Explosive Higher-Order Kuramoto Dynamics on Simplicial Complexes

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The higher-order interactions of complex systems, such as the brain, are captured by their simplicial complex structure and have a significant effect on dynamics. However, the existing dynamical models defined on simplicial complexes make the strong assumption that the dynamics resides exclusively on the nodes. Here we formulate the higher-order Kuramoto model which describes the interactions between oscillators placed not only on nodes but also on links, triangles, and so on. We show that higher-order Kuramoto dynamics can lead to an explosive synchronization transition by using an adaptive coupling dependent on the solenoidal and the irrotational component of the dynamics.

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From the brain [1–5] to social interactions [6–9] and complex materials [10,11], a vast number of complex systems have the underlying topology of simplicial complexes [12–14]. Simplicial complexes are topological structures formed by simplices of different dimension such as nodes, links, triangles, tetrahedra, and so on, and capture the many-body interactions between the elements of an interacting complex system. In the last years, simplicial complex modeling has attracted significant attention [15–18] revealing the fundamental mechanisms determining emergent network geometry [19] and the interplay between network geometry and degree correlations [16]. Modeling complex systems using simplicial complexes allows for the very fertile perspective of considering the role that higher-order interactions have on dynamical processes. For instance, recent works [6–9,20–24] on simplicial complex dynamics, including works on simplicial complex synchronization [21–24], reveal that the topology and geometry of the simplicial complexes and their many-body interactions induce cooperative phenomena that cannot be found in pairwise interaction networks.

In the last years, explosive synchronization [25,26] has been attracting increasing scientific interest. Different pathways to explosive synchronization have been explored in the framework of the Kuramoto dynamics of single and multilayer networks. These notably include correlating the intrinsic frequency of the nodes to their degree [27] or modulating the coupling between different oscillators adaptively using the local order parameter in single networks and in multiplex networks [28,29]. An outstanding open question is to establish the conditions that allow explosive synchronization on simplicial complexes.

Among the papers investigating synchonization dynamics beyond pairwise interactions [30,31], recent works [22,32] have proposed a many-body Kuramoto model where the phases associated with the nodes of the network can be coupled in triplets or quadruplets if the corresponding nodes share a triangle or a tetrahedron. Interestingly, in this context it has been shown [22] that the many-body Kuramoto dynamics can lead to explosive, i.e., discontinuous phase transitions. However this work, together with the vast majority of works that address the study of dynamics on simplicial complexes has the limitation that they associate a dynamic variable exclusively with nodes of a network. Here we are interested in a much more general scenario where the dynamics can be associated with the faces of dimension $n \geq 0$ of a simplicial complex. Indeed, dynamical processes might not just reside on nodes, instead they might be related directly to dynamics defined on higher-dimensional simplices leading to the definition of topological dynamical signals [33]. For instance, each link can be associated with a flux. Flow dynamics is relevant for biological transport networks including fungal networks [34], tree vascular networks [35], microvascular networks [36], or hemodynamic in the mammalian cortex [37], where there is some evidence that the dynamics can spontaneously give rise to oscillatory currents. Flow signals can also be used to analyze functional magnetic resonance image (fMRI) [38] and to study blood flow between different regions of the brain. More in general, the simplicial complex can be considered as a representation of interactions of different order. For instance, for any given networked structure, the line-graph construction [39,40] allows us to map links into nodes of the line graph, so that a dynamics defined on the links of a simplicial complex can be mapped on a node dynamics of its line graph. However, the original simplicial complex provides a definition of the many-body interactions solidly based on topology.

In this Letter, we formulate a higher-order Kuramoto dynamics where the dynamical variables are coupled oscillators associated with higher-dimensional simplices such as nodes, links, triangles, and so on. By using Hodge decomposition, we show that the dynamics defined on a n-dimensional simplex can be projected on the dynamics defined on $(n+1)$ and $(n-1)$-dimensional simplices. We propose a simple higher-order Kuramoto dynamics in which these two projected dynamics are decoupled and display a continuous phase transition. We then formulate the explosive higher-order Kuramoto dynamics which adaptively couples the two projected dynamics with a mechanism inspired by Ref. [28], showing that in this case the explosive higher-order Kuramoto dynamics leads to a discontinuous synchronization transition. This implies, for instance, that a dynamics defined on links can induce a simultaneous explosive synchronization on the dynamics projected on nodes and triangles. Therefore, our work elucidates an important mechanism leading to higher-order explosive Kuramoto dynamics.

Definition of simplicial complexes.—Simplicial complexes represent higher-order networks, which include interactions between two or more nodes described by simplices. A node is a zero-dimensional simplex, a link is a one-dimensional simplex, a triangle is a two-dimensional simplex, a tetrahedron is a three-dimensional simplex, and so on. The faces of a simplex α of dimension n are all the simplices α' of dimension $n' < n$ that can be constructed by taking proper subsets of the set of all the nodes forming the simplex α. A simplicial complex $\mathcal{K}$ is formed by a set of simplices that satisfy the condition of closure (given a simplex belonging to the simplicial complex, all its faces also belong to the simplicial complex). In this work, we will use the configuration model [16] of simplicial complexes, which naturally generalizes the configuration model of networks (see Supplemental Material [41] for other topologies). In particular, the d-dimensional configuration model generates simplicial complexes formed by gluing d-dimensional simplices such that every node is incident to a given number of d-dimensional simplices called its generalized degree.

In topology, simplices have also an orientation. An n-dimensional *oriented simplex* α is a set of ordered $n+1$ nodes

$$\alpha = [i_0, i_1, \ldots, i_n]. \tag{1}$$

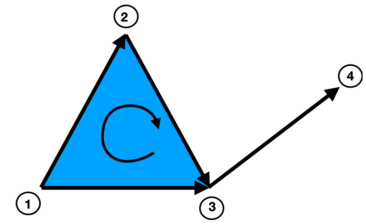

FIG. 1. An example of a small simplicial complex with the orientation of the simplices induced by the labeling of the nodes.

For instance, a link $\alpha = [i, j]$ has opposite sign of the link $[j, i]$, i.e., $[i, j] = -[j, i]$. Similarly to each higher-order simplex we associate an orientation such that

$$[i_0, i_1, \ldots, i_n] = (-1)^{\sigma(\pi)}[i_{\pi(0)}, i_{\pi(1)}, \ldots, i_{\pi(n)}], \tag{2}$$

where $\sigma(\pi)$ indicates the parity of the permutation π. Here we consider the orientation induced by the labeling of its nodes; i.e., for every simplex in a simplicial complex, we give positive orientation as the one provided by the increasing list of node labels (see Fig. 1).

In topology [33,45–47], the n-chains $\mathcal{C}_n$ are the elements of a free Abelian group with basis the n-dimensional simplices of a simplicial complex. The boundary map is a linear map ∂_n: $\mathcal{C}_n \to \mathcal{C}_{n-1}$ defined by its action on each simplex. Specifically, the boundary map maps every n-dimensional simplex α to a linear combination of the $(n-1)$-dimensional oriented faces at its boundary given by

$$\partial_n[i_0, i_1 \ldots, i_n] = \sum_{p=0}^{n}(-1)^p[i_0, i_1, \ldots, i_{p-1}, i_{p+1}, \ldots, i_n].$$

The boundary map satisfies the important property that $\partial_{n-1}\partial_n = 0$, that is usually expressed by saying that the boundary of a boundary is null (see Supplemental Material [41]). Given a simplicial complex with $N_{[n]}$ n-dimensional simplices, the boundary map ∂_n can be described using the $N_{[n-1]} \times N_{[n]}$ incidence matrix $\mathbf{B}_{[n]}$ (see Supplemental Material [41]). For instance, in Fig. 1 we show an example of a simplicial complex formed by the set of nodes $\{[1], [2], [3], [4]\}$, the set of links $\{[1, 2], [1, 3], [2, 3], [3, 4]\}$, and the set of triangles $\{[1, 2, 3]\}$. The incidence matrices [33,45–47] of this simplicial complex are given by

$$\mathbf{B}_{[1]} = \begin{array}{c|cccc} & [1,2] & [1,3] & [2,3] & [3,4] \\ \hline [1] & -1 & -1 & 0 & 0 \\ [2] & 1 & 0 & -1 & 0 \\ [3] & 0 & 1 & 1 & -1 \\ [4] & 0 & 0 & 0 & 1 \end{array} \qquad \mathbf{B}_{[2]} = \begin{array}{c|c} & [1,2,3] \\ \hline [1,2] & 1 \\ [1,3] & -1 \\ [2,3] & 1 \\ [3,4] & 0 \end{array}.$$

Higher-order Laplacians.—The graph Laplacian is widely used to study dynamical processes defined on the nodes of a network. It can be expressed in terms of the boundary matrix $\mathbf{B}_{[1]}$ as

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}_{[1]}^{\top}. \tag{3}$$

The higher-order Laplacian $\mathbf{L}_{[n]}$ [33,45–47], with $n > 0$, generalizes the graph Laplacian by describing diffusion taking place on n-dimensional faces. The nth Laplacian $\mathbf{L}_{[n]}$ is an $N_{[n]} \times N_{[n]}$ matrix given by

$$\mathbf{L}_{[n]} = \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]} + \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}. \tag{4}$$

The spectral properties of the higher-order Laplacian can be proven to be independent of the orientation of the simplices as long as the orientation is induced by a labeling of the nodes. The main property of the higher-order Laplacian is that the degeneracy of the zero eigenvalue of $\mathbf{L}_{[n]}$ is equal to the Betti number β_n, and that its corresponding eigenvectors localize around the corresponding n-dimensional cavities of the simplicial complex.

The higher-order Laplacians have notable spectral properties induced by the topological properties of the boundary map [33]. In fact, given that $\partial_{n-1}\partial_n = 0$, we have $\mathbf{B}_{[n-1]}\mathbf{B}_{[n]} = \mathbf{0}$ and, similarly, $\mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n-1]}^{\top} = \mathbf{0}$. Therefore the eigenvectors associated with the non-null eigenvalues of $\mathbf{L}_{[n]}^{[\mathrm{up}]} = \mathbf{B}_{[n+1]}\mathbf{B}_{[n+1]}^{\top}$ are orthogonal to the eigenvectors associated with the non-null eigenvalues of $\mathbf{L}_{[n]}^{[\mathrm{down}]} = \mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n]}$. It follows that the non-null eigenvalues of $\mathbf{L}_{[n]}$ are either the non-null eigenvalues of $\mathbf{L}_{[n]}^{[\mathrm{up}]}$ or the non-null eigenvalues of $\mathbf{L}_{[n]}^{[\mathrm{down}]}$. This property of the higher-order Laplacian can be exploited to prove that every vector $\mathbf{x}_{[n]}$ defined on n-dimensional simplices can be decomposed according to the *Hodge decomposition* [33] as

$$\mathbf{x}_{[n]} = \mathbf{x}_{[n]}^{H} + \mathbf{B}_{[n]}^{\top}\mathbf{z}_{[n-1]} + \mathbf{B}_{[n+1]}\mathbf{z}_{[n+1]}, \tag{5}$$

where $\mathbf{x}_{[n]}^{H}$ is the harmonic component that satisfies $\mathbf{B}_{[n+1]}^{\top}\mathbf{x}_{[n]}^{H} = \mathbf{0}$, $\mathbf{B}_{[n]}\mathbf{x}_{[n]}^{H} = \mathbf{0}$, the term $\mathbf{B}_{[n]}^{\top}\mathbf{z}_{[n-1]}$ is the irrotational component as we have $\mathbf{B}_{[n+1]}^{\top}\mathbf{B}_{[n]}^{\top}\mathbf{z}_{[n-1]} = \mathbf{0}$, and the third term $\mathbf{B}_{[n+1]}\mathbf{z}_{[n+1]}$ is the solenoidal component as we have $\mathbf{B}_{[n]}\mathbf{B}_{[n+1]}\mathbf{z}_{[n+1]} = \mathbf{0}$.

Higher-order Kuramoto dynamics.—The Kuramoto model [48] is a dynamical model for the vector $\boldsymbol{\theta}$ whose elements are the phases θ_i associated with the nodes of the simplicial complex. Each oscillator i has an internal frequency ω_i and is coupled pairwise to the oscillator of the connected nodes by the coupling constant σ. Interestingly, the Kuramoto dynamics can be interpreted as a dynamics defined on the nodes of a simplicial complex, i.e., simplices of dimension $n = 0$, indicated with label $i = 1, 2, \ldots, N_{[0]}$, and it can be expressed in terms of the incidence matrix $\mathbf{B}_{[1]}$ (see Supplemental Material [41]) as

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma\mathbf{B}_{[1]}\sin\mathbf{B}_{[1]}^{\top}\boldsymbol{\theta}, \tag{6}$$

where here and in the following, $\sin\mathbf{x}$ indicates the column vector where the sine function is taken elementwise, and $\boldsymbol{\omega}$ is the vector of internal frequencies ω_i associated with the nodes of the simplicial complex.

Here our goal is to extend the Kuramoto dynamics to describe synchronization among dynamical phases θ_α associated with each simplex α of dimension $n > 0$, i.e., links (for $n = 1$) or even higher-dimensional simplices. We assume that these dynamical signals are phases that oscillate with some internal frequency, and they can be coupled by higher-order interactions. The natural way to choose the coupling between n-dimensional phases is suggested by the generalization of the Kuramoto dynamics using the higher-order incidence matrices

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma\mathbf{B}_{[n+1]}\sin\mathbf{B}_{[n+1]}^{\top}\boldsymbol{\theta} - \sigma\mathbf{B}_{[n]}^{\top}\sin\mathbf{B}_{[n]}\boldsymbol{\theta}, \tag{7}$$

where $\boldsymbol{\theta}$ indicates the vector of phases θ_α and where $\boldsymbol{\omega}$ is the vector of intrinsic frequencies ω_α associated with each n-dimensional simplex α. Each internal frequency ω_α is drawn from a normal distribution with mean Ω and variance 1, i.e., $\omega \sim \mathcal{N}(\Omega, 1)$. The higher-order Kuramoto dynamics describes a dynamics of phases associated with simplices of dimension n as links ($n = 1$), triangles ($n = 2$), and so on (see Supplemental Material [41]).

An important question to ask is whether the dynamics associated with n-dimensional simplices induces a dynamics on lower- or higher-dimensional simplices. For instance, if we have a Kuramoto dynamics defined on links, what is the effect of this dynamics on nodes and triangles? It turns out that there is a simple way to project the dynamics defined on links into dynamics defined on nodes and triangles suggested by topology. More in general, we can project the dynamics defined on n simplices to the dynamics defined on simplices of dimension $n - 1$ and $n + 1$ by using the higher-order incidence matrices. To this end, let us indicate with $\boldsymbol{\theta}^{[+]}$ the vector of $N_{[n+1]}$ phases associated with each $n + 1$ simplex of the simplicial complex. This vector describes the projection of the dynamics on simplices of dimension $n + 1$. Similarly, let us indicate with $\boldsymbol{\theta}^{[-]}$ the vector of $N_{[n-1]}$ phases associated with each $n - 1$ simplex of the simplicial complex. This vector represents the projection of the dynamics on simplices of dimension $n - 1$. Topological considerations suggest the physical meaning of these projecting as $\boldsymbol{\theta}^{[+]}$ and $\boldsymbol{\theta}^{[-]}$ are, respectively, as the "discrete curl" and "discrete divergence" of $\boldsymbol{\theta}$, i.e.,

$$\boldsymbol{\theta}^{[+]} = \mathbf{B}_{[n+1]}^{\top}\boldsymbol{\theta},$$
$$\boldsymbol{\theta}^{[-]} = \mathbf{B}_{[n]}\boldsymbol{\theta}. \tag{8}$$

Using the Hodge decomposition, it is easy to show that $\boldsymbol{\theta}^{[+]}$ depends only on the solenoidal component of the dynamics defined on n-dimensional phases, whereas $\boldsymbol{\theta}^{[-]}$ depends only on the irrotational component. Since we have that $\mathbf{B}_{[n]}^{\top}\mathbf{B}_{[n-1]}^{\top} = \mathbf{0}$ and $\mathbf{B}_{[n-1]}\mathbf{B}_{[n]} = \mathbf{0}$, if $\boldsymbol{\theta}$ obeys the higher-Kuramoto dynamics, then the projected dynamical variables $\boldsymbol{\theta}^{[+]}$ and $\boldsymbol{\theta}^{[-]}$ evolve independently according to

$$\dot{\boldsymbol{\theta}}^{[+]} = \mathbf{B}_{[n+1]}^{\top}\boldsymbol{\omega} - \sigma\mathbf{L}_{[n+1]}^{[\mathrm{down}]}\sin(\boldsymbol{\theta}^{[+]}),$$
$$\dot{\boldsymbol{\theta}}^{[-]} = \mathbf{B}_{[n]}\boldsymbol{\omega} - \sigma\mathbf{L}_{[n-1]}^{[\mathrm{up}]}\sin(\boldsymbol{\theta}^{[-]}). \tag{9}$$

Therefore, the dynamics defined on n-dimensional simplices can naturally be decoupled into two noninteracting dynamics acting on $(n-1)$ and on $(n+1)$-dimensional simplices. The two order parameters for these two independent dynamics are $R^{[+]} = |\sum_{\alpha=1}^{N_{[n+1]}} e^{i\theta_{\alpha}^{[+]}}|/N_{[n+1]}$ and $R^{[-]} = |\sum_{\alpha=1}^{N_{[n-1]}} e^{i\theta_{\alpha}^{[-]}}|/N_{[n-1]}$ respectively.

In order to investigate the properties of the dynamics defined on n-dimensional simplices, we can consider the standard order parameter R given by $R = |\sum_{\alpha=1}^{N_{[n]}} e^{i\theta_{\alpha}}|/N_{[n]}$ and two additional order parameters $R^{[1]} = |\sum_{\alpha=1}^{N_{[n]}} e^{iy_{\alpha}^{[1]}}|/N_{[n]}$ and $R^{[2]} = |\sum_{\alpha=1}^{N_{[n]}} e^{iy_{\alpha}^{[2]}}|/N_{[n]}$, where $\mathbf{y}^{[1]} = \mathbf{L}_{[n]}^{[\mathrm{up}]}\boldsymbol{\theta} = \mathbf{L}_{[n]}^{[\mathrm{up}]}\mathbf{B}_{[n+1]}\mathbf{z}_{[n+1]}$ depends only on the solenoidal component of the dynamics on n-dimensional simplices, and $\mathbf{y}^{[2]} = \mathbf{L}_{[n]}^{[\mathrm{down}]}\boldsymbol{\theta} = \mathbf{L}_{[n]}^{[down]}\mathbf{B}_{[n]}^{\top}\mathbf{z}_{[n-1]}$ depends only on the irrotational component of the dynamics on n-dimensional simplices.

We have simulated the higher-order ($n = 1$) Kuramoto dynamics on the three-dimensional simplicial complexes produced by the configuration model with power-law generalized degree distribution of the nodes. These simplicial complexes have Betti numbers $\beta_1 > 0$, $\beta_2 = 0$. We observe that the projected dynamics on the two-dimensional simplices and the zero-dimensional simplices display a continuous synchronization transition (see Fig. 2). When we investigate the three order parameters for the dynamics defined on n-dimensional simplices, we observe that R does not capture the collective behavior of the phases due to the fact that the harmonic component of their dynamics is not coupled by the higher-order Kuramoto dynamics. However, the order parameters $R^{[1]}$ and $R^{[2]}$ are sensible to the synchronization of the solenoidal and irrotational component of the dynamics of the phases (see in Fig. 3). This suggests that this ordering in physical system can go unnoticed if the correct order parameters are not applied to the signal. A phenomenological analytical approach can show that while the projection of the phases

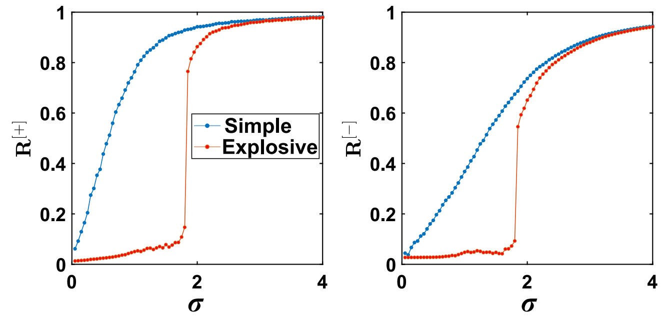

FIG. 2. The projection of the higher-order ($n = 1$) Kuramoto dynamics on $(n-1)$-dimensional faces and $(n+1)$-dimensional faces is investigated by plotting the order parameters $R^{[+]}$ (left panel) and $R^{[-]}$ (right panel), both for the simple (blue circles) and explosive (red squares) dynamics. Here both the simple and the explosive higher-order Kuramoto model have $\Omega = 2$ and are defined on a configuration model of $N_{[0]} = 1000$ nodes, $N_{[1]} = 5299$ links, and $N_{[2]} = 4147$ triangles with generalized degree of the nodes that is power-law distributed with power-law exponent $\gamma = 2.8$.

on the harmonic modes are decoupled, $\boldsymbol{\theta}^{[+]}$ and $\boldsymbol{\theta}^{[-]}$ have a continuous synchronization transition at $\sigma_c = 0$ (see Supplemental Material [41]). The nature of the phase transition does not change if we consider simplicial complexes with Poisson generalized degree distribution of the nodes and can be explained by an analytical framework (see Supplemental Material [41]).

Explosive higher-order Kuramoto dynamics.—In order to explore whether it is possible to enforce an explosive phase transition, we include a coupling between the equations determining the dynamics of $\boldsymbol{\theta}^{[+]}$ and $\boldsymbol{\theta}^{[-]}$. The way we coupled these two independent dynamics is inspired by the coupling of the dynamics of multiplex Kuramoto dynamics in Ref. [28]. However, while in the explosive multiplex Kuramoto dynamics the coupling between the phases in one layer is modulated by the local order parameter of each node in the other layer, here we consider a modulation of the coupling between the phases $\boldsymbol{\theta}^{[+]}$ and $\boldsymbol{\theta}^{[-]}$ given, respectively, by the global order parameters $R^{[-]}$ and $R^{[+]}$. This choice is driven by the fact that the $(n+1)$-dimensional faces are not in a one-to-one relation with the $(n-1)$-dimensional faces. Given these considerations, we propose the following explosive higher-order Kuramoto dynamics:

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \sigma R^{[-]}\mathbf{B}_{[n+1]}\sin\mathbf{B}_{[n+1]}^{\top}\boldsymbol{\theta}$$
$$- \sigma R^{[+]}\mathbf{B}_{[n]}^{\top}\sin\mathbf{B}_{[n]}\boldsymbol{\theta}. \tag{10}$$

This dynamics can be projected on the dynamics of $(n+1)$ and $(n-1)$-dimensional simplices producing now two equations coupled by the global order parameters $R^{[+]}$ and $R^{[-]}$:

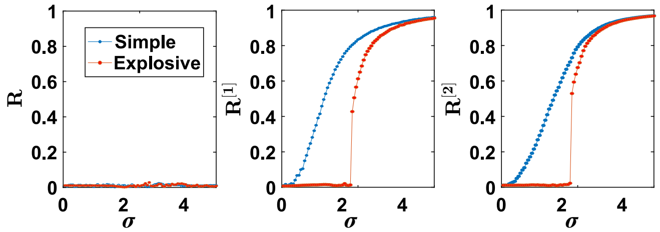

FIG. 3. The order parameters R, $R^{[1]}$, and $R^{[2]}$ of the simple (blue circles) and explosive (red squares) higher-order ($n = 1$) Kuramoto dynamics are plotted versus the coupling constant σ. The network parameters are the same as in Fig. 2.

$$\dot{\boldsymbol{\theta}}^{[+]} = \mathbf{B}_{[n+1]}^{\top}\boldsymbol{\omega} - \sigma R^{[-]}\mathbf{L}_{[n+1]}^{[\mathrm{down}]}\sin(\boldsymbol{\theta}^{[+]}),$$
$$\dot{\boldsymbol{\theta}}^{[-]} = \mathbf{B}_{[n]}\boldsymbol{\omega} - \sigma R^{[+]}\mathbf{L}_{[n]}^{[\mathrm{up}]}\sin(\boldsymbol{\theta}^{[-]}). \quad (11)$$

We have simulated the explosive higher-order Kuramoto dynamics on simplices of dimension $n = 1$ on the configuration model of simplicial complexes with power-law distribution of generalized degrees.

A discontinuous phase transition emerges in $R^{[+]}$ and $R^{[-]}$ (see Fig. 2). This transition is also reflected on the irrotational and solenoidal components of the dynamics on the n-dimensional phases captured by the order parameters $R^{[1]}$ and $R^{[2]}$, while due to the presence of the uncoupled harmonic component R remains close to zero (see Fig. 3). The nature of the phase transition does not change significantly if we consider simplicial complexes with Poisson generalized degree distribution (see Supplemental Material [41]). Our analytical framework (see Supplemental Material [41]) explains the physics behind this discontinuous phase transition.

Conclusions.—We have introduced the higher-order Kuramoto dynamics designed to characterize the coupling between phases associated with higher-dimensional simplices, such as links, triangles, and so on. This framework has allowed us to define a topologically projected dynamics on faces of dimension $n - 1$ and $n + 1$, which obey a dynamics of coupled oscillators. We have considered two versions of the higher-order Kuramoto dynamics, the simple and the explosive higher-order Kuramoto dynamics. We have found that the simple higher-order Kuramoto dynamics displays continuous phase transitions for the projected dynamics defined on $n + 1$ and $n - 1$ faces. Interestingly, however, when we introduced a coupling between the dynamics projected on the $n + 1$ and $n - 1$ dynamical phases, as in the explosive higher-order Kuramoto dynamics, the system then displayed an explosive synchronization transition. This work opens innovative perspectives on characterizing the Kuramoto dynamics on higher-dimensional simplices, and it shows that a higher-order synchronization dynamics defined on n-dimensional simplices (as, for example, links) can induce a simultaneous discontinuous transition on its projected dynamics defined on $(n - 1)$ and $(n + 1)$-dimensional simplices (i.e., nodes and triangles). In the future, the proposed dynamical model can be extended in different directions. For instance, one could explore coupling the dynamics of faces of different dimensions or other mechanisms leading to explosive synchronization.

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