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**A note on**

# Character Varieties

*Some of the stuff I've been learning for a while*

Preliminary version.
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# Contents

# Foreword

These notes[1] are an extended version of a series of mini-courses that I gave in the spring of 2022 at the KIT Karlsruhe and at the University of Heidelberg and in the spring of 2024 at Seoul National University. The main objective is to introduce the notions of *representation varieties* and *character varieties* from the perspective of both differential geometry and algebraic geometry. It is common to encounter different definitions of character varieties in the literature, depending on each author's "favourite quotient" which varies depending on the context and the applications. I will try to cover as many of these definitions as possible and explain how they relate to each other. The notes also aim at providing the reader with an introduction to the symplectic structure of surface groups representations which I will formulate in therms of group cohomology. Along the way, I also intend to elaborate on mapping class group dynamics on character varieties, as well as on some invariants, like the Euler or Toledo numbers, used to discriminate their connected components.

I will try as much as possible to provide precise references to the literature to help the readers find original statement and proofs. There are a couple options in the literature where one can find a broad introduction to character varieties (sometimes focusing on surface group representations; sometimes focusing on the algebraic aspects of the theory). These include, for instance, Bradlow-García–Prada-Goldman-Wienhard [BGPGW07], Labourie [Lab13], Marché [Mar22], Mondello [Mon16, §2], and Sikora [Sik12].

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# Chapter 1

# Background

## 1.1 Lie groups

The first of two ingredients that we need to define a character variety is a target group. This group essentially carries the topological structure of the character variety. It will always be a Lie group, sometimes even an algebraic group. We start by recalling these notions and their properties.

### 1.1.1 Generalities

**Definition 1.1.1.** A *Lie group* $G$ is a real smooth manifold with a group structure for which the operations of multiplication and inverse are smooth maps. A Lie group is called a *complex Lie group* if it has the structure of a complex manifold and the group operations are holomorphic.

Lie groups always admit an analytic atlas, unique up to analytic diffeomorphism, such that multiplication and inverse are analytic maps[1]. Lie groups are not necessarily connected. We will denote by $G^\circ$ the identity component of $G$. The *centralizer* of a subset $S \subset G$ is denoted $Z(S) := \{g \in G : gsg^{-1} = s, \forall s \in S\}$. It is a closed subgroup of $G$, hence a Lie subgroup of $G$. The *center of* $G$ is the Lie subgroup $Z(G) \subset G$.

**Example 1.1.2** (Linear Lie groups)**.** The standard examples of Lie groups are groups of invertible $n \times n$ matrices such as $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$, and all their closed subgroups, called *linear Lie groups*. These include the subgroups of determinant-1 matrices which we denote $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{C})$ and their subgroups $\mathrm{SO}(n)$ and $\mathrm{SO}(n, \mathbb{C})$, $\mathrm{SU}(p, q)$, or $\mathrm{Sp}(2n, \mathbb{R})$. The groups $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, and $\mathrm{SO}(n, \mathbb{C})$ are also examples of complex linear Lie groups.

The quotient of a Lie group $G$ by its center $Z(G)$ is also a Lie group called the *adjoint Lie group of* $G$. The adjoint Lie groups of linear groups are usually denoted by adding a P in front of the Lie group's name. For instance, the adjoint Lie group of $\mathrm{SL}(n, \mathbb{R})$ is written $\mathrm{PSL}(n, \mathbb{R})$.[2]

The Lie algebra of a Lie group $G$ is denoted $\mathfrak{g}$. Most of the time, we will think of $\mathfrak{g}$ as the tangent space to $G$ at the identity. In various places we will make use of the Lie theoretic exponential map

[1] This is a consequence of the Campbell-Hausdorff formula, see e.g. [Ser06, Part I, Chap. IV, §7-8].

[2] For more information on $\mathrm{PSL}(2, \mathbb{R})$, the reader is referred to Appendix A.

$\exp\colon \mathfrak{g} \to G$, which, in the case that $G$ is a linear Lie group, is the matrix exponential map. The *adjoint representation of* $G$ on $\mathfrak{g}$ is denoted by $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ and is defined by

$$\mathrm{Ad}(g)(\xi) := \left.\frac{d}{dt}\right|_{t=0} g\exp(t\xi)g^{-1}, \quad g \in G,\, \xi \in \mathfrak{g}.$$

By taking the derivative of Ad at the identity, we obtain the *adjoint representation of* $\mathfrak{g}$ which is commonly denoted by $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. If $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denotes the Lie bracket operation on $\mathfrak{g}$, then it holds that

$$\mathrm{ad}(\xi_1)(\xi_2) = [\xi_1, \xi_2], \quad \xi_1, \xi_2 \in \mathfrak{g}.$$

The kernel of the ad-representation is called the *center of* $\mathfrak{g}$ and is denoted by $\mathfrak{z}(\mathfrak{g}) := \mathrm{Ker}(\mathrm{ad})$. The center of $\mathfrak{g}$ can also be interpreted as the Lie algebra of $Z(G)$—the center of $G$.

### 1.1.2 Simple, semisimple, and reductive Lie groups

We will say that a Lie algebra $\mathfrak{g}$ is

- *simple* if it is not abelian and if its only proper ideal is the zero ideal. Since ideals of $\mathfrak{g}$ are in one-to-one correspondence with sub-representations of its adjoint representation, $\mathfrak{g}$ is simple if and only if its adjoint representation is irreducible and $\mathfrak{g}$ is not a one-dimensional abelian Lie algebra.

- *semisimple* if it has no nonzero abelian ideals. Equivalently, a Lie algebra is semisimple if it is a direct sum of simple Lie algebras [Bou98, Chap. I, §6.2, Cor. 1]. By Cartan's criterion, $\mathfrak{g}$ is semisimple if and only if its *Killing form*

$$\begin{aligned} K\colon \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R} \\ (\xi_1, \xi_2) &\mapsto \mathrm{Tr}(\mathrm{ad}(\xi_1)\,\mathrm{ad}(\xi_2)) \end{aligned}$$

  is non-degenerate [Bou98, Chap. I, §6.1, Thm. 1].

- *reductive* if it is the direct sum of an abelian and a semisimple Lie algebra. Equivalently, $\mathfrak{g}$ is reductive if and only if its adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is completely reducible[3], which is further equivalent to $\mathfrak{g}$ admitting a faithful, completely reducible, finite-dimensional representation [Bou98, Chap. I, §6.4, Prop. 5].

We call a connected Lie group *simple, semisimple* or *reductive* if its Lie algebra is simple, semisimple or reductive, respectively. Simple Lie groups are semisimple and semisimple Lie groups are reductive. The groups $\mathrm{SL}(n,\mathbb{R})$ for $n \geqslant 2$, $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathrm{SU}(p,q)$ for $p+q \geqslant 2$ are simple. The group $\mathrm{SO}(n)^\circ$ is simple for $n \geqslant 3, n \neq 4$ and semisimple for $n = 4$. In contrast, the group $\mathrm{GL}(n,\mathbb{R})^\circ$ is not semisimple for any $n \geqslant 1$ (its Killing form is degenerate). It is however reductive, because its Lie algebra is the direct sum of the simple Lie algebra of traceless matrices and the abelian Lie algebra of diagonal matrices. It is worth observing that a connected linear Lie group $G \subset \mathrm{GL}(n,\mathbb{R})$

[3] Recall that a *completely reducible* representation is a representation that decomposes as a direct sum of irreducible representations. Such representations are sometimes called *semisimple*.

is reductive if and only if the trace form

$$\begin{aligned}\operatorname{Tr}\colon \mathfrak{g}\times\mathfrak{g} &\to \mathbb{R}\\ (\xi_1,\xi_2) &\mapsto \operatorname{Tr}(\xi_1\xi_2)\end{aligned}$$

is non-degenerate. This can be seen as a consequence of the classification of semisimple Lie algebras and [Bou98, Chap. I, §6.4, Prop. 5]. The previous statement also holds for connected linear Lie groups $G \subset \mathrm{GL}(n,\mathbb{C})$. If the (in this case, complex-valued) trace form is non-degenerate, then so is its real part $\Re(\operatorname{Tr})\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ which gives a non-degenerate, symmetric, Ad-invariant, real-valued bilinear form.

### 1.1.3 Quadrable Lie groups

An important class of Lie groups for the purpose of these notes are those that admit a non-degenerate, symmetric and Ad-invariant pairing on their Lie algebra. Such Lie groups carry different names throughout the literature, see [Ova16] for an overview. We opt for the name quadrable.

**Definition 1.1.3** (Quadrable Lie groups)**.** A Lie group $G$ is called *quadrable* if there exists a bilinear form (also called pairing)

$$B\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$$

which is non-degenerate, symmetric and Ad-invariant.

Quadrable Lie groups are common among the standard Lie groups. For example, all semisimple Lie groups, and more generally all reductive Lie groups, are quadrable. An example of a non-degenerate, symmetric and Ad-invariant bilinear form on a reductive Lie algebra is given by taking the Killing form on the semisimple part and any non-degenerate, symmetric bilinear form on the abelian part. Alternatively, one may consider the trace form associated to a faithful, finite-dimensional representation[4] of $\mathfrak{g}$.

**Example 1.1.4.** For instance, $\mathrm{SL}(2,\mathbb{R})$ is quadrable. We usually chose to work with the pairing given by the trace form: $\operatorname{Tr}\colon \mathfrak{sl}_2\mathbb{R}\times\mathfrak{sl}_2\mathbb{R}\to\mathbb{R}$, $(\xi_1,\xi_2)\mapsto\operatorname{Tr}(\xi_1\xi_2)$. The trace of a matrix is invariant under conjugation, so the trace form is Ad-invariant. In the basis

$$\mathfrak{sl}_2\mathbb{R} = \left\langle \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\rangle,$$

the trace form is given by the pairing $2x_1x_2+y_1z_2+z_1y_2$. It is clearly symmetric and non-degenerate. Actually, in this case, the pairing $\operatorname{Tr}\colon \mathfrak{sl}_2\mathbb{R}\times\mathfrak{sl}_2\mathbb{R}$ has signature $(2,1)$.

**Example 1.1.5.** The Heisenberg group $H$ is an example of a non-quadrable Lie group. Recall that $H$ is defined to be the group of strictly upper triangular $3\times 3$ real matrices:

$$H = \left\{ \begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix} : a,b,c\in\mathbb{R} \right\}.$$

[4]The *trace form of a representation* $\rho\colon \mathfrak{g}\to\mathrm{GL}(n,\mathbb{R})$ is the symmetric bilinear form $\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ given by $(\xi_1,\xi_2)\mapsto\operatorname{Tr}(\rho(\xi_1)\rho(\xi_2))$. For instance, the Killing form is the trace form of the adjoint representation.

The Lie algebra $\mathfrak{h}$ of $H$ is generated by the three matrices

$$X := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A simple computation shows that $Z$ commutes with any element of $H$. Further

$$\mathrm{Ad}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}(X) = X - Z, \quad \mathrm{Ad}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}(Y) = Y, \tag{1.1.1}$$

and

$$\mathrm{Ad}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}(X) = X, \quad \mathrm{Ad}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}(Y) = Y + Z. \tag{1.1.2}$$

So, because of (1.1.1), any symmetric and Ad-invariant bilinear form $B\colon \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$, must satisfy

$$B(X, Z) = B(X - Z, Z) \text{ and } B(X, Y) = B(X - Z, Y)$$

which implies $B(Z, Z) = 0$ and $B(Y, Z) = 0$. Moreover, because of (1.1.2), it must also satisfy

$$B(X, Y) = B(X, Y + Z)$$

and thus $B(X, Z) = 0$. This shows that $B$ is degenerate.

If reductive Lie groups are always quadrable, it is not true that every quadrable Lie group is reductive. This was already pointed by Goldman in [Gol84, Footnote p. 204]. Here is an example.

**Example 1.1.6.** Let $G$ be the connected, simply connected Lie group whose Lie algebra is $\mathfrak{g} = \mathbb{R}^3 \oplus \mathbb{R}^3$ with the Lie bracket defined by $[(u_1, u_2), (v_1, v_2)] = (0, u_1 \times v_1)$, where $\times$ denotes the cross product on $\mathbb{R}^3$. We claim that $G$ is quadrable but not reductive.

Let us first prove that $G$ is not reductive. We will actually prove something stronger, namely that the center of $\mathfrak{g}$ (defined as the kernel of the adjoint representation of $\mathfrak{g}$ in Section 1.1) is $\mathfrak{z}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$. This will of course imply that $\mathfrak{g}$ is not reductive. To see that $\mathfrak{z}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$, first observe that for $u, v, w \in \mathfrak{g}$, we have

$$[u, [v, w]] = [u, (0, v_1 \times w_1)] = (0, u_1 \times 0) = (0, 0).$$

Conversely, if $\xi \in \mathfrak{g}$ is such that $[u, \xi] = (0, u_1 \times \xi_1) = (0, 0)$ for every $u \in \mathfrak{g}$, then it must have $\xi_1 = 0$ showing that $\xi \in [\mathfrak{g}, \mathfrak{g}]$.

In order to see that $\mathfrak{g}$ is quadrable, consider the bilinear form

$$\begin{aligned} B\colon \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R} \\ (u, v) &\mapsto \langle u_1, v_2 \rangle + \langle u_2, v_1 \rangle, \end{aligned}$$

where $\langle -, - \rangle$ denotes the standard scalar product on $\mathbb{R}^3$. Clearly, $B$ is symmetric and non-degenerate. We prove that $B$ is Ad-invariant in two steps. First, we prove that it is ad-invariant which means that for every $u, v, w \in \mathfrak{g}$, it holds that

$$B([u, v], w) + B(v, [u, w]) = 0.$$

This can be seen by the following computation:

$$B(v, [u, w]) = \langle v_1, u_1 \times w_1 \rangle = \langle w_1, v_1 \times u_1 \rangle = B(w, [v, u]) = -B([u, v], w).$$

Now, we explain how Ad-invariance follows from ad-invariance. Since the Lie exponential $\exp\colon \mathfrak{g} \to G$ is a local diffeomorphism at $0 \in \mathfrak{g}$ and $G$ (assumed to be connected here) is generated by a neighbourhood of the identity element, it's enough to check that $B$ is Ad-invariant on $\exp(g)$. Given $\xi, u, v \in \mathfrak{g}$, define

$$f(t) := B\big(\operatorname{Ad}(\exp(t\xi))u, \operatorname{Ad}(\exp(t\xi))v\big).$$

Then $f(0) = B(u, v)$ and

$$f'(0) = B([\xi, u], v) + B([\xi, v], u]) = 0$$

by ad-invariance. Using that $\exp((t+s)\xi) = \exp(s\xi)\exp(t\xi)$, we can easily that $f'(t) = 0$ for every $t$. This implies that $f$ is constant. The relation $f(1) = f(0)$ exactly shows that $B$ is Ad-invariant on $\exp(g)$.

This example of a quadrable, but not reductive, Lie algebra can be generalized as follows. From a quadrable Lie algebra $\mathfrak{g}$ with pairing $B$, construct the Lie algebra $\mathfrak{g} \times \mathfrak{g}$ with Lie bracket $[(\xi_1, \xi_2), (\zeta_1, \zeta_2)]_{\mathfrak{g}\times\mathfrak{g}} := (0, [\xi_1, \zeta_1]_{\mathfrak{g}})$. Similar arguments as above show that $\mathfrak{g} \times \mathfrak{g}$ is not reductive. To prove that however $\mathfrak{g} \times \mathfrak{g}$ is quadrable, consider the symmetric and non-degenerate pairing

$$\overline{B}\left((\xi_1, \xi_2), (\zeta_1, \zeta_2)\right) := B(\xi_1, \zeta_2) + B(\xi_2, \zeta_1).$$

Again, we can see that $\overline{B}$ is Ad-invariant by the reasoning as above.

## 1.2 Algebraic groups

**Definition 1.2.1.** A group $G$ is called an *algebraic group* if it is an algebraic variety[5] and if the operations are regular maps.

The Zariski closure of any subgroup of $G$ is an algebraic subgroup [Mil17, Lem. 1.40] and any algebraic subgroup of $G$ is Zariski closed [Mil17, Prop. 1.41]. For instance, the centralizer $Z(S)$ of a subset $S \subset G$ is Zariski closed, hence an algebraic subgroup. All algebraic groups over the fields of real or complex numbers, respectively called *real or complex algebraic groups*, are also Lie groups, see [Mil13, III, §2] and references therein.

[5] In the context of this work, an *algebraic variety* is understood to be the zero locus of a set of polynomial equations over $\mathbb{R}$ or $\mathbb{C}$ (in other words, algebraic varieties are always affine). We make no assumption about irreducibility and, in particular, we don't distinguish algebraic varieties and algebraic sets. Morphisms of algebraic varieties are restrictions of polynomial maps and are called *regular maps*.

**Example 1.2.2** (Linear algebraic groups)**.** Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. The group $\mathrm{GL}(n, \mathbb{K})$, and all its Zariski closed subgroups, such as $\mathrm{SL}(n, \mathbb{K})$, $\mathrm{Sp}(2n, \mathbb{K})$ or $\mathrm{SO}(n, \mathbb{K})$, are algebraic groups. They are called *linear algebraic groups.* Algebraic groups, however, are not necessarily linear (for instance, elliptic curves are non-linear algebraic groups). The group $\mathrm{SU}(p, q)$ is a real algebraic group, but is not a complex algebraic variety, see e.g. [SKKT00, Exercise 1.1.2].

Other examples of real algebraic groups include $\mathrm{PGL}(n, \mathbb{R})$ for every $n \geqslant 1$ as it can be seen as the group of automorphisms of the $n \times n$ real matrices, which is an algebraic subgroup of $\mathrm{GL}(n^2, \mathbb{R})$. For the same reason, $\mathrm{PGL}(n, \mathbb{C}) = \mathrm{PSL}(n, \mathbb{C})$ is a complex algebraic group for every $n \geqslant 1$. When $n$ is odd, then $\mathrm{PSL}(n, \mathbb{R}) = \mathrm{PGL}(n, \mathbb{R})$ and so $\mathrm{PSL}(n, \mathbb{R})$ is also algebraic. However, when $n$ is even, then $\mathrm{PSL}(n, \mathbb{R}) = \mathrm{PGL}(n, \mathbb{R})_0$ is only a *semi-algebraic group.*[6]

### 1.2.1 Reductive algebraic groups

**Definition 1.2.3.** Any algebraic group contains a unique maximal normal connected solvable subgroup called the *radical*, see [Mil17, Chap. 6, §h]. A *reductive algebraic group* is a connected algebraic group whose radical over $\mathbb{C}$ is an algebraic torus, i.e. isomorphic to $(\mathbb{C}^*)^n$ for some $n \geqslant 0$.

A reductive algebraic group over the fields of real or complex numbers is in particular a reductive Lie group in the sense of Section 1.1.2, hence quadrable [Mil13, II, §4].

**Example 1.2.4.** Connected linear algebraic groups $G \subset \mathrm{GL}(n, \mathbb{C})$ are reductive if and only if the trace form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $(\xi_1, \xi_2) \mapsto \mathrm{Tr}(\xi_1 \xi_2)$ is non-degenerate. In particular, $\mathrm{SL}(n, \mathbb{C})$ for $n \geqslant 2$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$ for $n \geqslant 3$ are reductive algebraic groups.

### 1.2.2 The groups $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{R})$

The group $\mathrm{SL}(2, \mathbb{C})$ is the group of complex $2 \times 2$ matrices with determinant 1. It is a reductive complex algebraic group of complex dimension 3. It is also a non-compact and simple complex Lie group. The group $\mathrm{SL}(2, \mathbb{C})$ is *irreducible* in the sense that it does preserve an proper subspace when it acts linearly on $\mathbb{C}^2$. Its center is $Z(\mathrm{SL}(2, \mathbb{C})) = \{\pm I\}$, where $I$ denotes the $2 \times 2$ identity matrix. The case of $\mathrm{SL}(2, \mathbb{C})$ is interesting because we have a complete understanding of its algebraic subgroups from Sit's classification.

**Theorem 1.2.5** ([Sit75])**.** *If $G$ is an infinite algebraic subgroup of $\mathrm{SL}(2, \mathbb{C})$, then one the following holds:*

1. $\dim_{\mathbb{C}} G = 3$ *and* $G = \mathrm{SL}(2, \mathbb{C})$.

2. $\dim_{\mathbb{C}} G = 2$ *and $G$ is conjugate to the parabolic subgroup of upper triangular matrices.*

3. $\dim_{\mathbb{C}} G = 1$ *and there are three possibilities:*

   (a) *$G$ is conjugate to*
$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a^n = 1,\, a, b \in \mathbb{C} \right\},$$

[6] Polynomials equalities are not enough to write $\mathrm{PSL}(n, \mathbb{R})$ when $n$ is even; we have to use polynomials inequalities too. When this is the case, we say that the group is *semi-algebraic.*

*and $G$ has $n$ connected components.*

*(b) $G$ is conjugate to* $\mathrm{SO}(2,\mathbb{C})$*, and $G$ is connected and abelian.*

*(c) $G$ is conjugate to* $\mathrm{SO}(2,\mathbb{C}) \cup i\,\mathrm{SO}(2,\mathbb{C})$*, and $G$ is abelian with two connected components.*

*Moreover, $G$ is irreducible if and only if $G = \mathrm{SL}(2,\mathbb{C})$ or if $G$ is conjugate to* $\mathrm{SO}(2,\mathbb{C}) \cup i\,\mathrm{SO}(2,\mathbb{C})$.

Recall that an algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$ of complex dimension 0 is necessarily finite (because algebraic varieties have finitely many connected components in the usual topology, as pointed out earlier). These groups are well-understood too, see e.g. [Sit75, Prop. 1.2]. Finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ are irreducible if they are non-abelian.

The real points of $\mathrm{SL}(2,\mathbb{C})$ give the real algebraic group $\mathrm{SL}(2,\mathbb{R})$, which is a simple and connected Lie group. The group $\mathrm{SL}(2,\mathbb{R})$ is irreducible in the sense that it does not preserve any proper subspace of $\mathbb{R}^2$. From the list of Theorem 1.2.5, we can obtain the list of algebraic subgroups of $\mathrm{SL}(2,\mathbb{R})$.

**Theorem 1.2.6.** *If $G$ is an infinite algebraic subgroup of* $\mathrm{SL}(2,\mathbb{R})$*, then one the following holds:*

1. $\dim_{\mathbb{R}} G = 3$ *and* $G = \mathrm{SL}(2,\mathbb{R})$.
2. $\dim_{\mathbb{R}} G = 2$ *and $G$ is conjugate to the parabolic subgroup of upper triangular matrices.*
3. $\dim_{\mathbb{R}} G = 1$ *and there are three possibilities:*
   *(a) $G$ is conjugate to*
   $$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\},$$
   *and $G$ is connected and abelian.*

   *(b) $G$ is conjugate to*
   $$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} : b \in \mathbb{R} \right\},$$
   *and $G$ is has two connected components.*

   *(c) $G$ is conjugate to* $\mathrm{SO}(2)$*, and $G$ is connected and abelian.*

   *(d) $G$ is conjugate to* $\mathrm{SO}(1,1)$*, and $G$ has two connected components.*

   *(e) $G$ is conjugate to*
   $$\mathrm{SO}(1,1) \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathrm{SO}(1,1),$$
   *and $G$ has four connected components.*

*Moreover, $G$ is irreducible if and only if $G = \mathrm{SL}(2,\mathbb{R})$ or if $G$ is conjugate to either* $\mathrm{SO}(2)$ *or to*

$$\mathrm{SO}(1,1) \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathrm{SO}(1,1).$$

## 1.3 Group (co)homology

This section is a short introduction to group (co)homology and relative group (co)homology. These notions are important because group cohomology is the natural language to describe the Zariski tangent spaces to character varieties. This note is a short summary of classical literature such as [Nos17, §7], [Löh10] and [BE78].

### 1.3.1 Definiton

We begin by recalling the definitions of group (co)homology. Group (co)homology is a functor from the category of discrete groups $\Gamma$ with a left $\Gamma$-module $M$ to the category of graded abelian groups:

$$H^*, H_*\colon \left(\begin{array}{c}\text{pairs of a discrete group}\\ \text{and a left module}\end{array}\right) \longrightarrow \left(\begin{array}{c}\text{graded abelian}\\ \text{groups}\end{array}\right).$$

By requiring $\Gamma$ to be discrete, we obtain a topological interpretation of group (co)homology. Recall that the natural topology on the fundamental group of a space that admits a universal cover is the discrete topology, because it is the coarser topology that makes the universal cover a principal bundle for the deck transformation action. Discrete groups have the following property.

**Theorem 1.3.1** (Classifying Space Theorem)**.** *If $\Gamma$ is a discrete group, then there is a unique connected space $B\Gamma$, up to canonical homotopy, called the* classifying space[7] *of $G$, such that*

$$\pi_1(B\Gamma) \cong \Gamma, \quad \pi_i(B\Gamma) = 0, \quad \forall i \geqslant 2.$$

A possible definition of the (co)homology of the pair $(\Gamma, M)$, where $\Gamma$ is a discrete group and $M$ is a left $\Gamma$-module, would be to say that it is the singular (co)homology of $B\Gamma$ with coefficients in $M$. We favour however a more intrinsic approach.

Let $\mathbb{Z}[\Gamma]$ be the *integral group ring* of $\Gamma$, i.e. the free $\mathbb{Z}$-module generated by the elements of $\Gamma$. Note that a $\Gamma$-module structure is by definition the same as a $\mathbb{Z}[\Gamma]$-module structure. Let $\varepsilon\colon \mathbb{Z}[\Gamma] \to \mathbb{Z}$ be the *augmentation map* defined by $g \mapsto 1$, $g \in \Gamma$, and extended $\mathbb{Z}$–linearly to $\mathbb{Z}[\Gamma]$. We denote by $\Delta$ the kernel of the augmentation map.

**Definition 1.3.2** (Group (co)homology)**.** The *group (co)homology* of the discrete group $\Gamma$ with coefficients in the left $\Gamma$-module $M$ is

$$H_*(\Gamma, M) := \mathrm{Tor}_*^{\mathbb{Z}[\Gamma]}(\mathbb{Z}, M), \quad H^k(\Gamma, M) := \mathrm{Ext}^*_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, M).$$

Definition 1.3.2 uses the derived functors Tor and Ext. What this really means is that group (co)homology can be computed with projective resolutions of $\mathbb{Z}[\Gamma]$-modules. Recall that a module $P$ is *projective* if it satisfies the following lifting property

$$\begin{tikzcd} & A \arrow[d, two heads, "\forall"] \\ P \arrow[ur, dashed, "\exists"] \arrow[r, "\forall"'] & B, \end{tikzcd}$$

[7]The names *Eilenberg-MacLane space* or $K(\Gamma, 1)$ *space* are also common.

by which we mean that every morphism $P \to B$ factors through every surjective morphism $A \to B$. Equivalently, $P$ is projective if every short exact sequence of modules

$$0 \longrightarrow A' \longrightarrow B' \xrightarrow{f} P \longrightarrow 0$$

splits, i.e. there exists a morphism of modules $h\colon P \to B'$, called *section map*, such that $f \circ h$ is the identity on $P$, see [Bou89, Chap. 2, §2, Prop. 4]. A *projective resolution* $\mathcal{P}$ of a module $C$ (not necessarily projective) is an exact sequence of projective modules ending in $C \to 0$:

$$\ldots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} C \longrightarrow 0 \quad \text{(exact)}.$$

A projective resolution is denoted $\mathcal{P} \twoheadrightarrow C$. The fundamental property of projective resolutions is

**Lemma 1.3.3.** *Any two projective resolutions of the same module are chain homotopic.*

The derived functors in Definition 1.3.2 mean that if $\mathcal{P} \twoheadrightarrow \Delta = \operatorname{Ker}(\varepsilon)$ is the projective resolution of $\mathbb{Z}[\Gamma]$-modules

$$\ldots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

then

$$H_*(\Gamma, M) = H_*(\mathcal{P} \otimes_\Gamma M), \quad H^*(\Gamma, M) = H^*\big(\operatorname{Hom}_\Gamma(\mathcal{P}; M)\big).$$

In particular, $H_0(\Gamma, M) = \Delta \otimes_\Gamma M$ and the negative-degree cohomology modules vanish. Similarly, $H^0(\Gamma, M) = \operatorname{Hom}_\Gamma(\Delta, M)$. Since any two projective resolutions of $\Delta$ are chain homotopic, group (co)homology is independent of the choice of the projective resolution $\mathcal{P} \twoheadrightarrow \Delta$.

**Example 1.3.4.** We compute the homology of free groups with coefficients in a trivial module $M$. Let $F_n = \langle \gamma_1, \ldots, \gamma_n \rangle$ be the free group on $n$ elements. We claim that $\Delta$ is the free $\mathbb{Z}[F_n]$–module given by $\Delta = \langle \gamma_1 - 1, \ldots, \gamma_n - 1 \rangle_{\mathbb{Z}[F_n]}$. The show the inclusion $\Delta \subset \langle \gamma_1 - 1, \ldots, \gamma_n - 1 \rangle_{\mathbb{Z}[F_n]}$, argument as follows. If $x \in \Delta$, then $x = \sum n_i h_i$ where $h_i \in F_n$ and the $n_i$ are integers whose sum is zero. An induction on the length of $h_i$ shows that $(h_i - 1) \in \langle \gamma_1 - 1, \ldots, \gamma_n - 1 \rangle_{\mathbb{Z}[F_n]}$. Now, since $x = \sum n_i h_i = \sum n_i (h_i - 1)$, we conclude that $x \in \langle \gamma_1 - 1, \ldots, \gamma_n - 1 \rangle_{\mathbb{Z}[F_n]}$. Since $\Delta$ is a free $\mathbb{Z}[F_n]$–module, then

$$0 \longrightarrow \Delta \longrightarrow \mathbb{Z}[F_n] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a free, hence projective, resolution of $\Delta$. In particular

$$H_k(F_n, M) = \begin{cases} M, & k = 0 \\ M^n, & k = 1 \\ 0, & k \geqslant 2 \end{cases}.$$

Note that this corresponds to the homology of a sphere with $n+1$ punctures, which is coherent by Theorem 1.3.1 since the fundamental group of a sphere with $n+1$ punctures is a free group on $n$ generators (see also Definition 1.4.1).

### 1.3.2 The bar resolution

Our favourite choice of projective resolution of $\Delta$ is the so-called *bar resolution*. It is defined by $P_k := \mathbb{Z}[\Gamma^{k+1}]$ for $k \geqslant 1$. Using the canonical isomorphism $M \otimes_\Gamma \mathbb{Z}[\Gamma^{k+1}] \cong M \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma^k]$, we obtain that the group homology of $\Gamma$ with coefficients in $M$ can be computed as the homology of the chain complex

$$C_k(\Gamma, M) := M \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma^k], \quad k \geqslant 0.$$

It is called the *bar chain complex* of $\Gamma$ and $M$. The differential $\partial_k \colon C_k(\Gamma, M) \to C_{k-1}(\Gamma, M)$ is defined by

$$\begin{aligned} \partial_k(a \otimes (g_1, \dots, g_k)) := & g_1 \cdot a \otimes (g_2, \dots, g_k) \\ & + \sum_{i=1}^{k-1} (-1)^i a \otimes (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_k) \\ & + (-1)^k a \otimes (g_1, \dots, g_{k-1}), \end{aligned} \tag{1.3.1}$$

where $a \in M$ and $(g_1, \dots, g_k) \in \Gamma^k$. The *bar cochain complex* is given by

$$C^k(\Gamma, M) := \mathrm{Map}(\Gamma^k, M), \quad k \geqslant 0,$$

where $\mathrm{Map}(\Gamma^k, M)$ is the $\Gamma$-module of set-theoretic functions from $\Gamma^k$ to $M$. The differential $\partial^k \colon C^{k-1}(\Gamma, M) \to C^k(\Gamma, M)$ is defined by

$$\begin{aligned} (\partial^k u)(g_1, \dots, g_k) := & g_1 \cdot u(g_2, \dots, g_k) \\ & + \sum_{i=1}^{k-1} (-1)^i u(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_k) \\ & + (-1)^k u(g_1, \dots, g_{k-1}), \end{aligned} \tag{1.3.2}$$

where $u \in \mathrm{Map}(\Gamma^{k-1}, M)$ and $(g_1, \dots, g_k) \in \Gamma^k$. One can easily check that the squares of the differentials $\partial_k$ and $\partial^k$ vanish. The sets of $k$-cocycles and $k$-coboundaries of the bar complex are denoted by $Z^k(\Gamma, M)$ and $B^k(\Gamma, M)$, respectively. For example, the 1-cocycles are

$$Z^1(\Gamma, M) := \{u \colon \Gamma \to M : u(g_1 g_2) = u(g_1) + g_1 \cdot u(g_2), \quad \forall g_1, g_2 \in \Gamma\}$$

and the 1-coboundaries are

$$B^1(\Gamma, M) := \{u \colon \Gamma \to M : \exists a \in M, \quad u(g) = g \cdot a - a, \quad \forall g \in \Gamma\}.$$

There is an obvious relation between the differentials (1.3.1) and (1.3.2) given by

$$(\partial^k u)(g_1, \dots, g_k) = \tilde{u}\big(\partial_k(1 \otimes (g_1, \dots, g_k))\big), \tag{1.3.3}$$

where $\tilde{u} \colon M \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma^{k-1}] \to M$ is the unique lift of the $\mathbb{Z}$-linear map $M \times \mathbb{Z}[\Gamma^{k-1}] \to M$, $(a, (g_1, \dots, g_k)) \mapsto a \cdot u(g_1, \dots, g_k)$.

### 1.3.3 Relative group (co)homology

Let $\Lambda = \{\Lambda_i : i \in I\}$ be a family of subgroups of $\Gamma$ stable under conjugation. We define the group (co)homology of $\Gamma$ relative to $\Lambda$ with coefficients in $M$. Let $\mathbb{Z}[\Gamma/\Lambda] := \bigoplus_{i\in I} \mathbb{Z}[\Gamma/\Lambda_i]$ be the direct sum of the free groups generated by the left cosets of $\Lambda_i$ in $\Gamma$. We denote by $\Delta$ the kernel of the augmentation map $\varepsilon\colon \mathbb{Z}[\Gamma/\Lambda] \to \mathbb{Z}$.

**Definition 1.3.5** (Relative group (co)homology)**.** The *relative (co)homology groups* of $\Gamma$ relative to $\mathcal{K}$ with coefficients in the $\Gamma$-module $M$ are defined by

$$H_*(\Gamma, \Lambda, M) := \mathrm{Tor}_{*-1}^{\mathbb{Z}[\Gamma]}(\mathbb{Z}, \Delta \otimes_\Gamma M),$$

$$H^*(\Gamma, \Lambda, M) := \mathrm{Ext}_{\mathbb{Z}[\Gamma]}^{*+1}(\mathbb{Z}, \mathrm{Hom}_\Gamma(\Delta, M)).$$

Observe that

$$H_*(\Gamma, \Lambda, M) = H_{*-1}(\Gamma, \Delta \otimes_\Gamma M), \tag{1.3.4}$$

$$H^*(\Gamma, \Lambda, M) = H^{*-1}(G, \mathrm{Hom}_\Gamma(\Delta, M)). \tag{1.3.5}$$

In particular, $H_0(\Gamma, \Lambda, M) = H^0(\Gamma, \Lambda, M) = 0$, $H_1(\Gamma, \Lambda, M) = \Delta \otimes_\Gamma M$ and $H^1(\Gamma, \Lambda, M) = \mathrm{Hom}_\Gamma(\Delta, M)$.

*Remark* 1.3.6. Definition 1.3.5 makes perfect sense even if $\Lambda$ is not assumed to be closed under conjugation. This gives a notion of group (co)homology relative to any family of subgroups. However, this notion is equivalent to the former in the following sense. If $\overline{\Lambda}$ denote the *conjugation closure* of $\Lambda$:

$$\overline{\Lambda} := \{g\Lambda_i g^{-1} : g \in \Gamma, \Lambda_i \in \Lambda\},$$

then there are canonical isomorphisms

$$H_*(\Gamma, \Lambda, M) \cong H_*(\Gamma, \overline{\Lambda}, M), \quad H^*(\Gamma, \Lambda, M) \cong H^*(\Gamma, \overline{\Lambda}, M). \tag{1.3.6}$$

Indeed, choose a set of coset representatives $\mathcal{X}$ for $\Gamma/\Lambda$. This gives an identification $\mathbb{Z}[\Gamma/\Lambda] \cong \mathbb{Z}[\Gamma/\overline{\Lambda}]$ which induces the desired isomorphisms. The resulting isomorphisms (1.3.6) are independent of the choice of $\mathcal{X}$, see [BE78, Proposition 7.5].

### 1.3.4 Bar resolution for relative (co)homology

The bar resolution for relative group (co)homology is obtained from the bar resolution for group (co)homology using the cone construction. Recall that if $A$ and $B$ are chain complexes and $f\colon B \to A$ is a morphism of chain complexes, then the *cone* of $f$ is the chain complex $C(f)$ with differential $d$ given by

$$C(f)_k := A_k \oplus B_{k-1}, \quad d(\alpha, \beta) := (-d\alpha + f(\beta), d\beta).$$

This construction produces an exact triangle of complexes $B \to A \to C(f) \to B[-1]$ where $B[-1]$ is the shifted complex obtained from $B$, also called the *suspension* of $B$. The exact triangle induces a long exact sequence in (co)homology. We adopt the shorthand notation

$$C_k(\Lambda, M) := \bigoplus_{i \in I} C_k(\Lambda_i, M), \quad C^k(\Lambda, M) := \prod_{i \in I} C^k(\Lambda_i, M).$$

The *relative bar chain complex* is given by the cone of the inclusion $K_i \subset \Gamma$, i.e.

$$\begin{aligned} C_k(\Gamma, \Lambda, M) &:= C_k(\Gamma, M) \oplus C_{k-1}(\Lambda, M), \\ &\cong M \otimes_\Gamma \big(\mathbb{Z}[\Gamma^k] \oplus \mathbb{Z}[\Lambda^{k-1}]\big). \end{aligned}$$

with differential $\partial_k \colon C_k(\Gamma, \Lambda, M) \to C_{k-1}(\Gamma, \Lambda, M)$ defined by

$$\partial_k(g, h) := \big( -\partial_k g + \sum_{i \in I} \imath_i h_i,\ \partial_{k-1} h \big), \tag{1.3.7}$$

where $g \in C_k(\Gamma, M)$ and $h = (h_i)_{i \in I} \in C_{k-1}(\Lambda, M)$. Recall that at most finitely many $h_i$ are nonzero so that the sum in (1.3.7) makes sense. The *relative bar cochain complex* is defined by

$$\begin{aligned} C^k(\Gamma, \Lambda, M) &:= C^k(\Gamma, M) \oplus C^{k-1}(\Lambda, M), \\ &\cong \operatorname{Map}\big(\mathbb{Z}[\Gamma^k] \oplus \mathbb{Z}[\Lambda^{k-1}], M\big). \end{aligned}$$

The differential $\partial^k \colon C^k(\Gamma, \Lambda, M) \to C^{k+1}(\Gamma, \Lambda, M)$ is given by

$$\begin{aligned} \partial^k(u, f) &:= \big(\partial^k u,\ u\,\imath_i - \partial^{k-1} f_i\big) \\ &= \big(u\,\partial_{k+1},\ u\,\imath_i - f_i \partial_k\big), \end{aligned} \tag{1.3.8}$$

where $u \in C^k(\Gamma, M)$ and $f = (f_i)_{i \in I} \in C^{k-1}(\Lambda, M)$. The second equality in (1.3.8) follows from the relation (1.3.3) which implies $u\partial_{k+1} = \partial^k u$ and $f\partial_k = \partial^{k-1} f$.

### 1.3.5 Long exact sequences

There are long exact sequences in group homology and cohomology that read

$$\ldots \longrightarrow H_k(\Lambda, M) \xrightarrow{\oplus(\imath_i)_\star} H_k(\Gamma, M) \xrightarrow{j} H_k(\Gamma, \Lambda, M) \xrightarrow{r} H_{k-1}(\Lambda, M) \longrightarrow \ldots \tag{1.3.9}$$

$$\ldots \longrightarrow H^{k-1}(\Lambda, M) \xrightarrow{r} H^k(\Gamma, \Lambda, M) \xrightarrow{j} H^k(\Gamma, M) \xrightarrow{\times(\imath_i)^\star} H^k(\Lambda, M) \longrightarrow \ldots \tag{1.3.10}$$

We used the shorthand notations $H_k(\Lambda, M) := \bigoplus_{i \in I} H_k(\Lambda_i, M)$ and $H^k(\Lambda, M) := \prod_{i \in I} H^k(\Lambda_i, M)$. The morphisms $j$ and $r$ are induced from the inclusion and restriction on the (co)chain complex level. The long exact sequences are obtained by applying the derived functors $\operatorname{Ext}^*_{\mathbb{Z}[\Gamma]}(-, M)$ and $\operatorname{Tor}_*^{\mathbb{Z}[\Gamma]}(-, M)$ to the short exact sequence

$$0 \longrightarrow \Delta \longrightarrow \mathbb{Z}[\Gamma/\Lambda] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

### 1.3.6 Relation to singular (co)homology

The purpose of this section is to explain how the singular (co)homology of a space relates to the group (co)homology of its fundamental group.

**Definition 1.3.7** (Eilenberg-MacLane pair)**.** A pair of topological spaces $(X, Y)$ with $Y \subset X$ is called an *Eilenberg-MacLane pair* of type $K(\Gamma, \Lambda, 1)$, if $X$ is a $K(\Gamma, 1)$ CW-complex and if $Y = \sqcup Y_i$ where each $Y_i$ is a $K(\Lambda_i, 1)$ subcomplex of $X$.

Equivalently, $(X, Y)$ is an Eilenberg-MacLane pair if each inclusion $Y_i \hookrightarrow X$ induces an injective homomorphism $\pi_1(Y_i, y_i) \hookrightarrow \pi_1(X, y_i)$ and if there exists an isomorphism $\varphi\colon \pi_1(X, y_i) \to \Gamma$ induced by a suitable choice of path connecting base points such that $\varphi(\pi_1(Y_i, y_i)) = \Lambda_i$

$$\begin{tikzcd} \pi_1(Y_i, y_i) \arrow[r, hook] \arrow[d, "\varphi"] & \pi_1(X, y_i) \arrow[d, "\varphi"] \\ \Lambda_i \arrow[r, hook, "\imath_i"] & \Gamma. \end{tikzcd}$$

The standard examples of Eilenberg-MacLane pairs are pairs $(X, Y)$ where $X$ is a $K(\Gamma, 1)$-space and $Y$ is the boundary of $X$.

**Theorem 1.3.8** ([BE78])**.** *Let $(X, Y)$ be an Eilenberg-MacLane pair of type $K(\Gamma, \Lambda, 1)$. Then there exist isomorphisms in (co)homology in every degree that relates the long exact sequences of the pairs $(X, Y)$ and $(\Gamma, \Lambda)$ such that the following diagram commutes (up to a minus sign for the middle square)*

$$\begin{tikzcd} H^{k-1}(\Lambda, M) \arrow[r] \arrow[d, dashed, "\cong"] & H^k(\Gamma, \Lambda, M) \arrow[r] \arrow[d, dashed, "\cong"] & H^k(\Gamma, M) \arrow[r] \arrow[d, dashed, "\cong"] & H^k(\Lambda, M) \arrow[d, dashed, "\cong"] \\ H^{k-1}(Y, M) \arrow[r] & H^k(X, Y, M) \arrow[r] & H^k(X, M) \arrow[r] & H^k(Y, M). \end{tikzcd}$$

*Remark* 1.3.9. Observe that if $(X, Y)$ is an Eilenberg-MacLane pair of type $K(\Gamma, \Lambda, 1)$, then it is also an Eilenberg-MacLane pair of type $K(\Gamma, \Lambda', 1)$ where $\Lambda'$ is obtained from $\Lambda$ by individually conjugating its elements. So, as a byproduct of Theorem 1.3.8, we get a natural isomorphism between the (co)homology of the pairs $(\Gamma, \Lambda)$ and $(\Gamma, \Lambda')$. This isomorphism corresponds to the one induced by (1.3.6). In addition there are natural isomorphisms

$$H_\star(X, Y, M) \cong H_\star(\Gamma, \overline{\Lambda}, M), \quad H^\star(X, Y, M) \cong H^\star(\Gamma, \overline{\Lambda}, M),$$

where $\overline{\Lambda}$ denotes the conjugation closure of $\Lambda$ introduced in Remark 1.3.6.

We refer the reader to [BE78, Thm. 1.3] for a proof of Theorem 1.3.8.

### 1.3.7 Cup product

We introduce the cup product in group cohomology using the bar cochain complex as in [Nos17, §7]. Let $\Gamma$ be a discrete group and $M, M'$ be two $\Gamma$-modules. Let $u \in C^k(\Gamma, M)$ and $v \in C^l(\Gamma, M')$. The *cup product* of $u$ and $v$ is defined as the cochain $u \smile v \in C^{k+l}(\Gamma, M \otimes_\Gamma M')$ defined by

$$u \smile v(g_1, \ldots, g_{k+l}) := u(g_1, \ldots, g_k) \otimes g_1 \cdots g_k \cdot v(g_{k+1}, \ldots, g_l). \tag{1.3.11}$$

**Lemma 1.3.10.** *The cup product satisfies the Leibniz rule:*

$$\partial^{k+l+1}(u \smile v) = \partial^{k+1}u \smile v + (-1)^k u \smile \partial^{l+1} v.$$

The Leibniz rule implies that the cup product descends to a well-defined $G$-invariant product on cohomology:

$$\smile : H^k(\Gamma, M) \otimes_G H^l(\Gamma, M') \to H^{k+l}(\Gamma, M \otimes_\Gamma M').$$

**Lemma 1.3.11.** *Up to the natural identification $M \otimes_\Gamma M' \cong M' \otimes_\Gamma M$, it holds that*

$$[u \smile v] = (-1)^{kl}[v \smile u], \quad \forall u \in Z^k(\Gamma, M), \forall v \in Z^l(\Gamma, M').$$

*Proof.* We treat the case $k = l = 1$. The other cases are similar. We start by computing the differential of $u \otimes v$ using (1.3.2)

$$\begin{aligned} -\partial^2(u \otimes v)(x, y) &= -u(x) \otimes v(x) + u(xy) \otimes v(xy) - x \cdot (u(y) \otimes v(y)) \\ &= u(x) \otimes x \cdot u(y) + x \cdot u(y) \otimes v(x) \\ &= u \smile v(x, y) + v \smile u(x, y), \end{aligned}$$

where in the second equality we used the cocycle property $u(xy) = u(x) + x \cdot u(y)$. This shows that $u \smile v + v \smile u$ is a coboundary. □

The cup product can be defined on relative cohomology as follows. Let $u \in C^k(\Gamma, M)$ and $f \in C^{k-1}(\Lambda, M)$, and $v \in C^l(\Gamma, M')$. Define the cup product of $(u, f)$ with $v$ to be the cochain

$$(u \smile v, f \smile v) \in C^{k+l}(\Gamma, \Lambda, M \otimes_\Gamma M').$$

It induces a cup product in relative cohomology

$$\smile : H^k(\Gamma, \Lambda, M) \otimes_\Gamma H^l(\Gamma, M') \to H^{k+l}(\Gamma, \Lambda, M \otimes_\Gamma M'). \tag{1.3.12}$$

### 1.3.8 Cap product and Poincaré duality

The purpose of [BE78] was to describe a notion of Poincaré duality for group pairs. This can be done as follows. Let $\mathcal{P} \twoheadrightarrow \mathbb{Z}$ be a projective resolution of $\Gamma$-modules. Then $\mathcal{P} \otimes_\Gamma \mathcal{P}$ is a projective resolution of $\mathbb{Z}$ for the diagonal $\Gamma$-action on $\mathcal{P} \otimes_\Gamma \mathcal{P}$. Let $g = p \otimes q \otimes a \in (\mathcal{P} \otimes_G \mathcal{P}) \otimes_G M$ and $u \in \mathrm{Hom}_G(\mathcal{P}, M')$. The *cap product* of $g$ and $u$ is defined to be

$$g \frown u := q \otimes (a \otimes u(p)) \in P \otimes_\Gamma (M \otimes_\Gamma M').$$

**Lemma 1.3.12.** *The cap product is a well-defined operation on complexes and satisfies the Leibniz rule*

$$\partial_k(g \frown u) = (-1)^l \partial_{k+l}\, g \frown u + g \frown \partial^l u.$$

The induced cap product on (co)homology is

$$\frown\colon H_{k+l}(\Gamma, M) \otimes_\Gamma H^k(G, M') \to H_l(\Gamma, M \otimes_\Gamma M')$$

The definition of the cap product in relative (co)homology uses the pairing

$$\begin{aligned} B\colon (\Delta \otimes_\Gamma M) \otimes_\Gamma \mathrm{Hom}_G(\Delta, M') &\to M \otimes_\Gamma M' \\ (g \otimes a) \otimes u &\mapsto a \otimes u(g). \end{aligned} \tag{1.3.13}$$

The *cap product on relative group (co)homology* is the dashed arrow that makes the following diagram commute.

$$\begin{tikzcd} H_{k+l}(\Gamma, \Lambda, M) \otimes_\Gamma H^l(\Gamma, \Lambda, M') \arrow[r, dashed, "\frown"] \arrow[d, equal] & H_l(\Gamma, M \otimes_\Gamma M') \\ H_{k+l-1}(\Gamma, \Delta \otimes_\Gamma M) \otimes_\Gamma H^{k-1}(\Gamma, \mathrm{Hom}_\Gamma(\Delta, M')) \arrow[ur, bend right, "B_* \frown"'] & \end{tikzcd}$$

The equality in the first column is an application of (1.3.4) and (1.3.5).

Using a modified version of the pairing (1.3.13), one can define a second variant of the cap product

$$\frown\colon H_{k+l}(\Gamma, \Lambda, M) \otimes_\Gamma H^k(\Gamma, M') \to H_l(\Gamma, \Lambda, M \otimes_\Gamma M').$$

The two versions of the cup product are natural operations in group (co)homology, see [BE78] for more details.

The cap product maps the long exact sequence in cohomology for the pair $(\Gamma, \Lambda)$ to its long exact sequence in homology. This commutes with the corresponding map in singular homology under the isomorphism of Theorem 1.3.8. Indeed, let $(X, Y)$ denote an Eilenberg-MacLane pair of type $K(\Gamma, \Lambda, 1)$. For any $e \in H_n(\Gamma, \Lambda, M)$, let $\overline{e} \in H_n(X, Y, M)$ be the image of $e$ under the isomorphism of Theorem 1.3.8. The following diagram commutes for $k = 0, \ldots, n$ (up to some minus signs depending on the degree of the two lower squares, see [BE78] for complete details)

$$\begin{tikzcd} H_{n-k}(\Gamma, \Lambda, M \otimes_\Gamma M') \arrow[r] & H_{n-k-1}(\Lambda, M \otimes_\Gamma M') \arrow[r] & H_{n-k-1}(\Gamma, M \otimes_\Gamma M') \\ H^k(\Gamma, M') \arrow[r] \arrow[u, "e\frown"] \arrow[d, "\cong"] & H^k(\Lambda, M') \arrow[r] \arrow[u, "r(e)\frown"] \arrow[d, "\cong"] & H^{k+1}(\Gamma, \Lambda, M') \arrow[u, "e\frown"] \arrow[d, "\cong"] \\ H^k(X, M') \arrow[r] \arrow[d, "\overline{e}\frown"] & H^k(Y, M') \arrow[r] \arrow[d, "r(\overline{e})\frown"] & H^{k+1}(X, Y, M') \arrow[d, "\overline{e}\frown"] \\ H_{n-k}(X, Y, M \otimes_\Gamma M') \arrow[r] & H_{n-k-1}(Y, M \otimes_\Gamma M') \arrow[r] & H_{n-k-1}(X, M \otimes_\Gamma M'). \end{tikzcd}$$

Here, $r$ denotes the connecting morphism of the long exact sequence (1.3.9). In particular, the following square commutes.

$$\begin{tikzcd} H^k(X, Y, M') \arrow[d, "\overline{e}\frown"] & H^k(\Gamma, \Lambda, M') \arrow[l, "\cong"'] \arrow[d, "e\frown"] \\ H_{n-k}(X, M \otimes_\Gamma M') & H_{n-k}(\Gamma, M \otimes_\Gamma M') \arrow[l, "\cong"'] \end{tikzcd}$$

Poincaré duality for de Rham cohomology says that if $X$ is a smooth, compact, connected manifold of dimension $n$, and $[X]$ is a generator of $H_n(X,\mathbb{Z}) \cong \mathbb{Z}$, then the cap product with $[X]$ is an isomorphism

$$[X]\frown\colon H^k_{dR}(X,\mathbb{R}) \xrightarrow{\cong} H_{n-k}(X,\mathbb{R}), \quad k = 0,\dots,n.$$

In the context of group (co)homology, one introduces the notion of Poincaré duality pairs.

**Definition 1.3.13** (Duality pairs)**.** The pair $(\Gamma,\Lambda)$ is called a *duality pair of dimension* $n$, in short a $D^n$*-pair*, if there exists a $G$–module $N$ and an element $e \in H_n(\Gamma,\Lambda,N)$ such that both

- $e\frown\colon H^k(\Gamma,M) \to H_{n-k}(\Gamma,\Lambda,N\otimes_\Gamma M)$
- $e\frown\colon H^k(\Gamma,\Lambda,M) \to H_{n-k}(\Gamma,N\otimes_\Gamma M)$

are isomorphisms for every $k = 0,\dots,n$ and for every $\Gamma$-module $M$. Moreover, if $N$ can be chosen to be isomorphic to $\mathbb{Z}$ as a group, then $(\Gamma,\Lambda)$ is called a *Poincaré duality pair of dimension* $n$, in short a $PD^n$*-pair.*

If $(\Gamma,\Lambda)$ is a $D^n$-pair, then by letting $M = \mathbb{Z}[\Gamma]$ and $k = n$, we obtain $H^n(\Gamma,\Lambda,\mathbb{Z}[\Gamma]) \cong H_0(\Gamma,N\otimes_\Gamma \mathbb{Z}[\Gamma]) \cong N$. Therefore, a duality pair determines a unique *dualizing module* $N$ up to isomorphism.

**Definition 1.3.14** (Fundamental class)**.** For a $PD^n$-pair we call each of the two generators of $H_n(\Gamma,\Lambda,N) \cong \mathbb{Z}$ a *fundamental class* of $(\Gamma,\Lambda)$.

**Example 1.3.15.** Let $X$ be a smooth, compact, connected, manifold of dimension $n$ with non-empty boundary $\partial X$. Let $[X,\partial X] \in H_n(X,\partial X,\mathbb{Z})$ be a fundamental class. Assume that $(X,\partial X)$ an Eilenberg-MacLane pair of type $K(\Gamma,\Lambda,1)$. Then $(\Gamma,\Lambda)$ is a $PD^n$-pair with fundamental class $[\Gamma,\Lambda]$ given by the image of $[X,\partial X]$ under the isomorphism of Theorem 1.3.8. In particular, the following diagram commutes

$$\begin{tikzcd}
H^n_{dR}(X,\partial X,\mathbb{R}) \arrow[r, "{[X,\partial X]\frown}"] & H_0(X,\mathbb{R}) \\
H^n(\Gamma,\Lambda,\mathbb{R}) \arrow[u, "\cong"] \arrow[r, "{[\Gamma,\Lambda]\frown}"] & H_0(\Gamma,\mathbb{R}). \arrow[u, "\cong"]
\end{tikzcd}$$

Here, $\mathbb{R}$ is the trivial $\Gamma$-module.

Observe that if $(\Gamma,\Lambda)$ is a $D^n$-pair, then there exists an induced isomorphism

$$r(e)\frown\colon \prod_{i\in I} H^k(\Lambda_i;M') \to \bigoplus_{i\in I} H_{n-k-1}(\Lambda_i,M\otimes_\Gamma M')$$

in every degree $k$ and for every $\Gamma$-modules $M$, $M'$. Therefore, $\Lambda$ must be a *finite* collection of subgroups.

**Lemma 1.3.16.** *Let $(\Gamma,\Lambda)$ be a $PD^n$-pair and $\mathbb{R}$ be the trivial $\Gamma$-module. The cap product in degree $n$ for the bar resolution is*

$$\begin{aligned}\frown\colon H_n(\Gamma,\Lambda,\mathbb{R})\otimes_\Gamma H^n(\Gamma,\Lambda,\mathbb{R}) &\to \mathbb{R}\\ [(g,h_1,\dots,h_m)]\otimes[(u,f_1,\dots,f_m)] &\mapsto u(g) - \sum_{i=1}^m f_i(h_i),\end{aligned} \tag{1.3.14}$$

*where* $u\colon \Gamma^n \to \mathbb{R}$ *and* $f_i\colon \Lambda_i^{n-1} \to \mathbb{R}$ *have been extended* $\mathbb{Z}$*-linearly to* $\mathbb{Z}[\Gamma^n]$*, respectively* $\mathbb{Z}[\Lambda_i^{n-1}]$*.*

*Proof.* We only check that (1.3.14) vanishes if $(g, h_1, \ldots, h_m)$ is exact. A complete proof is given in [KM96, Proposition 5.8].

The condition $\partial^n(u, f_1, \ldots, f_m) = 0$ as defined in (1.3.8) means that $\partial^n u = 0$ and $u\restriction_{\Lambda_i} - \partial^{n-1} f_i = 0$ for all $i$. Since $(g, h_1, \ldots, h_m)$ is assumed to be exact, there exist $(g', h_1', \ldots, h_m') \in C_{n+1}(\Gamma, \Lambda, \mathbb{R})$ such that

$$\begin{aligned}(g, h_1, \ldots, h_m) &= \partial_{n+1}(g', h_1', \ldots, h_m') \\ &= \left(\sum_{i=1}^m h_i' - \partial_{n+1} g',\, \partial_n h_1', \ldots, \partial_n h_m'\right).\end{aligned}$$

We compute

$$\begin{aligned}u(g) - \sum_{i=1}^m f_i(h_i) &= \sum_{i=1}^m u\restriction_{K_i}(h_i') - u(\partial_{n+1} g') - \sum_{i=1}^m f_i(\partial_n h_i') \\ &= \sum_{i=1}^m u\restriction_{K_i}(h_i') - \partial^n u(g') - \sum_{i=1}^m \partial^{n-1} f_i(h_i'),\end{aligned}$$

where in the second equality we applied the relation (1.3.3). The last expression vanishes because $(u, f_1, \ldots, f_m)$ is closed. □

### 1.3.9 Parabolic group cohomology

Parabolic group cohomology was introduced in the sixties by André Weil. We give a succinct introduction inspired from [GHJW97].

Let $\Gamma$ be a discrete group and $\Lambda = \{\Lambda_i : i \in I\}$ be a family of subgroups of $\Gamma$. Let $M$ be a $\Gamma$-module and $k \geqslant 0$ an integer. Define the set of *parabolic cocycles* in the bar complex to be the set $k$-cocycle $f\colon \Gamma^k \to M$ such that all the restrictions $f\restriction_{\Lambda_i}$ are exact, i.e. belong to $B^k(\Lambda_i, M)$. The set of parabolic cocycles in degree $k$ is denoted

$$Z^k_{par}(\Gamma, M) \subset Z^k(\Gamma, M).$$

Parabolic cocycles are thus cocycles that are exact on the boundary.

**Definition 1.3.17** (Parabolic group cohomology)**.** The *parabolic group cohomology* of $\Gamma$ with coefficients in the $\Gamma$-module $M$ is defined to be

$$H^*_{par}(\Gamma, M) := Z^*_{par}(\Gamma, M)/B^*(\Gamma, M) \subset H^*(\Gamma, M).$$

It follows from Definition 1.3.17 that parabolic group cohomology is related to relative group cohomology as follows.

**Lemma 1.3.18.** *Let* $j\colon H^k(\Gamma, \Lambda, M) \to H^k(\Gamma, M)$ *be the morphism of the long exact sequence* (1.3.10) *for the pair* $(\Gamma, \Lambda)$. *Then,*

$$H^k_{par}(\Gamma, M) = j\big(H^k(\Gamma, \Lambda, M)\big) \cong H^k(\Gamma, \Lambda, M)/\operatorname{Ker}(j).$$

The Leibniz rule of Lemma 1.3.10 implies that the kernel and the image of $j$ are orthogonal for the cup product (1.3.12). In particular, there is a non-degenerate induced product

$$\smile : H^k_{par}(\Gamma, M) \otimes_\Gamma H^l_{par}(\Gamma, M') \to H^{k+l}(\Gamma, \Lambda, M \otimes_\Gamma M'). \tag{1.3.15}$$

## 1.4 Finitely generated groups

We started this chapter by saying that the first ingredient to define a character variety is a target Lie group. Now, it is time to talk about the second ingredient: a finitely generated group $\Gamma$. In most examples and applications, $\Gamma$ will even be finitely presented. Finitely generated groups are always equipped with the discrete topology. In practice, it is often desirable to restrict to a particular class of finitely generated groups in order to obtain more precise statements. Typically, our favourite example of finitely presented groups are fundamental groups of oriented surfaces. These are traditionally called *surface groups*.

### 1.4.1 Surface groups

**Definition 1.4.1** (Surface groups)**.** Let $g \geqslant 0$ and $n \geqslant 0$ be two integers. A group is called a *surface group* if it is abstractly isomorphic to

$$\pi_{g,n} := \left\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \, : \, \prod_{i=1}^{g} [a_i, b_i] = \prod_{j=1}^{n} c_j \right\rangle, \tag{1.4.1}$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ denotes the commutator of $a_i$ and $b_i$. If $n = 0$, then it is called a *closed surface group.*

In particular, all free groups are surface groups since $\pi_{g,n}$ is isomorphic to the free group on $2g + n - 1$ generators whenever $n \geqslant 1$. Surface groups are almost never abelian as for instance $\pi_{g,0}$ is non-abelian for $g \geqslant 2$. The generators $c_i$ in (1.4.1) will play a central role later in Section 5.3 in the context of relative representation varieties. The name “surface group” is explained by the following lemma.

**Lemma 1.4.2.** *Let* $\Sigma_{g,n}$ *denote a connected orientable topological surface of genus* $g \geqslant 0$*, with* $n \geqslant 0$ *punctures. The fundamental group of* $\Sigma_{g,n}$ *is isomorphic to* $\pi_{g,n}$*.*

*Proof.* The proof for the case $n = 0$ is explained in [Lab13, Thm. 2.3.15]. Its generalization to punctured surfaces can be understood in two steps. First, observe that a sphere with $n \geqslant 1$ punctures is homotopy equivalent to the wedge of $n - 1$ circles. Hence, its fundamental group is the free group on $n - 1$ generators. Similarly, a surface of genus $g$ with one puncture is homotopy equivalent to the wedge of $2g$ circles. Thus, its fundamental group is the free group on $2g$ generators. Now, note that $\Sigma_{g,n}$ is the union of two sub-surfaces $\Sigma_{g,1}$ and $\Sigma_{0,n+1}$. The conclusion now follows from Van Kampen’s Theorem.

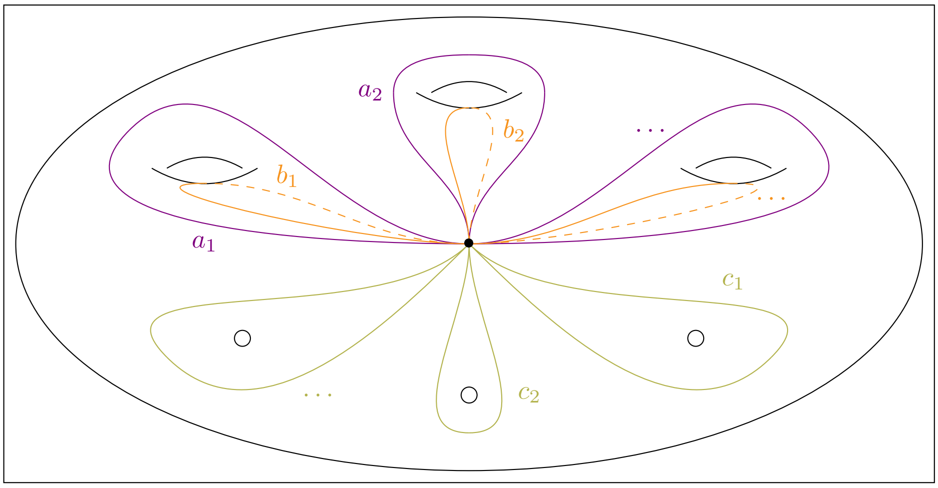



Figure 1.1: Illustration of a collection of generators of the fundamental group of a punctured surface: Two generators for each genus and one for each puncture. These are related by a single relation, namely that of (1.4.1).

□

We denote by $\hat{\Sigma}_{g,n}$ the surface with boundary obtained from $\Sigma_{g,n}$ by replacing each puncture by a boundary component. We also write $\partial\pi_{g,n}$ to denote the collection of 1-parameter subgroups of $\pi_{g,n}$ generated by $c_1, \dots, c_n$. With this notation in mind, we deduce from Lemma 1.4.2 that $(\hat{\Sigma}_{g,n}, \partial\hat{\Sigma}_{g,n})$ is an Eilenberg-MacLane pair of type $K(\pi_{g,n}, \partial\pi_{g,n}, 1)$ in the sense of Definition 1.3.7. We can thus compute the group (co)homology of $\pi_{g,n}$ relative to $\partial\pi_{g,n}$ from the (co)homology of the pair $(\hat{\Sigma}_{g,n}, \partial\hat{\Sigma}_{g,n})$ using Theorem 1.3.8. This shows for instance that the closed surface groups $\pi_{g,0}$ are pairwise non-isomorphic because their cohomology with real coefficients differs in degree 1. It also shows that $\pi_{g,0}$ is non-free for $g \geqslant 1$ since its homology differs from the homology of free groups computed in Example 1.3.4.

### 1.4.2 Fundamental class

It will be useful later to have an explicit fundamental class, in the sense of Definition 1.3.14, for the pair $(\pi_{g,n}, \partial\pi_{g,n})$ expressed in the relative bar complex. Recall from Example 1.3.15 that a fundamental class for the pair $(\pi_{g,n}, \partial\pi_{g,n})$ is a generator of $H_2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{Z}) \cong \mathbb{Z}$, where $\mathbb{Z}$ is the trivial $\pi_{g,n}$-module. For every $i = 1, \dots, g$, we consider the following 2-chains inside $C_2(\pi_{g,n}, \mathbb{Z}) = \mathbb{Z}[\pi_{g,n} \times \pi_{g,n}]$:

$$x_i \coloneqq \left(\prod_{j<i}[a_j, b_j], a_i\right), \quad y_i \coloneqq \left(\prod_{j<i}[a_j, b_j]a_i, b_i\right),$$

$$z_i \coloneqq \left(\prod_{j<i}[a_j, b_j]a_i b_i, a_i^{-1}\right), \quad w_i \coloneqq \left(\prod_{j<i}[a_j, b_j]a_i b_i a_i^{-1}, b_i^{-1}\right).$$

When we differentiate the sum $x_i + y_i + z_i + w_i$ using (1.3.1), we obtain

$$\partial_2(x_i + y_i + z_i + w_i) = a_i + a_i^{-1} + b_i + b_i^{-1} + \prod_{j<i}[a_i, b_i] - \prod_{j<i+1}[a_i, b_i].$$

So, if we further introduce $\varepsilon_i = (a_i, a_i^{-1}) + (b_i, b_i^{-1}) + 2(1,1)$, then we can compute $\partial_2\varepsilon_i = a_i + a_i^{-1} + b_i + b_i^{-1}$ and thus

$$\partial_2\left(\sum_{i=1}^{g} x_i + y_i + z_i + w_i - \varepsilon_i\right) = 1 - \prod_{i=1}^{g}[a_i, b_i].$$

Similarly, the 2-chain $\gamma \in C_2(\pi_{g,n}) = \mathbb{Z}[\pi_{g,n} \times \pi_{g,n}]$ defined by

$$\gamma \coloneqq (c_1, c_2) + (c_1c_2, c_3) + \ldots + (c_1 \cdots c_{n-1}, c_n) + (1,1)$$

satisfies $\partial_2\gamma = c_1 + \cdots + c_n - c_1 \cdots c_n + 1$. Recalling from (1.4.1) that $c_1 \cdots c_n = \prod_{i=1}^{g}[a_i, b_i]$, we obtain

$$\partial_2\left(\gamma - \sum_{i=1}^{g} x_i + y_i + z_i + w_i - \varepsilon_i\right) = \sum_{i=1}^{n} c_i.$$

From here, using (1.3.7), we can easily that the relative 2-chain

$$\left(\gamma - \sum_{i=1}^{g} x_i + y_i + z_i + w_i - \varepsilon_i, (c_1, \ldots, c_n)\right) \in C_2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{Z}) = \mathbb{Z}[\pi_{g,n} \times \pi_{g,n}] \oplus \mathbb{Z}[\partial\pi_{g,n}] \tag{1.4.2}$$

is closed. Its homology class will be denoted by $[\pi_{g,n}, \partial\pi_{g,n}] \in H_2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{Z})$. This notation is justified by the following lemma.

**Lemma 1.4.3.** *The homology class $[\pi_{g,n}, \partial\pi_{g,n}]$ is a fundamental class for the pair $(\pi_{g,n}, \partial\pi_{g,n})$.*

*Proof.* Recall that we defined $\partial\pi_{g,n}$ as the collection of 1-parameter subgroups of $\pi_{g,n}$ generated by $c_1, \ldots, c_n$. The subgroup generated by $c_i$ will be denoted by $\partial_i\pi_{g,n}$ and its inclusion inside $\pi_{g,n}$ by $\imath_i \colon \partial_i\pi_{g,n} \hookrightarrow \pi_{g,n}$. The long exact sequence (1.3.9) in group homology for the pair $(\pi_{g,n}, \partial\pi_{g,n})$ contains the subsequence

$$\ldots \to H_2(\pi_{g,n}, \mathbb{Z}) \longrightarrow H_2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{Z}) \xrightarrow{r} H_1(\partial\pi_{g,n}, \mathbb{Z}) \xrightarrow{\oplus\imath_i} H_1(\pi_{g,n}, \mathbb{Z}) \to \ldots.$$

Since $H_2(\pi_{g,n}, \mathbb{Z}) = 0$, the connecting morphism $r$ is injective and $\operatorname{Ker} \oplus\imath_i \cong H_2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{Z}) \cong \mathbb{Z}$. Recall from Section 1.3.5 that $r$ comes from the restriction on the chain complex level. This means that $r([\pi_{g,n}, \partial\pi_{g,n}]) = [(c_1, \ldots, c_n)] \in H_1(\partial\pi_{g,n}, \mathbb{Z})$. Also recall that $H_1(\partial\pi_{g,n}, \mathbb{Z}) = \bigoplus_{i=1}^{n} H_1(\partial_i\pi_{g,n}, \mathbb{Z})$. By definition of the relative bar complex, $H_1(\partial_i\pi_{g,n}, \mathbb{Z}) = \mathbb{Z}[\partial_i\pi_{g,n}]/(g + h - gh)$. Since $\partial_i\pi_{g,n}$ is the 1-parameter subgroup of $\pi_{g,n}$ generated by $c_i$, we conclude that $H_1(\partial_i\pi_{g,n}, \mathbb{Z}) \cong \mathbb{Z}$ and that under this identification $r([\pi_{g,n}, \partial\pi_{g,n}]) = (1, \ldots, 1) \in \mathbb{Z}^n$.

Similarly, we observe that $H_1(\pi_{g,n}, \mathbb{Z}) = \mathbb{Z}[\pi_{g,n}]/(g + h - gh)$ which gives $H_1(\pi_{g,n}, \mathbb{Z}) \cong \mathbb{Z}^{2g+n-1}$. Under these identifications, the morphism $\oplus\imath_i \colon \mathbb{Z}^n \to \mathbb{Z}^{2g+n-1}$ is the map $(a_1, \ldots, a_n) \mapsto (0, \ldots, 0, a_1 - a_n, \ldots, a_{n-1} - a_n)$. We conclude that the isomorphism $r \colon H_2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{Z}) \to \operatorname{Ker} \oplus\imath_i$, seen as a map $\mathbb{Z} \to \mathbb{Z}^n$, is $a \mapsto (a, \ldots, a)$. This shows that $r([\pi_{g,n}, \partial\pi_{g,n}])$ is a generator of $\mathrm{Ker} \bigoplus \imath_i$, from which we deduce that $[\pi_{g,n}, \partial\pi_{g,n}]$ is a fundamental class. □

Lemma 1.4.3 holds true for any $g \geqslant 0$ and any $n \neq 0$. For instance, when $n = 0$ we obtain a fundamental class for closed surface groups $[\pi_{g,0}] \in H_2(\pi_{g,0}, \mathbb{Z})$ given by the homology class of the 2-chain $\sum_{i=1}^{g} x_i + y_i + z_i + w_i - \varepsilon_i$. Also, the case $g = 0$ corresponds to the case of punctured spheres and the fundamental class $[\pi_{0,n}, \partial\pi_{0,n}] \in H_2(\pi_{g,0}, \mathbb{Z})$ is given by the homology class of the 2-chain $(\gamma, (c_1, \ldots, c_n))$. Similar computations of fundamental classes can be found in [GHJW97, Section 2] or for the case of closed surface groups in this mathoverflow question answered by Bertram Arnold.

# Chapter 2

# Representation varieties

Before introducing character varieties, we will first study representation varieties. In short, a representation variety is an analytic, sometimes algebraic, object associated to a finitely generated group $\Gamma$ and a Lie group $G$. It consists of all group homomorphisms from $\Gamma$ to $G$. We will explain where the analytic and algebraic structures of representation varieties come from and later discuss their Zariski tangent spaces, as well as their smooth points the case where $\Gamma$ is a surface groups.

## 2.1 Definition

**Definition 2.1.1** (Representation variety)**.** The *representation variety* associated to a finitely generated group $\Gamma$ and a Lie group $G$ is the set of group homomorphisms from $\Gamma$ to $G$ and is denoted by

$$\mathrm{Hom}(\Gamma, G).$$

The elements $\phi \in \mathrm{Hom}(\Gamma, G)$ are called *representations.*

The topology on the representation variety $\mathrm{Hom}(\Gamma, G)$ is defined to be the subspace topology induced by the compact-open topology on the space $G^\Gamma$ of all (necessarily continuous) functions $\Gamma \to G$. The resulting topology on $\mathrm{Hom}(\Gamma, G)$ can also be described using a system of generators as follows. For any set of generators $(\gamma_1, \dots, \gamma_n)$ of $\Gamma$, we introduce the subspace

$$X(\Gamma, G) \coloneqq \big\{ \big(\phi(\gamma_1), \dots, \phi(\gamma_n)\big) : \phi \in \mathrm{Hom}(\Gamma, G) \big\} \subset G^n.$$

**Lemma 2.1.2.** *Let $G$ be a Lie group equipped with an analytic atlas. The set $X(\Gamma, G)$ is an analytic subvariety*[1] *of $G^n$ and is homeomorphic to* $\mathrm{Hom}(\Gamma, G)$*. In particular,* $\mathrm{Hom}(\Gamma, G)$ *has a natural structure of analytic variety and the structure does not depend on the choice of generators of* $\Gamma$*.*

*Proof.* Let $R = \{r_i\}$ denote a (maybe infinite) set of relations for the generators $\gamma_1, \dots, \gamma_n$. Each relation $r_i$ defines an analytic map $r_i \colon G^n \to G$ because multiplication and inverse are assumed to

[1] An *analytic variety* is understood to be the zero locus of a set of analytic functions over $\mathbb{R}$ or $\mathbb{C}$.

be analytic operations on $G$. The map $r_i$ is called a *word map.* The set $X(\Gamma, G)$ is the analytic subset of $G^n$ cut out by the relations $r_i(g_1, \dots, g_n) = 1$ for every $i$.

Since a group homomorphism $\phi\colon \Gamma \to G$ is determined by the images of a set of generators of $\Gamma$, the map

$$\begin{aligned} \Pi\colon \operatorname{Hom}(\Gamma, G) &\to X(\Gamma, G) \\ \phi &\mapsto \big(\phi(\gamma_1), \dots, \phi(\gamma_n)\big) \end{aligned}$$

is a bijection. We prove that $\Pi$ is a homeomorphism. Recall that all the sets

$$V(K, U) := \{f : \Gamma \to G : K \subset \Gamma \text{ finite}, \ U \subset G \text{ open}, \ f(K) \subset U\}$$

form a sub-basis for the compact-open topology on $\operatorname{Hom}(\Gamma, G)$. To see that $\Pi$ is a continuous map, observe that, for a collection of open sets $U_1, \dots, U_n \subset G$,

$$\Pi^{-1}\left(X(\Gamma, G) \, \cap \, U_1 \times \ldots \times U_n\right) = \operatorname{Hom}(\Gamma, G) \cap \bigcap_{i=1}^{n} V(\{\gamma_i\}, U_i).$$

To prove that the inverse map $\Pi^{-1}$ is also continuous, note that any element $k \in \Gamma$, seen as a word in the generators $\gamma_1, \dots, \gamma_n$, determines an analytic function $k\colon G^n \to G$. Now, given a finite set $K \subset \Gamma$ and an open set $U \subset G$, we have

$$\Pi\left(\operatorname{Hom}(\Gamma, G) \, \cap \, V(K, U)\right) = X(\Gamma, G) \cap \bigcap_{k \in K} k^{-1}(U).$$

We conclude that both $\Pi$ and its inverse are continuous. Hence, $\Pi$ is a homeomorphism.

If $(\gamma'_1, \dots, \gamma'_{n'})$ is another set of generators of $\Gamma$ and $X'(\Gamma, G)$ is the associated space, then the map from $X(\Gamma, G)$ to $X'(\Gamma, G)$ defined as the composition

$$X(\Gamma, G) \to \operatorname{Hom}(\Gamma, G) \to X'(\Gamma, G)$$

is an isomorphism of analytic varieties. Indeed, the map sends $\big(\phi(\gamma_1), \dots, \phi(\gamma_n)\big)$ to $\big(\phi(\gamma'_1), \dots, \phi(\gamma'_{n'})\big)$. Now, since $\gamma'_i$ is a word in the generators $\gamma_1, \dots, \gamma_n$, it follows that $\phi(\gamma'_i)$ is a word in $\phi(\gamma_1), \dots, \phi(\gamma_n)$. This shows that the map is analytic because word maps are analytic by assumption on $G$. ☐

**Lemma 2.1.3.** *Assume that $G$ has the structure of a real or complex algebraic group, then $X(\Gamma, G)$ is an algebraic subset of $G^n$. In particular,* $\operatorname{Hom}(\Gamma, G)$ *has a natural structure of real or complex algebraic variety and the structure does not depend on the choice of generators of* $\Gamma$.

*Proof.* The argument is analogous to the proof of Lemma 2.1.2. The key observation is that the relations $R = \{r_i\}$ give regular maps $r_i\colon G^n \to G$ by assumption on $G$. ☐

*Remark* 2.1.4 (Finitely generated versus finitely presented). Since we assumed $\Gamma$ to be finitely generated, and not finitely presented, the set of equations that define $X(\Gamma, G)$ might be infinite. However, Hilbert's basis theorem implies that any algebraic variety over a field can be described as the zero locus of finitely many polynomial equations, see e.g. [SKKT00, §2.2].

*Remark* 2.1.5 (Standard topology versus Zariski topology). If $G$ is a real or complex algebraic group, then it is also a Lie group, as mentioned earlier. This means that the representation variety $\mathrm{Hom}(\Gamma, G)$ has both the structure of an analytic variety and of an algebraic variety. The underlying topology of the analytic structure is called the *standard topology* and that of the algebraic structure the *Zariski topology*. The standard topology on an algebraic variety is always Hausdorff. The Zariski topology is coarser than the standard topology. Indeed, Zariski open sets are open in the standard topology because polynomials are continuous functions. A nonempty Zarsiki open set is also dense in both the standard and the Zariski topology.

**Example 2.1.6.** One occasion where one may encounter representations of finitely generated groups into Lie groups in the nature is by studying $(G, X)$ structures. This is because the holonomy of a $(G, X)$ structure on a surface $\Sigma_{g,n}$ is a morphism $\pi_1\Sigma_{g,n} \to G$. Not all the representations $\pi_1\Sigma_{g,n} \to G$ are holonomies of $(G, X)$ structures on $\Sigma_{g,n}$. However, if $n = 0$, then the set of holonomies is an open subset of $\mathrm{Hom}(\pi_1\Sigma_{g,0}, G)$ [Gol21, Cor. 7.2.2]. In the case where $G = \mathrm{PSL}(2, \mathbb{R})$ and $X = \mathbb{H}$, then the holonomies of $(\mathrm{PSL}(2, \mathbb{R}), \mathbb{H})$ structures (commonly known as hyperbolic structures) on a closed surface $\Sigma_{g,0}$ with $g \geqslant 2$ are precisely the discrete and faithful representations of $\mathrm{Hom}(\pi_1\Sigma_{g,0}, \mathrm{PSL}(2, \mathbb{R}))$. It is interesting to note that holonomies of $(G, X)$ structures is only well-defined up to conjugation by an element of $G$, foreshadowing the notion of character varieties. For more information on $(G, X)$ structures, the reader may consult Goldman's book [Gol21].

In the vocabulary of category theory, we can say that a representation variety is a bifunctor from the product of the category of finitely generated groups and the category of Lie/algebraic groups to the category of analytic/algebraic varieties. This is a consequence of Lemmata 2.1.2 and 2.1.3, and of the following.

**Lemma 2.1.7.** *Let $\Gamma$ be a finitely generated group and $G$ be a Lie/algebraic group.*

1. *If $\tau\colon \Gamma_1 \to \Gamma_2$ is a morphism of finitely generated groups, then the induced map $\tau^*\colon \mathrm{Hom}(\Gamma_2, G) \to \mathrm{Hom}(\Gamma_1, G)$ is an analytic/regular map.*
2. *If $r\colon G_1 \to G_2$ is a morphism of Lie groups or of algebraic groups, then the induced map $r_*\colon \mathrm{Hom}(\Gamma, G_1) \to \mathrm{Hom}(\Gamma, G_2)$ is an analytic map or a regular map, respectively.*

*Proof.* The second assertion is immediate. To prove the first statement, note that if $(\gamma_1^1, \ldots, \gamma_n^1)$ is a set of generators for $\Gamma_1$ and $(\gamma_1^2, \ldots, \gamma_m^2)$ is a set of generators for $\Gamma_2$, then $(\tau^*\phi)(\gamma_i^1) = \phi(\tau(\gamma_i^1))$ is a word in $\phi(\gamma_1^2), \ldots, \phi(\gamma_m^2)$. Word maps are analytic, respectively regular, and thus so is $\tau^*$. □

## 2.2 Symmetries

The representation variety $\mathrm{Hom}(\Gamma, G)$ has two natural symmetries given by the right action of the group $\mathrm{Aut}(\Gamma)$ of automorphisms of $\Gamma$ by pre-composition and the left action of $\mathrm{Aut}(G)$ by post-composition:

$$\mathrm{Aut}(G) \circlearrowright \mathrm{Hom}(\Gamma, G) \circlearrowleft \mathrm{Aut}(\Gamma).$$

An immediate consequence of Lemma 2.1.7 is

**Lemma 2.2.1.** *The actions of the groups* $\mathrm{Aut}(\Gamma)$ *and* $\mathrm{Aut}(G)$ *on* $\mathrm{Hom}(\Gamma, G)$ *preserve its analytic/algebraic structure.*

There is a normal subgroup of $\mathrm{Aut}(G)$ that is of particular interest: namely, the subgroup of inner automorphisms of $G$, denoted $\mathrm{Inn}(G)$. Recall that an *inner automorphism* of $G$ is an automorphism given by conjugation by a fixed element of $G$. In particular, $\mathrm{Inn}(G) \cong G/Z(G)$, where $Z(G)$ denotes the centre of $G$ that we introduced in Section 1.1.

*Remark* 2.2.2. We want to point out that if $G$ is semisimple, then $\mathrm{Inn}(G)$ is a finite index subgroup of $\mathrm{Aut}(G)$. This can be seen as follows. First, assume that $G$ is also simply connected. In that case, the map $\mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ induced by derivation is an isomorphism of Lie groups, see e.g. [Ser06, Part II, Chap. V, §8, Thm. 1]. So, it is sufficient to prove the statement on the level of Lie algebras. If $\mathfrak{g}$ is semisimple, then the Lie algebras of $\mathrm{Inn}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{g})$ are isomorphic, see e.g. [HN12, Thm. 5.5.14]. Hence, $\mathrm{Inn}(\mathfrak{g})$ is a finite index subgroup of $\mathrm{Aut}(\mathfrak{g})$, and the same holds for $\mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$. If $G$ is not simply connected, then one considers the simply connected cover $\widetilde{G}$ of $G$ (see [HN12, Thm. §9.5]). Because of lifting properties, there is an injective map $\mathrm{Aut}(G)/\mathrm{Inn}(G) \hookrightarrow \mathrm{Aut}(\widetilde{G})/\mathrm{Inn}(\widetilde{G})$. This concludes the argument.

The action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ is relevant in many concrete cases. For instance, the holonomy representations mentioned in Example 2.1.6 are really defined up to conjugation by an element of $G$ and so it makes sense to see them as elements of the quotient

$$\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G). \tag{2.2.1}$$

The quotient (2.2.1) is our first prototype of character variety for the pair $(\Gamma, G)$.

The action of $\mathrm{Aut}(\Gamma)$ on the representation variety descends to an action of $\mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ on the quotient (2.2.1). The group $\mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ is denoted $\mathrm{Out}(\Gamma)$ and is called the group of *outer automorphisms* of $\Gamma$.

**Example 2.2.3.** The group of outer automorphisms of the surface group $\pi_1\Sigma_{g,n}$ has a particular significance. It contains the pure mapping class group of the surface $\Sigma_{g,n}$ as a subgroup. This is known as the Dehn–Nielsen-Baer Theorem. We develop this observation further in Section 7.2.

## 2.3 Zariski tangent spaces

In this section, we would like to determine the Zariski tangent spaces to representation varieties. We start by recalling the classical notion of Zariski tangent spaces for analytic varieties in $\mathbb{R}^n$.

**Definition 2.3.1** (Zariski tangent spaces)**.** Let $X \subset \mathbb{R}^n$ is an analytic variety defined as the zero locus of some analytic functions $f_1, \ldots, f_m \colon \mathbb{R}^n \to \mathbb{R}$. The *Zariski tangent space* at $x \in X$ is the kernel of the $m \times n$ Jacobi matrix

$$\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{i,j}. \tag{2.3.1}$$

Equivalently, the Zariski tangent space at $x$ consists of all tangent vectors $x'(0)$ tangent to a smooth path $x(t)$ inside $\mathbb{R}^n$ with $x(0) = x$ and that satisfies the relations $f_i = 0$ *up to first order* by which we mean that $f_i(x(0)) = 0$ and $\frac{d}{dt}\big|_{t=0} f_i(x(t)) = 0$.

To specialize to the case of representation varieties, we need a notion of Zariski tangent spaces for analytic varieties in the infinite product $G^\Gamma$. We follow the approach of [Kar92] and refer the reader to that paper for more details. The relevant notion here is that of real valued ringed space.

**Definition 2.3.2** (Real valued ringed space)**.** A *real valued ringed space* is a topological space with a sheaf of real valued continuous functions.

Examples of real valued ringed spaces include smooth manifolds together with the sheaf of smooth functions, analytic varieties together with the sheaf of analytic functions or algebraic varieties together with the sheaf of rational maps. There is a notion of *Zariski tangent space* for real valued ringed spaces that generalizes the notion of tangent spaces for manifolds and that of Zariski tangent spaces for analytic and algebraic varieties.

On the space $G^\Gamma$, one can define a notion of smooth functions. A function $F\colon G^\Gamma \to \mathbb{R}$ is called *locally smooth* if it is locally a smooth function of a finite number of coordinates. The space $G^\Gamma$, together with the sheaf of locally smooth real-valued functions on $G^\Gamma$, is a real valued ringed space. In the case of $G^\Gamma$, the Zariski tangent space at any point can be identified with $\mathfrak{g}^\Gamma$ via left translation.

The representation variety $\mathrm{Hom}(\Gamma, G)$ is the subspace of the space $G^\Gamma$ cut out by the equations

$$\phi(xy)\phi(y)^{-1}\phi(x)^{-1} = 1, \quad \forall x, y \in \Gamma.$$

As such, it has an induced ringed space structure. Previously, in the context of Lemma 2.1.2, we explained that $\mathrm{Hom}(\Gamma, G)$ inherits its structure from the embedding inside $G^n$ that depends on a choice of generators for $\Gamma$. In contrast, the embedding $\mathrm{Hom}(\Gamma, G) \subset G^\Gamma$ does not require to fix a set of generators for $\Gamma$. The disadvantage is that $G^\Gamma$, unlike $G^n$, is an infinite product.

**Lemma 2.3.3** ([Kar92])**.** *Fix a set of $n$ generators of $\Gamma$ and let $F_n$ be the free group on $n$ generators. The following diagram is a commutative diagram of real valued ringed spaces:*

$$\begin{tikzcd} & G^n \arrow[dr] & \\ \mathrm{Hom}(\Gamma, G) \arrow[ur] \arrow[dr] & & G^{F_n} \\ & G^\Gamma \arrow[ur] & \end{tikzcd}$$

*In particular, the structures induced by $G^n$ and $G^\Gamma$ on $\mathrm{Hom}(\Gamma, G)$ coincide.*

We refer the reader to [Kar92] for a proof of Lemma 2.3.3.

Working with the embedding $\mathrm{Hom}(\Gamma, G) \subset G^\Gamma$, we can determine the Zariski tangent space to the representation variety without referring to a presentation of $\Gamma$. Let $F_{x,y}\colon G^\Gamma \to G$ be defined by $F_{x,y}(f) := f(xy)f(y)^{-1}f(x)^{-1}$. The Zariski tangent space to $\mathrm{Hom}(\Gamma, G)$ at $\phi$ is the intersection of the kernels of the linear forms $D_\phi F_{x,y}\colon \mathfrak{g}^\Gamma \to \mathfrak{g}$ for all $x, y \in \Gamma$ (each tangent space to $G$ is naturally identified to $\mathfrak{g}$ via left translation).

**Lemma 2.3.4.** *It holds that*

$$D_\phi F_{x,y}(v) = v(xy) - v(x) - \mathrm{Ad}(\phi(x))v(y)$$

*for* $v \in \mathfrak{g}^\Gamma$ *and* $\phi \in \operatorname{Hom}(\Gamma, G)$*.*

*Proof.* By definition, we have that

$$\begin{aligned} D_\phi F_{x,y}(v) &= \frac{d}{dt}\bigg|_{t=0} F_{x,y}(\exp(tv)\phi) \\ &= \frac{d}{dt}\bigg|_{t=0} \exp(tv(xy))\phi(xy)\phi(y)^{-1}\exp(-tv(y))\phi(x)^{-1}\exp(-tv(x)) \\ &= v(xy) - v(x) - \operatorname{Ad}(\phi(x))v(y). \end{aligned}$$

Here $\exp\colon \mathfrak{g} \to G$ denotes the Lie theoretic exponential map. □

We conclude

**Corollary 2.3.5** ([Gol84], [Kar92])**.** *The Zariski tangent space to* $\operatorname{Hom}(\Gamma, G)$ *at* $\phi$ *is*

$$T_\phi \operatorname{Hom}(\Gamma, G) = \left\{ v \in \mathfrak{g}^\Gamma : v(xy) = v(x) + \operatorname{Ad}(\phi(x))v(y), \quad \forall x, y \in \Gamma \right\}.$$

Corollary 2.3.5 can be reformulated in terms of group cohomology.[2] A representation $\phi \in \operatorname{Hom}(\Gamma, G)$ equips $\mathfrak{g}$ with the structure of a $\Gamma$-module by

$$\Gamma \xrightarrow{\phi} G \xrightarrow{\operatorname{Ad}} \operatorname{Aut}(\mathfrak{g}).$$

The resulting $\Gamma$-module is denoted by $\mathfrak{g}_\phi$. The set of 1-cochains in the bar complex that computes the cohomology of $\Gamma$ with coefficients in $\mathfrak{g}_\phi$ is $\mathfrak{g}^\Gamma$, see Section 1.3.2 for more details on the bar complex. The space of 1-cocycles is

$$Z^1(\Gamma, \mathfrak{g}_\phi) := \left\{ v \in \mathfrak{g}^\Gamma : v(xy) = v(x) + \operatorname{Ad}(\phi(x))v(y), \quad \forall x, y \in \Gamma \right\}$$

and thus identifies with the Zariski tangent space to $\operatorname{Hom}(\Gamma, G)$ at $\phi$. The space of 1-coboundaries, defined by

$$B^1(\Gamma, \mathfrak{g}_\phi) := \left\{ v \in \mathfrak{g}^\Gamma : \exists \xi \in \mathfrak{g}, \quad v(x) = \xi - \operatorname{Ad}(\phi(x))\xi, \quad \forall x \in \Gamma \right\},$$

also plays a role in this context. They can be identified with the Zarisiki tangent space to the $\operatorname{Inn}(G)$-orbit of $\phi \in \operatorname{Hom}(\Gamma, G)$ at $\phi$ (recall from Section 2.2 that $\operatorname{Inn}(G)$ acts on the representation variety by post-composition). We denote this orbit by

$$\mathcal{O}_\phi \subset \operatorname{Hom}(\Gamma, G).$$

**Proposition 2.3.6** ([Gol84], [Kar92])**.** *The Zariski tangent space to* $\mathcal{O}_\phi$ *at* $\phi$ *is*

$$T_\phi \mathcal{O}_\phi = \left\{ v \in \mathfrak{g}^\Gamma : \exists \xi \in \mathfrak{g}, \quad v(x) = \xi - \operatorname{Ad}(\phi(x))\xi, \quad \forall x \in \Gamma \right\}.$$

*Proof.* The orbit $\mathcal{O}_\phi$ is a smooth manifold isomorphic to the quotient of $G$ by the stabilizer of $\phi$ for the conjugation action. The stabilizer of $\phi$ is the centralizer $Z(\phi) := Z(\phi(\Gamma))$ of $\phi(\Gamma)$ inside $G$,

[2] An introduction to group (co)homology, containing all the relevant notions for this work, is provided in Section 1.3.

which is a closed subgroup of $G$. In particular, the Zariski tangent space to $\mathcal{O}_\phi$ at $\phi$ coincides with the usual notion of tangent space.

A smooth deformation of $\phi$ inside $\mathcal{O}_\phi$ is of the form $\phi_t = g(t)\phi g(t)^{-1}$, where $g(t)$ is a smooth 1-parameter family inside $G$ with $g(0) = 1$. The tangent vector to $\phi_t$ at $t = 0$ is the coboundary $v(x) = \xi - \mathrm{Ad}(\phi(x))\xi$ where $\xi \in \mathfrak{g}$ is the tangent vector to $g(t)$ at $t = 0$. Conversely, for any $\xi \in \mathfrak{g}$, the coboundary $v(x) = \xi - \mathrm{Ad}(\phi(x))\xi$ is tangent to $\exp(t\xi)\phi\exp(-t\xi)$ at $t = 0$. ☐

Observe that $B^1(\Gamma, \mathfrak{g}_\phi)$ can be identified with the quotient $\mathfrak{g}/\mathfrak{z}(\phi)$, where $\mathfrak{z}(\phi)$ is the Lie algebra of $Z(\phi)$. In particular, it holds that

$$\dim B^1(\Gamma, \mathfrak{g}_\phi) = \dim \mathcal{O}_\phi = \dim G - \dim Z(\phi). \tag{2.3.2}$$

We mention that the quotient

$$H^1(\Gamma, \mathfrak{g}_\phi) = Z^1(\Gamma, \mathfrak{g}_\phi)/B^1(\Gamma, \mathfrak{g}_\phi)$$

is known as the first cohomology group of the group $\Gamma$ with coefficients in the $\Gamma$-module $\mathfrak{g}_\phi$ introduced in Definition 1.3.2.

**Example 2.3.7** (Surface groups)**.** In the special case of a closed surface group (Definition 1.4.1), one can obtain the conclusion of Corollary 2.3.5 from the embedding $\mathrm{Hom}(\pi_{g,0}, G) \subset G^{2g}$. Let $\phi \in \mathrm{Hom}(\pi_{g,0}, G)$ and let $A_i := \phi(a_i)$ and $B_i := \phi(b_i)$, where $a_i$ and $b_i$ are the generators of $\pi_{g,0}$ in the presentation (1.4.1). The Zariski tangent space to $\mathrm{Hom}(\pi_{g,0}, G)$ at $\phi$ is isomorphic to the kernel of the differential of the map

$$\begin{aligned} F\colon G^{2g} &\to G \\ (X_1, \ldots, X_g, Y_1, \ldots, Y_g) &\mapsto \prod_{i=1}^{g} [X_i, Y_i] \end{aligned}$$

at $(A_1, \ldots, A_g, B_1, \ldots, B_g)$. A simple computation shows that the kernel of $D_{(A_i,B_i)}F$ corresponds to the subset of $\mathfrak{g}^{2g}$ that consists of all those $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ such that

$$\begin{aligned} &\big(\alpha_1 + \mathrm{Ad}(A_1)\beta_1\big) - \mathrm{Ad}\left([A_1, B_1]\right)\big(\beta_1 + \mathrm{Ad}(B_1)\alpha_1\big) \\ &+ \mathrm{Ad}\left([A_1, B_1]\right)\big(\alpha_2 + \mathrm{Ad}(A_2)\beta_2\big) - \mathrm{Ad}\left([A_1, B_1][A_2, B_2]\right)\big(\beta_2 + \mathrm{Ad}(B_2)\alpha_2\big) \\ &+ \ldots \\ &= \sum_{i=1}^{g} \mathrm{Ad}\left(\prod_{j=1}^{i-1}[A_j, B_j]\right)\big(\alpha_i + \mathrm{Ad}(A_i)\beta_i\big) - \mathrm{Ad}\left(\prod_{j=1}^{i}[A_j, B_j]\right)\big(\beta_i + \mathrm{Ad}(B_i)\alpha_i\big) \end{aligned} \tag{2.3.3}$$

vanishes, compare [Lab13, Prop. 5.3.12]. Once again, we identified $T_{A_i}G \cong \mathfrak{g}$ and $T_{B_i}G \cong \mathfrak{g}$ via left translation.

To see the correspondence between this description of the Zariski tangent space and that of Corollary 2.3.5, we proceed as follows. First, if one defines $v\colon \pi_{g,0} \to \mathfrak{g}$ by $v(a_i) := \alpha_i$ and $v(b_i) := \beta_i$ for $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ that satisfy (2.3.3), and extend to $\pi_{g,0}$ using $v(xy) = v(x) + \mathrm{Ad}(\phi(x))v(y)$, then $v$ defines an element of $Z^1(\pi_{g,0}, \mathfrak{g}_\phi)$. Indeed, it is sufficient to check that

$v\big(\prod[a_i,b_i]\big)=0$. If one develops $v\big(\prod[a_i,b_i]\big)$ using $v(xy)=v(x)+\mathrm{Ad}(\phi(x))v(y)$ and $v([x,y])=v(xy)-\mathrm{Ad}(\phi([x,y]))v(yx)$, then one gets that $v\big(\prod[a_i,b_i]\big)=0$ is equivalent to (2.3.3) vanishing. Conversely, given $v\in Z^1(\pi_{g,0},\mathfrak{g}_\phi)$, then $(v(a_1),\ldots,v(a_g),v(b_1),\ldots,v(b_g))$ satisfies 2.3.3 by the same argument as above.

## 2.4 Smooth points

Smooth points of analytic varieties in $\mathbb{R}^n$ are defined as follows.

**Definition 2.4.1** (Smooth points)**.** A point $x$ of an analytic variety $X\subset\mathbb{R}^n$ is a *smooth point* if there is an open neighbourhood $U\subset X$ of $x$ such that $U$ is an embedded submanifold of $\mathbb{R}^n$.

Using the Implicit Function Theorem, we can reformulate the condition and say that $x$ is a smooth point of $X$ if and only if the rank of the Jacobi matrix (2.3.1) at $x$ is maximal. By the Rank-Nullity Theorem, this happens if and only if the dimension of the Zariski tangent space to $X$ at $x$ is minimal. If every point of an analytic variety is smooth, then it is an analytic manifold.

In the context of representation varieties, we will use the characterization of smooth points as the ones that minimize the dimension of the Zariski tangent space. For instance, if $\Gamma$ is a free group, then $\mathrm{Hom}(\Gamma,G)$ is an analytic manifold because of the absence of relations (recall from Lemma 2.1.2 that representation varieties are analytic varieties).

**Lemma 2.4.2.** *The set of smooth points of* $\mathrm{Hom}(\Gamma,G)$ *is invariant under the* $\mathrm{Inn}(G)$*-action.*

*Proof.* The action of $G$ on itself by conjugation is analytic. Therefore, it preserves smooth neighbourhoods of points inside $\mathrm{Hom}(\Gamma,G)$. We can give an alternative argument by observing that the Zariski tangent spaces at $\phi$ and $g\phi g^{-1}$ are isomorphic as $\Gamma$-modules, hence have the same dimension. The isomorphism is given by

$$\begin{aligned} Z^1(\Gamma,\mathfrak{g}_\phi)&\to Z^1(\Gamma,\mathfrak{g}_{g\phi g^{-1}})\\ v&\mapsto \mathrm{Ad}(g)v. \end{aligned}$$

□

### 2.4.1 Surface groups

It is hard to formulate a statement about smooth points of representation varieties for an arbitrary finitely generated group $\Gamma$. However, when $\Gamma=\pi_{g,0}$ is a closed surface group and $G$ is quadrable, it is possible to describe the smooth points of the representation variety explicitly.

**Proposition 2.4.3** ([Gol84])**.** *Let $G$ be a quadrable Lie group. The smooth points of* $\mathrm{Hom}(\pi_{g,0},G)$ *are those representations $\phi$ satisfying*

$$\dim Z(G)=\dim Z(\phi),$$

*where $Z(G)$ denotes the centre of $G$ and $Z(\phi)$ is the centralizer of $\phi(\pi_{g,0})$ inside $G$ (the dimensions are to be understood in terms of manifolds here).*

*Proof.* We compute the dimension of the Zariski tangent space to $\mathrm{Hom}(\pi_{g,0},G)$ at $\phi$. We use the identification with $Z^1(\pi_{g,0},\mathfrak{g}_\phi)$ provided by Corollary 2.3.5. Recall that the group cohomology of

$\pi_{g,0}$ with coefficients in $\mathfrak{g}_\phi$ is isomorphic to the de Rham cohomology of the surface $\Sigma_{g,0}$ with coefficients in the flat vector bundle $E_\phi$ associated to $\mathfrak{g}_\phi$ (i.e. the adjoint bundle of the principal $G$-bundle $(\widetilde{\Sigma}_{g,0} \times G)/\pi_{g,0}$ built from $\phi$, see [Gol84] for more details):

$$H^*(\pi_{g,0}, \mathfrak{g}_\phi) \cong H^*_{dR}(\Sigma_{g,0}, E_\phi).$$

In particular, it vanishes in degrees larger than 2.

The Euler characteristic

$$\dim H^0(\pi_{g,0}, \mathfrak{g}_\phi) - \dim H^1(\pi_{g,0}, \mathfrak{g}_\phi) + \dim H^2(\pi_{g,0}, \mathfrak{g}_\phi) \tag{2.4.1}$$

is independent of $\phi$. Indeed, since the spaces of cochains in simplicial cohomology with local coefficients are finite-dimensional in every degree, the quantity (2.4.1) can be expressed as the alternating sum of the dimensions of the spaces of cochains. The latter is independent of $\phi$, because the structure of $\pi_{g,0}$-module of $\mathfrak{g}_\phi$ only intervenes in the differential, see the definition of the bar resolution (1.3.2). If $\phi$ is the trivial representation, then $\mathfrak{g}_\phi$ is the trivial $\pi_{g,0}$-module and (2.4.1) is equal to the Euler characteristic of $\Sigma_{g,0}$ times the dimension of $G$. We conclude

$$\dim H^1(\pi_{g,0}, \mathfrak{g}_\phi) = (2g-2)\dim G + \dim H^0(\pi_{g,0}, \mathfrak{g}_\phi) + \dim H^2(\pi_{g,0}, \mathfrak{g}_\phi).$$

Poincaré duality (see Appendix 1.3.8) implies $H^2(\pi_{g,0}, \mathfrak{g}_\phi) \cong H^0(\pi_{g,0}, \mathfrak{g}_\phi^*)^*$. The existence of a non-degenerate, Ad-invariant, symmetric, bilinear form on $\mathfrak{g}$ implies that $\mathfrak{g}_\phi \cong \mathfrak{g}_\phi^*$ as $\pi_{g,0}$-modules. Hence, $\dim H^0(\pi_{g,0}, \mathfrak{g}_\phi) = \dim H^2(\pi_{g,0}, \mathfrak{g}_\phi)$. It is easy to see that $H^0(\pi_{g,0}, \mathfrak{g}_\phi)$ is the space of $\mathrm{Ad}(\phi)$-invariant elements of $\mathfrak{g}$, namely $\mathfrak{z}(\phi)$. Hence

$$\dim H^1(\pi_{g,0}, \mathfrak{g}_\phi) = (2g-2)\dim G + 2\dim Z(\phi).$$

Recall from (2.3.2) that the dimension of $B^1(\pi_{g,0}, \mathfrak{g}_\phi)$ is equal to $\dim G - \dim Z(\phi)$. Finally, we obtain

$$\dim Z^1(\pi_{g,0}, \mathfrak{g}_\phi) = (2g-1)\dim G + \dim Z(\phi).$$

Since $Z(G) \subset Z(\phi)$, it holds that $\dim Z(G) \leqslant \dim Z(\phi)$, and we conclude that $\phi$ minimizes the dimension of its Zariski tangent space if and only if $\dim Z(G) = \dim Z(\phi)$. $\square$

*Alternative proof.* Instead of using group cohomology (and the embedding of the representation variety in $G^\Gamma$), one can alternatively compute the dimension of the Zariski tangent space at a representation $\phi$ from the embedding $\mathrm{Hom}(\pi_{g,0}, G) \subset G^{2g}$, compare [Lab13, Prop. 5.3.12]. The infinitesimal kernel of the unique relation of a closed surface group is described by (2.3.3), where $A_i = \phi(a_i)$ and $B_i = \phi(b_i)$.

Consider the orthogonal complement $V$ in $\mathfrak{g}$, with respect to the Ad-invariant pairing $B$ coming from the quadrability of $G$, of the image of the map $\mu\colon \mathfrak{g}^{2g} \to \mathfrak{g}$ defined by (2.3.3). A simple computation leads to

$$\mu(\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g) = \sum_{i=1}^{g}\left(\prod_{j<i}\mathrm{Ad}\left([A_j,B_j]\right)\right)\left(\alpha_i - \mathrm{Ad}(A_iB_iA_i^{-1})\alpha_i\right)$$
$$- \sum_{i=1}^{g}\left(\prod_{j\leqslant i}\mathrm{Ad}\left([A_j,B_j]\right)\right)\left(\beta_i - \mathrm{Ad}(B_iA_iB_i^{-1})\beta_i\right).$$

The orthogonal complement of the Lie algebra of the centralizer $Z(g)$ of any element $g \in G$ is equal to the image of the map $\mathfrak{g} \to \mathfrak{g}$ given by $\xi \mapsto \xi - \mathrm{Ad}(g)\xi$. Therefore, using the general fact that $Z(ghg^{-1}) = gZ(h)g^{-1}$ for any $g, h \in G$, we obtain that $V$ must contain the Lie algebra of

$$\begin{aligned}
&\bigcap_{i=1}^{g}\prod_{j<i}\mathrm{Ad}\left([A_j,B_j]\right)\left(Z(A_iB_iA_i^{-1}) \cap Z(A_iB_iA_iB_i^{-1}A_i^{-1})\right)\\
&= \bigcap_{i=1}^{g}\prod_{j<i}\mathrm{Ad}\left([A_j,B_j]\right)\mathrm{Ad}(A_iB_i)\left(Z(B_i) \cap Z(A_i)\right)\\
&= \bigcap_{i=1}^{g}\prod_{j<i}\mathrm{Ad}\left([A_j,B_j]\right)\left(Z(B_i) \cap Z(A_i)\right)\\
&= \bigcap_{i=1}^{g}\left(Z(A_i) \cap Z(B_i)\right)\\
&= Z(\phi).
\end{aligned}$$

Hence, $\mathfrak{z}(\phi) \subset V$. The reverse inclusion is obvious. Using the Rank-Nullity Theorem, we conclude, as before, that the dimension of the Zariski tangent space at the representation $\phi$ is

$$\dim Z^1(\pi_{g,0},\mathfrak{g}_\phi) = \dim \mathrm{Ker}(\mu) = (2g-1)\dim G + \dim Z(\phi). \qquad \square$$

Proposition 2.4.3 applies to closed surface groups. In Proposition 5.3.5 below, we will discuss an analogous description of smooth points for fundamental groups of punctured surfaces.

# Chapter 3

# The conjugation action

In this section, we elaborate on the action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ by post-composition which was introduced in Section 2.2. It is quite common to refer to this action as the *the conjugation action of $G$* on the representation variety. We will stick the notation introduced previously and write $\mathcal{O}_\phi$ for the $\mathrm{Inn}(G)$-orbit of a representation $\phi \in \mathrm{Hom}(\Gamma, G)$.

## 3.1 Freeness

The action of $\mathrm{Inn}(G) \cong G/Z(G)$ on $\mathrm{Hom}(\Gamma, G)$ is never free, since the trivial representation is always a global fixed point. It is easy to see that the stabilizer of a representation $\phi \in \mathrm{Hom}(\Gamma, G)$ is $Z(\phi)/Z(G)$. In particular

**Lemma 3.1.1.** *The $\mathrm{Inn}(G)$-action is free on the $\mathrm{Inn}(G)$-invariant subset that consists of all the representations $\phi$ such that*

$$Z(G) = Z(\phi).$$

There is a neat characterization of the points where the action is locally free. Recall that the action of a topological group on a set $X$ is *locally free at $x \in X$* if the stabilizer of $x$ is discrete.

**Proposition 3.1.2** ([Gol84])**.** *The action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ is locally free at $\phi$ if and only if*

$$\dim Z(G) = \dim Z(\phi).$$

*Proof.* The action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ induces, for any representation $\phi$, a surjective linear map $\mathfrak{Inn}(G) \to T_\phi \mathcal{O}_\phi$, where $\mathfrak{Inn}(G)$ denotes the Lie algebra of $\mathrm{Inn}(G)$ and $\mathcal{O}_\phi$ the $\mathrm{Inn}(G)$-orbit of $\phi$. The map is given by

$$\xi \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)(\phi).$$

Observe that the action of $\mathrm{Inn}(G)$ on $\mathrm{Hom}(\Gamma, G)$ is locally free at $\phi$ if and only if the induced map $\mathfrak{Inn}(G) \to T_\phi \mathcal{O}_\phi$ is injective. Since the map is always surjective, this is equivalent to asking that both spaces $\mathfrak{Inn}(G)$ and $T_\phi \mathcal{O}_\phi$ have the same dimension. The dimension of $\mathfrak{Inn}(G)$ is $\dim G - \dim Z(G)$ and the dimension of $T_\phi \mathcal{O}_\phi$ is $\dim G - \dim Z(\phi)$, as computed in (2.3.2). Hence, the dimensions coincide if and only if $\dim Z(G) = \dim Z(\phi)$. $\square$

*Remark* 3.1.3. It is striking that the condition of Proposition 3.1.2 coincides with that of Proposition 2.4.3. This means that if $\Gamma = \pi_{g,0}$ is a closed surface group, then the smooth points of $\mathrm{Hom}(\pi_{g,0}, G)$ are precisely those where the action of $\mathrm{Inn}(G)$ is locally free.

### 3.1.1 Regular representations

Proposition 3.1.2 motivates the following definition.

**Definition 3.1.4** (Regular representations)**.** A representation $\phi \in \mathrm{Hom}(\Gamma, G)$ is called *regular* if

$$\dim Z(G) = \dim Z(\phi).$$

We denote by $\mathrm{Hom}^{\mathrm{reg}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of regular representations. If it further holds that $Z(G) = Z(\phi)$, we say that $\phi$ is *very regular*. The $\mathrm{Inn}(G)$-invariant subspace of very regular representations is denoted by $\mathrm{Hom}^{\mathrm{vReg}}(\Gamma, G)$.

We will see later that if $G$ is a reductive algebraic group in the sense of Definition 1.2.3, then most representations are regular, see Proposition 3.2.10.

**Example 3.1.5.** When $G = \mathrm{PSL}(2, \mathbb{R})$, the representations $\phi\colon \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ that are not regular are of a particular kind. We use the description of centralizers in $\mathrm{PSL}(2, \mathbb{R})$ provided by Lemma A.2.8 from Appendix A. It tells us that a non-regular representation $\phi\colon \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ is of one of the following kinds:

1. $\phi$ is the trivial representation.
2. The elements of $\phi(\Gamma)$ are rotations around the same point of $\mathbb{H}$ and $Z(\phi) \cong \mathrm{PSO}(2)$.
3. The elements of $\phi(\Gamma)$ fix a common geodesic in $\mathbb{H}$ and $Z(\phi) \cong \mathbb{R}_{>0}$.
4. The elements of $\phi(\Gamma)$ fix the same point in the boundary of $\mathbb{H}$ and $Z(\phi) \cong \mathbb{R}$.

As soon as the image of $\phi(\Gamma)$ contains, for instance, two elements of different nature (elliptic, hyperbolic or parabolic) or two elements of the same nature with different fixed points/geodesics, then $Z(\phi) = Z(\mathrm{PSL}(2, \mathbb{R}))$ is trivial and $\phi$ is regular, actually very regular. In particular, for $G = \mathrm{PSL}(2, \mathbb{R})$, every regular representation is also very regular.

## 3.2 Properness

The conjugation action of $G$ on $\mathrm{Hom}(\Gamma, G)$ is in general not proper.

**Example 3.2.1.** Consider the case where $\Gamma = \mathbb{Z}$ and $G = \mathrm{PSL}(2, \mathbb{R})$. Let $\phi_1\colon \mathbb{Z} \to \mathrm{PSL}(2, \mathbb{R})$ be the representation given by $\phi_1(1) = \mathrm{par}^+$ in the notation of (A.2.6). Let $\phi_2$ denote the trivial representation. Since the closure of the conjugacy class of any parabolic element of $\mathrm{PSL}(2, \mathbb{R})$ contains the identity, we observe that

$$\phi_2 \in \overline{\mathcal{O}_{\phi_1}} \smallsetminus \mathcal{O}_{\phi_1} \quad \text{and} \quad \{\phi_2\} = \mathcal{O}_{\phi_2}.$$

So, the orbits $\mathcal{O}_{\phi_1}$ and $\mathcal{O}_{\phi_2}$ cannot be separated by disjoint open sets in the (topological) quotient $\mathrm{Hom}(\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{Inn}(\mathrm{PSL}(2,\mathbb{R}))$. In particular, the quotient is not Hausdorff and the conjugacy action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathrm{Hom}(\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$ is not proper.

### 3.2.1 Irreducible representations

Example 3.2.1 hints at the pathological behaviour of representations whose image lies in a parabolic subgroup. This is essentially a worst case scenario, as we explain below.

**Definition 3.2.2** (Borel and parabolic subgroups)**.** A *Borel subgroup* of a complex algebraic group $G$ is a maximal, Zariski closed, solvable connected subgroup of $G$. A *parabolic subgroup* of a (real or complex) algebraic group $G$ is a Zariski closed subgroup of $G$ that contains a Borel subgroup over $\mathbb{C}$.

By definition, a Borel subgroup of $G$ is automatically a Borel subgroup of $G^\circ$. Similarly, $P$ is a parabolic subgroup of $G$ if and only if $P^\circ$ is a parabolic subgroup of $G^\circ$. If $G$ is connected, then all parabolic subgroups are connected [Mil17, Cor. 17.49].

**Example 3.2.3.** Let $G = \mathrm{GL}(n,\mathbb{C})$. The subgroup of upper triangular matrices is a Borel subgroup of $G$. More generally, the Borel subgroups of $\mathrm{GL}(n,\mathbb{C})$ are the ones that preserve a full flag in $\mathbb{C}^n$ and the parabolic subgroups are those that preserve a (partial) flag in $\mathbb{C}^n$ [Bou05, Chap. VIII, §13].

**Definition 3.2.4** (Irreducible representations)**.** Let $G$ be an algebraic group. A representation $\phi\colon \Gamma \to G$ is called *irreducible* if the image of $\phi$ does not lie in a proper parabolic subgroup of $G$. We denote by $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of irreducible representations.

*Remark* 3.2.5. The notion of irreducibility for representations does depend on the underlying field. There exist representations $\phi\colon \Gamma \to G$ that are irreducible over $\mathbb{R}$, but reducible over $\mathbb{C}$, see Example 3.2.24.

Observe that if $G = \mathrm{GL}(n,\mathbb{C})$, then $\phi$ being irreducible in the sense of Definition 3.2.4 is equivalent to $\mathbb{C}^n$ being an irreducible $\Gamma$-module (i.e. $\phi$ is an irreducible representation in the classical sense). This is a consequence of Example 3.2.3.

**Example 3.2.6.** Let $G = \mathrm{SL}(2,\mathbb{C})$. The irreducible representations into $\mathrm{SL}(2,\mathbb{C})$ can be characterized in terms of traces as stated by the following lemma.

**Lemma 3.2.7.** *A representation $\phi\colon \Gamma \to G$ is irreducible if and only there exists an element $\gamma \in [\Gamma,\Gamma] \subset \Gamma$ of the commutator subgroup of $\Gamma$ such that $\mathrm{Tr}(\phi(\gamma)) \neq 2$.*

A proof of Lemma 3.2.7 can be found in [CS83, Lem. 1.2.1]. The argument relies on the following observation: if $A, B \in \mathrm{SL}(2,\mathbb{C})$ are two upper-triangular matrices, then their commutator $[A,B]$ is upper-triangular and has trace 2 (i.e. upper-triangular with ones on the diagonal).

**Definition 3.2.8** (Irreducible subgroups)**.** A subgroup of an algebraic group $G$ is called *irreducible* if it is not contained in a proper parabolic subgroup of $G$.

Definition 3.2.8 is a generalization of the notion of irreducibility of subgroups of $\mathrm{SL}(2,\mathbb{C})$ introduced in Section 1.2.2. Observe that a representation $\phi\colon \Gamma \to G$ is irreducible if and only if its image is an irreducible subgroup of $G$. The centralizer of an irreducible subgroup in a reductive group $G$ is a finite extension of $Z(G)$ [Sik12, Prop. 15] (see also [Sik12, Cor. 17]). We obtain the following lemma.

**Lemma 3.2.9.** *Let $G$ be a reductive algebraic group. Irreducible representations into $G$ are regular:*

$$\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{reg}}(\Gamma, G).$$

It is important to note the following statement.

**Proposition 3.2.10.** *Let $G$ be a reductive algebraic group. The subspace of irreducible representations $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)$ is Zariski open in the representation variety $\mathrm{Hom}(\Gamma, G)$. Moreover, if $\Gamma = \pi_{g,n}$ is a surface group, then $\mathrm{Hom}^{\mathrm{irr}}(\pi_{g,n}, G)$ is dense in a nonempty set of irreducible components of $\mathrm{Hom}(\pi_{g,n}, G)$.*

We refer the reader to [Sik12, Prop. 27 & 29] for a proof. Observe nevertheless that the second part of Proposition 3.2.10 follows from the first assertion and from the existence of at least one irreducible representation. The main result of this section says that if one restricts to irreducible representations, then the conjugation action of $G$ becomes proper.

**Theorem 3.2.11** ([JM87])**.** *Let $G$ be a reductive algebraic group. The $\mathrm{Inn}(G)$-action on $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)$ is proper.*

We refer the reader to [JM87, Prop. 1.1] and references therein for a proof of Theorem 3.2.11.

### 3.2.2 Good representations

Following [JM87], we call *good* all the representations that are simultaneously irreducible and very regular.

**Definition 3.2.12** (Good representations)**.** Let $G$ be an algebraic group. A representation $\phi\colon \Gamma \to G$ is called *good*[1] if it is irreducible and very regular. We denote by $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of good representations.

Lemma 3.1.1 implies that the $\mathrm{Inn}(G)$-action on $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ is free and by Theorem 3.2.11 it is also proper. It is, however, not clear a priori whether good representations exist. However, one can prove the following

**Lemma 3.2.13** ([JM87])**.** *Let $G$ be a reductive algebraic group. The set of good representations $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ is Zariski open in the representation variety $\mathrm{Hom}(\Gamma, G)$.*

Lemma 3.2.13 is proven in [JM87, Prop 1.3 & Lem. 1.3]. In general, $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ might not be a smooth manifold. However, it is the case for closed surface groups by Proposition 2.4.3. We conclude from Theorem 3.2.11 and Lemma 3.1.1 that

[1] In [JM87] and [Sik12] a good representation is defined to be a very regular reductive representation (see Definition 3.2.17). If $G$ is reductive, then their definition is equivalent to ours (see Lemma 3.2.19).

**Corollary 3.2.14.** *Let $G$ be a reductive algebraic group. Let $\Gamma = \pi_{g,0}$ be a closed surface group. The space of good representations* $\mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G)$ *is an analytic manifold of dimension* $(2g - 1)\dim G + \dim Z(G)$*. The* $\mathrm{Inn}(G)$*-action on* $\mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G)$ *is proper and free, and the quotient*

$$\mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G)/\,\mathrm{Inn}(G)$$

*is an analytic manifold of dimension* $(2g - 2)\dim G + 2\dim Z(G)$*.*

Note that the dimension of the quotient in Corollary 3.2.14 is always even. This observation will be relevant later in Section 5 when we discuss the symplectic nature of character varieties.

### 3.2.3 Reductive representations

The notion of irreducible representations can be generalized to the notion of reductive representations, sometimes called completely reducible representations too.

**Definition 3.2.15** (Linearly reductive groups)**.** An algebraic group is called *linearly reductive* if all its finite-dimensional representations are completely reducible.

Equivalently, over the fields of real or complex numbers, an algebraic group $G$ is linearly reductive if and only if the algebraic subgroup that consists of the identity component for the Zariski topology is reductive [Mil17, Cor. 22.43].

**Definition 3.2.16** (Completely reducible subgroups)**.** A subgroup of an algebraic group is called *completely reducible* if its Zariski closure is linearly reductive.

**Definition 3.2.17** (Reductive representations)**.** Let $G$ be an algebraic group. A representation $\phi\colon \Gamma \to G$ is called *reductive* (or *completely reducible*) if $\phi(\Gamma) \subset G$ is completely reducible. We denote by $\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subspace of reductive representations.

In particular, a representation $\phi\colon \Gamma \to \mathrm{GL}(n, \mathbb{C})$ is reductive if and only if $\mathbb{C}^n$ is a completely reducible $\Gamma$-module (i.e. a direct sum of irreducible $\Gamma$-modules). Equivalently, $\phi\colon \Gamma \to \mathrm{GL}(n, \mathbb{C})$ is reductive if and only if every $\Gamma$-invariant subspace of $\mathbb{C}^n$ has a $\Gamma$-invariant complement.

We defined irreducible representations $\Gamma \to G$ to be those whose image is not contained in a parabolic subgroup of $G$ (Definition 3.2.4). Using the notion of *Levi subgroups*[2] of $G$ (see e.g. [Sik12, §3]), one could also define reductive representations as those representations $\phi\colon \Gamma \to G$ with the property that if $\phi(\Gamma)$ is contained in a parabolic subgroup $P$ of $G$, then it is actually contained in a Levi subgroup $L \subset P$.

**Lemma 3.2.18.** *Let $G$ be a reductive algebraic group. Irreducible representations $\phi\colon \Gamma \to G$ are reductive:*

$$\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{red}}(\Gamma, G).$$

*Proof.* The proof relies on the observation that irreducible subgroups of reductive algebraic groups are completely reducible. This is proved in [Sik12, §3] using the notion of Levi subgroups. □

[2]The reader who wishes to learn more about Levi subgroups could have a look at [Sik12, §3]. We recall nevertheless that when $G = \mathrm{GL}(n, \mathbb{C})$ and $P$ is a parabolic subgroup of $G$ stabilizing some flag $F_1 \subset \cdots \subset F_r$, then a Levi subgroup $L$ of $P$ consist of the elements of $P$ that preserve a decomposition $F_1 = E_1$, $F_2 = E_1 \oplus E_2$, …, $F_r = E_1 \oplus \cdots \oplus E_r$.

The converse of Lemma 3.2.18 is not true in general. However, the following holds.

**Lemma 3.2.19.** *Let $G$ be a reductive algebraic group. A reductive representation into $G$ is irreducible if and only if it is regular:*

$$\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) = \mathrm{Hom}^{\mathrm{red}}(\Gamma, G) \cap \mathrm{Hom}^{\mathrm{reg}}(\Gamma, G).$$

The reader is referred to [Sik12, Cor. 17] for a proof of Lemma 3.2.19. Reductive representations can be characterized as follows:

**Proposition 3.2.20.** *Let $G$ be a reductive algebraic group. A representation $\phi\colon \Gamma \to G$ is reductive if and only if the the $\mathrm{Inn}(G)$-orbit $\mathcal{O}_\phi$ of $\phi$ is closed in $\mathrm{Hom}(\Gamma, G)$.*

A proof of Proposition 3.2.20 can be found in [Sik12, Thm. 30], based on an argument of [JM87]. An immediate consequence of Proposition 3.2.20 is that the points of the topological quotient $\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/\,\mathrm{Inn}(G)$ are closed, i.e. it is a $\mathcal{T}_1$ space.[3]

**Proposition 3.2.21** ([RS90])**.** *Let $G$ be a reductive algebraic group. The topological quotient*

$$\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/\,\mathrm{Inn}(G)$$

*is Hausdorff.*

The reader is referred to [RS90, §7.3] and references therein for a proof of Proposition 3.2.21.

### 3.2.4 Zariski dense representations

Some authors favour the notion of Zariski dense representations over irreducible representations, see for instance [Lab13] or [Mon16].

**Definition 3.2.22** (Zariski dense representations)**.** Let $G$ be an algebraic Lie group. A representation $\phi \in \mathrm{Hom}(\Gamma, G)$ is called *Zariski dense* if $\phi(\Gamma)$ is a Zariski dense subgroup of $G$. It is called *almost Zariski dense* if the Zariski closure of $\phi(\Gamma)$ contains $G^\circ$. The $\mathrm{Inn}(G)$-invariant spaces of Zariski dense and almost Zariski dense representations are denoted $\mathrm{Hom}^{\mathrm{Zd}}(\Gamma, G)$ and $\mathrm{Hom}^{\mathrm{aZd}}(\Gamma, G)$, respectively.

Recall that a subgroup $H$ of an algebraic groups $G$ is Zariski dense if and only if any regular function that vanishes on $H$ also vanishes on $G$.

**Lemma 3.2.23.** *Let $G$ be an algebraic Lie group. Almost Zariski dense representations are irreducible:*

$$\mathrm{Hom}^{\mathrm{aZd}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{irr}}(\Gamma, G).$$

*Proof.* Let $\phi\colon \Gamma \to G$ be almost Zariski dense. By definition, the Zariski closure of $\phi(\Gamma)$ contains $G^\circ$. In particular, no proper parabolic subgroups of $G^\circ$ can contain the identity component of the Zariski closure of $\phi(\Gamma)$. Since parabolic subgroups are by definition Zariski closed, no proper parabolic subgroup of $G$ can contain $\phi(\Gamma)$. $\square$

[3] See Section 4 for a reminder of some notions of separability.

**Example 3.2.24.** Let $\alpha_1, \ldots, \alpha_n \in (0, 2\pi)^n$ be angles such that $\alpha_1 + \ldots + \alpha_n = 2k\pi$ for some integer $k$. Let $F_n = \langle a_1, \ldots, a_n \rangle$ denote the free group on $n$ generators. We consider the representation $\phi\colon F_n \to \mathrm{PSL}(2, \mathbb{R})$ defined by $\phi(a_i) = \mathrm{rot}_{\alpha_i}$ in the notation of (A.2.2). The representation $\phi$ is not Zariski dense because its image lies inside $\mathrm{PSO}(2, \mathbb{R})$ which is Zariski closed in $\mathrm{PSL}(2, \mathbb{R})$. However, $\phi$ is irreducible as one can check that $\phi(\Gamma)$ has no fixed point in $\mathbb{RP}^1 = \mathbb{R}^2/\mathbb{R}^\times$. Consider now the representation $\overline{\phi}$ defined as the composition of $\phi$ with the inclusion $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{PSL}(2, \mathbb{C})$. Observe that $\overline{\phi}\colon F_n \to \mathrm{PSL}(2, \mathbb{C})$ is reducible since it fixes $[1 : i] \in \mathbb{CP}^1 = \mathbb{C}^2/\mathbb{C}^\times$, and it is again not Zariski dense because its image lies inside $\mathrm{PSO}(2, \mathbb{C})$ which is Zariski closed in $\mathrm{PSL}(2, \mathbb{C})$.

*Remark* 3.2.25. It was established in Lemma 3.2.23 that Zariski dense representations into any algebraic group are irreducible. The converse statement for $\mathrm{SL}(2, \mathbb{C})$ can sometimes be found in the literature, see e.g. [Mon16, Rem. 2.13]. It is not true. For instance, given a finite non-abelian subgroup $G$ of $\mathrm{SL}(2, \mathbb{C})$ of order $g$, then there is a surjective group homomorphism $F_g \to G$, where $F_g = \langle \gamma_1, \ldots, \gamma_g \rangle$ is the free group on $g$ generators. The fundamental group of a closed surface of genus $g$ maps surjectively to $F_g$ by $a_i, b_i \mapsto \gamma_i$, where $a_i, b_i$ refer to the presentation (1.4.1). We obtain two irreducible representations $\pi_{g,0} \to \mathrm{SL}(2, \mathbb{C})$ and $F_g \to \mathrm{SL}(2, \mathbb{C})$ that are not Zariski dense.

**Lemma 3.2.26.** *Let $G$ be an algebraic group such that $Z(G) = Z(G^\circ)$. If $\phi \in \mathrm{Hom}^{\mathrm{aZd}}(\Gamma, G)$, then $\phi$ is very regular, i.e.*

$$Z(G) = Z(\phi).$$

*In particular, almost Zariski dense representations are good:*

$$\mathrm{Hom}^{\mathrm{aZd}}(\Gamma, G) \subset \mathrm{Hom}^{\mathrm{good}}(\Gamma, G).$$

*Proof.* The argument is taken from [Lab13, §5.3]. Denote by $Z(Z(\phi))$ the centralizer of $Z(\phi) = Z(\phi(\Gamma))$ in $G$. It is a Zariski closed subgroup of $G$ that contains $\phi(\Gamma)$. Hence, by almost Zariski density of $\phi(\Gamma)$, it holds $G^\circ \subset Z(Z(\phi))$ and thus $Z(\phi) \subset Z(G^\circ)$. Since we assumed $Z(G^\circ) = Z(G)$, we conclude that $Z(G) = Z(\phi)$. It now follows from 3.2.23 that almost Zarsiki dense representations are good. □

It follows from Theorem 3.2.11 and Lemma 3.2.23 that, for a reductive algebraic group $G$ (hence connected) and $\Gamma = \pi_{g,0}$ a closed surface group, the $\mathrm{Inn}(G)$-action on the subspace of Zariski dense representations is free and proper, compare [Lab13, Thm. 5.2.6] and [Mon16, Lem. 2.10]. It is interesting to note that the resulting quotient, at least in the case when $Z(G)$ is finite, has the same dimension as the quotient from Corollary 3.2.14.

### 3.2.5 Summary

We summarize all the different notions of representations introduced above in the form of a Venn diagram depicted in Figure 3.1.

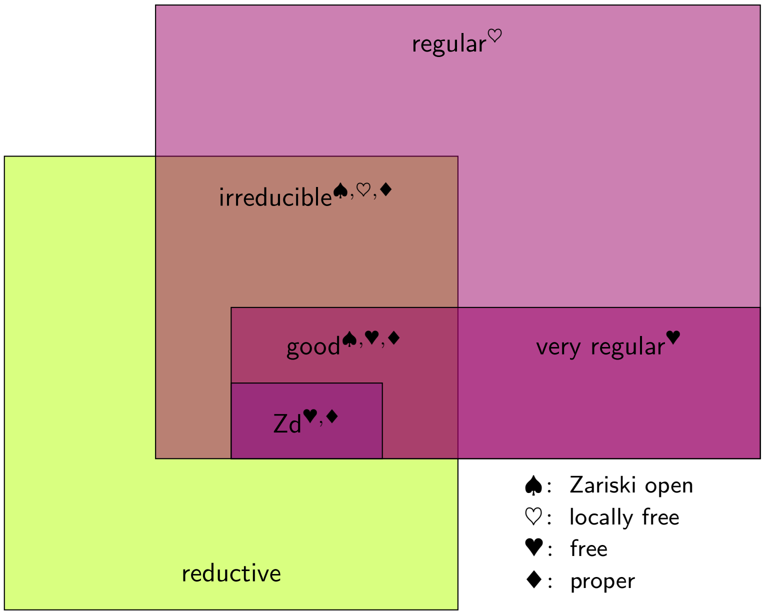



Figure 3.1: We assume for simplicity that $G$ is a reductive algebraic group (hence connected). The two largest families of representations are the regular and the reductive ones. Their intersection is the set of irreducible representations. A representation that is irreducible and very regular is called good. Zariski dense representations are good.

## 3.3 Invariant functions

**Definition 3.3.1** (Invariant functions)**.** We say that a function $\mathrm{Hom}(\Gamma, G) \to \mathbb{K}$ for some field $\mathbb{K}$ (typically $\mathbb{R}$ or $\mathbb{C}$) which is invariant under the conjugation action of $G$ is called an *invariant function* of the representation variety.

In this section, we will focus on algebraic groups $G$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Recall that when this is case, then $\mathrm{Hom}(\Gamma, G)$ inherits the structure of algebraic variety. The algebra of regular functions on the variety $\mathrm{Hom}(\Gamma, G)$— also known as its coordinate ring—is denoted $\mathbb{K}[\mathrm{Hom}(\Gamma, G)]$ and the subalgebra of invariant functions is denoted by

$$\mathbb{K}[\mathrm{Hom}(\Gamma, G)]^G.$$

We are interested in describing a generating family for $\mathbb{K}[\mathrm{Hom}(\Gamma, G)]^G$. In this context, "generate" should be understood in the algebraic sense; that is, a generating family is a collection of invariant functions such that any invariant function can be written as a polynomial expression in the generating functions.

*Remark* 3.3.2. It is worth recalling Nagata's Theorem at this stage which implies that, if $G$ is a reductive algebraic group over $\mathbb{C}$, then $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is finitely generated, see for instance [Dol03, Thm. 3.3].

There is standard way to construct invariant functions $\mathrm{Hom}(\Gamma, G) \to \mathbb{K}$ from a conjugacy invariant function $f : G \to \mathbb{K}$. Pick an element $\gamma \in \Gamma$ and define the function $f_\gamma : \mathrm{Hom}(\Gamma, G) \to \mathbb{K}$ by $f_\gamma(\phi) := f(\phi(\gamma))$. To make our life easier, we will only consider the case where $G$ is a linear algebraic group $G \subset \mathrm{GL}(m, \mathbb{K})$. In that case, classical examples of conjugacy invariant functions $G \to \mathbb{K}$ include the trace $\mathrm{Tr}\colon G \to \mathbb{K}$ or the determinant $\det\colon G \to \mathbb{K}^*$. The invariant functions constructed from the trace will play a central role.

**Definition 3.3.3** (Trace functions)**.** When $\gamma \in \Gamma$, we call the invariant function

$$\begin{aligned} \mathrm{Tr}_\gamma\colon \mathrm{Hom}(\Gamma, G) &\to \mathbb{K} \\ \phi &\mapsto \mathrm{Tr}(\phi(\gamma)). \end{aligned}$$

the *trace function of* $\gamma$. We denote by $\mathcal{T}(\Gamma, G)$ the subalgebra of $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ generated by trace functions.

**Example 3.3.4.** It is known since Fricke and Vogt that $\mathbb{C}[\mathrm{Hom}(\Gamma, \mathrm{SL}(2,\mathbb{C}))]^{\mathrm{SL}(2,\mathbb{C})}$ is generated by the trace functions $\mathrm{Tr}_\gamma$ for $\gamma \in \Gamma$. In this case, since $\mathrm{SL}(2,\mathbb{C})$ enjoys the trace relation $\mathrm{Tr}_{\gamma_1\gamma_2} + \mathrm{Tr}_{\gamma_1^{-1}\gamma_2} = \mathrm{Tr}_{\gamma_1}\mathrm{Tr}_{\gamma_2}$ for any $\gamma_1, \gamma_2 \in \Gamma$, we observe that $\mathbb{C}[\mathrm{Hom}(\Gamma, \mathrm{SL}(2,\mathbb{C}))]^{\mathrm{SL}(2,C)}$ is *linearly* generated by trace functions. We will elaborate on the case of $\mathrm{SL}(2,\mathbb{C})$ below in Example 3.3.10.

The next sections are dedicated to trying to understand better the relation between $\mathcal{T}(\Gamma, G)$ and $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$.

### 3.3.1 Procesi's fundamental theorems of invariants

Procesi studied the "invariants of $n$-tuples of $m \times m$ matrices" in [Pro76]. This can be made precise with the following notation. Let $\mathbb{K}$ denote either the field of real or complex numbers. We denote by $M_m(\mathbb{K})$ the algebra of $m \times m$ matrices with coefficients in $\mathbb{K}$. Let $M_m(\mathbb{K})^n = M_m(\mathbb{K}) \times \ldots \times M_m(\mathbb{K})$ and $\mathbb{K}[M_m(\mathbb{K})^n]$ be the algebra of polynomial functions in $n$ matrix variables $\xi_k = (x_{i,j}^k)_{i,j=1,\ldots,m}$. The group $\mathrm{GL}(m,\mathbb{K})$ acts diagonally on $M_m(\mathbb{K})^n$ by conjugation. For any subgroup $G \subset \mathrm{GL}(m, \mathbb{K})$, the subalgebra of $\mathbb{K}[M_m(\mathbb{K})^n]$ that consists of $G$-invariant polynomials is denoted $\mathbb{K}[M_m(\mathbb{K})^n]^G$ and called the *algebra of invariant*.

In the notation of representation varieties, $M_m(\mathbb{K})^n$ is replaced by $\mathrm{Hom}(F_n, M_m(\mathbb{K})$ where $F_n$ denotes the free group on $n$ generators, and $\mathbb{K}[M_m(\mathbb{K})^n]^G = \mathbb{K}[\mathrm{Hom}(F_n, M_m(\mathbb{K})]^G$. Depending on $G$, $\mathbb{K}[M_m(\mathbb{K})^n]^G$ may contain more or less functions. For instance, when $G = \mathrm{SO}(m,\mathbb{K})$, then the function $M_m(\mathbb{K})^2 \to \mathbb{K}$ defined by $(X, Y) \mapsto \mathrm{Tr}(XY^t)$ is invariant under $\mathrm{SO}(m, \mathbb{K})$ conjugation, but not under conjugation by the larger group $\mathrm{SL}(m, \mathbb{K})$. Procesi proved the following the following statement about system of generators for $M_m(\mathbb{K})^2 \to \mathbb{K}$ [Pro76, Thm. 3.4].

**Theorem 3.3.5** (Procesi's First Fundamental Theorem)**.** *The following hold:*

- *If* $G \in \{\mathrm{GL}(m,\mathbb{K}), \mathrm{SL}(m,\mathbb{K})\}$, *then* $\mathbb{K}[M_m(\mathbb{K})^n]^G$ *is finitely generated by trace polynomials* $\mathrm{Tr}(W)$, *where* $W$ *is a reduced word in* $\xi_1, \ldots, \xi_n$ *of length at most* $2^m - 1$.
- *If* $G \in \{\mathrm{O}(m,\mathbb{K}), \mathrm{SO}(m,\mathbb{K})\}$, *then* $\mathbb{K}[M_m(\mathbb{K})^n]^G$ *is finitely generated by trace polynomials* $\mathrm{Tr}(W)$, *where* $W$ *is a reduced word of length at most* $2^m - 1$ *in* $\xi_1, \ldots, \xi_n$ *and their orthogonal transposes.*[4]

[4]The *orthogonal transpose* of a matrix is the inverse of its transpose. The orthogonal group $\mathrm{O}(m, \mathbb{K})$ consists precisely of the matrices that are equal to their orthogonal transposes.

- *If $G = \mathrm{Sp}(2m, \mathbb{K})$, then $\mathbb{K}[M_{2m}(\mathbb{K})^n]^G$ is finitely generated by trace polynomials $\mathrm{Tr}(W)$, where $W$ is a reduced word of length at most $2^m - 1$ in $\xi_1, \dots, \xi_n$ and their symplectic transposes.*[5]

A proof of Theorem 3.3.5 can be found in [Pro76]. For a more recent account, the reader can consult [DCP17].

One could now ask what happens when $M_m(K)^n$ is replaced by a linear algebraic group $G \subset \mathrm{GL}(m, \mathbb{K})$ and what are the invariants of $G^n$. In other words, what would be a generating family for $\mathbb{C}[\mathrm{Hom}(F_n, G)]^G$. The answer will of course depends on $G$. For instance, the function $\det^{-1} \colon \mathrm{GL}(m, \mathbb{C}) \to \mathbb{K}^*$ is non-trivial and invariant under $\mathrm{GL}(m, \mathbb{K})$. Its restriction to $\mathrm{SL}(m, \mathbb{K})$ however is the constant function 1. Using the Cayley-Hamilton Theorem, it is possible to express the determinant of $X \in \mathrm{GL}(m, \mathbb{K})$ as polynomial in $\mathrm{Tr}(X^m), \dots, \mathrm{Tr}(X)$, so the inverse of the determinant can be expressed as a rational function of traces. It is explained in [Mar22, Sec. 2] how trace functions and the invariant functions associated to the inverse of the determinant generate the invariants of $n$ matrices in $\mathrm{GL}_m(\mathbb{K})$.

**Theorem 3.3.6.** *The algebra $\mathbb{C}[\mathrm{Hom}(F_n, \mathrm{GL}(m, \mathbb{K})]^{\mathrm{GL}(m,\mathbb{K})}$ is generated by the invariant functions $\mathrm{Tr}_\gamma$ and $\det_\gamma^{-1}$ for $\gamma \in F_n$. In particular, $\mathbb{C}[\mathrm{Hom}(F_n, \mathrm{SL}(m, \mathbb{K})]^{\mathrm{SL}(m,\mathbb{K})}$ is generated by the trace functions $\mathrm{Tr}_\gamma$, for $\gamma \in F_n$, only.*

There exist analogues statements to Theorem 3.3.6 for $\mathrm{Sp}(2m, \mathbb{K})$ and $\mathrm{SO}(2m+1, \mathbb{K})$. The story is slightly more subtle for $\mathrm{SO}(2m, \mathbb{K})$. We refer the reader to [Mar22, Sec. 2.3] for more details.

Obviously, trace functions on words are not algebraically independent. Luckily, Procesi also described a collection of generators for the ideal of relations among the generators of Theorem 3.3.5 in [Pro76, Thm. 4.5] (see also [DCP17, Thm. 4.13]). The bottom-line is that the ideal of relations is generated by trace identities.

**Theorem 3.3.7** (Procesi's Second Fundamental Theorem)**.** *The ideal of relations in $\mathbb{K}[M_m(\mathbb{K})^n]^{\mathrm{GL}_m(\mathbb{K})}$ for the generators $\mathrm{Tr}(W)$ from Theorem 3.3.5 is generated by $\mathrm{Tr}(1) - m$ and*

$$\sum_{\sigma \in S_{m+1}} \varepsilon(\sigma) \, \mathrm{Tr}^\sigma(W_0, W_1, \dots, W_m),$$

*where $W_0, W_1, \dots, W_m$ run over all possible reduced words in $\xi_1, \dots, \xi_n$. Here, $S_{m+1}$ denotes the symmetric group on $m+1$ symbols, $\varepsilon(\sigma)$ is the signature of $\sigma$, $T^\sigma = \prod T^{\sigma_l}$ for the decomposition of $\sigma$ into the product of cycles $\sigma_l$ (including trivial cycles), and finally, if $\sigma_l$ is the cycle $(i_1 \cdots i_k)$, then $\mathrm{Tr}^{\sigma_l}(W_0, W_1, \dots, W_m) = \mathrm{Tr}(W_{i_1} \cdots W_{i_k})$.*

Before illustrating Theorem 3.3.7 with Example 3.3.10 below, we explain what happens to the algebra of invariant functions of the presentation variety when $F_n$ is replaced by a finitely generated group $\Gamma$. So, let $\Gamma$ denote a finitely generated group with generating family $(\gamma_1, \dots, \gamma_n)$. The embedding $\imath \colon \mathrm{Hom}(\Gamma, G) \subset G^n = \mathrm{Hom}(F_n, G)$ induces a surjective morphism

$$\imath^* \colon \mathbb{C}[G^n] \twoheadrightarrow \mathbb{C}[\mathrm{Hom}(\Gamma, G)]. \tag{3.3.1}$$

[5] The *symplectic transpose* of a matrix $A \in M_{2m}(\mathbb{K})$ is the matrix $JA^tJ$, where $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ and $I_m$ is the $m \times m$ identity matrix. The symplectic group $\mathrm{Sp}(2m, \mathbb{K})$ consists precisely of the matrices that are equal to their symplectic transposes.

The morphism $\imath^*$ maps invariant functions to invariant functions and thus restricts to a morphism

$$(\imath^*)^G \colon \mathbb{C}[G^n]^G \to \mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G. \tag{3.3.2}$$

If we further assume $G$ to be reductive, then $(\imath^*)^G$ is surjective. This is a consequence of the existence of Reynolds operators, see [Sik13, Rem. 25] or [Hos15, Cor. 4.23]. The morphism $(\imath^*)^G$ maps trace functions to trace functions in the following sense.

**Lemma 3.3.8.** *Let $W$ be a reduced word in the matrices variables $\xi_1, \dots, \xi_n$. It holds that*

$$(\imath^*)^G(\mathrm{Tr}(W)) = \mathrm{Tr}_{W(\gamma_1,\dots,\gamma_n)}\,.$$

*Proof.* The word $W$ induces a word map $W \colon G^n \to G$. The trace function $\mathrm{Tr}(W) \colon G^n \to \mathbb{C}$ sends $(g_1, \dots, g_n)$ to $\mathrm{Tr}(W(g_1, \dots, g_n))$. The image $(\imath^*)^G(\mathrm{Tr}(W))$ is the invariant function $\mathrm{Hom}(\Gamma, G) \to \mathbb{C}$ given by $\phi \mapsto \mathrm{Tr}(W(\phi(\gamma_1), \dots, \phi(\gamma_n)))$. Because $\phi$ is a group homomorphism, it holds that $\mathrm{Tr}(W(\phi(\gamma_1), \dots, \phi(\gamma_n))) = \mathrm{Tr}(\phi(W(\gamma_1, \dots, \gamma_n))$, where we now think of $W$ as a function $W \colon \Gamma^n \to \Gamma$. We conclude that $(\imath^*)^G(\mathrm{Tr}(W)) = \mathrm{Tr}_{W(\gamma_1,\dots,\gamma_n)}$. □

**Lemma 3.3.9.** *Let $G \subset \mathrm{GL}(m, \mathbb{C})$ be a reductive linear algebraic group such as $\mathrm{SL}(m, \mathbb{C})$. If the algebra $\mathbb{C}[G^n]^G$ is generated by trace functions, then*

$$\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G = \mathcal{T}(\Gamma, G).$$

*Proof.* If $G$ is reductive, then $(\imath^*)^G$ is surjective and so $(\imath^*)^G(\mathbb{C}[G^n]^G) = \mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. Moreover, $(\imath^*)^G$ maps trace functions to trace functions, thus, if $\mathbb{C}[G^n]^G$ is generated by trace functions, then it holds $(\imath^*)^G(\mathbb{C}[G^n]^G) = \mathcal{T}(\Gamma, G)$. □

We have seen in Theorem 3.3.6 that $\mathbb{C}[G^n]^G$ is generated by trace functions for $G = \mathrm{SL}(m, \mathbb{C})$, but not for $G = \mathrm{GL}(m, \mathbb{C})$ since we need to account for the inverse of the determinant. In particular, we conclude for Lemma 3.3.9 that

$$\mathbb{C}[\mathrm{Hom}(\Gamma, \mathrm{SL}(m, \mathbb{C}))]^{\mathrm{SL}(m,\mathbb{C})} = \mathcal{T}(\Gamma, \mathrm{SL}(m, \mathbb{C})).$$

**Example 3.3.10.** When $G = \mathrm{GL}(2, \mathbb{C})$, Theorems 3.3.5 and 3.3.7 say that the algebra of invariant functions $\mathbb{C}[\mathrm{Hom}(\Gamma, \mathrm{GL}(2, \mathbb{C}))]^{\mathrm{GL}(2,\mathbb{C})}$ is generated by $\mathrm{Tr}_\gamma$ and $\det_\gamma^{-1}$ for $\gamma \in \Gamma$ and the ideal of relations is generated by $\mathrm{Tr}_1 - 2$ and

$$\mathrm{Tr}_\alpha \mathrm{Tr}_\beta \mathrm{Tr}_\gamma - \mathrm{Tr}_\alpha \mathrm{Tr}_{\beta\gamma} - \mathrm{Tr}_\beta \mathrm{Tr}_{\alpha\gamma} - \mathrm{Tr}_\gamma \mathrm{Tr}_{\alpha\beta} + \mathrm{Tr}_{\alpha\beta\gamma} + \mathrm{Tr}_{\alpha\gamma\beta}, \tag{3.3.3}$$

for $\alpha, \beta, \gamma \in \Gamma$.

If we further restrict $G$ to $\mathrm{SL}(2, \mathbb{C})$, then it is possible to simplify these relations to $\mathrm{Tr}_1 - 2$ and to the famous trace relation

$$\mathrm{Tr}_\alpha \mathrm{Tr}_\beta - \mathrm{Tr}_{\alpha\beta} - \mathrm{Tr}_{\alpha\beta^{-1}} \tag{3.3.4}$$

for $\alpha, \beta \in \Gamma$. In other words, there is an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[\mathrm{Hom}(\Gamma, \mathrm{SL}(2,\mathbb{C}))]^{\mathrm{SL}(2,\mathbb{C})} \cong \mathbb{C}[X_\gamma : \gamma \in \Gamma] \Big/ \left(X_1 - 2, X_{\gamma_1}X_{\gamma_2} - X_{\gamma_1\gamma_2} - X_{\gamma_1\gamma_2^{-1}}\right).$$

To see that, it suffices to recover (3.3.3) from (3.3.4). Note that by taking $\alpha = 1$ in (3.3.4), we obtain the relation $\mathrm{Tr}_\beta - \mathrm{Tr}_{\beta^{-1}}$. From (3.3.4), we obtain $\mathrm{Tr}_{\alpha\beta\gamma} = \mathrm{Tr}_\alpha \mathrm{Tr}_{\beta\gamma} - \mathrm{Tr}_{\alpha\gamma^{-1}\beta^{-1}}$. We further compute $\mathrm{Tr}_{\alpha\gamma^{-1}\beta^{-1}} = \mathrm{Tr}_{\alpha\gamma^{-1}} \mathrm{Tr}_\beta - \mathrm{Tr}_{\beta\alpha\gamma^{-1}}$. Now, $\mathrm{Tr}_{\alpha\gamma^{-1}} = \mathrm{Tr}_\alpha \mathrm{Tr}_\gamma - \mathrm{Tr}_{\alpha\gamma}$ and $\mathrm{Tr}_{\beta\alpha\gamma^{-1}} = \mathrm{Tr}_{\beta\alpha} \mathrm{Tr}_\gamma - \mathrm{Tr}_{\beta\alpha\gamma}$. Combining these relations lead to (3.3.3) which proves the claim.

## 3.4 Characters

The reader may have already wondered where the name "character variety" comes from. The notion of "characters" is some sort of dual to trace functions in the sense that a character is defined by fixing a representation and letting $\gamma \in \Gamma$ be the variable. We will assume here that $G \subset \mathrm{GL}(m,\mathbb{C})$ is a linear algebraic group.

**Definition 3.4.1** (Characters)**.** The *character* of a representation $\phi \in \mathrm{Hom}(\Gamma, G)$ is the function

$$\begin{aligned} \chi_\phi \colon \Gamma &\to \mathbb{C} \\ \gamma &\mapsto \mathrm{Tr}(\phi(\gamma)). \end{aligned}$$

In other words, $\chi_\phi(\gamma) = \mathrm{Tr}_\gamma(\phi)$. We denote by $\chi(\Gamma, G) \subset \mathbb{C}^\Gamma$ the set of all characters coming from representations in $\mathrm{Hom}(\Gamma, G)$. We equip it with the subspace topology inherited from the compact-open topology on $\mathbb{C}^\Gamma$.

Note that $\chi(\Gamma, G) \subset \mathbb{C}^\Gamma$ is automatically a Hausdorff space because $\mathbb{C}^\Gamma$ is a Hausdorff space.

**Theorem 3.4.2** ([CS83])**.** *If $G = \mathrm{SL}(2,\mathbb{C})$, then $\chi(\Gamma, \mathrm{SL}(2,\mathbb{C})) \subset \mathbb{C}^\Gamma$ is a closed algebraic variety.*

We refer the reader to Culler-Shalen's paper [CS83, Cor. 1.4.5] for a proof of Theorem 3.4.2. The map

$$\mathrm{Hom}(\Gamma, G) \to \chi(\Gamma, G)$$

is surjective by definition and factors through the quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$. We point out however that a character does not necessarily determine a unique conjugacy class of representations. For instance, the two representations of Example 3.2.1 are not conjugate but determine the same character. Nevertheless, we have the following statement.

**Proposition 3.4.3.** *Let $G \subset \mathrm{GL}(m,\mathbb{C})$ be a linear algebraic group. Conjugacy classes of irreducible representations are determined by their characters.*

Culler-Shalen provide a proof of Proposition 3.4.3 in [CS83, Prop. 1.5.2] for the case $G = \mathrm{SL}(2,\mathbb{C})$ and claim that the result still holds when $\mathrm{SL}(2,\mathbb{C})$ is replaced by $\mathrm{GL}(m,\mathbb{C})$. The analogous result for almost Zariski dense representations can be found in [Lab13, Cor. 5.3.7].

# Chapter 4

# Character varieties

## 4.1 Specification sheet

First via Example 2.1.6 and then through Chapter 3, we highlighted the relevance of the quotient space $\operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$. This will serve as our fist prototype of character variety. When we simply equip it with the quotient topology, we observed in Chapter 3 that there is no reason to expect this quotient to have any reasonably nice topological structure. This is explained by the conjugation action of $G$ on the representation variety being non-free and non-proper in general.

The goal of this section is to construct an alternative space to replace the topological quotient $\operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$ with some guarantees on its topology, including separability properties. This space should always come with a projection from $\operatorname{Hom}(\Gamma, G)$ that factors through $\operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$. In other words, we would like to construct the largest possible finer quotient of $\operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$ whose topology enjoys some regularity properties, or even has the structure of a variety or of a smooth manifold. The resulting space will be called the *character variety* of representations from ta finitely generated group $\Gamma$ into a Lie group $G$. We present several definitions of character varieties below, each of them guaranteeing increasingly richer structures for the price of requiring more assumptions on $\Gamma$ and $G$.

Often, when we complain about the bad properties of the topological quotient we mean its lack of separability. We will mainly focus on two notions of separability.

**Definition 4.1.1** (Separability)**.** A topological space $X$ is said to be

- $\mathcal{T}_1$ if for any pair of distinct points in $X$, each point lies in an open set that does not contain the other, or, equivalently, $X$ is $\mathcal{T}_1$ if the points of $X$ are closed.
- $\mathcal{T}_2$ or *Hausdorff* if for any pair of distinct points in $X$, there are two disjoint open sets such that each contains one of the two points.

## 4.2 Examples

We will start with a series of examples, trying to understand better what a character variety should be and where things go wrong.

### 4.2.1 Abelian target group

When $G$ is abelian, then the $\mathrm{Inn}(G)$ action on $\mathrm{Hom}(\Gamma, G)$ is trivial. In that case, the character variety of the pair $(\Gamma, G)$ is simply its representation variety $\mathrm{Hom}(\Gamma, G)$. Observe that in that case any representation $\Gamma \to G$ factorizes through the abelianization $\Gamma^{\mathrm{ab}} := \Gamma/[\Gamma, \Gamma]$ of $\Gamma$ and so

$$\mathrm{Hom}(\Gamma, G) = \mathrm{Hom}(\Gamma^{\mathrm{ab}}, G).$$

In the particular case where $G = \mathbb{R}$ and $\Gamma = \pi_1 X$ is the fundamental group of some connected topological space $X$, we have the following interpretation of $\mathrm{Hom}(\Gamma^{\mathrm{ab}}, G)$. By the Hurewicz Theorem, the abelianization of $\pi_1 X$ is isomorphic to the first homology group $H_1(X, \mathbb{R})$ of $X$. This shows that the character variety $\mathrm{Hom}(\pi_1 X, \mathbb{R})$ is naturally isomorphic to the vector space given by the first cohomology of $X$:

$$\mathrm{Hom}(\pi_1 X, \mathbb{R}) = \mathrm{Hom}(H_1(X, \mathbb{R}), \mathbb{R}) = H^1(X, \mathbb{R}).$$

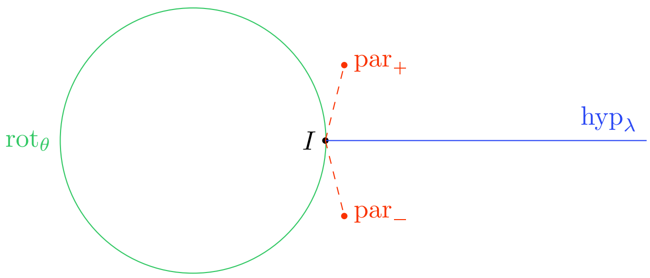
### 4.2.2 Free group on one generator

This example considers the case where $\Gamma = \mathbb{Z}$ is a the free group on one generator. In that case, the topological quotient $\mathrm{Hom}(\mathbb{Z}, G)/\mathrm{Inn}(G)$ is the space of conjugacy classes of $G$. Sometimes it is simply denoted by

$$G/G := \mathrm{Hom}(\mathbb{Z}, G)/\mathrm{Inn}(G).$$

So, defining what the character variety of the pair $(\mathbb{Z}, G)$ should be, amounts to defining a suitable notion for the space of conjugacy classes of $G$.

We started to study the case $G = \mathrm{PSL}(2, \mathbb{R})$ in Example 3.2.1. We already observed that $\mathrm{PSL}(2, \mathbb{R})/\mathrm{PSL}(2, \mathbb{R})$ is not a Hausdorff topological space. It is actually not even $\mathcal{T}_1$ since, in the notation of Example 3.2.1, the closure of the orbit of $\phi_1$ always contains the orbit of $\phi_2$. If we take all the conjugacy classes of $\mathrm{PSL}(2, \mathbb{R})$ into account, as illustrated on Figure A.1 from Appendix A, we obtain the following cartoon picture of $\mathrm{PSL}(2, \mathbb{R})/\mathrm{PSL}(2, \mathbb{R})$. The elliptic conjugacy classes




are parametrized by the parameter $\theta \in (0, 2\pi)$ with $\mathrm{rot}_\theta$ approaching the identity as $\theta$ approaches 0 or $2\pi$. The hyperbolic conjugacy classes are parametrized by the positive real number $\lambda$ and $\mathrm{hyp}_\lambda$ gets arbitrarily close to the identity as $\lambda$ approaches 0. Finally, the two parabolic conjugacy classes determined by $\mathrm{par}_+$ and $\mathrm{par}_-$ are the non-closed points of $\mathrm{PSL}(2, \mathbb{R})/\mathrm{PSL}(2, \mathbb{R})$. Their closure contains the identity. The lack of separability is denoted by the dashed line. It is worth observing that $\mathrm{par}_+$ and $\mathrm{par}_-$ corresponds to the only two representations $\mathbb{Z} \to \mathrm{PSL}(2, \mathbb{R})$ that are not reductive. In other words, $\mathrm{Hom}^{\mathrm{red}}(\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{Inn}(\mathrm{PSL}(2,\mathbb{R}))$ is the Hausdorff, though not smooth, space given by the circle and the blue half-line meeting at the identity.

## 4.3 Hausdorff quotient

The first approach consists in considering the Hausdorffization of the topological quotient. The Hausdorffization of a topological space $X$ is broadly speaking the largest Hausdorff quotient of $X$. Let us give a more precise definition.

**Definition 4.3.1** (Hausdorffization)**.** Consider the equivalence relation on $X$ given by $x \sim y$ if and only if $x \approx y$ for all equivalence relations $\approx$ on $X$ such that $X/\approx$ is Hausdorff (such a relation $\approx$ always exists, as one can identify all the points of $X$). The quotient

$$\mathrm{Haus}(X) := X/\sim$$

is the *Hausdorffization of* $X$.

**Lemma 4.3.2.** *The space* $\mathrm{Haus}(X)$ *is a Hausdorff topological space. Moreover, the space* $\mathrm{Haus}(X)$ *has the following universal property: If* $Y$ *is a Hausdorff topological space, then any continuous map* $X \to Y$ *factors uniquely through the projection* $X \to \mathrm{Haus}(X)$*.*

*Proof.* First we prove that $\mathrm{Haus}(X)$ is a Hausdorff space. Let $x, y \in X$ be two points with $x \nsim y$. By definition, there exists an equivalence relation $\approx$ on $X$ with Hausdorff quotient such that $x \not\approx y$. Since the projections of $x$ and $y$ in $X/\approx$ are separable and the map $X/\sim \to X/\approx$ is continuous, the projections of $x$ and $y$ are also separable in $X/\sim$.

Let now $Y$ be a Hausdorff space and $f\colon X \to Y$ be a continuous map. Define an equivalence relation on $X$ by $x \approx y$ if and only if $f(x) = f(y)$. The quotient $X/\approx$ is homeomorphic to the Hausdorff space $f(X) \subset Y$. This implies the existence of a continuous surjective map $\mathrm{Haus}(X) \to f(X)$ such that $f$ is the composition $X \to \mathrm{Haus}(X) \to f(X) \subset Y$. The factoring map is uniquely determined by $f$. □

**Corollary 4.3.3.** *If* $x$ *and* $y$ *are two points of* $X$ *such that* $\overline{\{x\}} \cap \overline{\{y\}} \neq \emptyset$*, then* $x \sim y$*.*

*Proof.* Since $\mathrm{Haus}(X)$ is Hausdorff, its points are closed. In particular, the conjugacy classes for the relation $\sim$ are closed subsets of $X$. If we assume that $x \nsim y$, then the conjugacy classes of $x$ and $y$ are disjoint closed subsets of $X$. This implies that the closures of $\{x\}$ and $\{y\}$ are disjoint. □

**Definition 4.3.4** (Hausdorff character variety)**.** The *Hausdorff character variety* of a finitely generated group $\Gamma$ and a Lie group $G$ is the Hausdorffization of the topological quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$ and is denoted

$$\mathrm{Rep}^{\mathcal{T}_2}(\Gamma, G) := \mathrm{Haus}\left(\mathrm{Hom}(\Gamma, G) \big/ \mathrm{Inn}(G)\right).$$

The construction of character varieties by Hausdorff quotients has the advantage to work in a broad sense: it makes sense for any finitely generated group $\Gamma$ and any Lie group $G$ (even for any topological group $G$). The downside of the Hausdorff quotient is its lack of concreteness. It is nevertheless a common choice in the literature, such as in [Mon16] for instance.

## 4.4 $\mathcal{T}_1$ quotient

An alternative to the Hausdorff quotient is the $\mathcal{T}_1$ quotient used in [RS90, §7]. Let us start with some notation. For a topological group $G$ acting on a space $X$, we denote the $G$-orbit of $x \in X$ by $\mathcal{O}_x$. We will assume that the action of $G$ on $X$ has the following crucial property:

$$\forall x \in X, \quad \overline{\mathcal{O}_x} \subset X \text{ contains a unique closed } G\text{-orbit.} \tag{4.4.1}$$

We write $X /\!/ G$ to denote the set of closed orbits for the action of $G$ on $X$ and define

$$\pi \colon X \to X /\!/ G$$

to be the map that sends $x$ to the unique closed orbit contained in $\overline{\mathcal{O}_x}$. A topology on $X /\!/ G$ is defined by declaring $\pi$ to be a quotient map, i.e $Z \subset X /\!/ G$ is closed if and only if $\pi^{-1}(Z) \subset X$ is closed.

Alternatively, one could consider the relation on $X$ defined by

$$x \approx y \quad \Leftrightarrow \quad \overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} \neq \varnothing.$$

It turns out that this is precisely the relation behind $X /\!/ G$.

**Lemma 4.4.1.** *Under the assumption (4.4.1), the relation $\approx$ is an equivalence relation and $X /\!/ G$ is homeomorphic to the quotient $X/\approx$.*

*Proof.* The relation $\approx$ is obviously symmetric and reflexive. We prove that it is also transitive. Assume that $x \approx y$ and $y \approx z$. In particular, $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$ is nonempty and thus contains an element $w$. Since $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$ is closed and $G$-invariant, it holds $\overline{O_w} \subset \overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$. We conclude that $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y}$ contains a unique closed orbit which is the one contained in $\overline{O_w}$. Similarly, $\overline{\mathcal{O}_y} \cap \overline{\mathcal{O}_z}$ contains a unique closed orbit. By uniqueness of the closed orbit contained in $\overline{O_y}$, the two must coincide. Hence, $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} \cap \overline{\mathcal{O}_z}$ contains $\overline{O_w}$ and is therefore nonempty. This shows that $x \approx z$.

To see that $X /\!/ G \cong X/\approx$, observe that, by the above argument, $\pi(x) = \pi(y)$ if and only if $x \approx y$. Both are quotients of $X$ and therefore homeomorphic. □

**Lemma 4.4.2.** *The space $X /\!/ G$ has the following universal property: for every $\mathcal{T}_1$ space $Y$, any continuous map $X \to Y$ that is constant on $G$-orbits factors uniquely through $\pi \colon X \to X /\!/ G$.*

*Proof.* Let $Y$ be $\mathcal{T}_1$ with a continuous map $f \colon X \to Y$ that is constant on $G$-orbits. Let $x \in X$. We want to prove that $f$ is constant on $\overline{O_x}$. Let $y = f(x)$. Since $Y$ is $\mathcal{T}_1$, the singleton $\{y\} \subset Y$ is closed and so is $f^{-1}(y)$. Therefore, $\overline{O_x} \subset f^{-1}(y)$ and $f$ is constant on $\overline{O_x}$. This shows that $f \colon X \to Y$ factors through $X /\!/ G$. The factoring map $\overline{f} \colon X /\!/ G \to Y$ is continuous and uniquely determined by $f$. □

In the case that $X /\!/ G$ is a $\mathcal{T}_1$ space, then Lemma 4.4.2 says that $X /\!/ G$ is the largest $\mathcal{T}_1$ quotient of $X$. There is a relation between $X /\!/ G$ and the Hausdorffization of the topological quotient $X/G$ as shown in the following lemma.

**Lemma 4.4.3.** *There is a natural surjective continuous map*

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d, two heads, "\pi"] & X/G \arrow[d, two heads] \\ X /\!/ G \arrow[r, dashed, two heads, "\exists"] & \mathrm{Haus}(X/G) \end{tikzcd}$$

*Proof.* Let $x$ and $y$ be two points of $X$. Lemma 4.4.1 says that if $\pi(x) = \pi(y)$, then $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} \neq \varnothing$. This means the closures of $\mathcal{O}_x$ and $\mathcal{O}_y$, seen as singletons in $X/G$, have a nonempty intersection. By Corollary 4.3.3, we conclude that $x$ and $y$ project to the same point in $\mathrm{Haus}(X/G)$. $\square$

**Corollary 4.4.4.** *If $X /\!/ G$ is Hausdorff, then it is homeomorphic to the Hausdorffization of $X/G$.*

**Definition 4.4.5** ($\mathcal{T}_1$ character variety)**.** If the conjugation action of $G$ on the representation variety $\mathrm{Hom}(\Gamma, G)$ satisfies property (4.4.1), we define the $\mathcal{T}_1$ *character variety* of $\Gamma$ and $G$ to be

$$\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) := \mathrm{Hom}(\Gamma, G) /\!/ \mathrm{Inn}(G).$$

Note that the $\mathcal{T}_1$ character variety of $\Gamma$ and $G$ might not be a $\mathcal{T}_1$ space, but always lies over any $\mathcal{T}_1$ quotient of $\mathrm{Hom}(\Gamma, G)$ by Lemma 4.4.2. In particular, by Lemma 4.4.3, there is a surjection

$$\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) \twoheadrightarrow \mathrm{Rep}^{\mathcal{T}_2}(\Gamma, G)$$

which is a homeomorphism when $\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G)$ is Hausdorff.

## 4.5 GIT quotient

When $G$ is a complex reductive algebraic group, such as $\mathrm{SL}(n, \mathbb{C})$ for instance, then it is possible to define a notion of character variety using the complex algebraic nature of the representation variety. The definition is based on geometric invariant theory—in short, GIT. The idea is to define a variety from what should be its algebra of regular functions. The reader may consult Sikora's notes [Sik12], or [Dre04, §2] and [Lou15, §B.5] for further details.

Recall that if $G$ is a complex algebraic group then the representation variety $\mathrm{Hom}(\Gamma, G)$ is an algebraic variety by Lemma 2.1.3. It is common in algebraic geometry to study a variety through its algebra of regular functions. In Section 3.3, we studied the algebra of regular functions of $\mathrm{Hom}(\Gamma, G)$ which was denoted by $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ and its subalgebra of $G$-invariant functions $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. We reminded the reader about Nagata's Theorem which implies that $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ is finitely generated when $G$ is reductive, see Remark 3.3.2. In that case, there is an algebraic variety denoted $\mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$, called the *spectrum* of $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$, whose algebra of regular functions is $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. More concretely, one may think of the spectrum of $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$ as the algebraic variety given by the set of points inside $\mathbb{C}^n$ that belong to the image of $\mathrm{Hom}(\Gamma, G)$ under a family of generators $(f_1, \ldots, f_n)$ of $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. Recall from Section 3.3 that when $G$ is the linear algebraic group $\mathrm{SL}(n, \mathbb{C})$, then a system of generators is provided by trace functions.

**Definition 4.5.1** (GIT character variety)**.** The *GIT character variety* of a finitely generated group

$\Gamma$ and a complex reductive algebraic group $G$ is defined to be

$$\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G) := \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G).$$

The GIT character variety is sometimes denoted by $\mathrm{Hom}(\Gamma, G)//G$ using the double quotient-bar notation.

In other words, the GIT character variety is the algebraic variety whose algebra of regular functions are the invariant functions of $\mathrm{Hom}(\Gamma, G)$. The GIT character variety has by definition the structure of a complex algebraic variety. As such, it is a Hausdorff space for the Euclidean topology. The inclusion $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G \subset \mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ induces a surjective morphism of algebraic varieties

$$p\colon \mathrm{Hom}(\Gamma, G) \twoheadrightarrow \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G).$$

We recall here some general properties of GIT quotients. The reader may consult [Dre04, §2] and [Lou15, §B.5], and references therein, for proofs.

**Lemma 4.5.2.** *The GIT quotient* $\mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$ *has the following universal property: for every algebraic variety* $Y$*, any morphism* $\mathrm{Hom}(\Gamma, G) \to Y$ *that is constant on* $G$*-orbits factors uniquely through* $p\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$.

**Lemma 4.5.3.** *The projection* $p\colon \mathrm{Hom}(\Gamma, G) \twoheadrightarrow \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$ *has the following properties.*

1. *For any two representations* $\phi_1, \phi_2 \in \mathrm{Hom}(\Gamma, G)$*, it holds that*

$$p(\phi_1) = p(\phi_2) \quad \Longleftrightarrow \quad \overline{\mathcal{O}_{\phi_1}} \cap \overline{\mathcal{O}_{\phi_2}} \neq \varnothing.$$

2. *Any fibre of* $p$ *contains a unique closed orbit (compare* (4.4.1)*).*

When we combine Lemma 4.5.3 and Lemma 4.4.1 we obtain that the underlying topological structure of the GIT character variety of $\Gamma$ and $G$ coincides with the $\mathcal{T}_1$ character variety. Since the GIT character variety is a Hausdorff space, it further coincides with the Hausdorff character variety by Corollary 4.4.4

$$\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G) \cong \mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) \cong \mathrm{Rep}^{\mathcal{T}_2}(\Gamma, G).$$

### 4.5.1 (Poly)stable representations

The GIT character variety can be described more concretely as follows.

**Definition 4.5.4** (Stability of representations)**.** Let $G$ be an algebraic group. A representation $\phi\colon \Gamma \to G$ is

- *polystable* if $\mathcal{O}_\phi$ is closed.
- *stable* if $\phi$ is polystable and regular.

The $\mathrm{Inn}(G)$-invariant subspace of polystable representations is denoted $\mathrm{Hom}^{\mathrm{ps}}(\Gamma, G)$ and the subspace of stable representations is denoted $\mathrm{Hom}^{\mathrm{s}}(\Gamma, G)$.

These notions are redundant if $G$ is a reductive complex algebraic group because of the following.

**Proposition 4.5.5.** *Let $G$ be a reductive complex algebraic group. Let $\phi \in \mathrm{Hom}(\Gamma, G)$ be a representation. Then*

1. *$\phi$ is reductive if and only if $\phi$ is polystable,*
2. *$\phi$ is irreducible if and only if $\phi$ is stable.*

The first assertion of Proposition 4.5.5 was already stated in Proposition 3.2.20. The second assertion is a consequence of Lemma 3.2.19.

**Theorem 4.5.6.** *Let $G$ be a reductive complex algebraic group. The topological quotient*

$$\mathrm{Hom}^{\mathrm{ps}}(\Gamma, G)/\,\mathrm{Inn}(G) = \mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/\,\mathrm{Inn}(G)$$

*is homeomorphic to $\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G)$. It contains, as an open subset, the topological quotient*

$$\mathrm{Hom}^{\mathrm{s}}(\Gamma, G)/\,\mathrm{Inn}(G) = \mathrm{Hom}^{\mathrm{irr}}(\Gamma, G)/\,\mathrm{Inn}(G)$$

*which is an orbifold whenever $Z(G)$ is finite.*

*Proof.* Polystable representations have a closed orbit under the $\mathrm{Inn}(G)$-action by definition. So, the first statement of Lemma 4.5.3 implies that the projection $p\colon \mathrm{Hom}(\Gamma, G) \to \mathrm{Spec}(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G)$ factors through an injective map

$$\mathrm{Hom}^{\mathrm{ps}}(\Gamma, G)/\,\mathrm{Inn}(G) \to \mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G).$$

We can use the second statement of Lemma 4.5.3 to see that this map is also surjective.

Recall now from Proposition 3.2.10 that $\mathrm{Hom}^{\mathrm{irr}}(\Gamma, G) = \mathrm{Hom}^{\mathrm{s}}(\Gamma, G)$ is open in $\mathrm{Hom}(\Gamma, G)$. To prove the orbifold statement, we use that an algebraic variety over the real or the complex numbers has a finite number of connected components in the usual topology, see e.g. [DK81, Thm. 4.1]. So, if $Z(G)$ is finite, then a polystable representation $\phi\colon \Gamma \to G$ is stable if and only if $Z(\phi)$ is finite. Equivalently, $\phi$ is stable if and only if it has a finite stabilizer for the $\mathrm{Inn}(G)$-action. This shows that the quotient is an orbifold since the $\mathrm{Inn}(G)$-action on $\mathrm{Hom}^{\mathrm{s}}(\Gamma, G)$ is proper by Theorem 3.2.11. □

### 4.5.2 Semi-algebraic quotient

Theorem 4.5.6 says that there is a natural structure of algebraic variety on the quotient of the space of reductive representations by the $\mathrm{Inn}(G)$-action, given that $G$ is a reductive complex algebraic group. The GIT theory is sadly not available when $G$ is a *real* algebraic group. It is not clear in that case what would be a good definition of "algebraic character variety". Attempting of defining it as the real points of the GIT character variety of representations into the complexification $G^{\mathbb{C}}$ of $G$ runs into the following issues:

- Real points of the GIT character variety of representations into $G^{\mathbb{C}}$ correspond to representations into one of the real forms of $G$. For instance, when $G^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$, then real points of an $\mathrm{SL}(2,\mathbb{C})$-character variety correspond to representations into $\mathrm{SL}(2,\mathbb{R})$ or into $\mathrm{SU}(2)$.

- Two non-conjugate elements of $G$ might be conjugate inside $G^{\mathbb{C}}$, so non-conjugate representations into $G$ might be identified when one looks at the real points of the $G^{\mathbb{C}}$-character variety. For instance, the rotation matrices $\mathrm{rot}_\theta$ and $\mathrm{rot}_{4\pi-\theta}$ are not conjugate in $\mathrm{SL}(2,\mathbb{R})$, but they are in $\mathrm{SL}(2,\mathbb{C})$ because they have the same trace.

It turns out that by restricting to reductive representations before taking the topological quotient, the result space has a natural structure of *semi-algebraic*[1] variety by the work of Richardson-Slowdowy.

**Theorem 4.5.7** ([RS90])**.** *Let $G$ be a real algebraic group. The topological quotient*

$$\mathrm{Hom}^{\mathrm{red}}(\Gamma, G)/\operatorname{Inn}(G)$$

*has a natural structure of real semialgebraic variety.*

Theorem 4.5.7 is proved in [RS90, Thm. 7.6].

## 4.6 Analytic quotient

If one is interested in defining a character variety that has the structure of a smooth analytic manifold, one can restrict to good representations which we introduced in Definition 3.2.12. This will work well for closed surface groups for instance since we saw in Corollary 3.2.14 that when $\Gamma = \pi_{g,0}$, then $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)$ is a nonempty analytic manifold. In that case, we explained that the topological quotient $\mathrm{Hom}^{\mathrm{good}}(\Gamma, G)/\operatorname{Inn}(G)$ is a smooth analytic manifold.

**Definition 4.6.1** (Analytic character variety)**.** The *analytic character variety* of a closed surface group $\Gamma = \pi_{g,0}$ and a reductive algebraic group $G$ is defined to be

$$\mathrm{Rep}^{\infty}(\pi_{g,0}, G) \coloneqq \mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G)/\operatorname{Inn}(G).$$

The topology of the analytic character variety is Hausdorff. The inclusion $\mathrm{Hom}^{\mathrm{good}}(\pi_{g,0}, G) \subset \mathrm{Hom}(\pi_{g,0}, G)$ induces an inclusion $\mathrm{Rep}^{\infty}(\pi_{g,0}, G) \subset \mathrm{Hom}(\pi_{g,0}, G)/\operatorname{Inn}(G)$. Since $\mathrm{Rep}^{\infty}(\pi_{g,0}, G)$ is Hausdorff, we obtain an inclusion

$$\mathrm{Rep}^{\infty}(\pi_{g,0}, G) \hookrightarrow \mathrm{Rep}^{\mathcal{T}_2}(\pi_{g,0}, G).$$

[1] A *semialgebraic variety* is defined to be a set of points satisfying polynomial equalities and inequalities.

# Chapter 5

# Symplectic structure of character varieties

Throughout this section we assume that $G$ is a quadrable Lie group. We also fix a non-degenerate, symmetric, Ad-invariant bilinear form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. Our goal is to give a detailed description of Goldman's natural symplectic structure on the character variety of representations of a closed surface group into a quadrable group. Recall that a symplectic form on a manifold is the data of a closed and non-degenerate 2-form.

## 5.1 Abstract definition

Assume for now that $\Gamma$ is any finitely generated group. We explained in Corollary 2.3.5 that the Zariski tangent space to $\mathrm{Hom}(\Gamma, G)$ at a representation $\phi$ can be identified with the set of closed 1-cochains $Z^1(\Gamma, \mathfrak{g}_\phi) \subset \mathfrak{g}^\Gamma$. To define a 2-form on the representation variety $\mathrm{Hom}(\Gamma, G)$ we use the cup product in group cohomology (1.3.11). Combined with the pairing $B$, this gives a map

$$\omega\colon Z^1(\Gamma, \mathfrak{g}_\phi) \times Z^1(\Gamma, \mathfrak{g}_\phi) \xrightarrow{\smile} Z^2(\Gamma, \mathfrak{g}_\phi \otimes \mathfrak{g}_\phi) \xrightarrow{B_*} Z^2(\Gamma, \mathbb{R}). \tag{5.1.1}$$

The map $\omega$ is bilinear and anti-symmetric because the cup product is anti-symmetric in degree 1 (Lemma 1.3.11) and $B$ is symmetric. Building up on the work of Goldman in [Gol84], Karshon proved the following statement.

**Theorem 5.1.1** ([Kar92])**.** *Let $\varphi\colon Z^2(\Gamma, \mathbb{R}) \to \mathbb{R}$ be any continuous linear function that vanishes on $B^2(\Gamma, \mathbb{R})$. Then, $\varphi \circ \omega$ is a closed 2-form on $\mathrm{Hom}(\Gamma, G)$.*

The main conclusion of Theorem 5.1.1 is the statement that the form $\varphi \circ \omega$ is closed. Karshon gives an elementary proof of the closeness via direct computations in group cohomology.

The cup product of coboundaries in $B^1(\Gamma, \mathfrak{g}_\phi)$ is itself a coboundary inside $B^2(\Gamma, \mathfrak{g}_\phi \otimes \mathfrak{g}_\phi)$, showing that the 2-form $\varphi \circ \omega$ is degenerate in general. Recall from Proposition 2.3.6 that the tangent space at $\phi$ to the $G$-orbit $\mathcal{O}_\phi \subset \mathrm{Hom}(\Gamma, G)$ can be identified with the 1-coboundaries $B^1(\Gamma, \mathfrak{g}_\phi) \subset \mathfrak{g}^\Gamma$. So, $\varphi \circ \omega$ is degenerate at least along the tangent directions to the $G$-orbit of $\phi$. In general, the kernel of $\varphi \circ \omega$ might contain more degenerate directions than those which arise from $\mathcal{O}_\phi$.

**Definition 5.1.2** (Goldman symplectic form)**.** In the case that the $G$-orbits are the only directions of degeneracy of $\varphi \circ \omega$, we denote by $(\omega_{\mathcal{G}})_\phi$ the induced non-degenerate pairings on cohomology:

$$(\omega_{\mathcal{G}})_\phi \colon H^1(\Gamma, \mathfrak{g}_\phi) \times H^1(\Gamma, \mathfrak{g}_\phi) \to \mathbb{R}.$$

In a slight abuse of language, we say that $\omega_{\mathcal{G}}$ is the *the Goldman symplectic form* on the quotient $\operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$.

The label $\mathcal{G}$ in index of $\omega$ refers to Goldman. We are abusing the terminology “symplectic form” here since the topological quotient $\operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$ does not need to be a variety in general. It is also equally abusive to say that the “Zariski tangent space” at $[\phi] \in \operatorname{Hom}(\Gamma, G)/\operatorname{Inn}(G)$ is the quotient space $H^1(\Gamma, \mathfrak{g}_\phi) = Z^1(\Gamma, \mathfrak{g}_\phi)/B^1(\Gamma, \mathfrak{g}_\phi)$. What we should say instead, if one wishes to be fromal, is that $\omega_{\mathcal{G}}$ is a 2-form on $\operatorname{Hom}(\Gamma, G)$ that is degenerate precisely along the orbits of the $\operatorname{Inn}(G)$-action.

## 5.2 Closed surface groups

We start by considering the case where $\Gamma = \pi_{g,0}$ is a closed surface group. Let $[\pi_{g,0}]$ be a fundamental class for $\pi_{g,0}$ which we defined in Definition 1.3.14 as a generator of $H_2(\pi_{g,0}, \mathbb{Z}) \cong \mathbb{Z}$ (where $\mathbb{Z}$ is the trivial $\pi_{g,0}$-module). We can also think of $[\pi_{g,0}]$ as a choice of orientation for the surface $\Sigma_{g,0}$ under the isomorphism $H_2(\pi_{g,0}, \mathbb{Z}) \cong H_2(\Sigma_{g,0}, \mathbb{Z})$ of Theorem 1.3.8. Integration against $[\pi_{g,0}]$ gives an isomorphism

$$[\pi_{g,0}] \frown \colon H^2(\pi_{g,0}, \mathbb{R}) \to \mathbb{R}.$$

Let $\varphi \colon Z^2(\pi_{g,0}, \mathbb{R}) \to \mathbb{R}$ be given by the composition of the quotient map $Z^2(\pi_{g,0}, \mathbb{R}) \to H^2(\pi_{g,0}, \mathbb{R})$ and the integration against $[\pi_{g,0}]$. Clearly, $\varphi$ vanishes on $B^2(\pi_{g,0}, \mathbb{R})$.

**Lemma 5.2.1.** *Let $\Gamma = \pi_{g,0}$ be a closed surface group. The composition of $\varphi \colon Z^2(\pi_{g,0}, \mathbb{R}) \to \mathbb{R}$ with the form $\omega$ of* (5.1.1) *defines a 2-form on $\operatorname{Hom}(\pi_{g,0}, G)$ whose kernel is $B^1(\pi_{g,0}, \mathbb{R})$.*

*Proof.* The proof relies on Poincaré duality in group cohomology for the group $\pi_{g,0}$. It implies that the cup product

$$H^1(\pi_{g,0}, \mathbb{R}) \times H^1(\pi_{g,0}, \mathbb{R}) \xrightarrow{\smile} H^2(\pi_{g,0}, \mathbb{R})$$

is a non-degenerate pairing. This means that the form $\varphi \circ \omega$ is degenerate on $B^1(\pi_{g,0}, \mathbb{R})$ only. □

The induced non-degenerate closed form $(\omega_{\mathcal{G}})_\phi \colon H^1(\pi_{g,0}, \mathfrak{g}_\phi) \times H^1(\pi_{g,0}, \mathfrak{g}_\phi) \to \mathbb{R}$ is the classical Goldman symplectic form for character varieties of closed surface groups representations. The original argument of Goldman in [Gol84] to prove that the $\omega_{\mathcal{G}}$ is closed is inspired by the work of Atiyah-Bott from [AB83] who considered the case where $G$ is compact. The proof involves an infinite dimensional symplectic reduction from the affine space of connections on some vector bundle, see [Gol84] and [Lab13, §6] for more details.

*Remark* 5.2.2. The Goldman symplectic form depends on the pairing $B$ that we chose for the Lie algebra of $G$. Different choices of pairing for the same Lie group $G$ may lead to different symplectic structures. Abusing once again of the term "symplectic manifold", one can say that Goldman's construction is a functor form the product category of the category of closed connected oriented surfaces $\Sigma_{g,0}$ with the category of quadrable Lie groups $G$ with a choice of a form pairing $B$ to the category of "symplectic manifold"

$$\big(\Sigma_{g,0}, (G, B)\big) \rightsquigarrow \big(\operatorname{Hom}(\pi_1(\Sigma_{g,0}), G)/\operatorname{Inn}(G), \omega_{\mathcal{G}}\big).$$

We point out that the quotients $\operatorname{Hom}(\pi_1(\Sigma_{g,0}), G)/\operatorname{Inn}(G)$ obtained for different choices of basepoints in $\Sigma_{g,0}$ are naturally isomorphic (the isomorphism does *not* depend on the choice of path connecting different basepoints).

**Example 5.2.3.** In the case where $G = \mathbb{R}$, we saw in Section 4.2.1 that the character variety of the pair $(\pi_{g,0}, \mathbb{R})$ can be naturally identified with the vector space $H^1(\Sigma_{g,0}, \mathbb{R})$. This vector space is of dimension $2g$ and carries a symplectic form given by the so-called intersection pairing. This form corresponds to the standard symplectic form on $\mathbb{R}^{2g} \cong H^1(\Sigma_{g,0}, \mathbb{R})$ and also to the wedge product of differential form when one thinks in terms of de Rham cohomology. It follows from the definition that, when the paring $B$ is taken to be the product of real numbers, then the Goldman symplectic form on the character variety of the pair $(\pi_{g,0}, \mathbb{R})$ corresponds to any of the above symplectic forms on $H^1(\Sigma_{g,0}, \mathbb{R})$.

## 5.3 General surface groups

We now consider the case where $\Gamma = \pi_{g,n}$ is a surface group with some punctures on the underlying surface $\Sigma_{n,g}$. We will assume in this section that $n > 0$. As mentioned earlier, in that case $\pi_{g,n}$ is a free group and the representation variety $\operatorname{Hom}(\pi_{g,n}, G)$ is isomorphic to the product $G^{2g+n-1}$.

### 5.3.1 Relative character varieties

Instead of looking at the whole space $\operatorname{Hom}(\pi_{g,n}, G)$ at once, we would rather decompose it as the disjoint union of so-called relative representation varieties.

**Definition 5.3.1** (Relative representation variety)**.** Let $\mathcal{C} = (C_1, \ldots, C_n)$ be an ordered collection of conjugacy classes in $G$. The *relative representation variety* associated to $\big(\pi_{g,n}, (G, \mathcal{C})\big)$ is the subspace of $\operatorname{Hom}(\pi_{g,n}, G)$ given by

$$\operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G) := \{\phi \in \operatorname{Hom}(\pi_{g,n}, G) : \phi(c_i) \in C_i,\ \forall i = 1, \ldots, n\},$$

where $c_1, \ldots, c_n$ refer to the generators of $\pi_{g,n}$ in the presentation (1.4.1).

If $G/G$ denotes the set of conjugacy classes in $G$, then

$$\operatorname{Hom}(\pi_{g,n}, G) = \bigsqcup_{\mathcal{C} \in (G/G)^n} \operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G).$$

When we realize $\pi_{g,n}$ as the fundamental group of a surface $\Sigma_{g,n}$, then a relative character variety is the collection of all representations $\pi_1\Sigma_{g,n} \to G$ that sends clockwise oriented loops around the punctures of $\Sigma_{g,n}$ to prescribed conjugacy classes inside $G$. The conjugation action of $G$ on $\mathrm{Hom}(\pi_{g,n}, G)$ restricts to $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$.

**Lemma 5.3.2.** *Let $G$ be a Lie group equipped with an analytic atlas. The relative representation variety $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is naturally an analytic subvariety of $G^{2g+n}$. If $G$ is a complex algebraic group, then $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is an algebraic subvariety of $\mathrm{Hom}(\pi_{g,n}, G)$. If $G$ is a real algebraic group, then $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is a semialgebraic subvariety of $\mathrm{Hom}(\pi_{g,n}, G)$.*

*Proof.* The proof is analogous to the proof of Lemma 2.1.2. A conjugacy class $C \in G/G$ is a smooth submanifold of $G$ isomorphic to $G/Z(c)$, where $c$ is any element of $C$ (recall that $Z(c)$ is a closed subgroup of $G$). It has a unique structure of real analytic manifold that makes the projection map $G \to G/Z(c)$ an analytic submersion. The relative representation variety $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is naturally identified with the subspace of $G^{2g} \times C_1 \times \ldots \times C_n$ cut out by the single relation of the surface group $\pi_{g,n}$ (see (1.4.1)). This shows that $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ is an analytic subvariety of $G^{2g+n}$. Observe now that, if $G$ is a complex algebraic group, then conjugacy classes in $G$ are algebraic subvarieties of $G$. This can be seen as a consequence of Chevalley's Theorem. Moreover, if $G$ is a real algebraic group, then conjugacy classes in $G$ are semialgebraic subvarieties of $G$.[1] This, in turn, is a consequence of Tarski-Seidenberg Theorem. □

We would like to determine the Zariski tangent space to relative character varieties. We follow the approach of [GHJW97, §4]. Let $\phi \in \mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$. The Zariski tangent space to $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ at $\phi$ is the space of all tangent vectors in $Z^1(\pi_{g,n}, \mathfrak{g}_\phi)$ tangent to a smooth deformation $\phi_t$ of $\phi$ inside $\mathrm{Hom}(\pi_{g,n}, G)$ that satisfies $\phi_t(c_i) \in C_i$ up to first order. Observe that the condition $\phi_t(c_i) \in C_i$ is equivalent to the existence of a smooth 1-parameter family $g_i(t) \in G$, with $g_i(0) = 1$, and

$$\phi_t(c_i) = g_i(t)\phi(c_i)g_i(t)^{-1}. \tag{5.3.1}$$

**Lemma 5.3.3.** *A vector $v \in Z^1(\pi_{g,n}, \mathfrak{g}_\phi)$ tangent to $\phi_t$ at $t = 0$ satisfies (5.3.1) up to first order if and only if*

$$v(c_i) = \dot{g}_i - \mathrm{Ad}(\phi(c_i))\dot{g}_i,$$

*where $\dot{g}_i \in \mathfrak{g}$ is the tangent vector to $g_i(t)$ at $t = 0$.*

*Proof.* We use $\frac{d}{dt}\big|_{t=0} \phi_t(c_i)\phi(c_i)^{-1} = v(c_i)$ and derive the relation (5.3.1). □

**Corollary 5.3.4** ([GHJW97])**.** *The Zariski tangent space to $\mathrm{Hom}_{\mathcal{C}}(\Gamma, G)$ at $\phi$ is*

$$T_\phi \mathrm{Hom}_{\mathcal{C}}(\Gamma, G) = \left\{v \in Z^1(\pi_{g,n}, \mathfrak{g}_\phi) \,:\, \forall i = 1, \ldots, n,\ \exists \xi_i \in \mathfrak{g},\ v(c_i) = \xi_i - \mathrm{Ad}(\phi(c_i))\xi_i\right\}.$$

The cocycles $v \in Z^1(\pi_{g,n}, \mathfrak{g}_\phi)$ that satisfy the property stated in the conclusion of Corollary 5.3.4 are called *parabolic 1-cocycles* and were introduced in Section 1.3.9. The subspace of parabolic cocycles is denoted

$$Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \subset Z^1(\pi_{g,n}, \mathfrak{g}_\phi).$$

[1] An example of conjugacy classes that are a semialgebraic subvarieties, but not algebraic subvarieties, are parabolic conjugacy classes inside $\mathrm{SL}(2, \mathbb{R})$.

The tangent space to the $G$-orbit $\mathcal{O}_\phi$ of $\phi \in \mathrm{Hom}_{\mathcal{C}}(\Gamma, G)$ still identifies with $B^1(\pi_{g,n}, \mathfrak{g}_\phi)$. The quotient of parabolic 1-cocycles by 1-coboundaries is the first parabolic group cohomology group of $\pi_{g,n}$ with coefficients in the $\pi_{g,n}$-module $\mathfrak{g}_\phi$:

$$H^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) = Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi)/B^1(\pi_{g,n}, \mathfrak{g}_\phi).$$

**Proposition 5.3.5.** *Let $G$ be a quadrable Lie group. The dimension of the Zariski tangent space to $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ at $\phi$ is*

$$(2g-1)\dim G + \sum_{i=1}^{n} \dim C_i + \dim Z(\phi).$$

*In particular, the smooth points of $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ are the representations $\phi$ such that*

$$\dim Z(G) = \dim Z(\phi).$$

*Proof.* We proceed as in the alternative proof of Proposition 2.4.3. Let $A_i = \phi(a_i)$, $B_i = \phi(b_i)$ and $R_i = \phi(c_i)$, where $a_i, b_i, c_i$ refer to the presentation (1.4.1). Consider the map $\mu\colon \mathfrak{g}^{2g+n} \to \mathfrak{g}$ obtained by differentiating the unique surface group relation:

$$\begin{aligned}\mu(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_n) = &\sum_{i=1}^{g} \left( \prod_{j<i} \mathrm{Ad}\left([A_j, B_j]\right) \right) (\alpha_i - \mathrm{Ad}(A_i B_i A_i^{-1})\alpha_i) \\ &- \sum_{i=1}^{g} \left( \prod_{j\leqslant i} \mathrm{Ad}\left([A_j, B_j]\right) \right) (\beta_i - \mathrm{Ad}(B_i A_i B_i^{-1})\beta_i) \\ &+ \prod_{k=1}^{g} \mathrm{Ad}\left([A_k, B_k]\right) \sum_{i=1}^{n} \left( \prod_{j=1}^{i-1} \mathrm{Ad}\left(R_j\right) \right) (\gamma_i - \mathrm{Ad}(R_i)\gamma_i).\end{aligned}$$

Let $V$ be the orthogonal complement of the image of $\mu$ with respect to the pairing $B$. Similarly as in the alternative proof of Proposition 2.4.3, we conclude that $V = \mathfrak{z}(\phi)$. The Rank-Nullity Theorem gives

$$\dim T_\phi \mathrm{Hom}_{\mathcal{C}}(\Gamma, G) = \dim \mathrm{Ker}(\mu) = (2g-1)\dim G + \sum_{i=1}^{n} \dim C_i + \dim Z(\phi). \qquad \square$$

*Remark* 5.3.6. We make a little digression on the dimension of conjugacy orbits inside Lie groups. Recall that $\mathcal{C} \in G/G$ of some element $g \in G$ is a smooth submanifold of $G$ diffeomorphic to the quotient $G/Z(g)$. If $G$ is quadrable, the pairing $B$ on $\mathfrak{g}$ can be used to identify coadjoint orbits in $\mathfrak{g}^*$ to adjoint orbits in $\mathfrak{g}$. Coadjoint orbits are naturally symplectic, see e.g. [CdS01, Homework 17]. The exponential map maps the adjoint orbit of $\xi \in \mathfrak{g}$ to the conjugacy orbit of $\exp(\xi)$ in $G$. Recall however that the Lie theoretic exponential map needs not be a local diffeomorphism at $\xi$. If it were, it would imply that the conjugacy orbit of $\exp(\xi)$ in $G$ has even dimension. Max Riestenberg pointed out to the author a class of examples of Lie groups that contain conjugacy classes of odd dimension. They consist of the group of all isometries of an odd-dimensional symmetric space $X$. In that case, the conjugacy class of the orientation-reversing isometry $s_p$ that reflects through a point $p$ is the set of all the orientation-reversing isometries $s_q$ for $q \in X$ and is therefore isomorphic to $X$.

**Question 5.3.7.** When does a conjugacy orbit in a quadrable Lie group $G$ have even dimension? Is it necessarily the case if it lies in the image of the exponential map?

We close the digression and go back to relative representation varieties. We would like to obtain an analogue of the Goldman symplectic form for general surface groups. We denote by $\partial_i \pi_{g,n}$ the subgroup of $\pi_{g,n}$ generated by $c_i$. We write $\partial \pi_{g,n}$ for the collection of subgroups $\{\partial_i \pi_{g,n}\}$. Observe that the cup product in group cohomology restricts to the product (1.3.15) in parabolic group cohomology. It gives an anti-symmetric bilinear form

$$\omega\colon Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \times Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \xrightarrow{\smile} Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathfrak{g}_\phi \otimes \mathfrak{g}_\phi) \xrightarrow{B_*} Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}).$$

Let $[\pi_{g,n}, \partial\pi_{g,n}]$ be a generator of $H_2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{Z}) \cong \mathbb{Z}$, that corresponds to a choice of orientation for the surface $\Sigma_{g,n}$. An explicit example was computed in Section 1.4.2. Integrating against the fundamental class $[\pi_{g,n}, \partial\pi_{g,n}]$ gives an isomorphism $H^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}$. Let $\varphi\colon Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \to \mathbb{R}$ be the composition of the quotient map $Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \to H^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R})$ with the integration against $[\pi_{g,n}, \partial\pi_{g,n}]$. Similarly as in the closed case, it was proven in [GHJW97, §3] that the 2-form $\varphi \circ \omega$ is degenerate precisely on $B^1(\pi_{g,n}, \mathfrak{g}_\phi)$ and is furthermore closed [GHJW97, Thm. 7.1] (see also [Law09]). We obtain

**Theorem 5.3.8** ([GHJW97])**.** *Let $\Gamma = \pi_{g,n}$ be a surface group. The composition of*

$$\omega\colon Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \times Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \to Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R})$$

*with $\varphi\colon Z^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \to \mathbb{R}$ gives a non-degenerate closed 2-form*

$$(\omega_{\mathcal{G}})_\phi\colon H^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \times H^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \to \mathbb{R}$$

**Definition 5.3.9** (Relative character varieties)**.** The Hausdorffization of the topological quotient

$$\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)/\operatorname{Inn}(G)$$

is called the *relative character variety* associated to $(\pi_{g,n}, G, \mathcal{C})$. The non-degenerate closed 2-form $\omega_{\mathcal{G}}$ is the *the Goldman symplectic form* on $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)/\operatorname{Inn}(G)$.

Depending on the properties of the group $G$, the definition of relative character variety can be refined in order to get a better control of its structure similarly as in Section 4.

*Remark* 5.3.10 (Poisson structure). The representation variety $\mathrm{Hom}(\pi_{g,n}, G)$ is the disjoint union of all the relative representation varieties $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ over all possible choices for $\mathcal{C} \in (G/G)^n$. The quotient of each relative representation variety by the $\operatorname{Inn}(G)$-action has a symplectic structure in the sense of Theorem 5.3.8. It turns out that these quotients are the symplectic leaves of a Poisson structure on the quotient of the representation variety by the $\operatorname{Inn}(G)$-action. The reader is referred to [BJ21] for a precise statement, a proof, and references to prior proofs.

## 5.4 Symplectic measure

**Definition 5.4.1** (Goldman symplectic measure)**.** Both in the case of character varieties for closed surfaces and in the case of relative character varieties for punctured surfaces, the Liouville measure associated to the Goldman symplectic form is denoted $\nu_{\mathcal{G}}$ and called the *Goldman symplectic measure*.

The Goldman symplectic measure is a strictly positive Borel measure. It means that open sets are measurable and always have positive measure if they are nonempty.

## 5.5 Case of a punctured sphere

When $\Gamma = \pi_{0,n}$ is the fundamental group of a sphere with $n$ punctures, then one can obtain fairly explicit formulae for the Goldman symplectic form on $\mathrm{Hom}_{\mathcal{C}}(\pi_{0,n}, G)$. For simplicity, we abbreviate $\pi_n := \pi_{0,n}$ in this section.

Let $u, v \in Z^1_{par}(\pi_n, \mathfrak{g}_\phi)$. By definition of parabolic cocycles, there exist $\xi_i, \zeta_i \in \mathfrak{g}$ such that

$$u(c_i) = \xi_i - \mathrm{Ad}(\phi(c_i))\xi_i, \quad v(c_i) = \zeta_i - \mathrm{Ad}(\phi(c_i))\zeta_i, \quad i = 1, \ldots, n.$$

The first step consists in computing a preimage of $u$ inside $Z^1(\pi_n, \partial\pi_n, \mathfrak{g}_\phi)$. Note that

$$\partial\xi_i(c_i) = \mathrm{Ad}(\phi(c_i))\xi_i - \xi_i = -u(c_i).$$

Hence, the 1-cochain $(u, -\xi_1, \ldots, -\xi_n)$ is closed and is a preimage of $u$.

To compute $\omega_{\mathcal{G}}(u, v)$, we proceed as follows:

1. Apply the cup product to $(u, -\xi_1, \ldots, -\xi_n)$ and $v$.
2. Apply the pairing $B$.
3. Using Lemma 1.3.16, compute the cap product with the fundamental form $[\pi_n, \partial\pi_n]$ from Section 1.4.2.

This gives

$$\omega_G(u, v) = B_*(u \smile v)(e) + \sum_{i=1}^{n} B_*(\xi_i \smile v)(c_i). \tag{5.5.1}$$

We develop each cup product according to (1.3.11) and plug in the value of $e$ computed in Section 1.4.2. The right-hand side of (5.5.1) becomes

$$\sum_{i=2}^{n} B(u(c_1 \cdot \ldots \cdot c_{i-1}) \cdot \mathrm{Ad}(\phi(c_1 \cdot \ldots \cdot c_{i-1}))v(c_i)) + \sum_{i=1}^{n} B(\xi_i \cdot v(c_i)). \tag{5.5.2}$$

We can further simplify (5.5.2) using to the Ad-invariance of $B$ and the formula $u(x^{-1}) = -\mathrm{Ad}(\phi(x^{-1}))u(x)$. It is useful to introduce the notation $b_{i-2} := c_{i-1}^{-1} \cdots c_1^{-1}$. In particular, $b_0 = c_1^{-1}$ and $b_{n-1} = 1$. We obtain

$$\omega_{\mathcal{G}}(u,v) = -\sum_{i=2}^{n} B(u(b_{i-2})\cdot v(c_i)) + \sum_{i=1}^{n} B(\xi_i \cdot v(c_i)). \tag{5.5.3}$$

Using that $\omega_G$ and the cup product are anti-symmetric, we get the following equivalent form of (5.5.3)

$$\omega_{\mathcal{G}}(u,v) = -\sum_{i=2}^{n} B(u(b_{i-2})\cdot v(c_i)) - \sum_{i=1}^{n} B(\zeta_i \cdot u(c_i)). \tag{5.5.4}$$

Formulae (5.5.1), (5.5.4), and (5.5.3), were already obtained in the proof of [GHJW97, Key Lemma 8.4]. We go one step further.

**Lemma 5.5.1.** *It holds that*

$$\omega_{\mathcal{G}}(u,v) = \sum_{i=1}^{n-2} B((\zeta_{i+1} - \zeta_{i+2})\cdot u(b_i)). \tag{5.5.5}$$

*Proof.* Using $v(c_i) = \zeta_i - \mathrm{Ad}(\phi(c_i))\zeta_i$ and the Ad-invariance of $B$, we get

$$B(u(b_{i-2})\cdot v(c_i)) = B(\zeta_i \cdot u(b_{i-2})) - B(\mathrm{Ad}(\phi(c_i^{-1}))u(b_{i-2})\cdot \zeta_i)$$

By construction, $b_{i-1} = c_i^{-1} b_{i-2}$ and thus $u(b_{i-1}) = u(c_i^{-1}) + \mathrm{Ad}(\phi(c_i^{-1}))u(b_{i-2})$. So,

$$B(u(b_{i-2})\cdot v(c_i)) = B(\zeta_i \cdot u(b_{i-2})) - B(\zeta_i \cdot u(b_{i-1})) + B(\zeta_i \cdot u(c_i^{-1})).$$

Therefore, (5.5.4) becomes

$$\begin{aligned}
\omega_{\mathcal{G}}(u,v) =& \sum_{i=2}^{n} B(\zeta_i \cdot u(b_{i-1})) - B(\zeta_i \cdot u(b_{i-2})) \\
&- B(\zeta_1 \cdot u(c_1)) - \sum_{i=2}^{n} B(\zeta_i \cdot (u(c_i^{-1}) + u(c_i)) \\
=& B(\zeta_2 \cdot u(b_1)) + \sum_{i=3}^{n} B(\zeta_i \cdot u(b_{i-1})) - B(\zeta_i \cdot u(b_{i-2})) \\
&- \sum_{i=1}^{n} B\big(\zeta_i \cdot (u(c_i^{-1}) + u(c_i))\big) \\
=& \sum_{i=1}^{n-2} B\big((\zeta_{i+1} - \zeta_{i+2})\cdot u(b_i)\big) - \underbrace{\sum_{i=1}^{n} B\big(\zeta_i \cdot (u(c_i^{-1}) + u(c_i))\big)}_{=:\Omega},
\end{aligned}$$

where in the second equality we used $b_0 = c_1^{-1}$ and in the third equality that $u(b_{n-1}) = u(1) = 0$. It remains to prove that $\Omega = 0$. Using $u(x^{-1}) = -\mathrm{Ad}(\phi(x^{-1}))u(x)$, we get

$$B(\zeta_i \cdot u(c_i^{-1})) = -B(\mathrm{Ad}(\phi(c_i))\zeta_i \cdot u(c_i)).$$

Therefore, using $v(c_i) = \zeta_i - \mathrm{Ad}(\phi(c_i))\zeta_i$, we conclude

$$\Omega = \sum_{i=1}^{n} B(u(c_i) \cdot v(c_i)).$$

By construction, $B(u(\cdot) \cdot v(\cdot))$ defines a 1-cocycle in $Z^1(\pi_n, \mathbb{R})$. Closeness can also be computed directly using (1.3.2), similarly as in the proof of Lemma 1.3.11. Therefore, $\Omega$ is equal to the evaluation of the 1-cocycle $B(u(\cdot) \cdot v(\cdot))$ on the 1-cycle $c_1 + \ldots + c_n$. The 1-cycle $\sum_{i=1}^n c_i$ vanishes in homology (this is a consequence of the fact that $\prod_{i=1}^n c_i = 1$). Hence, $\Omega = B(u(1) \cdot v(1)) = 0$ as desired. $\square$

# Chapter 6

# Euler and Toledo numbers

The topology of a representation variety is notably known to be complicated. The enumeration of its connected components is a non-trivial task. There exist some invariants that lets us approach this problem. The most classical one is the so-called Euler number, later generalized as Toledo number. The goal of this section is to recall their definitions. I take the occasion to acknowledge the contribution by Jacques Audibert and Xenia Flamm to the material presented in this chapter.

## 6.1 Preliminary observations

We will use the notation $\pi_0(X)$ to denote the number of connected components of a topological space $X$. Our first result relates the number of connected of components of a space and of its Hausdorffization which we introduced in Definition 4.3.1.

**Lemma 6.1.1.** *If $X$ denotes a topological space, then there is a bijection*

$$\pi_0(X) \cong \pi_0(\mathrm{Haus}(X))$$

*induced by the projection $X \to \mathrm{Haus}(X)$.*

*Proof.* Recall that $\mathrm{Haus}(X)$ is defined to be the quotient $X/\sim$, where $\sim$ is the equivalence relation on $X$ defined by $x \sim y$ if and only if $x \approx y$ for every equivalence relation $\approx$ such that $X/\approx$ is Hausdorff. Consider the equivalence relation $\approx$ defined by $x \approx y$ if and only if $x$ and $y$ are in same connected component of $X$. The quotient $X/\approx \cong \pi_0(X)$ is discrete, hence Hausdorff. The projections $X \to X/\sim \to X/\approx$ induce surjective maps

$$\pi_0(X) \to \pi_0(X/\sim) \to \pi_0(X/\approx) \cong \pi_0(X).$$

We conclude that $\pi_0(X) \cong \pi_0(X/\sim)$. ☐

The following standard fact will also come in handy later.

**Lemma 6.1.2.** *Let $X$ be a topological space and $G$ a topological group acting continuously on $X$. The quotient map $p\colon X \to X/G$ is open. Furthermore, if $G$ is finite, then $p$ is closed.*

*Proof.* Let $U \subseteq X$ be open. We need to prove that $p(U)$ is open in the quotient topology, i.e. we need to show that $p^{-1}(p(U))$ is open in $X$. We have

$$p^{-1}(p(U)) = \bigcup_{g \in G} gU,$$

which is open. By the same argument we can deduce that $p^{-1}(p(C)) = \bigcup_{g \in G} gC$ is closed if $C$ is closed and $G$ is finite. □

**Lemma 6.1.3.** *Let $X$ be a topological space with a continuous action of a connected topological group $G$. Then, there is a bijection*

$$\pi_0(X) \cong \pi_0(X/G)$$

*induced by the projection $X \to X/G$.*

*Proof.* Since $G$ is connected and the action is continuous, $G$ preserves each connected component of $X$, i.e. it acts trivially on $\pi_0(X)$. The quotient map $p\colon X \to X/G$ is an open (by Lemma 6.1.2), surjective, and continuous map. If $C_1$ and $C_2$ denote two distinct connected components of $X$, then $p(C_1)$ and $p(C_2)$ are disjoint because $G$ preserves each connected component of $X$. Since $p$ is an open map, $p(C)$ is open in $X/G$ for every connected component $C$ of $X$. Using that $p$ is surjective, we conclude that each connected component of $X/G$ is covered by disjoint open images of connected components of $X$. By connecetedness, there must be exactly one. We conlcude that $\pi_0(X) \cong \pi_0(X/G)$. □

When we apply the above observations to the context of character varieties, we conclude that the number of connected components of the representation variety for the pair $(\Gamma, G)$ is the same as for the Hausdorff character variety, assuming that $G$ is connected.

**Corollary 6.1.4.** *If $G$ is a connected Lie group, then*

$$\pi_0(\mathrm{Hom}(\Gamma, G)) \cong \pi_0\big(\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)\big) \cong \pi_0\left(\mathrm{Rep}^{\mathcal{T}_2}(\Gamma, G)\right).$$

## 6.2 Euler number

The idea behind the definition of the Euler number is to measure how hard it is to lift a representation $\Gamma \to G$ to a representation in the universal cover of $G$. The fundamental result to keep in mind is the following.

**Theorem 6.2.1.** *Let $\Gamma$ be a group (not necessarily finitely generated) and $G$ ne a path-connected topological group. Let $G'$ be a covering group of $G$. A homomorphism $\phi\colon \Gamma \to G$ lifts to a homorphism $\Gamma \to G'$ if and only if every homomorphism in the path-component of $\phi$ inside* $\mathrm{Hom}(\Gamma, G)$ *lifts.*

A proof of Theorem 6.2.1 can be found in [Cul86, Thm. 4.1] The construction of the Euler number is particularly interesting in the case when $\Gamma = \pi_{g,0}$ is a closed surface group (and, as we will see, is also particular to closed surface groups).

### 6.2.1 Definition

We will write $\widetilde{G}$ to denote the universal cover of $G$ and pick any $e \in \widetilde{G}$ that lifts $1 \in G$ (the choice of $e$ turns $\widetilde{G}$ into a topological group with neutral element $e$). There is a short exact sequence of topological groups

$$1 \to \pi_1(G) \to \widetilde{G} \to G \to 1.$$

The group $\pi_1(G)$ is equipped with the discrete topology and naturally embeds inside $\widetilde{G}$.

**Lemma 6.2.2.** *Let $G$ be a connected topological group and $N \subset G$ be a discrete normal subgroup of $G$. Then $N$ lies in the center $Z(G)$ of $G$.*

*Proof.* For $n \in N$ we consider the map

$$f_n\colon G \to N,\, g \mapsto gng^{-1}.$$

Then $f_n$ is continuous, and since $G$ is connected, so is the image of $f_n$. Since $N$ is discrete, $f_n$ is constant on $G$ and equal to $n$. Thus $N$ commutes with every element in $G$, hence $N \subset Z(G)$ is normal. □

**Corollary 6.2.3.** *The group $\pi_1(G)$ lies in $Z(\widetilde{G})$.*

**Definition 6.2.4** (Euler number)**.** Let $\Gamma = \pi_{g,0}$ be a closed surface group. The *Euler number* of a representation $\phi\colon \pi_{g,0} \to G$ is the element of $\pi_1(G)$ defined by

$$\mathrm{eu}(\phi) := \prod_{i=1}^{g} \left[\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)}\right],$$

where $\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)} \in \widetilde{G}$ are any lifts of $\phi(a_i), \phi(b_i)$.

The name Euler "number" comes from the case where $G = \mathrm{PSL}(2, \mathbb{R})$, in which case $\pi_1(\mathrm{PSL}(2, \mathbb{R})$ can be identified with $\mathbb{Z}$. Note that $\mathrm{eu}(\phi)$ is a lift of $1 \in G$ which explain why it is an element of $\pi_1(G) \subset \widetilde{G}$.

**Lemma 6.2.5.** *The Euler number is independent of the choice of the lifts.*

*Proof.* Different choices of lifts differ by elements of $\pi_1(G)$. Note that each generator and its inverse appear exactly once in the definition of the Euler invariant. So, since $\pi_1(G)$ lies inside $Z(\widetilde{G})$ by Lemma 6.2.2, the product is indeed independent of the choice of the lifts. □

**Lemma 6.2.6.** *The Euler number is a continuous function*

$$\mathrm{eu}\colon \mathrm{Hom}(\pi_g, G) \to \pi_1(G),$$

*that factors through $\pi_0(\mathrm{Hom}(\pi_g, G))$.*

*Proof.* The continuity is a consequence of Lemma 6.2.5 and the properties of universal covers. Since $\pi_1(G)$ is discrete, the Euler number is thus constant on connected components of $\mathrm{Hom}(\pi_g, G)$. □

**Lemma 6.2.7.** *The Euler number is natural in the sense that if $H$ is a subgroup of $G$, then the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Hom}(\pi_g, H) & \xrightarrow{\mathrm{eu}} & \pi_1(H) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\pi_g, G) & \xrightarrow{\mathrm{eu}} & \pi_1(G). \end{array}$$

*Remark* 6.2.8. In general, the Euler number $\mathrm{eu}\colon \pi_0(\mathrm{Hom}(\pi_g, G)) \to \pi_1(G)$ is neither injective nor surjective. It is for instance injective if $G = \mathrm{PSL}(2,\mathbb{R})$, but it fails to be injective for $G = \mathrm{SL}(2,\mathbb{R})$. It is also not surjective in both cases.

If $G$ is semi-simple and not of Hermitian type, then the Euler number is always surjective. This follows from Theorem 6.2.9, since $\pi_1(G) = \pi_1(K)$, where $K$ is a maximal compact subgroup of $G$. If $G$ is not of Hermitian type, then $K$ is semi-simple.

### 6.2.2 Compact and complex Lie groups

In some cases, it's possible to classify all the components of a representation variety using the Euler number only. This is the case when the Euler number is an injective function. This works when $G$ is either a connected semisimple compact Lie group or a connected semisimple complex Lie group. The first statement was proved by Atiyah-Bott and a proof of the second statement can be found in [Li93, Thm. 0.1].

**Theorem 6.2.9** ([AB83])**.** *If $G$ is a connected and compact semisimple Lie group, then the connected components of* $\mathrm{Hom}(\pi_{g,0}, G)$ *are classified by* $\pi_1(G)$ *via the Euler number.*

**Theorem 6.2.10.** *If $\mathcal{G}$ is a connected and complex semisimple Lie group, then connected components of* $\mathrm{Hom}(\pi_{g,0}, G)$ *are classified by* $\pi_1(G)$ *via the Euler number.*

### 6.2.3 The case of $\mathrm{SL}(n,\mathbb{R})$

There is another case where the Euler number can be used to classify the connected components of the representation variety, namely when $G = \mathrm{PSL}(2,\mathbb{R})$. In that case, the Euler number takes values in $\pi_1(\mathrm{PSL}(2,\mathbb{R})) \cong \mathbb{Z}$. The classification is due Goldman who proved the following.

**Theorem 6.2.11** ([Gol88])**.** *When $G = \mathrm{PSL}(2,\mathbb{R})$, then* $\mathrm{eu}\colon \pi_0(\mathrm{Hom}(\pi_{g,0}, \mathrm{PSL}(2,\mathbb{R}))) \to \mathbb{Z}$ *is injective and its image is $\mathbb{Z} \cap [2-2g, 2g-2]$. In particular,* $\mathrm{Hom}(\pi_{g,0}, \mathrm{PSL}(2,\mathbb{R}))$ *has $4g-3$ connected components.*

It is worth pointing out that not all representation $\pi_{g,0} \to \mathrm{PSL}(2,\mathbb{R})$ lift to representations in $\mathrm{SL}(2,\mathbb{R})$. Actually, we have the following statement.

**Lemma 6.2.12.** *A representation $\phi\colon \pi_{g,0} \to \mathrm{PSL}(2,\mathbb{R})$ lifts to a representation $\pi_{g,0} \to \mathrm{SL}(2,\mathbb{R})$ if and only if $\mathrm{eu}(\phi)$ is even. In other words, the image of*

$$p_*\colon \pi_0(\mathrm{Hom}(\pi_g, \mathrm{SL}(2,\mathbb{R}))) \to \pi_0(\mathrm{Hom}(\pi_g, \mathrm{PSL}(2,\mathbb{R})))$$

*is $\mathrm{eu}^{-1}(2\mathbb{Z})$, where $p\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ is the natural quotient map.*

*Proof.* The Lie group $\mathrm{SL}(2,\mathbb{R})$ is a double cover of $\mathrm{PSL}(2,\mathbb{R})$ because $Z(\mathrm{SL}(2,\mathbb{R})) = \{\pm I\}$. The universal cover of $\mathrm{PSL}(2,\mathbb{R})$ therefore identifies with the universal cover $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ of $\mathrm{SL}(2,\mathbb{R})$. There is a group homomorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ induced by

$$\pi_1(\mathrm{PSL}(2,\mathbb{R})) \longrightarrow \widetilde{\mathrm{PSL}(2,R)} = \widetilde{\mathrm{SL}(2,R)} \xrightarrow{\pi} \mathrm{SL}(2,\mathbb{R}).$$

Pick $\overline{e} \in \pi^{-1}(-I)$. Since $\overline{e}$ is in the kernel of $\widetilde{\mathrm{SL}(2,\mathbb{R})} \to \mathrm{PSL}(2,\mathbb{R})$, there is $z \in \pi_1(\mathrm{PSL}(2,\mathbb{R}))$ that is mapped to $\overline{e}$ under $\pi_1(\mathrm{PSL}(2,\mathbb{R})) \to \widetilde{\mathrm{SL}(2,R)}$. This shows that the induced homomorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is non-trivial and, hence, is the reduction of integers modulo 2.

Let $\phi\colon \pi_{g,0} \to \mathrm{PSL}(2,\mathbb{R})$ be a representation. If $\mathrm{eu}(\phi)$ is even, then $\phi$ lifts to $\overline{\phi}\colon \pi_{g,0} \to \mathrm{SL}(2,\mathbb{R})$ defined by $\overline{\phi}(a_i) := \pi(\widetilde{\phi(a_i)})$ and $\overline{\phi}(b_i) := \pi(\widetilde{\phi(b_i)})$, where $\widetilde{\phi(a_i)}, \widetilde{\phi(b_i)} \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ are any lifts of $\phi(a_i), \phi(b_i)$. Conversely, if $\phi$ lifts, then $\mathrm{eu}(\phi) \in p_*^{-1}(0)$ is an even number. ☐

A consequence of the proof of Lemma 6.2.12 is that the induced map $\pi_1(\mathrm{SL}(2,\mathbb{R})) \to \pi_1(\mathrm{PSL}(2,\mathbb{R}))$, seen as a map $\mathbb{Z} \to \mathbb{Z}$, is the multiplication by 2. This means that a representation $\phi\colon \pi_{g,0} \to \mathrm{PSL}(2,\mathbb{R})$ lifts if and only if $\mathrm{eu}(\phi)$ is in the image of $\pi_1(\mathrm{SL}(2,\mathbb{R})) \to \pi_1(\mathrm{PSL}(2,\mathbb{R}))$. We can go one step further and ask for the number of connected components of $\mathrm{Hom}(\pi_{g,0}, \mathrm{SL}(2,\mathbb{R}))$ that lie above a given connected component of $\mathrm{Hom}(\pi_{g,0}, \mathrm{PSL}(2,\mathbb{R}))$. We start with following general lemma.

**Lemma 6.2.13.** *We denote by* $\mathrm{Ad}(G) = G/Z(G)$ *the adjoint Lie group of a Lie group* $G$. *Assume that* $Z(G)$ *is finite and has cardinality* $m$. *Let* $C$ *be a connected component of* $\mathrm{Hom}(\pi_{g,0}, \mathrm{Ad}(G))$ *that lifts to* $\mathrm{Hom}(\pi_{g,0}, G)$. *Then* $p^{-1}(C)$ *is a* $m^{2g}$*-fold cover of* $C$.

*Proof.* Any $\phi \in C$ lifts in $m^{2g}$ different ways since there are exactly $m$ choices of lift for any of the $\phi(a_1), \phi(b_1), \ldots, \phi(a_g), \phi(b_g)$. ☐

**Corollary 6.2.14.** *Each of the two connected components of* $\mathrm{Hom}(\pi_{g,0}, \mathrm{PSL}(2,\mathbb{R}))$ *that correspond to Teichmüller space lift to* $2^{2g}$ *distinct connected components of* $\mathrm{Hom}(\pi_{g,0}, \mathrm{SL}(2,\mathbb{R}))$.

*Proof.* This follows form the fact that the Teichmüller components inside the character variety of $(\pi_{g,0}, \mathrm{PSL}(2,\mathbb{R}))$ are balls and hence simply connected. So, any finite degree cover of Teichmüller space must be trivial. ☐

Goldman proved each of the $2g-4$ connected components of $\mathrm{Hom}(\pi_{g,0}, \mathrm{PSL}(2,\mathbb{R}))$ with Euler number in $2\mathbb{Z} \cap [4-2g, 2g-4]$ lifts inside the sane connected component of $\mathrm{Hom}(\pi_{g,0}, \mathrm{SL}(2,\mathbb{R}))$.

**Theorem 6.2.15** ([Gol88])**.** *The number of connected components of* $\mathrm{Hom}(\pi_{g,0}, \mathrm{SL}(2,\mathbb{R}))$ *is* $2^{2g+1} + 2g - 3$.

Before closing this section, let us mention that Hitchin later computed (using different methods) the number of connected components in the case $G = \mathrm{PSL}(n,\mathbb{R})$ for $n \geqslant 3$.

**Theorem 6.2.16** ([Hit92])**.** *If* $G = \mathrm{PSL}(n,\mathbb{R})$, *then* $\mathrm{Hom}(\pi_{g,0}, \mathrm{PSL}(n,\mathbb{R}))$ *has* 3 *connected components if* $n$ *is odd and* 6 *connected components if* $n$ *is even.*

If we are interested in determining which representations lift to $\mathrm{SL}(n,\mathbb{R})$, we can proceed as follows. Note however that the question is trivial when $n$ is an odd integer since in that case $\mathrm{SL}(n,\mathbb{R}) = \mathrm{PSL}(n,\mathbb{R})$.

**Lemma 6.2.17.** *Let $n \geqslant 4$ be an even number. It holds that*

$$\pi_1(\mathrm{SL}(n,\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}, \quad \pi_1(\mathrm{PSL}(n,\mathbb{R})) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \text{if } n \equiv 0 \pmod 4, \\ \mathbb{Z}/4\mathbb{Z}, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

**Lemma 6.2.18.** *Let $n \geqslant 4$ be an even number. A representation $\phi\colon \pi_{g,0} \to \mathrm{PSL}(n,\mathbb{R})$ lifts to a representation $\pi_{g,0} \to \mathrm{SL}(n,\mathbb{R})$ if and only if $\mathrm{eu}(\phi)$ is in the image of $\pi_1(\mathrm{SL}(n,\mathbb{R})) \to \pi_1(\mathrm{PSL}(n,\mathbb{R}))$.*

The proof of Lemma 6.2.18 is analogous to that of Lemma 6.2.12.

## 6.3 Toledo number

The Toledo number was defined by Burger-Iozzi-Wienhard in [BIW10] for representations of surface groups $\Gamma = \pi_{g,n}$ into Hermitian Lie groups $G$. When $n \geqslant 1$, then $\pi_{g,n}$ is a free group, so every representation $\pi_{g,n} \to G$ will lift to the universal cover of $G$. In order to define a meaningful invariant, we need to somehow take into account some boundary data of $\pi_{g,n}$.

### 6.3.1 Hermitian Lie groups

Recall that a *Hermitian Lie group* $G$ is a semisimple Lie group, with finite center and no compact factors, such that its associated symmetric space $X$ is a Hermitian manifold. The Kähler form obtained from the unique $G$-invariant Hermitian metric of constant sectional curvature $-1$ on $X$ is denoted $\omega_X$. The classical examples of Hermitian Lie groups include $\mathrm{SU}(p,q)$ and $\mathrm{Sp}(2n,\mathbb{R})$. For instance, the symmetric space of $\mathrm{SL}(2,\mathbb{R})$ is the upper half-plane $X = \mathbb{H}$ on which $\mathrm{SL}(2,\mathbb{R})$ acts by Möbius transformations. More considerations can be found in Appendix A. The group of orientation-preserving isometries of $\mathbb{H}$ is $\mathrm{PSL}(2,\mathbb{R})$ and the associated Kähler form is $\omega_{\mathbb{H}} = (dx \wedge dy)/y^2$.

### 6.3.2 The area of a triangle

Let us now fix a Hermitian Lie group $G$ with symmetric space $X$. Given three points $z_1, z_2, z_3$ in $X$, we denote by $\Delta(z_1,z_2,z_3)$ the oriented geodesic triangle in $X$ with vertices $z_1, z_2, z_3$. Its signed area, computed with the area form associated to $\omega_X$, is denoted by

$$[\Delta(z_1,z_2,z_3)] \coloneqq \int_{\Delta(z_1,z_2,z_3)} \omega_X.$$

Fix a basepoint $z \in X$ and consider the function

$$\begin{aligned} c\colon G \times G &\to \mathbb{R} \\ (g_1,g_2) &\to \big[\Delta\big(z, g_1 z, g_1 g_2 z\big)\big]. \end{aligned} \tag{6.3.1}$$

**Lemma 6.3.1.** *The function $c$ satisfies the cocycle condition*

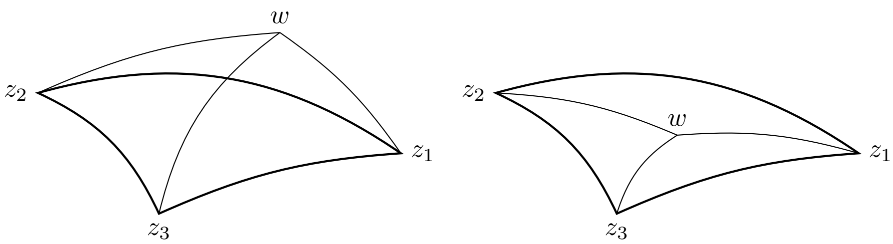
$$c(g_2,g_3) - c(g_1g_2,g_3) + c(g_1,g_2g_3) - c(g_1,g_2) = 0 \tag{6.3.2}$$

*for every* $g_1, g_2, g_3 \in G$*, compare* (1.3.2).

*Proof.* We need the following identity: if $z_1, z_2, z_3$ are any three points in $X$, then, for any fourth point $w \in X$,

$$[\Delta(z_1, z_2, z_3)] = [\Delta(z_1, z_2, w)] + [\Delta(z_2, z_3, w)] + [\Delta(z_3, z_1, w)]. \tag{6.3.3}$$

The following picture should convince the reader of (6.3.3).




In terms of triangle areas, the cocycle condition (6.3.2) is equivalent to

$$[\Delta(z, g_2z, g_2g_3z)] + [\Delta(z, g_1z, g_1g_2g_3z)]$$

being equal to

$$[\Delta(z, g_1g_2z, g_1g_2g_3z)] + [\Delta(z, g_1z, g_1g_2z)].$$

Since $g_1 \in G$ acts by isometry on $X$ and preserves the orientation, the latter is equivalent to

$$[\Delta(g_1z, g_1g_2z, g_1g_2g_3z)] + [\Delta(z, g_1z, g_1g_2g_3z)]$$

being equal to

$$[\Delta(z, g_1g_2z, g_1g_2g_3z)] + [\Delta(z, g_1z, g_1g_2z)].$$

This is precisely formula (6.3.3) applied to $z_1 = z$, $z_2 = g_1z$, $z_3 = g_1g_2z$ and $w = g_1g_2g_3z$. □

Lemma 6.3.1 implies that $c$ defines a cohomology class $\kappa := [c]$ inside $H^2(G, \mathbb{R})$. The function $c$ is bounded because the area of a geodesic triangle in $X$ is bounded. This means that the cohomology class $\kappa$ gives a class $\kappa \in H_b^2(G, \mathbb{R})$ in the second bounded cohomology group of $G$. We recommend [Löh10] for an introduction to bounded group cohomology.

**Lemma 6.3.2.** *The cohomology class* $\kappa$ *is independent of the choice of the basepoint* $z$ *involved in the definition of* $c$ *(whereas* $c$ *does depend on the point* $z$*).*

*Proof.* For the purpose of this proof, we will write $c_z$ instead of $c$ for the cocycle (6.3.1) to emphasize the dependence on the basepoint $z$. Given another basepoint $x \in X$, we prove that $c_z - c_x$ is a coboundary.

First, we develop $c_z(g_1, g_2) = [\Delta(z, g_1z, g_1g_2z)]$ using (6.3.3) with $w = g_1x$. We obtain

$$\begin{aligned} c_z(g_1, g_2) &= [\Delta(z, g_1z, g_1x)] + [\Delta(g_1z, g_1g_2z, g_1x)] + [\Delta(g_1g_2z, z, g_1x)] \\ &= -[\Delta(x, z, g_1^{-1}z)] + [\Delta(x, z, g_2z)] + [\Delta(g_1g_2z, z, g_1x)]. \end{aligned}$$

Now, we develop $[\Delta(g_1g_2z, z, g_1x)]$ using (6.3.3) with $w = x$. This gives

$$\begin{aligned}[\Delta(g_1g_2z, z, g_1x)] &= [\Delta(g_1g_2z, z, x)] + [\Delta(z, g_1x, x)] + [\Delta(g_1x, g_1g_2z, x)] \\ &= -[\Delta(x, z, g_1g_2z)] - [\Delta(z, x, g_1x)] + [\Delta(g_1x, g_1g_2z, x)].\end{aligned}$$

Finally, we develop $[\Delta(g_1x, g_1g_2z, x)]$ using (6.3.3) with $w = g_1g_2x$. We have

$$\begin{aligned}[\Delta(g_1x, g_1g_2z, x)] &= [\Delta(g_1x, g_1g_2z, g_1g_2x)] + [\Delta(g_1g_2z, x, g_1g_2x)] + [\Delta(x, g_1x, g_1g_2x)] \\ &= [\Delta(z, x, g_2^{-1}x)] - [\Delta(z, x, g_2^{-1}g_1^{-1}x)] + c_x(g_1, g_2).\end{aligned}$$

Consider the 1-cochain $v_{x,z}(g) \coloneqq [\Delta(x, z, gz)]$. It holds that

$$\partial v_{x,z}(g_1, g_2) = [\Delta(x, z, g_1z)] + [\Delta(x, z, g_2z)] - [\Delta(x, z, g_1g_2z)].$$

In particular, $\partial v_{x,z}(g, g^{-1}) = [\Delta(x, z, gz)] + [\Delta(x, z, g^{-1}z)]$. The previous computations show that

$$c_z(g_1, g_2) - c_x(g_1, g_2) = \partial v_{x,z}(g_1, g_2) - \partial v_{x,z}(g_1, g_1^{-1}) + \partial v_{z,x}(g_2^{-1}, g_1^{-1}) - \partial v_{z,x}(g_1, g_1^{-1}).$$

We conclude as predicted that $c_z - c_x$ is a coboundary. □

### 6.3.3 Group cohomological definition

Given a representation $\phi\colon \pi_{g,n} \to G$, we can pull back $\kappa$ to the class $\phi^*\kappa$ inside $H_b^2(\pi_{g,n}, \mathbb{R})$. An important property of the bounded cohomology of the group $\pi_{g,n}$ is that the map

$$j\colon H_b^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \to H_b^2(\pi_{g,n}, \mathbb{R}) \tag{6.3.4}$$

from the long exact sequence in cohomology for the pair $(\pi_{g,n}, \partial\pi_{g,n})$ is an isomorphism, see [Löh10, Thm. 2.6.14]. Recall finally that integrating along a fundamental class $[\pi_{g,n}, \partial\pi_{g,n}]$ gives an isomorphism $H^2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{R}) \cong \mathbb{R}$.

**Definition 6.3.3** (Toledo number, [BIW10])**.** Let $G$ be a Hermitian Lie group. The *Toledo number* of a representation $\phi\colon \pi_{g,n} \to G$ is the real number defined by

$$\mathrm{Tol}(\phi) \coloneqq j^{-1}(\phi^*\kappa) \frown [\pi_{g,n}, \partial\pi_{g,n}].$$

### 6.3.4 Properties

The first thing to point out is that the Toledo number is a generalization of the Euler number for representations of closed surface groups into $\mathrm{PSL}(2, \mathbb{R})$. This is explained in [BIW10]. Now, we proceed to other basic properties of the Toledo number.

**Lemma 6.3.4.** *The volume is invariant under the conjugation action of $G$ on $\mathrm{Hom}(\pi_{g,n}, G)$ and thus descends to a function*

$$\mathrm{Tol}\colon \mathrm{Hom}(\pi_{g,n}, G)/\mathrm{Inn}(G) \to \mathbb{R}.$$

*Proof.* Consider the cocycle $c$ defined in (6.3.1). The diagonal conjugation action of an element $g \in G$ on $G \times G$ amounts to a change of basepoint in the definition of $c$. Indeed, if $c_z$ denotes the cocycle (6.3.1) defined using the basepoint $z \in X$, then it holds that $c_z(gg_1g^{-1}, gg_2g^{-1}) = c_{g^{-1}z}(g_1, g_2)$. Since, by Lemma 6.3.2, the cohomology class $\kappa$ is independent of the choice of the basepoint defining $c$, we conclude that the volume is an invariant of conjugation. $\square$

The main properties of the volume are the following. We denote by $\chi(\Sigma_{g,n})$ the Euler characteristic of $\Sigma_{g,n}$.

**Theorem 6.3.5** ([BIW10])**.** *The volume, seen as a function* $\mathrm{Tol}\colon \mathrm{Hom}(\pi_{g,n}, G) \to \mathbb{R}$, *has the following properties:*

1. Tol *is a continuous function.*

2. Tol *is locally constant on each relative representation variety.*

3. (Milnor-Wood inequality) Tol *is bounded:*

$$|\,\mathrm{Tol}\,| \leqslant |\chi(\Sigma_{g,n})|\,\mathrm{rank}(G),$$

*moreover, if $n > 0$, then* Tol *is a surjective function onto the interval*

$$[|\chi(\Sigma_{g,n})|\,\mathrm{rank}(G),\, |\chi(\Sigma_{g,n})|\,\mathrm{rank}(G)]\,.$$

4. Tol *is additive: if $\Sigma_{g,n}$ is separated by a simple closed curve into two surfaces $S_1$ and $S_2$, then, for every $\phi \in \mathrm{Hom}(\pi_{g,n}, G)$,*

$$\mathrm{vol}(\phi) = \mathrm{vol}(\phi\restriction_{\pi_1(S_1)}) + \mathrm{vol}(\phi\restriction_{\pi_1(S_2)}).$$

The first and second statement in Theorem 6.3.5 imply that the set of representations of a given Toledo number forms a collection of connected components of each relative character variety. In general, there is no reason for this collection to contain a unique connected component. Recall that in the case of a closed surface group and $G = \mathrm{PSL}(2, \mathbb{R})$, the Euler number completely distinguishes the connected components of the character variety [Gol88].

The Toledo number has an interesting symmetry that comes from reversing the orientation of $X$. By definition, for each $z \in X$, there exists an orientation-reversing isometry $s_z$ of $X$ that fixes $z$. This gives an involutive outer automorphism $\sigma\colon G \to G$ defined by $\sigma(g) \coloneqq s_z \circ g \circ s_z$. Indeed, if $g \in G$ is an orientation-preserving isometry of $X$, then $s_z \circ g \circ s_z$ is again an orientation-preserving isometry of $X$, hence belongs to $G$. Using the functoriality of representation varieties (see Lemma 2.1.7), the involution $\sigma$ descends to an analytic involution

$$\sigma\colon \mathrm{Hom}(\pi_{g,n}, G) \to \mathrm{Hom}(\pi_{g,n}, G).$$

**Lemma 6.3.6.** *The involution $\sigma$ satisfies the following properties:*

1. *$\sigma$ preserves conjugacy classes of representations, and therefore descends to an involution*

$$\overline{\sigma}\colon \operatorname{Hom}(\pi_{g,n}, G)/\operatorname{Inn}(G) \to \operatorname{Hom}(\pi_{g,n}, G)/\operatorname{Inn}(G).$$

2. *$\sigma$ depends on the choice of $z \in X$ only up to conjugation, in particular, $\overline{\sigma}$ is independent of the choice of $z \in X$.*

3. *For any representation $\phi \in \operatorname{Hom}(\pi_{g,n}, G)$ it holds that*

$$\operatorname{Tol}(\sigma(\phi)) = -\operatorname{Tol}(\phi).$$

*Proof.* The first assertion follows from $\sigma(g\phi g^{-1}) = (s_z \circ g \circ s_z)\sigma(\phi)(s_z \circ g^{-1} \circ s_z)$ and the observation that $s_z \circ g \circ s_z$ is orientation-preserving. If $z' \in X$ is a second point, then it holds that $s_{z'} \circ g \circ s_{z'} = (s_{z'} \circ s_z)(s_z \circ g \circ s_z)(s_z \circ s_{z'})$, which proves the second assertion because $s_{z'} \circ s_z$ is orientation-preserving. Finally, note that $(\sigma(\phi))^*\kappa = \phi^*(\sigma^*\kappa)$ and $\sigma^*\kappa = -\kappa$ because $s_z$ reverses the orientation of $X$. □

**Example 6.3.7.** Consider the case $G = \mathrm{SL}(2,\mathbb{R})$. An example of orientation-reversing isometry of the upper half-plane is given by $z \mapsto -\overline{z}$. It fixes the imaginary axis. The associated involutive outer automorphism $\sigma$ of $\mathrm{SL}(2,\mathbb{R})$ is given by conjugation by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of determinant $-1$.

The involution $\sigma\colon \operatorname{Hom}(\pi_{g,n}, G) \to \operatorname{Hom}(\pi_{g,n}, G)$ maps the relative representation variety $\operatorname{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$ to the relative representation variety $\operatorname{Hom}_{\sigma(\mathcal{C})}(\pi_{g,n}, G)$. Since $G$ is of Hermitian type, it is by definition semisimple, hence quadrable. The Goldman symplectic form built from the Killing form on $\mathfrak{g}$ is invariant under $\sigma$. This is a consequence of the fact that the Killing form is invariant under automorphisms of $\mathfrak{g}$. In this case, the involution $\sigma\colon G \to G$ induces an automorphism $D\sigma\colon \mathfrak{g} \to \mathfrak{g}$.

### 6.3.5 Alternative definition via rotation numbers

A downside of Definition 6.3.3 is the lack of computability. Given a representation $\phi\colon \pi_{g,n} \to G$, computing $j^{-1}(\phi^*\kappa)$ means finding a primitive in $H^1(\partial_i\pi_{g,n}, \mathbb{R})$ for each restriction $\phi^*\kappa\restriction_{\partial_i\pi_{g,n}}$. This is a non-trivial task in general. There is an alternative definition of the volume of a representation that makes it easier to compute. It is based on a notion of rotation number that generalizes the classical notion of rotation number for homeomorphisms of the circle described for instance in [Ghy01]. The *rotation number*, as introduced by Burger-Iozzi-Wienhard, is a function $\rho\colon G \to \mathbb{R}/\mathbb{Z}$ that lifts to a quasimorphism $\widetilde{\rho}\colon \widetilde{G} \to \mathbb{R}$ of the universal cover of $G$. We explain the construction in the case $G = \mathrm{PSL}(2,\mathbb{R})$ and refer the reader to [BIW10, §7] for the general construction. The main result is the following.

**Theorem 6.3.8** ([BIW10])**.** *Let $\widetilde{\phi}\colon \pi_{g,n} \to \widetilde{G}$ be a group homomorphism that covers $\phi$. Then*

$$\operatorname{Tol}(\phi) = -\sum_{i=1}^{n} \widetilde{\rho}\left(\widetilde{\phi}(c_i)\right),$$

*where $c_i$ are the generators of $\pi_{g,n}$ from presentation* (1.4.1).

Let's study the case $G = \mathrm{PSL}(2,\mathbb{R})$. We fix a topological group structure on $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ by fixing a unit $e$ in the fibre over the identity. The action of $\mathrm{PSL}(2,\mathbb{R})$ on the circle $\mathbb{R}/\mathbb{Z}$ (see Lemma A.2.1) gives a group homomorphism $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z})$. The classical rotation number is a function $\mathrm{rot}\colon \mathrm{Homeo}^+(\mathbb{R}/2\pi\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ defined as follows. Given $f \in \mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z})$, lift it to an orientation-preserving $F\colon \mathbb{R} \to \mathbb{R}$, uniquely defined up to translation by an integer. The group of all such lifts is denoted by $\widetilde{\mathrm{Homeo}}^+(\mathbb{R}/2\pi\mathbb{Z})$. We can then compute the number

$$\mathrm{Rot}(F) := \lim_{n\to\infty} \frac{F^n(x) - x}{n}$$

where $x$ is any real number. The limit always exists and is independent of $x$. The rotation number of $f$ is then defined to be the projection of $\mathrm{Rot}(F)$ inside $\mathbb{R}/\mathbb{Z}$. Composing the map $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z})$ with $\mathrm{rot}\colon \mathrm{Homeo}^+(\mathbb{R}/2\pi\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ gives the *rotation number*

$$\rho\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{R}/\mathbb{Z}.$$

The action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{R}/\mathbb{Z}$ lifts to a faithful action of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ on the universal cover $\mathbb{R}/\mathbb{Z}$, defining a group homomorphism $\widetilde{\mathrm{PSL}}(2,\mathbb{R}) \to \widetilde{\mathrm{Homeo}}^+(\mathbb{R}/2\pi\mathbb{Z})$. When we compose it with $\mathrm{Rot}\colon \widetilde{\mathrm{Homeo}}^+(\mathbb{R}/2\pi\mathbb{Z}) \to \mathbb{R}$, we get a quasimorphism

$$\widetilde{\rho}\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{R}.$$

It can also be described as the unique lift of $\rho\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathbb{R}/\mathbb{Z}$ satisfying $\widetilde{\rho}(e) = 0$.

We can describe $\rho$ more explicitly by considering conjugacy classes in $\mathrm{PSL}(2,\mathbb{R})$. Recall that, if $\mathcal{E}$ denotes the set of elliptic conjugacy classes in $\mathrm{PSL}(2,\mathbb{R})$, then there is a well-defined angle function $\vartheta\colon \mathcal{E} \to (0,2\pi)$, see Lemma A.2.5. It extends to an upper semi-continuous function $\overline{\vartheta}\colon \mathrm{PSL}(2,\mathbb{R}) \to [0,2\pi]$ defined by

$$\overline{\vartheta}(A) := \begin{cases} \vartheta(A), & \text{if } A \text{ is elliptic,} \\ 0, & \text{if } A \text{ is hyperbolic or positively parabolic,} \\ 2\pi, & \text{if } A \text{ is the identity or negatively parabolic.} \end{cases} \tag{6.3.5}$$

The notions of positively and negatively parabolic refer to the two conjugacy classes of parabolic elements in $\mathrm{PSL}(2,\mathbb{R})$ represented by (A.2.6). The definition of the function $\overline{\vartheta}$ is ad hoc, however it satisfies $\overline{\vartheta} = \rho$ modulo $2\pi$. In particular, the correction term

$$k(\phi) := \left( \frac{1}{2\pi} \sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i)) - \sum_{i=1}^{n} \widetilde{\rho}\left(\widetilde{\phi}(c_i)\right)\right) \tag{6.3.6}$$

is an integer called the *relative Euler class* of $\phi$. It was introduced by Deroin-Tholozan in [DT19]. The definition of the relative Euler class very much depends on the choice of the extension $\overline{\vartheta}$ of $\vartheta$. Theorem 6.3.8 implies

$$k(\phi) = \mathrm{Tol}(\phi) + \sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i)).$$

The range of the relative Euler class over $\mathrm{Hom}(\pi_{g,n}, G)$ was studied in [DT19], where the following was proved.

**Proposition 6.3.9** ([DT19])**.** *Let $\phi\colon \pi_{g,n} \to \mathrm{PSL}(2,\mathbb{R})$ be a representation. Then*

$$k(\phi) \leqslant \max\left\{ |\chi(\Sigma_{g,n})|,\ \frac{1}{2\pi}\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i)) \right\}.$$

*Remark* 6.3.10. Observe that, as soon as $g \geqslant 1$, then $|\chi(\Sigma_{g,n})| \geqslant n \geqslant \frac{1}{2\pi}\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i))$ and thus the inequality $k(\phi) \leqslant |\chi(\Sigma_{g,n})|$ prevails. In the case $g = 0$, it is however possible that $\frac{1}{2\pi}\sum_{i=1}^{n} \overline{\vartheta}(\phi(c_i)) > |\chi(\Sigma_{0,n})|$. The representation which satisfy the latter have very interesting properties, such as being *totally elliptic*. For further considerations, the reader may consult [DT19] or [Mar21].

# Chapter 7

# Mapping class group dynamics

This chapter is an expansion on some results and remarks from Section 2.2 on the mapping class group action on character varieties. Recall that the $\mathrm{Aut}(\Gamma)$-action on the representation variety $\mathrm{Hom}(\Gamma, G)$ descends to an action of the outer automorphisms group $\mathrm{Out}(\Gamma)$ on the quotient $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$. This action preserves the analytic/algebraic structure of $\mathrm{Hom}(\Gamma, G)$ by Lemma 2.2.1. When $\Gamma = \pi_{g,n}$ is a surface group, then $\mathrm{Out}(\pi_{g,n})$ contains the pure mapping class group of the surface $\Sigma_{g,n}$ as a subgroup, compare Example 2.2.3. The induced action is the so-called *mapping class group action on character varieties.*

## 7.1 Remarks on the $\mathrm{Aut}(\Gamma)$-action

We start with some general considerations on the $\mathrm{Aut}(\Gamma)$-action on $\mathrm{Hom}(\Gamma, G)$ and then specialize to the case of a surface group.

**Lemma 7.1.1.** *The* $\mathrm{Aut}(\Gamma)$*-action on* $\mathrm{Hom}(\Gamma, G)$ *preserves the subspaces of (very) regular, reductive, irreducible, good and (almost) Zariski dense representations.*

*Proof.* All these particular notions of representations are defined in terms of the image of the representation. However, for any $\tau \in \mathrm{Aut}(\Gamma)$ and $\phi \in \mathrm{Hom}(\Gamma, G)$, it holds that $\phi(\Gamma) = (\phi \circ \tau)(\Gamma)$. $\square$

A consequence of Lemma 7.1.1 is that the $\mathrm{Out}(\Gamma)$-action on $\mathrm{Hom}(\Gamma, G)/\mathrm{Inn}(G)$ restricts to an action of $\mathrm{Out}(\Gamma)$ on the GIT character variety $\mathrm{Rep}^{\mathrm{GIT}}(\Gamma, G)$ (by Theorem 4.5.6, assuming $G$ is a reductive complex algebraic group) and on the analytic character variety $\mathrm{Rep}^{\infty}(\pi_{g,0}, G)$.

**Lemma 7.1.2.** *The* $\mathrm{Aut}(\Gamma)$*-action on* $\mathrm{Hom}(\Gamma, G)$ *preserves closed orbits.*

*Proof.* This is an immediate consequence of Lemma 2.2.1. $\square$

In particular, Lemma 7.1.2 implies that the $\mathrm{Aut}(\Gamma)$-action on $\mathrm{Hom}(\Gamma, G)$ induces an $\mathrm{Out}(\Gamma)$-action on the $\mathcal{T}_1$ character variety $\mathrm{Rep}^{\mathcal{T}_1}(\pi_{g,0}, G)$. It is not clear to the author whether there is an induced action of $\mathrm{Out}(\Gamma)$ on the Hausdorff character variety in general.

## 7.2 Generalities about mapping class groups

The *mapping class group* of a closed and oriented surface $\Sigma_{g,0}$ is the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,0}$. For punctured oriented surfaces $\Sigma_{g,n}$, the pure mapping class group is defined to be the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,n}$ that fix each puncture individually. It contrasts with the mapping class group where punctures can be permuted. Our notation for the pure mapping class group will be $\mathrm{Mod}(\Sigma_{g,n})$ and the isotopy class of an orientation-preserving homeomorphism $f\colon \Sigma_{g,n} \to \Sigma_{g,n}$ is denoted $[f] \in \mathrm{Mod}(\Sigma_{g,n})$. The group law is given by composition and the identity element correspond to the identity homeomorphism.

**Theorem 7.2.1.** *The mapping class group is finitely presented. Generators can be chosen to be Dehn twists along simple closed curves on $\Sigma_{g,n}$.*

More details about Theorem 7.2.1, including proof and explicit generating family, can be found in [FM12, §4]. A homeomorphism $f$ of $\Sigma_{g,n}$ induces a group isomorphism $\pi_1(\Sigma_{g,n}, x) \to \pi_1(\Sigma_{g,n}, f(x))$. After choosing a continuous path from $x$ to $f(x)$, we get an induced automorphism of the fundamental group of $\Sigma_{g,n}$ (that depends up to conjugation on the choice of the path). This gives a group homomorphism

$$\mathrm{Mod}(\Sigma_{g,n}) \to \mathrm{Out}(\pi_{g,n}).$$

The Dehn–Nielsen Theorem says that it is injective and provides a description of its image.

**Theorem 7.2.2** (Dehn–Nielsen Theorem)**.** *The mapping class group $\mathrm{Mod}(\Sigma_{g,0})$ is an index two subgroup of $\mathrm{Out}(\pi_{g,0})$ for $g \geqslant 1$ (and is trivial for $g = 0$). Moreover, if $\Sigma_{g,n}$ has negative Euler characteristic, then the mapping class group $\mathrm{Mod}(\Sigma_{g,n})$ is an index two subgroup of $\mathrm{Out}^\star(\pi_{g,n})$, where $\mathrm{Out}^\star(\pi_{g,n})$ is the subgroup of $\mathrm{Out}(\pi_{g,n})$ that consists of the outer automorphisms that act by conjugation on each of the generators $c_i$ of $\pi_{g,n}$ (for the presentation (1.4.1)).*

We refer the reader to [FM12, §8] for more considerations on the Dehn-Nielsen Theorem. Theorem 7.2.2 implies that the $\mathrm{Aut}(\pi_{g,0})$-action on the representation variety $\mathrm{Hom}(\pi_{g,0}, G)$ induces an action

$$\mathrm{Mod}(\Sigma_{g,0}) \circlearrowright \mathrm{Hom}(\pi_{g,0}, G)/\,\mathrm{Inn}(G).$$

The action is analytic/algebraic on the regular part of the quotient by Lemma 2.2.1. In the case of a punctured surface, the action of $\mathrm{Aut}(\pi_{g,n})$ on $\mathrm{Hom}(\pi_{g,n}, G)$ restricts to an action of $\mathrm{Aut}^\star(\pi_{g,n})$ on any relative representation variety $\mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$. This gives, by Theorem 7.2.2, an action

$$\mathrm{Mod}(\Sigma_{g,n}) \circlearrowright \mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)/\,\mathrm{Inn}(G),$$

for any choice of conjugacy classes $\mathcal{C} \in (G/G)^n$. These two actions are what we call the *mapping class group action on character varieties.*

## 7.3 Properties of the mapping class group action

The first property is that the mapping class group action preserves the Goldman symplectic form. We start with the case of a closed surface. Let $[f] \in \mathrm{Mod}(\Sigma_{g,0})$ and take any $\tau \in \mathrm{Aut}(\pi_{g,0})$

that lies over the image of $[f]$ inside $\mathrm{Out}(\pi_{g,0})$. We choose the generator $[\pi_{g,0}]$ of $H_2(\pi_{g,0}, \mathbb{Z})$ that corresponds to the orientation of the surface $\Sigma_{g,0}$. Since $f$ is orientation-preserving, it holds that $\tau_*[\pi_{g,0}] = [\pi_{g,0}]$. For any $\phi \in \mathrm{Hom}(\pi_{g,0}, G)$, the automorphism $\tau$ induces a map $(d\tau)_\phi \colon Z^1(\pi_{g,0}, \mathfrak{g}_\phi) \to Z^1(\pi_{g,0}, \mathfrak{g}_{\phi\circ\tau})$, $v \mapsto v \circ \tau$, on the Zariski tangent spaces to the representation variety.

**Lemma 7.3.1.** *If $\omega_\mathcal{G}$ denotes the Goldman symplectic form from Definition 5.1.2, then, for any $\phi \in \mathrm{Hom}(\pi_{g,0}, G)$, the following diagram commutes*

$$\begin{tikzcd}
Z^1(\pi_{g,0}, \mathfrak{g}_\phi) \times Z^1(\pi_{g,0}, \mathfrak{g}_\phi) \arrow[r, "(\omega_\mathcal{G})_\phi"] \arrow[d, "(d\tau)_\phi \times (d\tau)_\phi"'] & \mathbb{R} \\
Z^1(\pi_{g,0}, \mathfrak{g}_{\phi\circ\tau}) \times Z^1(\pi_{g,0}, \mathfrak{g}_{\phi\circ\tau}) \arrow[ur, bend right, "(\omega_\mathcal{G})_{\phi\circ\tau}"'] &
\end{tikzcd}$$

*In other words, it holds that*

$$\tau^*\omega_\mathcal{G} = \omega_\mathcal{G}.$$

*Proof.* Let $B \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the pairing used in the definition of $\omega_\mathcal{G}$. For any $v, w \in Z^1(\pi_{g,0}, \mathfrak{g}_\phi)$, we have

$$\begin{aligned}
(\omega_\mathcal{G})_{\phi\circ\tau}(v \circ \tau, w \circ \tau) &= B(v \circ \tau, w \circ \tau) \frown [\pi_{g,0}] \\
&= B(v, w) \frown \tau_*[\pi_{g,0}].
\end{aligned}$$

Since $\tau_*[\pi_{g,0}] = [\pi_{g,0}]$, we conclude $(\omega_\mathcal{G})_{\phi\circ\tau}(v \circ \tau, w \circ \tau) = (\omega_\mathcal{G})_\phi(v, w)$. □

As a consequence of Lemma 7.3.1, we obtain that the $\mathrm{Mod}(\Sigma_{g,0})$-action on the quotient $\mathrm{Hom}(\pi_{g,0}, G)/\mathrm{Inn}(G)$ preserves the Goldman symplectic measure $\nu_\mathcal{G}$ from Definition 5.4.1.

The situation is similar for punctured surfaces. Let $[f] \in \mathrm{Mod}(\Sigma_{g,n})$ and take any $\tau \in \mathrm{Aut}^\star(\pi_{g,n})$ that lies over the image of $[f]$ inside $\mathrm{Out}^\star(\pi_{g,n})$. The generator $[\pi_{g,n}, \partial\pi_{g,n}]$ of $H_2(\pi_{g,n}, \partial\pi_{g,n}, \mathbb{Z})$ is again chosen to correspond to the orientation of the surface $\Sigma_{g,n}$. Similarly as before, $\tau_*[\pi_{g,n}, \partial\pi_{g,n}] = [\pi_{g,n}, \partial\pi_{g,n}]$. Moreover, the map $(d\tau)_\phi$ restricts to to a map $(d\tau)_\phi \colon Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \to Z^1_{par}(\pi_{g,n}, \mathfrak{g}_{\phi\circ\tau})$. Indeed, note that if $v(c_i) = \xi_i - \mathrm{Ad}(\phi(c_i))\xi_i$ and $\tau(c_i) = g_i c_i g_i^{-1}$, then

$$(v \circ \tau)(c_i) = \big(v(g_i) + \mathrm{Ad}(\phi(g_i))\xi_i\big) - \mathrm{Ad}\,\big((\phi \circ \tau)(c_i)\big)\big(v(g_i) + \mathrm{Ad}(\phi(g_i))\xi_i\big).$$

**Lemma 7.3.2.** *If $\omega_\mathcal{G}$ denotes the Goldman symplectic form from Definition 5.3.9, then, for any $\phi \in \mathrm{Hom}_\mathcal{C}(\pi_{g,n}, G)$, the following diagram commutes*

$$\begin{tikzcd}
Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \times Z^1_{par}(\pi_{g,n}, \mathfrak{g}_\phi) \arrow[r, "(\omega_\mathcal{G})_\phi"] \arrow[d, "(d\tau)_\phi \times (d\tau)_\phi"'] & \mathbb{R} \\
Z^1_{par}(\pi_{g,n}, \mathfrak{g}_{\phi\circ\tau}) \times Z^1_{par}(\pi_{g,n}, \mathfrak{g}_{\phi\circ\tau}) \arrow[ur, bend right, "(\omega_\mathcal{G})_{\phi\circ\tau}"'] &
\end{tikzcd}$$

*In other words, it holds that*

$$\tau^* \omega_{\mathcal{G}} = \omega_{\mathcal{G}}.$$

The proof is analogous to the proof of Lemma 7.3.1.

The second property is that the mapping class group action also preserves the Toledo number of a representation. As before, let $[f] \in \mathrm{Mod}(\Sigma_{g,n})$ and take any $\tau \in \mathrm{Aut}^\star(\pi_{g,n})$ that lies over the image of $[f]$ inside $\mathrm{Out}^\star(\pi_{g,n})$. Again, $\tau_*[\pi_{g,n}, \partial \pi_{g,n}] = [\pi_{g,n}, \partial \pi_{g,n}]$.

**Lemma 7.3.3.** *Let $G$ be a Hermitian Lie group. For any $\phi \in \mathrm{Hom}_{\mathcal{C}}(\pi_{g,n}, G)$, it holds that*

$$\mathrm{Tol}(\phi \circ \tau) = \mathrm{Tol}(\phi).$$

*Proof.* We compute directly from Definition 6.3.3 that

$$\begin{aligned} \mathrm{Tol}(\phi \circ \tau) &= j^{-1}\big((\phi \circ \tau)^* \kappa\big) \frown [\pi_{g,n}, \partial \pi_{g,n}] \\ &= j^{-1}\big(\tau^* \phi^* \kappa\big) \frown [\pi_{g,n}, \partial \pi_{g,n}] \\ &= j^{-1}\big(\phi^* \kappa\big) \frown \tau_*[\pi_{g,n}, \partial \pi_{g,n}]. \end{aligned}$$

We conclude by using $\tau_*[\pi_{g,n}, \partial \pi_{g,n}] = [\pi_{g,n}, \partial \pi_{g,n}]$. □

# Appendix A

# The group $\mathrm{PSL}(2,\mathbb{R})$

## A.1 Generalities

We introduced $\mathrm{SL}(2,\mathbb{R})$ as the subgroup of $\mathrm{SL}(2,\mathbb{C})$ consisting of real matrices in Section 1.2.2. The group $\mathrm{SL}(2,\mathbb{R})$ is Zariski dense inside $\mathrm{SL}(2,\mathbb{C})$ by Theorem 1.2.5 (actually, even the group $\mathrm{SL}(2,\mathbb{Z})$ is Zariski dense in $\mathrm{SL}(2,\mathbb{C})$). The maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$ is $\mathrm{SO}(2)$. Note that $\mathrm{SO}(2)$ is Zariski closed inside $\mathrm{SL}(2,\mathbb{R})$ by Theorem 1.2.6, but the Zariski closure of $\mathrm{SO}(2)$ inside $\mathrm{SL}(2,\mathbb{C})$ is $\mathrm{SO}(2,\mathbb{C})$. The center of $\mathrm{SL}(2,\mathbb{R})$ is $Z(\mathrm{SL}(2,\mathbb{R}))=\{\pm I\}$. The center-free quotient $\mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ is the adjoint group of $\mathrm{SL}(2,\mathbb{R})$ and is traditionally denoted by $\mathrm{PSL}(2,\mathbb{R})$. If $A\in\mathrm{SL}(2,\mathbb{R})$, then we will denote by $\pm A$ its projection inside $\mathrm{PSL}(2,\mathbb{R})$. Even if $\mathrm{PSL}(2,\mathbb{R})$ is not group of $2\times 2$ matrices, it turns out that it can be realized as a liner Lie group of $\times 3$ matrices.

**Lemma A.1.1.** *The group* $\mathrm{PSL}(2,\mathbb{R})$ *can be identified with the conjugate of the matrix group* $\mathrm{SO}(2,1)^\circ$ *that consists of special linear transformations of* $\mathbb{R}^3$ *preserving the Hermitian form* $y^2-xz$ *via the map*

$$\pm\begin{pmatrix} a & b \\ c & d\end{pmatrix}\mapsto\begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2\end{pmatrix}.$$

Lemma A.1.1 highlights the hyperbolic nature of $\mathrm{PSL}(2,\mathbb{R})$. More precisely, $\mathrm{PSL}(2,\mathbb{R})$ can be identified with the group of orientation-preserving isometries of the upper half-plane $\mathbb{H}=\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}$. It acts on $\mathbb{H}$ by Möbius transformations

$$\pm\begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot z:=\frac{az+b}{cz+d}.$$

**Lemma A.1.2.** *The group* $\mathrm{PSL}(2,\mathbb{R})$ *has the topology of an open solid torus.*

*Proof.* The transitive action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}$ extends to a transitive action on the unit tangent bundle $T^1\mathbb{H}$. It is not too hard to see that the stabilizers of points for the action of $\mathrm{PSL}(2,\mathbb{R})$ on $T^1\mathbb{H}$ are trivial. We conclude that $\mathrm{PSL}(2,\mathbb{R})$ and $T^1\mathbb{H}$ are homeomorphic. □

## A.2 Conjugacy classes

The action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}$ extends to the boundary $\partial\mathbb{H}$.

**Lemma A.2.1.** *The action of* $\mathrm{PSL}(2,\mathbb{R})$ *on* $\partial\mathbb{H}$ *is isomorphic to the projective action of* $\mathrm{PSL}(2,\mathbb{R})$ *on* $\mathbb{RP}^1 = \mathbb{R}^2/\mathbb{R}^\times$.

*Proof.* Identifying $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$, one can define a homeomorphism $f\colon \partial\mathbb{H} \to \mathbb{RP}^1$ by $x \mapsto [1:x]$ and $\infty \mapsto [0:1]$. We claim that $f$ conjugates the two actions of $\mathrm{PSL}(2,\mathbb{R})$. Indeed, it is sufficient to compare stabilizers, and it is easy to see that the stabilizers of $[1:0] \in \mathbb{RP}^1$ and of $0 \in \partial\mathbb{H}$ coincide with the subgroup of upper triangular matrices in $\mathrm{PSL}(2,\mathbb{R})$. ☐

**Definition A.2.2.** The open subspace of $\mathrm{PSL}(2,\mathbb{R})$ consisting of elements whose trace in absolute value is smaller than 2 is called the subspace of *elliptic* elements of $\mathrm{PSL}(2,\mathbb{R})$. It is denoted $\mathcal{E} \subset \mathrm{PSL}(2,\mathbb{R})$. Equivalently, an element of $\mathrm{PSL}(2,\mathbb{R})$ is elliptic if and only if it has a unique fixed point in $\mathbb{H}$.

**Lemma A.2.3.** *If* $A = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is elliptic, then* $b \neq 0$ *and* $c \neq 0$.

*Proof.* If $b = 0$ or $c = 0$, then $\det(A) = ad = 1$. So, $\mathrm{Tr}(A)^2 = (a+d)^2 \geqslant 4ad = 4$ and $A$ is not elliptic. ☐

Let $A = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an elliptic element of $\mathrm{PSL}(2,\mathbb{R})$. The association of $A$ to its unique fixed point $\mathrm{fix}(A) \in \mathbb{H}$ defines a map $\mathrm{fix}\colon \mathcal{E} \to \mathbb{H}$.

**Lemma A.2.4.** *The unique fixed point of* $A$ *is*

$$\mathrm{fix}(A) = \frac{a-d}{2c} + i \cdot \frac{\sqrt{4-(a+d)^2}}{2|c|}, \tag{A.2.1}$$

*and the map* $\mathrm{fix}\colon \mathcal{E} \to \mathbb{H}$ *is analytic.*

*Proof.* The first assertion is a straightforward computation. Since $c \neq 0$ by Lemma A.2.3, the map $\mathrm{fix}\colon \mathcal{E} \to \mathbb{H}$ is analytic. ☐

The elliptic elements of $\mathrm{PSL}(2,\mathbb{R})$ that fix the complex unit $i \in \mathbb{H}$ are of the form

$$\mathrm{rot}_\vartheta := \pm\begin{pmatrix} \cos(\vartheta/2) & \sin(\vartheta/2) \\ -\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \tag{A.2.2}$$

for $\vartheta \in (0, 2\pi)$. Every $A \in \mathcal{E}$ is conjugate to a unique $\mathrm{rot}_{\vartheta(A)}$. This defines a function $\vartheta\colon \mathcal{E} \to (0,2\pi)$. The number $\vartheta(A) \in (0,2\pi)$ is called the *angle of rotation* of $A$.

**Lemma A.2.5.** *The angle of rotation of* $A$ *is*

$$\vartheta(A) = \arctan\left(\frac{-c}{|c|} \cdot \frac{a+d}{(a+d)^2-2}\sqrt{4-(a+d)^2}\right) + \varepsilon(A), \tag{A.2.3}$$

*where*

$$\varepsilon(A) := \begin{cases} 0, & \text{if } (a+d)^2 > 2 \text{ and } (a+d)\frac{-c}{|c|} > 0, \\ \pi, & \text{if } (a+d)^2 < 2, \\ 2\pi, & \text{if } (a+d)^2 > 2 \text{ and } (a+d)\frac{-c}{|c|} < 0. \end{cases}$$

*Moreover, the function $\vartheta\colon \mathcal{E} \to (0, 2\pi)$ is analytic.*

*Proof.* The number $\vartheta(A)$ can be computed as the complex argument of the complex number

$$\left.\frac{dA}{dz}\right|_{z=\operatorname{fix} A} = \left(\frac{(a+d)^2}{2} - 1\right) - i \cdot (a+d)\frac{c}{|c|}\frac{\sqrt{4-(a+d)^2}}{2}. \tag{A.2.4}$$

Observe that the imaginary part of (A.2.4) vanishes if and only if $a + d = 0$, in which case its real part is equal to $-1$. This means that the complex number defined by (A.2.4) takes values inside $\mathbb{C} \smallsetminus \mathbb{R}_{\geqslant 0}$. If we think of the complex argument of a number inside $\mathbb{C} \smallsetminus \mathbb{R}_{\geqslant 0}$ as a function $\mathbb{C} \smallsetminus \mathbb{R}_{\geqslant 0} \to (0, 2\pi)$, then it is analytic. This shows that $\vartheta\colon \mathcal{E} \to (0, 2\pi)$ is an analytic function. □

**Lemma A.2.6.** *The map*

$$(\operatorname{fix}, \vartheta)\colon \mathcal{E} \to \mathbb{H} \times (0, 2\pi)$$

*is an analytic diffeomorphism that identifies the subset of elliptic elements in* $\mathrm{PSL}(2,\mathbb{R})$ *with an open ball.*

*Proof.* We explained above that the map $(\operatorname{fix}, \vartheta)$ is analytic. The inverse map sends a point $z = x + i \cdot y \in \mathbb{H}$ and an angle $\vartheta \in (0, 2\pi)$ to the elliptic element

$$\operatorname{rot}_\vartheta(z) = \pm \begin{pmatrix} \cos(\vartheta/2) - xy^{-1}\sin(\vartheta/2) & (x^2y^{-1} + y)\sin(\vartheta/2) \\ -y^{-1}\sin(\vartheta/2) & \cos(\vartheta/2) + xy^{-1}\sin(\vartheta/2) \end{pmatrix}. \tag{A.2.5}$$

Indeed, an immediate computation gives

$$\begin{aligned} \operatorname{fix}(\operatorname{rot}_\vartheta(z)) &= \frac{-2xy^{-1}\sin(\vartheta/2)}{-2y^{-1}\sin(\vartheta/2)} + i \cdot \frac{2\sin(\vartheta/2)}{2y^{-1}\sin(\vartheta/2)} \\ &= x + iy, \end{aligned}$$

and

$$\begin{aligned} \vartheta(\operatorname{rot}_\vartheta(z)) &= \arg\left(\left(\frac{4\cos(\vartheta/2)^2}{2} - 1\right) - i \cdot (2\cos(\vartheta/2)) \cdot (-1) \cdot \frac{2\sin(\vartheta/2)}{2}\right) \\ &= \arg(\cos(\vartheta) + i\sin(\vartheta)) \\ &= \vartheta. \end{aligned}$$

□

**Definition A.2.7.** The elements of $\mathrm{PSL}(2,\mathbb{R})$ whose trace in absolute value is equal to 2 are called *parabolic*. Parabolic elements are those that have a unique fixed point of the boundary of $\mathbb{H}$. There are two conjugacy classes of parabolic elements represented by

$$\mathrm{par}^+ := \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{par}^- := \pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{A.2.6}$$

The elements conjugate to $\mathrm{par}^+$ are called *positively parabolic* and those conjugate to $\mathrm{par}^-$ *negatively parabolic*. Each conjugacy class of parabolic elements is an open annulus whose closures intersect at the identity.

The elements of $\mathrm{PSL}(2, \mathbb{R})$ with a trace larger than 2 in absolute value are called *hyperbolic*. Hyperbolic elements have precisely two fixed points on the boundary of $\mathbb{H}$. A hyperbolic element of $\mathrm{PSL}(2, \mathbb{R})$ is always conjugate to a diagonal element

$$\mathrm{hyp}_\lambda := \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

for a unique $\lambda > 0$. Hyperbolic conjugacy classes are open annuli.

Elliptic, parabolic, and hyperbolic conjugacy classes foliate $\mathrm{PSL}(2, \mathbb{R})$ in a way that is illustrated on Figure A.1.

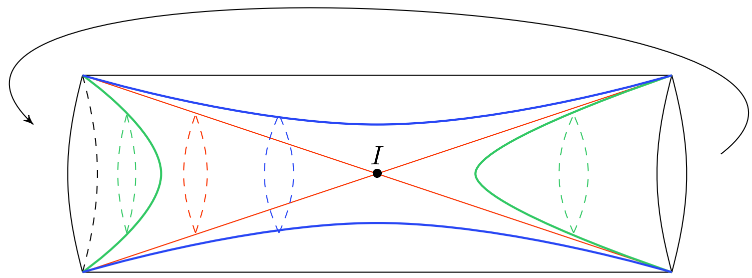



Figure A.1: The elliptic conjugacy classes are drawn in green. They foliate an open ball into disks. The open ball is bounded by the two parabolic conjugacy classes which have the shape of two red cones joined at the identity. The hyperbolic conjugacy classes foliate an open solid torus, bounded by the red cones, into blue annuli.

The next lemma describes the centralizers of elements of $\mathrm{PSL}(2, \mathbb{R})$ according to their conjugacy class.

**Lemma A.2.8.** *The centralizers of* $\mathrm{rot}_\vartheta$, $\mathrm{hyp}_\lambda$ *and* $\mathrm{par}^+$ *are given by*

1. $Z(\mathrm{rot}_\vartheta) = \{\mathrm{rot}_\theta : \theta \in [0, 2\pi)\} \cong \mathrm{PSO}(2, \mathbb{R})$.

2. $Z(\mathrm{hyp}_\lambda) = \{\mathrm{hyp}_t : t > 0\} \cong \mathbb{R}_{>0}$.

3. $Z(\mathrm{par}^+) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \cong \mathbb{R}$.

It is worth noticing that the centralizer of an element of $\mathrm{PSL}(2, \mathbb{R})$ always consists of the identity element and of elements of the same nature (i.e. elliptic, parabolic, and hyperbolic). In particular, two elements of $\mathrm{PSL}(2, \mathbb{R})$ different from the identity commute if and only if they have the same set of fixed points in $\mathbb{H} \cup \partial\mathbb{H}$.

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