TOTALLY ELLIPTIC SURFACE GROUP REPRESENTATIONS

ARNAUD MARET

Abstract. We characterize all totally elliptic surface group representations into $\mathrm{PSL}_2\,\mathbb{R}$ by showing that they are either representations into a compact subgroup or Deroin–Tholozan representations.

arXiv:submit/6034192 [math.GR] 29 Nov 2024

1. Introduction

1.1. **Main result.** Surface group representations are homomorphisms from the fundamental group of a surface Σ into a Lie group G. Their deformation spaces are called *character varieties*. A character variety is not necessarily a connected space. Dynamical behaviour and qualitative properties of representations may vary considerably between components. At one end of the spectrum are the higher Teichmüller spaces which are a generalization of the classical and extensively studied Teichmüller space of Σ. Higher Teichmüller spaces, as defined in [Wie18], consist of discrete and faithful representations and tend to behave like moduli spaces of geometric structures on Σ, although a precise correspondence is still unavailable. This article aims at drawing the reader's attention towards the other end of the spectrum where connected components are mostly made of non-discrete representations.

We'll focus on a particular prototype of representations that are antipodal to discrete and faithful ones: the so-called totally elliptic representations.

Definition 1.1. A surface group representation $\pi_1\Sigma \to G$ is *totally elliptic* if every simple closed curve on Σ is mapped to an elliptic element of G. By an *elliptic* element of G, we mean an element of a compact subgroup of G.[1]

The surface Σ will always be taken to be oriented and connected, and can be punctured. Recall that a *simple closed curve* on Σ is the free isotopy class of a closed loop on Σ that doesn't self-intersect and isn't homotopically trivial. A simple closed curve can be homotopic to a puncture, in which case we call it *peripheral*. By definition, the images of every peripheral curve by a totally elliptic representation are elliptic. We want to avoid the situation where peripheral curves are mapped to the identity and we say that a representation is *reduced* when this isn't the case (Definition 2.7).

The first instances of totally elliptic representations, though not the most interesting, are those into a compact subgroup of G. Interestingly, however, by requiring only simple closed curves to be mapped to elliptic elements (as opposed to all curves), one can construct Zariski dense examples of totally elliptic representations into non-compact Lie groups, such as every Hermitian Lie group (Section 1.3), whose deformations remain totally elliptic.

The initial case to investigate is when G is $\mathrm{PSL}_2\,\mathbb{R}$. In that case, the first totally elliptic representations with Zariski dense image were discovered by Benedetto–Goldman [BG99], and later generalized by Deroin–Tholozan [DT19]. Those representations were originally named *supra-maximal* in [DT19], but we'll refer to them as *DT representations* (Definition 2.10), to avoid any confusion with maximal representations from higher Teichmüller theory. DT representations only exist when Σ is a genus-0 surface with at least three punctures. As Deroin–Tholozan observed, their deformation spaces are compact components inside the corresponding relative character varieties and made exclusively of totally elliptic representations (Theorem 2.9). The question was raised whether they're the only totally elliptic representations into $\mathrm{PSL}_2\,\mathbb{R}$, besides representations into compact subgroups. We bring a positive answer to that question.

Theorem A. *Let Σ be a connected and oriented surface with genus $g \geq 0$ and $n \geq 0$ punctures. Reduced totally elliptic representations $\rho\colon \pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ can only be of two kinds:*

- *The image of ρ is contained in a compact subgroup of $\mathrm{PSL}_2\,\mathbb{R}$. In that case, we say that ρ is an* orthogonal representation.
- *If ρ is not orthogonal, then $g = 0$, $n \geq 3$, and ρ is a DT representation.*

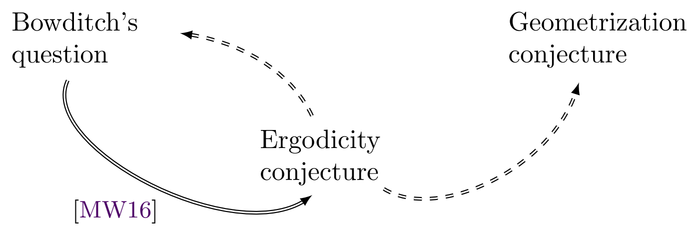
We prove Theorem A in Section 2.

[1] Those elements are sometimes called *compact* in the literature.

1.2. **Motivations.** Characterizing a certain type of surface group representations (such as totally elliptic ones) by the images of *simple* closed curves may seem baroque at first. Similar problems, however, have sparked substantial interest in recent decades. Bowditch famously raised the question of whether discrete and faithful representations of a closed surface group $\pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$—also known as *Fuchsian representations*—can be characterized by the property of mapping every simple closed curve on Σ to a hyperbolic element of $\mathrm{PSL}_2\,\mathbb{R}$ (an element whose trace is larger than 2 in absolute value) [Bow98, Question C]. If true, this would be a finer characterization of Fuchsian representations which are known to be the only representations that map every non-trivial element of $\pi_1\Sigma$ to hyperbolic elements of $\mathrm{PSL}_2\,\mathbb{R}$. Bowditch's question has been answered positively for surfaces of genus 2 by Marché–Wolff [MW16, MW19] and remains open in higher genera.

Of course, not every representation is Fuchsian. Goldman proved that the character variety of representations $\pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$, where Σ is a closed surface of genus $g \geq 2$, has $4g-3$ connected components [Gol88]. Two of the components are made of conjugacy classes of Fuchsian representations, forming two copies of the Teichmüller space of Σ; the other $4g-5$ components are known as *intermediate components* and they don't contain any Fuchsian representation. Bowditch's question is equivalent to asking whether every representation whose conjugacy class lies in an intermediate component maps at least one simple closed curve on Σ to a non-hyperbolic element of $\mathrm{PSL}_2\,\mathbb{R}$. Marché–Wolff also proved that a positive answer to Bowditch's question would imply that the mapping class group action on every intermediate component is ergodic [MW16, Theorem 1.6]. The latter statement is known as Goldman's ergodicity conjecture [Gol06, Conjecture 3.1] and is open, like Bowditch's question, for surfaces of genus at least 3.

Goldman also conjectured that almost every non-Fuchsian closed surface group representation $\pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ should be the holonomy of a *branched hyperbolic structure* on Σ which is the data of hyperbolic metric of Σ with several conical singularities of angle $2\pi k$ with $k \geq 2$. To be precise, according to Goldman, every dense representation whose conjugacy class lies in an intermediate component should be the holonomy of a branched hyperbolic structure [Gol06, Conjecture 3.9]. Mathews proved some partial implication from a positive answer to Bowditch's question to Goldman's geometrization conjecture for intermediate components of co-Euler class 1 [Mat12]. Observe as well that Goldman's ergodicity conjecture implies a "full measure version" of the geometrization conjecture, as well as a positive answer to Bowditch's question on a full measure subset of intermediate components.



Goldman's two conjectures and Bowditch's question, and the interplay among them, constitute a whole program to study intermediate components which we refer to as the *Bowditch–Goldman program.*

Totally elliptic representations, and more particularly DT representations, naturally arise as a prototype of surface group representations that fit perfectly in the Bowditch–Goldman program. By definition, they map every simple closed curve to an elliptic element of $\mathrm{PSL}_2\,\mathbb{R}$, and therefore satisfy the analogue of Bowditch's condition on intermediate representations in a strong sense. The ergodicity of the pure mapping class group action on the character variety of DT representations has been established in [Mar22]. As explained in [DT19, Section 4] and further detailed in [FM23], DT representations are also holonomies of hyperbolic cone structures with branch point singularities, thereby satisfying a precise version of the geometrization conjecture.

1.3. **Perspectives away from** $\mathrm{PSL}_2\,\mathbb{R}$**.** This note is also an opportunity to survey some recent developments about totally elliptic representations into Lie groups G different than $\mathrm{PSL}_2\,\mathbb{R}$. Several families of compact connected components of totally elliptic representations were identified when G is a Hermitian Lie group. They're all genus-0 surface group representations. The first examples were found by Tholozan–Toulisse when G is the Lie group $\mathrm{SU}(p,q)$, as well as when G is $\mathrm{Sp}(2n,\mathbb{R})$ or $\mathrm{SO}^*(2n)$ [TT21]. Analogous compact components have been identified by Feng–Zhang when G is the identity component of $\mathrm{SO}(2,n)$ [FZ23]. Their methods studies the topology of some moduli spaces of parabolic Higgs bundles which relate to relative character varieties via the non-abelian Hodge correspondence. In particular, they proved that for every representation $\rho\colon \pi_1\Sigma \to G$ whose conjugacy class belongs to one of the compact components they identified, and for every punctured Riemann sphere structure on Σ, there exists a ρ-equivariant holomorphic map from the universal cover $\widetilde{\Sigma}$ to the symmetric space G/K. The same property for DT representations was proven by Mondello [Mon16]. Surface group representations satisfying this condition are sometimes called *universal variations of Hodge structures*.

Unlike DT representations which are all Zariski dense, it's not clear yet whether every compact component of totally elliptic representations into Hermitian Lie groups other than $\mathrm{PSL}_2\,\mathbb{R}$ contains a Zariski dense representation (almost all of them do though, by the preponderance of Zariski dense representations in any character variety). The topology of those compact components is also yet to be determined, whereas DT components are known to be homeomorphic (even symplectomorphic) to complex projective spaces [DT19, Theorem 4].

There are however more explicit examples of compact components of totally elliptic representations discovered by Marc-Zwecker [MZ24]. Marc-Zwecker's examples are made of representations of a pair of pants into $\mathrm{SU}(2,1)$ where the peripheral curves are mapped to products of reflections through three complex geodesic lines in the complex hyperbolic space of real dimension four. The three complex geodesic lines are the *mirrors* and they're arranged as a triangle. The resulting components are homeomorphic to 2-dimensional spheres. The geometric nature of the representations in Marc-Zwecker's components may lead to other compact components for spheres with a higher number of punctures obtained by "chaining" the triangles of mirrors in the spirit of [Mar24].

A natural question to ask is whether the class of Lie groups for which there exist substantial connected components entirely made of totally elliptic genus-0 surface group representations is larger than the class of Hermitian Lie groups.

Question 1. *Are there semisimple real Lie groups other than Hermitian Lie groups for which there exist compact connected components (of maximal expected dimension) inside relative character varieties of genus-0 surface group representations made of totally elliptic representations and containing Zariski dense representations?*

There are reasons to believe that the key property is the presence of a nonempty open subset of elliptic elements in the target Lie group. This property is shared by all Hermitian Lie groups, but not only; for instance, Sugiura proved that $\mathrm{SO}(2p,q)$ has an open subset of elliptic elements [Sug59, Theorem 8], even though it's not Hermitian if $p \geq 2$ and $q \geq 3$. In contrast, the subset of elliptic elements in $\mathrm{SL}_n\,\mathbb{R}$ has empty interior for every $n \geq 3$. Furthermore, Kabenyuk proved that admitting a nonempty open subset of elliptic elements is equivalent to having a compact Cartan subgroup [Kab90].

One should nevertheless be aware that "large" Lie groups, unlike $\mathrm{SL}_2\,\mathbb{C}$, may admit subgroups Γ containing only elliptic elements but such that Γ is not contained into a compact subgroup. Bass gives an example of such a Γ which is isomorphic to a rank-2 free subgroup of $\mathrm{SL}_3\,\mathbb{C}$ [Bas80, Counterexample 1.10]. Bass' example is not Zariski dense and, in fact, it couldn't be as explained for instance in [Pra94]. However, when $\mathbb{C}$ is replaced by a non-Archimedean valued field $\mathbb{F}$, then the situation is similar to the case of $\mathrm{SL}_2\,\mathbb{C}$ as Parreau proved that a finitely generated subgroup of $\mathrm{SL}_n\mathbb{F}$ containing only elliptic elements is necessarily bounded [Par03, Théorème 1]. This doesn't rule out the existence of interesting totally elliptic surface group representations into $\mathrm{SL}_n\mathbb{F}$.

1.4. **Acknowledgments.** Many thanks to Samuel Bronstein and Nicolas Tholozan for enriching conversations on totally elliptic representations. I'm grateful to Bertrand Deroin for sparking my curiosity about the conjectures related to the Bowditch–Goldman program, and to Maxime Wolff for teaching me the elegant geometric argument used to prove Lemma 2.1. I also extend my gratitude to my friends Jacques Audibert and Xenia Flamm for their insightful comments on an earlier draft of this paper and for pointing out to me Parreau's paper.

2. Proof of Theorem A

2.1. **Overview.** The proof of Theorem A unfolds over the next subsections. We start by showing that when the genus of Σ is at least 1, then the only totally elliptic representations are orthogonal representations (Section 2.2). The arguments mostly rely on the observation that a commutator of elliptic elements in $\mathrm{PSL}_2\,\mathbb{R}$ is itself elliptic only when it's trivial (Lemma 2.1). We then briefly recall what DT representations are (Section 2.3) and prove that a totally elliptic genus-0 surface group representation that isn't orthogonal must be a DT representation (Sections 2.3.1—2.3.3).

2.2. **Positive genus.** We start by proving that a totally elliptic representation $\rho\colon \Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ of a surface Σ of genus $g \geq 1$ is necessarily an orthogonal representation (Propositions 2.5 and 2.8). Most of the reasoning will rely on the following standard results about $\mathrm{PSL}_2\,\mathbb{R}$, which we include for the sake of completeness.

According to Definition 1.1, the elliptic elements of $\mathrm{PSL}_2\,\mathbb{R}$ fall into two categories: the regular elliptic elements and the identity. The *regular* elliptic elements are those whose trace is smaller than 2 in absolute value. Geometrically, they act on the hyperbolic plane $\mathbb{H}$ by rotations around their unique fixed point. The subset of regular elliptic elements in $\mathrm{PSL}_2\,\mathbb{R}$ is homeomorphic to the open ball $(0, 2\pi) \times \mathbb{H}$, where the first factor records the *rotation angle* of a regular elliptic and the second its unique fixed point (the centre of rotation).

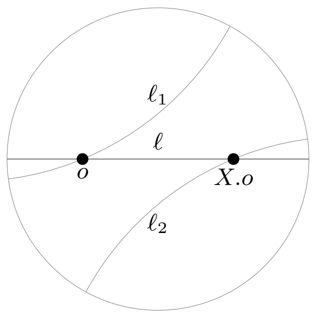
Lemma 2.1. *A regular elliptic element A of $\mathrm{PSL}_2\,\mathbb{R}$ is contained in a unique maximal compact subgroup which consists of all the elements of $\mathrm{PSL}_2\,\mathbb{R}$ which fix the unique fixed point of A. Furthermore, if X is any element of $\mathrm{PSL}_2\,\mathbb{R}$, then the commutator*

$$[A, X] = AXA^{-1}X^{-1}$$

is either the identity or a hyperbolic element of $\mathrm{PSL}_2\,\mathbb{R}$. The commutator is trivial if and only if X is elliptic and fixes the unique fixed point of A.

Proof. The unique maximal compact subgroup of $\mathrm{PSL}_2\,\mathbb{R}$ that contains A is its centralizer, which is conjugate to $\mathrm{PSO}(2)$. Since A is regular, any element that commutes with A is either the identity or a regular elliptic with the same fixed point as A.

Let's now consider the commutator $[A, X]$. As we just said, if $[A, X] = 1$, then X is elliptic and fixes the unique fixed point of A. If $[A, X] \neq 1$ and o denotes the unique fixed point of A, then $XA^{-1}X^{-1}$ is regular elliptic with fixed point $X.o \neq o$. Note that if α is the rotation angle of A, then the rotation angle of $XA^{-1}X^{-1}$ is $2\pi - \alpha$. Let ℓ denote the unique geodesic line in $\mathbb{H}$ through o and $X.o$. Similarly, let ℓ_1 and ℓ_2 denote the geodesic lines obtained by rotating ℓ anti-clockwise by an angle $\alpha/2$ around o, respectively $X.o$.



The orientation-reversing reflections of $\mathbb{H}$ through the lines ℓ, ℓ_1, and ℓ_2 are denoted by σ, σ_1, and σ_2, respectively. With this notation, we can write $A = \sigma_1\sigma$ and $XA^{-1}X^{-1} = \sigma\sigma_2$, so that $[A, X] = \sigma_1\sigma_2$. We conclude that $[A, X]$ preserves the common orthogonal geodesic line to ℓ_1 and ℓ_2, and is therefore hyperbolic. □

Lemma 2.1 plays a central role in our discussion because it tells us that the commutator of two elliptic elements is elliptic if and only if it is trivial. It turns out that the commutator of two simple closed curves (seen as elements of $\pi_1\Sigma$) sometimes represents a simple closed curve too. We must, however, be careful, because a simple closed curve on Σ is represented by many elements in $\pi_1\Sigma$, all conjugate to each other.

Lemma 2.2. *If two simple closed curves on Σ intersect once (their geometric intersection number is 1) and we represent them in the simplest way by two fundamental group elements a and b with basepoint being the intersection point, then $[a, b]$ also represents a simple closed curve on Σ.*

Proof. Pick two loops on Σ that only intersect at the basepoint and whose homotopy classes are a and b. Enlarge each loop by replacing it with a thin ribbon. The boundary of the union of the two ribbons is an embedded circle in Σ which is in the free homotopy class determined by the fundamental group element $[a, b]$. □

As we're about to see, Lemmas 2.1 and 2.2 together are the main obstruction for constructing totally elliptic surface group representations. Let's start by introducing some notation. We pick a *geometric generating family* for $\pi_1\Sigma$ which consists in $2g + n$ generators $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, and $c_1, \ldots, c_n$ satisfying the unique relation

$$\prod_{i=1}^{g} [a_i, b_i] = \prod_{i=1}^{n} c_i.$$

A possible choice of geometric generators are the homotopy classes of the loops illustrated on Figure 1.

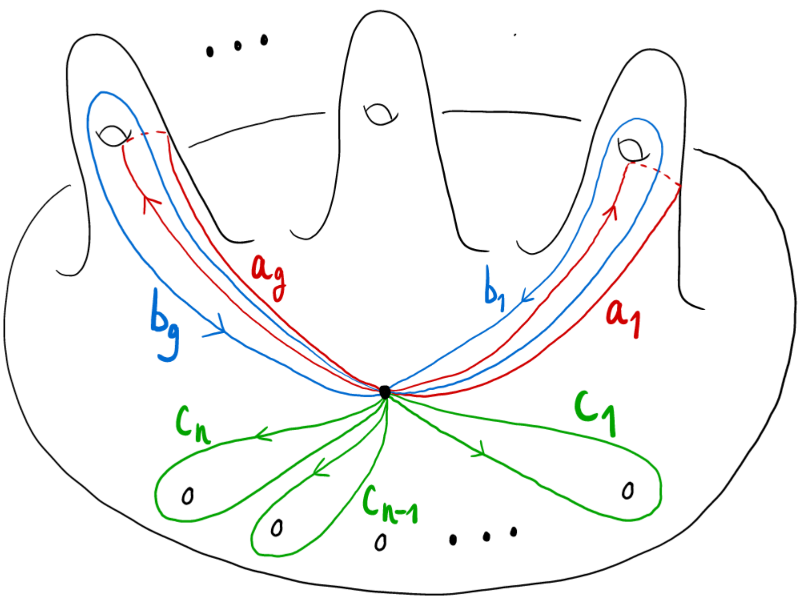


Figure 1. A system of geometric generators for a surface Σ of genus g with n punctures.

Remark 2.3. We adopt the convention of concatenating loops by starting with the rightmost loop. More precisely, if γ_1 and γ_2 are two loops on Σ based at the same point, then their concatenation $\gamma_1\gamma_2$ is the loop obtained by going along γ_2 first, and then along γ_1. This rule applies to fundamental group multiplications too.

Lemma 2.4. *If the genus g of Σ is at least 2 and $i \neq j$ are two indices in $\{1, \ldots, g\}$, then the following fundamental group elements represent simple closed curves on Σ:*

$$[a_i, b_i], \quad [a_j, b_j], \quad [b_i^{-1}b_j^{-1}, a_i], \quad [a_i^{-1}a_j^{-1}, b_j],$$
$$[b_j^{-1}b_i^{-1}, a_j], \quad [a_j^{-1}a_i^{-1}, b_j], \quad [b_i a_i^{-1} b_j, a_j], \quad [a_i b_i^{-1} a_j, b_j].$$

Proof. Each pair of fundamental group elements appearing in a given commutator represents two simple closed curves on Σ with geometric intersection number 1. So, by Lemma 2.2, we can expect their commutators to represent simple closed curves too. We must, however, be careful when we represent a simple closed curve by a fundamental group element because of basepoint issues. The choice we made ensures that each commutator indeed represents a simple closed curve on Σ. This can be seen by drawing the corresponding curves on Figure 1. $\square$

Proposition 2.5. *Assume that Σ is closed and has genus $g \geq 1$. If $\rho\colon \pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ is a totally elliptic representation, then ρ is an orthogonal representation.*

Proof. We'll write $A_i = \rho(a_i)$ and $B_i = \rho(b_i)$. If $g = 1$, then the image of ρ is generated by A_1 and B_1 which satisfy $[A_1, B_1] = 1$. So, the image of ρ is generated by two commuting elliptic elements, hence contained in a compact subgroup of $\mathrm{PSL}_2\,\mathbb{R}$.

Let's now consider the case $g \geq 2$. If ρ is the trivial representation, then it's orthogonal. If not, then there exists an index $i \in \{1, \ldots, g\}$ such that either $A_i \neq 1$ or $B_i \neq 1$. We'll look at the case where $A_i \neq 1$ (the case where $B_i \neq 1$ is identical). Let $j \neq i$ be another index. Because ρ is totally elliptic, it maps all the simple closed curves from Lemma 2.4 to the identity because of Lemma 2.1. In particular, we have $[A_i, B_i] = 1$ and $[A_i, B_i^{-1}B_j^{-1}] = 1$, implying that $[A_i, B_j] = 1$. Since $[A_i, B_i] = 1$, up to conjugating ρ, we may assume that A_i and B_i belong to $\mathrm{PSO}(2)$. From $[A_i, B_j] = 1$ and our assumption that $A_i \neq 1$, we conclude that $B_j \in \mathrm{PSO}(2)$ by Lemma 2.1.

We also infer from Lemmas 2.1 and 2.4 that $[A_j, B_j] = [A_j, B_j^{-1}B_i^{-1}] = [A_j, B_iA_i^{-1}B_j] = 1$. If $B_j \neq 1$, then we immediately conclude that $A_j \in \mathrm{PSO}(2)$ from $[A_j, B_j] = 1$ by Lemma 2.1. Since $[A_j, B_j] = 1$ and $[A_j, B_j^{-1}B_i^{-1}] = 1$, we deduce that $[A_j, B_i] = 1$ which also implies $A_j \in \mathrm{PSO}(2)$ as soon as $B_i \neq 1$. If both $B_i = 1$ and $B_j = 1$, then we alternatively observe that $[A_i^{-1}, A_j] = [B_iA_i^{-1}B_j, A_j] = 1$ which gives $A_j \in \mathrm{PSO}(2)$ as well, because $A_i \neq 1$.

In conclusion, we proved that $A_i, B_i, A_j, B_j \in \mathrm{PSO}(2)$. Since the indices i and j were picked arbitrarily, we conclude that the image of ρ is contained in $\mathrm{PSO}(2)$, proving that ρ is orthogonal. □

We now consider the case where Σ has punctures. We first identify more simple closed curves.

Lemma 2.6. *If Σ is a surface of genus $g \geq 1$ and with $n \geq 1$ punctures, then for any index $i \in \{1, \ldots, g\}$ and $j \in \{1, \ldots, n\}$ the following fundamental group elements represent simple closed curves on Σ:*

$$a_i^{-1}[a_i, b_i]c_j^{-1}a_ic_j \quad \text{and} \quad b_i[a_i, b_i]c_j^{-1}b_i^{-1}c_j. \tag{2.1}$$

If furthermore $n \geq 2$ and $k \in \{1, \ldots, n\}$ is another index such that $j > k$, then

$$b_ic_kb_i^{-1}c_jb_ia_i^{-1}b_i^{-1}c_k^{-1}b_i^{-1}c_j^{-1} \tag{2.2}$$

represents a simple closed curve too.

Proof. The first fundamental group element in (2.1) can be rewritten as $b_ia_i^{-1}b_i^{-1} \cdot c_j^{-1}a_ic_j$, which is the product of a conjugate of a_i^{-1} and a conjugate of a_i. A similar observation holds for the second fundamental group element in (2.1) which is equal to $b_ia_ib_ia_i^{-1}b_i^{-1} \cdot c_j^{-1}b_i^{-1}c_j$ and is thus a product of conjugates of b_i and b_i^{-1}. By drawing a rapid picture, we convince ourselves that both represent simple closed curves on Σ. We prove in the same way that the fundamental group element (2.2) represents a simple closed curve too. □

Definition 2.7. A surface group representation $\pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ is said to be *reduced* if it doesn't map any peripheral curve on Σ to the identity. In terms of the geometric generators from Figure 1, a representation $\pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ is reduced if it doesn't map any generator c_i to the identity.

Note that every non-reduced totally elliptic surface group representation $\pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ corresponds to a reduced totally elliptic representation $\pi_1\Sigma' \to \mathrm{PSL}_2\,\mathbb{R}$, where Σ' is obtained from Σ by removing the punctures whose peripheral curves are mapped to the identity.

Proposition 2.8. *Assume that Σ is a surface of genus $g \geq 1$ and with $n \geq 1$ punctures. If $\rho\colon \pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ is a reduced totally elliptic representation, then ρ is an orthogonal representation.*

Proof. As before, we'll write $A_i = \rho(a_i)$ and $B_i = \rho(b_i)$, as well as $C_j = \rho(c_j)$. Proposition 2.5 implies that all the A_i and B_i are contained in the same orthogonal subgroup of $\mathrm{PSL}_2\,\mathbb{R}$, which we can assume to be $\mathrm{PSO}(2)$ up to conjugating ρ. In particular, $[A_i, B_i] = 1$ for every index i.

Let's first assume that there exists an index $i \in \{1, \ldots, g\}$ such that $A_i \neq 1$ or $B_i \neq 1$. We'll consider the case where $A_i \neq 1$ (similar arguments apply when $B_i \neq 1$). Using the simple closed curves (2.1) from Lemma 2.6 and since ρ is totally elliptic, we conclude that $A_i^{-1}[A_i, B_i]C_j^{-1}A_iC_j$ is elliptic for every index j. Because $[A_i, B_i] = 1$, we even have that $[A_i^{-1}, C_j^{-1}]$ is elliptic. As both A_i and C_j are elliptic by total ellipticity of ρ, Lemma 2.1 implies that $[A_i, C_j] = 1$ and thus $C_j \in \mathrm{PSO}(2)$ because $A_i \neq 1$. This holds for every index j and therefore the image of ρ is contained in $\mathrm{PSO}(2)$.

Now, let's consider the case where $A_i = B_i = 1$ for every $i \in \{1, \ldots, g\}$. Since we're assuming ρ to be reduced, we must have $n \geq 2$. Using the simple closed curve (2.2) from Lemma 2.6, we deduce that $B_iC_kB_i^{-1}C_jB_iA_i^{-1}B_i^{-1}C_k^{-1}B_i^{-1}C_j^{-1}$ is elliptic for every index i and $j > k$. Since $A_i = B_i = 1$, we even have $[C_k, C_j] = 1$ for every $j > k$ in $\{1, \ldots, n\}$. This implies that ρ is orthogonal for the same reasons as before. □

Propositions 2.5 and 2.8 say that the only examples of reduced totally elliptic representations $\pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ when Σ has positive genus and an arbitrary number of punctures are orthogonal representations.

2.3. **Genus zero.** In order to finish the proof of Theorem A, we shall prove that a reduced totally elliptic genus-0 surface group representation $\rho\colon \pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ is either orthogonal or a DT representation. Clearly, if Σ has at most two punctures, then ρ is orthogonal. Let's now assume that Σ is a sphere with a set $\mathcal{P}$ of $n \geq 3$ punctures. Recall that the simple closed curves on Σ that are homotopic to a puncture are called peripheral. By definition, a reduced totally elliptic representation maps every peripheral curve on Σ to a regular elliptic element. In other words, the conjugacy class $[\rho]$ of a reduced totally elliptic representation $\rho\colon \pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ belongs to an *α-relative character variety*

$$\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$$

which consists of all conjugacy classes of representations mapping peripheral curves around every puncture $p \in \mathcal{P}$ to regular elliptic elements in $\mathrm{PSL}_2\,\mathbb{R}$ of fixed rotation angle $\alpha_p \in (0, 2\pi)$. The numbers α_p are the components of the angle vector $\alpha \in (0, 2\pi)^{\mathcal{P}}$. We'll write

$$|\alpha| = \sum_{p\in\mathcal{P}} \alpha_p.$$

Unlike in the positive genus case, there are Zariski dense totally elliptic representations $\pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ when Σ is an n-punctured sphere. The first examples were discovered via computer simulations by Benedetto–Goldman in the case $n = 4$. Their construction was later generalized by Deroin–Tholozan for spheres with an arbitrary number of punctures.

Theorem 2.9 ([BG99, DT19])**.** *If $|\alpha| < 2\pi$ or $|\alpha| > 2\pi(n-1)$, then $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ contains a compact connected component isomorphic to $\mathbb{CP}^{n-3}$ and entirely made of conjugacy classes of Zariski dense totally elliptic representations that don't contain any simple closed curve in their kernel.*

Definition 2.10. A representation whose conjugacy class belongs to one of the compact components of Theorem 2.9 will be called a *DT representation*. The components themselves will be referred to as *DT components*.

According to Theorem 2.9, DT representations are more than just totally elliptic in the sense of Definition 1.1; they are "regularly totally elliptic" by which we mean that every simple closed curve is mapped to a regular elliptic element of $\mathrm{PSL}_2\,\mathbb{R}$.

DT representations can be characterized by their *Toledo number* in the sense of Burger–Iozzi–Wienhard [BIW10]. Namely, according to Deroin–Tholozan's original definition, a conjugacy class of representations in $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ belongs to a DT component if and only if its Toledo number is

$$\begin{cases} 1 - \frac{|\alpha|}{2\pi}, & \text{when } |\alpha| < 2\pi, \\ (n-1) - \frac{|\alpha|}{2\pi}, & \text{when } |\alpha| > 2\pi(n-1). \end{cases} \tag{2.3}$$

It's worth noticing that the Toledo number of a DT representation is always contained in the interval $(-1, 1) \setminus \{0\}$ and can be arbitrarily close to 0, highlighting another strong contrast with Fuchsian representations.

The topology of relative $\mathrm{PSL}_2\,\mathbb{R}$ character varieties has been studied by Mondello. Among other things, he characterized all their compact components, proving that the only compact connected components that aren't isolated points are DT components.

Corollary 2.11 ([Mon16, Corollary 4.17])**.** *The relative character variety $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ contains at most one compact connected component and it does precisely in the following cases:*

- *$|\alpha| \in 2\pi\,\mathbb{Z}$ and the compact component is an isolated point corresponding to an orthogonal representation.*
- *$|\alpha| \in (0, 2\pi) \cup (2\pi(n-1), 2\pi n)$ and the compact component is the DT component from Theorem 2.9.*

In the context of punctured spheres, Theorem A can now be reformulated as follows. If an element $[\rho] \in \mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ is the conjugacy class of a reduced totally elliptic representation, then α is one of the angle vectors from Corollary 2.11 and $[\rho]$ belongs to the unique compact connected component of $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$. Actually, Deroin–Tholozan already observed that totally elliptic representations belong to a bounded region of $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$.

Proposition 2.12 ([DT19, Proposition 2.5])**.** *The closure of the subset of all conjugacy classes of totally elliptic representations inside $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ is compact.*

Deroin–Tholozan's argument to prove Proposition 2.12 uses that all the representations in a sequence of totally elliptic representations are dominated by the same Fuchsian representation and therefore correspond to a sequence of Lipschitz equivariant maps $\mathbb{H} \to \mathbb{H}$ by a result of Guéritaud–Kassel [GK17]. The conclusion then follows from the Arzelà–Ascoli theorem (see [DT19] for more details).

Even though totally elliptic representations are contained in a bounded region, it could nevertheless be possible *a priori* to have a compact subset of a non-compact component of

$\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ made of totally elliptic representations. To finish the proof of Theorem A, we shall prove that no such compact subsets exist. We'll do so by going over all possible values of n inductively (recall that n is the number of punctures of Σ).

2.3.1. *The case* $n = 3$. A 3-punctured sphere Σ only contains three non-trivial simple closed curves: the three peripheral curves. So any point in $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ is automatically the conjugacy class of a totally elliptic representation. It turns out that when $n = 3$, the relative character variety $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ is either empty or a singleton. More precisely, let's pick a geometric presentation of $\pi_1\Sigma$ with generators c_1, c_2, c_3 satisfying $c_1c_2c_3 = 1$. Now, if ρ is a reduced totally elliptic representation, then the unique fixed points of $\rho(c_1)$, $\rho(c_2)$, and $\rho(c_3)$ form a triangle in $\mathbb{H}$ with vertices (C_1, C_2, C_3). Denote the rotation angles of $\rho(c_1)$, $\rho(c_2)$, and $\rho(c_3)$ by $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (0, 2\pi)^3$. When we study the location of the points (C_1, C_2, C_3), we observe that the triangle they form can only be in three possible configurations, as explained in [Mar24, Section 3.1 and Table 1].

- $|\alpha| \in \{2\pi, 4\pi\}$ and the triangle is degenerate with $C_1 = C_2 = C_3$.
- $|\alpha| < 2\pi$ and the triangle (C_1, C_2, C_3) is anti-clockwise oriented with interior angles given by $(\alpha_1/2, \alpha_2/2, \alpha_3/2)$.
- $|\alpha| > 4\pi$ and the triangle (C_1, C_2, C_3) is clockwise oriented with interior angles given by $(\pi - \alpha_1/2, \pi - \alpha_2/2, \pi - \alpha_3/2)$.

If $|\alpha| \in \{2\pi, 4\pi\}$, then ρ is orthogonal. When $|\alpha| < 2\pi$ or when $|\alpha| > 4\pi$, then ρ is a DT representation. (Note that when $|\alpha| \in (2\pi, 4\pi)$, then $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ is empty.) So, Theorem A holds for 3-punctured spheres.

2.3.2. *The case* $n = 4$. Throughout this section, Σ will denote a 4-punctured sphere. We start by considering the case of representations with a non-peripheral simple closed curve in their kernel.

Lemma 2.13. *If* $\rho\colon \pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ *is a totally elliptic representation with a non-peripheral simple closed curve in its kernel, then* ρ *is orthogonal.*

Proof. We work with a geometric presentation of $\pi_1\Sigma$ with generators c_1, c_2, c_3, c_4 satisfying $c_1c_2c_3c_4 = 1$. Let's assume that c_1c_2 is a fundamental group element that represents the non-peripheral simple closed curve in the kernel of ρ. This means that there exist two elements X, Y of $\mathrm{PSL}_2\,\mathbb{R}$ such that $\rho(c_1) = X^{-1}$, $\rho(c_2) = X$, $\rho(c_3) = Y$, and $\rho(c_4) = Y^{-1}$. Now, observe that the fundamental group element $c_2c_3c_1c_3^{-1}$ represents a simple closed curve on Σ (it's the image of the simple closed curve represented by c_1c_2 under the Dehn twist along c_2c_3). Since ρ is totally elliptic, we conclude that $\rho(c_2c_3c_1c_3^{-1}) = [X, Y]$ is elliptic. By Lemma 2.1, this is possible only if $[X, Y] = 1$, proving that ρ is orthogonal. □

It was already observed in [Mar22, Remark 2.8] that if $|\alpha| < 2\pi$ or if $|\alpha| > 6\pi$, then every totally elliptic representation in $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ is a DT representation. It turns out that a stronger statement is true: a representation that maps a single non-peripheral simple closed curve to a regular elliptic element is necessarily a DT representation. The proof of this statement uses the characterization of DT representations in terms of Toledo number (given by (2.3)) and the additive properties of the Toledo number (see [Mar22] for more details). When $|\alpha| \in [2\pi, 6\pi]$, however, there exist representations outside of the compact component of $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ that map *some* non-peripheral simple closed curves to elliptic elements, but are not totally elliptic (an explicit example is provided by Palesi in [Pal14, Section 5.4]).

We now give a more general argument that applies for every value of $|\alpha|$ and proves that totally elliptic representations always belong to a compact component of $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$. Let $\rho\colon \pi_1\Sigma \to \mathrm{PSL}_2\,\mathbb{R}$ be a totally elliptic representation and consider the pure mapping class group orbit of $[\rho]$ inside $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ (we refer the reader to [Mar22] for a detailed description of this action). The orbit of $[\rho]$ is a bounded subset of $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ by Proposition 2.12. Bounded orbits for 4-punctured spheres have been studied by Cantat–Loray in the more general context of $\mathrm{SL}_2\,\mathbb{C}$ representations. Their work has the following implication.

Corollary 2.14 ([CL09, Theorem C])**.** *Let* Σ *be a 4-punctured sphere. The relative character variety* $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$ *contains an infinite bounded mapping class group orbit if and only if* $|\alpha| < 2\pi$ *or* $|\alpha| > 6\pi$*. Moreover, each infinite bounded orbit is dense in the DT component of* $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\,\mathbb{R})$*.*

This means that if the mapping class group orbit of $[\rho]$ is infinite, then $|\alpha| < 2\pi$ or $|\alpha| > 6\pi$, and ρ is a DT representation. If instead the orbit of $[\rho]$ is finite, then we can consult Lisovyy–Tykhyy's classification of finite mapping class group orbits for 4-punctured spheres from [LT14] and observe that ρ is either a representation into $\mathrm{PSO}(2)$ (which corresponds to the isolated points of $\mathrm{Rep}_\alpha(\Sigma, \mathrm{PSL}_2\mathbb{R})$ when $|\alpha| \in \{2\pi, 4\pi, 6\pi\}$) or a DT representation. Altogether, this proves Theorem A for 4-punctured spheres.

2.3.3. *The case $n \geq 5$.* Let now Σ denote a sphere with $n \geq 5$ punctures and $\rho\colon \pi_1\Sigma \to \mathrm{PSL}_2\mathbb{R}$ be a reduced totally elliptic representation. We'll see that the proof of Theorem A in this case can be brought back to the case of 4-punctured spheres (Section 2.3.2). As before, we start with the case of representations with a non-peripheral simple closed curve in their kernel.

Lemma 2.15. *If $\rho\colon \pi_1\Sigma \to \mathrm{PSL}_2\mathbb{R}$ is a reduced totally elliptic representation with a non-peripheral simple closed curve in its kernel, then ρ is orthogonal.*

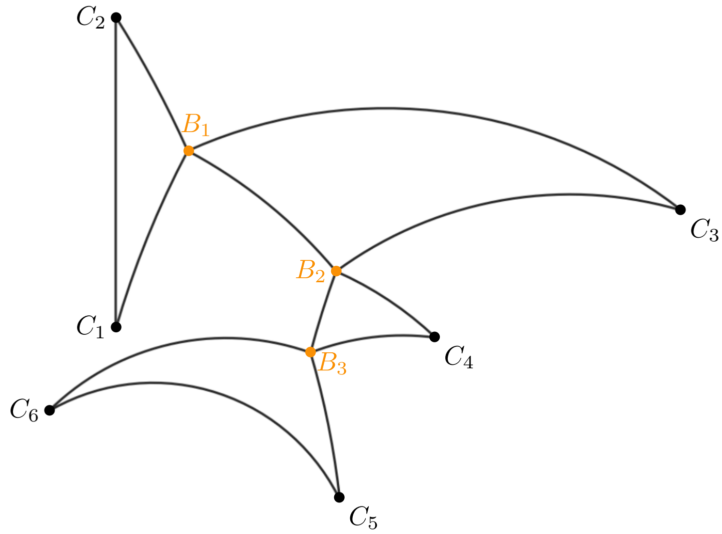
Proof. We choose a geometric generating family of $\pi_1\Sigma$ consisting of generators $c_1, \ldots, c_n$ satisfying $c_1 \cdots c_n = 1$. They can be chosen in a way that the simple closed curve in the kernel of ρ is represented by the fundamental group element $c_1 \cdots c_{i+1}$ for some $i \in \{2, \ldots, n-3\}$. This implies that $\rho(c_{i+1}) = \rho(c_1 \cdots c_i)^{-1}$ and $\rho(c_{i+2}) = \rho(c_{i+3} \cdots c_n)^{-1}$. Arguing as in the proof of Lemma 2.13, we know that we can conjugate ρ so that $\rho(c_{i+1})$ and $\rho(c_{i+2})$ both belong to $\mathrm{PSO}(2)$.

Now, note that the fundamental group element $c_{i+1}c_j(c_1 \cdots c_i)c_j^{-1}$ represents a simple closed curve for every $j \geq i+2$ and similarly for $c_{i+2}^{-1}c_j^{-1}(c_{i+3} \cdots c_n)^{-1}c_j$ when $j \leq i+1$. The first one is the image of $c_1 \cdots c_{i+1}$ by the Dehn twist along $c_{i+1}c_j$ and the second one is the image of $(c_{i+2} \cdots c_n)^{-1}$ by the Dehn twist along c_jc_{i+2}. Since ρ is totally elliptic, we conclude that $\rho(c_{i+1}c_j(c_1 \cdots c_i)c_j^{-1}) = [\rho(c_{i+1}), \rho(c_j)]$ and $\rho(c_{i+2}^{-1}c_j^{-1}(c_{i+3} \cdots c_n)^{-1}c_j) = [\rho(c_{i+2})^{-1}, \rho(c_j)^{-1}]$ are both elliptic, hence trivial by Lemma 2.1. Because we're assuming ρ to be reduced, it holds that $\rho(c_{i+1}) \neq 1$ and $\rho(c_{i+2}) \neq 1$. We conclude that $\rho(c_j)$ belongs to $\mathrm{PSO}(2)$ for every j again by Lemma 2.1. This proves that ρ is orthogonal. $\square$

From now on, let's assume that ρ is a reduced totally elliptic representation with no simple closed curves in its kernel (ρ is "regularly totally elliptic" in the terminology introduced after Definition 2.10). To such a representation corresponds a *chain of triangles* in the hyperbolic plane $\mathbb{H}$. It's constructed from a *chained pants decomposition* $\mathcal{B}$ of Σ, by which we mean a pants decomposition of Σ in which each pair of pants contains at least one puncture. We then choose a geometric presentation of $\pi_1\Sigma$ with generators $c_1, \ldots, c_n$ satisfying $c_1 \cdots c_n = 1$, such that the $n-3$ pants curves of $\mathcal{B}$ are represented by the fundamental group elements $b_i = (c_1 \cdots c_{i+1})^{-1}$ for $i = 1, \ldots, n-3$. This is always possible as explained in [Mar24, Appendix B]. Since ρ is reduced and has no simple closed curves in its kernel, all the elliptic elements $\rho(c_1), \ldots, \rho(c_n)$ and $\rho(b_1), \ldots, \rho(b_{n-3})$ are regular, and have a unique fixed point in $\mathbb{H}$ which we denote by $C_1, \ldots, C_n$ and $B_1, \ldots, B_{n-3}$. The triangle chain of ρ consists of the $n-2$ triangles

$$(C_1, C_2, B_1), (B_1, C_3, B_2), \ldots, (B_{n-4}, C_{n-2}, B_{n-3}), (B_{n-3}, C_{n-1}, C_n).$$

We call it the *$\mathcal{B}$-triangle chain of ρ*. For instance, when $n = 6$, the $\mathcal{B}$-triangle chain of ρ may look like this.



For more details on triangle chains, the reader is referred to [Mar24].

Every triangle in the chain of ρ is associated to one of the $n-2$ pairs of pants determined by $\mathcal{B}$ in the same way as we described in Section 2.3.1. So, each triangle can be either degenerate to a single point, or clockwise or anti-clockwise oriented. It turns out that being a DT representation can be characterized by the orientations of the triangles.

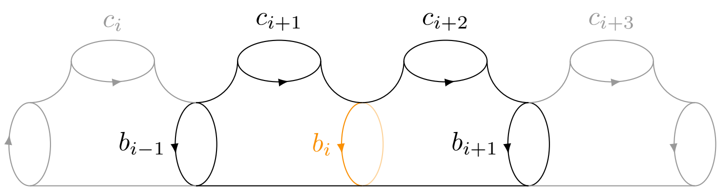
Proposition 2.16 ([Mar24, Lemma 3.5]). *If there exists a chained pants decomposition $\mathcal{B}$ such that all the triangles in the $\mathcal{B}$-triangle chain of ρ are non-degenerate and have the same orientation, then ρ is a DT representation. Conversely, if ρ is a DT representation, then for every chained pants decomposition $\mathcal{B}$, all non-degenerate triangles in the $\mathcal{B}$-triangle chain of ρ have the same orientation.*

If the $\mathcal{B}$-triangle chain of ρ only contains degenerate triangles, then all $\rho(c_i)$ fix the same point. This means that ρ is an orthogonal representation. Instead, if at least one triangle is non-degenerate, then the elliptic elements $\rho(c_i)$ are not all rotations around the same point in $\mathbb{H}$ and ρ is not orthogonal. In the latter case, we can actually replace $\mathcal{B}$ by a new chained pants decomposition of Σ and find a compatible geometric presentation of $\pi_1\Sigma$ such that the new triangle chain of ρ only consists of non-degenerate triangles. This new pants decomposition can be constructed using the same algorithm as in [FM23, Proposition 2] and the compatible geometric presentation of $\pi_1\Sigma$ is obtained by the same reasoning as in [FM23, Proposition 3]. From now on, we'll therefore assume that all the triangles in the $\mathcal{B}$-triangle chain of ρ are non-degenerate.

We'll denote the rotation angles of the images of the pants curve $b_1, \ldots, b_{n-3}$ in $\mathcal{B}$ under ρ by $\beta_1, \ldots, \beta_{n-3} \in (0, 2\pi)$. Recall that $\rho(b_1), \ldots, \rho(b_{n-3})$ are all regular elliptic elements because we're assuming that the kernel of ρ doesn't contain any simple closed curve.

Proposition 2.17. *If ρ is a reduced totally elliptic representation with no simple closed curves in its kernel, then all the triangles in the $\mathcal{B}$-triangle chain of ρ have the same orientation. Hence, ρ is a DT representation by Proposition 2.16.*

Proof. For each $i = 1, \ldots, n-3$, we consider the 4-punctured spheres $\Sigma^{(i)} \subset \Sigma$ with peripheral curves $((b_{i-1})^{-1}, c_{i+1}, c_{i+2}, b_{i+1})$.



The sphere Σ admits $n-3$ such sub-spheres, one for each pants curve in $\mathcal{B}$. We follow the convention of writing $b_0 = c_1^{-1}$ and $\beta_0 = 2\pi - \alpha_1$, as well as $b_{n-2} = c_n$ and $\beta_{n-2} = \alpha_n$. The conjugacy class of the restriction of ρ to $\Sigma^{(i)}$ is a point in the relative character variety

$$\operatorname{Rep}_{\alpha^{(i)}}(\Sigma^{(i)}, \operatorname{PSL}_2 \mathbb{R})$$

where $\alpha^{(i)} = (2\pi - \beta_{i-1}, \alpha_{i+1}, \alpha_{i+2}, \beta_{i+1})$. The restriction of ρ to $\Sigma^{(i)}$ is obviously still totally elliptic.

The standard pants decomposition of $\Sigma^{(i)}$ associated to the geometric system of generators $((b_{i-1})^{-1}, c_{i+1}, c_{i+2}, b_{i+1})$ of $\pi_1\Sigma^{(i)}$ consists of one curve: b_i. We'll write it $\mathcal{B}_i$. The $\mathcal{B}_i$-triangle chain of the restriction of ρ to $\Sigma^{(i)}$ is made of the two triangles that share the vertex B_i in the $\mathcal{B}$-triangle chain of ρ. These two triangles are non-degenerate because we're assuming that every triangle in the $\mathcal{B}$-triangle chain of ρ is non-degenerate. In particular, this means that none of the restrictions of ρ to any of the spheres $\Sigma^{(i)}$ is a representation into $\operatorname{PSO}(2)$. So, by the discussion of Section 2.3.2 about 4-punctured spheres, they're all DT representations because they are totally elliptic. This means that the two triangles in the $\mathcal{B}_i$-triangle chain of ρ have the same orientation by Proposition 2.16. Now, because the $\mathcal{B}$-triangle chain of ρ is made of the $\mathcal{B}_i$-triangle chain of each restriction, with an overlap of one non-degenerate triangle for consecutive indices, we conclude that all the triangles in the $\mathcal{B}$-triangle chain of ρ have the same orientation. □

Proposition 2.17 finishes the proof of Theorem A.

References

[Bas80] Hyman Bass, *Groups of integral representation type*, Pac. J. Math. **86** (1980), 15–51 (English).
[BG99] Robert L. Benedetto and William M. Goldman, *The topology of the relative character varieties of a quadruply-punctured sphere*, Exp. Math. **8** (1999), no. 1, 85–103 (English).
[BIW10] Marc Burger, Alessandra Iozzi, and Anna Wienhard, *Surface group representations with maximal Toledo invariant*, Ann. Math. (2) **172** (2010), no. 1, 517–566 (English).
[Bow98] B. H. Bowditch, *Markoff triples and quasifuchsian groups*, Proc. Lond. Math. Soc. (3) **77** (1998), no. 3, 697–736 (English).

[CL09] Serge Cantat and Frank Loray, *Dynamics on character varieties and Malgrange irreducibility of Painlevé VI equation*, Ann. Inst. Fourier **59** (2009), no. 7, 2927–2978 (English).

[DT19] Bertrand Deroin and Nicolas Tholozan, *Supra-maximal representations from fundamental groups of punctured spheres to* $\mathrm{PSL}(2,\mathbb{R})$., Ann. Sci. Éc. Norm. Supér. (4) **52** (2019), no. 5, 1305–1329 (English).

[FM23] Aaron Fenyes and Arnaud Maret, *The geometry of deroin-tholozan representations*, arXiv preprint arXiv:2312.09199v1 (2023).

[FZ23] Yu Feng and Junming Zhang, *Compact relative* $\mathrm{SO}_0(2,q)$*-character varieties of punctured spheres*, arXiv preprint arXiv:2309.15553v1 (2023).

[GK17] François Guéritaud and Fanny Kassel, *Maximally stretched laminations on geometrically finite hyperbolic manifolds*, Geom. Topol. **21** (2017), no. 2, 693–840 (English).

[Gol88] William M. Goldman, *Topological components of spaces of representations*, Invent. Math. **93** (1988), no. 3, 557–607 (English).

[Gol06] ______, *Mapping class group dynamics on surface group representations*, Problems on mapping class groups and related topics, Providence, RI: American Mathematical Society (AMS), 2006, pp. 189–214 (English).

[Kab90] M. I. Kabenyuk, *Compact elements and Cartan subgroups of connected Lie groups*, Ukr. Math. J. **42** (1990), no. 2, 145–148 (English).

[LT14] Oleg Lisovyy and Yuriy Tykhyy, *Algebraic solutions of the sixth Painlevé equation*, J. Geom. Phys. **85** (2014), 124–163 (English).

[Mar22] Arnaud Maret, *Ergodicity of the mapping class group action on Deroin-Tholozan representations*, Groups Geom. Dyn. **16** (2022), no. 4, 1341–1368. MR 4536432

[Mar24] ______, *Action-angle coordinates for surface group representations in genus zero*, J. Symplectic Geom. **22** (2024), no. 5, 937–999 (English).

[Mat12] Daniel V. Mathews, *Hyperbolic cone-manifold structures with prescribed holonomy. II: Higher genus*, Geom. Dedicata **160** (2012), 15–45 (English).

[Mon16] Gabriele Mondello, *Topology of representation spaces of surface groups in* $\mathrm{PSL}_2(\mathbb{R})$ *with assigned boundary monodromy and nonzero Euler number*, Pure Appl. Math. Q. **12** (2016), no. 3, 399–462. MR 3767231

[MW16] Julien Marché and Maxime Wolff, *The modular action on* $\mathrm{PSL}_2(\mathbb{R})$*-characters in genus 2*, Duke Math. J. **165** (2016), no. 2, 371–412 (English).

[MW19] ______, *Six-point configurations in the hyperbolic plane and ergodicity of the mapping class group*, Groups Geom. Dyn. **13** (2019), no. 2, 731–766 (English).

[MZ24] Arielle Marc-Zwecker, *Relative* $\mathrm{SU}(2,1)$*-character varieties and decomposable complex hyperbolic triangle groups*, Geom. Dedicata **218** (2024), no. 5, 23 (English), Id/No 103.

[Pal14] Frédéric Palesi, *Dynamics of the modular group action and Markov triples*, Actes de Séminaire de Théorie Spectrale et Géométrie. Année 2012–2014, St. Martin d'Hères: Université de Grenoble I, Institut Fourier, 2014, pp. 137–161 (French).

[Par03] Anne Parreau, *Elliptic subgroups of linear groups over a valued field*, J. Lie Theory **13** (2003), no. 1, 271–278 (French).

[Pra94] Gobal Prasad, $\mathbb{R}$*-regular elements in Zariski-dense subgroups*, Q. J. Math., Oxf. II. Ser. **45** (1994), no. 180, 541–545 (English).

[Sug59] Mitsuo Sugiura, *Conjugate classes of Cartan subalgebras in real semisimple Lie algebras*, J. Math. Soc. Japan **11** (1959), 374–434 (English).

[TT21] Nicolas Tholozan and Jérémy Toulisse, *Compact connected components in relative character varieties of punctured spheres*, Épijournal de Géom. Algébr., EPIGA **5** (2021), 37 (English), Id/No 6.

[Wie18] Anna Wienhard, *An invitation to higher Teichmüller theory*, Proceedings of the international congress of mathematicians 2018, ICM 2018, Rio de Janeiro, Brazil, August 1–9, 2018. Volume II. Invited lectures, Hackensack, NJ: World Scientific; Rio de Janeiro: Sociedade Brasileira de Matemática (SBM), 2018, pp. 1013–1039 (English).

Université de Strasbourg, IRMA, 7 rue Descartes, 67000 Strasbourg, France
Email address: maret.arnaud@unistra.fr