Scalar curvature (ignoring terms of $\Gamma^{\cdot}{}_{\cdot\cdot,\cdot}$):

$$R = g^{ab} R_{ab} = g^{ab} \left(\Gamma^{c}{}_{ab} \Gamma^{d}{}_{cd} - \Gamma^{c}{}_{ad} \Gamma^{d}{}_{bc} \right)$$

$$g^{ab}\left(\Gamma^{c}{}_{ab} \Gamma^{d}{}_{cd}\right) = \frac{1}{4} g^{ab} g^{cm} g^{dn} \left(g_{am,b} + g_{mb,a} - g_{ab,m} \right)\left(g_{cn,d} + g_{nd,c} - g_{cd,n} \right)$$

(la permutació dels índexs determina el signe)

En total 9 termes:

$$g^{ab} g^{cm} g^{dn} g_{am,b}\, g_{cn,d} \overset{(bm)(dn)}{=} g^{am} g^{cb} g^{nd} g_{ab,m}\, g_{cd,n}$$

$$g^{ab} g^{cm} g^{dn} g_{am,b}\, g_{nd,c} \overset{(bm)(cn)}{=} g^{am} g^{nb} g^{dc} g_{ab,m}\, g_{cd,n}$$

$$-g^{ab} g^{cm} g^{dn} g_{am,b}\, g_{cd,n} \overset{(bm)}{=} -g^{am} g^{cb} g^{dn} g_{ab,m}\, g_{cd,n}$$

$$g^{ab} g^{cm} g^{dn} g_{mb,a}\, g_{cn,d} \overset{(am)(dn)}{=} g^{mb} g^{ca} g^{nd} g_{ab,m}\, g_{cd,n}$$

$$g^{ab} g^{cm} g^{dn} g_{mb,a}\, g_{nd,c} \overset{(am)(cn)}{=} g^{mb} g^{na} g^{dc} g_{ab,m}\, g_{cd,n}$$

$$-g^{ab} g^{cm} g^{dn} g_{mb,a}\, g_{cd,n} \overset{(am)}{=} -g^{mb} g^{ca} g^{dn} g_{ab,m}\, g_{cd,n}$$

$$-g^{ab} g^{cm} g^{dn} g_{ab,m}\, g_{cd,n} \overset{(nd)}{=} -g^{ab} g^{cm} g^{nd} g_{ab,m}\, g_{cd,n}$$

$$-g^{ab} g^{cm} g^{dn} g_{ab,m}\, g_{nd,c} \overset{(cn)}{=} -g^{ab} g^{nm} g^{dc} g_{ab,m}\, g_{cd,n}$$

$$g^{ab} g^{cm} g^{dn} g_{ab,m}\, g_{cd,n} = g^{ab} g^{cm} g^{dn} g_{ab,m}\, g_{cd,n}$$

En total:

$$g^{ab}\left(\Gamma^{c}{}_{ab} \Gamma^{d}{}_{cd}\right) = (2-2)D + (3-1)C - A = 2C - A$$

$$g^{ab}\left(\Gamma^{c}{}_{ad}\,\Gamma^{d}{}_{bc}\right) = \frac{1}{4} g^{ab} g^{cm} g^{dn}\left(g_{am,d} + g_{md,a} - g_{ad,m}\right)\left(g_{bn,c} + g_{nc,b} - g_{bc,n}\right)$$

El signe el determina la permutació

Uns altres 9 termes:

(mbcnd)

$$g^{ab} g^{cm} g^{dn} g_{am,d}\, g_{bn,c} = g^{ac} g^{nb} g^{md} g_{ab,m}\, g_{cd,n}$$

(mbncd)

$$g^{ab} g^{cm} g^{dn} g_{am,d}\, g_{bn,c} = g^{an} g^{db} g^{mc} g_{ab,m}\, g_{cd,n}$$

(mbcd)

$$-g^{ab} g^{cm} g^{dn} g_{am,d}\, g_{bc,n} = -g^{ac} g^{db} g^{mn} g_{ab,m}\, g_{cd,n}$$

(ma)(dbcn)

$$g^{ab} g^{cm} g^{dn} g_{md,a}\, g_{bn,c} = g^{mc} g^{na} g^{bd} g_{ab,m}\, g_{cd,n}$$

(ma)(dbnc)

$$g^{ab} g^{cm} g^{dn} g_{md,a}\, g_{nc,b} = g^{mn} g^{da} g^{bc} g_{ab,m}\, g_{cd,n}$$

(ma)(dbc)

$$-g^{ab} g^{cm} g^{dn} g_{md,a}\, g_{bc,n} = -g^{mc} g^{da} g^{bn} g_{ab,m}\, g_{cd,n}$$

(dbcn)

$$-g^{ab} g^{cm} g^{dn} g_{ad,m}\, g_{bn,c} = -g^{ac} g^{nm} g^{bd} g_{ab,m}\, g_{cd,n}$$

(dbnc)

$$-g^{ab} g^{cm} g^{dn} g_{ad,m}\, g_{nc,b} = -g^{an} g^{dm} g^{bc} g_{ab,m}\, g_{cd,n}$$

(dbc)

$$g^{ab} g^{cm} g^{dn} g_{ad,m}\, g_{bc,n} = g^{ac} g^{dm} g^{bn} g_{ab,m}\, g_{cd,n}$$

En total: $g^{ab}\left(\Gamma^{c}{}_{ad}\,\Gamma^{d}{}_{bc}\right) = (4-2)E + (1-2)B = 2E - B$

Tenim 18 termes de la forma $g^{\cdot\cdot}g^{\cdot\cdot}g^{\cdot\cdot}g_{ab,m}g_{cd,n}$.

Com que estan contrets amb un factor amb simetries, tenim les simetries $a \leftrightarrow b$, $c \leftrightarrow d$ i $abm \leftrightarrow cdn$. Això dóna 5 termes diferents (també que cada $g^{\cdot\cdot}$ és simètric)

A $g^{ab}g^{cd}g^{mn}$

B $g^{ac}g^{bd}g^{mn}$

C $g^{am}g^{bn}g^{cd} = g^{cm}g^{dn}g^{ab}$

D $g^{am}g^{cn}g^{bd}$

E $g^{an}g^{cm}g^{bd}$

Per tant

$$R = \frac{1}{4}\left(2C - A + B - 2E\right)$$

$$= \frac{1}{4}\underbrace{\left(g^{ac}g^{bd}g^{mn} - g^{ab}g^{cd}g^{mn} + 2g^{am}g^{bn}g^{cd} - 2g^{an}g^{cm}g^{bd}\right)}_{g^{abm\,cdn}} g_{ab,m}\,g_{cd,n}$$

Aleshores:

$$\mathcal{L} = \rho(R - \lambda_m z^m) \qquad (\rho = \sqrt{|g|})$$

$$\frac{\partial \mathcal{L}}{\partial g_{ab,m}} = \rho \frac{\partial R}{\partial g_{ab,m}} = \frac{1}{4}\rho\left(S^{\alpha\beta\mu\gamma\delta\nu}\frac{\partial g_{\alpha\beta,\mu}}{\partial g_{ab,m}} g_{\gamma\delta,\nu} + S^{\alpha\beta\mu\gamma\delta\nu} g_{\alpha\beta,\mu}\frac{\partial g_{\gamma\delta,\nu}}{\partial g_{ab,m}}\right)$$

$$= \frac{1}{4}\rho\left(S^{\alpha\beta\mu\gamma\delta\nu}\delta^{ab}_{\alpha\beta}\delta^{m}_{\mu} g_{\gamma\delta,\nu} + S^{\alpha\beta\mu\gamma\delta\nu}\delta^{ab}_{\gamma\delta}\delta^{m}_{\nu} g_{\alpha\beta,\mu}\right)$$

$$= \frac{1}{4}\rho\,\frac{n(a,b)}{2}\left(2S^{(ab)m\gamma\delta\nu} g_{\gamma\delta\nu} + 2S^{\alpha\beta\mu(ab)m} g_{\alpha\beta,\mu}\right)$$

$$= \frac{1}{4}\rho\,\frac{n(a,b)}{2}\left(2S^{(ab)m\alpha\beta\mu} + 2S^{\alpha\beta\mu(ab)m}\right) g_{\alpha\beta,\mu}.$$

$$\frac{\partial \mathcal{L}}{\partial z^m} = -\rho\lambda_m$$

El terme adicional és:

$$\frac{\partial \mathcal{L}}{\partial z^m}\frac{\partial \mathcal{L}}{\partial g_{ab,m}} = -\rho^2\frac{n(a,b)}{2}\left(\frac{2}{4}S^{(ab)m\alpha\beta\mu} + \frac{2}{4}S^{\alpha\beta\mu(ab)m}\right) g_{\alpha\beta,\mu}\lambda_m.$$

Això dona un objecte amb índexos a dalt. Els baixem:

$$2S^{(kl)m\,\alpha\beta\mu} + 2S^{\alpha\beta\mu\,(kl)m} =$$

$$g^{k\alpha}g^{l\beta}g^{m\mu} + g^{l\alpha}g^{k\beta}g^{m\mu} - 2g^{kl}g^{\alpha\beta}g^{m\mu} + 2g^{km}g^{l\mu}g^{\alpha\beta} + 2g^{lm}g^{k\mu}g^{\alpha\beta} - 2g^{k\mu}g^{\alpha m}g^{l\beta} - 2g^{l\mu}g^{\alpha m}g^{k\beta}$$

$$+ g^{\alpha k}g^{\beta l}g^{\mu m} + g^{\alpha l}g^{\beta k}g^{\mu m} - 2g^{\alpha\beta}g^{kl}g^{\mu m} + 4g^{\alpha\mu}g^{\beta m}g^{kl} - 2g^{\alpha m}g^{k\mu}g^{\beta l} - 2g^{\alpha m}g^{l\mu}g^{\beta k}$$

$$= 2\left(g^{k\alpha}g^{l\beta} + g^{l\alpha}g^{k\beta}\right)g^{m\mu} + 2\left(g^{km}g^{l\mu} + g^{lm}g^{k\mu}\right)g^{\alpha\beta} - 4\left(g^{k\mu}g^{l\beta} + g^{l\mu}g^{k\beta}\right)g^{\alpha m}$$

$$+ 4\left(g^{\alpha\mu}g^{\beta m} - g^{\alpha\beta}g^{\mu m}\right)g^{kl}$$

baixem els índexos:

$$2g_{ak}g_{bl}\left(S^{(kl)m\,\alpha\beta\mu} + S^{\alpha\beta\mu\,(kl)m}\right) = 2\left(\delta^\alpha_a\delta^\beta_b + \delta^\alpha_b\delta^\beta_a\right)g^{m\mu} + 2\left(\delta^m_a\delta^\mu_b + \delta^m_b\delta^\mu_a\right)g^{\alpha\beta}$$

$$- 4\left(\delta^\mu_a\delta^\beta_b + \delta^\mu_b\delta^\beta_a\right)g^{\alpha m} + 4\left(g^{\alpha\mu}g^{\beta m} - g^{\alpha\beta}g^{\mu m}\right)g_{ab}.$$

Contraiem amb $g_{\alpha\beta,\mu}\lambda_m$:

$$\frac{2}{4}\left(S_{(ab)}{}^{m\alpha\beta\mu} + S^{\alpha\beta\mu}{}_{(ab)}{}^{m}\right)g_{\alpha\beta,\mu}\lambda_m =$$

$$= g^{m\mu}g_{ab,\mu}\lambda_m + \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,b}\lambda_a + \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,a}\lambda_b - g^{\alpha m}g_{\alpha b,a}\lambda_m - g^{\alpha m}g_{\alpha a,b}\lambda_m$$

$$+ g_{ab}\left(g^{\alpha\mu}g^{\beta m}g_{\alpha\beta,\mu}\lambda_m - g^{\alpha\beta}g^{\mu m}g_{\alpha\beta,\mu}\lambda_m\right)$$

L'objecte que surt a Lazo 2018 és

$$K_{ab} = \Gamma^{m}{}_{ab}\lambda_m - \frac{1}{2}\left(\Gamma^{m}{}_{am}\lambda_b + \Gamma^{m}{}_{mb}\lambda_a\right)$$

$$\Gamma^{m}{}_{ab}\lambda_m = \frac{1}{2} g^{mk}\left(g_{ak,b} + g_{kb,a} - g_{ab,k}\right)\lambda_m$$

$$\Gamma^{m}{}_{am}\lambda_b + \Gamma^{m}{}_{mb}\lambda_a = \frac{1}{2} g^{mk}\left(\cancel{g_{ak,m}}\lambda_b + g_{km,a}\lambda_b - \cancel{g_{am,k}}\lambda_b + g_{mk,b}\lambda_a + \cancel{g_{kb,m}}\lambda_a - \cancel{g_{mb,k}}\lambda_a\right)$$

$$= \frac{1}{2} g^{mk}\left(g_{km,a}\lambda_b + g_{mk,b}\lambda_a\right)$$

Per tant la traça de K_{ab} és:

$$g^{ab}K_{ab} = \frac{1}{2} g^{mk} g^{ab}\left(g_{ak,b} + g_{kb,a} - g_{ab,k}\right)\lambda_m - \frac{1}{4} g^{ab} g^{mk}\overbrace{\left(g_{km,a}\lambda_b + g_{mk,b}\lambda_a\right)}^{\text{iguals quan contraiem amb } g^{ab}}$$

$$= \frac{1}{2} g^{ab} g^{mk}\Big(\underbrace{g_{ak,b}\lambda_m + g_{kb,a}\lambda_m}_{\text{iguals intercanviant } a\leftrightarrow b} - \underbrace{g_{ab,k}\lambda_m - g_{km,a}\lambda_b}_{\text{iguals intercanviant } ab\leftrightarrow km}\Big)$$

$$= g^{ab} g^{mk}\left(g_{ak,b}\lambda_m - g_{ab,k}\lambda_m\right)$$

La part escalar que tenim és K:

$$g_{ab}\left(g^{\alpha\mu} g^{\beta m} g_{\alpha\beta,\mu}\lambda_m - g^{\alpha\beta} g^{\mu m} g_{\alpha\beta,\mu}\lambda_m\right) =$$

$$= g_{ab}\, g^{\alpha\beta} g^{\mu m}\left(g_{\alpha\mu,\beta} - g_{\alpha\beta,\mu}\right)\lambda_m = g_{ab}\, g^{\alpha\beta} K_{\alpha\beta} = g_{ab} K.$$

La part tensorial és $-2K_{ab}$:

$$2K_{ab} = g^{mk} g_{ak,b}\lambda_m + g^{mk} g_{kb,a}\lambda_m - g^{mk} g_{ab,k}\lambda_m - \frac{1}{2} g^{mk} g_{km,a}\lambda_b - \frac{1}{2} g^{mk} g_{mk,b}\lambda_a$$

$$\frac{2}{4}\left(S_{(ab)}{}^{m\alpha\beta\mu} + S^{\alpha\beta\mu}{}_{(ab)}{}^{m}\right) g_{\alpha\beta,\mu}\lambda_m =$$

$$= g^{m\mu} g_{ab,\mu}\lambda_m + \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,b}\lambda_a + \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,a}\lambda_b - g^{\alpha m} g_{\alpha b,a}\lambda_m - g^{\alpha m} g_{\alpha a,b}\lambda_m$$

És a dir:

$$\frac{2}{p^2 n(a,b)} \frac{\partial \mathcal{L}}{\partial z^{\mu}} \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,\mu}} = -2K_{ab} + g_{ab} K$$

Aquest terme apareix a E-L amb un signe −, per quadrar del tot amb Lazo 2018 falta un factor $\frac{1}{2}$.