# Riemannian Geometry

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Chapter 1

# Riemannian Manifolds

## 1.1 Inner products

Recall that an inner product on a real vector space $V$ is a symmetric bilinear form $\phi\colon V \times V \to \mathbb{R}$ which is positive definite, meaning that for all $v \in V$ one has $\phi(v, v) > 0$. An inner product is also often written as $\langle u, v\rangle$. Equipping a vector space with an inner product is, in some sense, giving it a notion of geometry. Following the usual euclidean case, we define the *length* or *norm* of a vector as $\|v\| = \sqrt{\langle v, v\rangle}$ and the *angle* between two vectors by

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}.$$

Positive-definiteness is essential for both of these definitions. In fact, as a consequence of Sylvester's theorem, the geometry of a vector space with an inner product isn't much different from that of ordinary euclidean space. Indeed, for any inner product there exists an *orthonormal* basis, i.e. a basis $e_1, \ldots, e_n$ such that

$$\langle e_i, e_j\rangle = \delta_{ij}.$$

## 1.2 Riemannian Metrics

Riemannian manifolds are smooth manifolds with extra structure which allows us to speak of, not only their topology, but also their geometry. Think of the difference between a balloon which can be deformed freely and a sphere with a rigid shape. The first one is $S^2$ as a naked smooth manifold and the second one is $S^2$ as a Riemannian manifold. What is this extra structure? A Riemannian metric. Informally, a Riemannian metric is an inner product at every tangent space of the manifold which varies smoothly along the manifold.

**Definition 1.1** (Riemannian metric)**.** A Riemannian metric $g$ on a manifold $M$ is a smooth section of the bundle $\operatorname{Sym}^2 M$ of the symmetric bilinear forms on $M$ such that for every $x \in M$ $g_x$ is positive definite. △

### 1.2.1 Components of a metric in local coordinates

Given a local chart $(U, \phi)$

### 1.2.2 The pullback of a metric

Given an immersion $f\colon M \to N$ and a

The pullback gives a succint characterisation of local isometries: a local diffeomorphism $f\colon (M, g) \to (N, h)$ is a local isometry if and only if $f^*h = g$.

**Example 1.1.** By considering manifolds embedded into $\mathbb{R}^n$ and pulling back the euclidean metric onto them we obtain many examples of Riemannian manifolds. Consider the $n$-sphere $S^n$ embedded into $\mathbb{R}^{n+1}$ as

$$S^n = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

The pullback of the euclidean metric along this inclusion gives $S^n$ a Riemannian metric which we will refer to as its *standard* or *canonical* metric. ▽

This also shows that any manifold admits a metric. By Whitney's Embedding Theorem, we know that any manifold can be immersed into $\mathbb{R}^n$ for a sufficiently large $n$, and then we just pull back the euclidean metric along this immersion. We can give an alternative proof of this fact which does not rely on Whitney's Theorem.

**Theorem 1.1.** *Any smooth manifold admits a Riemannian metric.*

*Proof.* Consider a smooth manifold $M$ and a local chart $(U, \phi)$. It is clear that $\phi$ gives an immersion of $U$ into $\mathbb{R}^n$, so we can pull back the euclidean metric onto $U$. What we need, however, is a way of turning this into a global metric. The tool for this job are partitions of unity. Then let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be a locally finite atlas of $M$ and $\rho_\alpha$ a partition of unity subordinate to this atlas. Then we define $g$ as

$$g_x(X, Y) = \sum_{\alpha \in I} \rho_\alpha(x)(\phi_\alpha^* g_{\mathrm{st}})_x(X, Y)$$

for all $x \in M$ and $X, Y \in T_xM$. This is a well-defined sum since $\rho$ is a partition of unity. Since every $\phi_\alpha^* g_{\mathrm{st}}$ is a symmetric bilinear form then so is $g$. And $g$ is also positive definite. Indeed, for every non-zero $X \in T_xM$ there exists at least $\alpha_0 \in I$ such that $(\phi^*_{\alpha_0} g_{\mathrm{st}})_x(X,X) > 0$ which means

$$g_x(X,X) \geq \rho_{\alpha_0}(x)(\phi^*_{\alpha_0} g_{\mathrm{st}})_x(X,X) > 0.$$

It remains to be shown that $g$ is globally smooth. Consider at the point $x$ the coordinate vectors $\frac{\partial}{\partial x_i}|_x$ corresponding to the chart $\phi_{\alpha_0}$. Then we have

$$g_x\left(\frac{\partial}{\partial x_i}|_x, \frac{\partial}{\partial x_j}|_x\right) = \sum_{\alpha \in I} \rho_\alpha(x)(\phi^*_\alpha g_{\mathrm{st}})_x \left(\frac{\partial}{\partial x_i}|_x, \frac{\partial}{\partial x_j}|_x\right)$$

□

### 1.2.3 The product metric

**Definition 1.2** (Product metric)**.** Given two Riemannian manifolds $(M,g)$ and $(N,h)$, we define the *product metric* on the product manifold $M \times N$ by

$$(g \times h)_{(x,y)}\Big((X_1,Y_1),(X_2,Y_2)\Big) := g_x(X_1,X_2) + h_y(Y_1,Y_2)$$

where $x \in M$, $y \in N$, $X_1, X_2 \in T_xM$ and $Y_1, Y_2 \in T_yN$. △

**Proposition 1.2.** *The product metric is indeed a Riemannian metric.*

*Proof.* Binearity and symmetry □

**Example 1.2.** Consider the cylinder as submanifold of $\mathbb{R}^3$, so the set

$$\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^3$$

with the corresponding induced metric. Another representation of the cylinder is as the product $S^1 \times \mathbb{R}$. The product metric for this manifold can be written as $dg_0 + dt^2$ where $dg_0$ is the standard metric for $S^1$ and $dt^2$ is the euclidean metric on $\mathbb{R}$. These two representations are isometric. ▽

### 1.2.4 Quotient metrics

**Definition 1.3** (Riemannian covering map)**.** Given two Riemannian manifolds $(M,g)$ and $(N,h)$, a covering map $p\colon M \to N$ is called *Riemannian* provided

$(i)$ $p$ is a smooth covering map, meaning $p$ is a surjection and for every $y \in N$ there is an open neighbourhood $U$ of $y$ such that $p^{-1}(U)$ is a disjoint union of open subsets of $M$ all diffeomorphic to $U$, i.e.

$$p^{-1}(U) = \bigsqcup_{\alpha \in I} U_\alpha$$

and $p|_{U_\alpha} : U_\alpha \to U$ is a diffeomorphism for every $\alpha \in I$

$(ii)$ and more than that, the $U_\alpha$ are all isometric to $U$, which can be written ad $p^*h = g$.

$\triangle$

Typically, covering map arise through the action of a group on a manifold, as the following result shows.

**Proposition 1.3.** *Let $(M, g)$ be a Riemannian manifold and $G$ a subgroup of the isometry group of $M$ such that its action is*

$(i)$ *free, so that if $g \in G$ has a fixed point then $g$ must be the identity,*

$(ii)$ *proper*[1]*, meaning for every $x, y \in M$ there is an open neighbourhood $V$ of $x$ and $W$ of $y$ such that for every $g \in G$*

$$gV \cap W = \emptyset.$$

*Then the quotient manifold $M/G$ admits a unique Riemannian metric $g_G$ such that the projection map*

$$p\colon (M, g) \to (M/G, g_G)$$

*is a Riemannian covering map.*

*Proof.* It is a result from differential topology that $p$ is indeed a smooth covering map. We will show the construction of the induced metric. Take a point $x \in M/G$. Then it has a preimage $y \in M$ and there is a neighbourhood $U$ of $y$ such that $p|_U$ is a diffeomorphism. Then define a metric on $x$ by $p^{-1}|_U^* g$ where $g$ is the metric on $M$. This construction does not depend on the choice of the preimage $y$. Indeed,

We know that the map $p$ is a smooth covering. Consider $x \in M/G$ and two of its preimages, $y_1 \in U_1 \subseteq M$ and $y_2 \in U_2 \subseteq M$. Since $p(y_1) = p(y_2) = x$, these two points are

SKETCH:

1. Define the metric at $x \in M/G$ as the metric at one of its preimages.

2. The fact that every preimage of $x$ has isometric neighbourhoods shows well-definition. $\square$

**Example 1.3** (The metric of $\mathbb{PR}^n$)**.** Recall that the projective space $\mathbb{PR}^n$ can be constructed as the quotient of $S^n$ by the action of the group $\langle a \rangle$ where $a$ is the antipodal map. If we give $S^n$ its standard metric then it is easy to show that $a$

[1]This condition guarantees that the quotient will be a Hausdorff space

is not only a diffeomorphism but also an isometry. The action is free and proper so $\mathbb{PR}^n = S^n/\langle a \rangle$ is a smooth manifold. The induced metric on $\mathbb{PR}^n$ is called the standard or canonical metric on $\mathbb{PR}^n$. $\triangledown$

**Example 1.4** (Flat tori)**.** The $n$-dimensional torus can be given its standard metric as the product metric of the spheres.

Translations are isometries of euclidean space —so $\mathbb{R}^n$ with the standard metric— . If $v_1, \ldots, v_n$ is a basis of $\mathbb{R}^n$ then the group of translations $\Gamma = \langle T_{v_1}, \ldots, T_{v_n} \rangle$ is a group of isometries of $\mathbb{R}^n$. Furthermore, it is easy to show that its action is free and proper, so that $\mathbb{R}^n/\Gamma$ is a smooth manifold which we equip with its induced metric. Topologically, this space is homeomorphic to an $n$-torus. $\triangledown$

## 1.3 Lengths in a Riemannian manifold

Once we have a Riemannian metric on a manifold we can begin to speak of length and volume.

### 1.3.1 Length of a curve

**Definition 1.4** (Length of a curve)**.** Let $(M, g)$ be a connected Riemannian metric and $\gamma\colon [a, b] \to M$ a a piecewise differentiable path[2]. Then the length of $\gamma$ is

$$L(\gamma) := \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\, \mathrm{d}t$$

where $\dot{\gamma}(t)$ is $T_t\gamma(1)$, the vector tangent to the curve at the point $t$. The quantity $\sqrt{g_x(X, X)}$ is of course the norm of the vector $X$ induced by the metric $g$ and is usually written $\|X\|_g$. $\triangle$

In the case where $M = \mathbb{R}^n$ and the metric is the euclidean metric this coincides with the usual notion of the length of a curve obtained by taking the supremum of the length of every rectification of the curve.

**Proposition 1.4.** *The length of a curve is independent of its parametrisation.*

*Proof.* Take a curve $\gamma\colon [a, b] \to M$. A reparemetrisation of $\gamma$ is another curve $\gamma \circ h$

[2]Meaning $\gamma$ is continous and there exists a partition $a = a_0 < a_1 < \cdots < a_n = b$ of the interval $[a, b]$ such that $\gamma|_{[a_i, a_{i+1}}$ for $0 \leq i \leq n-1$

where $h\colon [c,d] \to [a,b]$ is a diffemorphism. Then

$$\begin{aligned} L(\gamma \circ h) &= \int_c^d \|(\gamma \circ h)'(t)\|_g \, \mathrm{d}t \\ &= \int_c^d \|h'(t)\dot{c}(h(t))\|_g \, \mathrm{d}t \\ &= \int_{h^{-1}(a)}^{h^{-1}(b)} \|\dot{\gamma}(h(t))\|_g \, |h'(t)| \, \mathrm{d}t \\ &= \int_a^b \|\dot{\gamma}(t)\|_g \, \mathrm{d}t \\ &= L(\gamma) \end{aligned}$$

where in the last step we used the Change of Variable Theorem. □

**Definition 1.5** (Energy of a curve)**.** Following ideas from Physics one defines the *energy* of a curve by

$$E(\gamma) := \int_a^b g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) \, \mathrm{d}t.$$

As opposed to the length, the energy does depend on the parametrisation. △

A quick application of the Cauchy-Schwarz inequality gives the result

$$L(\gamma)^2 \leq E(\gamma)(b-a)$$

with equality if a nd only if $\|\dot{\gamma}(t)\|_g$ is constant along the curve.

### 1.3.2 The distance defined by a metric

**Theorem 1.5.** *Let $(M,g)$ be a connected Riemannian manifold. The map*

$$\begin{aligned} d_g \colon M \times M &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto \inf\{L(\gamma) \mid \forall \gamma\colon [a,b] \to M \colon \gamma(a) = x, \gamma(b) = y\} \end{aligned}$$

*is a distance and the metric space $(M, d_g)$ has the same topology as the original topology of $M$.*

*Proof.* Let's first show that $d_g$ is well-defined, meaning it is finite for any pair $x, y \in M$.

Consider the set $U_x$ of all points of $M$ such that there exists a piecewise differentiable curve between it and $x$. We will show that it is both open and closed and therefore equal to $M$ since $M$ is connected[3].

[3] This is essentially a proof that a connected manifold is path-connected

We first show $U_x$ is open. Take $y \in U_x$. Then there is a chart $(U, \phi)$ around $y$ such that $\phi(U)$ is connected, therefore path-connected. Then for any $y' \in U$ there is a path $\gamma\colon : [b, b+\epsilon] \colon \phi(U)$ such that $\gamma(b) = \phi(y)$ and $\gamma(b+\epsilon) = \phi(y')$. Then $\phi^{-1} \circ \gamma$ is a path in $U$ which connects $y$ and $y'$. Then by concatenating the path that connects $x$ and $y$ with $\phi^{-1} \circ \gamma$ we get a path connecting $x$ to $y'$. This means $U \subseteq U_x$ and so $U$ is an open neighbourhood of $y$ in $U_x$, therefore $U_x$ is open.

Now

$$M - U_x = \bigcup_{y \notin U_x} U_y$$

which is open, therefore $U_x$ is closed. Therefore it is both open and closed and since it is not empty ($x \in U_x$) it must be equal to $M$.

Since $M$ is path connected, there is a piecewise smooth path between any two points $x$ and $y$ in $M$, call it $\gamma$. Then

$$d_g(x, y) \leq L(\gamma) < \infty.$$

We now show that $d_g$ is indeed a distance. For any pair $x$ and $y$, if $\gamma\colon [a, b] \to M$ is a path from $x$ to $y$, then the path $-\gamma\colon [-b, -a] \to M$ defined by $-\gamma(t) := \gamma(-t)$ satisfies $-\gamma(-b) = y$ and $-\gamma(-a) = x$ and of course has the same length as $\gamma$ since it is just a reparametrisation of it. Then, by the definition of $d_g$, for every $\epsilon > 0$, there exists a path $\gamma$ from $x$ to $y$ such that $L(\gamma) \leq d_g(x, y) + \epsilon$. But then

$$d_g(y, x) \leq L(-\gamma) = L(\gamma) \leq d_g(x, y) + \epsilon$$

which implies $d_g(y, x) = d_g(x, y)$.

Let's show the triangle inequality, i.e. for any $x, y, z \in M$

$$d_g(x, z) \leq d_g(x, y) + d_g(y, z).$$

For any $\epsilon > 0$ there exists paths $\gamma_1$ from $x$ to $y$ and $\gamma_2$ from $y$ to $z$ such that

$$L(\gamma_1) \leq d(x, y) + \epsilon$$

and

$$L(\gamma_2) \leq d(y, z) + \epsilon.$$

Consider the concatenation $\gamma_1 + \gamma_2$ of these two paths[4]. Then this is a piecewise differentiable path from $x$ to $z$ and its length (as can be shown by a simple computation) is $L(\gamma_1 + \gamma_2) = L(\gamma_1) + L(\gamma_2)$. Thus

$$d(x, z) \leq L(\gamma_1 + \gamma_2) \leq d(x, y) + d(y, z) + 2\epsilon$$

[4]For this to work we may need to potentially alter the parametrisations of the paths, but this will not alter their lengths

which implies the triangle inequality since it works for arbitrarily small $\epsilon$. Finally we prove that if $x$ and $y$ are not equal, then $d(x, y) > 0$. Take a chart $(U, \phi)$ around $x$. Then, since $M$ is a Hausdorff space, there exists $\epsilon > 0$ such that $y \notin \phi^{-1}\left(\bar{B}_\epsilon(\phi(x))\right)$.

Let $\gamma\colon [a, b] \to M$ be a piecewise smooth path from $x$ to $y$. Then there exists $t_0 \in (a, b]$ such that $\gamma(t) \in \phi^{-1}\left(\bar{B}_\epsilon(\phi(x))\right)$ for all $t \in [a, t_0]$ and $\|\phi(\gamma(t_0)) - \phi(x)\| = \epsilon$ (so the first time the path intersects the preimage of the ball). Then

$$
\begin{aligned}
L(\gamma) \geq L(\gamma|_{[a,t_0]}) &= \int_a^{t_0} \|\dot{\gamma}(t)\|_g \,\mathrm{d}t \\
&= \int_a^{t_0} \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\,\mathrm{d}t \\
&= \int_a^{t_0} \sqrt{g_{ij}(\phi(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t)} \\
&\geq \int_a^{t_0} \sqrt{\lambda} \left\|\dot{x}^i(t)e_i\right\| \,\mathrm{d}t \\
&= \sqrt{\lambda} \int_a^{t_0} \|\partial_t(\phi \circ \gamma)(t)\| \,\mathrm{d}t \\
&\geq \sqrt{\lambda} \left\| \int_a^{t_0} \partial_t(\phi \circ \gamma)(t) \right\| \,\mathrm{d}t \\
&= \sqrt{\lambda}\, \|\phi(\gamma(t_0) - \phi(x)\| = \sqrt{\lambda}\epsilon > 0.
\end{aligned}
$$

So we have $(M, d_g)$ is a metric space, and consequently a topological space. Lastly we show this topology agrees with the topology $M$ carries by the fact it is a manifold. We just showd that for any $\epsilon > 0$ there is a $\lambda > 0$ such that

$$
\{y \in M \mid d_g(x, y) < \sqrt{\lambda}\epsilon\} \subseteq \phi^{-a}\left(B_\epsilon(\phi(x))\right)
$$

The sets of the form on the left are a basis for the topology of $M$ as a metric space, whereas those on the right are a basis for the topology of $M$ as a manifold. We need the other inclusion. For any $y \in \phi^{-1}\left(B_\epsilon(\phi(x))\right)$, define the path

$$
\begin{aligned}
\gamma_y\colon [0, 1] &\longrightarrow M \\
t &\longmapsto \phi^{-1}\left(t\phi(y) + (1-t)\phi(x)\right)
\end{aligned}
$$

so the preimage of the straight line segment between $\phi(x)$ and $\phi(y)$. Then

$$
\begin{aligned}
d_g(x, y) \leq L(\gamma_y) &= \int_0^1 \sqrt{g_{ij}(\phi(\gamma_y(t)))\dot{x}^i(t)\dot{x}^j(t)}\,\mathrm{d}t \\
&\leq \int_0^1 \sqrt{\mu} \left\|\dot{\phi_\gamma}(t)\right\| \,\mathrm{d}t \\
&\sqrt{\mu}\, \|\phi(y) - \phi(x)\| < \sqrt{\mu}\epsilon
\end{aligned}
$$

$\square$

The computation we did in the previous proof can be used to show that the distance induced by the Euclidean metric is exactly the same as the distance given by the Euclidean norm, for for any path $\gamma\colon [a,b] \to \mathbb{R}^n$ from $x$ to $y$ we have

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \, \mathrm{d}t \geq \left\| \int_a^b \dot{\gamma}(t) \, \mathrm{d}t \right\| = \|\gamma(b) - \gamma(b)\| = \|x - y\|$$

which means $d(x,y) \geq \|x-y\|$. But the segment between $x$ and $y$ is a path between $x$ and $y$ with length $\|x-y\|$ which shows we have equality. In particular, we just showed that in euclidean space there always exists a path of minimum length between two points, the segment (duh).

## 1.4 Volume in a Riemannian manifold

### 1.4.1 Orientability

Let's recall the definition of an orientable manifold.

**Definition 1.6** (Orientable manifold)**.** A smooth manifold $M$ is called *orientable* if it admits an orientable atlas, that is an atlas whose every transition map is orientation preserving (so that the sign of its jacobian is positive). If no such atlas exists then the manifold is called *non orientable*.

If $M$ is orientable, choosing an orientable atlas determines an orientation for $M$. If the union of two such atlases is orientable then we say they determine the same orientation. △