\chi}m\left(\chi^{'},\chi\right)x\left(\chi^{'},\chi\right)dG_\chi(\chi') \tag{B.2}$$

$$\int_{S_\chi}l(\chi)\,dG_\chi(\chi)=L-L_f \tag{B.3}$$

where L_f is taken as given.

Denoting the Lagrange multipliers on constraints (B.2) and (B.3) by $\left(\frac{U}{\Delta_H}\right)^{\frac{1}{\sigma}}\eta(\chi)\,G_\chi(\chi)$ and $\left(\frac{U}{\Delta_H}\right)^{\frac{1}{\sigma}}$ respectively, the first-order conditions for the planner's problem can be expressed as:

$$x_H(\chi)=\Delta_H\delta^{\sigma-1}\eta(\chi)^{-\sigma} \tag{B.4}$$

$$l(\chi)=X(\chi)\,\eta(\chi)^{\sigma}\,\phi^{\sigma-1} \tag{B.5}$$

$$x\left(\chi,\chi^{'}\right)=X(\chi)\,\eta(\chi)^{\sigma}\,\alpha^{\sigma-1}\eta\left(\chi^{'}\right)^{-\sigma} \tag{B.6}$$

Substituting these equations into (B.1) and (B.2), one obtains:

$$\Phi(\chi)=\phi^{\sigma-1}+\alpha^{\sigma-1}\int_{S_\chi}m\left(\chi,\chi^{'}\right)\Phi\left(\chi^{'}\right)dG_\chi\left(\chi^{'}\right) \tag{B.7}$$

$$\Delta(\chi)=\delta^{\sigma-1}+\alpha^{\sigma-1}\int_{S_\chi}m\left(\chi^{'},\chi\right)\Delta\left(\chi^{'}\right)dG_\chi\left(\chi^{'}\right) \tag{B.8}$$

where $\Phi(\chi)\equiv\eta(\chi)^{1-\sigma}$and $\Delta(\chi)\equiv\frac{1}{\Delta_H}X(\chi)\,\eta(\chi)^{\sigma}$.

Note that equations (B.4)-(B.8) are identical to equations (2.2), (2.7), (2.8), (2.15), and (2.16) respectively only when $\mu=1$. This tells us that the static market equilibrium allocation is identical to the planner's allocation if and only if the markups charged by all firms are equal to one. With a finite elasticity of substitution σ, the static market equilibrium is therefore inefficient relative to the planner's allocation because of the monopoly markup distortion.

B.2 Endogenous network market equilibrium efficiency

To study the efficiency properties of the endogenous network market equilibrium, we consider the problem of a social planner that chooses the set of active relationships so as to maximize the value of household welfare in each period, subject to the same relationship costs faced by firms in the market equilibrium. To isolate the efficiency properties of this extensive margin decision from the intensive margin inefficiency characterized above in section B.1, we assume that the planner makes this choice taking into account that once the network is given, firms will behave as in the exogenous network market equilibrium while charging a constant markup μ. The case in which $\mu = 1$ then corresponds to the complete social planner's problem in which the planner chooses both the allocations of labor and output as well as the set of active relationships.

First, note that the planner's choice about which relationships to form is equivalent to a choice over the values$\{\xi_{max,t}(\chi,\chi')\}_{(\chi,\chi')\in S_\chi^2}$, where $\xi_{max,t}(\chi,\chi')$ specifies the maximum value of the idiosyncratic relationship cost shock component for which $\chi - \chi'$ firm pair relationships are accepted. Formally, the planner's problem can therefore be written as:

$$\max_{\{\xi_{max}(\chi,\chi')\}_{(\chi,\chi')\in S_\chi^2}} U \tag{B.9}$$

where the maximization is subject to $\xi_{max}(\chi,\chi') \geq 0$ for all $(\chi,\chi') \in S_\chi^2$, as well as the following constraints:

$$U = \mu^{-\sigma}(L - L_f)\frac{\left[\int_{S_\chi}\Phi(\chi)\delta^{\sigma-1}dG_\chi(\chi)\right]^{\frac{\sigma}{\sigma-1}}}{\int_{S_\chi}\Delta(\chi)\phi^{\sigma-1}dG_\chi(\chi)} \tag{B.10}$$

$$\Phi(\chi) = \phi^{\sigma-1} + \left(\frac{\alpha}{\mu}\right)^{\sigma-1}\int_{S_\chi} m(\chi,\chi')\Phi(\chi')dG_\chi(\chi') \tag{B.11}$$

$$\Delta(\chi) = \mu^{-\sigma}\delta^{\sigma-1} + \mu^{-\sigma}\alpha^{\sigma-1}\int_{S_\chi} m(\chi',\chi)\Delta(\chi')dG_\chi(\chi') \tag{B.12}$$

$$m(\chi,\chi') = G_\xi[\xi_{max}(\chi,\chi')] \tag{B.13}$$

$$L_f = \int\int_{S_\chi}\int_0^{\xi_{max}(\chi,\chi')}\xi dG_\xi(\xi)dG_\chi(\chi)dG_\chi(\chi') \tag{B.14}$$

For brevity, denote $\xi^*_{max} \equiv \xi_{max}(\chi^*,\chi^{*'})$ and $m^* \equiv m(\chi^*,\chi^{*'})$ for a given firm pair $(\chi^*,\chi^{*'})$. The first step in solving the planner's problem is to find an expression for the derivative of U with respect to ξ^*_{max}. First, we differentiate (B.14) with respect to ξ^*_{max} to get:

$$\frac{dL_f}{d\xi^*_{max}} = H^* \xi^*_{max} \tag{B.15}$$

where $H^* \equiv g_\chi(\chi^*) \, g_\chi\left(\chi^{*'}\right) g_\xi(\xi^*_{max})$ is the product of three probability densities. Next, differentiating (B.13) for $\left(\chi, \chi^{'}\right) = \left(\chi^*, \chi^{*'}\right)$ with respect to ξ^*_{max} gives:

$$\frac{dm^*}{d\xi^*_{max}} = g_\xi(\xi^*_{max}) \tag{B.16}$$

Differentiating the functional equation (B.11) with respect to ξ^*_{max}, we then obtain:

$$\frac{d\Phi(\chi)}{d\xi^*_{max}} = \frac{d\Phi(\chi)}{dm^*}\frac{dm^*}{d\xi^*} \tag{B.17}$$

$$= g_\xi(\xi^*_{max}) \left[\left(\frac{\alpha}{\mu}\right)^{\sigma-1} \Phi\left(\chi^{*'}\right) \mathbf{1}_{\chi^*}(\chi) + \left(\frac{\alpha}{\mu}\right)^{\sigma-1} \int_{S_\chi} m\left(\chi, \chi^{'}\right) \frac{d\Phi\left(\chi^{'}\right)}{d\xi^*_{max}} dG_\chi\left(\chi^{'}\right) \right] \tag{B.18}$$

$$= H^* \left[\sum_{d=0}^{\infty} \left(\frac{\alpha}{\mu}\right)^{d(\sigma-1)} m^{(d)}\left(\chi, \chi^*\right) \right] \left(\frac{\alpha}{\mu}\right)^{\sigma-1} \Phi\left(\chi^{*'}\right) \tag{B.19}$$

where $\mathbf{1}_{\chi^*}(\chi)$ is the indicator function that equals 1 if $\chi = \chi^*$ and 0 otherwise. Similarly, differentiating (B.12) with respect to ξ^*_{max}, we obtain:

$$\frac{d\Delta(\chi)}{d\xi^*_{max}} = H^* \left[\sum_{d=0}^{\infty} \mu^{-d} \left(\frac{\alpha}{\mu}\right)^{d(\sigma-1)} m^{(d)}\left(\chi, \chi^*\right) \right] \mu^{-\sigma}\alpha^{\sigma-1}\Delta(\chi^*) \tag{B.20}$$

Note that equations (B.19) and (B.20) summarize the effect of a change in the mass of connections between $\chi^* - \chi^{*'}$ firm pairs on the network productivities and demands of all firms that are downstream of χ^* firms and upstream of $\chi^{*'}$ firms respectively.

Now, differentiating the expression for household welfare (B.10) with respect to ξ^*_{max} and using (B.15), (B.19), and (B.20), we then get:

$$\frac{dU}{d\xi^*_{max}} = \frac{UH^*}{L - L_f}\left[\pi^{SP}\left(\chi^*, \chi^{*'}\right) - \xi^*_{max}\right] \tag{B.21}$$

where π^{SP} is the planner's analog of the profit function:

$$\pi^{SP}\left(\chi^*, \chi^{*'}\right) \equiv f^{SP}(\mu)\,\alpha^{\sigma-1}\Delta_H \Delta(\chi^*)\,\Phi\left(\chi^{*'}\right) \tag{B.22}$$

and $f^{SP}(\mu)$ is as defined in Proposition 4. Note that in deriving (B.21) and (B.22), we have used the expressions for Φ, Δ, and Δ_H given by equations (2.17), (2.18), and (2.28) respectively in the main text.

Equation (B.21) then immediately establishes that the optimal choice of ξ^*_{max} for the planner is equal to the expression in (B.22).