

SUPPLY NETWORK FORMATION AND FRAGILITY

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ABSTRACT. We model the production of complex goods in a large supply network. Firms source several essential inputs through relationships with other firms. Due to the risk of such supply relationships being idiosyncratically disrupted, firms multisource inputs and invest to make relationships with suppliers stronger. In equilibrium, aggregate production is robust to idiosyncratic disruptions. However, depending on parameters, the supply network may be robust or arbitrarily sensitive to small aggregate shocks that affect the functioning of relationships. We give conditions under which the equilibrium network is driven to a fragile configuration, where arbitrarily small aggregate shocks cause discontinuous losses. We use the model to provide a unified account of a number of stylized facts about complex economies.

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1. INTRODUCTION

Complex supply networks among firms are a central feature of the modern economy. Consider, for instance, a product such as an airplane. It consists of multiple parts, each of which is essential for its production, and many of which are sourced from suppliers. The parts themselves are produced using multiple inputs, and so on.¹ The resulting interdependencies among firms can, on the one hand, propagate shocks. On the other hand, businesses are sophisticated and invest considerable effort in optimizing their supply networks so that they function well and are robust to risks. Moreover, if there are many supply chains, no single one will be very important. It is far from obvious then, what role—if any—complex supply networks play in the large-scale propagation and amplification of shocks. We develop a theory in which many firms strategically form supply networks to insure against idiosyncratic risk. We show that equilibrium supply networks under endogenous investment can be robust yet fragile, exhibiting the following four features: (i) the economy is robust to idiosyncratic risk; yet (ii) small shocks that systematically affect the functioning of supply relationships are massively amplified; (iii) the successful functioning of many unrelated supply chains is highly correlated; and (iv) the complexity of production is key to the level of aggregate volatility.

To motivate our modeling, consider first the idiosyncratic risks each firm faces and their immediate consequences. In a supply chain such as that of an airplane, firms rely on their particular suppliers to deliver particular, customized inputs. For instance, Rolls-Royce designed and developed its Trent 900 engine for the Airbus A380; Airbus could not just buy the engine it requires off-the-shelf. Such inputs are customized to meet the customer’s specifications, and there are only a small number of potential suppliers that a given manufacturer contracts with. The same is true at the subpart level, and so on. Thus, production of the airplane is not exposed just to market-level shocks in the availability of each needed type of part, but rather to idiosyncratic shocks in the operation of the few particular supply relationships that can deliver the part. As a result, a given idiosyncratic disruption in a supply chain—a delay in shipment, a fire at a factory, a misunderstanding by a supplier that delivers an unsuitable component, industrial action by workers—can cause damage that cascades through the supply chain and affects many downstream firms (Kremer, 1993). Indeed, Carvalho et al. (2016) provide evidence of shocks propagating through supply networks following the Great East Japan Earthquake of 2011. In Section 2.5, we discuss further evidence that such disruptions are practically important, and can be very damaging to particular firms.

Nevertheless, such risk does not, on its own, imply much about aggregate volatility. First, if production is risky (e.g., because it depends on the availability of a particular customized part), an optimizing firm will invest to reduce this risk (e.g., by multisourcing so that it is not reliant on just one supplier, and by monitoring relationships carefully).² Airbus, for example, has two engine suppliers for its A380 airplanes. Second, even if each supply chain is down some of the time due to

¹For example, An Airbus A380 has millions of parts produced by more than a thousand companies (Slutsken, 2018). In addition to the physical components involved, many steps of production require specific contracts and relationships with logistics firms, business services, etc. to function properly.

²Strategic responses to risk in networks is a topic that has attracted considerable attention recently. See, for instance, Bimpikis, Candogan, and Ehsani (2019), Blume, Easley, Kleinberg, Kleinberg, and Tardos (2011), Talamàs and Vohra (2018), and Erol and Vohra (2018).

cascading failures, there will be other, separate, supply chains that operate; thus, total production can be quite stable, by standard diversification arguments.³ For aggregate production, idiosyncratic failures wash out, and what matters is the fraction of supply chains functioning. This fraction depends on parameters that affect the whole supply network—that is, aggregate parameters.

Suppose, for example, that the institutions that help uphold contracts and facilitate business transactions suddenly decline in quality. Each supply relationship then becomes more prone to the aforementioned idiosyncratic disruptions. Indeed, Blanchard and Kremer (1997) present evidence that the former Soviet Union suffered this kind of shock when it transitioned to a market-based economy and argue that this contributed to the subsequent drop in output. For another concrete interpretation of the kinds of aggregate shocks we have in mind, consider a small shock to the availability of credit for businesses in the supply network. The shock matters for firms that are on the margin between getting and not getting credit that is essential for them to deliver on a commitment. The effect of such a credit shock can be modeled as any given supply relationship being slightly less likely to function (depending on ex ante uncertain realizations of whether a firm is on the relevant margin). Similarly, for the 2019-20 outbreak of the novel coronavirus, Covid-19, there is much uncertainty about how different supply relationships might be affected and this can be captured by modeling it as a systematic increase in the risk that suppliers will be unable to deliver the inputs required from them.

Even if the shock to the operation of supply relationships is slight, we show that it can unravel production in many supply chains simultaneously. Indeed, our main finding gives conditions under which, *in equilibrium, supply networks are systematically driven to a fragile configuration where the fraction of supply chains functioning is very sensitive to certain arbitrarily small aggregate shocks*. Moreover, somewhat paradoxically, this fragility is a consequence of precisely the decentralized strategic “insurance” that firms undertake to protect against idiosyncratic risks. This is because firms’ investments in their supply relationships systematically push many supply chains into what we term a *fragile regime*. Supply chains in the fragile regime are extremely sensitive to *any* systematic and persistent increase in the rate of disruptions, and the impact of such a shock on them causes aggregate production to decrease discontinuously. If the strength of relationships were set exogenously, such a fragile configuration would be extremely unlikely. Nevertheless, under conditions we describe, firms entering the market and investing in their supply networks are drawn systematically toward the precipice. This fragility is highly inefficient: a social planner would never locate a supply network near such a precipice.

The novelty of our framework comes from the *combination* of two features that are essential for our results. The first is complexity: In the supply networks we study, firms require multiple essential inputs that cannot be purchased off-the-shelf. The second is the presence of idiosyncratic disruptions. Jointly, these phenomena create the possibility of precipices, which underlie our analysis.

We now outline the model and results more formally. There are many products and many firms specialized in each product; each firm is small relative to the overall supply network. These firms

³In our model no firms are systemically important. On the complementary issue of when individual firms can be systemically important, see Gabaix (2011).

form supply relationships with suppliers to secure the produced inputs they require.⁴ A firm may have multiple relationships with different producers of a given input. Each of these relationships may fail, independently, with some probability. Relationships are said to be *stronger* if they are more likely to operate successfully. For a firm to source an input from a given supplier, it must be that (i) the relationship with that supplier operates successfully and (ii) the supplier is able to produce the desired input. **The key parameters for describing the mechanics of a supply network are: the number of distinct inputs required in each production process (a measure of complexity); the number of potential suppliers of each input (a measure of the availability of multisourcing); and the strength of each relationship.** Our first set of results examines the mechanics of such a system when the strength of the relationships is exogenous. It shows that when production is complex (i.e., firms have multiple essential inputs they need to source) there is a discontinuity in aggregate productivity: when relationship strength in a certain supply network falls below a certain threshold, which we call a *precipice*, production there drops discontinuously to zero. This raises the prospect of fragility: a small negative shock to relationship strengths (e.g., a public event that makes contracts less likely to be enforced) can lead to considerable economic damage.

We next show that a social planner will always choose costly relationship strengths so that the supply network is away from a precipice. A natural question is then whether a supply network will be near a precipice when relationship strengths are determined by equilibrium choices rather than by a planner.⁵ To analyze this, we model the incentives of firms that are attempting to produce. Firms that succeed in producing sell their (differentiated) output to consumers to make a gross profit.⁶ They choose a level of costly investment toward making relationships stronger—i.e., likelier to operate.⁷ These investment decisions, which depend on the gross profit that firms expect to make, determine the strength of relationships in equilibrium. The gross profit is itself endogenous: it is determined by the number of firms competing to sell a given product—i.e. on the crowdedness of the market. When more firms sell a product, each earns a smaller gross profit.

The basic force that can push the supply network toward a precipice is as follows. When the supply network is reliable and gross profits are high, firms want to enter. Competition drives down gross profits and makes it less appealing to pay costs to make relationships strong. So relationships get weaker. (Indeed, because firms do not internalize the full benefit of their reliability for other firms, there is inefficient underinvestment.⁸) The question is where this dynamic stops. The precipice can be a natural stopping point, because near it reliability suffers so much that entry is no longer appealing. An interesting effect of the precipice is that it can ration entry while those entering obtain positive profits. This causes supply networks to get stuck at the precipice for a range of parameters.

⁴Firms may also use labor or other unproduced factors as inputs to their production processes, as well as commodities that can be acquired without specific relationships (see Appendix B).

⁵On the importance of the strength of contracts in supply relationships, see, among others, Antràs (2005) and Acemoglu, Antràs, and Helpman (2007).

⁶Gross of fixed entry costs, that is.

⁷This can be interpreted in two ways: (1) investment on the intensive margin, e.g. to anticipate and counteract exogenous risks or improve contracts; (2) on the extensive margin, to find more partners out of a set of potential ones.

⁸There is also a business stealing effect that creates incentives for overinvestment, but this turns out to be dominated by the free riding effect.

We study how the outcome depends on an aggregate parameter of the supply network, which can be either a productivity shifter (which multiplies the value of production) or a cost shifter (a parameter, such as access to financing or the quality of institutions, that affects relationship quality). The supply network in equilibrium can end up in one of three configurations: (i) a *noncritical equilibrium* where the equilibrium investment by entering firms is enough to keep relationship strength away from the precipice; (ii) a *critical equilibrium* where the number of firms entering is on the verge of critically undermining investment incentives and the equilibrium relationship strength is on the precipice; and (iii) an unproductive equilibrium where positive investment cannot be sustained. These regimes are ordered. As the productivity of the supply network (or, equivalently, the quality of institutions) decreases from a high to a low level, the regimes occur in the order just given. Each regime occurs for a positive interval of values of the parameters. Equivalently, for an economy comprised of many supply networks distributed will full support over the parameter space, a positive measure of them will be in the fragile regime. The critical equilibrium is fragile in the following sense. Suppose a small shock occurs that affects the strengths of all relationships. Then output falls drastically: indeed, production falls off the precipice.

We then explore some implications of our modeling, especially as it relates to heterogeneity within and across supply networks. First, we study how fragility manifests in a supply network with rich heterogeneity across multiple dimensions (number of inputs required, amount of multisourcing possibilities, directed multisourcing efforts, profitability, etc.). We find that the basic fragility in production that we identify is not dependent on homogeneity, only complexity—that production relies on multiple relationships working. Moreover, a supply network is only as strong as its weakest links: as one product enters the fragile regime, all products that depend on it directly or indirectly are simultaneously pushed into the fragile regime. Second, we show how the supply networks we have studied can be embedded in a larger economy with intersectoral linkages. It turns out that even at a macroeconomic scale, the fragilities we have identified are not smoothed away. Indeed, our model yields a new channel for the propagation of shocks across sectors, and their stark amplification.⁹ Third, while the focus of our analysis is on linking complex supply networks to aggregate volatility, we also discuss how the model can provide a unified perspective on some stylized facts concerning industrial development (see Section 2.5.5). In Section 7 after presenting our results, we discuss in detail how they fit into the most closely related literatures.

2. COMPLEX SUPPLY NETWORKS AND DISCONTINUITIES IN RELIABILITY

In this section, we describe the structure of the supply network. In Sections 2.1–2.3, we present the basic mechanics of the supply network and examine how aggregate production depends on a parameter measuring the strength of supply relationships. This gives the first and simplest manifestation of precipices. We describe other features of the economy needed to fill out the economic model only informally and at a high level, leaving a more complete description for Section 3. Section 2.4 considers the problem of a planner who sets all relationship strengths in the economy at a cost, and characterize the efficient level of investment. Finally, Section 2.5 contains several

⁹This ties into concerns of Baqaee (2018), which focuses on the macroeconomic implications of such propagation, and Baqaee and Farhi (2019), which focuses on the macroeconomics of nonlinearities in production functions more broadly. Theirs is probably the work in macroeconomics closest to ours. We build on it by showing how supply chains naturally give rise to extreme nonlinearities—discontinuities—and then studying the implications.

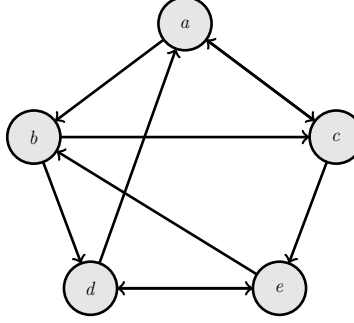


FIGURE 1. An example of a production network, describing the interdependencies between products. An arrow from node i to j represents that to produce i , it is necessary to procure product j .

discussions. First, it presents facts motivating the modeling. It then demonstrates how the key ingredients of the model act in tandem to generate the precipices that are at the center of our subsequent analysis. Finally, it discusses how our framework can be used in a rudimentary theory of the development of complex industries.

2.1. The supply network. We first define a *product network* that describes technological relationships in the economy. Its nodes are a finite set \mathcal{I} of *products*. Product i 's production function requires as inputs a set of other products, which we denote by $\mathcal{I}_i \subseteq \mathcal{I}$. If a firm producing product i procures none of a required input product, it is unable to produce any output.

The product network describes the relationships only among *produced* factors: there may also be unproduced factors, such as labor, that are a substantial part of production. For a concrete production function with these properties, take any CES production function with an elasticity of substitution equal to or greater than 1. (See Appendix B for details.)

A regular-network assumption. For simplicity, here and in much of our exposition, we take the product network to be *regular*: the number of different inputs to produce a given product is the same number m across products (i.e., $|\mathcal{I}_i| = m$ for each i). We call m the *complexity* of the network. The advantage of this assumption is that it allows us to study the mechanics and equilibrium outcomes of the system using intuitive one-dimensional fixed-point equations amenable to exact analysis. The substantive conclusions continue to hold in much richer versions of the model, where different industries require different numbers of inputs, have different (and endogenous) levels of investment in robustness, and where supply chains may end after a finite number of tiers. We highlight the relevant extensions in Section 2.5.4.

Figure 1 provides a very simple illustrative example of a regular product network involving five products. Product a requires products b and c as inputs; product b requires products d and c as inputs; and so on.

The *supply network* describes relationships among firms rather than products. For each product i , there is a continuum of firms \mathcal{F}_i , with the same production function. We denote a typical firm if , where i is the product and $f \in \mathbb{R}$ is the label of the firm. A firm making product \mathcal{I}_i requires the products in \mathcal{I}_i as inputs. Thus a firm if has n *potential supply relationships* for each product in \mathcal{I}_i . Only some of these relationships will turn out to be operational. (The firm multisources for its inputs precisely because link operation is uncertain, as we will see.) The firms that if has supply

relationships with for input $j \in \mathcal{I}$, are drawn independently, according to an atomless distribution, from the continuum of firms \mathcal{F}_j who produce the required product.

Figure 2 depicts a small subpart of the supply network—a firm, its suppliers, and their suppliers. Consider a firm $a1$. According to the product network, making product a requires two input products— b and c . The firm sources these via supply relationships with particular firms specialized in these products. For each of b and c , the firm has potential supply relationships with two producers of each input it requires. The suppliers of our focal firm, $a1$ are in a position similar to that of $a1$ itself.¹⁰ These supply relationships are directed edges, from the firm that is doing the procuring or ordering to its supplier. Thus, orders flow upstream, while products flow downstream.

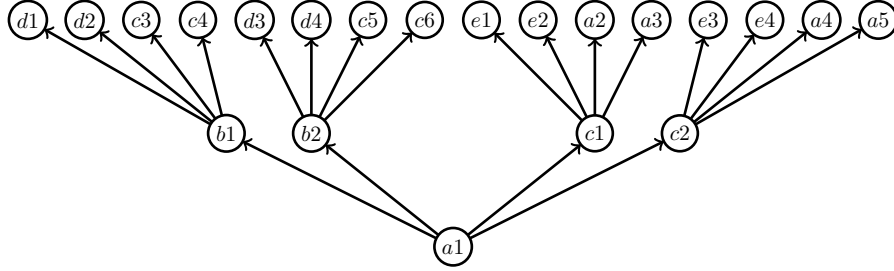


FIGURE 2. This figure illustrates the first three tiers of the supply network for producer $a1$, corresponding to the product network shown in Figure 1. Firms higher up are upstream of $a1$, and a directed edge from one firm to another reflects that the first may try to source from the second. (Thus, orders or sourcing attempts go in the direction of the arrows, and products are delivered in the opposite direction, downstream.)

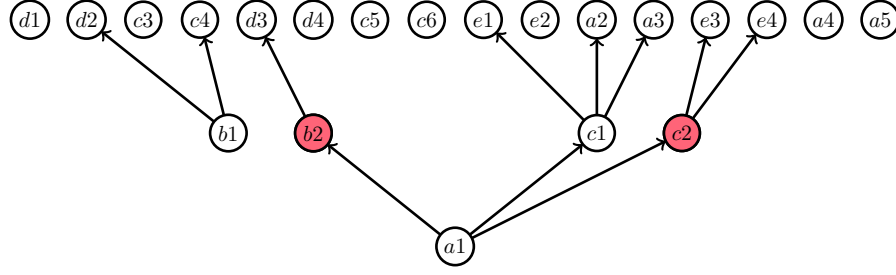
2.2. Uncertainty in sourcing and firm functionality. We now introduce the uncertainty of sourcing inputs, coming from the stochastic failure of some supply relationships. Each of a firm’s supply relationships may be *operational* or not—a binary random outcome. There is a parameter x , called *relationship strength* (for now exogenous and homogeneous across the supply network) which is the probability that any relationship is operational. All these realizations are independent.¹¹ We can think of such a realization as describing whether the “pipe” meant to ship inputs from the particular (upstream) supplier to its (downstream) customer is working. The actual supply network is then obtained by keeping each potential supply link independently with probability x . We illustrate this in Figure 3.

For a firm to be *functional*, it must be able to source *each* input. This means that, for each input, at least one of the suppliers of that input must itself be functional. (Conditional on at least one supplier being able to supply, it can fulfill the needed quantity, so the possibility of failure is the only reason to multisource.) Thus, functionality is interdependent—a point we will return to soon.

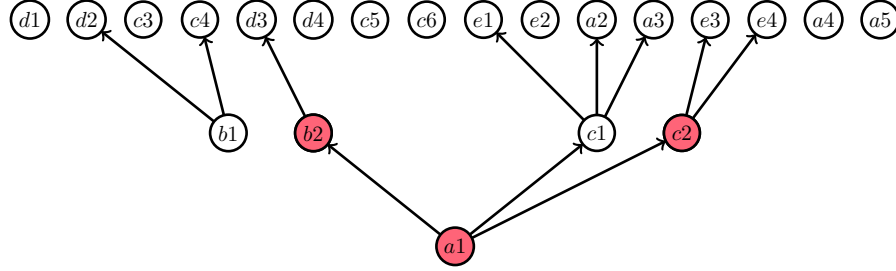
Continuing with our running example, take the firm $a1$ for illustration. For this firm to be functional, it must be that for any input product (say, b), there is at least one of a ’s potential suppliers ($b1$ or $b2$) of that product such that (i) the link that $a1$ has to source from this supplier is operational (the pipe works) and (ii) this supplier is functional itself (able to source its required

¹⁰Because there is a continuum of firms and only finitely many (randomly sampled) firms appear upstream of any given firm, the probability of any firm appearing multiple times upstream of a given firm is 0.

¹¹For a formal construction of the potential supply network and the functionality realizations of the links, see Appendix A.



(A) Stage 1: firms $b2$ and $c2$ are removed from $\hat{\mathcal{F}}(0)$. They can't produce because they can't get all the essential inputs they need. This gives $\hat{\mathcal{F}}(1)$.



(B) Stage 2: now firm $a1$ is removed from $\hat{\mathcal{F}}(1)$. As $b2 \notin \hat{\mathcal{F}}(1)$, $a1$ is unable to source a b -product from a functional supplier and so cannot produce. This gives $\hat{\mathcal{F}}(2)$.

FIGURE 3. An illustration of the algorithm for determining the maximal set of functional firms. At stage s , only those firms with at least one operational supply relationship to a firm in the set of still-functional firms ($\hat{\mathcal{F}}(s-1)$) for each input required remain functional. Firms that are not functional are shaded red. The algorithm terminates after stage 2. The set of functional firms is $\hat{\mathcal{F}}(2)$.

inputs and produce, so there is something to ship down the pipe). Whether (ii) holds depends on other functionality realizations throughout the supply network.

2.3. Which firms can consistently function? We are interested in the physical production possibilities frontier. Which firms *can* produce, given the supply link realizations, and what is the size of this set?

For a given realization of which supply links are operational we need to determine which firms are functional. As the functionality of firms is interdependent, it can be consistent for different sets of firms to be functional. We focus throughout the paper on the maximal set of firms that can be consistently functional.

A simple algorithm determines the maximal functional set for a given realization of which supply links are operational. (1) Initialize the set of (putatively) functional firms, $\hat{\mathcal{F}}(0)$, to be all firms—i.e., the union of all the \mathcal{F}_i . (2) At any stage of the algorithm $s = 1, 2, \dots$ let $\hat{\mathcal{F}}(s)$ be the set of firms in $\hat{\mathcal{F}}(s-1)$ that, for each input, have an operational link for that input to some firm in $\hat{\mathcal{F}}(s-1)$.

Intuitively, this algorithm begins by optimistically assuming that all firms are functional, and removing firms only when they have no operational links supplying some needed input.

This procedure gives a decreasing sequence of sets. Define $\bar{\mathcal{F}}$ to be the intersection (i.e., limit) of these sets. By a standard application of the Tarski fixed-point theorem, this turns out to be the

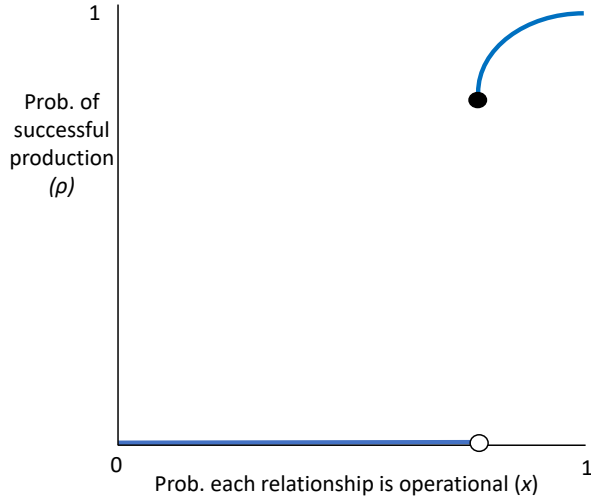


FIGURE 4. The reliability of a supply network, $\rho(x)$, which is the probability that a random firm can successfully produce, plotted as the probability any given relationship is operational, x , varies.

maximal set of firms such that it is consistent for all firms in the set to produce.¹² The algorithm is simply an intuitive way to describe this set. This determines the physical production possibilities of the supply network.¹³ For an illustration of this process, see Figure 3.

2.3.1. The reliability function. Next we are interested in the mass of firms in $\overline{\mathcal{F}}$ producing any given product; by symmetry it does not depend on the product. We denote this quantity by $\rho(x)$ and call it the *reliability* of the supply network; it can be interpreted as the probability that a randomly selected firm can produce. The reliability depends on relationship strength x , which is the probability of any given supply link being operational.

Recall that each firm requires m different inputs and has n potential suppliers of each of these inputs. In Figure 4 we plot reliability as a function of the probability x of each relationship being operational; we use the $m = n = 2$ case here, as in our illustrations above. The key fact about this plot is that reliability is *discontinuous* in relationship strength x , jumping at a value that we will call x_{crit} . The probability of successful production is 0 when $x < x_{\text{crit}}$, but then increases discontinuously to more than 70% at this threshold.

This property of $\rho(x)$ is general—for any number $m \geq 2$ of required inputs and any potential multisourcing level $n \geq 1$, $\rho(x)$ has a unique point of discontinuity at x_{crit} , and for all lower levels of x , production always fails ($\rho(x) = 0$). This is summarized in the following result:

Proposition 1. Let the complexity of production m be at least 2 and the number n of potential suppliers for each input be at least 1. The reliability of the supply network (share of firms in $\overline{\mathcal{F}}$), denoted by $\rho(x)$, is a nondecreasing function with the following properties.

¹²Formally, given a realization of which supply links are operational, it is the maximal set $\widehat{\mathcal{F}}$ of firms such that: for every firm i in $\widehat{\mathcal{F}}$, and each input j necessary to produce product i , the firm has at least one operational supply relationship to a supplier of product j , which is also in $\widehat{\mathcal{F}}$. In Appendix B we discuss more formally the production that takes place conditional on the set of functional firms.

¹³It is worth noting that the maximal fixed point is the fixed point selected in the limit of finite versions of our model (see Section SA1 of the Supplementary Appendix).

- (1) There are numbers $x_{\text{crit}}, r_{\text{crit}} > 0$ such that ρ has a discontinuity at a critical level of relationship strength x_{crit} , where it jumps from 0 to r_{crit} and is strictly increasing after that.
- (2) If $n = 1$, we have that $x_{\text{crit}} = 1$; otherwise $x_{\text{crit}} < 1$.
- (3) If $x_{\text{crit}} < 1$, then as the relationship strength x approaches x_{crit} from above, the derivative $\rho'(x)$ tends to ∞ .

The result is proved in Appendix D. We explain the intuition behind the result in the next subsection.

This result already has some stark implications. As relationship strength increases, a threshold is passed at which production becomes possible. Moreover, reliability (the probability of successfully producing the good) jumps abruptly from 0 to a positive number (in the $n = m = 2$ case, approximately 0.8) as x improves around the threshold x_{crit} . This implies that small improvements in relationship strength x , for example through the improvement of institutions, can have large payoffs for an economy, and the net marginal returns on investment in x can change sharply from being negative to being positive and very large.

As supply networks become large and decentralized, one might think that the impact of uncertainty on the probability of successful production would be smoothed somehow by averaging. We find the opposite: in aggregating up the uncertainty through the interdependencies of the supply network, we get a very sharp sensitivity of aggregate productivity to relationship strength. This is in contrast to standard production network models (e.g., Baqaee and Farhi (2019)), where the aggregate production function is differentiable at any point. The difference comes from the fact that we model the failure of nodes to produce when they do not receive (enough) inputs.

2.3.2. Deriving the reliability function: Analyzing the mass of functional firms. We now explain the reasoning behind Proposition 1.

Recall the algorithm described in Section 2.3 for determining the mass of functional firms. Fix x and an arbitrary stage s of the algorithm. Let r_s be the probability that a randomly selected firm is in $\hat{\mathcal{F}}(s)$. Now let us consider the probability that a fixed arbitrary firm is in $\hat{\mathcal{F}}(s+1)$. We claim that this number is

$$\mathcal{R}(r_s; x) = (1 - \underbrace{(1 - x r_s)^n}_{\text{probability a given input cannot be acquired}})^m. \quad (1)$$

By definition, being in $\hat{\mathcal{F}}(s)$ requires that for each input, our focal firm has an operational link to some firm in $\hat{\mathcal{F}}(s)$ producing that input. Consider the first input of our focal firm. For a given supplier of that input, the probability that supplier is in the set $\hat{\mathcal{F}}(s-1)$ is r , and the probability that the link the supplier is operational is x . The probability that both events happen is xr_s . The probability that this combination of events happens for at least one of the n potential suppliers of the first input is therefore $1 - (1 - xr_s)^n$. Finally, the probability that for all m inputs our focal firm has an operational link to some firm in $\hat{\mathcal{F}}(s)$ producing that input is $(1 - (1 - xr_s)^n)^m$.

The above discussion shows that $r_{s+1} = \mathcal{R}(r_s; x)$. (We will often drop the x argument.) To initialize this sequence, we observe that $r_0 = 1$, since by definition the algorithm is initialized with

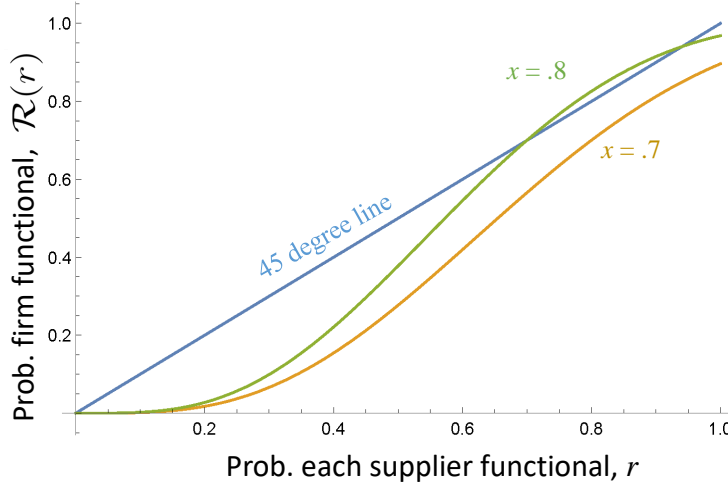


FIGURE 5. The probability, $\mathcal{R}(r)$, that a focal firm is functional as a function of r , the probability that a random supplier is functional. Here we use the parameters $n = 4$ and $m = 3$.

$\mathcal{F}(0)$ consisting of all firms. The sequence $(r_s)_{s=1}^{\infty}$ converges to the largest fixed point of equation (1), which is, by definition $\rho(x)$. This $\rho(x)$ solves $\mathcal{R}(\rho(x)) = \rho(x)$.¹⁴

This discussion allows us to describe x_{crit} in Proposition 1: it is the smallest x for which the equation has a nonzero solution r , and r_{crit} is that solution.

The last step is to understand why the largest fixed point $\rho(x)$ jumps to a positive level discontinuously as we vary x . Consider Figure 5, which shows the shape of \mathcal{R} as x is varied. We see that beyond a certain value of x , this curve has a nontrivial intersection with the 45-degree line, but not before. The essential feature of the curve that leads to this shape is its convexity for low values of r , which occurs whenever the exponent m is at least 2. In that case, the curve \mathcal{R} is initially bounded above by a quadratic function in x . This means that it cannot have an intersection with the 45-degree line close to zero.¹⁵ Thus, any positive fixed point must emerge discontinuously as x increases. The fact that $m \geq 2$ —that is, that the supply network is complex—is crucial, as we will see again in Section 2.5.2.

2.3.3. How production unravels when relationship strength is too low. Figure 4 shows that when x drops below x_{crit} , the mass of firms that can consistently function falls discontinuously to $\rho(x) = 0$. While we will typically just work with the fixed point as the outcome of interest, the transition will not be instantaneous in practice. How then might the consequences of a shock to x actually play out?

In Figure 6, we work through a toy illustration to shed some light on the dynamics of collapse. Using the same parameters as our previous example, suppose relationship strength starts out at $x = 0.8$. The higher curve in panel (a) is $\mathcal{R}(\cdot; x)$ for this value of x . The reliability of the economy here is r_0 , a fixed point of \mathcal{R} , which is mass of functioning firms. Now suppose that a shock occurs, and all relationships become weaker, operating with the lower probability $x = 0.7$. The \mathcal{R} curve now shifts, becoming the lower curve.

¹⁴We can interpret that fixed-point condition in an intuitive way. By the same reasoning we have given above, when all firms in the supply network—and in particular, all one’s potential suppliers—are functioning with probability $\rho(x)$, the probability of sourcing all inputs successfully—i.e., functioning—is the right-hand side $\mathcal{R}(\rho(x))$.

¹⁵Since $\mathcal{R}(r) < Cr^2$ for some C , and the right-hand side is much smaller than r for small r .

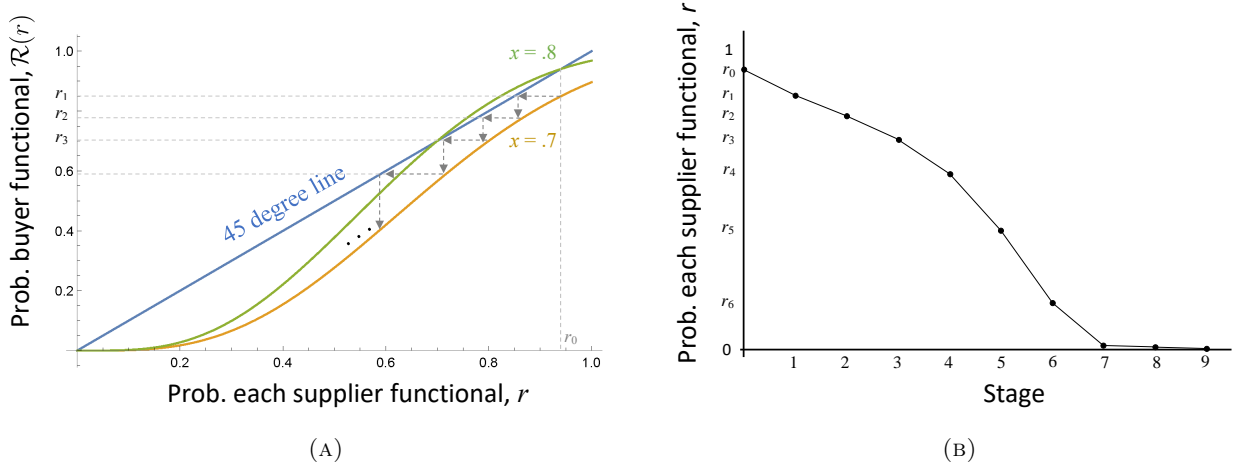


FIGURE 6. The dynamics of unraveling (with the same parameters as in Figure 5, as discussed in Section 2.3.3).

To consider the dynamics of how production responds, we must specify a few more details. We sketch one dynamic, and only for the purposes of this subsection. We interpret idiosyncratic link operation realizations as whether a given relationship works in a given period. For example, firms might hold inventories that enable them to maintain production for a month, even when unable to source an essential input. Before the shift in x , a fraction r_0 of the firms are functional. Let $\tilde{\mathcal{F}}(0)$ be the random set of functional firms at the time of the shock to x . Now x shifts to 0.7; we can view this as a certain fraction of formerly functional links failing, at random. Then firms begin reacting over a sequence of stages. Let us suppose that at stage s a firm can source its inputs if it has a functional link to a supplier who was functional in the last stage, $s - 1$.¹⁶ Let $\tilde{\mathcal{F}}(s)$ be the set of these functional firms. By the same reasoning as in the previous subsection, we can see that the mass of $\tilde{\mathcal{F}}(s)$, which we call r_s , is $\mathcal{R}(r_{s-1})$. Iterating the process leads to more and more firms being unable to produce as their suppliers fail to deliver essential inputs. After stage 1, the first set of firms that lost access to an essential input run out of stock and are no longer functional. This creates a new set of firms that cannot access an essential input, and these firms will be unable to produce at the end of the subsequent month, and so on. The mass of each r_s can be described via the graphical procedure of Figure 6: take steps between the \mathcal{R} curve and the 45-degree line.

This discussion helps make three related points. First, even though the disappearance of the positive fixed point—and thus the possibility of a positive mass of consistently functional firms—is sudden, the implications can play out slowly under natural dynamics.¹⁷ The first few steps may look like a few firms being unable to produce, rather than a sudden and total collapse of output.

The second point is more subtle. Suppose that when the dynamic of the previous paragraph reaches r_2 , the shock is reversed, x again becomes 0.8, and \mathcal{R} again becomes the higher curve.

¹⁶Even if firms engage in just-in-time production and do not maintain inventories of essential inputs, there can be a lag between shipments being sent and arriving.

¹⁷Indeed, in a more realistic dynamic, link realizations might be revised asynchronously, in continuous time, and firms would stop operating at a random time when they can no longer go without the supplier (e.g., when inventory runs out). Then the dynamics would play out “smoothly,” characterized by differential equations rather than discrete iteration.

Then, with some supply links reactivated, some of the firms that were made non-functional as the supply chain unravelled will become functional again, and this will allow more firms to become functional, and so on. Such dynamics could take the system back to the r_0 fixed point if sufficiently many firms remain functional at the time the shock is reversed. Thus, our theory predicts that *sufficiently persistent* shocks to relationship strength lead to eventual collapses of production, but, depending on the dynamics, the system may also be able to recover from sufficiently transient shocks.

The third point builds on the second. Suppose that a shock is anticipated and expected to be temporary. Then firms may take actions that slow the unravelling to reduce their amount of downtime. For example, they may build up stockpiles of essential inputs. If all firms behave in this way, the dynamics can be substantially slowed down and the possibility of recovery will improve.

Having illustrated some of the basic forces and timing involved in unraveling, we do not pursue here a more complete study of the dynamics of transient shocks, endogenous responses, etc.—an interesting subject in its own right. Instead, from now on we will focus on the size, $\rho(x)$, of the consistent functional set, which is the steady-state outcome under a relationship strength x .

2.4. A planner's problem. We consider now a planner who chooses the value of x , which can be thought of as the quality of institutions. The planner's problem is

$$\max_{x \in [0,1]} \kappa \rho(x) - c(x), \quad (2)$$

where $c(x)$ is a convex function representing the cost of maintaining institutions of quality x . We assume that $c(0) = 0$, $c'(0) = 0$, and $\lim_{x \rightarrow 1} c'(x) = \infty$. Here $\kappa > 0$ is a parameter we use to shift the value of production. Thus the planner seeks to maximize expected social surplus, which is the total surplus produced by the firms that are functional minus the cost of maintaining institutions.

Define the correspondence $x^{SP}(\kappa) = \operatorname{argmax}_{x \in [0,1]} \kappa(1 - (1 - x\rho(x))^n)^m - c(x)$. This gives the values of x that solve the social planner's problem for a given κ .

Proposition 2. Fix any $n \geq 2$ and $m \geq 2$. Then there exists a $\kappa_{\text{crit}} > 0$ such that

- (i) for all $\kappa < \kappa_{\text{crit}}$, $x^{SP}(\kappa) = 0$.
- (ii) for all $\kappa > \kappa_{\text{crit}}$, all values of $x^{SP}(\kappa)$ are strictly greater than x_{crit} .
- (iii) for $\kappa = \kappa_{\text{crit}}$, all values of $x^{SP}(\kappa)$ are either strictly greater than x_{crit} or else equal to 0.

The first part of Proposition 2 says that when κ is sufficiently low, it is too costly for the social planner to invest anything in the quality of institutions. As κ increases, a threshold κ_{crit} is reached and at this value of κ it first becomes optimal to invest in institutional quality. At this threshold, the social planner's investment increases discontinuously. Moreover, it immediately increases to a level strictly *above* x_{crit} , and for all larger κ all solutions stay above x_{crit} .

It is worth emphasizing that the planner never chooses to invest at the critical level x_{crit} . An efficient supply network never operates at the point of discontinuity. This is intuitive: at $x = x_{\text{crit}}$ the marginal social benefits of investing are infinite, as can be seen in Figure 4, while marginal costs at x_{crit} are finite, and so the social planner can always do better by increasing investment a little. In contrast, Section 4 will show that individual investment choices *will* sometimes put the supply network on the precipice in equilibrium.

2.5. Discussion. The key features of our model are that (i) each firm’s production relies on multiple non-commodity inputs and (ii) that these are sourced through failure-prone relationships with particular suppliers. In this section we first discuss the motivation and interpretation of these assumptions. We then highlight the importance of each by comparing the model to two benchmarks. In the first benchmark, feature (i) is not present: we consider the production of simple goods, where each firm requires only one non-commodity input. In the second benchmark, feature (ii) is not present: we consider market-based, as opposed to relationship-based, sourcing of all inputs. Next, we highlight aspects of our model that are *not* important to the key phenomena: symmetry in firms’ positions, “infinite” production processes, and “all or nothing” link failures, . Finally, we discuss some first implications of Proposition 1 for industrial development.

2.5.1. Comments on key concepts and assumptions. The fact that some key inputs are supplied via a limited number of supply relationships is a key assumption in our model. While a firm may use labor and other commodities as in a standard model of production, some inputs require relationships.

As we have discussed in the introduction, specific sourcing relationships are a central feature of the modern economy, and firms are often constrained in the number of supply relationships they can maintain—for example, by technological compatibility, geography, trust, understanding, etc. Supplier relationships have been found to play important roles in many parts of the economy—for relationship lending between banks and firms see Petersen and Rajan (1994, 1995); for traders in Madagascar see Fafchamps and Minten (1999); for the New York apparel market see Uzzi (1997), for food supply chains see Murdoch, Marsden, and Banks (2000), for the diamond industry see Bernstein (1992), for Japanese electronics manufacturers see Nishiguchi (1994), and so on. Indeed, even in fish markets, a setting where we might expect relationships to play a minor role, they seem to be important (Kirman and Vriend, 2000; Graddy, 2006).

The parameter x that we have called relationship strength, which is the probability that a supply relationship is operational, can capture a variety of considerations: uncertainty regarding compatibility, whether delivery can happen on time, possible misunderstanding about the required input, access to credit that may be needed to deal with unexpected costs, etc. It will depend on the context or environment in which production occurs, and also (as we explicitly model below) on the investments agents make.

We view the presence of idiosyncratic shocks, which can be only imperfectly insured against, as a realistic feature. While some potential incidents may be mitigated by inventories, many are not, and these shocks can cause cascading damage. For example, a fire at a Philips Semiconductor plant in March 2000 halted production, preventing Ericsson from sourcing critical inputs, causing its production to also stop (*The Economist*, 2006). Ericsson is estimated to have lost hundreds of millions of dollars in sales as a result, and it subsequently exited the mobile phone business (Norrman and Jansson, 2004). In another example, two strikes at General Motors parts plants in 1998 led 100 other parts plants, and then 26 assembly plants, to shut down, reducing GM’s earnings by \$2.83 billion (Snyder et al., 2016). Carvalho, Nirei, Saito, and Tahbaz-Salehi (2016) quantify the cascading disruptions in supply chains following the Great East Japan Earthquake of 2011 and resulting tsunami, and the large associated costs.

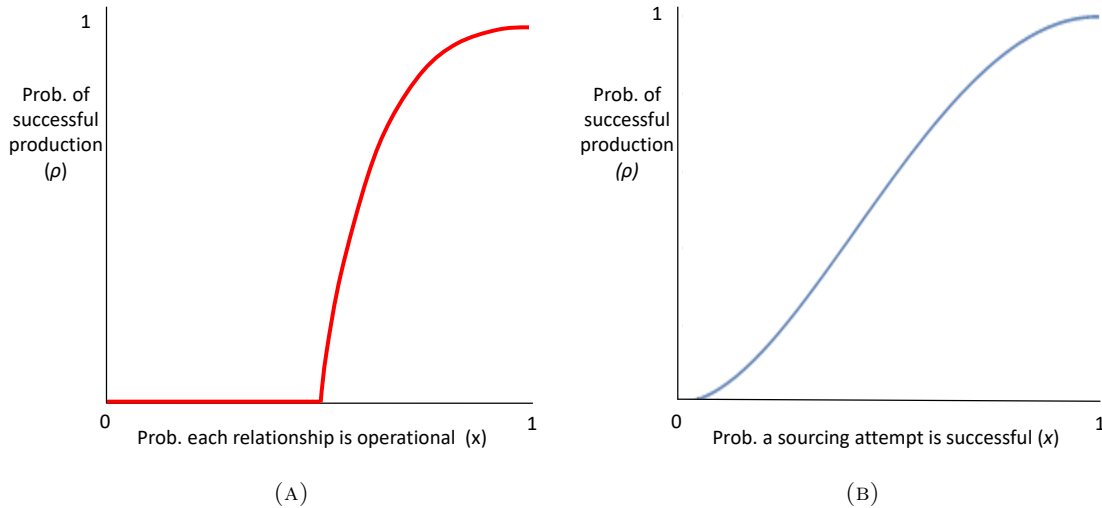


FIGURE 7. (a) The probability of successful production of a simple good as relationship strength varies. (b) The probability of successful production for a firm as market-based sourcing attempts become more likely to succeed.

Though the above cases are particularly dramatic and well-documented, disruptions to supply networks occur frequently. In a survey of studies on this subject in operations and management, Snyder et al. (2016) write, “It is tempting to think of supply chain disruptions as rare events. However, although a given type of disruption (earthquake, fire, strike) may occur very infrequently, the large number of possible disruption causes, coupled with the vast scale of modern supply chains, makes the likelihood that some disruption will strike a given supply chain in a given year quite high.” An industry study found 1,069 supply chain disruption events globally during a six-month period in 2018 (Supply Chain Quarterly, 2018). In the model, a given supply chain is operating with some probability, which can be interpreted as some fraction of the time. Thus, as in reality, disruption events are frequent.

2.5.2. Contrast with sourcing for simple production. To contrast with the case of complex products, we consider a supply network where each firm requires only a single relationship-sourced input ($m = 1, n = 2$). We call such production simpler in that each firm requires only one type of risky input relationship to work.¹⁸ We plot how the probability of successful production varies with relationship strength in Figure 7(a). In comparison to the case of complex production illustrated in Figure 4, there is a stark difference. For values of $x < 0.5$ the probability of successful production is 0 and for values of $x > 0.5$ the probability of successful production is strictly positive, but the change about this point is continuous.

The change at $x = 0.5$ is abrupt with the derivative changing discontinuously. The intuition for this change is one that is familiar from the networks literature and in particular from studies of contagion (see, for example, Elliott, Golub, and Jackson (2014) in the context of financial contagion). For a given producer, production will be successful if the supply network doesn’t die out after a finite number of steps. This depends on whether the rate at which new branches in

¹⁸As a matter of interpretation, there may be more than one physical input at each stage. The key assumption is that all but one are sourced as commodities rather than through relationships, and so are not subject to disruption.

the network are created is higher or lower than the rate at which existing branches die out due to failure. It turns out that when $x > 0.5$, a supply tree grows without bound in expectation, while when $x < 0.5$ it dies out.¹⁹ The kink in the probability of successful production around the key threshold of 0.5 is related to the emergence of a giant component in an Erdős–Rényi random graph. That is not the case, as we have seen, for complex production. That is a different sort of phase transition, reliant on the need for multiple inputs at each stage. As we’ll see, this difference has stark economic implications once investments in reliability are endogenous.

2.5.3. Contrast with market-based sourcing. In this benchmark, each firm sources all its inputs through markets, rather than requiring pre-established relationships. The market is populated by those potential suppliers that are able to successfully produce the required input. However, upon approaching a supplier there is still a chance that sourcing fails for one reason or another. (A shipment might be lost or defective, or a misunderstanding could lead the wrong part to be supplied.) In the metaphor we introduced earlier, we now assume each firm extends pipes *only* to functional suppliers, but we still keep the randomness in whether the pipes work.²⁰

Let the probability a given attempt at sourcing an input succeeds be x , independently. As before, in view of this idiosyncratic risk, each firm *if* multisources by contracting with two potential suppliers of each input. The probability that both potential suppliers of a given input type fail to provide the required input is $(1 - x)^2$, and the probability that at least one succeeds is $1 - (1 - x)^2$. As the firm needs access to all its required inputs to be able to produce, and it requires 2 different input types, the probability the firm is able to produce is $(1 - (1 - x)^2)^2$. In Figure 7(B) we plot how the probability that a given firm is able to produce varies with the probability their individual sourcing attempts are successful. This probability increases smoothly as x increases.

2.5.4. Dispensable assumptions: symmetries and all-or-nothing failure. One might suspect that the regularity of the network structure, or some other kind of symmetry, plays an important role in generating the discontinuity in the probability of successful production. In contrast with the key roles played by complexity and specific sourcing, symmetries are not important to our main points. The discontinuity we have documented persists in the presence of asymmetric networks where different products require different numbers of inputs, there are different numbers of suppliers for each input, and relationship strengths vary by input. These results are discussed in Section 6.

Relatedly, there can be asymmetry among firms in that some firms do not rely on *any* produced, non-commodity inputs. Our substantive findings—both the mechanics discussed in this section and our later economic conclusions—hold for such production as well, as long as supply chains are long enough; see Section 4.5.2. More fundamentally, as we also discuss in that section, the discontinuities in production do not rely on products being assembled in an infinite number of steps.

¹⁹Given that each producer has two potential suppliers for the input, and each of these branches is operational with probability x , the expected number of successful relationships a given firm in this supply network has is $2x$. When $x < 1/2$, each firm links to on average less than 1 supplier, and so the rate at which branches in the supply tree fail is faster than the rate at which new branches are created. The probability that a path in the supply network reaches beyond a given tier l then goes to 0 as l gets large and production fails with probability 1. On the other hand, when $x > 1/2$, the average number of children each node has is greater than 1 and so new branches appear in the supply tree at a faster rate than they die out, leading production to be successful with strictly positive probability.

²⁰Formally, in the construction of the random graph, we allow each firm’s potential supply links to be formed conditional on the realization of its supplier’s functionality; in particular, these links are extended only to functional suppliers.

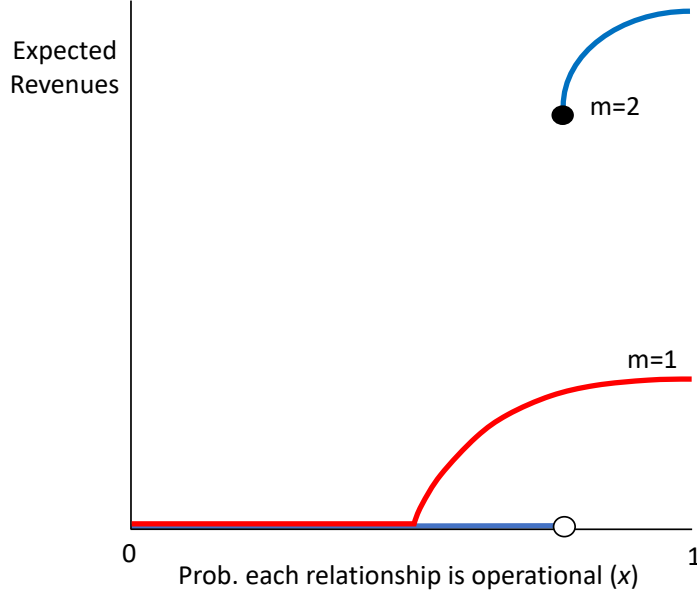


FIGURE 8. A contrast of the $m = 1$ and $m = 2$ cases with the degree of multisourcing being held at $n = 2$. Expected revenues are on the vertical axis. This is a product of the probability of successful production and price of goods. The case in which the complex good retails for a price of 1 while the simple good retails for a price of $1/4$ is illustrated.

Finally, we assume that a firm can produce as long as it has an operational link to at least one functioning producer of each input. More realistically, it may be that to produce, a sufficient quantity or quality of each input is needed, and the shock is to whether the requisite level is reached. The shock need not destroy all the output, but may destroy or reduce the value of the output by some amount (say, a random fraction of the output). We consider the starkest case for simplicity; the key force is robust to these sorts of extensions.

2.5.5. An implication for industrial development. We can use the comparison between the production of complex and simple products to sketch a rudimentary theory of industrial development. Suppose a complex good ($m = 2$) can be sold for more in the market than a simple one ($m = 1$), and (just for simplicity) the level of multisourcing is the same in both cases ($n = 2$). We then vary x exogenously, which can be thought of as varying quality of commercial institutions. As illustrated in Figure 8, as x increases there is a first threshold at which it will become possible to produce a simple good with $m = 1$, and the probability of such production being successful then increases in a continuous way in x . At some higher threshold value of x , it will become possible to produce a complex good. Hence, at the point where its institutions become good enough to support complex production, the value-weighted productivity of the economy can jump discontinuously. At the same time, with more complex production, there is also a higher rate of supply chain disruptions.

Even this rudimentary theory of development and industrialization fits a number of stylized facts. First industrialization, when it occurs, is rapid and economic output increases dramatically. Within our model, this can be interpreted simply as crossing the discontinuity as institutions develop. Second, at the same time, the production technology changes and the share of the value of total production that can be attributed to intermediate inputs increases quickly (Chenery, Robinson,

and Syrquin, 1986). This is consistent with our model as well: as institutions improve, it becomes possible to support the more complex production technology, with higher-value (more complex) intermediate inputs. Next, the quality of institutions, and particularly those related to contracting, help explain cross-country differences in development (Acemoglu and Johnson, 2005). In our model, differences in institutional quality can determine whether production of a certain complexity (m) is economically viable or not.²¹ A final stylized fact is that, as in our model, more complex supply chains are associated with higher rates of disruption (Craighead, Blackhurst, Rungtusanatham, and Handfield, 2007).

3. SUPPLY NETWORKS WITH ENDOGENOUS RELATIONSHIP STRENGTH AND ENTRY

We now introduce our full model. This builds on the mechanics of production described in the previous section. However, relationship strength, as well as decisions to enter, are now both endogenous choices. After presenting the formal structure of the economy formally and briefly, we interpret our modeling choices and relate them to real-world motivations in Section 3.4.

3.1. Entry decisions and the supply network. There is a finite set \mathcal{I} of *products*. For each i , the set $\mathcal{I}_i \subseteq \mathcal{I}$ is the set of input products necessary for the production of i .

There is a continuum of firms $\tilde{\mathcal{F}}_i$ in each product $i \in \mathcal{I}$ that are *potential entrants*. Each $\tilde{\mathcal{F}}_i$ is a copy of the interval $[0, 1]$; its firms are labeled if , where i is the product label and $f \in [0, 1]$ is the firm label. Each firm if may enter or not; this is captured by $e_{if} \in \{0, 1\}$. Firm if has a cost $\Phi_i(f)$ of entering, where $\Phi_i : [0, 1] \rightarrow \mathbb{R}_+$ is an increasing function.²²

Assumption 1. For each i , $\Phi_i(f)$ is a strictly increasing function with $\Phi_i(0) = 0$.

First, firms decide whether to pay the fixed cost to enter²³ or not, and if a firm does choose to enter, it also chooses its investment in relationship strength. To produce one unit of its good, firm if must source inputs from some supplier of product j for each $j \in \mathcal{I}_i$. A firm obtains the inputs it needs to produce by forming relationships with suppliers of those inputs. We assume that for each $if \in \mathcal{F}_i$, and each input $j \in \mathcal{I}_i$, there is a finite set $\mathcal{F}_{if,j}$ of firms that, exogenously, are *potential suppliers* of good j to firm if . We suppose the identities (i.e., indices f) of these firms are drawn uniformly at random from among the producers of product j that enter.²⁴ The supply relationships that can be used by if to source inputs will come from this set. A (random) *potential supply network* \mathcal{G} is a directed graph whose nodes are $\mathcal{F} = \{if : e_{if} = 1\}$, consisting of all firms that enter, and where each node has directed links to all of its suppliers $\mathcal{F}_{if,j}$.

3.2. Investment decisions. The key choice each firm makes conditional on entry is how much to invest to increase the probability with which each of its potential supply relationships is *operational*, delivering any goods produced by the supplier. In Section 3.4.1 we discuss evidence suggesting that these decisions matter in practice, and provide some alternative interpretations for

²¹The differences in what kinds of production different economies can support is, of course, related to explaining the enormous differences in wages and productivities across countries (Jones, 2011).

²²The set of firms producing each product is endowed with the Lebesgue measure and so the c.d.f. of the fixed costs of entry is Φ^{-1} .

²³The assumption that the lowest entry cost is zero is not necessary for our results, as we discuss in Section 4.

²⁴Without losing tractability we could also dispense with the conditioning on who enters and allow a firm's potential relationships to include some firms that don't end up in the market.

the investments (both in terms of both searching for new suppliers and ensuring good relations with existing suppliers). For the simplest description of the investment decisions, we make some strong symmetry assumptions. First, we assume $|\mathcal{I}_i| = m$ for each product so that m different inputs are required to produce each good. Second, we assume that $|\mathcal{F}_{if,j}| = n$ for all firms if and all required inputs $j \in \mathcal{I}_i$. Third, we take $\Phi_i = \Phi$ for all i . As was the case for the basic analysis of discontinuities, these assumptions facilitate a simple analysis of key forces via one-dimensional fixed-point equations, but they are not crucial, and are relaxed in Section 6.

With the symmetry assumptions made, we can formulate a simple model of the formation of production links. The firm chooses an investment level $y_{if} \geq 0$, which has a private cost $c(y_{if})$ and results in a relationship strength

$$x_{if} = \underline{x} + y_{if}.$$

The intercept $\underline{x} \geq 0$ is a baseline probability of success that occurs absent any costly investment, which might, for example, reflect the quality of institutions. The main purpose of this baseline level is as a simple channel to shock relationship strength (and other specifications, such as a multiplicative one, could be used). We will typically think of firms directly choosing the strength of their relationships, $x_{if} \in [\underline{x}, 1]$, and paying the corresponding investment cost $c(x_{if} - \underline{x})$.

Recalling the definition of x_{crit} from Proposition 1, we make the following assumptions:

Assumption 2.

- (i) $\underline{x} < x_{\text{crit}}$;
- (ii) c' is increasing and weakly convex, with $c(0) = 0$;
- (iii) the Inada conditions hold: $\lim_{y \downarrow 0} c'(y) = 0$ and $\lim_{y \uparrow 1 - \underline{x}} c'(y) = \infty$.

The first part of this assumption ensures that baseline (free) relationship strength is not so high that the supply network is guaranteed to be productive even without any investment. The second assumption imposes assumptions on investment costs that help guarantee agents' optimization problems are well-behaved. The Inada conditions, as usual, will be important in making investments interior.

Given the chosen relationship strengths x_{if} , we can define a random *realized supply network* $\mathcal{G}' \subseteq \mathcal{G}$ that consists of those links in \mathcal{G} that are operational, as introduced in Section 2. For this construction, let a link from if to a supplier jj' be operational with probability x_{if} , which is determined by the investment of the sourcing firm. These realizations are effectively²⁵ independent across links.

Given the realization of \mathcal{G}' , Section 2.3 describes which firms are capable of producing—*functional*.²⁶ We denote by $F_{if} \in \{0, 1\}$ the random (\mathcal{G}' -determined) realization of whether firm if is functional. Finally, we denote by r_i the probability that a producer of product i , selected uniformly at random among the entering producers of product i , is functional, which we call the *reliability* of that product.

This formulation assumes that a firm cannot produce anything if it fails to source any one of its inputs. This is a feature of many standard production functions. For example, all CES production functions with an elasticity of substitution less than or equal to 1 have this feature (e.g., the Leontief and Cobb-Douglas production functions).

²⁵There are some technical subtleties here arising from the continuum of agents in our model. See Appendix A.

²⁶We also give a formal definition in Appendix A.

3.3. Payoffs and equilibrium. We now turn to a firm’s payoffs, which we will specify only for symmetric behavior of other firms. Indeed, for our study of symmetric equilibria, it will suffice to examine the case where firms $[0, \bar{f}]$ enter for each product i . Recall that r_i is the probability that a firm, selected uniformly at random from producers of product i that enter, is functional. In a symmetric equilibrium all producers of all products have the same reliability. Denote this reliability by r . Then the mass of functional firms in the supply network is $\bar{f}r|\mathcal{I}|$. Conditional on being functional, we assume a firm earns a profit—gross of fixed entry costs and the cost of investment—of $G(\bar{f}r) = \kappa g(\bar{f}r)$; where $g : [0, 1] \rightarrow \mathbb{R}_+$ is a decreasing function. That is, functional firms each earn greater gross profits when the market of functional firms is less crowded. We interpret the multiplier $\kappa > 0$ as a quantity that shifts total factor productivity, and use it to explore the comparative statics of the supply network as productivity is varied. For some microfoundations giving rise to such profits, see the discussion in Section 3.4.2.

Thus, conditional on entering and making an investment y_{if} , the net expected profit of firm if is

$$\Pi_{if} = \underbrace{\mathbf{E}[F_{if}]}_{\text{prob. functional}} \underbrace{G(\bar{f}r)}_{\text{gross profit}} - \underbrace{c(y_{if})}_{\text{cost of investment}} - \underbrace{\Phi(f)}_{\text{entry cost}}. \quad (3)$$

If a firm does not enter, its net profit is 0. Note that the distribution of F_{if} depends on others’ relationship strengths (and hence investment decisions) as well as one’s own.

To summarize, the timing is:

1. Firms make their entry decisions.
2. Firms simultaneously choose their investment levels to maximize Π_{if} .
3. The realized supply network \mathcal{G}' is drawn, and payoffs are enjoyed.

A *firm outcome* is given by entry decisions e_{if} and relationship strengths x_{if} for all firms if .²⁷

Definition 1. A firm outcome is an *equilibrium* if the following conditions hold:

- Optimal investment: conditional on all entry decisions and all others’ investment decisions, each firm sets its y_{if} to maximize the expectation of its net profit Π_{if} .
- Optimal firm entry: Correctly anticipating subsequent investment decisions, and conditional on others’ entry decisions, no positive mass of firms can increase its profit by making a different entry decision.²⁸

3.4. Comments on key concepts and assumptions.

3.4.1. Interpretation of investment. Firms actively seek to mitigate supply chain disruptions in practice. For example, ethnographic work by Uzzi (1997) provides a firsthand account of the systematic efforts and investments made by New York garment manufacturers and their suppliers to maintain good relations. Such efforts make sense given that supply chain disruptions cause lasting damage to firms. Hendricks and Singhal (2003, 2005a,b) examine hundreds of supply chain

²⁷For an analysis with richer and more explicit dynamics, and the coevolution of supply network reliability and the production technology, see Brummitt, Huremović, Pin, Bonds, and Vega-Redondo (2017). A major focus there is on the best transition path to complex production.

²⁸Our entry condition is formulated in terms of a positive measure of firms. In Appendix E we tighten our equilibrium definition to require that no firms can profitably enter the market in equilibrium, and show that all our results are robust when we also introduce a competitive banking sector to finance (and hence ration) entry.

problems reported in the business press. Even minor disruptions are associated with significant and long-lasting declines in sales growth and stock returns.

Our modeling of investments into supply relationships is compatible with multiple interpretations. The first interpretation is that the set of possible suppliers is fixed, and the investment works on the intensive margin to improve the quality of these relationships (by reducing misunderstandings and so on). The second interpretation is that the investment works on the extensive margin—i.e., firms work to find a supplier willing and able to supply a given required input type, but their success is stochastic. In this interpretation there is a fixed set of n potential suppliers capable of supplying the required input to be found, and each one of them is found independently with probability x_{if} .²⁹ Conditional on being found, the relationship is operational. In Appendix C we discuss a richer extensive-margin interpretation, and also one that permits separate efforts to be directed to the extensive and intensive margins simultaneously.

3.4.2. Allocation of gross profits from production. We now briefly summarize one very simple foundation for the specification of gross profits that we have assumed, the key feature of which is that **profits fall as entry increases**. For simplicity, in this foundation we suppose firms make gross profits on their sales to consumers, but not on their sales of intermediate goods. For a variety of assumptions on market structure, sales to consumers will be at a markup, and this markup will depend on competition from other firms selling the same type of product. The key assumption is that **expected gross profits from these sales are decreasing in the number of operating firms in the same product**. In product i , this number is given by $r_i \bar{f}_i$ (where \bar{f}_i is the mass of entering producers of that product and r_i is the share of these firms that are functional; in a symmetric equilibrium these are the same for all i). In Appendix B we lay out one simple microfoundation for such a gross profit function, with firms selling differentiated goods and monopolistically competing. Many other microfoundations could produce the key comparative static.

Firms make profits on their sales to consumers, and we assume that all the rents associated with the sales of final products to consumers accrue to the producers of the final goods (rather than being distributed throughout their supply chains). This is consistent with, for example, a bargaining model in which the buyers of inputs can make take-it-or-leave-it offers to the sellers. Proceeding in this way ensures the transactions between firms are ex post efficient given the supply network of operational links; it avoids possible distortions, double marginalization, etc.—and associated strategic considerations—that could arise by taking an alternative approach to price formation. By using our specification we make the point that pricing or bargaining frictions in interactions among firms play no role in the sources of fragility we identify.

Of course, bargaining distortions in a complex supply network environment present interesting questions, and our model provides a framework to consider these. We leave these questions for future work but we do present a robustness check that shows our main conclusions would have a manifestation in such richer environments: In Appendix F we demonstrate that even allowing substantial ex post heterogeneity in both network position and bargaining power among firms, the key results on fragility survive.³⁰

²⁹This search technology is similar to ball-and-urn models of search, and is compatible with a matching function exhibiting constant returns to scale (see, for example, Hall (1979)).

³⁰See also Section 6 on the mechanics of heterogeneous supply networks.

4. EQUILIBRIUM SUPPLY NETWORKS AND THEIR FRAGILITY

We now study the equilibrium of our model: its productivity and its robustness. This section builds up to a main result: Theorem 1. Recalling that we parameterize gross profits as $G(\bar{f}r) = \kappa g(\bar{f}r)$, we show that as we increase κ , there are three regimes: unproductive, where positive investment cannot be sustained; critical, where equilibrium investment is x_{crit} and arbitrarily small shocks to relationship strength lead to discontinuous drops in production; and noncritical, where equilibrium investment is above x_{crit} and the economy is robust to small shocks. These regimes occur in that order, and we describe the cutoffs governing the transitions.

In order to establish this main result, we break the equilibrium problem down into two steps. First we study firms' equilibrium investment decisions conditional on a given level of entry. This lets us describe equilibrium relationship strengths. Then, building on that, we study equilibrium entry decisions and how they depend on the parameters of the economy.

4.1. Symmetric investment equilibria for a given level of entry. Suppose a measure \bar{f} of the producers of each product $i \in \mathcal{I}$ enter; such entry decisions will be the relevant ones for our further analysis. We will now analyze investment decisions in this situation for any \bar{f} : essentially the subgame at stage 2 in our timing described in Section 3.3. We focus on *symmetric investment equilibria*.

To define these, it is helpful to write a more explicit profit function: suppose all entering firms choose relationship strength x . We often refer to firms choosing strength $x_{if} \geq \underline{x}$ directly and paying the corresponding investment cost $c(x_{if} - \underline{x})$. Then firm if 's expected profit conditional on entering the market and choosing strength x_{if} is:

$$\Pi_{if}(x_{if}; x, \bar{f}) = \underbrace{\kappa g(\bar{f}\rho(x))}_{G(\bar{f}\rho(x))} \underbrace{(1 - (1 - x_{if}\rho(x))^n)^m}_{\mathbf{E}[F_{if}]} - c(x_{if} - \underline{x}) - \Phi(f), \quad (4)$$

where $\rho(x)$ is the reliability of any of a firm's suppliers when all firms in the supply network have relationship strength x .³¹ This comes from applying (3) and using the same reasoning as in Section 2 to give an explicit formula for $\mathbf{E}[F_{if}]$. We write $P(x_{if}; r)$ for $\mathbf{E}[F_{if}]$ when other firms' reliability is r .

Definition 2. We say $x \geq \underline{x}$ is a *symmetric investment equilibrium* for \bar{f} if $x_{if} = x$ maximizes $\Pi_{if}(x_{if}; x, \bar{f})$.³²

Note that a symmetric investment equilibrium is defined by the level of relationship strength x realized in it, rather than the level of investment. This turns out to be more convenient.

Because firms symmetrically choosing investments $y_{if} = 0$ results in a reliability of $\rho(\underline{x}) = 0$ (using, first, the fact that they then obtain relationship strengths $\underline{x} < x_{\text{crit}}$ by Assumption 2, and second, the characterization of ρ in Proposition 1), there is always a symmetric investment equilibrium (in fact a zero-investment equilibrium) with relationship strengths \underline{x} . Our next proposition says that either (i) this is the only possibility for a symmetric investment equilibrium, or (ii) there

³¹Note that because there is a continuum of firms the probability that a firm appears in its potential supply network upstream of itself is 0. Thus the reliability of if 's suppliers does not depend on x_{if} .

³²Formally, the maximization problem that a firm if is solving depends on f through $\Phi(f)$. However, by this time entry costs have been sunk and the value of f does not matter.

is, in addition to \underline{x} , at most one other symmetric investment equilibrium with relationship strengths $x > \underline{x}$. The proposition will also describe how the equilibrium relationship strength depends on \bar{f} .

Before stating the result, we deal with some technicalities. First note that when production is complex ($m \geq 2$), if there is no scope for multisourcing ($n = 1$) then $x_{\text{crit}} = 1$. By the Inada condition on the investment cost function (Assumption 2, part (iii)) there cannot then be a symmetric investment equilibrium in which there is a positive probability of successful production. We thus focus on the case in which $n \geq 2$.

In analyzing positive symmetric investment equilibria, it is helpful to make an assumption on the environment that ensures that local optimality implies global optimality.

Assumption 3. For any $x \geq x_{\text{crit}}$ and \bar{f} , the function $\Pi_{if}(x_{if}; x, \bar{f})$ has a unique interior local maximum.³³

Assumptions 1–3 will be maintained in the sequel. We now give a simple condition on primitives that is sufficient for Assumption 3 to hold.

Lemma 1. For any $m \geq 2$, $n \geq 2$, there is a number³⁴ \hat{x} , depending only on m and n , such that (i) $\hat{x} < x_{\text{crit}}$; and (ii) if $\underline{x} \geq \hat{x}$, then Assumption 3 is satisfied.

Consider any environment where $\underline{x} \in [\hat{x}, x_{\text{crit}})$. Part (i) of the lemma guarantees that the interval $[\hat{x}, x_{\text{crit}})$ is nonempty, and part (ii) guarantees that Assumption 3 is satisfied. For any such \underline{x} , a symmetric zero-investment equilibrium continues to exist. However, as we have said, there may also be a positive symmetric investment equilibrium; in this case, Assumption 3 guarantees that any such equilibrium must satisfy a simple first-order condition.³⁵

We now characterize the behavior of positive investment equilibria.³⁶

Proposition 3. Fix any $n \geq 2$ and $m \geq 3$, and any κ and g consistent with the maintained assumptions. Then there is a unique function $x^*(\bar{f})$ such that:

- (1) if $x^*(\bar{f}) > \underline{x}$, then this value is the unique symmetric investment equilibrium with positive investment;
- (2) if $x^*(\bar{f}) = \underline{x}$, the only symmetric investment equilibrium is \underline{x} .

There is a value $\bar{f}_{\text{crit}} \in [0, 1]$ (depending on g , κ , n , and m) such that, for all $\bar{f} \leq \bar{f}_{\text{crit}}$, the investment $x^*(\bar{f}) \geq x_{\text{crit}}$ and is strictly decreasing in \bar{f} , while for all $\bar{f} > \bar{f}_{\text{crit}}$ we have $x^*(\bar{f}) = \underline{x}$.

Proposition 3 shows that either there is no positive symmetric investment equilibrium, or else such an equilibrium is unique. Further, the function $x^*(\bar{f})$ gives the highest possible value of relationship strength that can be realized in a symmetric investment equilibrium (for a given entry level \bar{f}). The function $x^*(\bar{f})$ is decreasing in the mass of firms that enter (\bar{f}) and drops discontinuously to \underline{x} (i.e., zero investment for all firms) when entry is above a threshold \bar{f}_{crit} as illustrated in Figure 9.

³³The assumption permits another local maximum at a corner. We rule this out separately.

³⁴In the proof, we give an explicit description of \hat{x} in terms of the shape of the function $P(x_{if}; r)$.

³⁵While conditions we identify in Lemma 1 are sufficient for satisfying Assumption 3, as we will see later (for example, in Figure 12) they are not necessary.

³⁶Throughout this section, we restrict attention to the case of $m \geq 3$. It is essential for our results that supply networks are complex and $m \geq 2$, but the case of $m = 2$ generates some technical difficulties for our proof technique so we consider $m \geq 3$. In numerical exercises, our conclusions seem to also hold for $m = 2$.

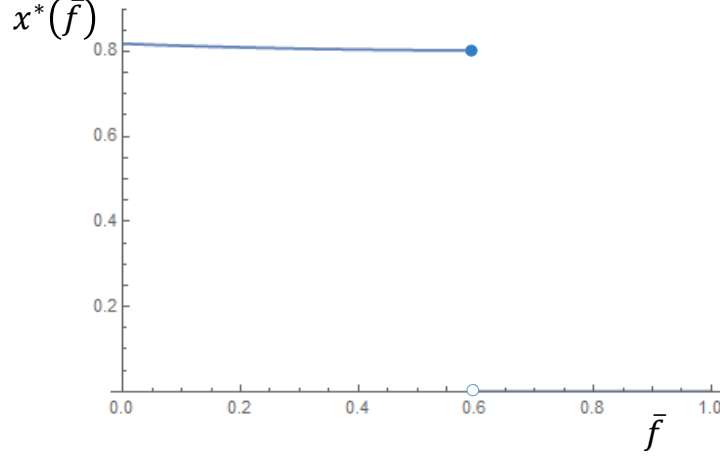


FIGURE 9. The relationship strength that occurs in a symmetric investment equilibrium is plotted as entry \bar{f} is varied. This is illustrated for $m = 5$, $n = 3$, $\underline{x} = 0$, $c(x_{if} - \underline{x}) = x_{if}^2$, and $G(\bar{f}\rho(x)) = 2(1 - \frac{\bar{f}\rho(x)}{2})$.

Establishing this result involves several challenges, which we explain before outlining the argument. First, the reliability of supply relationships at a symmetric investment level is only known to satisfy a certain fixed point condition. Second, investment incentives are nuanced, and non-monotonic. When a firm's suppliers are very unreliable there is little incentive to invest in stronger relationships with them—there is no point in having a working pipe when the suppliers have nothing to put in it. On the other hand, when a firm's suppliers are extremely reliable, a firm can free-ride on this reliability and invest relatively little in its pipes, knowing that as long as it has one working pipe for each input it requires, it is very likely to be able to source the inputs it requires. Investments in relationships strength are strategic complements in some regions of the parameter space, and strategic substitutes in others. Proposition 3 shows that, nevertheless, equilibrium investment has nice uniqueness and monotonicity properties.

The complete proof of Proposition 3 is in the appendix. We give the main ideas here. First, if the relationship strengths of all firms are symmetric (whether in or out of equilibrium), the investment problem each firm faces is identical, because—generalizing the example of Section 2—the supply network upstream of any firm looks identical (up to product labels that do not matter for reliability).

A physical consistency condition generalizing the fixed-point equation (1) from our example characterizes reliability for any symmetric level of investment. To make an outcome a symmetric investment equilibrium, a second condition must hold: firms' choices of relationship strength, x_{if} , must be optimal given the reliability of each of their suppliers. As there are a continuum of firms, firm if appears upstream in its own supply chain with probability 0, and so i 's investment choice x_{if} has no impact on the reliability of its suppliers. Given this, and that the same investment choice x is made by the other firms, the reliability of each of if 's suppliers is $\rho(x)$, which firm if takes as given. These conditions—physical consistency a best-response equation for choosing one's own relationship strength—together determine symmetric investment equilibria, and allow us to characterize when there is one with positive investment. We now expand on each of these.

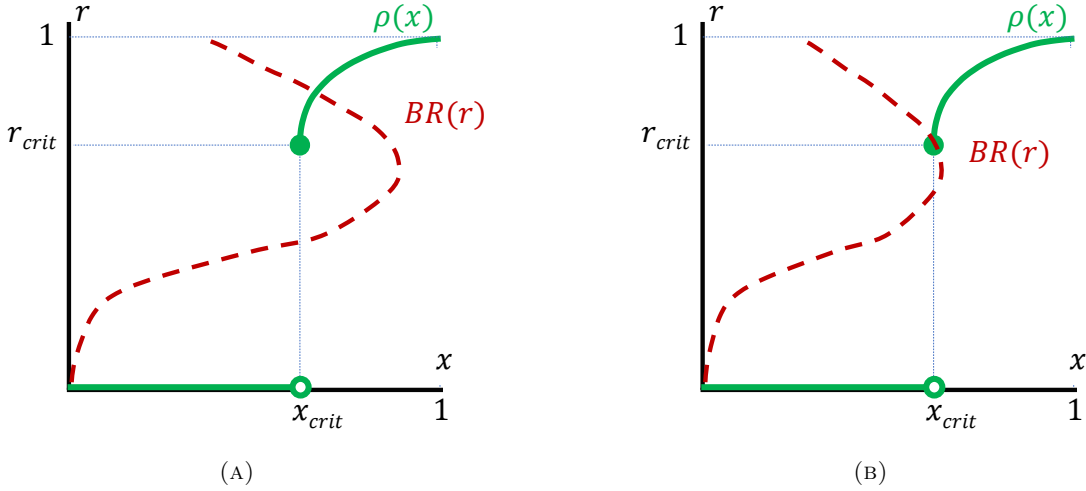


FIGURE 10. Panel (a) shows an investment equilibrium with $x_* > x_{crit}$. Panel (b) shows an investment equilibrium with $x_* = x_{crit}$.

4.1.1. *Physical consistency.* Suppose the strength of all relationships is x . From Proposition 1 and the argument we gave accompanying it, the *physical consistency* of the relationship strengths with the reliability r requires r to satisfy the equation

$$r = (1 - \underbrace{(1 - xr)^n}_{\text{probability a given input cannot be acquired}})^m, \quad (\text{PC})$$

and, moreover, to be the *largest* solution to this equation for the given x (recall Section 2.3). We denote this solution by $\rho(x)$. For a sketch of (PC), see the solid function in Figure 10(a).

4.1.2. *Best-response investment.* In a positive symmetric investment equilibrium, each entering firm chooses a relationship strength x_{if} that maximizes its profits, given the relationship strength x (or, equivalently, the reliability) chosen by other firms. Let $BR(r; \bar{f}, \kappa)$ be the set of best-response values of x_{if} when other firms have (symmetric) reliability r and entry cutoff \bar{f} , and when the productivity parameter is κ .³⁷

Recall from (2) the profit function $\Pi_{if}(x_{if}; x, \bar{f})$. Setting its derivative in x_{if} to zero, and $r = \rho(x)$ gives the first-order condition

$$\underbrace{G(r\bar{f})}_{\text{profits given successful production}} = \underbrace{\frac{\partial P(x_{if}; r)}{\partial x_{if}}}_{\text{marginal increase in probability } if \text{ can produce}} = \underbrace{c'(x_{if} - \underline{x})}_{\text{marginal cost}}. \quad (\text{OI})$$

Our assumptions guarantee that the first-order conditions are necessary and sufficient for global optimality of x_{if} as a response to the others' behavior. The condition (OI) implicitly defines the best response correspondence $BR(r)$.

It is easy to write out $\partial P(x_{if}; r)/\partial x_{if}$, the derivative, in x_{if} of the probability that firm if successfully produces.³⁸ The (gross) profits that if receives conditional on successful production,

³⁷We will sometimes drop arguments after the first on this function and related ones.

³⁸Specifically, $\frac{\partial P(x_{if}; r)}{\partial x_{if}} = rn(1 - x_{if}r)^{n-1}m(1 - (1 - x_{if}r)^n)^{m-1}$.

$G(r\bar{f})$, do not vary with if 's investment. Finally, $c'(x_{if} - \underline{x})$ is the strictly increasing marginal cost that firm if faces to increase its relationship strength x_{if} .

The following definitions will be helpful.

$$MB(x_{if}; r, \bar{f}, \kappa) = G(r\bar{f}) \frac{\partial P(x_{if}; r)}{\partial x_{if}} \quad (5)$$

$$MC(x_{if}) = c'(x_{if} - \underline{x}) \quad (6)$$

Recall that G also depends on κ .

4.1.3. Combining physical consistency and optimal investment. We can depict both (PC) and (OI) on the same plot, as shown in Figure 10 for a particular example. The physical consistency condition (PC)—the solid function, $\rho(x)$ —is determined only by the parameters n and m . The dashed function depicts the best-response function $BR(r)$ (where we view the vertical axis as the independent variable). A symmetric investment equilibrium (x_*, r_*) must correspond to an intersection of the two functions: the investment level x_* which if chosen by everyone generates the corresponding $r_* = \rho(x_*)$ through the mechanics of reliability; and, moreover, that x_* is a best-response to symmetric reliability r_* .

Proposition 3 asserts that there is at most one intersection of these functions. Moreover, it asserts that as we increase \bar{f} , the point of intersection decreases. At some point, when $\bar{f} = \bar{f}_{\text{crit}}$ the point of intersection is just at $(x_{\text{crit}}, r_{\text{crit}})$ and, for \bar{f} higher than that, there is no positive investment equilibrium.

These conclusions are far from immediate. First, we do not know very much about the shape of $BR(r)$. We do not know, for instance, that it is concave or even single-valued. Moreover, as we already mentioned just before Section 4.1.1, investment choices are neither strategic substitutes nor strategic complements. Nevertheless, in the proof of Proposition 3 we can establish that there is at most one intersection between the BR and ρ functions. This is the key fact. It can easily be seen that when \bar{f} increases, the BR curve shifts leftward. Intuitively, as entry increases profit margins decline and firms' incentives to invest in their supply relationships are diminished, all else equal. From this and uniqueness of the intersection (when one exists) it follows that this intersection (continuously) moves leftward as \bar{f} increases. Finally, the proof shows that there continues to be an intersection until the critical point $(x_{\text{crit}}, r_{\text{crit}})$ is reached, and then there is no intersection and no positive investment equilibrium.

Proving that the function (OI) and (PC) have the required properties to interact as we have described is fairly involved. Note that PC is a fixed point condition, while OI only implicitly defines the best response function. Indeed, there is no way we know of to argue directly from what we know of the shape of the (PC) and (OI) functions that they have a unique intersection. Instead, the proof makes progress by transforming the problem into one of characterizing the solution of an equation whose behavior is better understood. This is carried out in the appendix.

4.2. Two possible types of positive symmetric equilibria. We now close the model by making the entry cutoff endogenous. To study firm behavior, first note that optimal behavior by firms entails monotonicity of entry strategies: if firm if finds it profitable to enter given others' strategies, any firm $f' < f$ finds it strictly profitable (note that as firms are zero measure a firm's entry decision has no impact on the aggregate environment). Thus, without loss, in any equilibrium entry decisions

can be summarized by a cutoff \bar{f}_i^* for each product i . A symmetric equilibrium is associated with a pair $(\bar{f}^*, x^*(\bar{f}^*))$, where \bar{f}^* is the entry cutoff for producers of every product and $x^*(\bar{f}^*)$ is the equilibrium relationship strength. Indeed, recall by Proposition 3, when a measure \bar{f} of the producers of each product enter, there is at most one positive symmetric investment equilibrium $x^*(\bar{f})$. We call an equilibrium $(\bar{f}^*, x^*(\bar{f}^*))$ positive if $\bar{f}^* > 0$ and $x^*(\bar{f}^*) > \underline{x}$. We will show that there is at most one positive symmetric equilibrium, and that it can take one of two forms.

To simplify the analysis we make a further assumption, guaranteeing that there is not a corner solution to the entry problem in which all firms enter:

Assumption 4. $\Phi(1) > G(\rho(x^*(1)))\rho(x^*(1)) - c(x^*(1) - \underline{x})$.

Assumption 4 guarantees that if all firms enter and play the investment equilibrium $x^*(1)$ (corresponding to all firms entering), then the highest-cost firm of those that enter makes a loss.

Proposition 4. Fix any $n \geq 2$ and $m \geq 3$, and any g consistent with the maintained assumptions. Then:

- (i) For any κ , there is at most one positive symmetric equilibrium.
- (ii) If there is a positive symmetric equilibrium, then, for generic κ , it takes one of two forms:
 - $x^*(\bar{f}^*) = x_{\text{crit}}$ and $\Pi_{if} > 0$ for all entering firms; we call this a *critical equilibrium*.
 - $x^*(\bar{f}^*) > x_{\text{crit}}$ and marginal firms $i\bar{f}^*$ make zero profits ($\Pi_{if^*} = 0$), for all $i \in \mathcal{I}$; we call this a *noncritical equilibrium*.

Part (i) of Proposition 4 shows the uniqueness of a positive symmetric equilibrium when one exists. Part (ii) of Proposition 4 shows that when a positive symmetric equilibrium exists, generically we are in one of two regimes—either equilibrium relationship strength is at the critical level and entry is restricted such that all firms make positive profits, or equilibrium relationship strength is above the critical level and all firms that could make a positive profit by entering do.³⁹

Proposition 4 is silent on existence of the positive symmetric equilibrium. We show when such equilibria exist, and show which regime we will be in and when, in Theorem 1. The critical regime turns out *not* to be a knife-edge case.

We defer the proof of Proposition 4 to the appendix. To gain some intuition suppose we start from an (out of equilibrium) entry level \bar{f} in each market, such that all entering firms make strictly positive profits and relationship strength is above the critical level (i.e., $x^*(\bar{f}) > x_{\text{crit}}$). Some firms not entering the market are then able to enter the market and make strictly positive profits. As additional firms enter the market the incentives of firms to invest in reliability diminish ($x^*(\bar{f})$ is decreasing in \bar{f}). There are then two possibilities. First, entry may increase until a zero profit condition is satisfied. In this case we end up with an equilibrium $x^*(\bar{f}^*) > x_{\text{crit}}$ —what we term non-critical. Alternatively, before a zero profit condition is reached, we may increase entry to the point where firms would choose a relationship strength $x^*(\bar{f}^*) = x_{\text{crit}}$. This is then an equilibrium. Even though the marginal entering firm makes a strictly positive profit, there is no positive mass of

³⁹There is an additional case that occurs for a non-generic κ . At a specific level of κ we transition between the two regimes and the marginal firm makes zero profits *and* relationship strength is at the critical level.

non-entering firms that can enter because they tip the system over the precipice, causing production to fail and rendering these entry choices unprofitable.⁴⁰

The proposition says that only one of these two situations can occur for a given set of parameter values, and thus the environment uniquely determines the type of positive equilibrium that occurs, when one exists.

4.3. Dependence on productivity parameter κ : an ordering of regimes. Critical equilibria are important because they create the possibility of fragility: small shocks to relationship strengths via a reduction in \underline{x} can result in a collapse of production. We show now, that when a productivity parameter κ is below a certain threshold, there is no equilibrium with positive production. However, for κ above this threshold and up to another strictly higher threshold, there is a critical positive equilibrium as described in Proposition 4. In such an equilibrium, all firms make positive profits and relationship strength is at the critical level. Finally, for κ above the second threshold, there is a non-critical positive equilibrium, where the marginal firm entering the market makes zero profits.

Theorem 1. Fix any $n \geq 2$ and $m \geq 3$, and any g consistent with the maintained assumptions. There exist thresholds $\underline{\kappa}, \bar{\kappa}$ (which depend on n and m) satisfying $0 < \underline{\kappa} < \bar{\kappa} < \infty$ such that

- (i) For $\kappa \leq \underline{\kappa}$, no firms enter in equilibrium: $\bar{f}^* = 0$.
- (ii) For $\kappa \in (\underline{\kappa}, \bar{\kappa}]$: a positive fraction of firms enter, so that $\bar{f}^* > 0$. The level of investment is critical, i.e., $x^*(\bar{f}^*) = x_{\text{crit}}$, and, for $\kappa \neq \bar{\kappa}$, the profit $\Pi_{if} > 0$ for all entering firms.
- (iii) For $\kappa > \bar{\kappa}$: a positive fraction of firms enter, so that $\bar{f}^* > 0$. The level of investment is above the critical level, i.e. $x^*(\bar{f}^*) > x_{\text{crit}}$, and $\Pi_{i\bar{f}^*} = 0$ for all $i \in \mathcal{I}$.

To bring out the key point in this theorem, fix the complexity m and multisourcing number n and suppose that we have two different supply networks, with respective productivity parameters $\kappa \neq \kappa'$. With exogenous entry and equilibrium investment, it would be very unlikely that either of them is at a precipice, and impossible for both to be there.⁴¹ However, if entry is endogenous, the theorem states that as long as both κ and κ' are in the intermediate range $(\underline{\kappa}, \bar{\kappa}]$, then both of them will be at the precipice. Even with different n, m parameters, it is possible, and in fact not a knife-edge circumstance, for both supply networks to be at their respective precipices in equilibrium.⁴² In short, very different supply chains settle, in equilibrium, on their precipices, as long as their κ 's fall in the intermediate range.

Theorem 1 shows that there exists an equilibrium with positive production if and only if κ is above a threshold that we call $\underline{\kappa}$. It also describes how the unique productive equilibrium changes as κ increases beyond this threshold. In Figure 11 we illustrate these changes in more detail with an example.

While the proof of Theorem 1 is relegated to the appendix, we now provide some intuition. When κ is low—less than some cutoff $\underline{\kappa}$ —the markup $\kappa g(r_{\text{crit}}f)$ is sufficiently small (for all f) that even if all other entering firms have reliability r_{crit} , an entering firm if would have a best-response investment level $x_{if} < x_{\text{crit}}$. In other words, all firms want to free-ride so much that positive

⁴⁰Recall that our equilibrium definition just requires that a positive mass of firms cannot enter profitably. We show in Appendix E that we can tighten our equilibrium definition to require that no firms can profitably enter the market in equilibrium without affecting our results, by adding a competitive banking sector to the model.

⁴¹This is because, with positive reliability, equilibrium investment is strictly increasing in κ .

⁴²This occurs as long as κ and κ' are both in the intermediate, fragile intervals defined for their m and n parameters.

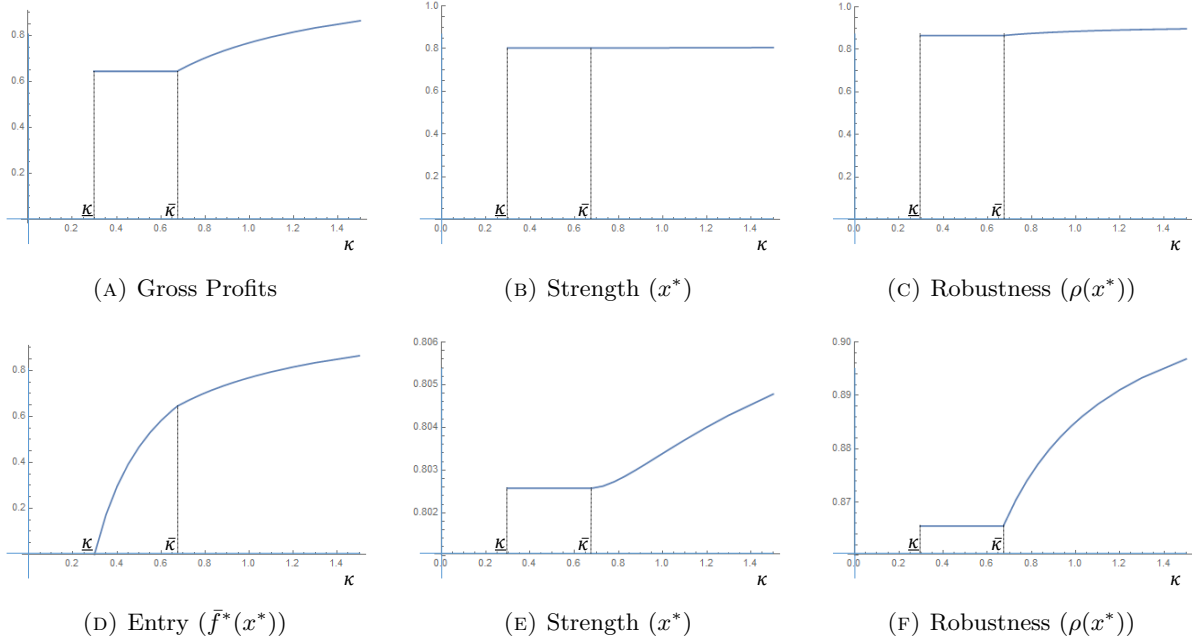


FIGURE 11. Equilibrium values of different variables as κ changes for the case of $m = 5$, $n = 3$, $g(\bar{f}\rho(x)) = 5(1 - \bar{f}\rho(x))$, $c(x_{if} - \underline{x}) = (x_{if} - \underline{x})^2$ and firms of each type distributed uniformly over the unit interval. For simplicity, we set $\underline{x} = 0$. Panel (A) considers the gross profits that entering firms obtain. Panels (B) and (E) show how equilibrium relationship strength changes (with different scales for the vertical axis). Panels (C) and (F) show how the equilibrium probability of successful production changes (also with different scales for the vertical axis). Finally, Panel (D) reports how equilibrium entry changes.

production is impossible. In contrast, when $\kappa = \underline{\kappa}$ this changes. For the first time, an entering firm if with $f = 0$ best-responds to others choosing investments x_{crit} by also choosing $x_{if} = x_{\text{crit}}$. This is shown in Figure 12, which plots the gross profits of firm if for different values of x_{if} and a markup $\underline{\kappa}g(0) = \max_f \underline{\kappa}g(r_{\text{crit}}f)$. Given these investments, production is successful with probability r_{crit} and the entering firms receive strictly positive gross profits.⁴³ It may seem a bit puzzling that these profits are strictly positive despite the fact that entering firms are indifferent to making investments x_{crit} ; the reason is that entering firms benefit from the positive externalities of other firms' supply chain investments.

With $\kappa = \underline{\kappa}$, the firms at $f = 0$ —since they have no entry costs—find it strictly profitable to enter. But then—holding investment fixed—so will a positive mass of other firms with sufficiently low entry costs. However, were a positive mass of such firms to also enter, markups would decline (recall that $g(\bar{f}\rho(x))$ is strictly decreasing in \bar{f}). This in turn would reduce the incentives for firms to invest in their supply network robustness and now the entering firms would no longer be willing to make the investments required to form a functioning supply network. (The discontinuity is key to this: it is not possible to sustain production with a slightly smaller investment, so this small diminution in incentives results in a large drop in production.) As a result, it is actually not profitable for any positive mass of firms to enter, and the only equilibrium involves an entry threshold at $f = 0$.

⁴³Even though all firms entering have $f = 0$ at this nongeneric value of κ , one can still take there to be a continuum of them.

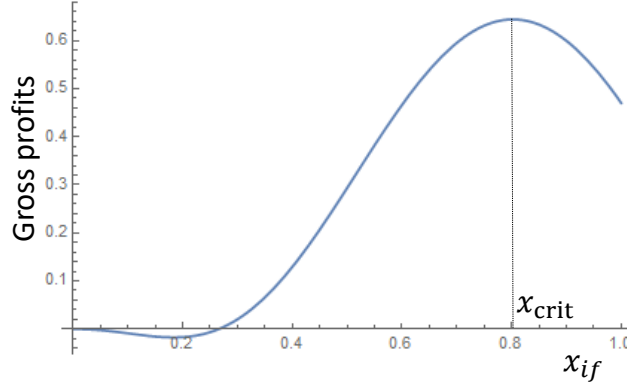


FIGURE 12. The gross profits that firm if can achieve by choosing different relationship strength x_{if} , when all other firms have relationship strength x_{crit} and $\kappa = \underline{\kappa}$.

As κ increases above $\underline{\kappa}$, the incentives to invest in supply chain robustness improve. If entry were held fixed, we would then have an investment equilibrium $x^* > x_{crit}$. However, the increase in κ allows more firms to enter without tipping the supply network over the edge of the precipice. This continues until firms have relationship strengths x_{crit} . An illustration of this is shown in Panels (B) and (E) of Figure 11: for the values of κ between $\underline{\kappa}$ and $\bar{\kappa}$, equilibrium relationship strength remains constant at x_{crit} . Likewise, as shown in Panels (C) and (F), the probability of successful production remains fixed at r_{crit} for these values of κ .

On the one hand, as κ increases, the profits of those firms already in the market directly increase. On the other hand, increased entry reduces profits. These two effects must exactly offset each other. Entry must occur up until the point where firms have exactly the same investment incentives as before (and so again choose relationship strength x_{crit}), and, at this point, these firms must receive exactly the same mark-up as before. Note that although all entering firms receive this constant level of gross profits as κ increases, the marginal entering firm receives ever lower *net* profits (because it has a higher entry cost). This can be seen by comparing panels (A) and (D) in Figure 11.

When $\kappa = \bar{\kappa}$, the marginal entering firm receives zero profits. From this point forward the equilibrium regime changes. As κ increases further, entry increases, but only until the zero profit condition is satisfied. This additional entry is now insufficient to keep gross profits constant, and gross profits increase. This increases investment incentives for the entering firms and investment now increases above x_{crit} .

4.4. Criticality and fragility. We now formalize the idea that supply networks in the critical regime will be fragile. We do so by explicitly examining how the supply network responds to a shock to the baseline quality of institutions \underline{x} , which for simplicity are taken to have zero probability (though the analysis is robust to anticipated shocks that happen with sufficiently small probability).

Definition 3 (Equilibrium fragility).

- A productive equilibrium⁴⁴ with investments y^* is *fragile* if *any* negative institutional shock, such that \underline{x} decreases to $\underline{x} - \epsilon$ for $\epsilon > 0$, results in output falling to 0 (i.e., $\rho(\underline{x} - \epsilon + y^*) = 0$).
- A productive equilibrium is *robust*, if it is not *fragile*.

⁴⁴By this we mean an equilibrium in which a positive mass of firms enter and are functional with positive probability.

In a fragile equilibrium we hold firms' investment decisions and entry choices fixed. Implicitly, we are assuming that investments in supply relationships and entry decisions are made over a sufficiently long time frame that firms cannot change the quality of their supply relationships or their entry decisions in response to a shock.

Proposition 5. If $\kappa \leq \bar{\kappa}$, then any productive equilibrium is fragile and if $\kappa > \bar{\kappa}$, then any productive equilibrium is robust.

Proposition 5 follows immediately from the definition of a fragile equilibrium.⁴⁵

4.4.1. *Interpretation.* There are many ways in which the small shock discussed in this section might arise in practice. Recall the example in the introduction. A small shock to credit markets makes many supply relationships slightly more likely to fail—in the states of the world where they happen to require short-term financing to deal with a disruption. Another, similar, example is uncertainty about which relationships will be affected by possible compliance issues related to new trade regulations, thereby making supply relationships more prone to disruption. For a third example, consider an increasing backlog in commercial courts—a circumstance that makes contracts more costly to enforce. This again decreases the probability that contracts function in some states of the world—an uncertainty that can affect many players in the supply chain. For a final example, suppose there is a new pandemic outbreak of a disease. This will likely affect some supply relationships, but not others; uncertainty around this can be captured by a small reduction in the probability that a given supply relationship is disrupted.

4.5. **Robustness of results.** We now discuss the robustness of our results along several dimensions.

4.5.1. *Varying costs rather than gross profits.* While we focus on a total productivity shifter (κ), the methods also imply analogous comparative statics for other exercises of interest. For example, we could posit that

$$\Pi_{if}(x_{if}; x, \bar{f}) = G(\bar{f}\rho(x)) \underbrace{(1 - (1 - x_{if}\rho(x))^n)^m}_{P(x_{if}; \rho(x))} - \frac{1}{\kappa} c(x_{if} - \underline{x}) - \Phi(f), \quad (7)$$

so that the parameter we are varying affects the costs of obtaining a certain level of relationship strength. If we want to consider how a given supply chain will change as an institution supporting contracting improves—for example, the functioning of commercial courts—this parameterization is the more appropriate one. With some adjustments, the arguments we have made would establish the ordering of regimes discussed in Theorem 1, with the same implications for fragility.

4.5.2. *Finite production processes.* In this section we discuss how to interpret our model in terms of real-life production processes. A natural interpretation suggested by some of our examples is that physical goods are assembled in stages. Here it is natural to take the number of stages to be finite. In this subsection, we discuss robustness checks showing that our main findings hold in a model with a finite number of production stages. We also contrast this perspective with a fixed-point

⁴⁵In Appendix G we show that even if investments can adjust in response to a shock, an equilibrium is fragile if $\kappa \leq \bar{\kappa}$ and robust if $\kappa > \bar{\kappa}$.

interpretation of interdependencies (which has been our main focus so far), and explain when the latter is realistic.

Suppose that a good (such as the A380 airliner) is being assembled in stages. Suppose also that the interdependencies in our model describe only requirements for physical inputs that are incorporated into increasingly complex products. Thus, the final assembly combines large parts (engines, fuselage); the tier upstream of that makes parts such as the engine out of smaller parts, etc. For this type of process, we can think of the diagram of Figure 2 as depicting the structure of a supply chain, and is realistic to take the number of tiers involved, which we will call T , to be some finite number. In the T -tier model, production works as we have described at all tiers requiring inputs, but the most upstream tier does not require any specific sourcing: it can produce using commodity inputs alone. We analyze such processes in Appendix SA1. We first show that as T grows large, reliability $\rho_T(x)$ depends continuously on x ; as $T \rightarrow \infty$, the function and its derivatives converge to the discontinuous function we have studied. Indeed, the shape is quite similar even for quite moderate T . For example, if $m = 5$, $n = 4$, and $T = 7$, then a decrease in the investment level from $x = 0.66$ to $x = 0.61$ causes the probability of successful production to drop from over 80% to below 10% (see Figure 13). For higher complexity m , the convergence (as we increase T) is even faster.

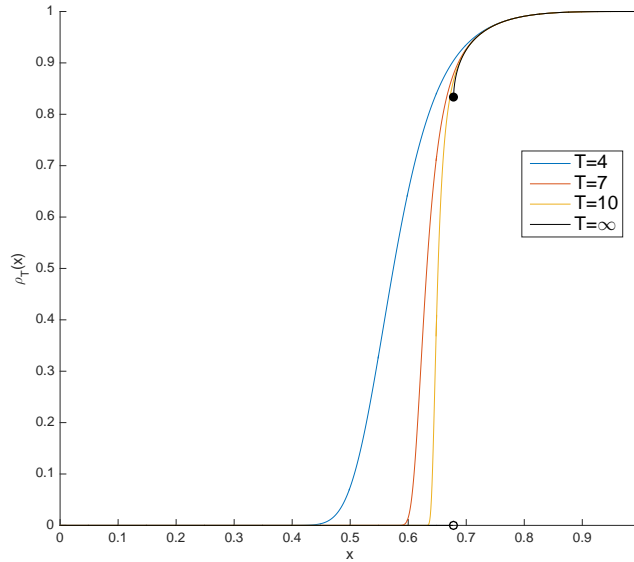


FIGURE 13. Successful production probability $\rho_T(x)$ for different finite numbers of tiers T ($m = 5$ and $n = 4$).

In the T -tier version of the model, we can also study endogenous investment. The main finding there is that, as $T \rightarrow \infty$, equilibrium investments x^* converge to the equilibrium investment levels we study in our benchmark model, and arbitrarily small shocks can have arbitrarily large effects if the economy is sufficiently (but finitely) complex. We conclude that our main economic insights will continue to hold with complex but finite supply trees.

However, for some kinds of interdependencies, the literal interpretation of tiered assembly doesn't apply and the baseline model from Section 3, with a fixed point interpretation (and no analogue

of T), is a better fit. Which interpretation is more appropriate will depend on the application; our main findings hold either way. For an example where a static perspective is more appropriate, consider, for example firms providing specialized auditing services (product i). In order to operate, each such firm contracts with specialized legal firms to procure a particular kind of tax filing (product⁴⁶ j). In turn, the law firms that produces the tax filings j contracts with accounting firms producing i for their compliance needs. If an i -service is unavailable to a j -firm in a given quarter, then the firm cannot operate, and vice versa. Note that an auditing report produced by an i firm is *not* a physical complex good being literally assembled in stages (using a certain amount of tax filings, which then use a certain amount of audits etc.). There is no close analogy with the airplane example, nor would it make sense to ask how many “stages” this process takes in order to produce an audit report. Nevertheless, the operation of some j firms that supply i (along with other suppliers) is necessary for the operation of i , and vice versa. Thus, the solution of the model comes down to asking what share of firms can operate in a consistent way. This is the fixed point problem we have been studying. Indeed, it is analogous to debt-clearing equilibria in models such as Eisenberg and Noe (2001) or Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015); as in those cases, there is no dynamic inherent in the solution.⁴⁷

4.5.3. *Heterogeneity.* We show in Section 6, supported by Appendix SA2, that our key economic insights also continue to hold when we introduce heterogeneity in many dimensions. In this section we: (i) permit different products to require different numbers of inputs; (ii) allow the number of potential suppliers to differ by the input being sourced; (iii) accommodate product specific mark-ups and entry costs; (iv) allow the cost of achieving a given strength of supply relationship to differ depending on the product being sourced; and (v) allow firms to make different investments into sourcing different inputs. We find that production exhibits the same discontinuities and solve two examples numerically in which the production of some but not all products occurs in the critical regime. Finally, using insights from these examples, we show that there is a *weakest link* phenomenon that holds between certain products that are intertwined in the supply network. Such products are either all in the critical regime or else none of them are.

5. WELFARE IMPLICATIONS

We now briefly discuss inefficiencies and welfare considerations within our model. We begin with some thought experiments about the planner intervening on only one margin (investment given entry, or entry only) and then consider richer policies. Throughout, we work with the microfoundation sketched in Appendix B.

A basic point is that there may be underinvestment in robustness because of positive externalities: each firm’s functionality contributes to the profits of many other firms. Any model where firms don’t fully appropriate the social surplus of their production will have this property.

To analyze this inefficiency, let us for a moment hold entry fixed to isolate effects arising from underinvestment, and study the investment equilibrium only. There are actually two effects that occur when firms become more reliable: one is the positive externality we have just described, but

⁴⁶Here we use the terms “product” and “service” interchangeably.

⁴⁷Of course, we can understand the static fixed points better by thinking about certain iterative processes or calculations, as in Section 2.3. But that is the business of the analyst, not the supply network itself.

another effect is that, because each produces with a higher probability, there is more competition in the product market, which can exert a negative pecuniary externality on other firms. However, total surplus and consumer surplus unambiguously increase, since both are increasing in total sales. So, on the whole, there is too little investment in reliability. Indeed, holding entry fixed, because of the discontinuity in ρ , the consequences of firms' failing to internalize the positive externalities can be extremely stark, yielding no production in equilibrium. Another sense in which there is too little investment is that lower x makes the supply network more susceptible to shocks, and thus more fragile.

Now we consider a separate question, about entry: starting at an equilibrium, would the planner want to drive more firms into the market, knowing that reliability will be determined in an investment equilibrium subsequently? If reliability is held *fixed*, more entry increases total surplus after entry (because more goods are produced). Moreover, the resulting social gain covers the entry cost of the new firms. This is because at the status quo, the marginal firm was just indifferent to paying its entry cost, but it was appropriating only a part of its contribution to social surplus. So far we have held reliability fixed. But of course reliability is not fixed: new entrants reduce all firms' gross profits and therefore reduce investment in reliability. This effect could overpower the first. Indeed, in parallel to what we have said above, our most distinctive observation here is that a small amount of new entry, as long as it pushes up production by a positive amount (which would be needed for the positive effect on welfare we have just discussed) can destroy the positive investment equilibrium completely, or at least damage it severely if the status quo is on the steep part of the ρ curve. So unless existing firms' gross profits can somehow be protected (which would prevent them from wanting to decrease investment), subsidizing entry can be dangerous.

We turn now to some other policy options. Given what we have said, an obvious goal for the planner to focus on is increasing investment in robustness. This is particularly valuable at a critical equilibrium, where the supply network is fragile. But, as we will now explain, the fragile regime is one in which it is hard to effect beneficial changes on this margin.

A tempting solution is to subsidize effort to increase robustness, for example making the cost of a given investment $1 - \theta$ times its original cost. More precisely, we make the gross (i.e., after-entry) profit equal to

$$\Pi_{if} = \underbrace{\mathbf{E}[F_{if}]}_{\text{prob. functional}} \underbrace{G(\bar{f}r)}_{\text{gross profit}} - \underbrace{(1 - \theta)c(y_{if})}_{\text{cost of effort}}.$$

Suppose the status quo was a critical equilibrium. It can readily be seen that a subsidy of this form cannot change the equilibrium reliability as long as the equilibrium remains critical—which it will for an open set of θ . The reason is that at a critical equilibrium, the relationship strength and reliability are at $(x_{\text{crit}}, r_{\text{crit}})$. The subsidy will increase the profitability of the marginal firm and allow the market to bear more entrants while providing sufficient incentives to keep reliability at r_{crit} . So, in the fragile regime, the reliability subsidy is entirely dissipated by increased entry.⁴⁸

Given the limitations of subsidies, a potentially appealing alternative, especially at a fragile equilibrium, is to impose a minimum required level of reliability for any firm that enters. Whether this is practical will depend heavily on the setting. If a reliability requirement above r_{crit} *can* be

⁴⁸At a non-fragile equilibrium, this strong non-responsiveness will not hold. Holding entry fixed, a reliability subsidy will increase reliability and make everyone better off. However, this will be partly undone by increased entry, which will drive down gross profits and incentives to invest.

imposed, then holding entry fixed, it will move the outcome away from the precipice. As argued above when we examined whether there is too little investment, this leads to an unambiguous improvement in total and consumer surplus. However, what if entry is now allowed to respond in a world with this new regulation? If *more* firms enter in equilibrium, then consumer and total surplus go up relative to the status quo, since both factors in the expression $r\bar{f}$ (total output) increase in equilibrium: r increases because of the new regulation, and \bar{f} increases by hypothesis. Could equilibrium entry decrease? This will not be a concern in the fragile regime: all entering firms make strictly positive profits at the status quo (by the properties of the fragile regime we have catalogued). Thus, when the equilibrium is critical, quality regulation is an unambiguous improvement both for consumers and for total surplus.

Regulating reliability directly may not be feasible: especially in complex markets with many supply relationships, most of the relevant outcomes might not be contractible or observable to an outsider. In this case, within our model, the only policy lever that remains is to regulate entry. The costs of collapse or volatility are not explicitly in our model; but if these are in the planner's objective, the planner will be willing to sacrifice some of the surplus from the market to avoid these costs. Thus, limiting entry only to firms that can earn considerable rents could be justified by macroprudential considerations.

6. HETEROGENEITY

We now study a version of our supply network with heterogeneity. We take the model as described in Section 3.1, and enrich it to accommodate asymmetries. Our analysis so far has relied on symmetry. Even in formulating the equation characterizing the reliability function, symmetry was important. This raises the possibility that it is the symmetry that is driving the discontinuities central to our analysis. In this section we show this is not the case. Paralleling the analysis of the homogeneous case, we show that the reliability of the supply network as a function of link strength is discontinuous. We then consider an analogue of the model of Section 3, with firms investing in the strength of their relationships with their suppliers. Through numerical examples, we then show that heterogeneous economies behave in a similar way to the homogeneous ones we have studied. Moreover, we show that supply networks in the presence of heterogeneity feature a *weakest link* property. When the production of one product is fragile, this makes the production of all products that rely directly or indirectly on it fragile as well.

We now formalize the different kinds of heterogeneity that we introduce. (1) The set of producers of product i , \mathcal{I}_i , is an arbitrary set with some cardinality m_i . Thus, we no longer require the input requirements to be symmetric. (2) For each product i and input product $j \in \mathcal{I}_i$, there is a number n_{ij} of potential suppliers of product j that each firm has; thus n_{ij} replaces the single multisourcing parameter n . (3) Investment and link strength are input-specific: For a firm if and an input $j \in \mathcal{I}_i$, there is a relationship strength $x_{if,j}$ which replaces the single number x_{if} . The cost of effort is $c_{ij}(x_{ij} - \underline{x}_{ij})$. (4) The gross profit conditional on producing product i is $G_i(\bar{f}_i r_i)$. Here G_i is a product-specific function (which can capture many different features of different product markets that affect their profitabilities), \bar{f}_i is the fraction of producers of product i that enter, and r_i is the reliability of producers of product i . (5) The distribution of entry costs is product-specific: there is a function Φ_i such that the cost of entry for firm if is $\Phi_i(f)$.

6.1. Discontinuities in reliability with exogenous link strength. Our first result shows that the basic physical implications of supply chain complexity are robust to heterogeneity. In the general environment we have described, there is an analogue of Proposition 1.

To formalize this we introduce just for simplicity a single parameter ξ reflecting institutional quality. We posit that $x_{if,j} = x_{ij}(\xi)$, where x_{ij} are strictly increasing, differentiable, surjective functions $[0, 1] \rightarrow [0, 1]$.

Proposition 6. Suppose that for all products i , the complexity m_i is at least 2. Moreover, suppose whenever $j \in \mathcal{I}_i$, the number n_{ij} of potential suppliers for each firm is at least 1. For any product i , the measure of the set of functional firms $\bar{\mathcal{F}}_i$, denoted by $\rho_i(\xi)$, is a nondecreasing function with the following properties.

- (1) There is a number ξ_{crit} and a vector $\mathbf{r}_{\text{crit}} > 0$ such that $\boldsymbol{\rho}$ has a discontinuity at ξ_{crit} , where it jumps from 0 to \mathbf{r}_{crit} and is strictly increasing in each component after that.
- (2) If $n_{ij} = 1$ for all i and j , we have that $\xi_{\text{crit}} = 1$; otherwise $\xi_{\text{crit}} < 1$.
- (3) If $\xi_{\text{crit}} < 1$, then as ξ approaches ξ_{crit} from above, the derivative $\rho'_i(x)$ tends to ∞ in some component.

The idea of this result is simple, and generalizes the graphical intuition of Figure 5 in Section 2.3.2. For any $\mathbf{r} \in [0, 1]^{|\mathcal{I}|}$, define $\mathcal{R}^\xi(\mathbf{r})$ to be the probability, under the parameter ξ , that a producer of product i is functional given that the reliability vector for producers of other products is given by \mathbf{r} . This can be written explicitly:

$$[\mathcal{R}^\xi(\mathbf{r})]_i = \prod_{j \in \mathcal{I}_i} [1 - (1 - r_j x_{ij}(\xi))^{n_{ij}}].$$

By the same reasoning as in Section 2, there is a pointwise largest fixed point of this function and this reflects the shares of firms that can produce each product.

Near 0, the map $\mathcal{R}^\xi : [0, 1]^n \rightarrow [0, 1]^n$ is bounded above by a quadratic function (as a consequence of $m_i \geq 2$ for all i). Therefore it cannot have any fixed points near 0. Thus, analogously to Figure 5, fixed points disappear abruptly as ξ is reduced past a critical value ξ_{crit} .

6.2. Numerical examples. We now provide two numerical examples. Details regarding how we numerically solve these examples are in Appendix SA2.

Example 1. There are seven products. Only product a is used as an input into its own production. Products a to d all use inputs from each other, products e to g all use inputs from each other, and products e to g also require product a as an input. Figure 14 shows the input dependencies between these products. We let products a and b have three potential suppliers for each of their required inputs, while products c to f have only two potential suppliers for each of their required inputs. We let the profitability of the seven products differ systematically. All else equal, product a is associated with the highest margins, then b , then c and so on. Specifically, we let $G_i(r_i \bar{f}_i) = \alpha_i(1 - r_i \bar{f}_i)$ and set

$$\alpha = [40, 30, 15, 10, 3.5, 3, 2.8].$$

We let the cost of a producer of product i from investing in supplier relationships with producers of product j be $c_{ij}(x_{ij} - \underline{x}_{ij}) = \frac{1}{2}\gamma_{ij}x_{ij}^2$ (where, for simplicity, we have set $\underline{x}_{ij} = 0$), and we set, for

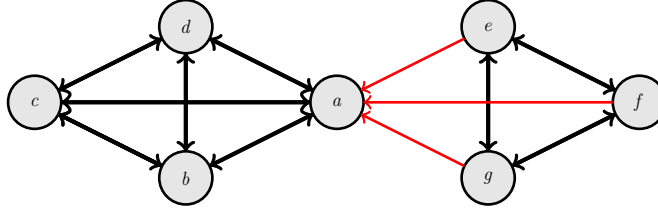


FIGURE 14. Supply dependencies: Black bold arrows represent reciprocated supply dependencies in which both products require inputs from each other. A red arrow from one product to another means that the product at the origin of the arrow uses as an input the product at the end of the arrow (e.g. product e requires product a as an input, but not the other way around). Product a also depends on itself, but the corresponding self-link is not shown.

now, $\gamma_{ij} = 1$ for all product pairs ij . Finally, we use an entry cost function $\Phi_i(f) = \beta_i f$, and set

$$\beta = [40.44, 39.85, 2.30, 2.28, 0.30, 0.40, 0.50].$$

The equilibrium investments that firms must make are pinned down by equating the marginal costs and benefits of each investment for each firm. Equilibrium entry levels are then pinned down by either a zero profit condition, or else by the requirement that investments are at the critical level. For this configuration, production of products e , f and g is critical, while production is non-critical for products a , b , c and d and entry is pinned down by a zero profit condition. We report the equilibrium levels of entry and investment, along with gross and net profits, in Appendix SA2.

We then model a small unanticipated shock to the cost of firms producing product i investing in their relationship strength with suppliers of product j , by increasing γ_{ij} from 1 to $1 + \epsilon$. After the shock to γ_{ij} , producers of product i choose to exert less effort sourcing input j . For γ_{ij} such that $i = a, b, c$ or d , the impact of this is minor. The probability of successful production for those products (r_i) only drops continuously and for a small shock the change will be small. Nevertheless, this is still sufficient for the output of the firms producing products e, f and g to collapse to 0 since they are ‘critical’ and they source (directly or indirectly) products a, b, c and d .

Similarly, if the shock occurs to the sourcing efforts of firms producing products e, f or g , output of these products will collapse to 0 since they are ‘critical’. On the other hand, output of products a, b, c and d will not be affected since these producers do not require inputs e, f or g .

Example 2. We adjust the configuration of the previous example by letting the vector of product profitabilities be

$$\alpha = [4, 5, 6, 7, 10, 15, 20],$$

and setting

$$\beta = [10.00, 4.00, 0.20, 0.20, 1.36, 1.36, 1.43].$$

Everything else, remains the same as before.

Given these parameters we numerically solve for the equilibrium (see Appendix SA2). For these parameter values production of products a, b, c and d is now critical, while production of products e, f and g is not.

Consider now a shock to γ_{ij} for $i \in \{a, b, c, d\}$. Output of the shocked product i then collapses to 0, and thus so will the output of the firms producing products $\{a, b, c, d\}$. Output of products e, f and g will then also collapse to 0 since those producers all source (directly or indirectly) inputs a, b, c and d .

Consider now a shock to γ_{ij} for $i \in \{e, f, g\}$. Output of the shocked product will adjust to accommodate the shock and the probability of successful production for the affected product will fall continuously. For a small shock, this decrease in output will be small. Products a, b, c and d will be unaffected even though they are critical, since their firms do not source products e, f and g .

Examples 1 and 2 illustrate that the relationships between the failures of production across products in the same supply network can be subtle. We now investigate this more systematically. For this section, we use $x^* = (x_1^*, x_2^*, \dots, x_{|\mathcal{I}|}^*)$ to describe equilibrium investment profiles for the different products. As before we identify the profile with the relationship strength achieved. We let $x_{-i} = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_{|\mathcal{I}|})$. Note that for this section $x_i^* = \{x_{i,j}^*\}_{j \in \mathcal{I}_i}$ is a vector, whereas elsewhere it is a scalar. An investment profile x_i is critical if, fixing the relationship strengths of the producers of other products x_{-i} , the probability of successful production of product j would be zero at any profile $\hat{x}_i < x_i$, where $\hat{x}_i < x_i$ means that each entry of \hat{x}_i is weakly lower than the corresponding entry of x_i and at least one such entry is strictly lower. Product i is critical in an equilibrium when the equilibrium investment profile of producers of product i is critical.

Proposition 7. Denote the cost for a firm in industry i of investing in a relationship with a supplier in industry j by $c_{ij}(x_{ij} - \underline{x}_{ij}) = \gamma_{ij}\tilde{c}_{ij}(x_{ij} - \underline{x}_{ij})$.

- (i) Suppose product i is critical, then any other product j on a directed path from j to i (on the product interdependencies graph) will fail following a shock $\epsilon > 0$ to γ_{ik} (which increases γ_{ik} to $\gamma_{ik} + \epsilon$) for any $k \in \mathcal{I}_i$.
- (ii) Let $\mathcal{I}^{SC} \subset \mathcal{I}$ be a set of products that are part of a strongly connected component of the product interdependencies graph. Then any equilibrium with positive effort is such that either all producers of products $i \in \mathcal{I}^{SC}$ have critical relationship strengths or no producers of products $i \in \mathcal{I}^{SC}$ have critical relationship strengths.

Proposition 7 shows that supply networks suffer from a weakest link phenomenon. First, if a product is critical, then a shock to it causes the production of other products that use it as an input, directly or indirectly, to also fail. Second, if we take a strongly connected component of products where none of them are critical and, say, reduce entry costs for the producers of one product until it becomes critical, then all products in that component will also become critical at the same time. The component is only as strong as its weakest link.

7. RELATED LITERATURE

A large literature has examined the economic implications of complementarities in production. Kremer (1993) and subsequent work have argued that complementarities can help provide a unified account of many economic phenomena. These include very large cross-country differences in production technology and aggregate productivity; rapid output increases during periods of industrialization; the macroeconomic propagation of idiosyncratic shocks; and the structure of production networks and international trade flows; see, among many others, Ciccone (2002), Acemoglu, Antràs, and Helpman (2007), Levchenko (2007), Jones (2011) and Levine (2012). Jovanovic (1987) shows how strategic interdependencies or complementarities can produce aggregate volatility in endogenous variables despite only seemingly “diversifiable” idiosyncratic volatility in exogenous variables.

We take the strong (and nested) complementarities as a foundation for our analysis; the precipice phenomena that are at the core of our results are not examined in this literature.

Perhaps the closest paper to ours in this literature is Blanchard and Kremer (1997). After the fall of the Soviet Union, the institutions in place for a planned economy were lost and market-based institutions had not been built up to adequately take their place. As we’ve stated in the introduction, this can be viewed as a shock to relationship strengths throughout the economy. Blanchard and Kremer (1997) argue that this type of shock can explain the large drop in aggregate output that occurred. Their explanation is focused on distinctive features of the state-run economy, including single-sourcing. In contrast, we focus on economies with well developed market-based institutions. We show that such economies can endogenously arrange themselves in a way that makes them, too, extremely sensitive to institutional shocks.⁴⁹ Instead of single-sourcing playing a key role in fragility, the interaction between endogenous multisourcing and complex production is what matters for the fragility of our decentralized market economy.

There is also a vibrant literature in macroeconomics on production networks. Carvalho (2014) provides a comprehensive survey covering much of this literature. Two recent developments in the literature are particularly relevant to our work: (i) the modeling of the endogenous determination of the input-output structure; and (ii) a firm-level approach as opposed considering inter-industry linkages at a more aggregated level. Some of the most relevant work on these issues includes Atalay, Hortacsu, Roberts, and Syverson (2011), Oberfield (2012), Carvalho and Voigtländer (2014), Acemoglu and Azar (2019), Taschereau-Dumouchel (2017), Boehm and Oberfield (2018), Tintelnot, Kikkawa, Mogstad, and Dhyne (2018), and König, Levchenko, Rogers, and Zilibotti (2019). Baqaee and Farhi (2019) and Baqaee and Farhi (2017) are focused specifically on the implications of nonlinearities, and discuss how nonlinearities in firms’ production functions aggregate up. Whereas they focus on smooth nonlinearities, we show that especially extreme nonlinearities—discontinuities—naturally come from complex supply networks. Endogenous firm entry and its interaction with markups plays an important role in our analysis—see, e.g., Melitz and Ottaviano (2008). We discuss below how our explicit modeling of sourcing failures at the micro level gives rise to new effects.

There has been considerable recent interest in markets with non-anonymous trade mediated through relationships.⁵⁰ The work most closely related to ours in this area also studies network formation in the presence of shocks. This includes work in the context of production (e.g., Fafchamps (2002), Levine (2012), Brummitt, Huremović, Pin, Bonds, and Vega-Redondo (2017), Bimpikis, Candogan, and Ehsani (2019), Yang, Scoglio, and Gruenbacher (2019), Amelkin and Vohra (2019)), work on financial networks (e.g., Cabrales, Gottardi, and Vega-Redondo (2017), Elliott, Hazell, and Georg (2018), Erol (2018), Erol and Vohra (2018), Jackson and Pernoud (2019)), and work in varied other contexts (e.g., Blume, Easley, Kleinberg, Kleinberg, and Tardos (2011), Jackson, Rodriguez-Barraquer, and Tan (2012), Talamàs and Vohra (2018)). More broadly, the aggregate implications of non-anonymous trade have been studied across a variety of settings. For work on thin financial

⁴⁹Indeed, in our model aggregate production drops discontinuously in response to arbitrarily small but persistent shocks to relationship strength, in contrast to the continuous behavior of the Blanchard and Kremer (1997) model.

⁵⁰A literature in sociology emphasizes the importance of business relationships, see for example Granovetter (1973) and Granovetter (1985). For a survey of related work in economics see Goyal (2017).

markets see, for example, Rostek and Weretka (2015), for buyer-seller networks see, e.g., Kranton and Minehart (2001), and for intermediation see, e.g., Gale and Kariv (2009).

Our model combines complementarities in production and strategic network formation choices. Agents make a continuous choice that determines the probability of their supply relationships operating successfully. The links that form may, however, fail in a “discrete” (i.e., non-marginal) way. The first feature makes the model tractable, while the second one yields discontinuities in the aggregate production function and distinguishes the predictions from models where the aggregate production function is differentiable. It might be thought that aggregating over many supply chains, these discontinuities would be smoothed out at the level of the macroeconomy; we show they are not.

Mathematically, our work is related to a recent applied mathematics literature on so-called *multilayer networks* and their phase transitions (Buldyrev, Parshani, Paul, Stanley, and Havlin, 2010). Discontinuities such as the one we study are termed first-order phase transitions in this literature.⁵¹ Buldyrev, Parshani, Paul, Stanley, and Havlin (2010), and subsequent papers in this area such as Tang, Jing, He, and Stanley (2016) and Yang, Scoglio, and Gruenbacher (2019), study quite different network processes, typically with exogenous networks. We point out that discontinuities of this kind arise even in the most standard models of networked production (e.g., that of Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012)), once specific sourcing relationships are taken into account, and, more importantly, that endogenous forces can put equilibria on a precipice. Predating the recent literature on multilayer networks, Scheinkman and Woodford (1994) used insights from physics models on self-organized criticality to provide a “sandpile” model of the macroeconomy in which idiosyncratic shocks have large aggregate effects.⁵² The setup and behavior of the model are rather different from ours: the main point of commonality is in the concern with endogenous fragility. In our work, the supply network is robust to idiosyncratic shocks but very sensitive to arbitrarily small aggregate shocks to relationship strength.

8. CONCLUDING DISCUSSIONS

8.1. Aggregate volatility. So far we have focused on a single complex supply network with particular parameters. The larger economy can be thought of as consisting of many such supply networks, each one small relative to the economy. We study whether the forces making some of these small supply networks fragile can lead to fragility in the aggregate for such an economy.

Suppose there are many supply networks operating independently of each other, with heterogeneity across supply networks but, for simplicity, homogeneity within each network. The parameters of these different supply networks, including their complexities m and multisourcing numbers n are drawn from a distribution. We know from the above that a small shock to relationship strength can discontinuously reduce the production of some of these supply networks. We now point out that a small shock can have a large macroeconomic effect, and that previous results on the fragile regime, especially Proposition 3, are essential to this.

⁵¹These can be contrasted with second-order phase transitions such as the emergence of a giant component in a communication network, which have been more familiar in economics—see Jackson (2008).

⁵²Endogenously, inventories reach a state analogous to a sandpile with a critical slope, where any additional shock (grain dropped on the sandpile) has a positive probability of leading to an avalanche.

For simplicity, fix the functions $c(\cdot)$ and $\Phi(\cdot)$.⁵³ A given supply network is then described by a tuple $\mathfrak{s} = (m, n, \kappa)$. We consider the space of these networks induced by letting the parameters m, n and κ vary. In particular, we let \mathcal{M} be the set of possible values of m , the set of integers between 1 and M ; we let \mathcal{N} be the set of possible values of n , integers between 1 and N , and we allow $\kappa \in \mathcal{K} = [0, K]$. The space of possible supply networks is now $\mathcal{S} = \mathcal{M} \times \mathcal{N} \times \mathcal{K}$. We let Ψ be a distribution over this space, and assume that it has full support.

In some supply networks there will not exist a positive equilibrium (for example, when κ is sufficiently low fixing the other parameters). Consider now those supply networks for which there is a positive equilibrium. There are two possibilities. It may be that the only supply networks for which there is a positive equilibrium have $m = 1$.⁵⁴ That is, the only supply networks with positive reliability are simple. In this case, there is no aggregate fragility.

But if, in contrast, \mathcal{S} contains supply networks where production is *not* simple, then we will have macroeconomic fragility. Indeed, an immediate consequence of Theorem 1 is that if there are some complex ($m \geq 2$) supply networks with positive equilibria, then some of the lower- κ networks with the same (n, m) —which are included in \mathcal{S} —are in the fragile regime. The measure that Ψ assigns to supply networks in the fragile regime is positive. Thus, a shock to relationship strengths will cause a discontinuous drop in expected aggregate output.

8.2. Interdependent supply networks and cascading failures. So far we have looked at the case in which the different supply networks operate independently and all business-to-business transactions occur through supply relationships confined to their respective supply networks. We think of these relationships as mediating the supply of inputs that are tailored to the specifications of the business purchasing them. They are not products that can be purchased off-the-shelf. For example, most business use computers, and many of them do not maintain failure-prone specific relationships with computer manufacturers. **So far we have abstracted from any interdependencies between businesses created by such purchases.** However, these interdependencies might matter. A failure of one product could reduce the productivity of others that purchase this product off-the-shelf. If this reduction in productivity is long-lasting it can reduce the incentives of firms to invest in their supply relationships. A shock that causes one fragile supply network to fail might then precipitate the failures of others, even though these others are disconnected in our supply network. Indeed, there can then be cascades of failures of the following sort: producers in supply networks that are not *initially* in the fragile regime are put into the fragile regime by others' failures, and then also are disabled by a small shock. We flesh out this idea in Appendix G.1. The discontinuous drops in output caused by fragility, combined with macroeconomic interdependence, come together to give a new foundation for a type of amplification whose implications are studied in Elliott, Golub, and Jackson (2014) (in a financial context) and Baqaee (2018) (in a macroeconomic context).

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⁵³These could also be drawn from a distribution, but the notation would be more cumbersome.

⁵⁴For example, it might be that $M = 1$ so that only simple production is feasible, or K might be sufficiently low that only simple production has a positive probability of being successful.

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APPENDIX A. FORMAL CONSTRUCTION OF THE SUPPLY NETWORK

Fix a product network as described in Section 2: a finite set \mathcal{I} of products and, for each i , a set \mathcal{I}_i of products which are inputs necessary to produce this project. Let \mathcal{P} denote the product network specified by these data.

Next we specify the nodes of the supply network. For each $i \in \mathcal{I}_i$ there is a set $\mathcal{F}_i = \{if : f \in [0, \bar{f}_i]\}$ of firms producing product i .⁵⁵ We endow this set with the Borel σ -algebra.

A *supply network with product network \mathcal{P} and multisourcing number n* is a random graph $\mathcal{G} = (\mathcal{F}, \mathcal{E})$ satisfying the following properties.

- Its nodes are the set \mathcal{F} ;
- Edges are ordered pairs (if, jf') where $j \in \mathcal{I}_i$ —the meaning that if sources from jf' , and we depict such an edge as an arrow from if to jf' .
- For any $if \in \mathcal{F}$, for each $j \in \mathcal{I}_i$ there are n edges (if, jf') to n distinct producers f' of product j , i.e. elements of \mathcal{F}_j . For any firm, define its neighborhood $N_{if} = \{jf' : (if, jf') \in \mathcal{E}\}$.
- For any firm if , the elements of $N_{if} \cap \mathcal{F}_j$ are independently drawn from an atomless distribution, and these realizations are independent across j .
- For any countable set of firms \mathcal{F}' , neighborhoods N_{if} are independent.

Define \mathcal{G}' to be a random subgraph of \mathcal{G} in which each edge is kept independently, with probability x . More formally, define for every edge e a random variable $O_e \in \{0, 1\}$ (whether the edge is operational) such that

- $\mathbf{P}[O_e = 1 \mid \mathcal{G}] = x$ for every $e \in \mathcal{G}$ and,
- for any countable subset E of edges in \mathcal{G} , the random variables $(O_e)_{e \in E}$ are independent conditional on \mathcal{G} .

A subset $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ is defined to be *consistent* if, for each $if \in \widehat{\mathcal{F}}$, the following holds: for all products j that if requires as inputs ($\forall j \in \mathcal{I}_i$) there is a potential supplier of project j for firm if ($\exists jf' \in \mathcal{F}_{if,j}$) so that if has an operational link to that supplier (if, jf') (s.t. $(if, jf') \in \mathcal{G}'$) with if and $jf' \in \widehat{\mathcal{F}}$. There may be many consistent sets, but by a standard argument, there will be a maximal one, $\overline{\mathcal{F}}$, which is a superset of any other consistent set. This set can be found via the algorithm in Section 2.

APPENDIX B. SIMPLE MICROFOUNDATIONS

In the main text, we have described the production in the supply network at an abstract level. Here we give a more detailed foundation. We make the simplest choices possible at each stage, but remark where generalizations are straightforward.

B.1. Production possibilities and allocations. We take the parameters laid out in Appendix A and the realized supply network \mathcal{G}' as constructed there, and then define the following environment based on these data. There is a representative household that inelastically supplies \bar{L} units of labor. Labor is the only factor (unproduced input).

There is one consumption good, produced as a CES aggregate of *consumption components* $(C_i)_{i \in \mathcal{I}}$, one for each product. We turn now to the production of these components. Letting c_{if} be the amount of firm if 's output contributed to that component, we have the CES production of consumption component i :

$$C_i = \left(\int_0^1 c_{if}^{\frac{\sigma-1}{\sigma}} df \right)^{\frac{\sigma}{\sigma-1}}.$$

An allocation is described by specifying

- (1) firm sourcing of intermediates: for each if and jf' , the amount that if buys from jf' ; called $z_{if,jf'}$; this is 0 if $(if, jf') \notin \mathcal{G}'$ and may be 0 or positive otherwise;
- (2) an amount $\ell_{if} \geq 0$ of labor contributed by the household to each firm;
- (3) firm output: the output Y_{if} produced by each producer of product i , which is constrained to be 0 if $e_{if} = 0$;

⁵⁵Here we have chosen a particular set of labels, namely numbers in $[0, 1]$; in the illustrations of the main text we have used real numbers outside this interval for convenience.

(4) an amount $c_{if} \geq 0$ of firm if 's output that allocated to consumption.

Each firm $if \in \mathcal{F}_i^e$ has a production technology that gives

$$\mathcal{Y}_i(\ell_{if}, (z_{if,jf'})_{j \in \mathcal{I}_i, jf' \in \mathcal{F}^e})$$

units of output when its labor and intermediate inputs are the ones given in the arguments. For simplicity only, we will work with a Leontief technology.⁵⁶ All our calculations would be very similar for any symmetric CES production technology at each step, for example Cobb-Douglas.

For fixed constants $0 < \epsilon, \lambda, \iota < 1$

$$\mathcal{Y}_i(\ell_{if}, (z_{if,jf'})_{jf' \in \mathcal{F}^e}) = \min \left\{ \frac{\ell_{if}}{\lambda}, \left\{ \frac{z_{if,jf'}}{\iota/m} \right\}_{j \in \mathcal{I}_i, jf' \in \mathcal{F}^e} \right\}.$$

In words, to produce a unit of its product, a firm needs λ units of labor and ι units of intermediates, the latter needing to come equally from its inputs.

B.2. Payments. We now describe how the surplus from production is allocated. The allocation is efficient. Rents accrue when firms sell their goods to produce the consumption components (i.e., effectively to the consumer). Firms sell these differentiated goods at a markup of σ , as in standard monopolistic competition models.

In sales of intermediates among firms, all bargaining power is allocated to the buyer at each stage of production. This avoids any distortions, double marginalization, etc., arising from markups by suppliers.⁵⁷ By using this specification we make the point that pricing or bargaining frictions play no role in the sources of fragility we identify.

B.3. The equilibrium. The technology implies that the ultimate factor requirement for one unit of production of any good is $\lambda/(1 - \iota)$ units of labor. By the bargaining power assumption, all intermediate inputs are sold at marginal cost, which is ultimately their factor content. We normalize the wage to 1.

Using all the labor produces $(1 - \iota)/\lambda$ units of output. Noting also that, by the technology, producing any unit of output requires $1/(1 - \iota)$ units of some intermediates, we can calculate that the units of output that go to consumption (as opposed to intermediates) is

$$\gamma = \frac{(1 - \iota)^2}{\lambda(2 - \iota)}.$$

Focusing on symmetric equilibria of the entry game, suppose firms $f \leq \bar{f}$ have entered for each product. In any efficient outcome, due to the convexity in the production of the consumption good, a share γ/n is output that comes from product i , and $c_{if} = \gamma/(nr\bar{f})$ is contributed by each firm. (Here r is the fraction of firms that are functional.)

These calculations imply that the total amount of consumption good C_i that comes out of the corresponding CES aggregator in equilibrium is

$$C_i = \frac{\gamma}{n} (r\bar{f})^{\frac{1}{\sigma-1}}.$$

The profits of firm i come from earning a markup of $\frac{\sigma}{\sigma-1}$ over their marginal cost. This yields, after some calculation, gross (i.e., post-entry) profits of

$$G = \frac{\gamma}{nr\bar{f}} \cdot \frac{1}{1 - \sigma}.$$

Thus, we have that consumer surplus is strictly increasing in $r\bar{f}$, the mass of functional firms that have entered, and the profits to each firm are strictly decreasing in this quantity.

⁵⁶In particular, the use of Leontief production in this example plays no role in generating discontinuities.

⁵⁷In Appendix F we show that our basic conclusions are robust to any distribution of the post-entry surplus generated by efficient production.

APPENDIX C. INTERPRETATION OF INVESTMENT

C.1. A richer extensive margin model. In Section 3.4.1 we gave an extensive margin search effort interpretation of x_{if} . In some ways this interpretation was restrictive. Specifically, it required there to be exactly n suppliers capable for supplying the input and that each such supplier be found independently with probability x_{if} . This alternative interpretation is a minimal departure from the intensive margin interpretation, which is why we gave it. However, it is also possible, through a change of variables, to see that our model encompasses a more general and standard search interpretation.

Fixing the environment a firms faces, specifically the probability other firms successfully produce $r > 0$ and a parameter n that will index the ease of search, suppose we let each firm if choose directly the probability that, through search, it finds an input of given type. When $r = 0$ we suppose that all search is futile and that firms necessarily choose $\hat{x}_{if} = 0$. Denote the probability firm if finds a supplier of a given input type by \hat{x}_{if} . Conditional on finding an input, we let it be successfully sourced with probability 1 so all frictions occur through the search process. Implicitly, obtaining a probability \hat{x}_{if} requires search effort, and we suppose that cost of achieving probability \hat{x}_{if} is $\hat{c}(\hat{x})$, where \hat{c} is a strictly increasing function with $\hat{c}(0) = 0$.

We suppose firms choose \hat{x}_{if} taking the environment as given. In particular, firms take as given the probability that suppliers of the inputs they require successfully produce. When many potential suppliers of an input produce successfully we let it be relatively easy to find one, and if none of these suppliers produce successfully then it is impossible to find one. In addition, the parameter n shifts how easy it is to find a supplier.

Given this set up we can let the probability of finding a supplier have the functional form $\hat{x} := 1 - (1 - x_{if}r)^n$, and the cost of achieving this probability be given by $\hat{c}(\hat{x}) := c\left(\frac{1 - (1 - \hat{x})^{1/n}}{r}\right)$. Although these functional form assumptions might seem restrictive, we still have freedom to use any function c satisfying our maintained assumptions. This degree of freedom is enough for the model to be quite general as all that matters is the size of the benefits of search effort relative to its cost, and not the absolute magnitudes. Further, these functional form assumptions satisfy all the desiderata we set out above. As $1 - (1 - x_{if}r)^n$ is the key probability throughout our analysis, all our results then go through with this interpretation.

C.2. Effort on both the extensive and intensive margins. This section supports the claims made in Section 3.4.1 that our model is easily extended to allow firms to make separate multi-sourcing effort choices on the intensive margin (quality of relationships) and the extensive margin (finding potential suppliers).

Suppose a firm if chooses efforts $\hat{e}_{if} \geq 0$ on the extensive margin and effort $\tilde{e}_{if} \geq 0$ on the intensive margin, and suppose that $x_{if} = h(\hat{e}_{if}, \tilde{e}_{if})$. Let the cost of investment be a function of $\hat{e}_{if} + \tilde{e}_{if}$ instead of y_{if} . This firm problem can be broken down into choosing an overall effort level $e_{if} = \hat{e}_{if} + \tilde{e}_{if}$ and then a share of this effort level allocated to the intensive margin, with the remaining share allocated to the extensive margin. Fixing an effort level e , a firm will choose $\hat{e}_{if} \in [0, e]$, with $\tilde{e}_{if} = e - \hat{e}_{if}$, to maximize x_{if} . Let $\hat{e}_{if}^*(e)$ and $\tilde{e}_{if}^*(e) = e - \hat{e}_{if}^*(e)$ denote the allocation of effort across the intensive and extensive margins that maximizes x_{if} given overall effort e . Given these choices, define $h^*(e) := h(\hat{e}_{if}^*(e), \tilde{e}_{if}^*(e))$. As h^* is strictly increasing in e , choosing e is then equivalent to choosing x_{if} directly, with a cost of effort equal to $c(h^{*-1}(e))$. Thus, as long as the cost function $\hat{c}(e) := c(h^{*-1}(e))$ continues to satisfy our maintained assumptions on c , everything goes through unaffected.

APPENDIX D. OMITTED PROOFS

D.1. The shape of the reliability function. We start with an important lemma about ρ , which is proved in Section SA3.1 of the Supplementary Appendix, and of which Proposition 1 is a corollary.

Lemma 2. Suppose the complexity of the economy is $m \geq 2$ and there are $n \geq 1$ potential input suppliers of each firm. For $r \in (0, 1]$ define

$$\chi(r) := \frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}. \quad (8)$$

Then there are values $x_{\text{crit}}, r_{\text{crit}} \in (0, 1]$ such that:

- (i) $\rho(x) = 0$ for all $x < x_{\text{crit}}$;
- (ii) ρ has a (unique) point of discontinuity at x_{crit} ;
- (iii) ρ is strictly increasing for $x \geq x_{\text{crit}}$;
- (iv) the inverse of ρ on the domain $x \in [x_{\text{crit}}, 1]$, is given by χ on the domain $[r_{\text{crit}}, 1]$, where $r_{\text{crit}} = \rho(x_{\text{crit}})$;
- (v) χ is positive and quasiconvex on the domain $(0, 1]$;
- (vi) $\chi'(r_{\text{crit}}) = 0$.

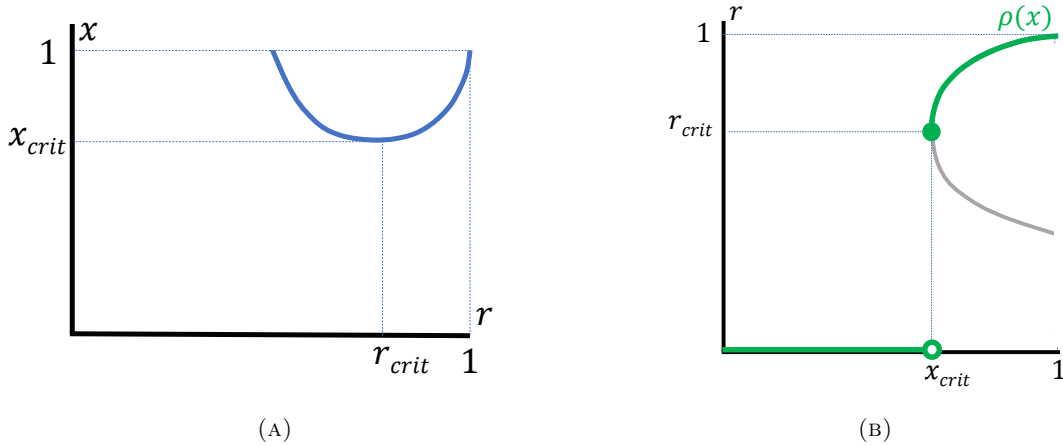


FIGURE 15. Panel (A) plots the function χ as r varies, and then in Panel (B) we show how switching the axes and taking the largest r value on the graph (corresponding to the largest solution of equation (PC)) generates $\rho(x)$.

Figure 15(a) depicts χ as a function of r . Since by definition the *largest* r satisfying (PC) is the one that determines reliability $\rho(x)$, it follows that the increasing part of the function, where $r \in [r_{\text{crit}}, 1]$ is the part relevant for determining equilibrium reliability—see Figure 15(b), where the light gray branch is not part of ρ .

D.2. Proof of Proposition 1. Proposition 1 is a direct corollary of Lemma 2 above.

D.3. Proof of Lemma 1. We establish some notation. Recall that

$$P(x_{if}, r) = (1 - (1 - x_{if}r)^n)^m.$$

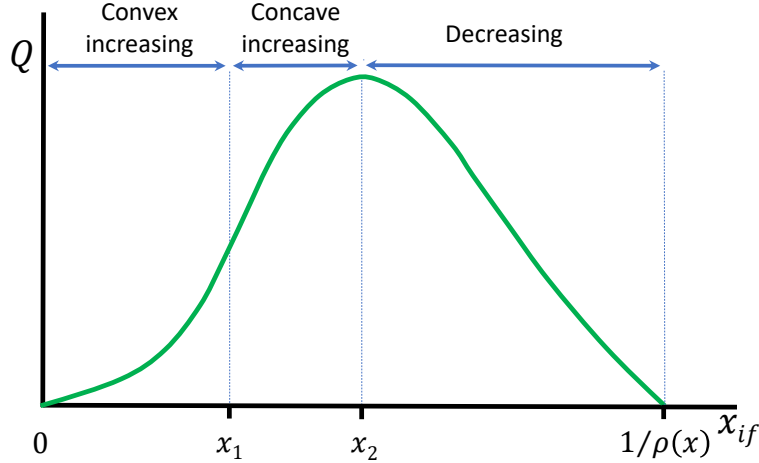
For the extended domain $x_{if} \in [0, 1/\rho(x)]$, we define

$$Q(x_{ik}; x) := \frac{\partial}{\partial x_{if}} P(x_{if}; \rho(x)). \quad (9)$$

We will need two steps to prove Lemma 1. The first step consists of establishing Lemma 3 on the basic shape of $Q(x_{if}; x)$.

Lemma 3. Fix any $m \geq 2$, $n \geq 2$, and $x \geq x_{\text{crit}}$. There are uniquely determined real numbers x_1, x_2 (depending on m, n , and x) such $0 \leq x_1 < x_2 < 1/\rho(x)$ and so that:

- 0. $Q(0; x) = Q(1/\rho(x); x) = 0$ and $Q(x_{if}; x) > 0$ for all $x_{if} \in (0, 1/\rho(x))$;

FIGURE 16. The shape of $Q(x_{if}, x)$.

1. $Q(x_{if}; x)$ is increasing and convex in x_{if} on the interval $[0, x_1]$;
2. $Q(x_{if}; x)$ is increasing and concave in x_{if} on the interval $(x_1, x_2]$;
3. $Q(x_{if}; x)$ is decreasing in x_{if} on the interval $(x_2, 1]$.
4. $x_1 < x_{\text{crit}}$.

The proof of Lemma 3 is in Section SA3.2 of the Supplementary Appendix. Figure 16 illustrates the shape of $Q(x_{if}, x)$ implied by Lemma 3.

We now complete the proof of Lemma 1 by setting $\hat{x} = x_1$. By Lemma 3 property 4, the interval $(\hat{x}, x_{\text{crit}})$ is non-empty. Thus we just need show to that Assumption 3 is satisfied when $\underline{x} \in (\hat{x}, x_{\text{crit}})$. Note that by Assumption 2 $c'(0)$ is 0 and increasing and weakly convex otherwise. Since, by Lemma 3, $P'(x_{if}; \rho(x))$ is first concave and increasing (possibly for the empty interval) and then decreasing (possibly for the empty interval) over the range $x_{if} \in [\underline{x}, 1]$, it follows that there is at most a single crossing point between the curves $P'(x_{if}; \rho(x))$ and $c'(x_{if} - \underline{x})$. This crossing point corresponds to the first-order condition $P'(x_{if}; \rho(x)) - c'(x_{if} - \underline{x})$, yielding the unique maximizer of $\Pi(x_{if}, x)$, as illustrated in Fig. 17. If such a crossing does not exist, $y_{if} = 0$ is a local and global maximizer of the profit function.

D.4. Proof of Proposition 2. Recall that $x^{SP}(\kappa)$ is the set of all values of x maximizing the planner's objective.

By Lemma 2, $x_{\text{crit}} > 0$, and for all $x < x_{\text{crit}}$, we have $\rho(x) = 0$. Thus no value of $x \in (0, x_{\text{crit}})$ can be a solution to the social planner's problem. Further, for all values of κ sufficiently small, and for all values of $x \in [0, 1]$,

$$c(x_{\text{crit}}) > \kappa \geq \kappa \rho(x).$$

Thus for sufficiently small values of κ the unique solution to the social planner's problem is $x^{SP}(\kappa) = \{0\}$.

Next note that for all values of κ sufficiently large,

$$c(x_{\text{crit}}) < \kappa x_{\text{crit}},$$

and the social planner strictly prefers choosing $x = x_{\text{crit}}$ to $x = 0$. Thus, when κ is sufficiently high all values in $x^{SP}(\kappa)$ will be weakly greater than x_{crit} .

Define $\kappa_{\text{crit}} := \sup_{\kappa: 0 \in x^{SP}(\kappa)} \kappa$. As the social planner's unique solution is $x = 0$ for values of κ sufficiently low, and $x = 0$ is not a solution for values of κ sufficiently high, κ_{crit} is a finite, strictly positive number. This establishes part (i).

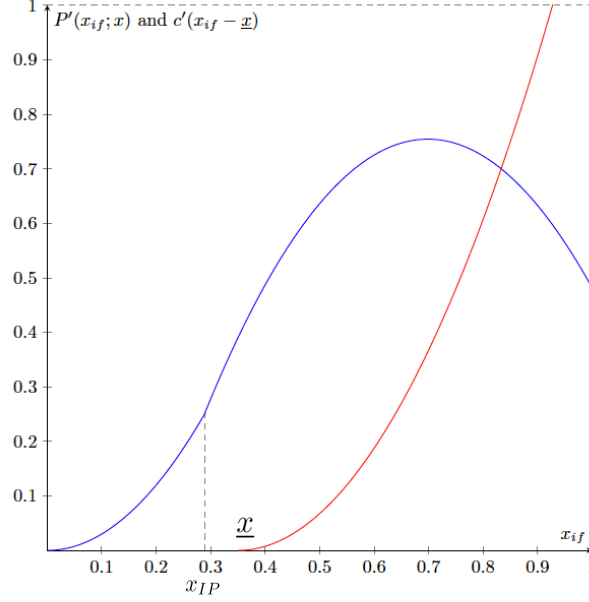


FIGURE 17. $P'(x_{if}; \rho(x))$ is in blue and $c'(x_{if} - \underline{x})$ is in red. There can be only one crossing point between the two curves, which corresponds to the maximizer of $\Pi(x_{if}; x)$.

We now show that $1 \notin x^{SP}(\kappa)$ for all values of κ . As κ is bounded, $\lim_{x \rightarrow 1} \frac{1}{\kappa} c'(x) = \infty$ by our assumption that $c'(x) = \infty$. Moreover, as $n \geq 2$, we have that $x_{\text{crit}} < 1$, so $\rho'(x)$ is bounded for values of x near 1. Hence, at $x = 1$ the social planner can always do a little better by reducing investment.

Thus for values of $\kappa > \kappa_{\text{crit}}$ all solutions to the social planner's problem must be interior and so the following first-order condition must hold:

$$\kappa \rho'(x) = c'(x).$$

By Lemma 2, part (vi), $\lim_{x \downarrow x_{\text{crit}}} \rho'(x) = \infty$. As by assumption $c'(x)$ is bounded for interior values of x , there cannot be an interior equilibrium at $x = x_{\text{crit}}$. This implies that all solutions to the social planner's problem for $\kappa > \kappa_{\text{crit}}$ are at values of x strictly greater than x_{crit} , establishing part (ii) and part (iii).

D.5. Proof of Proposition 3. In the following argument, we defer technical steps to lemmas, which are proved in the Supplementary Appendix.

We begin by sketching the idea of the argument.

We first want to show that there is at most one solution to (OI) and (PC) simultaneously for $r \in [r_{\text{crit}}, 1]$. Consider any positive symmetric equilibrium⁵⁸ x_\star with reliability $r_\star = \rho(x_\star)$. Recall that physical consistency entails

$$r = (1 - \underbrace{(1 - x r)^n}_{\text{probability a given input cannot be acquired}})^m. \quad (\text{PC})$$

By Lemma 2(i), if $x_\star < x_{\text{crit}}$, then $r = 0$ and marginal benefits from investing are 0, an impossibility by the optimal investment condition in Definition 1. Therefore, $x_\star \geq x_{\text{crit}}$ and so by Lemma 2(iv) we have $x_\star = \chi(r_\star)$. (See see Panel (a) of Figure 18.) Finally, by Lemma 2(iii)

$$r_\star \geq r_{\text{crit}} := \rho(x_{\text{crit}}). \quad (10)$$

⁵⁸Note that we are using a notation distinct from the x^* notation we later introduce for the function $x^* : \bar{f} \mapsto x_\star$ that gives the positive investment equilibrium (if there is one) for a given level of firm entry.

Now, given Assumption 3, the optimal investment condition OI says that we have

$$MB(x_\star; r_\star) = MC(x_\star),$$

where

$$MB(x_{if}; r) = \kappa g(r\bar{f})rn(1 - xr)^{n-1}m(1 - (1 - xr)^n)^{m-1} \quad (11)$$

and $MC(x) = c'(x - \underline{x})$. Since we recently deduced $x_\star = \chi(r_\star)$, we can substitute out x_\star to find that the following equation holds:

$$MB(\chi(r_\star); r_\star) = MC(\chi(r_\star)).$$

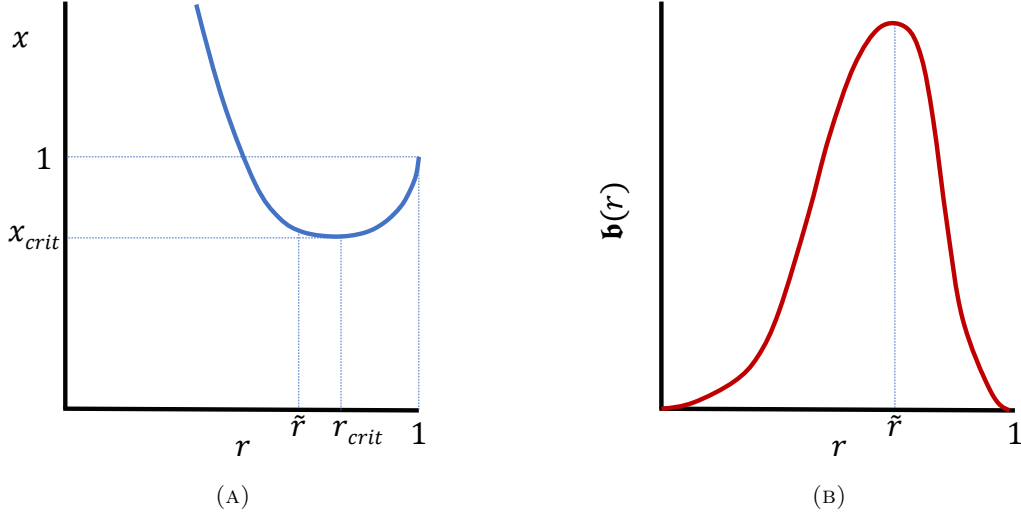


FIGURE 18. Panel (a) shows the relationship between r and x implied by physical consistency. Panel (b) plots the function $\mathfrak{b}(r)$ discussed in the proof.

Define two auxiliary functions of r :

$$\begin{aligned} \mathfrak{b}(r) &= MB(\chi(r); r) \\ \mathfrak{c}(r) &= MC(\chi(r)). \end{aligned}$$

We now know that the equation

$$\mathfrak{b}(r) = \mathfrak{c}(r) \quad (12)$$

is satisfied for any positive equilibrium reliability $r = r_\star$ and moreover (recall inequality (10)) $r_\star \geq r_{\text{crit}}$. More explicitly,

$$\underbrace{G(r\bar{f})mnr^{2-\frac{1}{m}}\left(1 - r^{1/m}\right)^{1-\frac{1}{n}}}_{\mathfrak{b}(r)} = c'\left(\underbrace{\frac{1 - \left(1 - r^{\frac{1}{m}} - \underline{x}\right)^{\frac{1}{n}}}{r}}_x\right) \quad (13)$$

Thus, to show there is at most one solution to (OI) and (PC), it suffices to show that there is at most one solution of equation (12) on the domain $r \in [r_{\text{crit}}, 1]$.⁵⁹ The right-hand side of the equation is increasing in r .⁶⁰ If we could establish that the left-hand side, which we call $\mathfrak{b}(r)$, is decreasing in r , it would follow that there is a unique r solving (13). Unfortunately, $\mathfrak{b}(r)$ is not

⁵⁹We note that the functions on both sides of the equation are merely constructs for the proof. In particular, when we sign their derivatives, these derivatives do not have an obvious economic meaning.

⁶⁰This follows directly: χ is increasing on the relevant domain and c' is increasing by assumption. (see Panel (a) of Figure 18).

decreasing in r . However, we can show that it *is* decreasing in r for $r \geq r_{\text{crit}}$, which is sufficient.⁶¹ Panel (b) of Figure 18 gives a representative depiction of $\mathfrak{b}(r)$, reflecting that it is decreasing to the right of r_{crit} .

By plugging in $x = \chi(r)$ into (11) and differentiating in r , we have

$$\mathfrak{b}(r) = \kappa g(r\bar{f}) m n r^{2-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{1-\frac{1}{n}}. \quad (14)$$

The following Lemma completes the proof.

Lemma 4. \mathfrak{b} is strictly decreasing on the domain $[r_{\text{crit}}, 1)$.

To prove lemma 4 we write $\mathfrak{b}(r)$ as a product of two pieces, $\alpha(r) := \kappa g(r\bar{f})$ and

$$\beta(r) := m n r^{2-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{1-\frac{1}{n}}.$$

Note that the function $\beta(r)$ is positive for $r \in (0, 1)$. We will show that it is also strictly decreasing on $[r_{\text{crit}}, 1)$. By assumption, $g(r\bar{f})$ is positive and strictly decreasing in its argument, so $\alpha(r)$ is also positive and decreasing in r . Thus, because \mathfrak{b} is the product of two positive, strictly decreasing functions on $[r_{\text{crit}}, 1)$, it is also strictly decreasing on $[r_{\text{crit}}, 1)$. It remains only to establish that $\beta(r)$ is strictly decreasing on the relevant domain. Two additional lemmas are helpful.

Lemma 5. The function $\beta(r)$ is quasiconcave and has a maximum at $\hat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^m$.

Lemma 6. For all $n \geq 2$ and $m \geq 3$, we have that $\hat{r} < r_{\text{crit}}$.

Lemmas 5 and 6 are proved in Sections SA3.3 and SA3.4 of the Supplementary Appendix. Together these show that $\beta(r)$ is strictly increasing and then strictly decreasing in r for $r \in (0, 1)$, with a turning point in the interval $(0, r_{\text{crit}})$. Thus $\beta(r)$ is strictly decreasing on the domain $[r_{\text{crit}}, 1)$, the final piece required to prove Lemma 4.

D.6. A lemma on how investment equilibrium depends on entry. We next establish our remaining claim about x^* —namely that it is decreasing when $x^*(\bar{f}) > \bar{f}_{\text{crit}}$. Consider the class of outcomes where the cutoff for entry is \bar{f} for each product. Here we study the shape of the investment equilibrium function, $x^*(\bar{f})$.

Lemma 7. The function $x^*(\bar{f})$ is continuously decreasing in \bar{f} on any interval of values of \bar{f} where $x^*(\bar{f}) > x_{\text{crit}}$. Moreover, $G(r\bar{f})$ is also decreasing in \bar{f} on an interval of values of \bar{f} where r is positive.

The lemma also implies that until $x^*(\bar{f})$ continuously decreases to x_{crit} rather than jumping to 0 before reaching x_{crit} .

The proof of Lemma 7 is in Section SA3.5 of the Supplementary Appendix.

D.7. Proof of Proposition 4.

Proof. We begin with an overview of the proof.

Overview:

The first step is to construct a function $H(\bar{f}) : [0, 1] \rightarrow [0, 1]$, defined as follows. For any (symmetric) cutoff \bar{f} for entry that may be in effect, supposing that all firms choose relationship strength $x^*(\bar{f})$ conditional on entering, what mass of firms find it profitable to enter and invest optimally? This is the value $H(\bar{f})$. Step 1 establishes some basic facts about the shape of H that are useful in the sequel. It also notes that in any symmetric equilibrium (and therefore, any positive symmetric equilibrium), all firms *if* with locations f less than a certain cutoff enter, justifying our focus on cutoff entry rules. A value \bar{f} is defined to be an *entry fixed point* if $H(\bar{f}) = \bar{f}$.

⁶¹We do this by showing that the global maximum of $\mathfrak{b}(r)$ is achieved at a number \tilde{r} that we can prove is smaller than r_{crit} .

Then we consider two possibilities. The first is that there is an entry fixed point (so that free entry determines the number of entering producers of each product). In this case, using what we have established about H , we show that fixing the parameters of the model (κ being the key one) there is a *unique* entry fixed point \bar{f}^* , and that $x^*(\bar{f}^*) \geq x_{\text{crit}}$. From these facts we deduce there is a unique positive symmetric equilibrium, and it has an entry cutoff \bar{f}^* and relationship strength $x^*(\bar{f}^*)$. The free-entry equilibrium is critical if and only if $x^*(\bar{f}^*) = x_{\text{crit}}$, which we show can happen for exactly one value of κ .

The second possibility is that there is no entry fixed point. Then we show any positive symmetric equilibrium must have an entry cutoff \bar{f}^* such that $H(\bar{f}^*) > \bar{f}^*$ and $x^*(\bar{f}^*) = x_{\text{crit}}$. We again show uniqueness of the positive symmetric equilibrium. These arguments use the facts about the shape of H established in Step 1.

This concludes the overview. We now proceed with the proof.

For notational simplicity, we let $\underline{x} = 0$ throughout this proof, so that investment and relationship strength are the same. The proof is essentially identical for $\underline{x} > 0$.

Step 1. To define H , we suppose that a mass of firms \bar{f} is entering the market and consider which entry levels are consistent with equilibrium. Since equilibrium entails investments $x = x^*(\bar{f})$, firm if will want to enter the market if $\Pi_{if}(x; x, \bar{f}) \geq 0$, i.e.,

$$G(\bar{f}\rho(x^*(\bar{f})))\rho(x^*(\bar{f})) - c(x^*(\bar{f})) - \Phi(f) \geq 0. \quad (15)$$

Rearranging equation (15), a firm if located at f will find it profitable to enter if and only if

$$f \leq \Phi^{-1}(G(\bar{f}\rho(x^*(\bar{f})))\rho(x^*(\bar{f})) - c(x^*(\bar{f}))).$$

We then define H as follows,

$$H(\bar{f}) := \Phi^{-1}(\max\{G(\bar{f}\rho(x^*(\bar{f})))\rho(x^*(\bar{f})) - c(x^*(\bar{f})), 0\}). \quad (16)$$

Lemma 8 catalogs some useful properties of $H(\bar{f})$.

Lemma 8. H has the following properties:

- $H(0) > 0$;
- $H(1) < 1$;
- $H(\bar{f})$ is strictly decreasing for all \bar{f} such that $x^*(\bar{f}) > 0$, and $H(\bar{f}) = 0$ for all \bar{f} such that $x^*(\bar{f}) = 0$.

The proof of Lemma 8 is in Section SA3.6 of the Supplementary Appendix. An entry fixed point is defined by

$$\bar{f} = H(\bar{f}). \quad (17)$$

It will be helpful to define an entry level \bar{f}_{crit} above which production fails for sure, if one exists. If $x^*(1) = 0$, then

$$\bar{f}_{\text{crit}} = \sup\{f : x^*(f) \geq x_{\text{crit}}\}.$$

On the other hand, if $x^*(1) > 0$ then for all $\bar{f} < 1$, $x^*(\bar{f}) > x_{\text{crit}}$. By Assumption 4 there cannot be entry $\bar{f} = 1$ in a positive symmetric equilibrium, so if $x^*(1) > 0$ there can be no positive symmetric equilibrium with investments x_{crit} . There is then no relevant critical entry level. Nevertheless, so that \bar{f}_{crit} is always well defined, in this case we let $\bar{f}_{\text{crit}} = 2$, an infeasibly high entry level (and any infeasibly high entry level would do).

There are two cases to consider.

- Case 1 (noncritical equilibrium): There exists a solution \bar{f}^* to equation (17) such that $\bar{f}^* < \bar{f}_{\text{crit}}$.
- Case 2 (critical equilibrium): There does not exist a solution \bar{f}^* to equation (17) such that $\bar{f}^* < \bar{f}_{\text{crit}}$.

Step 2: In this step of the proof we consider Case 1. In this case, there exists a \bar{f}^* such that

$$\Phi(\bar{f}^*) = G(\bar{f}^*\rho(x^*(\bar{f}^*)))\rho(x^*(\bar{f}^*)) - c(x^*(\bar{f}^*)).$$

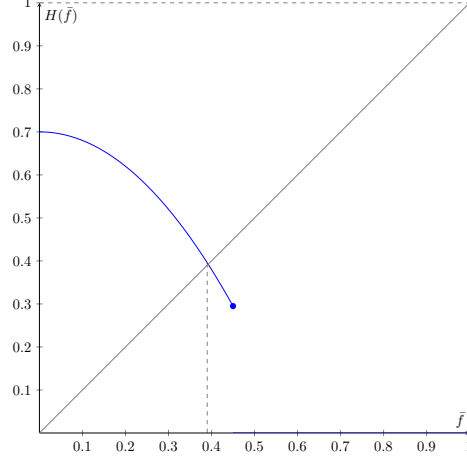


FIGURE 19. Case 1 (noncritical) equilibrium

Hence inequality (15) binds and the equilibrium expected profits is zero for any firm $i\bar{f}^*$ located at \bar{f}^* (for any product i). Moreover, since Φ is increasing in its argument, it follows that any firm if located at $f < \bar{f}^*$ will be able to obtain strictly positive equilibrium expected profits by entering, while any firm located at $f > \bar{f}^*$ will find it unprofitable to enter.

By the properties of $H(\bar{f})$ documented in Lemma 8, there can exist at most one solution \bar{f}^* to equation (17). (See Fig. 19 for an illustration.) We have shown that at this solution the equilibrium profits of firm $i\bar{f}^*$ are zero. Moreover, since $\bar{f}^* < \bar{f}_{\text{crit}}$, it follows from Lemma 7 that $x^*(\bar{f}^*) > x^*(\bar{f}_{\text{crit}}) = x_{\text{crit}}$.

Finally, note that there is no positive mass of firms that can profitably change their entry decisions. For a positive mass of additional firms to enter, some firms with entry costs strictly greater than $\Phi(\bar{f}^*)$ would have to enter. As firms located at \bar{f}^* make zero profits, these firms would make a loss.

Step 3: We now consider Case 2, that of the critical equilibrium. As we can never have a solution to equation (17) with $\bar{f}^* > \bar{f}_{\text{crit}}$ (because then $x^*(\bar{f}^*) = 0$), this case can be broken down into two subcases. There could exist a solution $\bar{f}^* = \bar{f}_{\text{crit}}$ to equation (17) such that the equilibrium is fragile (as illustrated in Figure 20(a)). Note that this subcase is non-generic. If \bar{f}_{crit} solves equation (17) for a given value of κ , then for any other value of κ the H function would shift and there would no longer be a solution with $\bar{f}^* = \bar{f}_{\text{crit}}$.

The second subcase is that there may not exist a solution to equation (17). Then by Lemma 8 the only way for $H(\bar{f})$ to not intersect the 45-degree line is for there to be a discontinuity before it crosses it. However, as $G(\cdot)$ is a continuous function and $c(\cdot)$ is a continuous function the only discontinuities in $H(\bar{f})$ must be where $\rho(\cdot)$ is discontinuous, and by Lemma 2 there is only one discontinuity in $\rho(x)$, and it is at x_{crit} . Thus, given that investment is optimal, the only possible discontinuity in $H(\bar{f})$ is at \bar{f}_{crit} . We must therefore have that $\bar{f}_{\text{crit}} \in (0, 1)$ and $H(\bar{f}_{\text{crit}}) > \bar{f}_{\text{crit}}$ (as illustrated in Figure 20(b)).

We show now that in these cases there is an equilibrium with $\bar{f}^* = \bar{f}_{\text{crit}}$ such that $x^*(\bar{f}^*) = x_{\text{crit}}$. As

$$H(\bar{f}_{\text{crit}}) = \Phi^{-1}(G(\bar{f}_{\text{crit}}\rho(x^*(\bar{f}_{\text{crit}})))\rho(x^*(\bar{f}_{\text{crit}}))) - c(x^*(\bar{f}_{\text{crit}})) > \bar{f}_{\text{crit}},$$

it follows that

$$G(\bar{f}_{\text{crit}}\rho(x^*(\bar{f}_{\text{crit}})))\rho(x^*(\bar{f}_{\text{crit}})) - c(x^*(\bar{f}_{\text{crit}})) > \Phi(\bar{f}_{\text{crit}})$$

and any firm $f \leq \bar{f}_{\text{crit}}$ achieves positive expected profits from entering of

$$G(\bar{f}_{\text{crit}}\rho(x^*(\bar{f}_{\text{crit}})))\rho(x^*(\bar{f}_{\text{crit}})) - c(x^*(\bar{f}_{\text{crit}})) - \Phi(f) > 0.$$

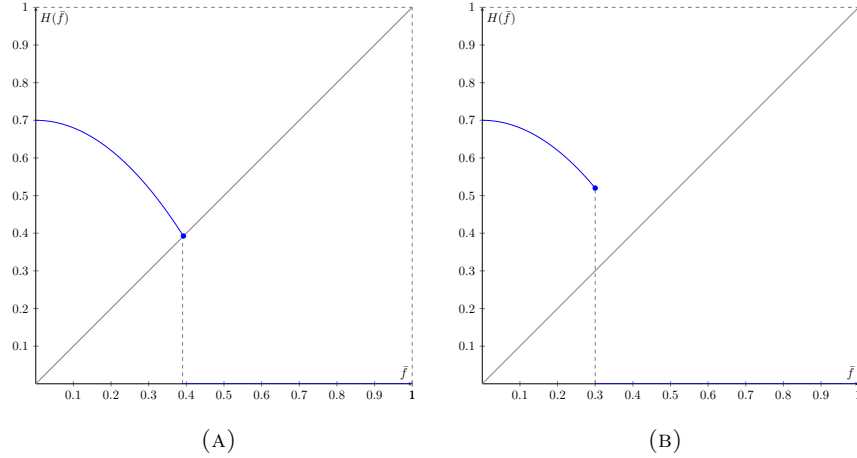


FIGURE 20. Case 2 (critical) equilibrium: (a) first subcase; (b) second subcase

To show that the pair $(x_{\text{crit}}, \bar{f}_{\text{crit}})$ is an equilibrium in this case, note that were a positive mass of non-entering firms to deviate and enter there would no longer be a positive production equilibrium and thus the deviation would be unprofitable. Therefore, in equilibrium, $\bar{f}^* = \bar{f}_{\text{crit}}$ and $x^*(\bar{f}^*) = x^*(\bar{f}_{\text{crit}}) = x_{\text{crit}}$. □

D.8. Proof of Theorem 1.

Proof. For notational simplicity, we let $\underline{x} = 0$ here. The proof is essentially identical for $\underline{x} > 0$.

We will prove the different parts of the proposition statement in reverse order. By Proposition 4 there is at most one equilibrium with positive entry. We will first show that when κ is sufficiently high (i.e., $\kappa > \bar{\kappa}$) there is an equilibrium with positive entry and equilibrium investment is increasing in κ , while $\rho(x^*) > r_{\text{crit}}$. We'll then consider lower values of κ (i.e., $\kappa \in (\underline{\kappa}, \bar{\kappa}]$) and show that for this range of κ there continues to be an equilibrium with positive entry and production, but with $\rho(x^*) = r_{\text{crit}}$. Finally, we'll show that when $\kappa \leq \underline{\kappa}$ there is no equilibrium with positive entry (or production).

Part (iii): We begin by considering the cases in which $\kappa > \bar{\kappa}$. First we will find the threshold $\bar{\kappa}$, and construct the unique positive entry equilibrium for this value of κ . We will then show what happens in equilibrium as κ increases.

Fixing the mass of entering firms at \bar{f} , Assumption 3 along with the Inada condition implies that equation (OI) identifies a necessary and sufficient condition for a positive investment equilibrium. Away from the precipice a no-profit condition must also be satisfied. Thus, firms are best-responding to each other in their entry (\bar{f}) and investment decisions (x) when the following two equations are satisfied:

$$MB(x; \rho(x), \bar{f}, \kappa) = MC(x) \quad (18)$$

$$\kappa g(\bar{f} \rho(x)) \rho(x) - c(x) - \Phi(\bar{f}) = 0 \quad (19)$$

The first equation equates firms' marginal benefits and marginal costs of investment given entry \bar{f} . The second equation requires that the marginal entering firm makes 0 profits when entry is \bar{f} and investments are x .

We look for a entry level (\bar{f}) and κ pair such that when \bar{f} firms enter, all entering firms choose investments x_{crit} and the marginal entering firm receives zero profits. We denote this pair by \bar{f}_{crit} and $\bar{\kappa}$, and they satisfy the following two equations

$$MB(x_{\text{crit}}; r_{\text{crit}}, \bar{f}_{\text{crit}}, \bar{\kappa}) = MC(x_{\text{crit}}),$$

$$\bar{\kappa} g(\bar{f}_{\text{crit}} r_{\text{crit}}) r_{\text{crit}} - c(x_{\text{crit}}) - \Phi(\bar{f}_{\text{crit}}) = 0,$$

To see that there is a unique pair $(\bar{\kappa}, \bar{f}_{\text{crit}})$ solving these equations, and these solutions are valid (i.e. $\bar{\kappa} > 0$ and $\bar{f}_{\text{crit}} \in [0, 1]$), note that the first equation implies that

$$\underbrace{\bar{\kappa}g(\bar{f}_{\text{crit}}r_{\text{crit}})mn r_{\text{crit}}(1 - x_{\text{crit}}r_{\text{crit}})^{n-1}(1 - (1 - x_{\text{crit}}r_{\text{crit}})^n)^{m-1}}_{MB(x_{\text{crit}}; r_{\text{crit}}, \bar{f}_{\text{crit}}, \bar{\kappa})} = \underbrace{c'(x_{\text{crit}})}_{MC(x_{\text{crit}})}, \quad (20)$$

and so

$$\bar{\kappa} = \frac{c'(x_{\text{crit}})}{g(\bar{f}_{\text{crit}}r_{\text{crit}})mn r_{\text{crit}}(1 - x_{\text{crit}}r_{\text{crit}})^{n-1}(1 - (1 - x_{\text{crit}}r_{\text{crit}})^n)^{m-1}}.$$

As long as $\bar{f}_{\text{crit}} > 0$ (as we show below), $g(\bar{f}_{\text{crit}}r_{\text{crit}}) > 0$. Thus, as we also have $c'(x_{\text{crit}}) > 0$, we must have $\bar{\kappa} > 0$.

The second equation implies that

$$\begin{aligned} \bar{f}_{\text{crit}} &= \Phi^{-1} \left(\underbrace{\bar{\kappa}g(\bar{f}_{\text{crit}}\rho(x_{\text{crit}}))r_{\text{crit}} - c(x_{\text{crit}})}_{\text{gross profits}} \right) \\ &= \Phi^{-1} \left(\frac{c'(x_{\text{crit}})r_{\text{crit}}}{mn r_{\text{crit}}(1 - x_{\text{crit}}r_{\text{crit}})^{n-1}(1 - (1 - x_{\text{crit}}r_{\text{crit}})^n)^{m-1}} - c(x_{\text{crit}}) \right). \end{aligned} \quad (21)$$

Note that this expression for \bar{f}_{crit} is *not* a fixed point condition— \bar{f}_{crit} does not appear on the right hand side.⁶² As Φ is a strictly increasing function, this inverse is well defined and by Assumption 4, $\bar{f}_{\text{crit}} < 1$. Moreover, by Assumption 3, when $\kappa = \bar{\kappa}$ and $\bar{f} = \bar{f}_{\text{crit}}$ if other entering firms choose $x = x_{\text{crit}}$, an entering firm *if* strictly prefers also choosing investment level $x_{if} = x_{\text{crit}}$ to choosing $x_{if} = 0$. Thus, by choosing $x = x_{\text{crit}}$ firm *if* must be making strictly positive *gross* profits (as choosing $x_{if} = 0$ guarantees 0 gross profits). As $\Phi(0) = 0$, this implies that $\bar{f}_{\text{crit}} > 0$.

By construction, $(x_{\text{crit}}, \bar{f}_{\text{crit}})$ is a solution to the system consisting of equations (18) and (19) when $\kappa = \bar{\kappa}$. Thus, these investment and entry levels constitute an equilibrium.

We consider now values of $\kappa > \bar{\kappa}$. Consider the system of equations comprised of the optimal investment condition $MB(x; \rho(x), \bar{f}, \kappa) = MC(x)$ and a zero profit condition:

$$\kappa g(\bar{f}\rho(x))mn\rho(x)^{2-\frac{1}{m}} \left(1 - \rho(x)^{1/m}\right)^{1-\frac{1}{n}} = c'(x) \quad (22)$$

$$\Phi^{-1} \left(\max\{\kappa g(\bar{f}\rho(x))\rho(x) - c(x), 0\} \right) = \bar{f}. \quad (23)$$

By Proposition 4 there is at most one pair $(x, \bar{f}) \in \mathbb{R}_{++} \times (0, 1]$ that solves this system. We show in Lemma 9 that this solution changes in a systematic way with κ .

Lemma 9. If for $\kappa = \hat{\kappa}$, there is a solution (\hat{x}, \hat{f}) to the system of equations (22) and (23), then for any $\tilde{\kappa} > \kappa$ there exists a unique solution (\tilde{x}, \tilde{f}) to equations (22) and (23) and it satisfies $\tilde{x} > \hat{x}$.

We prove Lemma 9 below.

As $(x_{\text{crit}}, \bar{f}_{\text{crit}})$ solves equations (18) and (19) when $\kappa = \bar{\kappa}$, by Lemma 9 there is also a solution to this system for $\kappa > \bar{\kappa}$ and the solution is such that $x^* > x_{\text{crit}}$. As before this constitutes an equilibrium. Note that the marginal firm makes zero profits in this equilibrium by equation (19) and so this equilibrium corresponds to a Case 1 (non-critical regime) equilibrium.

Part (ii):

There is a unique $\underline{\kappa} > 0$ that solves $MB(x_{\text{crit}}; r_{\text{crit}}, \bar{f} = 0, \underline{\kappa}) = MC(x_{\text{crit}})$. This value of $\underline{\kappa}$ is

$$\underline{\kappa} = \frac{c'(x_{\text{crit}})}{g(0)mn r_{\text{crit}}(1 - x_{\text{crit}}r_{\text{crit}})^{n-1}(1 - (1 - x_{\text{crit}}r_{\text{crit}})^n)^{m-1}}.$$

As $g(0) > 0$ and $c'(x_{\text{crit}}) > 0$, we have $\underline{\kappa} > 0$.

Thus, in the limit as the mass of entering firms converges to zero, if $\kappa = \underline{\kappa}$, then by Assumption 3 and equation (OI), if all firms other than firm *if* choose investments x_{crit} (resulting in a probability

⁶²Recall that r_{crit} and x_{crit} are defined independently of \bar{f}_{crit} .

of successful production equal to r_{crit}), firm i has a unique best response to choose $x_{if} = x_{\text{crit}}$. In particular, we must have

$$\underbrace{\kappa g(0) m n r_{\text{crit}} (1 - x_{\text{crit}} r_{\text{crit}})^{n-1} (1 - (1 - x_{\text{crit}} r_{\text{crit}})^n)^{m-1}}_{MB(x_{\text{crit}}; r_{\text{crit}}, \bar{f}=0, \kappa)} = \underbrace{c'(x_{\text{crit}})}_{MC(x_{\text{crit}})}.$$

Comparing this equation to equation (20) implies that $\kappa g(0) = \bar{\kappa} g(\bar{f}_{\text{crit}} r_{\text{crit}})$. Thus, as $\bar{f}_{\text{crit}} > 0$, and g is a strictly decreasing function, $\bar{\kappa} > \kappa$.

Consider now the case in which $\kappa \in (\kappa, \bar{\kappa})$. For this range, the equilibrium is such that investment is x_{crit} . Firms will enter up until an entry level \bar{f} , where \bar{f} is such that $MB(x_{\text{crit}}; r_{\text{crit}}, \bar{f}, \kappa) = MC(x_{\text{crit}})$ for firms to choose investments x_{crit} . Entry up to this point is strictly profitable and any further positive mass of entry will result in investments less than x_{crit} . Hence, entry adjusts to keep firms' markups just high enough to sustain an investment of x_{crit} .

We can rearrange equation (18) to write an explicit expression for the level of entry required to keep marginal benefits equal to marginal costs as κ varies. This gives us the function $\tilde{f} : (\kappa, \bar{\kappa}) \rightarrow [0, 1]$, where

$$\tilde{f}(\kappa) := \frac{1}{r_{\text{crit}}} g^{-1} \left(\frac{c'(x_{\text{crit}})}{\kappa m n \rho(x_{\text{crit}})^{2-\frac{1}{m}} (1 - \rho(x_{\text{crit}})^{1/m})^{1-\frac{1}{n}}} \right). \quad (24)$$

When $\kappa \in (\kappa, \bar{\kappa})$ we will show that there is an equilibrium in which firms located in the interval $[0, \tilde{f}(\kappa)]$ enter and receive positive profits, $x^* = x_{\text{crit}}$ and $\rho(x^*) = r_{\text{crit}}$. Note that such an equilibrium is in the critical regime.

We now verify that this is an equilibrium. By construction in such an outcome firms are best-responding with their investment choices to others' investment choices. To see that all firms are making strictly positive profits we compare the profits of the marginal entering firm $i\tilde{f}$ at $\kappa \in (\kappa, \bar{\kappa})$ to the profits of the marginal entering firm $i\bar{f}_{\text{crit}}$ when $\kappa = \bar{\kappa}$ (we established above the latter is 0).

$$\begin{aligned} \tilde{\Pi}_{i\tilde{f}} &= \tilde{\kappa} g(\tilde{f} \rho(x_{\text{crit}})) \rho(x_{\text{crit}}) - c(x_{\text{crit}}) - \Phi(\tilde{f}) && \text{by definition} \\ &> \tilde{\kappa} g(\bar{f}_{\text{crit}} \rho(x_{\text{crit}})) \rho(x_{\text{crit}}) - c(x_{\text{crit}}) - \Phi(\bar{f}_{\text{crit}}) && \text{because } \bar{f}_{\text{crit}} > \tilde{f}. \\ &= \bar{\kappa} g(\bar{f}_{\text{crit}} \rho(x_{\text{crit}})) \rho(x_{\text{crit}}) - c(x_{\text{crit}}) - \Phi(\bar{f}_{\text{crit}}) && x_{\text{crit}} \text{ still profit maximizing} \\ &= \Pi_{i\bar{f}_{\text{crit}}} && \text{by definition} \\ &= 0. && \text{as shown above} \end{aligned}$$

As entering firms are making strictly positive profits, their entry decisions are optimal. No other positive mass of firms can profitably enter as after their entry positive investment could not be sustained.

Part (i):

Finally, we consider $\kappa \leq \kappa$. Recall that for $\kappa > \kappa$ in the regime just considered, as $\kappa \rightarrow \kappa$ from above the mass of firms entering went to 0 while entering firms continued to make investments x_{crit} (i.e., $MB(x_{\text{crit}}; r_{\text{crit}}, \bar{f} = 0, \kappa) = MC(x_{\text{crit}})$) resulting in a probability of successful production equal to r_{crit} . By Lemma 7, $x^*(\bar{f})$ is strictly decreasing in \bar{f} . Thus, with $\kappa = \kappa$, if a positive mass of firms entered, then equilibrium investments would be below x_{crit} and production would fail for sure, and all firms with positive entry costs would make losses.

Consider now $\kappa < \kappa$. In comparison to $\kappa = \kappa$, at any fixed entry level and any investment choice, the marginal benefits a firm i receives from investing are then strictly lower (while the marginal costs remain the same). This can be seen immediately from equation (OI) as $g(\cdot)$ is a strictly decreasing function and, by assumption, $\kappa < \kappa$. Thus any intersection of marginal benefits and marginal costs must occur at an investment level $x < x_{\text{crit}}$. This implies that $\rho(x^*) = 0$, and so the only possible equilibrium is at $x^* = 0$. We conclude that there is no positive investment equilibrium for any $\kappa < \kappa$.

□

D.9. Proof of Lemma 9.

Proof. For notational simplicity, We assume here that $\underline{x} = 0$. The proof is otherwise identical.

Let $\tilde{r} = \rho(\tilde{x})$, $\hat{r} = \rho(\hat{x})$, and $\tilde{\Pi}_{if}$ denote the equilibrium profits of firm if when $\kappa = \tilde{\kappa}$ and a mass f of firms enter. Finally, let $\hat{\Pi}_{if}$ denote the equilibrium profits of firm if when $\kappa = \hat{\kappa}$ and a mass f of firms enter.

We will first assume that $\tilde{x} < \hat{x}$, and show this yields a contradiction. We'll then assume that $\tilde{x} = \hat{x}$ and show that this also yields a contradiction, allowing us to conclude that $\tilde{x} > \hat{x}$ as claimed.

First, towards a contradiction, suppose that $\tilde{x} < \hat{x}$. As $\tilde{x} < \hat{x}$ we have $\tilde{\kappa}g(\tilde{f}\tilde{r}) < \hat{\kappa}g(\hat{f}\hat{r})$ by equation (22). Thus, as $\tilde{\kappa} > \hat{\kappa}$, it must be that $g(\tilde{f}\tilde{r}) < g(\hat{f}\hat{r})$, which, as $g(\cdot)$ is a decreasing function implies that $\tilde{f}\tilde{r} > \hat{f}\hat{r}$. Then, as $\tilde{x} < \hat{x}$, by part (iii) of Lemma 2, $\tilde{r} < \hat{r}$ and so we must have $\tilde{f} > \hat{f}$.

When $\kappa = \tilde{\kappa}$, as \tilde{x}, \tilde{f} solve the system of equations (22) and (23), firm \tilde{f} is the marginal entering firm and so all firms with lower entry costs also enter. Thus all firms located at points $f < \tilde{f}$ enter. As shown above, $\tilde{x} < \hat{x}$ implies that $\hat{f} < \tilde{f}$. Consider then the equilibrium profits of a firm located at \hat{f} when $\kappa = \tilde{\kappa}$. This firm enters the market and receives profits less than 0. To see this, consider the following sequence of inequalities which we explain below.

$$\begin{aligned} \tilde{\Pi}_{i\hat{f}} &= \tilde{\kappa}g(\tilde{f}\tilde{r})\tilde{r} - c(\tilde{x}) - \Phi(\hat{f}) \\ &< \hat{\kappa}g(\hat{f}\hat{r})\tilde{r} - c(\tilde{x}) - \Phi(\hat{f}) \\ &< \hat{\kappa}g(\hat{f}\hat{r})(1 - (1 - \tilde{x}\hat{r})^n)^m - c(\tilde{x}) - \Phi(\hat{f}) \\ &= \hat{\Pi}_{i\hat{f}} \\ &= 0. \end{aligned}$$

The first inequality is implied by $\hat{\kappa}g(\hat{f}\hat{r}) > \tilde{\kappa}g(\tilde{f}\tilde{r})$, as argued above. The second inequality is implied by $\hat{r} > \tilde{r}$ (also as argued above). The third inequality is implied by the fact that \hat{x} solves equation (22), when $\kappa = \hat{\kappa}$ and every other firm chooses investments \hat{x} . Firm $i\hat{f}$ must thus strictly prefer choosing \hat{x} to \tilde{x} . As, combining these inequalities, $\tilde{\Pi}_{i\hat{f}} < 0$ the firm located at \hat{f} has a profitable deviation to not enter the market, which is a contradiction.

The only remaining case to consider is $\tilde{x} = \hat{x}$. Towards a contradiction suppose that $\tilde{x} = \hat{x}$. From equation (23) note that,

$$\hat{f} = \Phi^{-1}\left(\max\{\hat{\kappa}g(\hat{f}\hat{r})\hat{r} - c(\hat{x}), 0\}\right) \quad (25)$$

$$\tilde{f} = \Phi^{-1}\left(\max\{\tilde{\kappa}g(\tilde{f}\tilde{r})\tilde{r} - c(\tilde{x}), 0\}\right). \quad (26)$$

However, as $\tilde{x} = \hat{x}$ we must have (as essentially argued above) that $\tilde{\kappa}g(\tilde{f}\tilde{r}) = \hat{\kappa}g(\hat{f}\hat{r})$. Also, by the definition of $\rho(x)$, it must be that $\hat{r} = \tilde{r}$. Substituting these conditions into equations (25) and (26), and as $\Phi(\cdot)$ is a strictly increasing function, $\hat{f} = \tilde{f}$. But then, as $\tilde{\kappa}g(\tilde{f}\tilde{r}) = \hat{\kappa}g(\hat{f}\hat{r})$, we must have $\hat{\kappa} = \tilde{\kappa}$ which is a contradiction. \square

D.10. Proof of Proposition 6. For any $\mathbf{r} \in [0, 1]^{|I|}$, define $\mathcal{R}^\xi(\mathbf{r})$ to be the probability, under the parameter ξ , that a producer of product i is functional given that the reliability vector for all products is given by \mathbf{r} . This can be written explicitly:

$$[\mathcal{R}^\xi(\mathbf{r})]_i = \prod_{j \in \mathcal{I}_i} [1 - (1 - r_j x_{ij}(\xi))^{n_{ij}}].$$

Let $\boldsymbol{\rho}(\xi)$ be the elementwise largest fixed point of \mathcal{R}^ξ , which exists and corresponds to the mass of functional firms by the same argument as in Lemma 2.

(1) It is clear that $\boldsymbol{\rho}(1) = \mathbf{1}$.

(2) Next, there is an $\epsilon > 0$ such that if $\|\mathbf{r}\| < \epsilon$, then for all ξ , the function $\mathcal{R}^\xi(\mathbf{r}) < \mathbf{r}$ elementwise. So there are no fixed points near $\mathbf{0}$.

(3) For small enough ξ , the function \mathcal{R}^ξ is uniformly small, so $\boldsymbol{\rho}(\xi) = \mathbf{0}$.

These facts together imply that $\boldsymbol{\rho}$ has a discontinuity where it jumps up from $\mathbf{0}$. Let ξ_{crit} be the infimum of the ξ where $\boldsymbol{\rho}(\xi) \neq \mathbf{0}$.

Define

$$\Gamma(\mathcal{R}^\xi(\mathbf{r})) = \{(\mathbf{r}, \mathcal{R}^\xi(\mathbf{r})) : \mathbf{r} \in [0, 1]^{|I|}\}$$

to be the graph of the function. What we have just said corresponds to the fact that this graph intersects the diagonal when $\xi = \xi_{\text{crit}}$, but not for values $\xi < \xi_{\text{crit}}$. Suppose now, toward a contradiction, that the derivative $\boldsymbol{\delta}(\xi)$ of $\boldsymbol{\rho}(\xi)$ is bounded in every coordinate as $\xi \downarrow \xi_{\text{crit}}$. Then, passing to a convergent subsequence and using the smoothness of \mathcal{R} , we find that the derivative $\boldsymbol{\delta}(\xi_{\text{crit}})$ of $\boldsymbol{\rho}$ is well-defined at $\xi = \xi_{\text{crit}}$. But that contradicts our earlier deduction that $\boldsymbol{\rho}$ is discontinuous at ξ_{crit} .

D.11. Proof of Proposition 7. *Part (i):* Let \mathcal{P} be a directed path of length T from node T to node 1 and denote product i by 1. Since product 1 is critical, then following a shock $\epsilon > 0$ to γ_{ij} , $r_1^{*'} = 0$, where the ‘prime’ notation denotes the equilibrium quantity after the shock.

For any product $t + 1$ that sources input t and such that $r_t^{*'} = 0$, we have that

$$\begin{aligned} r_{t+1}^{*'} &= \prod_{l \in \mathcal{I}_{t+1}} (1 - (1 - x_{t+1,l}^* r_l^{*'})^{n_{t+1,l}}) \\ &= 0 \end{aligned}$$

since $t \in \mathcal{I}_{t+1}$.

Since $r_1^{*'} = 0$, it then follows by induction that the production of all products $t \in \mathcal{P}$ will fail.

Part (ii): Suppose production of some product $i \in \mathcal{I}^{SC}$ is critical and consider another product $k \in \mathcal{I}^{SC}$ that is an input for the production of product i (that is, $k \in \mathcal{I}_i$). As an investment equilibrium x^* is being played, the strength x_{kj} of a link from a producer of product k to a supplier of input $j \in \mathcal{I}_k$ must satisfy the following condition

$$MB_{kj} = G(\bar{f}_k r_k^*) \prod_{l \in \mathcal{I}_k, l \neq j} (1 - (1 - x_{kl} r_l^*)^{n_{kl}}) n_{kj} (1 - x_{kj} r_j^*)^{n_{kj}-1} r_j = \gamma_{kj} \tilde{c}'(x_{kj} - \underline{x}_{kj}) = MC_{kj}$$

Rearranging this equation yields

$$\frac{G(\bar{f}_k r_k^*)}{\gamma_{kj}} \prod_{l \in \mathcal{I}_k, l \neq j} (1 - (1 - x_{kl}^* r_l^*)^{n_{kl}}) n_{kj} r_j^* = \frac{\tilde{c}'(x_{kj}^* - \underline{x}_{kj})}{(1 - x_{kj}^* r_j^*)^{n_{kj}-1}}.$$

The right hand side is strictly increasing in x_{kj}^* , while the left hand side is constant.

Consider now a shock $\epsilon > 0$ that changes the value of γ_{kj} to $\gamma'_{kj} = \gamma_{kj} + \epsilon$. By an argument similar to the proof of Lemma 7 in Section SA3.5 the Supplementary Appendix, this strictly reduces $G(\bar{f}_k r_k^*)/\gamma_{kj}$, and hence the new equilibrium investment level satisfies $x_{kj}^{*'} < x_{kj}^*$. This in turn implies that $r_k^{*'} < r_k^*$, and so

$$r_i^{*'} = \prod_{l \in \mathcal{I}_i} (1 - (1 - x_{i,l}^* r_l^{*'})^{n_{i,l}}) \quad (27)$$

$$< \prod_{l \in \mathcal{I}_i} (1 - (1 - x_{i,l}^* r_l^*)^{n_{i,l}}) \quad (28)$$

$$= r_i^* \quad (29)$$

since $k \in \mathcal{I}_i$. Thus $r_i^{*'} = 0$ and the production of product i fails.

Now since both products k and i are part of a strongly connected component, there is also a directed path from k to i . From part (i), it follows that every product t on such a directed path (i.e. every product that uses input i either directly or indirectly through intermediate products)

will also have $r_t^* = 0$. This is true namely for product k and thus $r_k^* = 0$. We therefore conclude that, following a small decrease in its sourcing effort from the initial x_k^* , production of product k fails. Product k was thus necessarily critical and we must have had x_k^* be critical.

Proceeding similarly for any other product $e \in \mathcal{I}_k$ such that $e \in \mathcal{I}^{SC}$, we get that e is also critical. By induction it follows that all products in a strongly connected component \mathcal{I}^{SC} of the product interdependencies graph are critical when one of them is. We conclude that either production of all the products in \mathcal{I}^{SC} are critical or else production of all the products in \mathcal{I}^{SC} are noncritical.

APPENDIX E. ADDING A BANKING SECTOR

Our equilibrium definition requires that a positive mass of firms cannot profitably enter. Arguably, a more reasonable equilibrium definition would be to stipulate that no firm can profitably enter the market. This is problematic. Take a critical equilibrium under the current equilibrium definition. As firms are zero measure, it is possible for an additional firm to enter without changing the mass of firms in the market, and thus without changing investment decisions. Thus, when such firms contemplate entering, they anticipate that their deviation, holding the entry decisions of others fixed, will not affect investments and the economy will not fall off the precipice. Hence, entry for such firms is strictly profitable. So, the critical equilibria we found would not be equilibria under a more stringent equilibrium definition in which no firm can have a profitable deviation from changing its entry choice. Moreover, this problem cannot be resolved by allowing some more firms to enter as there is a positive mass of firms that would want to enter holding fixed the entry decisions of others in any critical equilibrium.

To overcome these technical concerns, in this section we adjust our model by adding a competitive banking sector to it. We now suppose that firms must now pay the fixed entry cost before they produce, and that they do not have cash at this time. Thus they borrow to cover their fixed entry costs. Firms can pledge their profits to repay debt. There is a competitive banking sector consisting of a finite number of banks. Before firms enter, each bank b simultaneously posts an interest rate r_b and a debt limit $\bar{\ell}_b$: it will not lend to a firm that has more than $\bar{\ell}_b$ debt in total. Each bank can service any measure of firms at these terms. Banks' cost of capital is normalized to 0.

As before, conditional on entry, firms maximize net profits Π_{if} .⁶³

To summarize, the timing is:

0. Banks simultaneously set terms: interest rates and lending limits $(r_b, \bar{\ell}_b)$.
1. Firms observe these terms; simultaneously take any loans consistent with the terms; and make their entry decisions.
2. Firms simultaneously choose their effort levels to maximize Π_{if} .
3. The supply tree and payoffs are realized.

Given a profile of bank terms, a *firm outcome* is now determined by entry decisions e_{if} and investment decisions x_{if} for all firms if , as well as of a specification of how much the firm borrows from each bank (which is purely a firm choice, given bank terms). We do not write it explicitly, but the net interest payments that a firm makes to the banks, if it is nonzero, comes out of its profits.⁶⁴

Definition 4. A firm outcome combined with a profile of bank terms is said to be a *full equilibrium* if the firm outcome is an equilibrium and we have two final conditions that are satisfied:

- Feasible entry: a firm if enters only if it takes a loan of sufficient size to pay the fixed cost of entry $\Phi(f)$.
- Optimal bank behavior: no bank can deviate to an interest rate and lending limit $(r'_b, \bar{\ell}'_b)$ that strictly increases its payoff given that an equilibrium is played conditional on these terms.

⁶³This amounts to assuming contracts are good enough to avoid any agency frictions between the firm and bank.

⁶⁴As we will see, given the competitive banking sector, interest payments are zero in equilibrium.

We show now that the equilibrium decisions of firms we found before can be embedded in a full equilibrium of this extended model.

Consider a critical equilibrium. Before, in a critical equilibrium a mass \bar{f}_{crit} of firms enter and choose investments x_{crit} , resulting in reliability r_{crit} . We show that there is a full equilibrium in which firms make the same entry and investment choices, while banks set a lending limit of $\Phi(\bar{f}_{\text{crit}})$ and charge an interest rate of 0.

Given these lending limits and interest rates, the firm entry and investment problem is unchanged from before so we just need to check that the banks have no incentives to deviate.

First, given all other banks set a lending limit of $\Phi(\bar{f}_{\text{crit}})$ and charge an interest rate of 0 it is a best response for a bank b to also set a lending limit of $\Phi(\bar{f}_{\text{crit}})$ and charge an interest rate of 0. A necessary condition for any deviation to be strictly profitable is that a strictly positive interest rate is set. Consider first deviations in which bank b sets a weakly lower lending limit (tighter lending constraint), and charges an interest rate strictly greater than 0. In this case, no firms would seek credit from b following the deviation and it would not be profitable.

Consider now the remaining possible profitable deviations—a deviation in which b sets a strictly higher lending limit and charges a strictly positive interest rate. The only firms willing to pay this higher interest rate must be located at $f > \bar{f}_{\text{crit}}$ (as they otherwise have access to lower interest credit). Moreover, if a firm located at $f > \bar{f}_{\text{crit}}$ finds it profitable to enter, all firms located at $f < \bar{f}_{\text{crit}}$ will also find it profitable to enter. Thus the mass of entering firms must increase above \bar{f}_{crit} . However, in this case, by Lemma 7, equilibrium investment choices after entry would decline below x_{crit} . As such the only possible investment equilibrium choice of entering firms is $x^* = 0$, and at this investment level, the deviating bank would not be able to recover its investment.

Consider a non-critical equilibrium. Before, in a non-critical equilibrium a mass \bar{f} of firms enter such that the marginal entering firm makes zero net profits, and firms choose investments $x > x_{\text{crit}}$, resulting in reliability $\rho > r_{\text{crit}}$. We show that there is a full equilibrium in which firms make the same entry and investment choices, while banks set a lending limit of $\Phi(\bar{f}_{\text{crit}})$ and charge an interest rate of 0.

Given these lending limit and interest rates, firms entry and investment problem is unchanged from before so, again, we just need to check that the banks have no incentives to deviate. Note now that, regardless of the lending limit set by a bank, no firm will seek credit from it if it changes an interest rate above 0. As all firms that can profitably enter the market already have access to credit, a bank that deviates by setting an interest rate above 0 will not be able to extend any credit. Thus no bank has a profitable deviation and we have a full equilibrium.

APPENDIX F. DISTRIBUTION OF PROFITS IN SUPPLY TREES

In this section we relax the assumption that positive profits are only made on final sales to consumers. However, we also make stronger assumptions on the symmetry of technology with respect to different products. For simplicity, we assume that the technology graph is vertex transitive.⁶⁵ This means that for any two vertices there is a graph automorphism that maps one vertex to the other.

In this setting we reinterpret $G(\bar{f}\rho(x))$ to be the overall producer surplus generated by sales of good i by a given firm, conditional on successful production, aggregated over all firms involved directly or indirectly in the successful production of the good. We let this surplus be distributed among suppliers in the successful (realized) supply tree, which we select uniformly at random from among the possible successful supply trees given the realization of relationship specific shocks, with minimal restriction. For each possible successful supply tree $S \in \mathcal{S}$ let $p(S)$ be the different positions in that supply tree, and let $\phi_{kl} : \mathcal{S} \rightarrow \Delta^{|p(S)|}$ be a measurable function mapping successful supply trees into an allocated share of profits for each position in that tree, where Δ^n is the n -dimensional simplex. For example, following Goyal and Vega-Redondo (2007), ϕ might specify that only critical

⁶⁵We leave the exercise of relaxing this assumption to future work.

firms (i.e., those firms in all possible successful supply trees) extract an equal share of profits, while the other firms collectively extract nothing.

Note that by construction a firm kl can only be allocated a positive share of profits if produces successfully. Consider then the aggregate profits obtained for goods of type $i \in \mathcal{I}$ and $j \in \mathcal{I}$. Suppose that a symmetric investment equilibrium is being played with entry \bar{f} and investment \bar{x} . Consider the frequency with which type i products appear as intermediate goods in a given position in the technology tree. As the technology graph is vertex transitive, there is graph automorphism that maps type i into type j , type j products must be used as intermediate goods in such a position with exactly the same frequency. Moreover, given that all links fail with the same probability in a type symmetric equilibrium, and as the provider of each input is selected uniformly at random from those producing that good, the probability a given type i firm appears in any position in a successful supply tree is equal to the probability any type j firm appears in that position in a successful supply tree.

Let $H(i, f, \bar{f}, \rho(x))$ be the expected profits of firm if , for $f \leq \bar{f}$, aggregated over all successful supply trees conditional on if 's successful production. Thus firm if receives expected overall profits of

$$H(i, f, \bar{f}, \rho(x))\mathbf{E}[F_{if}] - c(x_{if} - \underline{x}) - \Phi(f).$$

However, by the above argument, $H(j, f', \bar{f}\rho(x)) = H(i, f, \bar{f}\rho(x)) = H(\bar{f}\rho(x))$ for all j and all $f' \leq \bar{f}$. Moreover, as ϕ distributes profits $G(\bar{f}\rho(x))$ for each successfully produced product, the preservation of these profits implies that $H(\bar{f}\rho(x)) = G(\bar{f}\rho(x))$.

APPENDIX G. ADJUSTING INVESTMENTS IN RESPONSE TO SHOCKS

In this section we consider an alternative definition of a fragile equilibrium in which firms are able to adjust their investment choices in response to a shock. Interestingly, this has no impact on our characterization of fragile equilibria.

Definition 5 (Equilibrium fragility').

- A productive equilibrium is *fragile'* if, holding the fraction of firms in the market, \bar{f}^* , fixed, any negative shock $\epsilon > 0$ such that \underline{x} decreases to $\underline{x} - \epsilon$ results in equilibrium output falling to 0 (such that $\rho(x^*(\underline{x} - \epsilon)) = 0$).
- A productive equilibrium is *robust'*, if it is not *fragile'*.

In a *fragile'* equilibrium, firms are allowed to adjust their multisourcing efforts following the negative shock, but entry decisions are sunk (i.e. the number of firms \bar{f}^* in the market cannot change).

Proposition 8. If $\kappa \leq \bar{\kappa}$, then any productive equilibrium is *fragile'*. If $\kappa > \bar{\kappa}$, then any productive equilibrium is *robust'*.

We prove Proposition 8 below in Section G.2. It might be hoped that by adjusting their investment decisions the entering firms could absorb the shock and avoid production failing for sure. Proposition 8 shows that this does not happen. A severe free-riding problem prevents the entering firms from collectively increasing their investment to offset the shock. On the other hand, if (perhaps in the longer term) firms can re-optimize their entry decisions, then the probability of successful production need not collapse to 0.

G.1. Interdependent supply networks and cascading failures. We now posit an interdependence among supply networks wherein each firm's profit depends on the *aggregate* level of output in the economy, in addition to the functionality of the suppliers with whom it has supply relationships. Formally, suppose, $\kappa_s = K_s(Y)$, where K_s is a strictly increasing function and Y is the integral across all sectors of equilibrium output:

$$Y = \int_{\mathcal{S}} \bar{f}_s^* \rho(x_s^*) d\Phi(\mathfrak{s}).$$

Here we denote by (\bar{f}_s^*, x_s^*) the unique positive equilibrium in sector s . The output in the sector is the mass of entering firms, \bar{f}_s^* multiplied by the reliability in that sector, $\rho(x_s^*)$.

The interpretation of this is as follows: When a firm depends on a different sector, a specific supply relationship is not required, so the idiosyncratic failure of a given producer in the different sector does not matter—a substitute product can be readily purchased via the market. (Indeed, it is precisely when substitute products are not readily available that the supply relationships we model are important.) However, if some sectors experience a sudden drop in output, then other sectors suffer. They will not be able to purchase inputs, via the market, from these sectors in the same quantities or at the same prices. For example, if financial markets collapse, then the productivity of many real businesses that rely on these markets for credit are likely to see their effective productivity fall. In these situations, dependencies will result in changes to other sectors' profits even if purchases are made via the market. Our specification above takes interdependencies to be highly symmetric, so that only aggregate output matters, but in general these interdependencies would correspond to the structure of an intersectoral input-output matrix, and K would be a function of sector level outputs, indexed by the identity of the sourcing sector.

This natural interdependence can have very stark consequences. Consider an economy characterized by a distribution Ψ in which the subset of sectors with $m \geq 2$ has positive measure, and some of these have positive equilibria. Suppose that there is a small shock to \underline{x} . As already argued, this will directly cause a positive measure of sectors to fail. The failure of the fragile sectors will cause a reduction in aggregate output. Thus $\kappa_s = K_s(Y)$ will decrease in other sectors *discontinuously*. This will take some other sectors out of the robust regime. Note that this occurs due to the other supply chains failing and not due to the shock itself. As these sectors are no longer robust, they topple too following an infinitesimal shock to \underline{x} . Continuing this logic, there will be a domino effect that propagates the initial shock. This domino effect could die out quickly, but need not. A full study of such domino effects is well beyond our scope, but the forces in the very simple sketch we have presented would carry over to more realistic heterogeneous interdependencies.

Fig. 21 shows⁶⁶ how an economy with 100 interdependent sectors responds to small shocks to \underline{x} . In this example, sectors differ only in their initial κ 's. The technological complexity is set to $m = 5$ and the number of potential suppliers for each firm is set to $n = 3$. The cost function⁶⁷ for any firm if is $c(x_{if}) = x_{if}^2$ while the gross profit function is $g(\bar{f}\rho(x)) = 5(1 - \bar{f}\rho(x))$ and the entry cost function is $\Phi(f) = f$. This setup yields values $\underline{\kappa} = 0.298$ and $\bar{\kappa} = 0.673$ delimiting the region corresponding to critical (and therefore fragile) equilibria, as per Theorem 1 and Proposition 8.

In Fig. 21(a), the productivity shifter of a given sector is distributed uniformly, i.e. $\kappa_s \sim U(\underline{\kappa}, 3.5)$, so that many sectors have high enough productivity to be in a robust equilibrium while a small fraction have low enough productivity to be in a fragile equilibrium. A small shock to the \underline{x} of all sectors thus causes the failure of the fragile sectors (13 in total). This then decreases the output Y across the whole economy, but only to a small extent (as seen in the right panel). The resulting decrease in the productivities of the robust sectors is thus not enough to bring them into the fragile regime and thus to cause them to fail as well upon a small shock.

In contrast, Fig. 21(b) shows an economy where $\kappa_s \sim U(\underline{\kappa}, 1.9)$, so that more sectors have low enough productivity to be in a fragile equilibrium. A small shock to the \underline{x} of all sectors causes the failure of the fragile sectors (now 24). These have a larger effect on decreasing the output Y across

⁶⁶Note that in this example, the cascade dynamics is as follows: At step 1, firms in sectors with a κ in the fragile range fail due to an infinitesimal shock to \underline{x} . The initial economy-wide output Y_1 is then decreased to Y_2 and the κ 's are updated using an updating function $K(Y)$ increasing in Y . Only then, are the firms in the surviving sectors allowed re-adjust both f and x_{if} . At step 2, infinitesimal shocks hit again and the firms newly found in the fragile regime fail. This process goes on at each step until no further firm fails, at which point the cascade of failures stops. Note that we could also prevent firms from adjusting entry at all throughout all steps of the cascade. This would obviously worsen the number of sectors that fail. This example is thus a conservative estimate of the firms that could fail due to the interdependence of the sectors in an economy.

⁶⁷For simplicity, we set $\underline{x} = 0$. An infinitesimal shock to \underline{x} has the effect of causing the firms of sectors in the fragile regime to fail, but does not affect the value of \underline{x} , which remains at 0.

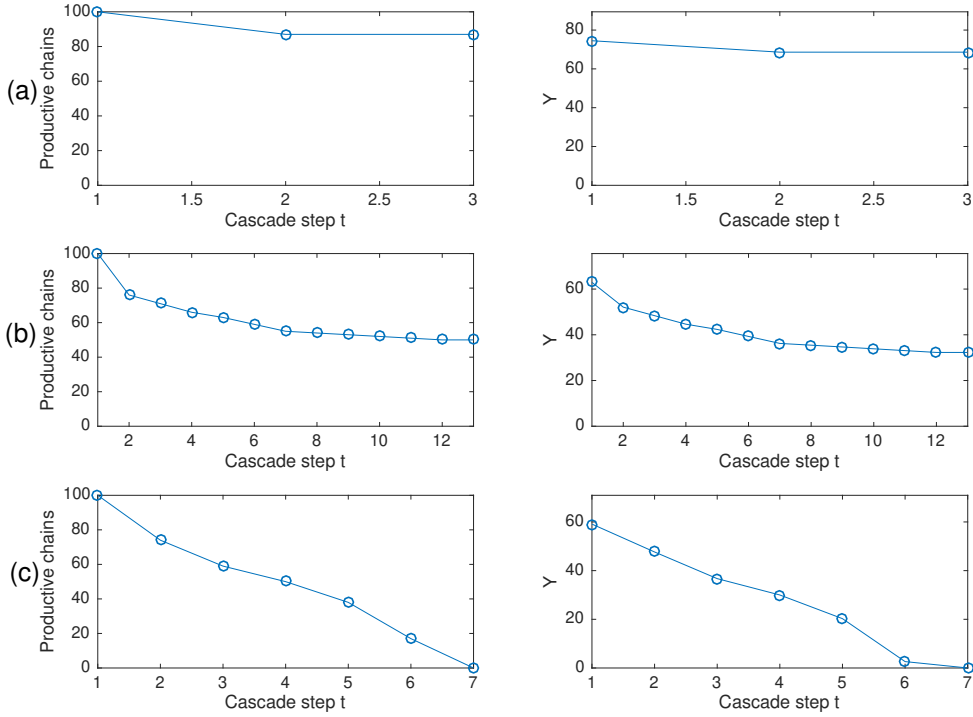


FIGURE 21. Number of sectors that remain productive (left) and economy-wide output Y (right) for each step of a cascade of failures among 100 interdependent sectors. For all sectors: $n = 3$, $m = 5$, $c(x_{if}) = x_{if}^2$, $g(\bar{f}\rho(x)) = 5(1 - \bar{f}\rho(x))$ and $\Phi(f) = f$. This yields $\underline{\kappa} = 0.298$ and $\bar{\kappa} = 0.673$. In row (a), 100 sectors have κ_s initially distributed according to $U(\underline{\kappa}, 3.5)$; In row (b), 100 sectors have κ_s initially distributed according to $U(\underline{\kappa}, 1.9)$; In row (c), 100 sectors have κ_s initially distributed according to $U(\underline{\kappa}, 1.6)$.

the whole economy (as seen in the right panel). The resulting decrease in the productivities of the robust sectors is now enough to bring some of them into the fragile regime and to cause them to fail too upon an infinitesimal shock to \underline{x} . This initiates a cascade of sector failures, ultimately resulting in 50 sectors ceasing production.

Fig. 21(c) shows an economy where $\kappa_s \sim U(\underline{\kappa}, 1.6)$, so that even more sectors have low enough productivity to be in a fragile equilibrium. A small shock to the \underline{x} of all sectors causes the failure of the fragile sectors (now 26 in total) and this initiates a cascade of sector failures which ultimately brings down all 100 sectors of the economy.

The discontinuous drops in output caused by fragility, combined with the simple macroeconomic interdependence that we have outlined, come together to form an amplification channel reminiscent, e.g., of Elliott, Golub, and Jackson (2014) and Baqaee (2018). Thus, the implications of those studies apply here: both the cautions regarding the potential severity of knock-on effects, as well as the importance of preventing first failures before they can cascade.

G.2. Proof of Proposition 8.

Proof. Consider a reduction in \underline{x} to $\underline{x} - \epsilon$ for $\epsilon > 0$. From Theorem 1, we know that if $\kappa \leq \bar{\kappa}(\underline{x})$, either $\kappa \leq \underline{\kappa}(\underline{x})$, in which case there is no productive equilibrium, or $\underline{\kappa}(\underline{x}) \leq \kappa \leq \bar{\kappa}(\underline{x})$, in which case, any productive equilibrium is a critical equilibrium.

First observe that $\bar{\kappa}(\underline{x})$ is strictly increasing in \underline{x} . Recall that to save on notation we set $\underline{x} = 0$ for many of our proofs. This is also true in equation (20), but once \underline{x} is reintroduced we have,

$$\bar{\kappa}(\underline{x}) = \frac{c'(x_{\text{crit}} - \underline{x})}{g(\bar{f}_{\text{crit}} r_{\text{crit}}) m n r_{\text{crit}} (1 - x_{\text{crit}} r_{\text{crit}})^{n-1} (1 - (1 - x_{\text{crit}} r_{\text{crit}})^n)^{m-1}},$$

and so $\bar{\kappa}(\underline{x})$ decreases continuously in \underline{x} . Similarly, recalling that

$$\underline{\kappa} = \frac{c'(x_{\text{crit}} - \underline{x})}{g(0) m n r_{\text{crit}} (1 - x_{\text{crit}} r_{\text{crit}})^{n-1} (1 - (1 - x_{\text{crit}} r_{\text{crit}})^n)^{m-1}},$$

it is immediate that $\underline{\kappa}$ decreases continuously in \underline{x} .

Consider first $\kappa \leq \underline{\kappa}$ and the corresponding equilibrium $(\bar{f}^*, x^*(\bar{f}^*))$, where, by Theorem 1, $x^*(\bar{f}^*) = 0$. Fixing the fraction of firms in the market at \bar{f}^* , as $\bar{\kappa}(\underline{x})$ is decreasing in \underline{x} , after the shock to \underline{x} we must have $\tilde{x}^*(\bar{f}^*) = 0$

Consider now $\underline{\kappa} \leq \kappa \leq \bar{\kappa}$ and the corresponding equilibrium $(\bar{f}^*, x^*(\bar{f}^*))$, where, by Theorem 1, $x^*(\bar{f}^*) = x_{\text{crit}}$. Fixing the fraction of firms in the market at \bar{f}^* , as $\bar{\kappa}(\underline{x})$ is decreasing in \underline{x} , $\tilde{x}^*(\bar{f}^*) \leq x_{\text{crit}}$. By equation (11), the marginal costs from investments evaluated at x_{crit} are strictly increasing in \underline{x} , while the marginal benefits of investment do not depend on \underline{x} . Thus, following *any* shock $\epsilon > 0$ to \underline{x} marginal benefits are the same and marginal costs are strictly higher at the same investment choice x_{crit} . As shown in the proof of Proposition 3, marginal benefits are (strictly) decreasing in x while marginal costs are (strictly) increasing in x . Thus, as \underline{x} has decreased the effort level must drop to keep marginal benefits equal to marginal costs. Denote by $\tilde{x}^*(\bar{f}^*)$, the optimal effort level (where marginal benefit equals marginal cost) after \underline{x} as decreased while keeping \bar{f}^* unchanged:

$$MB(\tilde{x}; \rho(\tilde{x}), \bar{f}^*, \underline{x} - \epsilon) = MC(\tilde{x}).$$

In the new investment equilibrium we therefore have $\tilde{x} = \tilde{x}^*(\bar{f}^*) < x^*(\bar{f}^*) = x$. Since in a critical equilibrium, $x = x_{\text{crit}}$, it follows that the after-shock, adjusted effort level, \tilde{x} , is smaller than x_{crit} and thus $\rho(\tilde{x}) = 0$ (the probability of producing collapses to zero). From Definition 3, the equilibrium $(\bar{f}^*, x^*(\bar{f}^*))$ is thus *fragile*'.

Now consider $\kappa > \bar{\kappa}$ and the corresponding equilibrium $(\bar{f}^*, x^*(\bar{f}^*) = x)$. Then, fixing entry \bar{f}^* , due to the continuity of the marginal costs in $x^*(\bar{f}^*) = 0$ and continuity of $\bar{\kappa}$ in \underline{x} , there exists a small-enough shock $\epsilon > 0$ such that $\bar{\kappa} < \tilde{\kappa}$ and the new investment equilibrium is such that $x > \tilde{x} > x_{\text{crit}}$. Thus $\rho(\tilde{x}) > \rho(x_{\text{crit}}) > 0$ and, by Definition 3, the equilibrium $(\bar{f}^*, x^*(\bar{f}^*))$ is *robust*'. Only large enough shocks could cause production to fail. \square