

ON REMARKABLE REPRESENTATIONS

$$\pi_1 \left(\text{Diagram of a surface with boundary components} \right) \rightarrow \mathrm{PSL}_2 \mathbb{R}$$

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Our plan today

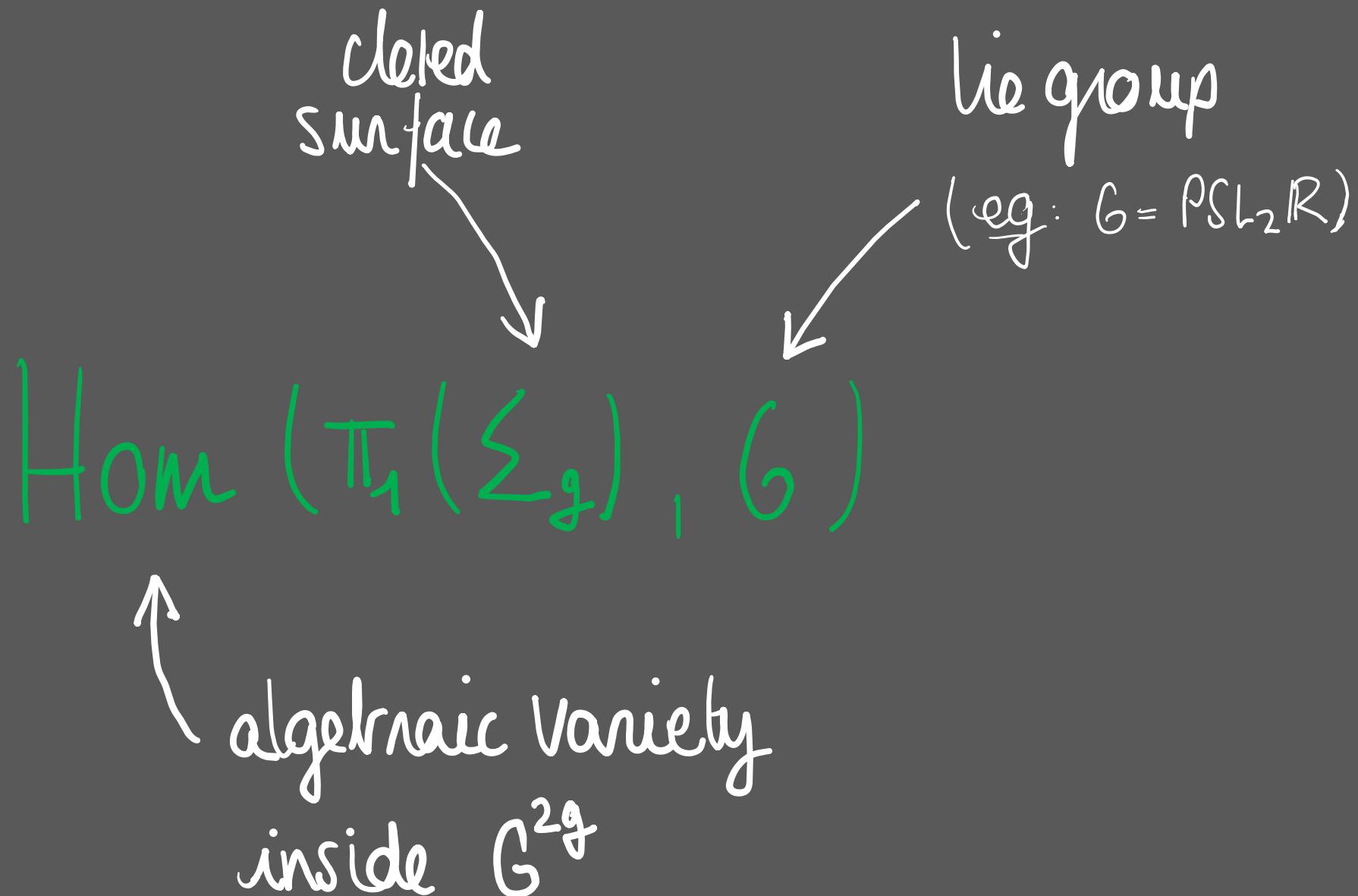
- (1) Character varieties
- (2) Dernin - Tholezan representations
- (3) A combinatorial model

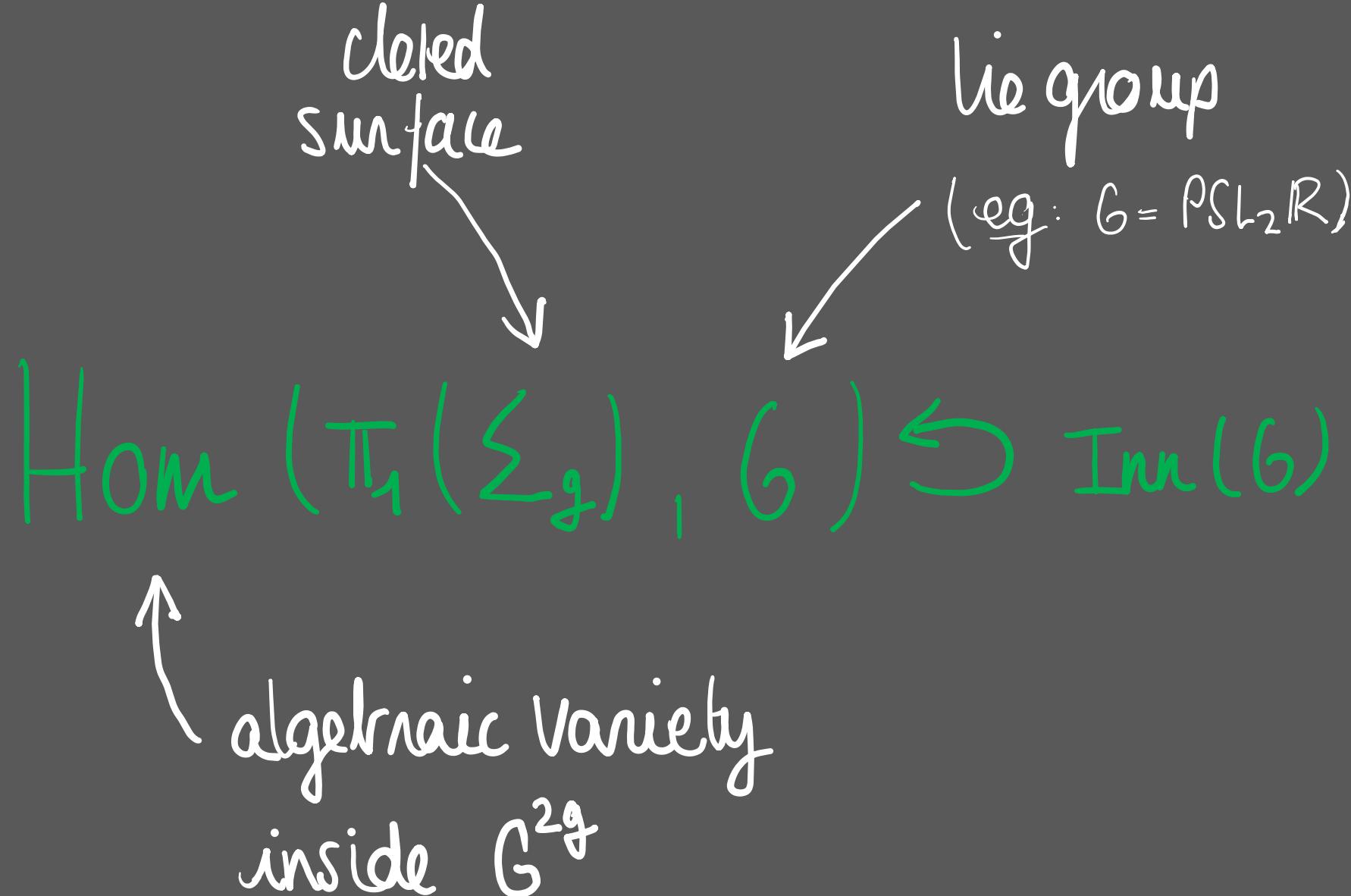
(1) Character Varieties

$$\text{Hom}(\pi_1(\Sigma_g), G)$$

clerical surface

lie group
(eg: $G = \mathrm{PSL}_2 \mathbb{R}$)





$$\text{Rep}(\Sigma_g, G) := \frac{\text{Hom}(\pi_1(\Sigma_g), G)}{\text{Inn}(G)}$$

↑ character variety
of (Σ_g, G)

punctured
surface

lie group

$$\text{Hom}(\pi_1(\Sigma_{g,n}), G)$$

punctured
surface

Lie group

$$\text{Hom}(\pi_1(\Sigma_{g,n}), G)$$

$$\pi_1 \left(\text{punctured surface} \rightarrow \text{Lie group} \right) = \langle a_1, \dots, a_g, b_1, \dots, b_g, c_1, c_2, \dots, c_n \mid \prod_{i=1}^g [a_i, b_i] = \prod_{j=1}^n c_j \rangle$$

The diagram shows a surface with two blue handles (punctures) and several green curves (cycles). Red numbers a_1, \dots, a_g are placed above the handles, and blue numbers b_1, \dots, b_g are placed below them. Green numbers c_1, c_2, \dots, c_n are placed above the green curves.

punctured
surface

Lie group

$$\text{Hom}(\pi_1(\Sigma_{g,n}), G)$$

$$\pi_1 \left(\text{punctured surface} \cup \text{Lie group} \right) \cong F_{2g+n-1}$$

The diagram shows a surface with genus g represented by two blue ovals. There are n punctures, each represented by a red circle with a small white loop inside. The punctures are located on the surface. Red curves labeled a_1, \dots, a_g represent the boundary curves of the genus components. Green curves labeled c_1, c_2, \dots, c_n represent the boundary curves of the punctures.

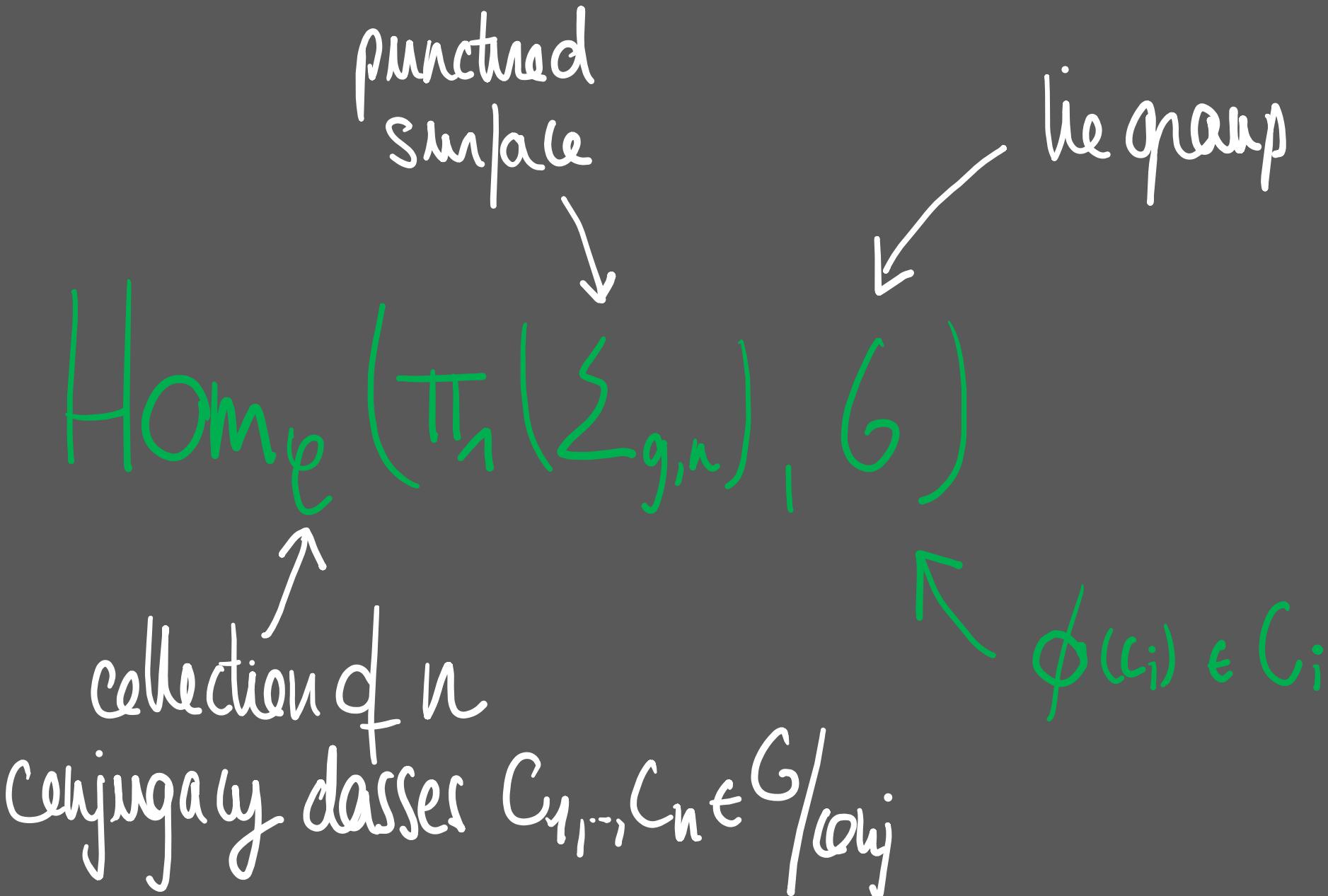
punctured
surface

Lie group

$$\text{Hom}(\pi_1(\Sigma_{g,n}), G) \cong G^{2g+n-1}$$

$$\pi_1 \left(\text{punctured surface} \right) \cong F_{2g+n-1}$$

The diagram shows a surface with genus g represented by two blue ovals at the bottom. Above them, red loops labeled a_1, \dots, a_g represent the handle curves. The surface then splits into several components, each ending in a puncture. The punctures are represented by green ovals labeled c_1, c_2, \dots, c_n . A white arrow points from the punctured surface diagram up to the Homomorphism equation.



punctured surface

lie group

collection of n

Cohomology classes $c_1, c_n \in G/\text{colj}$

$$\text{Rep}_{\mathcal{E}}(\Sigma_{g,n}, G) := \frac{\text{Hem}_{\mathcal{E}}(\pi_1(\Sigma_{g,n}), G)}{\text{Inn}(G)}$$



relative character
variety of $(\Sigma_{g,n}, G)$

main task : study

the topology / geometry
of (relative) character varieties

Refe ($\Sigma_{g,n}$, G)

main task : study

the topology / geometry
of (relative) character varieties

Refe ($\Sigma_{g,n}$, G)

(1) Goldman symplectic
structure

main task : study

the topology / geometry
of (relative) character varieties

$\text{Rep}_\mathbb{C}(\Sigma_{g,n}, G)$

(1) Goldman symplectic
structure

(2) Toledo number

even dimensional
manifold

symplectic
manifold

(M, ω)

closed & non-degenerate
2-form

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graph LR; A[even dimensional manifold] --> B["(M, ω)"]; C[symplectic manifold] --> B; B --> D[closed & non-degenerate 2-form]
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even dimensional
manifold

closed & non-degenerate
2-form

(M, ω)

symplectic
manifold

Section of T^*M $\xleftrightarrow{\omega}$ Section of TM

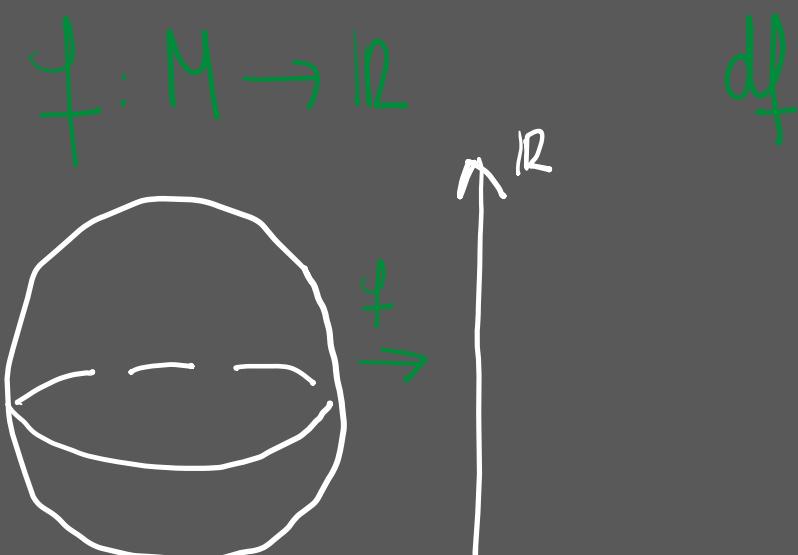
even dimensional
manifold

symplectic
manifold

(M, ω)

closed & non-degenerate
2-form

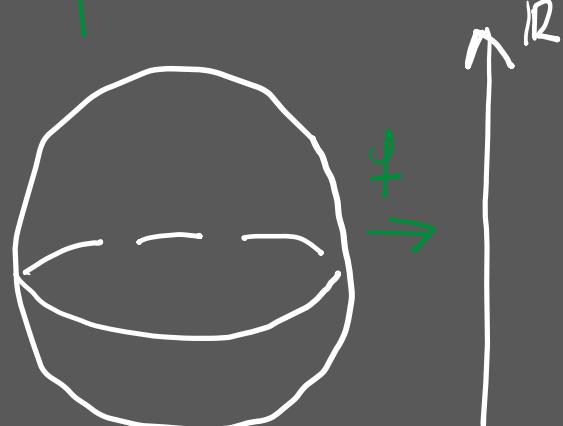
smooth
function \rightsquigarrow section of T^*M $\xleftrightarrow{\omega}$ section of TM



even dimensional
 manifold $\xrightarrow{\quad}$ (M, ω) $\xleftarrow{\quad}$ closed & non-degenerate
 symplectic manifold
 2-form

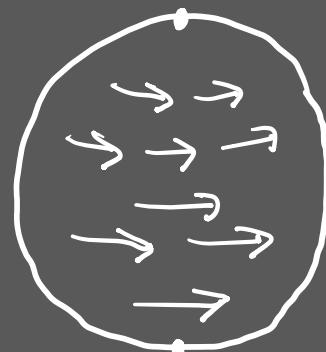
smooth function \rightsquigarrow section of $T^*M \xleftrightarrow{\omega}$ section of $TM \rightsquigarrow$ flow

$$f: M \rightarrow \mathbb{R}$$

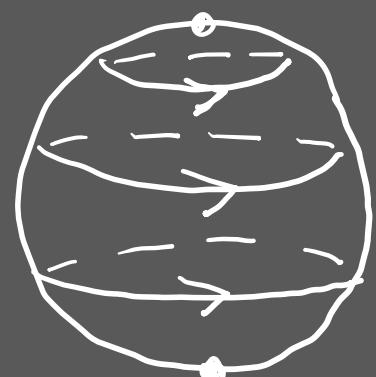


$$df$$

$$X_f$$



$$\Phi_t^t : M \rightarrow M, t \in \mathbb{R}$$



$$\text{Tol} : \text{Rep}(\Sigma_{g,n}, G) \rightarrow \mathbb{R}$$

Toledo number
 [Burger-Torzi-Wienkard]

- * continuous
- * locally constant or relative character variety

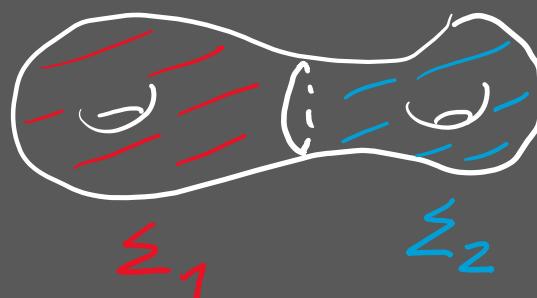
* (additivity)

$$\text{Tol}(\phi) = \text{Tol}(\phi|_{\pi_1(\Sigma_1)}) + \text{Tol}(\phi|_{\pi_1(\Sigma_2)})$$

* (Nilmer-Wood inequality)

$$|\text{Tol}| \leq |\chi(\Sigma_{g,n})| \cdot \text{rank}(G)$$

* (...)



(relative) character
variety

= symplectic manifold
associated to
 $(\mathcal{E}_{g,n}, \mathcal{G})$

$(Rep_{\mathcal{C}}(\mathcal{E}_{g,n}, \mathcal{G}), \omega_{Goddman})$

(2) Denain - Thibzan Representations

A remarkable example : Derafsh-Tahoori representations

$$\text{surface} = \Sigma_{0,n}$$



$$\text{lie group} = \mathrm{PSL}_2 \mathbb{R}$$

compact
component

$$\mathrm{Rep}_{\alpha}^{\mathrm{DT}}(\Sigma_{0,n}, \mathrm{PSL}_2 \mathbb{R}) \subseteq \mathrm{Rep}_{\alpha}(\Sigma_{0,n}, \mathrm{PSL}_2 \mathbb{R})$$

↪ elliptic conjugacy classes

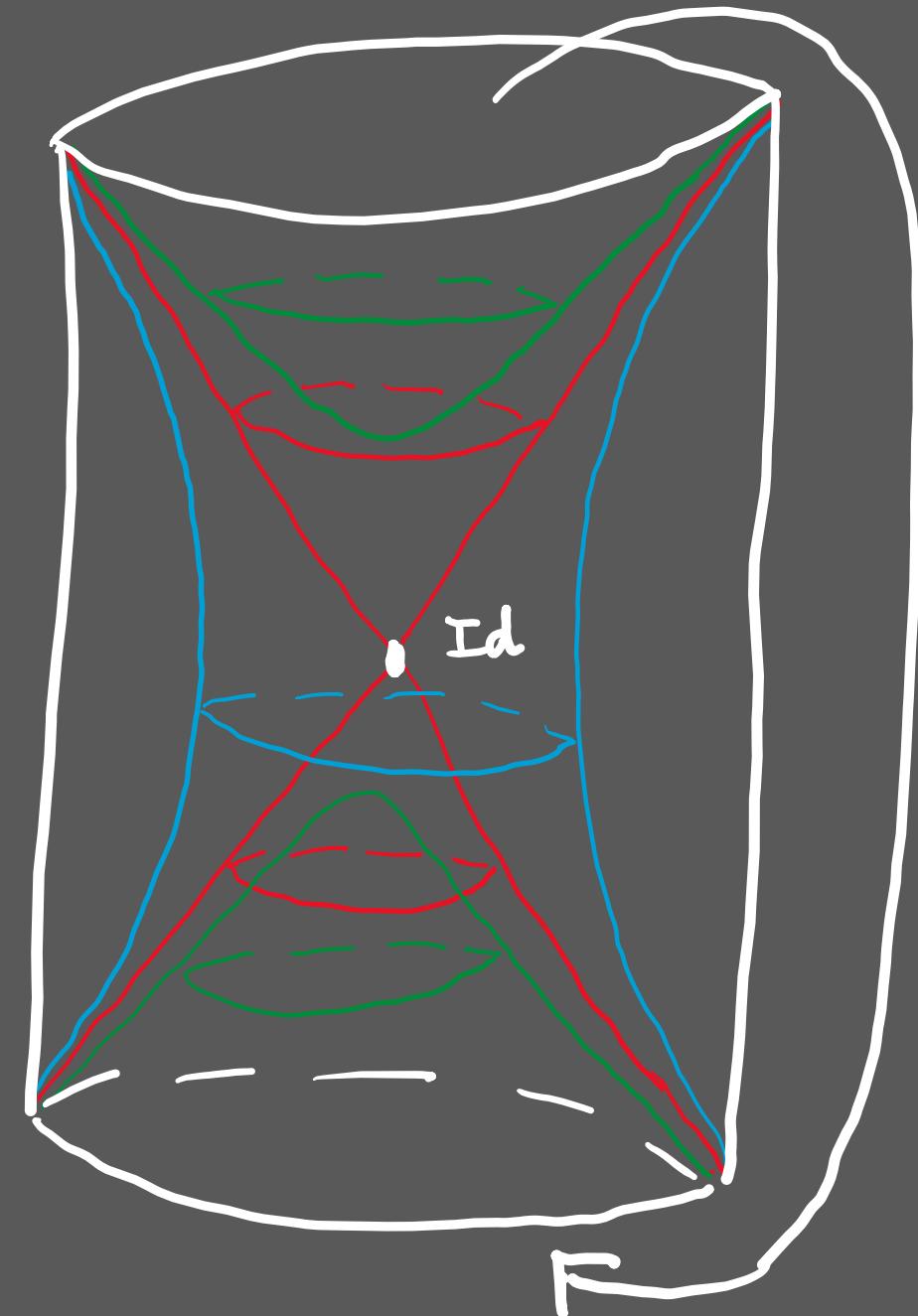
$PSL_2 \mathbb{H}^2$

elliptic = {unique fixed point in \mathbb{H}^2 }

$\cong \mathbb{H}^2 \times (0, 2\pi)$

fixed point

ϑ = angle of rotation

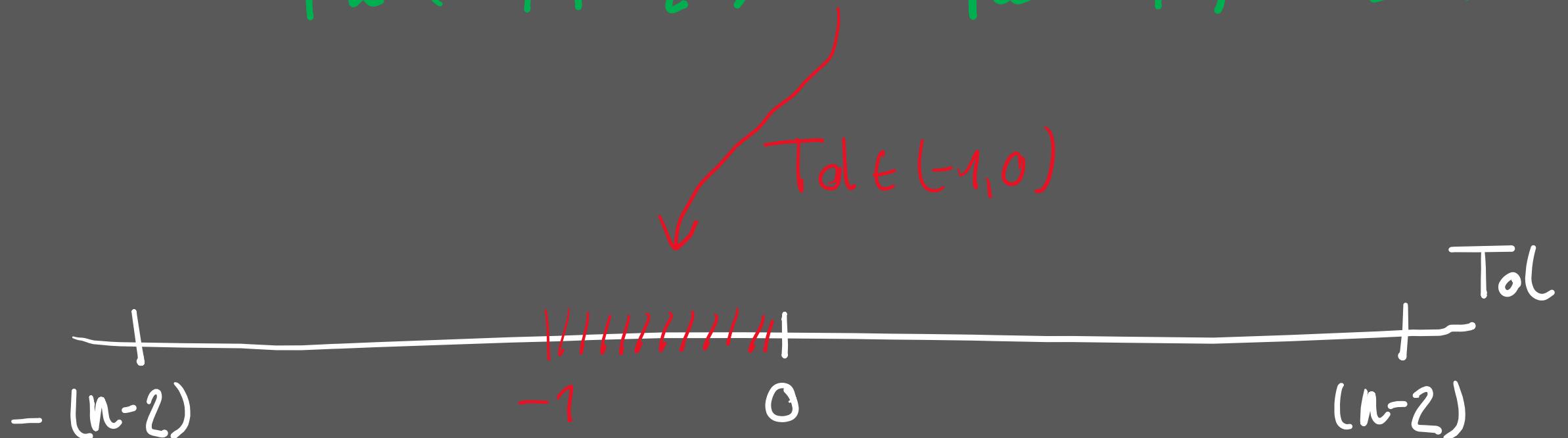


$$\alpha = (\alpha_1, \dots, \alpha_n) \in (0, 2\pi)^n$$

← crucial angles condition

$$2\pi \cdot n > \alpha_1 + \dots + \alpha_n > 2\pi \cdot (n-1)$$

$$\text{Rep}_\alpha^{\text{DT}}(\Sigma_{o,n}, \text{PSL}_2 \mathbb{R}) \subseteq \text{Rep}_\alpha(\Sigma_{o,n}, \text{PSL}_2 \mathbb{R})$$



THM [Benoist - Tholozan]

$$\left(\text{Rep}_{\mathbb{Q}}^{\text{DT}}(\Sigma_{0,n}, \text{PSL}_2 \cdot \mathbb{R}) \right)_{\text{W}_{\text{Goldman}}} \cong \left(\mathbb{C}P^{n-3}, \lambda \cdot \omega_{\text{FS}} \right)$$

$$\begin{aligned}\lambda &= \alpha_1 + \dots + \alpha_n - 2\pi(n-1) \\ &= -\frac{1}{2\pi} \text{Td}(\phi)\end{aligned}$$

THM [Benoist - Tholozan]

$$\left(\text{Rep}_{\mathbb{Q}}^{\text{PT}}(\Sigma_{0,n}, \text{PSL}_2 \cdot \mathbb{R}) \right)_+ w_{\text{Goldman}} \cong (\mathbb{C}P^{n-3}, \lambda \cdot \omega_{\text{FS}})$$

↑
compact

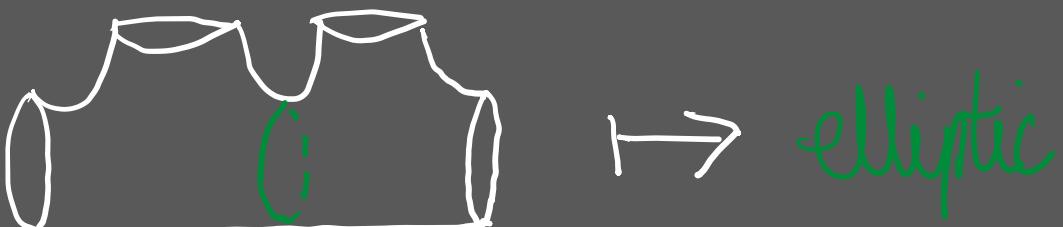
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THM [Benoist - Tholozan]

$$\left(\text{Rep}_{\mathbb{Q}}^{\text{OT}}(\Sigma_{0,n}, \text{PSL}_2 \cdot \mathbb{R}) \right)_{\text{W}_{\text{Goldman}}} \cong (\mathbb{C}\mathbb{P}^{n-3}, \lambda \cdot w_{\text{FS}})$$

↑ ↑
compact totally elliptic

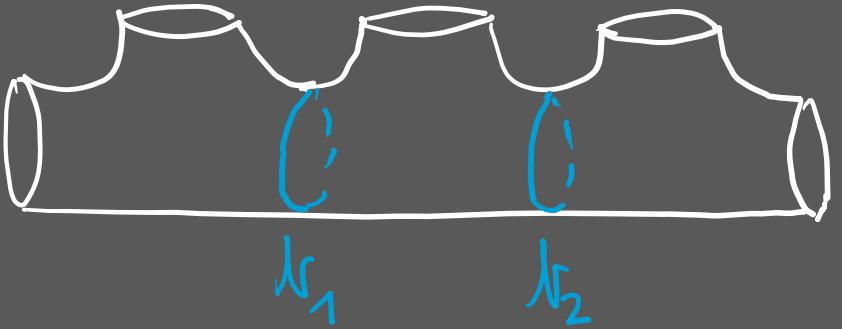
$$\begin{aligned}\lambda &= \alpha_1 + \dots + \alpha_n - 2\pi(n-1) \\ &= -\frac{1}{2\pi} \text{Td}(\phi)\end{aligned}$$



Proof: $T^{n-3} \hookrightarrow (\text{Rep}_\alpha^{\text{DT}}, w_{\text{Goldman}})$

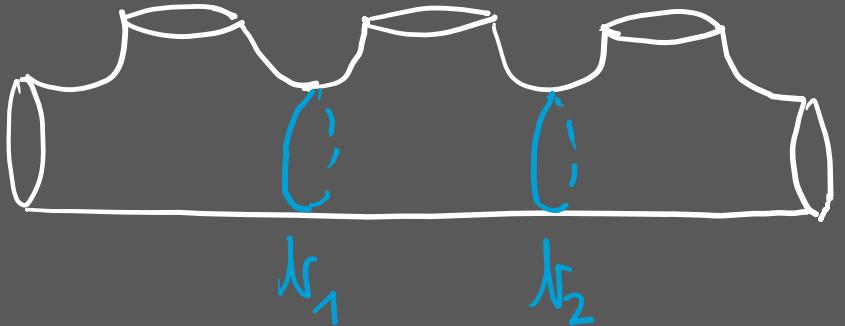
Proof: $T^{n-3} \hookrightarrow (\text{Rep}_\alpha^{\text{DT}}, \omega_{\text{Goldman}})$

($n=5$)



Proof: $T^{n-3} \hookrightarrow (\text{Rep}_{\alpha}^{\text{DT}}, \omega_{\text{Goldman}})$

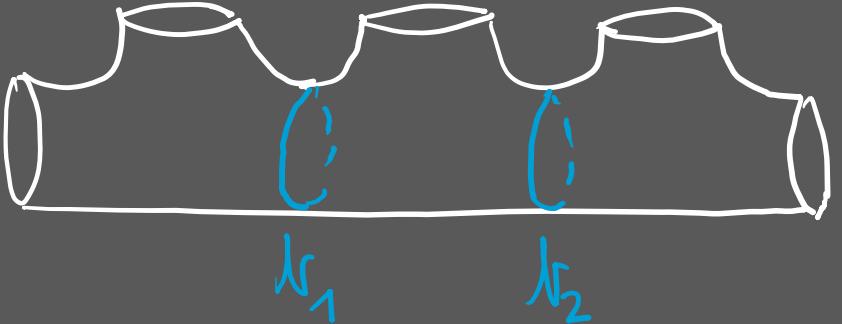
(n=5)



$\rightsquigarrow \mathcal{V}_{h_i}: \text{Rep}_{\alpha}^{\text{DT}} \rightarrow (0, 2\pi)$

Proof: $T^{n-3} \hookrightarrow (\text{Rep}_{\alpha}^{\text{DT}}, \omega_{\text{Goldman}})$

(n=5)



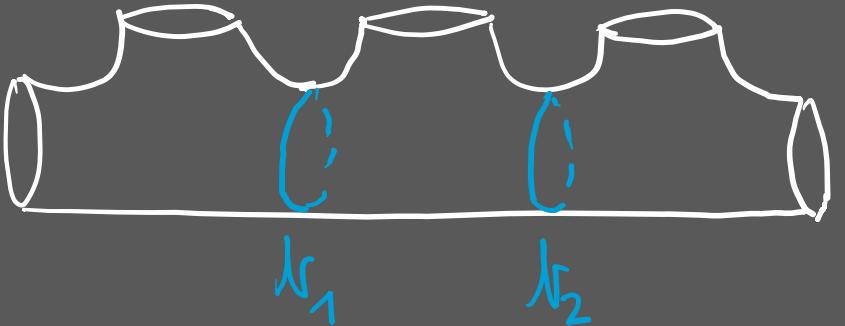
$\rightsquigarrow \vartheta_{h_i}: \text{Rep}_{\alpha}^{\text{DT}} \rightarrow (0, 2\pi)$

\rightsquigarrow periodic & commutative

flows $\Phi_{h_i}: \mathbb{R} \times \text{Rep}_{\alpha}^{\text{DT}} \rightarrow \text{Rep}_{\alpha}^{\text{DT}}$

Proof: $T^{n-3} \hookrightarrow (\text{Rep}_{\alpha}^{\text{DT}}, \omega_{\text{Goldman}})$

(n=5)



$\rightsquigarrow \vartheta_{k_i} : \text{Rep}_{\alpha}^{\text{DT}} \rightarrow (0, 2\pi)$

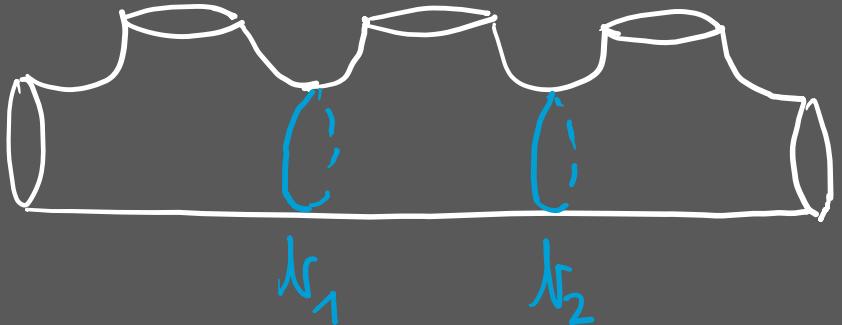
$\rightsquigarrow \underline{\text{periodic \& commutative}}$

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$= T^{n-3} \hookrightarrow (\text{Rep}_{\alpha}^{\text{DT}}, \omega_{\text{Goldman}})$

Proof: $T^{n-3} \hookrightarrow (\text{Rep}_{\alpha}^{\text{DT}}, \omega_{\text{Goldman}})$

(n=5)



$\rightsquigarrow \vartheta_{k_i}: \text{Rep}_{\alpha}^{\text{DT}} \rightarrow (0, 2\pi)$

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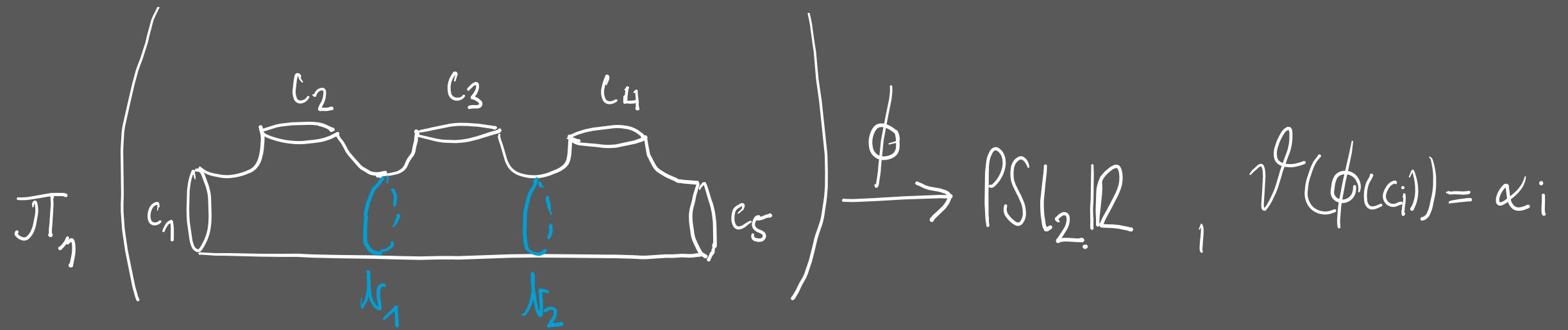
flows $\Phi_{k_i}: \mathbb{R} \times \text{Rep}_{\alpha}^{\text{DT}} \rightarrow \text{Rep}_{\alpha}^{\text{DT}}$

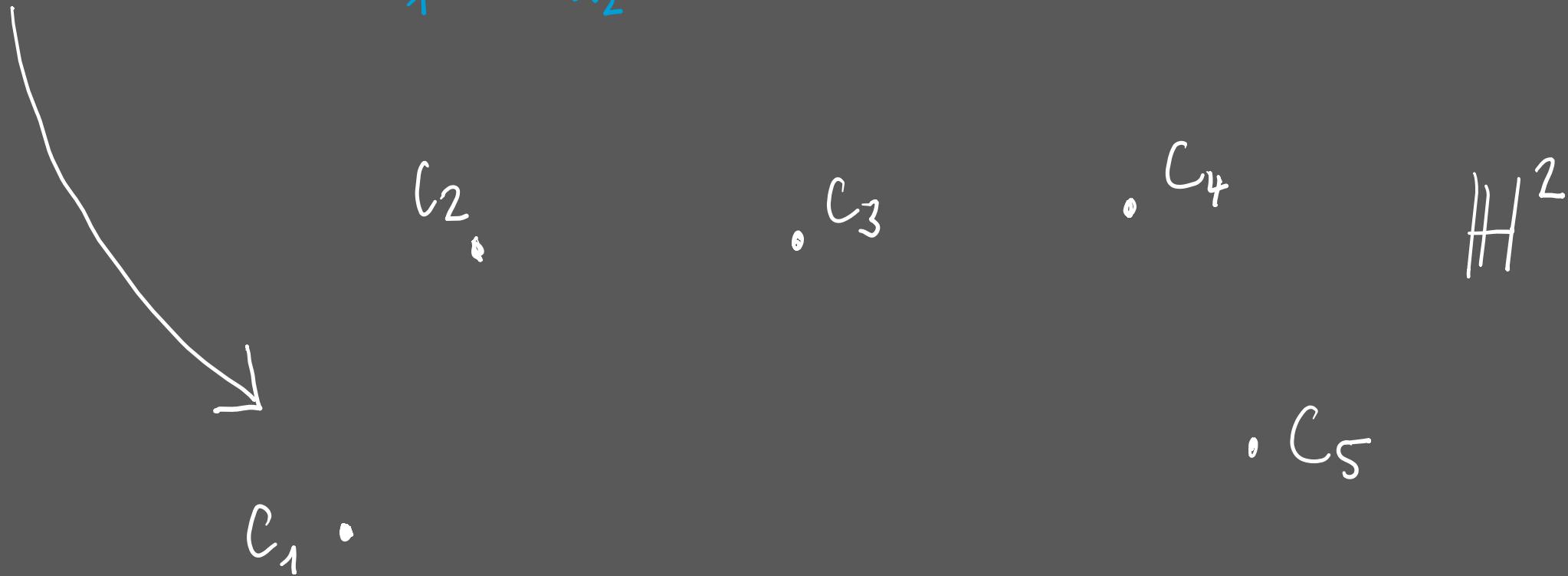
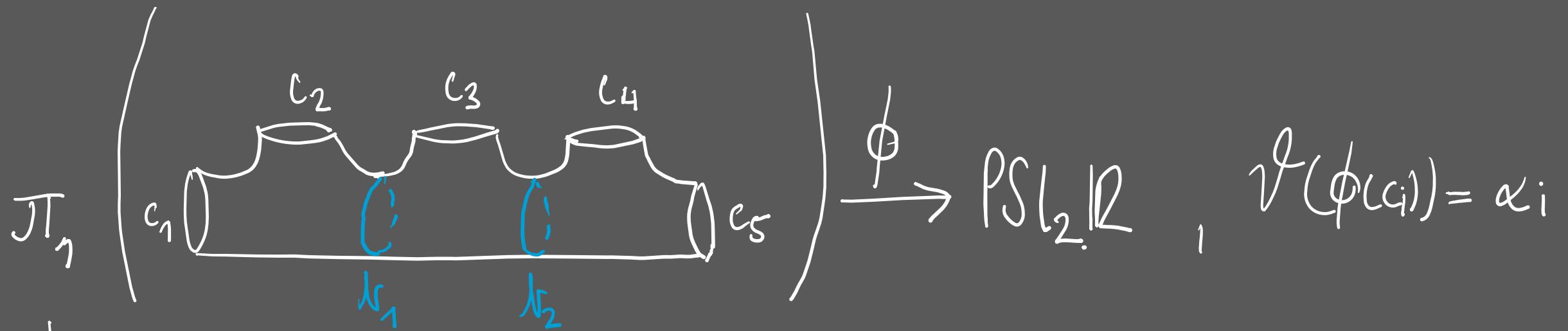
$= T^{n-3} \hookrightarrow (\text{Rep}_{\alpha}^{\text{DT}}, \omega_{\text{Goldman}})$

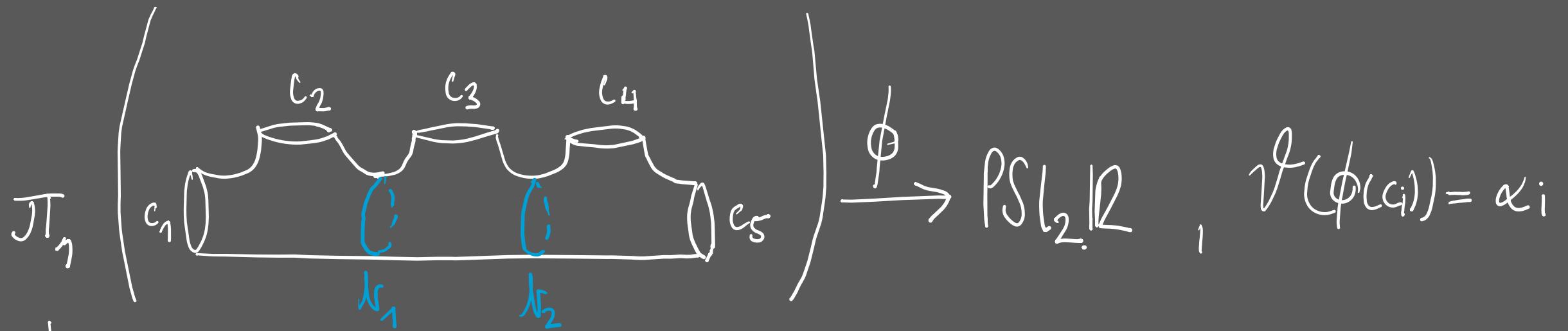
Delzant's classification $\Rightarrow \text{Rep}_{\alpha}^{\text{DT}} \cong \mathbb{C}P^{n-3}$

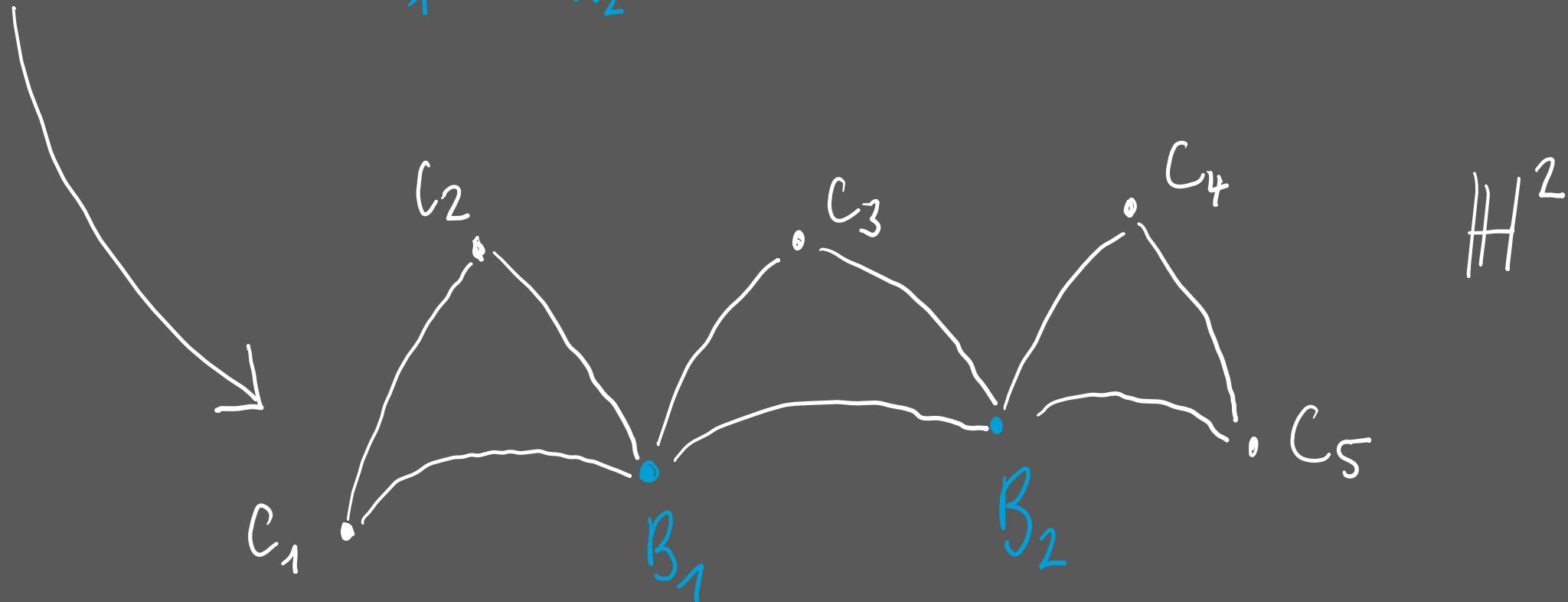
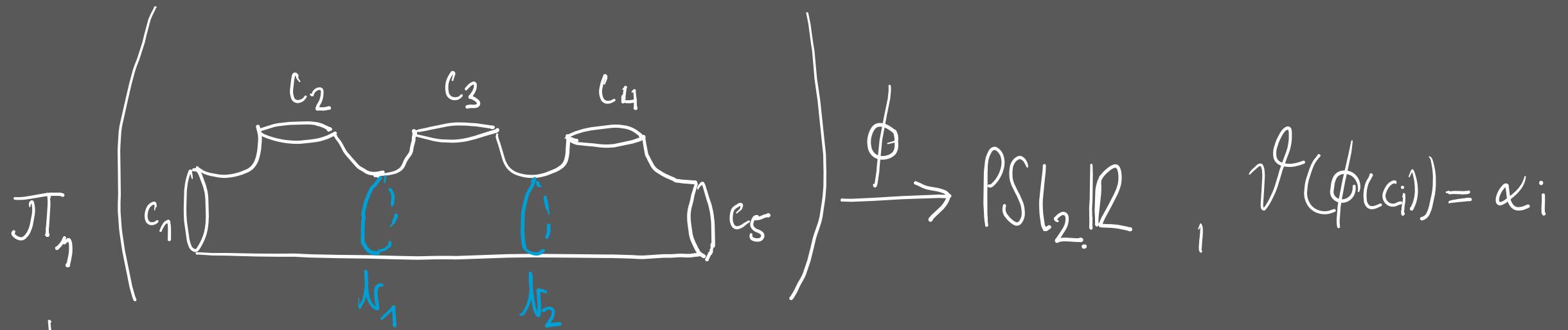
□

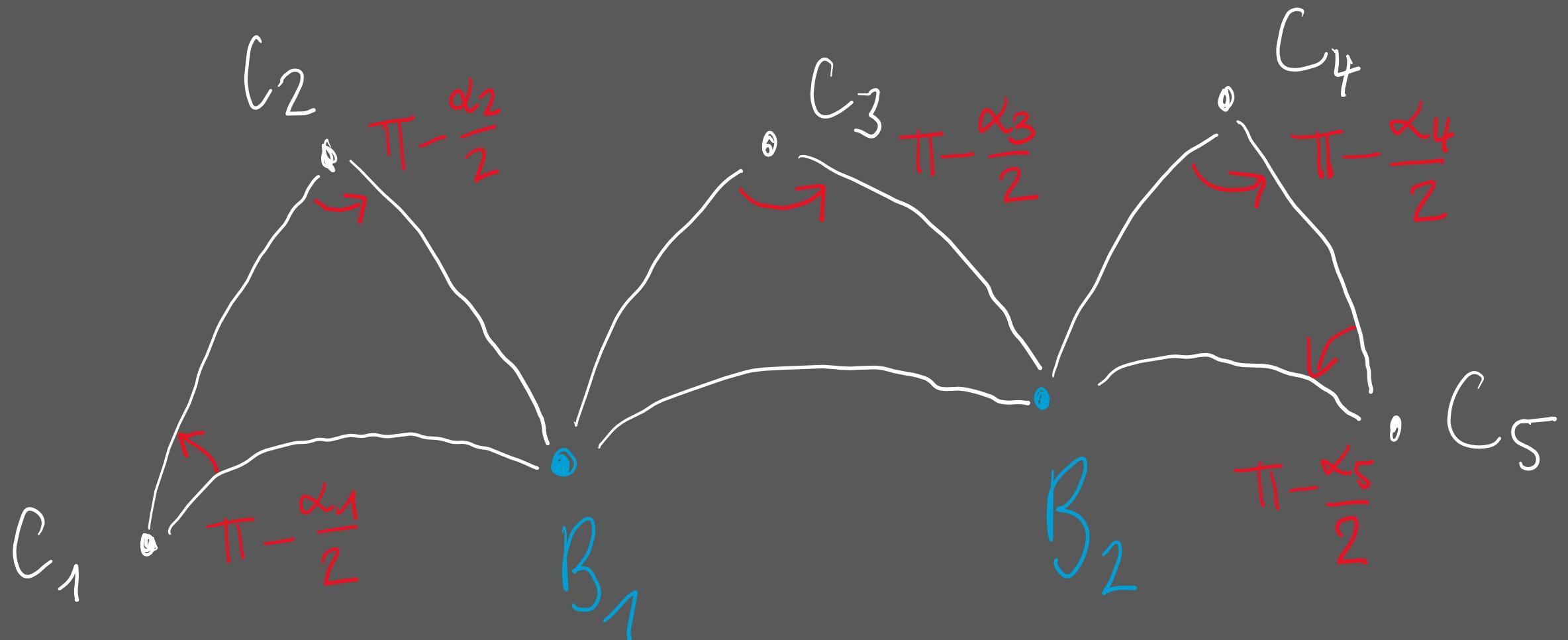
(3) A combinatorial model

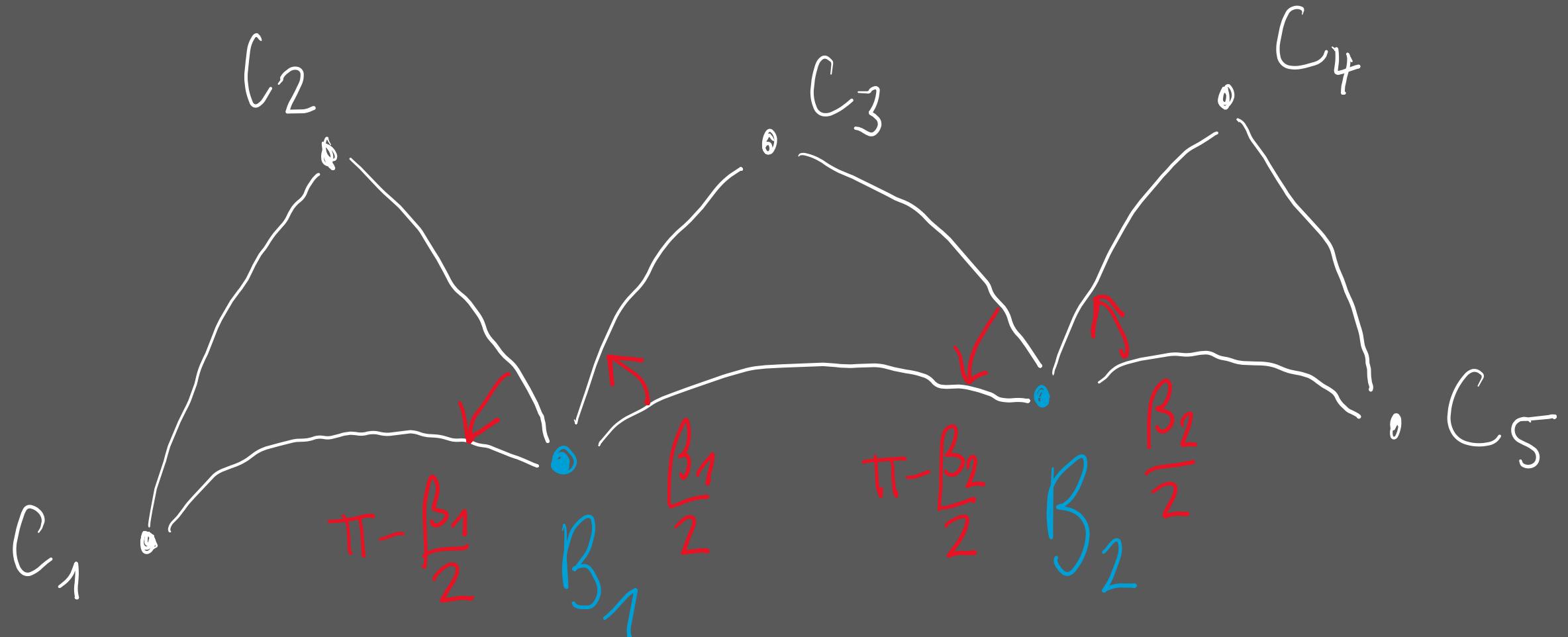




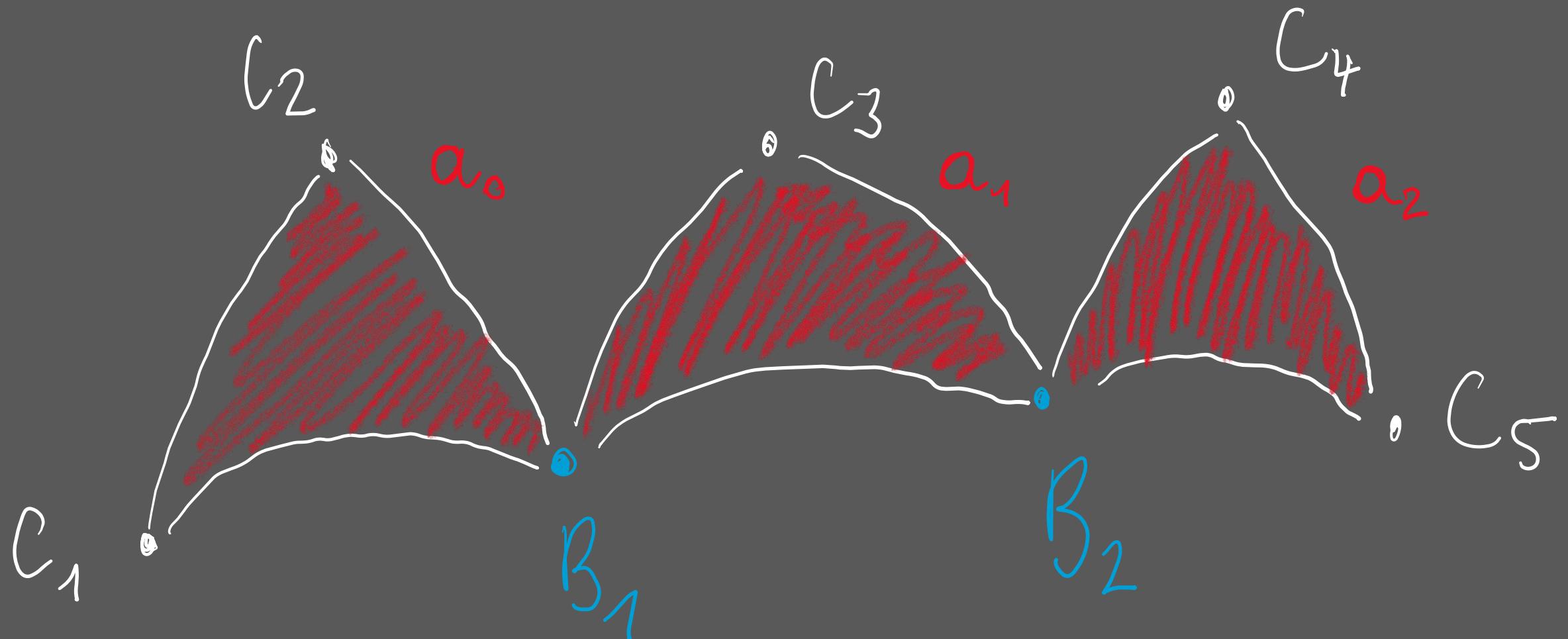




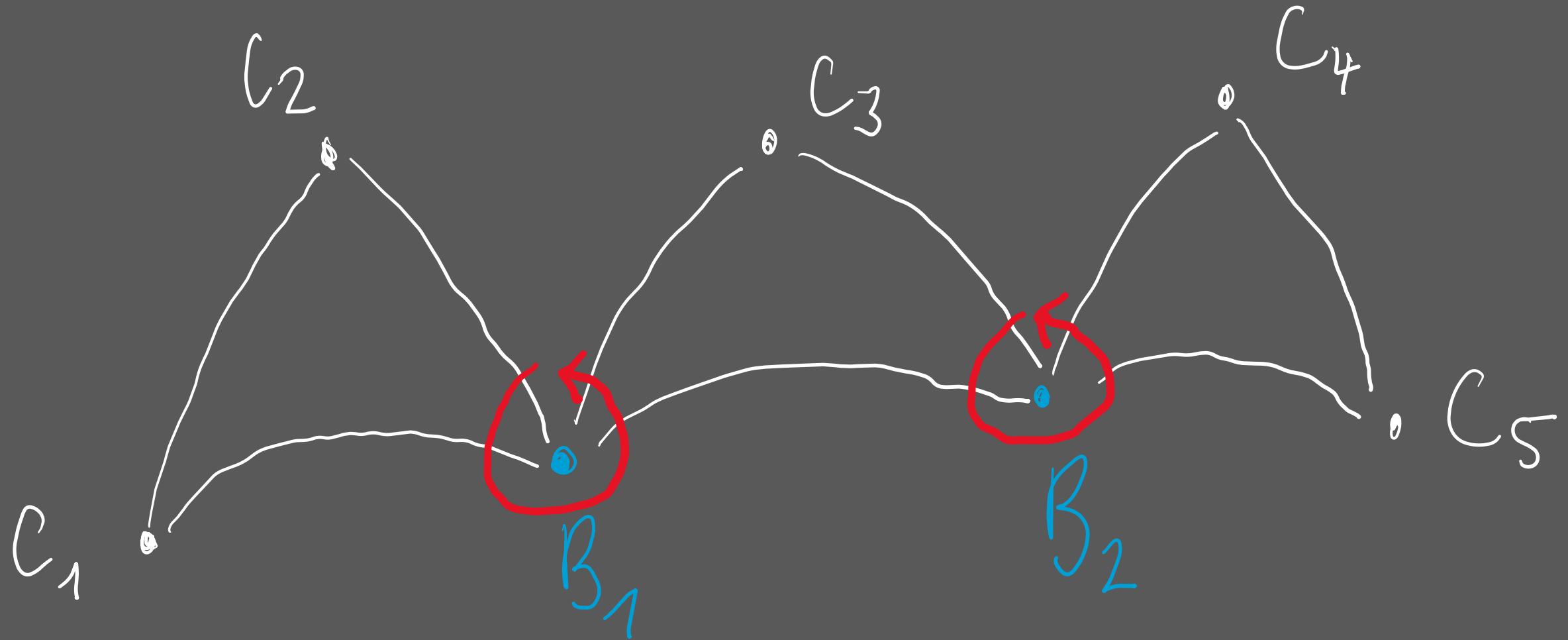


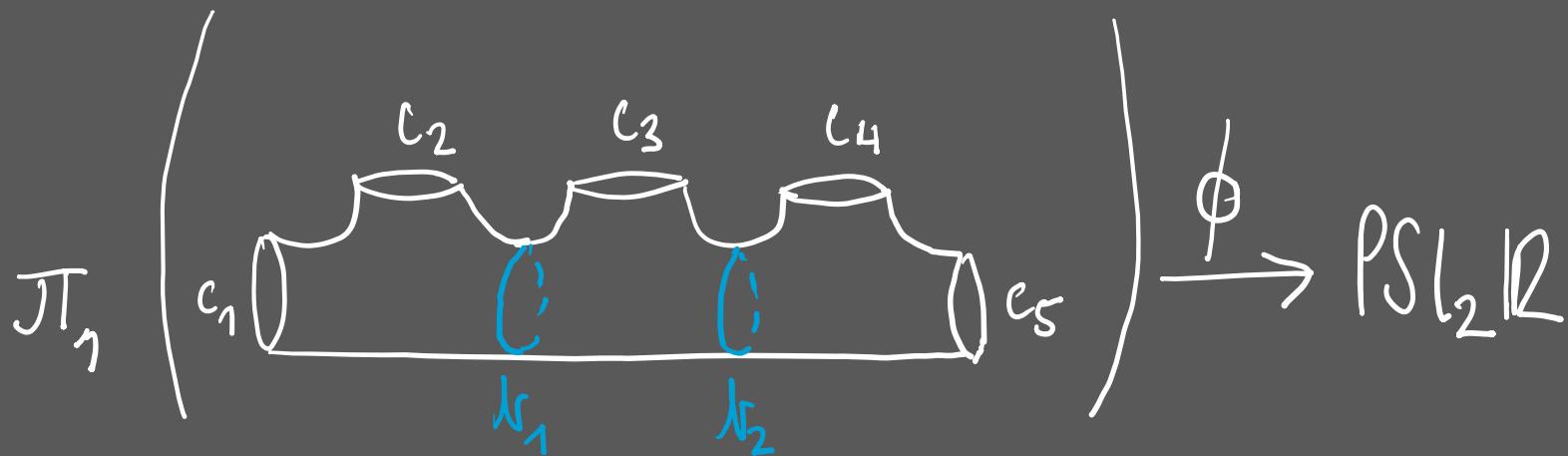


$$\beta_1 = \vartheta(\phi(l_1)), \quad \beta_2 = \vartheta(\phi(l_2))$$

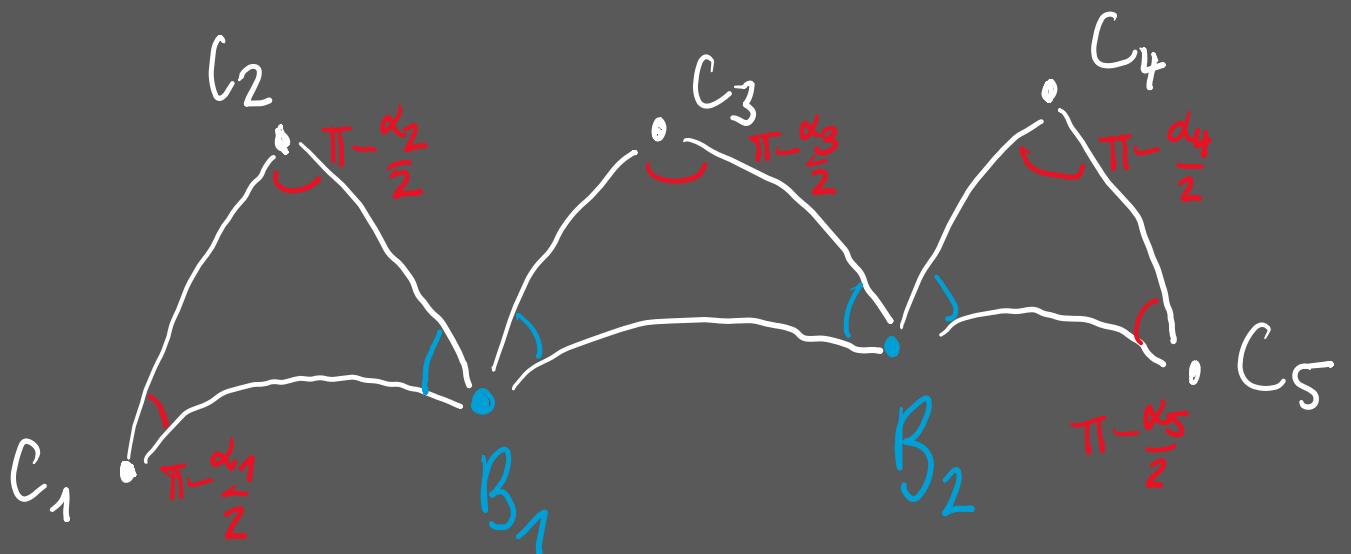


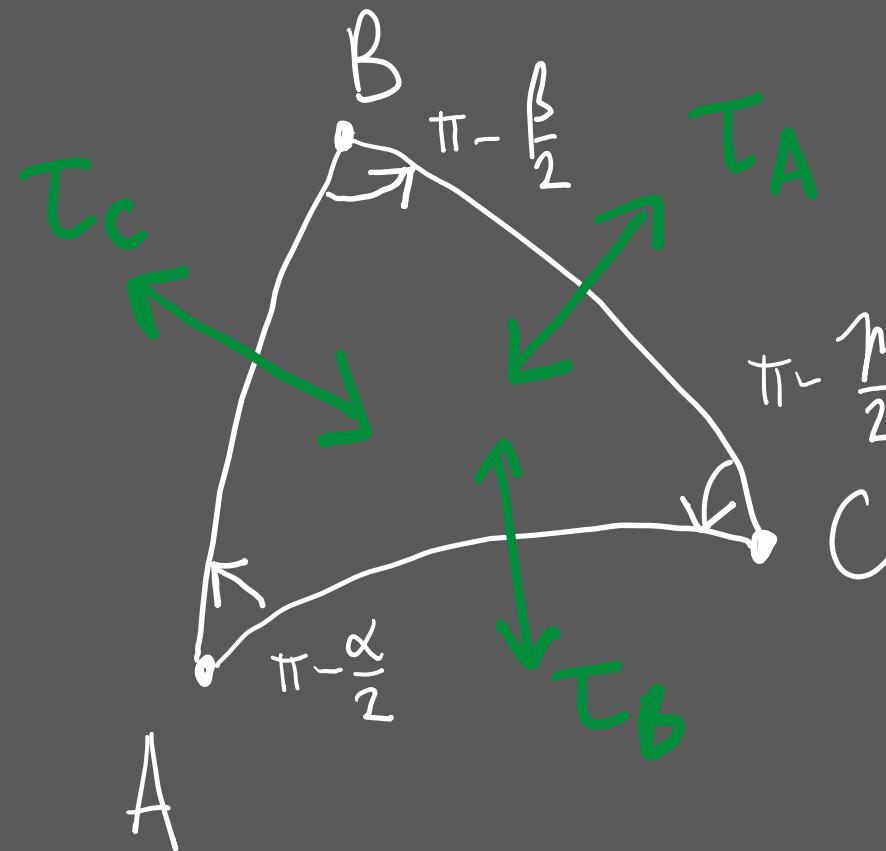
$$a_0 + a_1 + a_2 = \frac{\lambda}{2} = -\frac{1}{4\pi} \text{Td}(\phi)$$



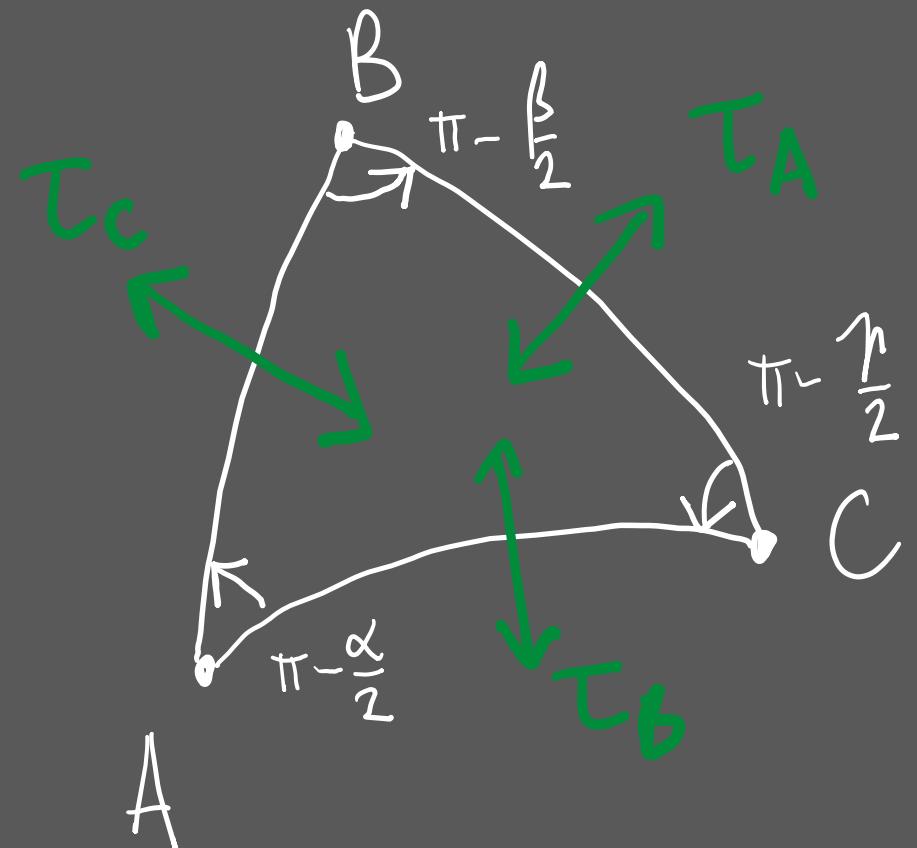


$\phi(c_i) :=$
 rotation of angle
 α_i around C_i

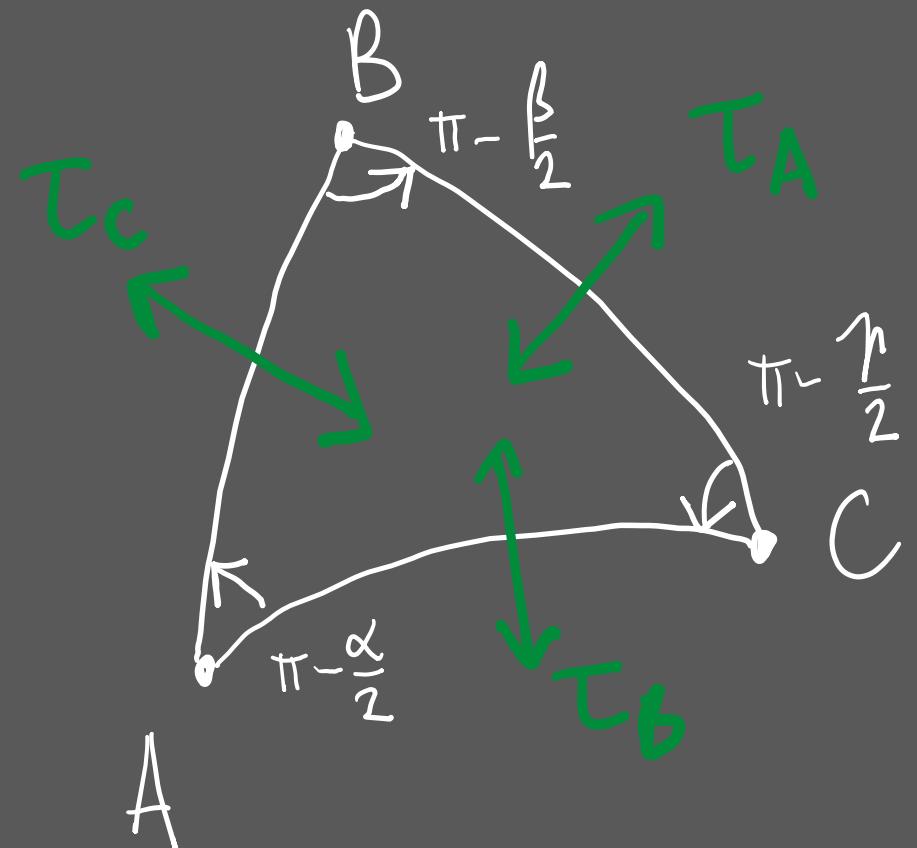




$$\tau_c \tau_A =$$

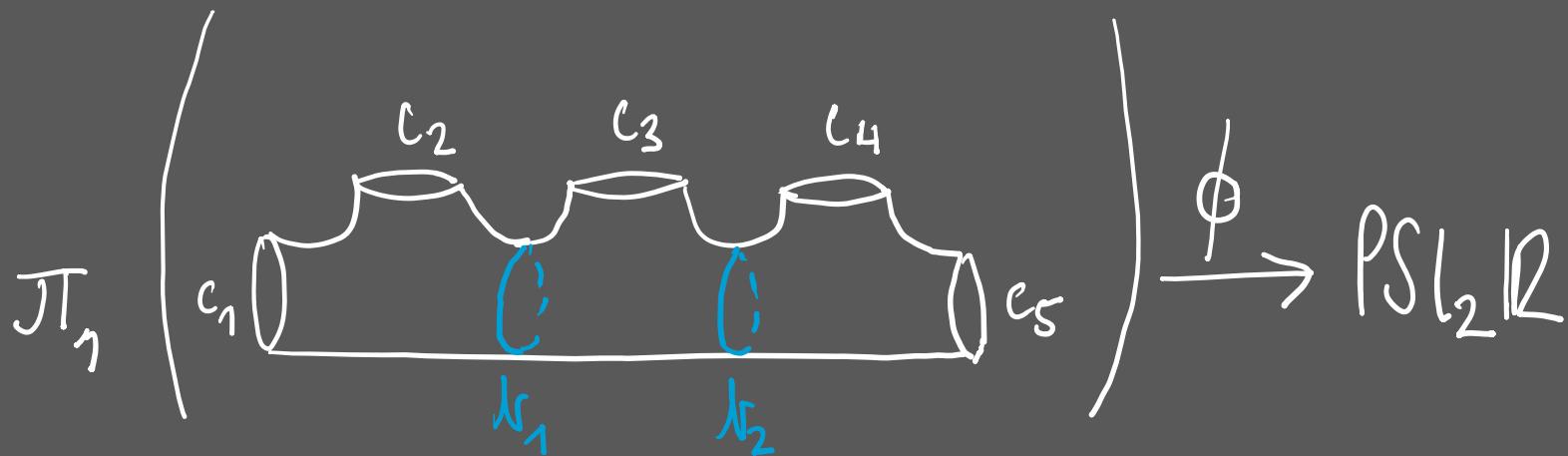


clockwise rotation
 $\tau_C \tau_A =$ of angle β
around B

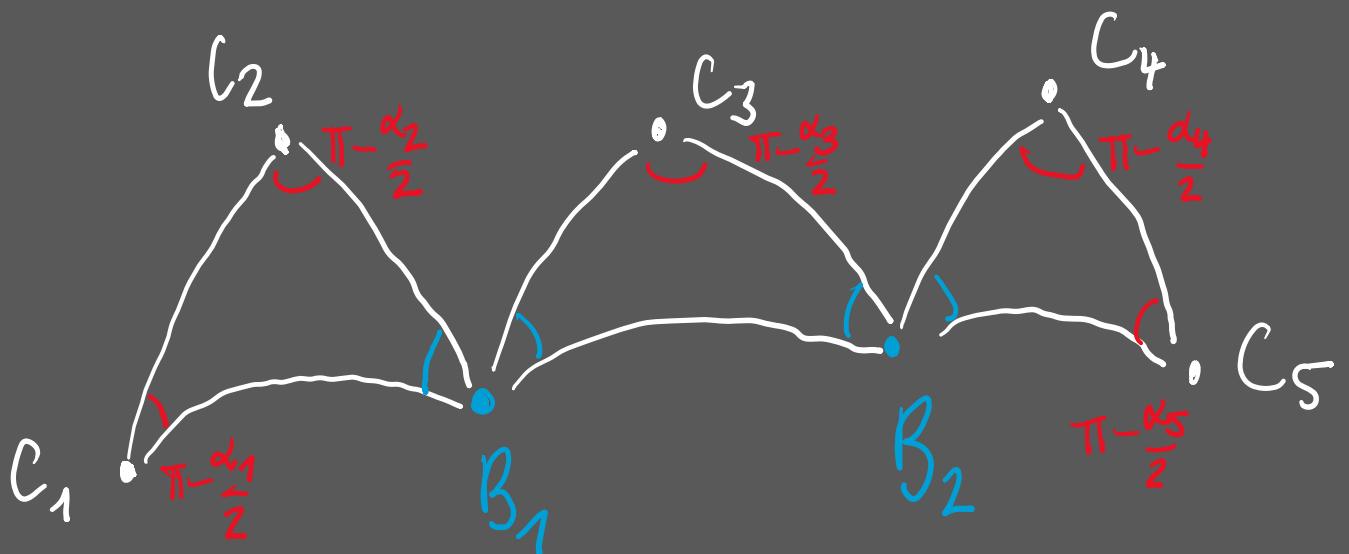


clockwise rotation
 $\tau_c \tau_a =$ of angle β
 around B

$$(\tau_c \tau_a) \cdot (\tau_a \tau_b) \cdot (\tau_b \cdot \tau_c) = 1$$



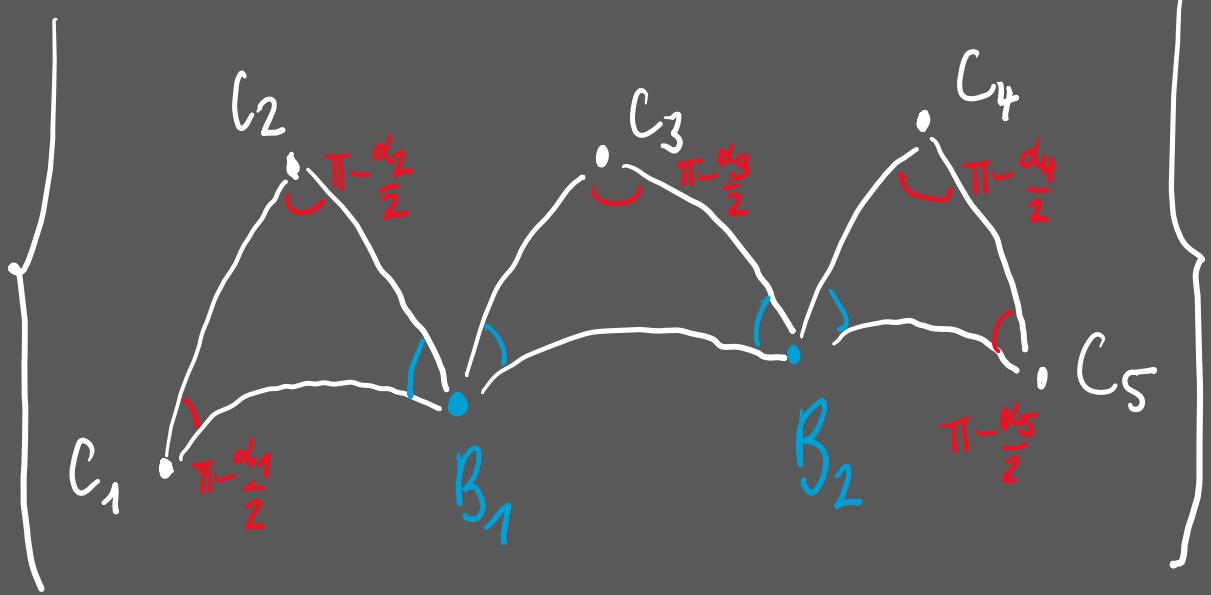
$\phi(c_i) :=$
 rotation of angle
 α_i around C_i



THM [M.]

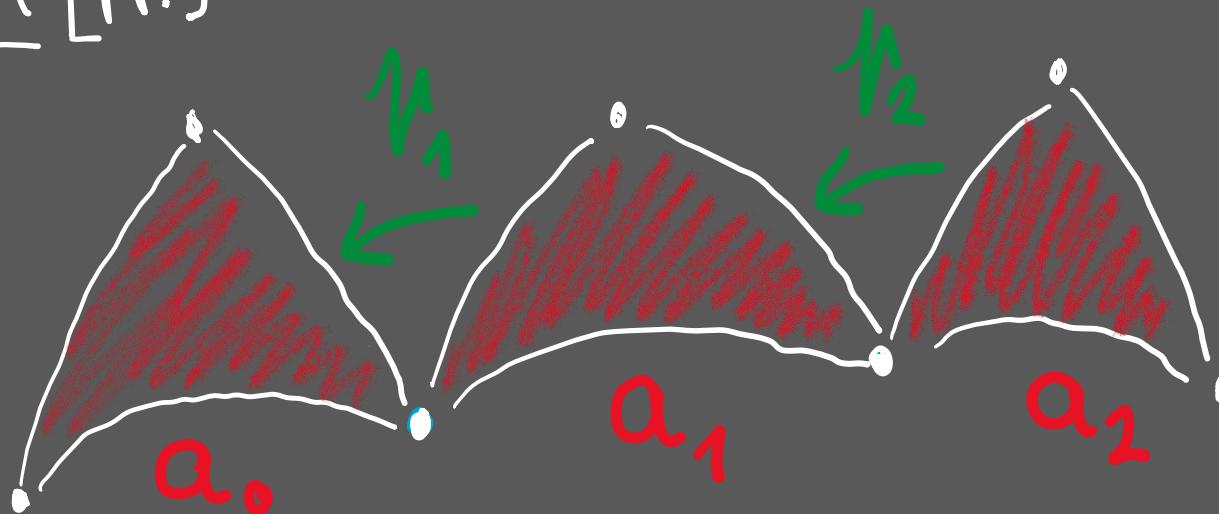
$$\text{Rep}_{\alpha}^{\text{DT}}(\Sigma_{0,n}, \text{PSL}_2|\mathbb{R})$$

1-1



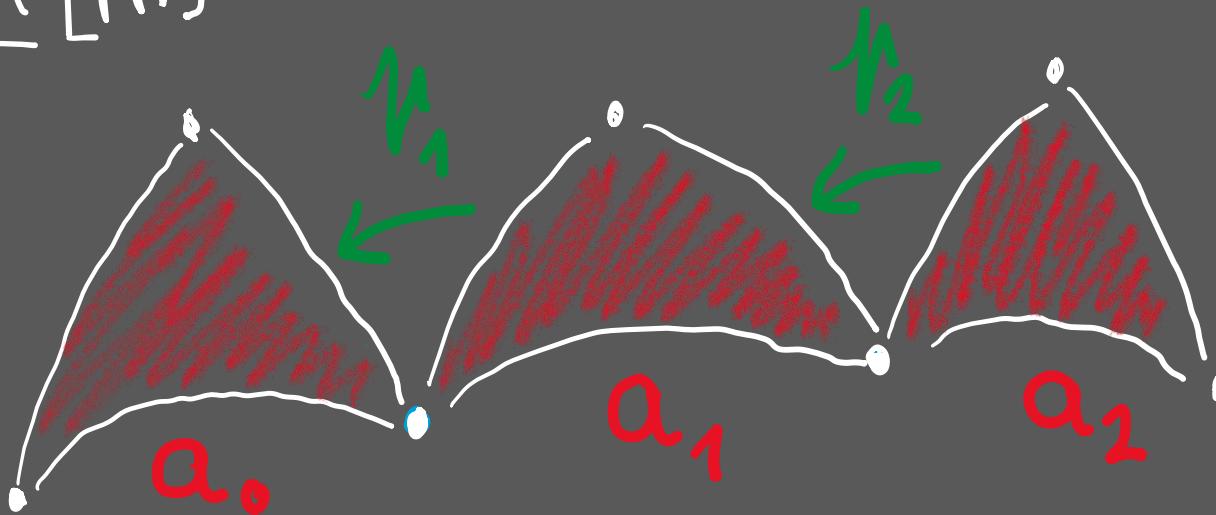
$$\begin{array}{c} \pi \\ \mathbb{H}^n \times \mathbb{H}^{n-3} \\ \diagdown \\ \text{PSL}_2|\mathbb{R} \end{array}$$

THM [M.]



$$\sigma_i := \mu_1 + \dots + \mu_i$$

THM [M.]



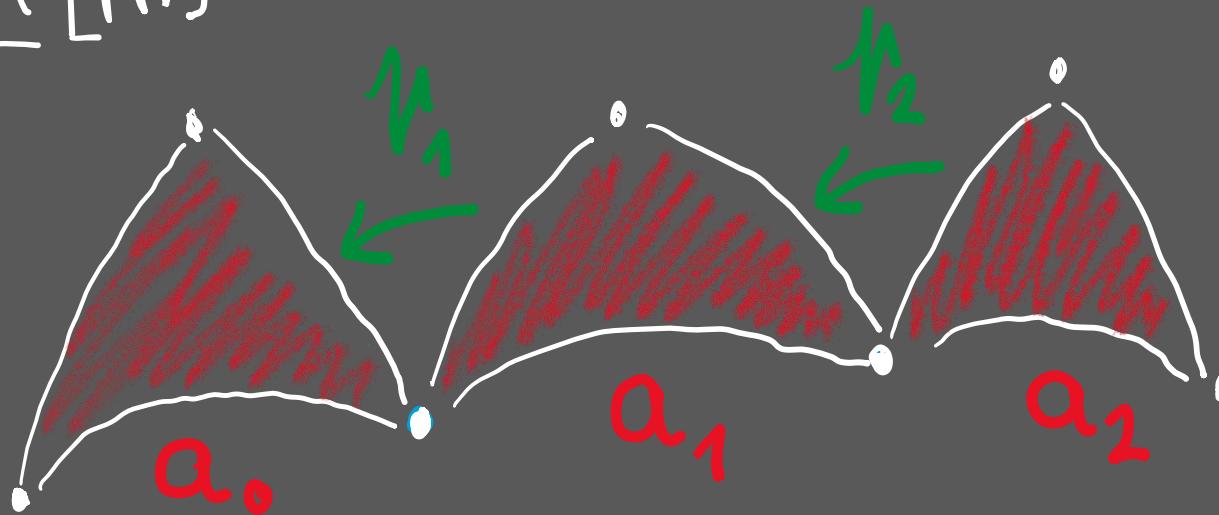
$$\sigma_i := \mu_1 + \dots + \mu_i$$

$$\text{Rep}_{\alpha}^{\text{DT}}(\sum_{o,n}, \text{PSL}_2\mathbb{R}) \rightarrow \mathbb{C}\mathbb{P}^{n-3}$$

equivariant
symplect.

$$[\phi] \mapsto [\int_{a_0} : e^{i\sigma_1} \int_{a_1} : \dots : e^{i\sigma_n} \int_{a_n}]$$

THM [M.]



$$\sigma_i := \mu_1 + \dots + \mu_i$$

$$\omega_{\text{Goldman}} = \frac{1}{2} \sum_{i=1}^{n-3} da_i \wedge d\sigma_i$$

Further remarks:

(1) \exists generalization for $G = \mathrm{SU}(p, q), \mathrm{Sp}(2n, \mathbb{R}), \dots$

[Thebaud - Toulisse]

(2) $\mathrm{Mod}(\Sigma_{0,n}) \hookrightarrow \mathrm{Rep}_{\alpha}^{\mathrm{DT}}(\Sigma_{0,n}, \mathrm{PSL}_2 \mathbb{R})$

[M.]

↑ ergodic