The Herglotz principle for fields

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Contents

The Herglotz principle provides a variational formulation for contact Hamiltonian systems. There are several attempts [3, 4] to generalize this theory to fields. In this paper we will derive this principle in a more general geometric language and compare it to the existing approaches. In order to do that, we will review different formulations of the Herglotz principle for mechanics and how they lead to distinct principles once we try to generalize it to higher dimensions.

1 The Herglotz principle in mechanics

The Herglotz principle, in simple terms might be explained as follows. Given a configuration manifold Q, we may consider a Lagrangian function $L : TQ \times \mathbb{R} \to \mathbb{R}$ that depends on the positions q^i, the velocities $\dot{q}^i$ and an extra variable z that we can think of as the *action*, but we will soon discuss its meaning with more detail. The Herglotz variational principle, then states that the trajectory of the system $c(t)$ is the one that is a critical point of the action $\zeta(1)$, satisfying $c(0) = q_0$, $c(1) = q_1$, $\zeta(0) = z_0$ and

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = L(c, \dot{c}, \zeta), \tag{1}$$

$$\zeta(0) = z_0. \tag{2}$$

We note that the action is given by

$$\zeta(1) = \int_0^1 \frac{\mathrm{d}\zeta}{\mathrm{d}t}\mathrm{d}t + \zeta(0) = \int_0^1 L(c(t), \dot{c}(t), \zeta(t))\mathrm{d}t + z_0, \tag{3}$$

which, in the case that the Lagrangian does not depend on z, it coincides with the usual Hamilton's action up to a constant.

A slight modification on this principle, that is the one we will prefer in this paper, is to consider as the action the increment of z, that is,

$$\zeta(1) - \zeta(0) = \int_0^1 \frac{\mathrm{d}\zeta}{\mathrm{d}t}\mathrm{d}t = \int_0^1 L(c(t), \dot{c}(t), \zeta(t))\mathrm{d}t, \tag{4}$$

which coincides exactly with Hamilton's action in the case that L does not depend on z. Since both definitions of the action differ only by a constant z_0, the critical curves are the same. Indeed, they are the curves c such that $(c, \dot{c}, \zeta)$ satisfy Herglotz's equations:

$$\frac{\partial L}{\partial q^i} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z}. \tag{5}$$

Being more precise, we distinguish two possible equivalent interpretations of this principle. We can either understand it as an implicit action principle for curves c on Q, or a constrained but explicit action principle for curves (c, ζ) on $Q \times \mathbb{R}$.

1.1 The Herglotz principle: implicit version

For the first interpretation, we consider the (infinite dimensional) manifold $\Omega(q_0, q_1)$ of curves $c : [0, 1] \to Q$ with endpoints $q_0, q_1 \in Q$. The tangent space of $T_c\Omega(q_0, q_1)$, is the space of vector fields along c vanishing at the endpoints. That is,

$$T_c\Omega(q_0, q_1) = \{\delta c \mid \delta c(t) \in T_{c(t)}Q, \delta c(0) = 0,\, \delta c(1) = 0\}. \tag{6}$$

We fix a real number $z_0 \in \mathbb{R}$ and consider the following operator:

$$\mathcal{Z}_{z_0} : \Omega(q_0, q_1) \to \mathcal{C}^\infty([0, 1] \to \mathbb{R}), \tag{7}$$

which assigns to each curve c the function $\mathcal{Z}_{z_0}(c)$ that solves the following ODE:

$$\begin{cases} \dfrac{\mathrm{d}\mathcal{Z}_{z_0}(c)}{\mathrm{d}t} & = L(c, \dot{c}, \mathcal{Z}_{z_0}(c)), \\ \mathcal{Z}_{z_0}(c)(0) & = z_0, \end{cases} \tag{8}$$

that is, it assigns to each curve on the base space, its action as a function of time. This map is well-defined because the Cauchy problem Eq. (8) always has a unique solution.

Now, the *contact action functional* maps each curve $c \in \Omega(q_0, q_1)$ to the increment of the solution of the ODE:

$$\begin{aligned} \mathcal{A}_{z_0} : \Omega(q_0, q_1) &\to \mathbb{R}, \\ c &\mapsto \mathcal{Z}_{z_0}(c)(1) - \mathcal{Z}_{z_0}(c)(0). \end{aligned} \tag{9}$$

Note that, by the fundamental theorem of calculus,

$$\mathcal{A}_{z_0}(c) = \int_0^1 L(c(t), \dot{c}(t), \mathcal{Z}_{z_0}(c)(t))\mathrm{d}t. \tag{10}$$

The critical points of this action functional are precisely the solutions to Herglotz equation [1]:

Theorem 1 (Herglotz variational principle, implicit version)**.** *Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian function and let $c \in \Omega(q_0, q_1)$ and $z_0 \in \mathbb{R}$. Then, $(c, \dot{c}, \mathcal{Z}_{z_0}(c))$ satisfies the Herglotz equations:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z},$$

if and only if c is a critical point of $\mathcal{A}_{z_0}$.

Proof. In order to simplify the notation, let $\psi = T_c\mathcal{Z}(\delta v)$.

Consider a curve $c_\lambda \in \Omega(q_0, q_1)$ (that is, a smoothly parametrized family of curves in Q with fixed endpoints q_0, q_1 such that

$$\delta c = \frac{\mathrm{d}c_\lambda}{\mathrm{d}\lambda}\Big|_{\lambda=0}$$

Since $\mathcal{Z}(c_\lambda)(0) = z_0$ for all λ, then $\psi(0) = 0$.

We compute the derivative of ψ by interchanging the order of the derivatives using the ODE defining $\mathcal{Z}$:

$$\begin{aligned}
\dot{\psi}(t) &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Z}(c_\lambda(t))|_{\lambda=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda}L(c_\lambda(t), \dot{c}_\lambda(t), \mathcal{Z}(c_\lambda)(t))|_{\lambda=0} \\
&= \frac{\partial L}{\partial q^i}(\chi(t))\delta c^i(t) + \frac{\partial L}{\partial \dot{q}^i}(\chi(t))\delta \dot{c}^i(t) + \frac{\partial L}{\partial z}(\chi(t))\psi(t).
\end{aligned}$$

Hence, the function ψ is the solution to the ODE above. Since $\psi(0) = 0$, necessarily,

$$\psi(t) = \frac{1}{\sigma(t)}\int_0^t \sigma(\tau)\left(\frac{\partial L}{\partial q^i}(\chi(\tau))\delta c^i(\tau) + \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau))\delta \dot{c}^i(\tau)\right)\mathrm{d}\tau, \tag{11}$$

where

$$\sigma(t) = \exp\left(-\int_0^t \frac{\partial L}{\partial z}(\chi(\tau))\mathrm{d}\tau\right) > 0. \tag{12}$$

Integrating by parts and using and that the variation vanishes at the endpoints, we get the following expression:

$$\begin{aligned}
T_c\mathcal{Z}(\delta c)(t) = \psi(t) = \delta c^i(t)&\frac{\partial L}{\partial \dot{q}^i}(\chi(t)) + \\
&\frac{1}{\sigma(t)}\int_0^\tau \delta c^i(\tau)\left(\sigma(\tau)\frac{\partial L}{\partial q^i}(\chi(\tau)) - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sigma(\tau)\frac{\partial L}{\partial \dot{q}^i}(\chi(\tau))\right)\right)\mathrm{d}\tau \\
&= \delta c^i(t)\frac{\sigma(\tau)}{\sigma(t)}\frac{\partial L}{\partial \dot{q}^i}(\chi(t)) + \int_0^\tau \delta c^i(\tau)\frac{\delta L}{\delta c^i}(\chi(\tau))\mathrm{d}\tau
\end{aligned}$$

where we used that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(t) = -\frac{\partial L}{\partial z}(\chi(t))\sigma(t). \tag{13}$$

We end this proof by noticing $T_c\mathcal{A}(\delta c) = T_c\mathcal{Z}(\delta c)(1)$ and that $\delta c(1) = 0$. ☐

1.2 The Herglotz principle: constrained version

Another way to understand this principle is as a constrained variational principle for curves on $Q \times \mathbb{R}$. This time, we will work on the manifold $\bar{\Omega}(q_0, q_1, z_0)$ of curves $\bar{c} = (c, \zeta) : [0,1] \to Q \times \mathbb{R}$ such that $c(0) = q_0$, $c(1) = q_1$, $\zeta(0) = z_0$. We do not constraint $\zeta(1)$. The tangent space at the curve c is given by

$$T_c\bar{\Omega}(q_0, q_1, z_0) = \{\delta\bar{c} = (\delta c, \delta\zeta) \in T(Q \times \mathbb{R}) \mid \delta c(0) = 0,\ \delta c(1) = 0, \delta z(0) = 0\}. \tag{14}$$

In this space, the action functional $\bar{\mathcal{A}}$ can be defined as an integral

$$\begin{aligned} \bar{\mathcal{A}} : \bar{\Omega}(q_0, q_1, z_0) &\to \mathbb{R}, \\ \bar{c} &\mapsto \zeta(1) - \zeta(0) = \int_0^1 \dot{\zeta}(t)\mathrm{d}t. \end{aligned} \tag{15}$$

We will restrict this action to the set of paths that satisfy $\dot{\zeta} = L$. For this, we consider the paths at the zero set of the constraint function ϕ_L:

$$\phi_L(q, \dot{q}, z, \dot{z}) = \dot{z} - L(q, \dot{q}, z). \tag{16}$$

That is, we consider

$$\bar{\Omega}_L(q_0, q_1, z_0) = \{\bar{c} = (c, \zeta) \in \bar{\Omega}(q_0, q_1, z_0) \mid \phi_L \circ \dot{\bar{c}} = \dot{\zeta} - L(c, \dot{c}, \zeta) = 0\}. \tag{17}$$

We notice that, since the Cauchy problem Eq. (8) has a unique solution, the elements $(c, \zeta) \in \bar{\Omega}_L(q_0, q_1, z_0)$ are precisely $(c, \mathcal{Z}_{z_0})$, where $c \in \Omega(q_0, q_1)$. That is, the map $\mathrm{Id} \times \mathcal{Z}_{z_0} : \Omega(q_0, q_1) \to \bar{\Omega}_L(q_0, q_1, z_0)$ given by $(\mathrm{Id} \times \mathcal{Z}_{z_0})(c) = (c, \mathcal{Z}_{z_0}(c))$ is a bijection, with inverse $(\mathrm{pr}_Q)_*(c, \zeta) = c$. Moreover, the following diagram commutes

$$\begin{tikzcd} & \mathbb{R} & \\ \Omega(q_0, q_1) \arrow[ur, "\mathcal{A}"] \arrow[rr, "\mathrm{Id} \times \mathcal{Z}_{z_0}"] & & \bar{\Omega}_L(q_0, q_1, z_0) \arrow[ul, "\bar{\mathcal{A}}"'] \end{tikzcd} \tag{18}$$

Hence $(c, \zeta) \in \bar{\Omega}_L(q_0, q_1, z_0)$ is a critical point of $\bar{\mathcal{A}}$ if and only if c is a critical point of $\mathcal{A}$. So the critical points of $\mathcal{A}$ restricted to $\bar{\Omega}_L(q_0, q_1, z_0)$ are precisely the curves that satisfy the Herglotz equations. Indeed:

Theorem 2 (Herglotz variational principle, constrained version)**.** *Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian function and let $(c, \zeta) \in \bar{\Omega}_L(q_0, q_1, z_0)$. Then, $(c, \dot{c}, \zeta)$ satisfies the Herglotz equations:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial \dot{q}^i}\frac{\partial L}{\partial z},$$

if and only if (c, ζ) is a critical point of $\bar{\mathcal{A}}|_{\Omega_L(q_0, q_1, z_0)}$.

We will also provide an alternate proof. We find directly the critical points of $\bar{\mathcal{A}}$ restricted to $\bar{\Omega}_L(q_0, q_1, z_0) \subseteq \bar{\Omega}(q_0, q_1, z_0))$ using the following infinite-dimensional version of the Lagrange multiplier theorem [**Arnold1997**].

Theorem 3 (Lagrange multiplier Theorem). *Let M be a smooth manifold and let E be a Banach space such that $g : M \to E$ is a smooth submersion, so that $A = g^{-1}(\{0\})$ is a smooth submanifold. Let $f : M \to \mathbb{R}$ be a smooth function. Then $p \in A$ is a critical point of $f|_A$ if and only if there exists $\hat{\lambda} \in E^*$ such that p is a critical point of $f + \hat{\lambda} \circ g$.*

Proof of Herglotz variational principle, constrained version. We will apply this result to our situation. In the notation of the theorem, $M = \bar{\Omega}(q_0, q_1, z_0)$ is the smooth manifold. We pick the Banach space $E = L^2([0,1] \to \mathbb{R})$ of square integrable functions. This space is, indeed, a Hilbert space with inner product

$$\langle \alpha, \beta \rangle = \int_0^1 \alpha(t)\beta(t) \mathrm{d}t. \tag{19}$$

We remind that, by the Riesz representation theorem, there is a bijection between $L^2([0,1] \to \mathbb{R})$ and its dual such that for each $\hat{\alpha} \in L^2([0,1] \to \mathbb{R})^*$ there exists $\alpha \in L^2([0,1] \to \mathbb{R})$ such that $\hat{\alpha}(\beta) = \langle \alpha, \beta \rangle$ for all $\beta \in L^2([0,1] \to \mathbb{R})$.

Our constraint function is

$$\begin{aligned} g : \bar{\Omega}(q_0, q_1, z_0) &\to L^2([0,1] \to \mathbb{R}), \\ \bar{c} &\mapsto (\phi_L) \circ (\bar{c}, \dot{\bar{c}}), \end{aligned} \tag{20}$$

where ϕ_L is a constraint locally defining N. Note that $A = g^{-1}(0) = \bar{\Omega}_L(q_0, q_1, z_0)$.

By Theorem 3, c is a critical point of $f = \bar{\mathcal{A}}$ restricted to $\bar{\Omega}_L(q_0, q_1, z_0)$ if and only if there exists $\hat{\lambda} \in L^2([0,1] \to \mathbb{R})^*$ (which is represented by $\lambda \in L^2([0,1] \to \mathbb{R})$) such that c is a critical point of $\bar{\mathcal{A}}_\lambda = \bar{\mathcal{A}} + \hat{\lambda} \circ g$.

Indeed,

$$\bar{\mathcal{A}}_\lambda = \int_0^1 L_\lambda(\bar{c}(t), \dot{\bar{c}}(t)) \mathrm{d}t, \tag{21}$$

where

$$L_\lambda(q, z, \dot{q}, \dot{z}) = \dot{z} - \lambda \phi_L(q, z, \dot{q}, \dot{z}). \tag{22}$$

Since the endpoint of ζ is not fixed, the critical points of this functional $\bar{\mathcal{A}}_\lambda$ are the solutions of the Euler-Lagrange equations for L_λ that satisfy the natural boundary condition:

$$\frac{\partial L_\lambda}{\partial \dot{z}}(\bar{c}(1), \dot{\bar{c}}(1)) = 1 - \lambda(1) \frac{\partial \phi_L}{\partial \dot{z}}(\bar{c}(1), \dot{\bar{c}}(1)) = 0. \tag{23}$$

For $\phi_L = \dot{z} - L$, this condition reduces to $\lambda(1) = 1$.

The Euler-Lagrange equations of $\mathcal{L}$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda(t) \frac{\partial \phi_L(\bar{c}(t), \dot{\bar{c}}(t))}{\partial \dot{q}^i} \right) - \lambda(t) \frac{\partial \phi_L(\bar{c}(t), \dot{\bar{c}}(t))}{\partial q^i} = 0 \tag{24a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda(t) \frac{\partial \phi_L(\bar{c}(t), \dot{\bar{c}}(t))}{\partial \dot{z}} \right) - \lambda(t) \frac{\partial \phi_L(\bar{c}(t), \dot{\bar{c}}(t))}{\partial z} = 0, \tag{24b}$$

since $\phi_L = \dot{z} - L$, the equation (24b) for z is just

$$\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} = -\lambda(t) \frac{\partial L}{\partial z}. \tag{25}$$

Substituting on (24a) and dividing by λ, we obtain the Herglotz equations. □

2 The Herglotz principle for fields

We will discuss the approach considered by [4] to the Herglotz principle to fields. In the literature, there exists also a non-covariant approach [3], but, as it is pointed out in [4], it can be seen as a particular case of the fist one.

Given a Lagrangian $L(x^\mu, u^a, u^a_\mu, z^\mu)$, which depends on the positions x^μ in an m-dimensional spacetime M, the values of fields u^a, their derivatives u^a_i at the point x and the variables z^μ that, in this context do not represent the action, but the *action density*. In order to compute the action of a local field σ defined on $D \subseteq M$, we find a vector field ζ^μ such that

$$\partial_\mu \zeta^\mu = L. \tag{26}$$

Then, the action is then

$$\int L \mathrm{d}^n x = \int \partial_\mu \zeta^\mu \mathrm{d}^n x = \int_{\partial D} \zeta^\mu \eta_\mu \mathrm{d}\sigma, \tag{27}$$

where η_μ is the the unit normal vector to the surface and $\mathrm{d}\sigma$ is the surface differential. The last equality follows from Stoke's Theorem. We note that in the case that M is 1-dimensional, the action is just $\zeta(1) - \zeta(0)$. Hence, we recover the Herglotz action for mechanics.

The critical points of this action among the local fields σ with the same values on the boundary would be the solutions two the Herglotz equations for fields:

$$\partial_\mu \left(\frac{\partial L}{\partial u^a_\mu} \right) - \frac{\partial L}{\partial u^a} = \frac{\partial L}{\partial u^a} \frac{\partial L}{\partial z^\mu}. \tag{28}$$

This equations are obtained through an implicit argument, in a similar spirit to the proof of Theorem 1. We also remark that the Lagrangian theory of k-contact fields [2] provides similar equations.

However, we find two issues on this derivation of the variational principle. First of all, the definition of z^μ in Eq. (26) depends on a metric on M in order to compute its divergence. This can be easily fixed by taking z^μ to be components of a $(k-1)$-form instead of a vector field.

The second issue issue is more subtle. The solution of Eq. (26) is not unique, hence the action is not well-defined. This is not a problem in the case that the Lagrangian does not depend on ζ^μ, because then all solutions to Eq. (27) differ only by an exact term that has 0 integral and does not contribute to the action, but that does not happen in general. Indeed, ζ may appear on the equation Eq. (27). This does not occur on the examples present in [4], since the Lagrangians only depend linearly on z, but this is apparent for more complex Lagrangians. Moreover Eq. (26) might not have solution, hence we will need to add more equations to ensure the existence.

One way to fix this problem is to prescribe boundary conditions on Eq. (26) that makes the solution unique. Moreover Eq. (26) might not have solution, hence we will need to add more equations to ensure the existence. However, we will avoid this problem choosing a "constrained formulation" of this problem, in the same spirit of Theorem 2, instead of the "implicit" approach used in [4].

2.1 Herglotz principle for fields: constrained version

Let M be our spacetime manifold and consider a fiber bundle $E \to M$. The fields $\sigma^a(x_\mu)$ are local sections of this bundle. Let j^1E be the first jet bundle of E, with natural coordinates (x^μ, u^a, u^a_μ). The action densities will be $(m-1)$-forms on M, hence the Lagrangian density will take values on the bundle of $(m-1)$-forms in addition to j^1E. That is $\mathcal{L} : j^1E \oplus_M \Lambda^{m-1}M \to \Lambda^m M$ is a fiber bundle morphism over M.

Given in a local coordinates x^μ of M, we can construct a basis $*(\mathrm{d}x)_\mu = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{\mu-1} \wedge \mathrm{d}x^{\mu+1} \wedge \ldots \wedge \mathrm{d}x^n$ of the $(m-1)$-forms. This basis induces a set of coordinates z^μ on Λ^{m-1}. If, in local coordinates, $\mathcal{L}(x^\mu, u^a, u^a_\mu, z^\mu) = L(x^\mu, u^a, u^a_\mu, z_a)\mathrm{d}^n x$, and $\zeta = \zeta^\mu * (\mathrm{d}x)_\mu$ is an m-form. If $\sigma \in \Gamma_M E$, its 1-jet is $j^1\sigma = (\sigma^a, \partial_\mu \sigma^a)$. Then Eq. (26) may be written intrinsically as

$$\mathrm{d}\zeta = \mathcal{L} \circ (j^1\sigma, \zeta), \tag{29}$$

That is, in local coordinates this equation is just

$$\partial_\mu \zeta^\mu(x^\mu) = L(x^\mu, \sigma^a, \partial^a_\mu, z_a). \tag{30}$$

The situation is described by the following commutative diagram

$$\begin{tikzcd}
J^1E \arrow[d] & & J^1E \bigoplus_M \Lambda^{m-1}M \arrow[ll] \arrow[d] \arrow[dr, "\mathcal{L}"] & \\
E \arrow[drr] & & \Lambda^{m-1} \arrow[d] & \Lambda^M M \\
& & M \arrow[uull, "j^1\sigma"] \arrow[ull, bend left, "\sigma"] \arrow[uu, bend right, "(j^1\sigma, d\zeta)"'] \arrow[u, bend right, "\zeta"'] \arrow[ur, "d\zeta"'] &
\end{tikzcd} \tag{31}$$

Now,let $D \subseteq M$ be a compact manifold with boundary diffeomorphic to the n-dimensional closed ball. We fix a section $\bar{\rho} = (\rho, \tau) : \partial D \to E \oplus_M \Lambda^{m-1}M$ of boundary values. We consider the space of sections of $E \oplus_M \Lambda^{m-1}M$ satisfying the boundary condition

$$\bar{\Omega}(\bar{\rho}) = \{\bar{\sigma} = (\sigma, \zeta) \in \Gamma_D(E \oplus_M \Lambda^{m-1}M \mid \bar{\sigma}|_{\partial D} = \bar{\rho}\}. \tag{32}$$

Thus, its tangent space on $\bar{\sigma}$ is given by vector fields over $\bar{\sigma}$ that vanish on the boundary, that is

$$\begin{aligned} T_{\bar{\sigma}}\bar{\Omega}(\bar{\rho}) = \{\delta\bar{\sigma} = (\delta\sigma, \delta\rho) : D \to T(E \oplus_M \Lambda^{m-1}M) \mid \delta\bar{\sigma}|_{\partial D} = 0, \\ \delta\bar{\sigma}(x) \in T_{\bar{\sigma}(x)}(E \oplus_M \Lambda^{m-1}M) \text{ for all } x \in M\}. \end{aligned} \tag{33}$$

We now constrain the space of sections so that the condition on the action flow Eq. (29) is satisfied. That is

$$\bar{\Omega}_{\mathcal{L}}(\bar{\rho}) = \{(\sigma, \zeta) \in \bar{\Omega}(\bar{\rho}) \mid \mathrm{d}\zeta = \mathcal{L}(j^1\sigma, \zeta)\} \tag{34}$$

Now we define the action $\bar{\mathcal{A}} : \bar{\Omega}(\bar{\rho}) \to \mathbb{R}$ as

$$\bar{\mathcal{A}}(\zeta, \rho) = \int_D \mathrm{d}\zeta = \int_{\partial D} \zeta. \tag{35}$$

Notice that for a constrained section $(\sigma,\zeta)\in\bar{\Omega}_{\mathcal{L}}(\bar{\rho})$ one also has that

$$\bar{\mathcal{A}}(\zeta,\rho)=\int_D \mathcal{L}\circ(j^1\sigma,\zeta), \tag{36}$$

hence this action coincides with the usual definition for Lagrangian densities independent on z_μ.

The critical points of this action satisfy the following

Theorem 4 (Herglotz variational principle, constrained version)**.** *Let $\mathcal{L}: j^1E\oplus_M \Lambda^{m-1}M\to\Lambda^m M$ be a Lagrangian density and let $(\sigma,\zeta)\in\bar{\Omega}_{\mathcal{L}}(\bar{\rho})$. Then, $(j^1\sigma,\zeta)$ satisfies the Herglotz equations for fields:*

$$\partial_\mu\left(\frac{\partial L}{\partial u^a_\mu}\right)-\frac{\partial L}{\partial u^a}=\frac{\partial L}{\partial u^a}\frac{\partial L}{\partial z_\mu}. \tag{37}$$

$$\partial_\nu\frac{\partial L}{\partial z^\mu}=\partial_\mu\frac{\partial L}{\partial z^\nu}. \tag{38}$$

if and only if (c,ζ) is a critical point of $\bar{\mathcal{A}}|_{\Omega_{\mathcal{L}}(\bar{\rho})}$.

Proof. By an argument based on the Lagrange multiplier theorem, similar to the one presented in the proof of Theorem 2, we can deduce that $(\sigma,\zeta)\in\bar{\Omega}_{\mathcal{L}}(\bar{\rho})$ is a critical point of

$$\bar{\mathcal{A}}_\lambda=\int_D \mathcal{L}_\lambda(j^1\bar{\sigma}), \tag{39}$$

for some $\lambda: D\to\mathbb{R}$, where $\mathcal{L}_\lambda=L_\lambda \mathrm{d}^n x$,

$$L_\lambda(x^\mu,u^a,u^a_\mu,\zeta^\mu,\zeta^\mu_\nu)=\zeta^\mu_\mu-\lambda(x^\mu)\phi_L(x^\mu,u^a,u^a_\mu,\zeta^\mu,\zeta^\mu_\nu), \tag{40}$$

and

$$\phi_{\mathcal{L}}(x^\mu,u^a,u^a_\mu,\zeta^\mu,\zeta^\mu_\nu)=\zeta^\mu_\mu-L(x^\mu,u^a,u^a_\mu,\zeta^\mu) \tag{41}$$

are the constraints imposed by (29).

The Euler-Lagrange equations of $\mathcal{L}_\lambda$ are given by

$$\partial_\mu\left(\lambda(x^\mu)\frac{\partial\phi_{\mathcal{L}}(j^1\bar{\sigma})}{\partial u^a_\mu}\right)-\lambda(x^\mu)\frac{\partial\phi_{\mathcal{L}}(j^1\bar{\sigma})}{\partial u^a}=0 \tag{42a}$$

$$\partial_\mu\left(\lambda(x^\mu)\frac{\partial\phi_{\mathcal{L}}(j^1\bar{\sigma})}{\partial \zeta^\nu_\mu}\right)-\lambda(x^\mu)\frac{\partial\phi_{\mathcal{L}}(j^1\bar{\sigma})}{\partial \zeta^\nu}=0, \tag{42b}$$

Substituting the value of $\phi_{\mathcal{L}}$ in Eq. (24b) we obtain

$$\partial_\mu\lambda(x^\mu)=-\lambda\frac{\partial L}{\partial z^\mu}(x^\mu,j^1\sigma(x^\mu),\zeta(x^\mu)). \tag{43}$$

If non-vanishing solutions to this equation exist, then expanding Eq. (24a) and dividing by λ we obtain the Herglotz's equation. However, some extra condition must be fulfilled. Taking $g=\log(|\lambda|)$, Eq. (43) gets converted into

$$\mathrm{d}g=\pm\frac{\partial L}{\partial z^\mu}\circ(j^1\sigma,\zeta) \tag{44}$$

This has solution if and only if the right hand side is closed. That is,

$$\partial_\nu \frac{\partial L}{\partial z^\mu} = \partial_\mu \frac{\partial L}{\partial z^\nu}. \tag{45}$$

If this condition is fulfilled, since D is a ball then,

$$\frac{\partial L}{\partial z^\mu} \circ (j^1\sigma, \zeta) = \mathrm{d}h. \tag{46}$$

So we pick $g = h$.

□

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