Real Analysis

Arnau Mas

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Chapter 1

Measure Theory

1.1 Measure Spaces

Definition 1.1 (σ-algebra)**.** We say a family of subsets $\mathcal{A} \subseteq \mathcal{P}(X)$ of a set X is a σ-algebra over it if

(*i*) $\emptyset, X \in \mathcal{A}$,

(*ii*) $\mathcal{A}$ is closed under countable unions, i.e. if there is a countable set $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$ then $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{A}$,

(*iii*) $\mathcal{A}$ is closed under countable intersections, i.e. if there is a countable set $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$ then $\bigcap_{i\in\mathbb{N}} A_i \in \mathcal{A}$,

(*iv*) $\mathcal{A}$ is closed under complements, i.e. if $A \in \mathcal{A}$ then $X - A \in \mathcal{A}$.

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Example 1.1. The following are all examples of σ-algebras.

(*i*) For any set X, $\mathcal{P}(X)$ is a σ-algebra called the *discrete σ-algebra*. It is the *finest* σ-algebra since any other possible σ-algebra over X is contained in it.

(*ii*) On the other hand, the *coarsest* σ-algebra over any set X is simply $\{\emptyset, X\}$, meaning any other possible σ-algebra contains it. It is called the *trivial σ-algebra*.

(*iii*) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-algebras over a set X then so is $\mathcal{A}_1 \cap \mathcal{A}_2$.

(*iv*) Given a family of subsets $S \subseteq \mathcal{P}(X)$ then the σ-algebra generated by it is the intersection of all σ-algebras that contain it and it is the smallest σ-algebra that contains S. We write it $\sigma(S)$.

(*v*) The *Borel σ-algebra* over a topological space X is the σ-algebra generated by the open sets of X, written $\mathcal{B}(X)$. Since a closed set is the complement of an open set the family of closed sets also generates the Borel σ-algebra.

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The pair formed by a set and its σ-algebra is called a *measurable space*.

Definition 1.2 (Measure)**.** Let $(X, \mathcal{A})$ be a measurable space. A measure is a map $\mu\colon A \to [0, \infty]$ such that the following are true

(*i*) $\mu(\emptyset) = 0$.

(*ii*) If $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$ is a family of pairwise disjoint sets of finite measure then

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i).$$

△

A measurable space equipped with a measure is called a *measure space*.

Example 1.2. The following are all examples of measures.

(*i*) Consider a measurable space X with the discrete σ-algebra. Then, for any subset $A \subseteq X$ we define $\mu(A) = |A|$ if A is finite and $\mu(A) = \infty$ if A is infinite. This is the *counting measure*, for obvious reasons.

(*ii*) On a finite measurable space X with the discrete σ-algebra we define for any subset $A \subseteq X$

$$\mu(A) = \frac{|A|}{|X|}.$$

This is a special case of a probability measure since $\mu(X) = 1$. In fact a probability is exactly a measure satisfying $\mu(X) = 1$.

(*iii*) On a measurable space X with any σ-algebra fix a point $x \in X$ and define $\mu(A) = 1$ if $x \in A$ and $\mu(A) = 0$ otherwise. This is called the *Dirac measure*.

▽

1.2 The Lebesgue measure

A question worth asking is whether any measurable space can be made into a measure space. Carathódory's extension theorem gives an affirmative answer to the question provided we give a starting point for the measure. Roughly speaking the starting point consists of specifying the measure of a collection of subsetsa, subject to some requirements, which can then be extended to a measure on the whole of the σ-algebra. In the case of $\mathbb{R}^n$,