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Arnau Mas Dorca

Supervised by

Dr Jordi Gaset

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Chapter 1

The Herglotz variational problem

In this chapter we develop the theory of *action-dependent Lagrangians*. The main appeal of this formalism is that it allows the description of non-conservative systems in terms of a variational principle, which is in general not possible with standard Lagrangian mechanics. The problem of finding the stationary paths of the action given by a Lagrangian of this sort is known as the Herglotz problem. The main difficulty of this variational problem is that, as opposed to the standard variational problem of Lagrangian mechanics, is a constrained optimisation problem. One way to approach this problem is the use of Lagrange multipliers. We show how this leads to the equations of motions for this kind of systems, the Herglotz equations, and how it can also be applied to field theory to derive the field theoretical Herglotz equations.

1.1 The implicit formulation of the Herglotz problem

Let's briefly describe what we will refer to as the *implicit formulation of the Herglotz problem*, as presented in [Laz+18]. Let us first clarify what we mean by an action-dependent Lagrangian. The idea is to consider the action as a dynamical quantity that changes along the path, and then allow the Lagrangian to depend on it. Naively, we would write the following. Starting with some path $q^\mu \colon [a, b] \to M$ in some configuration space M[1] we would write something like

$$S[q^\mu] = \int_a^b L(q^\mu(t), \dot{q}^\mu(t), S(t))\, \mathrm{d}t$$

where $S(t)$ is the action of the path until time t. Of course this makes no sense since we are defining S on the left-hand side, and it appears on the right-hand side! We can turn

[1] M could be space in the context of classical mechanics or spacetime in the context of relativity

this expression into something sensible if we add the time dependence of the action on the left-hand side, so that we write

$$S[q^\mu](t) = \int_a^t L(q^\mu(s), \dot{q}^\mu(s), S[q^\mu](s))\,\mathrm{d}s \tag{1.1}$$

and if we differentiate with respect to time we actually get an ODE for $S[q^\mu]$! Indeed

$$\dot{S}[q^\mu](t) = L(q^\mu(t), \dot{q}^\mu(t), S[q^\mu](t)). \tag{1.2}$$

Notice that Equation (1.1) actually forces the initial condition $S[q^\mu](a) = 0$. We can even drop this requirement, since all we are interested in is the difference of values of S:

$$S[q^\mu](b) - S[q^\mu](a) = \int_a^b L(q^\mu(t), \dot{q}^\mu(t), S[q^\mu](t))\,\mathrm{d}t. \tag{1.3}$$

What we have here is a functional which, for every path, is defined by an ODE. To find the stationary paths of this functional we would, in principle, have to solve Equation (1.2) for any possible path and among all of them find which ones yield extrema. This is the *Herglotz variational problem*. This approach is in general not feasible. However, just like the variational problem of classical Lagrangian mechanics can be turned into a set of ODEs, the Euler-Lagrange equations, so can the Herglotz problem be turned into a set of ODEs. These are known as the Herglotz equations, which can be written down as

$$\frac{\partial L}{\partial q^\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\mu} + \frac{\partial L}{\partial \dot{q}^\mu}\frac{\partial L}{\partial z} = 0 \tag{1.4}$$

where the Lagrangian is $L(q^\mu, \dot{q}^\mu, z)$, z being the action dependence. These equations differ from the Euler-Lagrange equations only by one term. And in fact, if L is action-independent, thus $\frac{\partial L}{\partial z} = 0$, we recover exactly the Euler-Lagrange equations.

See [Laz+18] or §3.2 of [LL21] for a detailed derivation of the Herglotz equations following the implicit approach.

1.1.1 Example: the damped harmonic oscillator

Before we move forward, let's see what kind of equations of motion we get from the Herglotz equations. The simplest action-dependence we can introduce —other than no action-dependence at all, of course— is a term linear in z. We will call this linear dissipation for reasons that will become clear in a moment. Let's see what happens when we add this term to the simple harmonic oscillator. Explicitly, consider the Lagrangian

$$L(q, \dot{q}, z) = \tfrac{1}{2}m\dot{q}^2 - \tfrac{1}{2}m\omega^2 q^2 - \gamma z.$$

A straightforward computation shows that for this Lagrangian, Equation (1.4) becomes

$$-m\omega^2 q - m\ddot{q} - \gamma m\dot{q} = 0$$

which after some rearranging can be turned into

$$\ddot{q} + \gamma\dot{q} + \omega^2 q = 0, \tag{1.5}$$

which one recognises as the equation of motion of the damped harmonic oscillator. The Herglotz principle delivers on its promise: we have just derived the equations of motion of a fundamentally non-conservative system from a variational problem! This also clarifies why we called the action dependence a dissipation term: the coefficient γ is related to the damping of the system and governs the rate at which energy is lost after every cycle.

1.2 The Herglotz problem as constrained optimisation

We have derived the Herglotz equations, so it would seem we have already solved the theory of action-dependent Lagrangians. However, general relativity is a field theory, so if we wish to understand an action-dependent variant of it we have to know the form the Herglotz equations take for field theory. If we try to apply the implicit method to field theory we run into a number of problems. For one, it is just not very elegant. The action functional is defined implicitly through an ordinary differential equation, one for every path. What's more, this equation is no longer an ODE in field theory, but rather it becomes a PDE. PDEs are notoriously much more difficult to solve thant ODEs, so we would like a way to circumvent this issue.

The idea we describe in what follows is being developed by Manuel Lainz, a PhD student at ICMAT. This is currently in a preprint stage, to which I have been graciously given access. It is part of broader ongoing research on the mathematics underpinning the theory of action-dependent field theory. This approach is a more elegant way of deriving the Herglotz equations, but as we will see, it also allows one to perform a direct variation of the action even for second-order theories such as general relativity.

The fundmanetal insight comes from framing the Herglotz problem as a constrained optimisation problem. We describe what this looks like for mehcanics. First, instead of considering paths in spacetime, $q^\mu \colon [a,b] \to M$, we enlarge the configuration space with one extra quantity, which we will call z. At this point z simply tracks a quantity that changes along the path, but we will later require that z actually match the action of the path at each time. This will be the constraint.

So, we have paths of the form $(q^\mu, z) \colon [a,b] \to M \times \mathbb{R}$. We define a functional on these paths as

$$S[q^\mu, z] = z(a) - z(b) = \int_a^b \dot{z}(t)\,\mathrm{d}t, \tag{1.6}$$

so S is just the change in z along the trajectory. This functional as it stands has no stationary paths, since we can find trajectories with arbitrarily large changes in z, both positive and negative. So we constrain the possible paths. Namely, we require that z actually represent the action. So we try to find the paths that extremise S only among those that satisfy the constraint

$$\dot{z}(t) = L(q^\mu(t), \dot{q}^\mu(t), z(t)), \tag{1.7}$$

where L is the action-dependent Lagrangian that describes the system. Notice that this is very similar to Equation (1.2).

Say(q^μ, z) is a trajectory that satisfies Equation (1.7). Then

$$S[q^\mu, z] = z(a) - z(b) = \int_a^b \dot{z}(t)\,\mathrm{d}t = \int_a^b L(q^\mu(t), \dot{q}^\mu(t), z(t))\,\mathrm{d}t.$$

So, for paths that satisfy the constraint, the functional S is indeed the action functional, understood as the integral of the Lagrangian along the path.

1.2.1 Lagrangian multipliers

This is all well and good, but how does one actually go about solving a constrained optimisation problem? It turns out, we can use an infinite dimensional analog of Lagrange multipliers to turn this into a regular optimisation problem to which we can apply the tools of the calculus of variations.

Recall, given some function $f \colon \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$ is an extremum of f subject to m constraints $G_k \colon \mathbb{R}^n \to \mathbb{R}$ —i.e. $g_k(x) = 0$ — if and only if there exist numbers $\lambda_k \in \mathbb{R}$ such that x is an extremum of the function

$$F = f - \sum_{k=1}^{m} \lambda_k g_k \tag{1.8}$$

without any constraints. The numbers λ_k are called the Lagrange multipliers. It can be shown that this result generalises to infinite dimensional spaces and infinitely many constraints. In our case, the function we are trying to find the extrema of is the functional S. Our constraints are parameterised by $t \in [a, b]$:

$$G_t[q^\mu, z] = \dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t)).$$

Thus, replacing sums with integrals in Equation (1.8), the extrema of Equation (1.6) subject to Equation (1.7) will be those that extremise the following functional:

$$\begin{aligned}
\tilde{S}[q^\mu, z, \lambda] &= S[q^\mu, z] - \int_a^b \lambda_t g_t[q^\mu, z]\,\mathrm{d}t \\
&= \int_a^b \dot{z}(t)\,\mathrm{d}t - \int_a^b \lambda_t\Big[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\Big]\,\mathrm{d}t \\
&= \int_a^b (1-\lambda_t)\dot{z}(t) + \lambda_t\Big(L(q^\mu(t), \dot{q}^\mu(t), z(t))\Big)\,\mathrm{d}t.
\end{aligned} \tag{1.9}$$

A couple of observations. First, we are thinking of the Lagrange multipliers as real numbers parameterised by t, but we could equivalently think of them as a function of t and write $\lambda(t)$ instead of λ_t. We will do this. Additionally, we introduced λ as a dynamical variable of the action functional. When we take the variation of the action with respect to λ we will actually recover the constraint.

1.2.2 Deriving the Herglotz equations

If we write the integrand of Equation (1.9) as

$$\tilde{L}(q^\mu(t), \dot{q}^\mu(t), z(t), \dot{z}(t)) = \Big(1 - \lambda(t)\Big)\dot{z}(t) + \lambda(t)\Big(L(q^\mu(t), \dot{q}^\mu(t), z(t))\Big)$$

one should be able to recognise $\tilde{S}$ as something that looks just like a regular old action functional defined by the integral of a regular old Lagrangian, except it is now defined on expanded trajectories (q^μ, z). So we should be able to use the Euler-Lagrange equations to write down the equations of motion of its extremal paths! The equation for z reads

$$0 = \frac{\partial \tilde{L}}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{z}} = \lambda\frac{\partial L}{\partial z} + \dot{\lambda}$$

or equivalently

$$\dot{\lambda} = -\lambda\frac{\partial L}{\partial z}. \tag{1.10}$$

The Euler-Lagrange equations for the positions then are

$$0 = \frac{\partial \tilde{L}}{\partial q^\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}^\mu} = \lambda \frac{\partial L}{\partial q^\mu} - \dot{\lambda}\frac{\partial L}{\partial \dot{q}^\mu} - \lambda \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\mu}$$

and after substituting in Equation (1.10) and dividing through by λ we find

$$0 = \frac{\partial L}{\partial q^\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\mu} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}^\mu} \tag{1.11}$$

which are exactly the Herglotz equations.

Let us for completeness write down the equation that results from taking the variation with respect to λ, which is

$$-\dot{z}(t) + L\left(q^\mu(t), \dot{q}^\mu(t), z(t)\right) = 0$$

which is exactly the constraint.

1.3 Action-dependent field theory

We have seen how to derive the Herglotz equations in a more elegant way using Lagrange multipliers. More importantly, we have written down a modified Lagrangian which gives rise to them through standard calculus of variations techniques. This will be very useful as, in some cases (general relativity being one of them), it is easier to derive the equations of motion of a system by direct variation of the action, rather than writing down the Euler-Lagrange equations. We will be able to do exactly this once we have the field theoretic version of Equation (1.9) in our hands.

1.3.1 Classical field theory and Lagrangian densities

First off, we need to set the stage for Lagrangian field theory. The parameter space is no longer just time, but rather all of spacetime, M. Fields are the assignment of some value to each point in spacetime, so we could have scalar fields, vector fields, or, as is the case in general relativity, tensor fields. Let us first fix some notation. We will denote by ϕ some field configuration. If ϕ is a scalar field then it carries no indices. If ϕ is a vector field then it carries one upper index, if it is a tensor field then it carries multiple indices. The metric carries two lower indices since it is a $(0,2)$ tensor field. In almost all that follows we will assume ϕ is a vector field, but the results we find transfer to tensors of other rank. Of course ϕ depends on spacetime, which we will sometimes write explicitly as $\phi^a(x^\mu)$.

As a convention, we will use latin indices for the indices of the field, and reserve greek indices for spacetime coordinates. The Einstein summation convention is assumed to be in place unless otherwise stated.

The Lagrangian of a field theory is a function of the field and of its derivatives. If it only contains first derivatives the theory is called a *first order theory*. General relativity is actually a second order theory, as we will discuss later. However, most of the following discussion still applies to general relativity.

The Lagrangian in field theory does not take value in the real numbers. To define the action we must integrate the Lagrangian, but as opposed to in mechanics where one integrates over time, in field theory this integration is performed over a patch of spacetime. Because spacetime is in general a curved manifold we will need to use the language of differential forms, which are the objects that can be integrated over manifolds. In general, a differential k-form[2] can be integrated over a manifold of dimension k. So if spacetime has dimension n, the Lagrangian has to be a differential n-form, alsow known as a top form.

With some care, however, we can still think of the Lagrangian as a function with real values. It turns out that any two top forms differ only by an overall factor, i.e., given two top forms ω_1 and ω_2, there exists a unique scalar function $f\colon M \to \mathbb{R}$ spacetime M such that $\omega_1 = f\omega_2$. So what this means is that, for a given Lagrangian $\mathcal{L}$, once we pick a distinguished top form there is a unique scalar function L such that $\mathcal{L} = L\omega$. This distinguished top form will in most cases be the top form induced by the coordinates we are working in, which we will write as $\mathrm{d}^n x$. Sometimes we will use Lagrangian density to refer to $\mathcal{L}$ and Lagrangian to refer to L.

The setup in classical field theory is as follows. Given some Lagrangian, which encodes the system we are studying, we define the action functional on all the possible field configurations as

$$S[\phi^a] = \int_D \mathcal{L}(\phi^a(x^\mu), \partial_\mu \phi^a(x^\mu)) = \int_D L(\phi^a(x^\mu), \partial_\mu \phi^a(x^\mu))\, \mathrm{d}^n x,$$

where D is some region of spacetime where this integral makes sense.

Using the calculus of variations one can show that the stationary configurations of this action functional satisfy the Euler-Lagrange equations of field theory

$$\frac{\partial L}{\partial \phi^a} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi^a)} = 0.$$

[2]A differential k-form is a k-multilinear alternating form that acts on tangent vectors

1.3.2 The action flux

How do we generalise this to an action dependent Lagrangian? The most naive approach would be to try to replicate the Herglotz equations from mechanics wholesale and write down

$$\frac{\partial L}{\partial \phi^a} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi^a)} + \frac{\partial L}{\partial(\partial_\mu \phi^a)} \frac{\partial L}{\partial z} = 0.$$

However this will not work because the last term has a pesky free μ index. This seems to suggest that we need to modify the nature of z. We had claimed before that z represented the action along the path, but if we look at Equation (1.6) we see this is not quite right. In mechanics the analog of D is $[a, b]$. The difference $z(a) - z(b)$ can also be written as $\int_{\partial[a,b]} z$, since the boundary of $[a, b]$ is just a and b. This seems to indicate that the correct analog of Equation (1.6) for field theory should be

$$S[\phi^a, z] = \int_{\partial D} z.$$

What kind of obejct should z be then? ∂D has dimension $n - 1$, so z has to be something we can integrate over an $(n-1)$-dimensional manifold, i.e. a differential $(n-1)$-form. Strictly speaking, then, z is an object with $n - 1$ lower indices. In the case of 4-dimensional spacetime, z would have 3 antisymmetrised indices:

$$z = z_{[\mu\nu\eta]}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\eta.$$

As it turns out, $(n-1)$-forms can be identified with vector fields. The idea is that instead of labeling the components of z by three indices, we label them by the missing index. Some signs appear because of the antisymmetry, which are encoded by the Levi-Civita symbol:

$$z^\mu = \epsilon^{\mu\nu\eta\alpha} z_{[\nu\eta\alpha]}.$$

So we have the identity

$$z = z^\mu\, \mathrm{d}x_\mu = z_{[\mu\nu\eta]}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\eta.$$

where

$$\mathrm{d}x^\mu = \epsilon_{\mu\nu\eta\alpha}\, \mathrm{d}x^\nu \wedge \mathrm{d}x^\eta \wedge \mathrm{d}x^\alpha.$$

But z is *not* a vector, since its components do not transform as the components of a vector, but rather as the components of a 3-form. This is very similar to the distinction between vectors and pseudo-vectors in 3-space.

Let's have a closer look at what z represents. We know that $\int_{\partial D} z$ has dimensions of action. The volume element of ∂D in natural units has dimensions of length^3, so the components of z have dimensions of action/length^3. This means that z is the *action flux*. From Stokes' theorem,

$$S[\phi^a, z^\mu] = \int_{\partial D} z = \partial_D \, \mathrm{d}z,$$

so $\mathrm{d}z$ has to be the action density, i.e. the Lagrangian. The constraint we want to impose is therefore

$$\mathrm{d}z(x^\mu) = \mathcal{L}(\phi^a(x^\mu), \partial_\mu \phi^a(x^\mu), z^\mu(x^\mu)) \tag{1.12}$$

analogous to Equation (1.7).

$\mathrm{d}z$ is the exterior derivative of z, which is a top form, so this equality makes sense. In coordinates it is easy to show that $\mathrm{d}z = \partial_\nu z^\nu \, \mathrm{d}^n x$, so that Equation (1.12) reads in coordinates as

$$\partial_\nu z^\nu(x^\mu) = L(\phi^a(x^\mu), \partial_\mu \phi^a(x^\mu)). \tag{1.13}$$

For field configurations that satisfy this constraint then one has, applying Stokes' theorem

$$S[\phi^a, z^\mu] = \int_{\partial D} z = \int_D \mathrm{d}z = \int_D \mathcal{L}(\phi^a, \partial_\mu \phi^a, z^\mu).$$

So we have arrived at the right formulation of the Herglotz variational problem for field theory.

1.3.3 Constrained optimisation in field theory

So just like before, we will turn this constrained optimisation problem into an unconstrained one using Lagrange multipliers. The expanded action, in analogy with Equation (1.9) will be

$$\tilde{S}[\phi^a, z^\mu] = \int_D (1-\lambda)\, \mathrm{d}z + \lambda \mathcal{L}(\phi^a, \partial_\mu \phi^a, z^\nu) = \int_D \mathrm{d}^n x \Big[(1-\lambda)\partial_\mu z^\mu + \lambda L(\phi^a, \partial_\mu \phi^a, z^\nu)\Big] \tag{1.14}$$

Note that the Lagrange multiplier λ is a function of spacetime. Like in mechanics, the Lagrange multiplier is technically also a dynamical quantity, but variation with respect to it just gives back the constraint. Let us write down the integrand of Equation (1.14) as an expanded Lagrangian:

$$\tilde{\mathcal{L}}(\phi^a, \partial_\mu \phi^a, z^\nu, \partial_\mu z^\nu) = \tilde{L}(\phi^a, \partial_\mu \phi^a, z^\nu, \partial_\mu z^\nu)\, \mathrm{d}^n x = \Big[(1-\lambda)\partial_\mu z^\mu + \lambda L(\phi^a, \partial_\mu \phi^a, z^\nu)\Big]\, \mathrm{d}^n x. \tag{1.15}$$

The Euler-Lagrange equations for this Lagrangian are the Herglotz equations for the theory. We write them down in the next section. Note that they are actually the Herglotz equations for a first-order field theory. If the theory is second order then the correct equations are the second-order Euler-Lagrange equations, which include additional terms. Alternatively, we could still perform a direct variation of the action to get the field equations, without worrying about the order of the theory. This is the procedure we will follow in the next chapter.

1.3.4 The Herglotz equations for field theory

Finally, we can derive the Herglotz equations for field theory from the expanded Lagrangian we have just obtained in Equation (1.15). The equations for the action flux are

$$0 = \frac{\partial \tilde{L}}{\partial z^\nu} - \partial_\mu \frac{\partial \tilde{L}}{\partial(\partial_\mu z^\nu)} = \lambda \frac{\partial L}{\partial z^\nu} + \partial_\mu(\lambda \delta^\mu_\nu) = \lambda \frac{\partial L}{\partial z^\nu} + \partial_\nu \lambda.$$

So, rearranging,

$$\partial_\nu \lambda = -\lambda \frac{\partial L}{\partial z^\nu}. \tag{1.16}$$

And for the values of the field

$$0 = \frac{\partial \tilde{L}}{\partial \phi^a} - \partial_\mu \frac{\partial \tilde{L}}{\partial(\partial_\mu \phi^a)} = \lambda \frac{\partial L}{\partial \phi^a} - (\partial_\mu \lambda) \frac{\partial L}{\partial(\partial_\mu \phi^a)} - \lambda \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi^a)}$$

and, when plugging in Equation (1.16) and dividing through by λ we arrive at the field theoretical Herglotz equations

$$\frac{\partial L}{\partial \phi^a} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi^a)} + \frac{\partial L}{\partial z^\mu} \frac{\partial L}{\partial(\partial_\mu \phi^a)} = 0. \tag{1.17}$$

Chapter 2

2.1 The Einstein-Hilbert Lagrangian

As is well-known, the Einstein field equations can be derived by means of a variational principle. The Lagrangian that gives rise to these equations is the Einstein-Hilbert Lagrangian. Let's write it down in the language of Section 1.3.1. The Einstein-Hilbert action acts on metrics. And given a metric g, one can construct a top form, which we will write ω_g. Choosing coordinates one has $\omega_g = \sqrt{g}\, \mathrm{d}^4x$, where $\sqrt{g}$ is the square root of the determinant of the expression of g in the coordinates that induce d^4x. This form is not the Einstein-Hilbert Lagrangian. The other element is the scalar curvature $R = g^{ab}R_{ab}$, which is a Lorentz invariant scalar that encodes the curvature associated to g. The Einstein-Hilbert Lagrangian is

$$\mathcal{L}_{\text{E-H}}(g_{ab}, \partial_\mu g_{ab}, \partial_\mu\partial_\nu g_{ab}) = R\omega_g = R\sqrt{g}\, \mathrm{d}^4x. \tag{2.1}$$

From now on we will write $g_{ab,\mu}$ and $g_{ab,\mu\nu}$ instead of $\partial_\mu g_{ab}$ and $\partial_\mu\partial_\nu g_{ab}$.

The Einstein-Hilbert action is therefore

$$S_{\text{E-H}}[g_{ab}] = \int_D \mathcal{L}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}) = \int_D R\sqrt{g}\, \mathrm{d}^4x. \tag{2.2}$$

A variation of this action leads one to the Einstein field equations, which in natural units are

$$R_{ab} - \tfrac{1}{2}g_{ab}R = 0. \tag{2.3}$$

More precisely, these are the Einstein field equations in a vacuum, since one can add various matter terms to the Einstein-Hilbert Lagrangian which leads to the Einstein equations in the presence of matter,

$$R_{ab} - \tfrac{1}{2} g_{ab} R = T_{ab}. \tag{2.4}$$

The object T_{ab} is the energy-momentum tensor and collects all of the terms coming from the presence of matter. See §4 of [Car97] for a derivation of the Einstein field equations.

2.2 An action dependent Einstein-Hilbert Lagrangian

What kind of action dependence can we incorporate into the Einstein-Hilbert Lagrangian? The simplest one is perhaps the following

$$\mathcal{L}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, z^\mu) = R\omega_g - \theta \wedge z. \tag{2.5}$$

Recall, z is the action flux which is an $(n-1)$-differential form (so a 3-form in general relativity), so, if θ is a given covector field (a 1-form), the wedge $\theta \wedge z$ is a 4-form and Equation (2.5) makes perfect sense.

In coordinates we have

$$\theta \wedge z = (\theta_\mu \, \mathrm{d}x^\mu) \wedge (z^\nu \, \mathrm{d}x_\nu) = \theta_\mu z^\nu \, \mathrm{d}x^\mu \wedge \, \mathrm{d}x_\nu = \theta_\mu z^\nu \delta^\mu_\nu \, \mathrm{d}^4 x = \theta_\mu z^\mu \, \mathrm{d}^4 x$$

so

$$\mathcal{L}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, z^\mu) = (R\sqrt{g} - \theta_\mu z^\mu) \, \mathrm{d}^4 x. \tag{2.6}$$

This Lagrangian does not quite match the one proposed in eq. (9) of [Laz+17]. It can be shown that they are in fact one and the same. If instead of choosing the basis $\mathrm{d}x^\mu$ for the 3- forms we expand with respect to the basis $\omega_{g,\mu} = \sqrt{g} \, \mathrm{d}x_\mu$ we have

$$z = \zeta^\mu \omega_{g,\mu} = \zeta^\mu \sqrt{g} \, \mathrm{d}x_\mu$$

which implies $z^\mu = \sqrt{g}\zeta^\mu$. So, with these new coordinates for the action flux, Equation (2.6) looks like

$$\mathcal{L}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, \zeta^\mu) = (R\sqrt{g} - \theta_\mu \zeta^\mu \sqrt{g}) \, \mathrm{d}^4 x = (R - \theta_\mu \zeta^\mu)\omega_g \tag{2.7}$$

which is the same Lagrangian proposed in [Laz+17].

We now write down the constraint in Equation (1.12) for this Lagrangian. In the original coordinates for the action flux we have

$$\mathrm{d}z = \partial_\mu z^\mu \, \mathrm{d}^4 x$$

so, in coordinates

$$\partial_\mu z^\mu = R\sqrt{g} - \theta_\mu z^\mu. \tag{2.8}$$

If instead we choose the other coordinates for the action flux, we see

$$\mathrm{d}z = \partial_\mu(\sqrt{g}\zeta^\mu)\,\mathrm{d}^4x = \nabla_\mu\zeta^\mu\sqrt{g}\,\mathrm{d}^4x = \nabla_\mu\zeta^\mu\omega_g$$

where ∇ is the covariant derivative induced by g. In these coordinates the constraint takes the form

$$\nabla_\mu\zeta^\mu = R - \theta_\mu\zeta^\mu \tag{2.9}$$

which is the same form that appears in [Laz+17].

2.3 The expanded action

So we have seen what the Herglotz problem looks like for an Einstein-Hilbert Lagrangian with a linear action dependence. We will apply the method of Lagrange multipliers, as described in previous chapter, to derive a modified version of Einstein's equations. The expanded Lagrangian is

$$\tilde{\mathcal{L}}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, z^\nu, \partial_\mu z^\nu) = \Big[(1-\lambda)\partial_\mu z^\mu + \lambda(R\sqrt{g} - \theta_\mu z^\mu)\Big]\,\mathrm{d}^4x.$$

Let's therefore compute the variation of the action given by this Lagrangian

$$\begin{aligned}\delta\tilde{S}[g_{ab}, z^\nu] &= \int_D \Big[(1-\lambda)\delta\partial_\mu z^\mu + \lambda(\delta(R\sqrt{g}) - \theta_\mu\delta z^\mu)\Big]\,\mathrm{d}^4x \\ &= \int_D (1-\lambda)\partial_\mu\delta z^\mu - \lambda\theta_\mu\delta z^\mu\,\mathrm{d}^4x + \int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x \\ &= \int_D \partial_\mu\Big((1-\lambda)\delta z^\mu\Big)\,\mathrm{d}^4x + \int_D (\partial_\mu\lambda - \lambda\theta_\mu)\delta z^\mu\,\mathrm{d}^4x + \int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x.\end{aligned} \tag{2.10}$$

The first integral is a boundary term coming from an integration by parts. Since we are assuming the variations vanish at the boundary of D so must this boundary term also vanish. From the second integral we can read off, using the fundamental theorem of the calculus of variations, that

$$\partial_\mu\lambda = \lambda\theta_\mu. \tag{2.11}$$

Coordinate free this can also be written as $\mathrm{d}\lambda = \lambda\theta$. This has an interesting implication for θ since

$$\mathrm{d}(\lambda\theta) = \mathrm{d}\lambda\wedge\theta + \lambda\,\mathrm{d}\theta = \lambda\theta\wedge\theta + \lambda\,\mathrm{d}\theta = \lambda\,\mathrm{d}\theta$$

and

$$\lambda\,\mathrm{d}\theta = \mathrm{d}(\lambda\theta) = \mathrm{d}^2\lambda = 0$$

so we conclude $\mathrm{d}\theta = 0$, i.e. θ cannot be any 1-form, it must be a a *closed form.* This means that in coordinates $\partial_\mu\theta_\nu = \partial_\nu\theta_\mu$.

We retake the calculation from Equation (2.10). Since the integrals involving z and g decouple, we can just consider the last term. We will follow the derivation in [Car97] for as long as we can. Since $R\sqrt{g} = g^{ab}R_{ab}\sqrt{g}$, from the product rule its variation results in three terms:

$$\int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x = \int_D \lambda\delta g^{ab}R_{ab}\sqrt{g}\,\mathrm{d}^4x + \int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x + \int_D \lambda R\delta\sqrt{g}\,\mathrm{d}^4x \tag{2.12}$$

The first term is already in the form required to apply the fundamental theorem of the calculus of variations. For the third one uses the standard result

$$\delta\sqrt{g} = -\tfrac{1}{2}\sqrt{g}g_{ab}\delta g^{ab}.$$

The first and third terms of Equation (2.12) can be combined into

$$\int_D \lambda(R_{ab} - \tfrac{1}{2}Rg_{ab})\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x. \tag{2.13}$$

In the standard derivation of Einstein's equations, one shows that the middle integral of Equation (2.12) actually vanishes, so that Equation (2.13) must vanish for any variation δg_{ab}, or equivalently for any variation of the inverse metric δg^{ab}. Therefore the integrand itself must vanish, which gives Einstein's equations. In the presence of λ, however, the middle integral doesn't vanish and actually contributes additional terms to the equations.

The variation of the Ricci curvature can be shown to be

$$g^{ab}\delta R_{ab} = g^{ab}(\nabla_m\delta\Gamma^m{}_{ab} - \nabla_a\delta\Gamma^m{}_{mb}) = \nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb}) \tag{2.14}$$

so

$$\int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x = \int_D \lambda\nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x$$

and if λ weren't there this integral would vanish because of the divergence theorem and the fact that the variations vanish on the boundary of D. But λ is there, so we must work on this integral some more:

$$\int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x =$$

$$
\begin{aligned}
&= \int_D \lambda \nabla_n (g^{ab} \delta \Gamma^n{}_{ab} - g^{nb} \delta \Gamma^m{}_{mb}) \sqrt{g} \, \mathrm{d}^4 x \\
&= \int_D \lambda \partial_n \left(\sqrt{g} (g^{ab} \delta \Gamma^n{}_{ab} - g^{nb} \delta \Gamma^m{}_{mb}) \right) \, \mathrm{d}^4 x \\
&= \int_D \partial_n \left(\lambda \sqrt{g} (g^{ab} \delta \Gamma^n{}_{ab} - g^{nb} \delta \Gamma^m{}_{mb}) \right) \, \mathrm{d}^4 x - \int_D (\partial_n \lambda)(g^{ab} \delta \Gamma^n{}_{ab} - g^{nb} \delta \Gamma^m{}_{mb}) \sqrt{g} \, \mathrm{d}^4 x.
\end{aligned}
$$

The first integral vanishes because it is the integral of a divergence and the variations vanish on the boundary of D. So we are left with just the second term, which we split into two integrals. The variation of the Christoffel symbols can be shown to be

$$
\delta \Gamma^a{}_{bc} = \tfrac{1}{2} g^{am} (\nabla_c \delta g_{bm} + \nabla_b \delta g_{mc} - \nabla_m \delta g_{bc}) \tag{2.15}
$$

Using this and Equation (2.11) we compute for the first integral

$$
- \int_D (\partial_n \lambda) g^{ab} \delta \Gamma^n{}_{ab} \sqrt{g} \, \mathrm{d}^4 x = -\tfrac{1}{2} \int_D \lambda \theta_n g^{ab} g^{nk} (\nabla_b \delta g_{ak} + \nabla_a \delta g_{kb} - \nabla_k \delta g_{ab}) \sqrt{g} \, \mathrm{d}^4 x. \tag{2.16}
$$

The presence of g^{ab} means the indices a and b are symmetrised, so

$$
g^{ab} \nabla_b \delta g_{ak} = g^{ab} \nabla_a \delta g_{kb}.
$$

This means Equation (2.16) simplifies to

$$
\begin{aligned}
&- \int_D (\partial_n \lambda) g^{ab} \delta \Gamma^n{}_{ab} \sqrt{g} \, \mathrm{d}^4 x = \\
&= - \int_D \lambda \theta_n g^{ab} g^{nk} \nabla_b \delta g_{ak} \sqrt{g} \, \mathrm{d}^4 x + \tfrac{1}{2} \int_D \lambda \theta_n g^{ab} g^{nk} \nabla_k \delta g_{ab} \sqrt{g} \, \mathrm{d}^4 x \\
&= - \int_D \lambda \theta_n \nabla_b (g^{ab} g^{nk} \delta g_{ak}) \sqrt{g} \, \mathrm{d}^4 x + \tfrac{1}{2} \int_D \lambda \theta_n \nabla_k (g^{ab} g^{nk} \delta g_{ab}) \sqrt{g} \, \mathrm{d}^4 x.
\end{aligned} \tag{2.17}
$$

Let's try to perform an integration by parts for the first integral. We have to be a bit careful. Introducing the shorthand $X^{bn} = g^{ab} g^{nk} \delta g_{ak}$, we compute

$$
\nabla_c (\lambda \theta_n X^{bn}) = \nabla_c (\lambda \theta_n) X^{bn} + \lambda \theta_n \nabla_c X^{bn}
$$

so

$$
\begin{aligned}
- \int_D \lambda \theta_n \nabla_b (g^{ab} g^{nk} \delta g_{ak}) \sqrt{g} \, \mathrm{d}^4 x &= - \int_D \lambda \theta_n \nabla_b X^{bn} \sqrt{g} \, \mathrm{d}^4 x \\
&= - \int_D \nabla_b (\lambda \theta_n X^{bn}) \sqrt{g} \, \mathrm{d}^4 x + \int_D \nabla_b (\lambda \theta_n) X^{bn} \sqrt{g} \, \mathrm{d}^4 x.
\end{aligned}
$$

The first integral is the integral of a divergence, so it vanishes. We are left with the second which we can expand into

$$\begin{aligned}\int_D \nabla_b(\lambda\theta_n)X^{bn}\sqrt{g}\,\mathrm{d}^4x &= \int_D (\partial_b\lambda\theta_n + \lambda\nabla_b\theta_n)(g^{ab}g^{nk}\delta g_{ak})\sqrt{g}\,\mathrm{d}^4x \\ &= \int_D \lambda(\theta_b\theta_n + \nabla_b\theta_n)(g^{ab}g^{nk}\delta g_{ak})\sqrt{g}\,\mathrm{d}^4x\end{aligned}$$

As a last step, we use the identity

$$\delta g^{ab} = -g^{am}g^{bn}\delta g_{mn}$$

to write our integral as a variation with respect to the inverse metric.

$$\int_D \lambda(\theta_b\theta_n + \nabla_b\theta_n)(g^{ab}g^{nk}\delta g_{ak})\sqrt{g}\,\mathrm{d}^4x = -\int_D \lambda(\theta_b\theta_n + \nabla_b\theta_n)\delta g^{bn}\sqrt{g}\,\mathrm{d}^4x.$$

Without going through the details again, the other integral in Equation (2.17) can be brought to the form

$$\begin{aligned}\tfrac{1}{2}\int_D \lambda\theta_n\nabla_k(g^{ab}g^{nk}\delta g_{ab})\sqrt{g}\,\mathrm{d}^4x &= -\tfrac{1}{2}\int_D \nabla_k(\lambda\theta_n)g^{ab}g^{nk}\delta g_{ab}\sqrt{g}\,\mathrm{d}^4x \\ &= \tfrac{1}{2}\int_D \lambda(\theta_k\theta_n + \nabla_k\theta_n)g^{ab}g^{nk}g_{ma}g_{lb}\delta g^{ml}\sqrt{g}\,\mathrm{d}^4x \\ &= \tfrac{1}{2}\int_D \lambda g^{nk}(\theta_k\theta_n + \nabla_k\theta_n)g_{ml}\delta g^{ml}\sqrt{g}\,\mathrm{d}^4x\end{aligned}$$

There is still another integral we need to evaluate, namely

$$\int_D (\partial_n\lambda)g^{nb}\delta\Gamma^m{}_{mb}\sqrt{g}\,\mathrm{d}^4x = \tfrac{1}{2}\int_D \lambda\theta_n g^{nb}g^{mk}(\nabla_b\delta g_{mk} + \nabla_m\delta g_{kb} - \nabla_k\delta g_{mb})\sqrt{g}\,\mathrm{d}^4x. \quad (2.18)$$

Because m and k are symmetrised, the second and third terms cancel, leaving us with

$$\begin{aligned}\tfrac{1}{2}\int_D \lambda\theta_n g^{nb}g^{mk}\nabla_b\delta g_{mk}\sqrt{g}\,\mathrm{d}^4x &= -\tfrac{1}{2}\int_D \nabla_b(\lambda\theta_n)g^{nb}g^{mk}\delta g_{mk}\sqrt{g}\,\mathrm{d}^4x &(2.19)\\ &= \tfrac{1}{2}\int_D \lambda(\theta_b\theta_n + \nabla_b\theta_n)g^{nb}g^{mk}g_{am}g_{lk}\delta g^{al}\sqrt{g}\,\mathrm{d}^4x &(2.20)\\ &= \tfrac{1}{2}\int_D \lambda g^{nb}(\theta_b\theta_n + \nabla_b\theta_n)g_{al}\delta g^{al}\sqrt{g}\,\mathrm{d}^4x. &(2.21)\end{aligned}$$

We have calculated all the integrals we need. Before we put them all together, let us make the following observation:

$$\nabla_a\theta_b = \partial_a\theta_b - \Gamma^m{}_{ab}\theta_m = \partial_b\theta_a - \Gamma^m{}_{ba}\theta_m = \nabla_b\theta_a$$

which uses the fact that θ must be closed. Therefore we can define the follwoing symmetric

(0,2) tensor

$$K_{ab} = \theta_a \theta_b + \nabla_{(a}\theta_{b)}. \tag{2.22}$$

So, after liberal relabeling of indices, we find that Equation (2.10) becomes

$$\delta \tilde{S}[g_{ab}, z^\mu] = \int_D (\partial_\mu \lambda - \lambda \theta_\mu)\delta z^\mu \, \mathrm{d}^4 x + \int_D \lambda (R_{ab} - \tfrac{1}{2} R g_{ab} - K_{ab} + K g_{ab}) \delta g^{ab} \sqrt{g} \, \mathrm{d}^4 x. \tag{2.23}$$

Applying the fundamental theorem of the calculus of variations, the action will be stationary if and only if the integrands of both terms vanish. From the first integral we get Equation (2.11), which we have already used. And from the second one we get the modified Einstein field equations

$$R_{ab} - \tfrac{1}{2} R g_{ab} - K_{ab} + K g_{ab} = 0 \tag{2.24}$$

with K_{ab} defined as in Equation (2.22) and $K = g^{mn} K_{mn}$ its trace.

Note that K is indeed a tensor. One could see this by showing that it transforms like one, or alternatively by observing that K_{ab} are the components of $\theta \otimes \theta + \nabla \theta$, which is certainly a tensor.

Chapter 3

Consequences of the equations

Let us recap what we did in the previous chapter. We have shown, by computing the variation of the corresponding action, that the field Equations of an Einstein-Hilbert Lagrangian with linear dissipation, namely

$$L(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, z^\mu) = R(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu})\sqrt{g} - \theta_\mu z^\mu \tag{3.1}$$

are

$$R_{ab} - \tfrac{1}{2}Rg_{ab} - K_{ab} + Kg_{ab} = 0 \tag{3.2}$$

where $K_a b$ is the $(0,2)$ symmetric tensor defined as

$$K_{ab} = \nabla_{(a}\theta_{b)} + \theta_a\theta_b. \tag{3.3}$$

These equations are not the ones obtained in [Laz+17]. For the same Lagrangian, the equations derived are

$$R_{ab} + \tilde{K}_{ab} - \tfrac{1}{2}g_{ab}(R + \tilde{K}) = 0 \tag{3.4}$$

where

$$\tilde{K}_{ab} = \theta_m\Gamma^m{}_{ab} - \tfrac{1}{2}\left(\theta_a\Gamma^m{}_{mb} + \theta_b\Gamma^m{}_{am}\right). \tag{3.5}$$

We will make to case as to why these equations are not the correct ones. The first issue is that $\tilde{K}_{ab}$ is not a tensor. To see this, we will compute $\tilde{K}_{ab}$ in two different coordinate systems and show that it does not transform as a $(0,2)$-tensor would.

The Christoffel symbols of the flat Minkowski vanish if we take cartesian coordinates. Therefore, for any dissipation form we might consider, K_{ab} would vanish. Now, if K_{ab} actually were a tensor then it would vanish in any other coordinate system. But we can show this is not the case. If instead of cartesian coordinates we choose spherical coordinates then the Christoffel symbols don't vanish. Specifically, the non-vanishing ones are

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^\theta{}_{r\theta} = \frac{1}{r} \qquad \Gamma^\phi{}_{r\phi} = \frac{1}{r}$$
$$\Gamma^r{}_{\phi\phi} = -r\sin\theta^2 \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi{}_{\theta\phi} = \frac{1}{\tan\theta}.$$

This means that, for example,

$$\tilde{K}_{tr} = 0 - \tfrac{1}{2}(\theta_t\Gamma^m{}_{mr} + 0) = -\frac{\theta_t}{2r}$$

which is certainly non-zero if θ_t does not vanish. This shows that $\tilde{K}_{ab}$ is not a tensor, or rather $\tilde{K}_{ab}$ do not represent the components of a tensor, since if it were then if it vanishes in some coordinate system it must do so in any other coordinate system, and we have just exhibited a coordinate system in which it vanishes and another one in which at least one of its components does not. In other words, the object derived in [Laz+17] is not coordinate independent so it cannot possibly represent meaningful physics.

On the contrary, the object K_{ab} that we found in Equation (2.22) is indeed a tensor. We could compute explicitly its transformation law and see that it is the one of a $(0,2)$ tensor. But a simpler way is writing it out in a coordinate-free manner. Specifically, K_{ab} are the components of the object $K = \theta\otimes\theta + \nabla\theta$. Indeed, given two vector fields X and Y

$$\begin{aligned} K(X,Y) &= (\theta\otimes\theta)(X,Y) + (\nabla_X\theta)Y \\ &= \theta_a\theta_b X^a Y^b + X^a\nabla_a\theta_b Y^b \\ &= (\theta_a\theta_b + \nabla_{(a}\theta_{b)})X^a Y^b \end{aligned}$$

where in the last step we used that $\nabla_a\theta_b = \nabla_{(a}\theta_{b)}$ because θ is closed. This shows K_{ab} are the components of a tensor since $\theta\otimes\theta$ and $\nabla\theta$ are both $(0,2)$ tensors (they are bilinear).

There is another reason that indicates that the equations in [Laz+17] are not the correct ones. When we wrote down the Herglotz equations for the harmonic oscillator with linear dissipation we obtained equations linear in the dissipation coefficient (see Equation (1.5)). However, the Lagrangian for this system is first order, whereas, as we had already discussed, the Einstein-Hilbert Lagrangian is actually second order. The equations of motion for a second order Lagrangian with linear dissipation, called the damped Pais-Uhlenbeck oscillator, are derived in [**Leon2021a**], and they are in fact not linear in the dissipation coefficient, but rather quadratic. In our case, the dissipation form plays the role of the dissipation coefficient so by analogy we would expect the equations to be quadratic in θ, not linear. And this is indeed the case for the equations we derived, whereas the equations in [Laz+17] are linear in θ.

So these are reasons for why the equations derived in [Laz+17] are not the right ones, but we can actually point at why they derived them in the first place. One of the simplifying assumptions they made was to take a simplified version of the Ricci curvature. Specifically, the Ricci curvature consists of four terms. Two of them are contracions of the Christoffel symbols with themselves, the other two are derivatives of the Christoffel symbols. It can be shown that if one drops these last two when writing down the Einstein-Hilbert action, the resulting equations remain unchanged. The justification is that the terms with derivatives are a divergence, so they leave the action unchanged. However, this justification fails for action dependent Lagrangians, which is ultimately what dooms the whole computation.

Chapter 4

Conclusions

Let us now summarise the most relevant points of this thesis.

4.1 Further research

In this thesis we have only explored the derivation of the field equations. Of course, if we want this theory to produce relevant results we must be able to make testable predictions. As we have discussed before, the main motivation at the moment to explore and test modified theories of gravitation is their application to cosmology. Therefore, the natural next step is to understand the cosmology that follows from them. One of the driving assumptions of cosmology for the pasat decades has been the hypothesis of homogeneity and isotropy of space, at least over large scales. There is now mounting evidence to the contrary, however. The presence of the dissipation form in the equations we have derived provides a way of breaking the homogeneity and isotropy in the models one studies. What's more, these equations are not just the addition of some ad hoc terms to the Einstein field Equations, but instead they come from a variational principle, which is always preferable in physics.

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