# Workshop on Interplay between symplectic geometry and cluster theory

IWH Heidelberg

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Cluster algebras were introduced by Fomin and Zelevinsky in 2000 in the context of Lie theory. In the last twenty-two years the theory of cluster algebras gave life to several fascinating applications between different fields of mathematics such as quiver representations, Calabi-Yau categories, Teichmüller theory, Poisson geometry, and many others.
An interesting example stems from the work of Vianna [1], where a connection between Markov triples and not Hamiltonian isotopic Lagrangian tori is established. Motivated by this result, we would like to further understand the relations between cluster structures, quiver representations and almost toric fibrations.
The aim of this workshop is to gain a deeper understanding of these interplay thanks to two mini-courses held by experts as well as some research talks. The IWH will provide us with a charming environment for discussions towards further explorations and perspectives.

## 1 Minicourses

**Cluster algebras and representation theory**, by Jenny August

> **Abstract.** In these lectures, we will give an introduction to cluster algebras, and how representation theory has proved to be a valuable tool in their study. We will use quivers (aka directed graphs) to define cluster algebras and describe some of their basic properties, before then introducing quiver representations and the related cluster category, and then finally linking the two settings together.

**Introduction to almost toric fibrations**, by Jonny Evans

> **Abstract.** This will be a sequence of three lectures in which I will introduce almost toric fibrations from scratch. I will start by explaining the integral affine structure on the base of a Lagrangian torus fibration and give some simple examples with nontrivial monodromy. Then I will define almost toric fibrations more generally and introduce the basic operations (nodal trades, slides and mutations) which will allow us to construct many interesting examples like the Vianna fibrations on $\mathbb{CP}^2$ associated with Markov triples. If there is time, I

will also introduce the Floer superpotential and explain how this connects with Galkin-Usnich mutation of Laurent polynomials via the Pascaleff-Tonkonog theorem.

**Applications of almost toric fibrations**, by Felix Schlenk

**Abstract.** We use almost toric fibrations to prove several results in 4-dimensional symplectic topology. Some of these results were known earlier, but are immediate with ATFs, some were proven so far only by ATFs:

1. Full symplectic fillings of the ball by ellipsoids
2. Existence of many different monotone Lagrangian tori in $B^4$, $\mathbb{CP}^2$, $S^2 \times S^2$
3. Nonisotopic symplectic embeddings of cubes into the 4-ball
4. Pinwheels as Lagrangian barriers
5. Symplectic almost squeezing

As a main tool I will introduce Hofer's displacement energy, that will also be basic for Joé's subsequent talk.

## 2 Research talks

**Symplectic Markov Numbers**, by Zachary Greenberg

**Abstract.** The tree of Markov triples can be understood as the exchange graph of the cluster structure on a punctured torus. From a purely combinatorial perspective the Markov triples are recovered by taking an initial cluster with every variable set to 1. The cluster structure also parameterized representations into $\mathrm{SL}(2, \mathbb{Z})$ and the Markov numbers are recovered as traces. We will discuss generalizing these correspondences to noncommutative cluster algebras and explore concrete examples in the $\mathrm{Sp}(4, \mathbb{R})$ case.

**Symplectic embeddings of Hirzebruch surfaces**, by Nicki Magill

**Abstract.** The four dimensional ellipsoid embedding function of a toric symplectic manifold $M$ measures when a symplectic ellipsoid embeds into $M$. It generalizes the Gromov width and ball packing numbers. In 2012, McDuff and Schlenk computed this function for a ball. The function has a delicate structure known as an infinite staircase. This implies infinitely many obstructions are needed to know when an embedding can exist. Based on various work with McDuff, Pires, and Weiler, we will discuss the classification of which Hirzebruch surfaces have infinite staircases. The argument relies on a correspondence between constructing embeddings via almost toric fibrations and finding obstructions via exceptional spheres. The talk will focus on explaining this correspondence.

**DT invariants of some 3CY quotients**, by Sergey Moskovoy

**Abstract.** Crepant resolutions of quotients of $\mathbb{C}^3$ by finite groups can be interpreted using the corresponding McKay quiver and the canonical potential over it. I will discuss refined Donaldson-Thomas invariants of the induced Jacobian algebra, with an emphasis on the quotient $\mathbb{C}^3/Z_3$.

**From ATFs to exotic tori and back**, by Joé Brendel

**Abstract.** Lagrangian submanifolds do not have local symplectic invariants, but what about global ones? In this talk, we will be interested in Lagrangian embeddings which are not equivalent up to a symplectomorphism of the ambient space. They are sometimes called exotic. Since Vianna's work, it became apparent that one can construct different Lagrangian torus fibrations on the same space and hope for their fibres to be exotic. After reviewing the four-dimensional case, we give a concrete construction of exotic tori in dimension six (recovering a result by Auroux) and try to turn the above approach on its head by extending the tori to a fibration of the ambient space. We hope that this gives us some ideas about what ATFs in higher dimensions should look like.

**Special Foldings of Cluster Algebras and Applications**, by Dani Kaufman

**Abstract.** A folding of a cluster algebra produces a new algebra akin to the usual folding of Dynkin diagrams. I will present some work in progress on "Special Foldings" which do not have an analogue in Dynkin diagrams. These new algebras are not cluster algebras by the usual definition, but they share most of the properties one expects. As an application, I show that special foldings can be used to circumvent the "Remove 2-cycles" portion of the construction of a cluster algebra from a surfaces. This has applications to non commutative cluster algebras and (possibly) to mutation of Lagrangian skeleta in the sense of Shende, Treumann and Williams.

# References

[1] Renato Ferreira de Velloso Vianna. Infinitely many exotic monotone Lagrangian tori in $\mathbb{CP}^2$. *J. Topol.*, 9(2):535–551, 2016.