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Title

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Chapter 1

The Herglotz variational problem

We claim that a way of describing non-conservative systems with a variational principle is by what we will call an action-dependent Lagrangian. In all of the standard examples from classical mechanics, the Lagrangian is assumed to be a function of the positions and velocities of the particle. An *action-dependent Lagrangian* is a Lagrangian that is also allowed to depend on the action of the path. However, recall that the action is defined as the integral of the Lagrangian function along the path so this seems awfully circular! Thus, great care has to be taken when formulating the theory of action-dependent Lagrangians, in order to circumvent issues of circularity. The problem of finding the stationary paths of the action functional corresponding to an action-dependent Lagrangian is known as the Herglotz variational problem. In this chapter we will first discuss an *implicit approach* to this problem and some of its shortcomings. Then we will frame the Herglotz problem as a constrained optimisation problem and show how this different approach generalises to field theory.

1.1 The implicit formulation of the Herglotz problem

Let's briefly describe what we will refer to as the *implicit formulation of the Herglotz problem*, as presented in [Laz+18]. We start by writing down a naive equation for the action corresponding to an action dependent Lagrangian. Namely, given a path $q^\mu\colon [a,b] \to M$ we would write something like

$$S[q^\mu] = \int_a^b L(q^\mu(t), \dot{q}^\mu(t), S(t))\,\mathrm{d}t$$

where $S(t)$ is the action of the path until time t. Of course this makes no sense since we are defining S on the left-hand side, and it appears on the right-hand side! However, if, given a path then we define its action as a function of time then we could write something like

$$S[q^\mu](t) = \int_a^t L(q^\mu(s), \dot{q}^\mu(s), S[q^\mu](s))\, \mathrm{d}s \tag{1.1}$$

and if we differentiate with respect to time we actually get an ODE for $S[q^\mu]$! Indeed

$$\dot{S}[q^\mu](t) = L(q^\mu(t), \dot{q}^\mu(t), S[q^\mu](t)). \tag{1.2}$$

Notice that Equation (1.1) actually forces the initial condition $S[q^\mu](a) = 0$. We can even drop this requirement, since all we are interested in is the difference of values of S:

$$S[q^\mu](b) - S[q^\mu](a) = \int_a^b L(q^\mu(t), \dot{q}^\mu(t), S[q^\mu](t))\, \mathrm{d}t. \tag{1.3}$$

What we have here is a functional which, for every path, is defined by an ODE. To find the stationary paths of this functional we would, in principle, have to solve Equation (1.2) for any possible path and among all of them find which ones yield extrema. This is the *Herglotz variational problem*. This approach is in general not feasible. However, just like the variational problem of classical Lagrangian mechanics can be turned into a set of ODEs, the Euler-Lagrange equations, so can the Herglotz problem be turned into a set of ODEs. These are known as the Herglotz equations, which can be written down as

$$\frac{\partial L}{\partial q^\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\mu} + \frac{\partial L}{\partial \dot{q}^\mu}\frac{\partial L}{\partial z} = 0 \tag{1.4}$$

where the Lagrangian is $L(q^\mu, \dot{q}^\mu, z)$, z being the action dependence. These equations differ from the Euler-Lagrange equations only by one term. And in fact, if L is action-independent, thus $\frac{\partial L}{\partial z} = 0$, we recover exactly the Euler-Lagrange equations.

See [Laz+18] §3.2 of [LL21] or for a derivation of the Herglotz equations following the implicit approach.

1.1.1 Example: the damped harmonic oscillator

Before we move forward, let's see what kind of equations of motion we get from the Herglotz equations. The simplest action-dependence we can introduce —other than no action-dependence at all, of course— is a term linear in z. Let's see what happens when we add this term to the simple harmonic oscillator. Explicitly, consider the Lagrangian

$$L(q, \dot{q}, z) = \tfrac{1}{2}m\dot{q}^2 - \tfrac{1}{2}m\omega^2 q^2 - \gamma z.$$

A straightforward computation shows that for this Lagrangian, Equation (1.4) becomes

$$-m\omega^2 q - m\ddot{q} - \gamma m\dot{q} = 0$$

which after some rearranging can be turned into

$$\ddot{q} + \gamma\dot{q} + \omega^2 q = 0,$$

which one recognises as the equation of motion of the damped harmonic oscillator. The Herglotz principle delivers in its promise: we have just derived the equations of motion of a fundamentally non-conservative system from a variational problem!

1.2 The Herglotz problem as a constrained optimisation problem

We have derived the Herglotz equations, so it would seem we have already solved the theory of action-dependent Lagrangians. However, general relativity is a field theory, so if we wish to understand an action-dependent variant of it we have to know the form the Herglotz equations take for field theory. If we try to apply the implicit method to field theory we run into a number of problems. For one, it is just not very elegant. The action functional is defined implicitly through an ordinary differential equation, one for every path. What's more, this equation is no longer an ODE in field theory, but rather it becomes a PDE. PDEs are notoriously much more difficult to solve thant ODEs, so we would like a way to circumvent this issue.

We can achieve this if we frame the Herglotz problem as a constrained optimisation problem. Not only that, we also get a much clearer and more elegant formulation. We describe what this looks like for mehcanics. First, instead of considering paths in spacetime, $q^\mu\colon [a,b] \to M$, we enlarge the configuration space with one extra quantity, which we will call z. At this point z simply tracks a quantity that changes along the path, but we will later require that z actually match the action of the path at each time. This will be the constraint.

So, we have paths of the form $(q^\mu, z)\colon [a,b] \to M \times \mathbb{R}$. We define a functional on these paths as

$$S[q^\mu, z] = z(a) - z(b) = \int_a^b \dot{z}(t)\,\mathrm{d}t, \tag{1.5}$$

so S is just the change in z along the trajectory. This functional as it stands has no stationary paths, since we can find trajectories with arbitrarily large changes in z, both positive and negative. So we constrain the possible paths. Namely, we require that z

actually represent the action. So we try to find the paths that extremise S only among those that satisfy the constraint

$$\dot{z}(t) = L(q^\mu(t), \dot{q}^\mu(t), z(t)), \tag{1.6}$$

where L is the action-dependent Lagrangian that describes the system. Notice that this is very similar to Equation (1.2).

Say(q^μ, z) is a trajectory that satisfies Equation (1.6). Then

$$S[q^\mu, z] = z(a) - z(b) = \int_a^b \dot{z}(t)\,\mathrm{d}t = \int_a^b L(q^\mu(t), \dot{q}^\mu(t), z(t))\,\mathrm{d}t.$$

So, for paths that satisfy the constraint, the functional S is indeed the action functional, understood as the integral of the Lagrangian along the path.

This is all well and good, but how does one actually go about solving a constrained optimisation problem? It turns out, we can use an infinite dimensional analog of Lagrange multipliers to turn this into a regular optimisation problem to which we can apply the tools of the calculus of variations.

Recall, given some function $f\colon \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$ is an extremum of f subject to m constraints $g_k\colon \mathbb{R}^n \to \mathbb{R}$ —i.e. $g_k(x) = 0$ — if and only if there exist numbers $\lambda_k \in \mathbb{R}$ such that x is an extremum of the function

$$F = f - \sum_{k=1}^{m} \lambda_k g_k \tag{1.7}$$

without any constraints. The numbers λ_k are called the Lagrange multipliers. It can be shown that this result generalises to infinite dimensional spaces and infinitely many constraints. In our case, the function we are trying to find the extrema of is the functional S. Our constraints are parameterised by $t \in [a, b]$:

$$g_t[q^\mu, z] = \dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t)).$$

Thus, replacing sums with integrals in Equation (1.7), the extrema of Equation (1.5) subject to Equation (1.6) will be those that extremise the following functional:

$$\begin{aligned}
\tilde{S}[q^\mu, z] &= S[q^\mu, z] - \int_a^b \lambda_t g_t[q^\mu, z]\,\mathrm{d}t \\
&= \int_a^b \dot{z}(t)\,\mathrm{d}t - \int_a^b \lambda_t\Big[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\Big]\,\mathrm{d}t \\
&= \int_a^b (1 - \lambda_t)\dot{z}(t) + \lambda_t\Big(L(q^\mu(t), \dot{q}^\mu(t), z(t))\Big)\,\mathrm{d}t.
\end{aligned} \tag{1.8}$$

A couple of observations. First, we are thinking of the Lagrange multipliers as real numbers parameterised by t, but we could equivalently think of them as a function of t and write $\lambda(t)$ instead of λ_t. We will do this. Secondly, and more importantly, if we write the integrand of Equation (1.8) as

$$\tilde{L}(q^\mu(t), \dot{q}^\mu(t), z(t), \dot{z}(t)) = \Big(1 - \lambda(t)\Big)\dot{z}(t) + \lambda(t)\Big(L(q^\mu(t), \dot{q}^\mu(t), z(t))\Big)$$

one should be able to recognise $\tilde{S}$ as something that looks just like a regular old action functional defined by the integral of a regular old Lagrangian, except it is now defined on expanded trajectories (q^μ, z). So we should be able to use the Euler-Lagrange equations to write down the equations of motion of its extremal paths! The equation for z reads

$$0 = \frac{\partial \tilde{L}}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{z}} = \lambda \frac{\partial L}{\partial z} + \dot{\lambda}$$

or equivalently

$$\dot{\lambda} = -\lambda \frac{\partial L}{\partial z}. \tag{1.9}$$

The Euler-Lagrange equations for the positions then are

$$0 = \frac{\partial \tilde{L}}{\partial q^\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}^\mu} = \lambda \frac{\partial L}{\partial q^\mu} - \dot{\lambda}\frac{\partial L}{\partial \dot{q}^\mu} - \lambda \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\mu}$$

and after substituting in Equation (1.9) and dividing through by λ we find

$$0 = \frac{\partial L}{\partial q^\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\mu} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}^\mu} \tag{1.10}$$

which are exactly the Herglotz equations.

1.3 Action-dependent field theory

We have seen how to derive the Herglotz equations in a more elegant way using Lagrange multipliers. More importantly, we have written down a modified Lagrangian which gives rise to them through standard calculus of variations techniques. This will be very useful as, in some cases (general relativity being one of them), it is easier to derive the equations of motion of a system by direct variation of the action, rather than writing down the Euler-Lagrange equations. We will be able to do exactly this once we have the field theoretic version of Equation (1.8) in our hands.

1.3.1 Classical field theory and Lagrangian densities

First off, we need to set the stage for Lagrangian field theory. The parameter space is no longer just time, but rather all of spacetime, M. Fields are the assignment of some value to each point in spacetime, so we could have scalar fields, vector fields, or, as is the case in general relativity, tensor fields. Let us first fix some notation. We will denote by ϕ some field configuration. If ϕ is a scalar field then it carries no indices. If ϕ is a vector field then it carries one upper index, if it is a tensor field then it carries multiple indices. The metric carries two lower indices since it is a $(0,2)$ tensor field. In almost all that follows we will assume ϕ is a vector field, but the results we find transfer to tensors of other rank. Of course ϕ depends on spacetime, which we will sometimes write explicitly as $\phi^a(x^\mu)$. As a convention, we will use latin indices for the indices of the field, and reserve greek indices for spacetime coordinates.

The Lagrangian of a field theory is a function of the field and of its derivatives. It is not, however, a function of real values, but rather an n-dimensional —where n is the dimension of spacetime—, differential form, also known as a top form. This is because, the action is defined as the integral of the Lagrangian over some region of spacetime, and the objects we can integrate over spacetime are precisely the top forms.

It turns out, however, that any two top forms differ only by an overall factor, i.e., given two top forms ω_1 and ω_2, there exists a scalar function of spacetime M such that $\omega_1 = f\omega_2$. This means, that if we think of the Lagrangian as some top form $\mathcal{L}$, once we pick a distinguished top form there is a unique scalar function L such that $\mathcal{L} = L\omega$. This distinguished top form will in most cases be the top form induced by the coordinates we are working in, which we will write as $\mathrm{d}^n x$. Sometimes we will use Lagrangian density to refer to $\mathcal{L}$ and Lagrangian to refer to L.

The setup in classical field theory is as follows. Given some Lagrangian, which encodes the system we are studying, we define the action functional on all the possible field configurations as

$$S[\phi^a] = \int_D \mathcal{L}(\phi^a(x^\mu), \partial_\mu \phi^a(x^\mu)) = \int_D L(\phi^a(x^\mu), \partial_\mu \phi^a(x^\mu))\, \mathrm{d}^n x,$$

where D is some region of spacetime where this integral makes sense.

Using the calculus of variations one can show that the stationary configurations of this action functional satisfy the Euler-Lagrange equations of field theory

$$\frac{\partial L}{\partial \phi^a} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi^a)} = 0$$

where a summation convention is implied over μ.

1.3.2 The action flux

How do we generalise this to an action dependent Lagrangian? The most naive approach would be to try to replicate the Herglotz equations from mechanics wholesale and write down

$$\frac{\partial L}{\partial \phi^a} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi^a)} + \frac{\partial L}{\partial(\partial_\mu \phi^a)} \frac{\partial L}{\partial z} = 0.$$

However this will not work because the last term has a pesky free μ index. This seems to suggest that we need to modify the nature of z. We had claimed before that z represented the action along the path, but if we look at Equation (1.5) we see this is not quite right. In mechanics the analog of D is $[a, b]$. The difference $z(a) - z(b)$ can also be written as $\int_{\partial[a,b]} z$, since the boundary of $[a, b]$ is just a and b. This seems to indicate that the correct analog of Equation (1.5) for field theory should be

$$S[\phi^a, z] = \int_{\partial D} z.$$

What kind of obejct should z be then? ∂D has dimension $n-1$, so z has to be something we can integrate over an $(n-1)$-dimensional manifold, i.e. a differential $(n-1)$-form. As it turns out, $(n-1)$-forms behave almost like vector fields. Specifically, by expanding in the basis $\mathrm{d}x_\mu$ we have $z = z^\mu \, \mathrm{d}x_\mu$. Thus, what z represents is the action flux. We encode this in a constraint perfectly analogous to Equation (1.6),

$$\mathrm{d}z(x^\mu) = \mathcal{L}(\phi^a(x^\mu), \partial_\mu \phi^a(x^\mu), z^\mu(x^\mu)). \tag{1.11}$$

$\mathrm{d}z$ is the exterior derivative of z, which is a top form, so this equality makes sense. In coordinates it is easy to show that $\mathrm{d}z = \partial_\nu z^\nu \, \mathrm{d}^n x$, so that Equation (1.11) reads in coordinates as

$$\partial_\nu z^\nu(x^\mu) = L(\phi^a(x^\mu), \partial_\mu \phi^a(x^\mu)). \tag{1.12}$$

This expresses the fact that $\mathrm{d}z$ has to be the action density. Indeed, for field configurations that satisfy this constraint then one has, applying Stokes' theorem

$$S[\phi^a, z^\mu] = \int_{\partial D} z = \int_D \mathrm{d}z = \int_D \mathcal{L}(\phi^a, \partial_\mu \phi^a, z^\mu).$$

So we have arrived at the right formulation of the Herglotz variational problem for field theory.

1.3.3 Constrained optimisation in field theory

So just like before, we will turn this constrained optimisation problem into an unconstrained one using Lagrange multipliers. The expanded action, in analogy with Equation (1.8) will be

$$\tilde{S}[\phi^a, z^\mu] = \int_D (1-\lambda)\,\mathrm{d}z + \lambda\mathcal{L}(\phi^a, \partial_\mu\phi^a, z^\nu) = \int_D \mathrm{d}^n x\Big[(1-\lambda)\partial_\mu z^\mu + \lambda L(\phi^a, \partial_\mu\phi^a, z^\nu)\Big] \tag{1.13}$$

Note that the Lagrange multiplier λ is a function of spacetime. Let us write down the integrand of Equation (1.13) as an expanded Lagrangian:

$$\tilde{\mathcal{L}}(\phi^a, \partial_\mu\phi^a, z^\nu, \partial_\mu z^\nu) = \tilde{L}(\phi^a, \partial_\mu\phi^a, z^\nu, \partial_\mu z^\nu)\,\mathrm{d}^n x = \Big[(1-\lambda)\partial_\mu z^\mu + \lambda L(\phi^a, \partial_\mu\phi^a, z^\nu)\Big]\,\mathrm{d}^n x. \tag{1.14}$$

1.3.4 The Herglotz equations for field theory

Finally, we can derive the Herglotz equations for field theory from the expanded Lagrangian we have just obtained in Equation (1.14). The equations for the action flux are

$$0 = \frac{\partial\tilde{L}}{\partial z^\nu} - \partial_\mu\frac{\partial\tilde{L}}{\partial(\partial_\mu z^\nu)} = \lambda\frac{\partial L}{\partial z^\nu} + \partial_\mu(\lambda\delta^\mu_\nu) = \lambda\frac{\partial L}{\partial z^\nu} + \partial_\nu\lambda.$$

So, rearranging,

$$\partial_\nu\lambda = -\lambda\frac{\partial L}{\partial z^\nu}. \tag{1.15}$$

And for the values of the field

$$0 = \frac{\partial\tilde{L}}{\partial\phi^a} - \partial_\mu\frac{\partial\tilde{L}}{\partial(\partial_\mu\phi^a)} = \lambda\frac{\partial L}{\partial\phi^a} - (\partial_\mu\lambda)\frac{\partial L}{\partial(\partial_\mu\phi^a)} - \lambda\partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi^a)}$$

and, when plugging in Equation (1.15) and dividing through by λ we arrive at the field theoretical Herglotz equations

$$\frac{\partial L}{\partial\phi^a} - \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi^a)} + \frac{\partial L}{\partial z^\mu}\frac{\partial L}{\partial(\partial_\mu\phi^a)} = 0. \tag{1.16}$$

Chapter 2

2.1 The Einstein-Hilbert Lagrangian

As is well-known, the Einstein field equations can be derived by means of a variational principle. The Lagrangian that gives rise to these equations is the Einstein-Hilbert Lagrangian. Let's write it down in the language of Section 1.3.1. The Einstein-Hilbert action acts on metrics. And given a metric g, one can construct a top form, which we will write ω_g. Choosing coordinates one has $\omega_g = \sqrt{g}\,\mathrm{d}^4x$, where $\sqrt{g}$ is the square root of the determinant of the expression of g in the coordinates that induce d^4x. This form is not the Einstein-Hilbert Lagrangian. The other element is the scalar curvature $R = g^{ab}R_{ab}$, which is a Lorentz invariant scalar that encodes the curvature associated to g. The Einstein-Hilbert Lagrangian is

$$\mathcal{L}_{\text{E-H}}(g_{ab}, \partial_\mu g_{ab}, \partial_\mu \partial_\nu g_{ab}) = R\omega_g = R\sqrt{g}\,\mathrm{d}^4x. \tag{2.1}$$

From now on we will write $g_{ab,\mu}$ and $g_{ab,\mu\nu}$ instead of $\partial_\mu g_{ab}$ and $\partial_\mu \partial_\nu g_{ab}$.

The Einstein-Hilbert action is therefore

$$S_{\text{E-H}}[g_{ab}] = \int_D \mathcal{L}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}) = \int_D R\sqrt{g}\,\mathrm{d}^4x. \tag{2.2}$$

A variation of this action leads one to the Einstein field equations, which in natural units are

$$R_{ab} - \tfrac{1}{2}g_{ab}R = 0. \tag{2.3}$$

More precisely, these are the Einstein field equations in a vacuum, since one can add various matter terms to the Einstein-Hilbert Lagrangian which leads to the Einstein equations in the presence of matter,

$$R_{ab} - \tfrac{1}{2} g_{ab} R = T_{ab}. \tag{2.4}$$

The object T_{ab} is the energy-momentum tensor and collects all of the terms coming from the presence of matter. See §4 of [Car97] for a derivation of the Einstein field equations.

2.2 An action dependent Einstein-Hilbert Lagrangian

What kind of action dependence can we incorporate into the Einstein-Hilbert Lagrangian? The simplest one is perhaps the following

$$\mathcal{L}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, z^\mu) = R\omega_g - \theta \wedge z. \tag{2.5}$$

Recall, z is the action flux which is an $(n-1)$-differential form (so a 3-form in general relativity), so, if θ is a given covector field (a 1-form), the wedge $\theta \wedge z$ is a 4-form and Equation (2.5) makes perfect sense.

In coordinates we have

$$\theta \wedge z = (\theta_\mu \, \mathrm{d}x^\mu) \wedge (z^\nu \, \mathrm{d}x_\nu) = \theta_\mu z^\nu \, \mathrm{d}x^\mu \wedge \, \mathrm{d}x_\nu = \theta_\mu z^\nu \delta^\mu_\nu \, \mathrm{d}^4 x = \theta_\mu z^\mu \, \mathrm{d}^4 x$$

so

$$\mathcal{L}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, z^\mu) = (R\sqrt{g} - \theta_\mu z^\mu) \, \mathrm{d}^4 x. \tag{2.6}$$

This Lagrangian does not quite match the one proposed in eq. (9) of [Laz+17]. It can be shown that they are in fact one and the same. If instead of choosing the basis $\mathrm{d}x^\mu$ for the 3- forms we expand with respect to the basis $\omega_{g,\mu} = \sqrt{g} \, \mathrm{d}x_\mu$ we have

$$z = \zeta^\mu \omega_{g,\mu} = \zeta^\mu \sqrt{g} \, \mathrm{d}x_\mu$$

which implies $z^\mu = \sqrt{g}\zeta^\mu$. So, with these new coordinates for the action flux, Equation (2.6) looks like

$$\mathcal{L}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, \zeta^\mu) = (R\sqrt{g} - \theta_\mu \zeta^\mu \sqrt{g}) \, \mathrm{d}^4 x = (R - \theta_\mu \zeta^\mu)\omega_g \tag{2.7}$$

which is the same Lagrangian proposed in [Laz+17].

We now write down the constraint in Equation (1.11) for this Lagrangian. In the original coordinates for the action flux we have

$$\mathrm{d}z = \partial_\mu z^\mu \, \mathrm{d}^4 x$$

so, in coordinates

$$\partial_\mu z^\mu = R\sqrt{g} - \theta_\mu z^\mu. \tag{2.8}$$

If instead we choose the other coordinates for the action flux, we see

$$\mathrm{d}z = \partial_\mu(\sqrt{g}\zeta^\mu)\,\mathrm{d}^4x = \nabla_\mu\zeta^\mu\sqrt{g}\,\mathrm{d}^4x = \nabla_\mu\zeta^\mu\omega_g$$

where ∇ is the covariant derivative induced by g. In these coordinates the constraint takes the form

$$\nabla_\mu\zeta^\mu = R - \theta_\mu\zeta^\mu \tag{2.9}$$

which is the same form that appears in [Laz+17].

2.3 The expanded action

So we have seen what the Herglotz problem looks like for an Einstein-Hilbert Lagrangian with a linear action dependence. We will apply the method of Lagrange multipliers, as described in previous chapter, to derive a modified version of Einstein's equations. The expanded Lagrangian is

$$\tilde{\mathcal{L}}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, z^\nu, \partial_\mu z^\nu) = \Big[(1-\lambda)\partial_\mu z^\mu + \lambda(R\sqrt{g} - \theta_\mu z^\mu)\Big]\,\mathrm{d}^4x.$$

Let's therefore compute the variation of the action given by this Lagrangian

$$\begin{aligned}\delta\tilde{S}[g_{ab}, z^\nu] &= \int_D \Big[(1-\lambda)\delta\partial_\mu z^\mu + \lambda(\delta(R\sqrt{g}) - \theta_\mu\delta z^\mu)\Big]\,\mathrm{d}^4x \\ &= \int_D (1-\lambda)\partial_\mu\delta z^\mu - \lambda\theta_\mu\delta z^\mu\,\mathrm{d}^4x + \int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x \\ &= \int_D \partial_\mu\Big((1-\lambda)\delta z^\mu\Big)\,\mathrm{d}^4x + \int_D (\partial_\mu\lambda - \lambda\theta_\mu)\delta z^\mu\,\mathrm{d}^4x + \int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x.\end{aligned} \tag{2.10}$$

The first integral is a boundary term coming from an integration by parts. Since we are assuming the variations vanish at the boundary of D so must this boundary term also vanish. From the second integral we can read off, using the fundamental theorem of the calculus of variations, that

$$\partial_\mu\lambda = \lambda\theta_\mu. \tag{2.11}$$

Coordinate free this can also be written as $\mathrm{d}\lambda = \lambda\theta$. This has an interesting implication for θ since

$$\mathrm{d}(\lambda\theta) = \mathrm{d}\lambda\wedge\theta + \lambda\,\mathrm{d}\theta = \lambda\theta\wedge\theta + \lambda\,\mathrm{d}\theta = \lambda\,\mathrm{d}\theta$$

and

$$\lambda\,\mathrm{d}\theta = \mathrm{d}(\lambda\theta) = \mathrm{d}^2\lambda = 0$$

so we conclude $\mathrm{d}\theta = 0$, i.e. θ cannot be any 1-form, it must be a a *closed form*. This means that in coordinates $\partial_\mu\theta_\nu = \partial_\nu\theta_\mu$.

We retake the calculation from Equation (2.10). Since the integrals involving z and g decouple, we can just consider the last term. We will follow the derivation in [Car97] for as long as we can. Since $R\sqrt{g} = g^{ab}R_{ab}\sqrt{g}$, from the product rule its variation results in three terms:

$$\int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x = \int_D \lambda\delta g^{ab}R_{ab}\sqrt{g}\,\mathrm{d}^4x + \int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x + \int_D \lambda R\delta\sqrt{g}\,\mathrm{d}^4x \tag{2.12}$$

The first term is already in the form required to apply the fundamental theorem of the calculus of variations. For the third one uses the standard result

$$\delta\sqrt{g} = -\tfrac{1}{2}\sqrt{g}g_{ab}\delta g^{ab}.$$

The first and third terms of Equation (2.12) can be combined into

$$\int_D \lambda(R_{ab} - \tfrac{1}{2}Rg_{ab})\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x. \tag{2.13}$$

In the standard derivation of Einstein's equations, one shows that the middle integral of Equation (2.12) actually vanishes, so that Equation (2.13) must vanish for any variation δg_{ab}, or equivalently for any variation of the inverse metric δg^{ab}. Therefore the integrand itself must vanish, which gives Einstein's equations. In the presence of λ, however, the middle integral doesn't vanish and actually contributes additional terms to the equations.

The variation of the Ricci curvature can be shown to be

$$g^{ab}\delta R_{ab} = g^{ab}(\nabla_m\delta\Gamma^m{}_{ab} - \nabla_a\delta\Gamma^m{}_{mb}) = \nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb}) \tag{2.14}$$

so

$$\int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x = \int_D \lambda\nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x$$

and if λ weren't there this integral would vanish because of the divergence theorem and the fact that the variations vanish on the boundary of D. But λ is there, so we must work on this integral some more:

$$\int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x =$$

$$
\begin{aligned}
&= \int_D \lambda \nabla_n (g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&= \int_D \lambda \partial_n \left(\sqrt{g}(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\right)\,\mathrm{d}^4x \\
&= \int_D \partial_n \left(\lambda\sqrt{g}(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\right)\,\mathrm{d}^4x - \int_D (\partial_n\lambda)(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x.
\end{aligned}
$$

The first integral vanishes because it is the integral of a divergence and the variations vanish on the boundary of D. So we are left with just the second term, which we split into two integrals. The variation of the Christoffel symbols can be shown to be

$$
\delta\Gamma^a{}_{bc} = \tfrac{1}{2}g^{am}(\nabla_c\delta g_{bm} + \nabla_b\delta g_{mc} - \nabla_m\delta g_{bc}) \tag{2.15}
$$

Using this and Equation (2.11) we compute for the first integral

$$
-\int_D (\partial_n\lambda)g^{ab}\delta\Gamma^n{}_{ab}\sqrt{g}\,\mathrm{d}^4x = -\tfrac{1}{2}\int_D \lambda\theta_n g^{ab}g^{nk}(\nabla_b\delta g_{ak} + \nabla_a\delta g_{kb} - \nabla_k\delta g_{ab})\sqrt{g}\,\mathrm{d}^4x. \tag{2.16}
$$

The presence of g^{ab} means the indices a and b are symmetrised, so

$$
g^{ab}\nabla_b\delta g_{ak} = g^{ab}\nabla_a\delta g_{kb}.
$$

This means Equation (2.16) simplifies to

$$
\begin{aligned}
&-\int_D (\partial_n\lambda)g^{ab}\delta\Gamma^n{}_{ab}\sqrt{g}\,\mathrm{d}^4x = \\
&= -\int_D \lambda\theta_n g^{ab}g^{nk}\nabla_b\delta g_{ak}\sqrt{g}\,\mathrm{d}^4x + \tfrac{1}{2}\int_D \lambda\theta_n g^{ab}g^{nk}\nabla_k\delta g_{ab}\sqrt{g}\,\mathrm{d}^4x \\
&= -\int_D \lambda\theta_n \nabla_b(g^{ab}g^{nk}\delta g_{ak})\sqrt{g}\,\mathrm{d}^4x + \tfrac{1}{2}\int_D \lambda\theta_n \nabla_k(g^{ab}g^{nk}\delta g_{ab})\sqrt{g}\,\mathrm{d}^4x.
\end{aligned} \tag{2.17}
$$

Let's try to perform an integration by parts for the first integral. We have to be a bit careful. Introducing the shorthand $X^{bn} = g^{ab}g^{nk}\delta g_{ak}$, we compute

$$
\nabla_c(\lambda\theta_n X^{bn}) = \nabla_c(\lambda\theta_n)X^{bn} + \lambda\theta_n\nabla_c X^{bn}
$$

so

$$
\begin{aligned}
-\int_D \lambda\theta_n\nabla_b(g^{ab}g^{nk}\delta g_{ak})\sqrt{g}\,\mathrm{d}^4x &= -\int_D \lambda\theta_n\nabla_b X^{bn}\sqrt{g}\,\mathrm{d}^4x \\
&= -\int_D \nabla_b(\lambda\theta_n X^{bn})\sqrt{g}\,\mathrm{d}^4x + \int_D \nabla_b(\lambda\theta_n)X^{bn}\sqrt{g}\,\mathrm{d}^4x.
\end{aligned}
$$

The first integral is the integral of a divergence, so it vanishes. We are left with the second which we can expand into

$$\begin{aligned}\int_D \nabla_b(\lambda\theta_n)X^{bn}\sqrt{g}\,\mathrm{d}^4x &= \int_D(\partial_b\lambda\theta_n + \lambda\nabla_b\theta_n)(g^{ab}g^{nk}\delta g_{ak})\sqrt{g}\,\mathrm{d}^4x \\ &= \int_D \lambda(\theta_b\theta_n + \nabla_b\theta_n)(g^{ab}g^{nk}\delta g_{ak})\sqrt{g}\,\mathrm{d}^4x\end{aligned}$$

As a last step, we use the identity

$$\delta g^{ab} = -g^{am}g^{bn}\delta g_{mn}$$

to write our integral as a variation with respect to the inverse metric.

$$\int_D \lambda(\theta_b\theta_n + \nabla_b\theta_n)(g^{ab}g^{nk}\delta g_{ak})\sqrt{g}\,\mathrm{d}^4x = -\int_D \lambda(\theta_b\theta_n + \nabla_b\theta_n)\delta g^{bn}\sqrt{g}\,\mathrm{d}^4x.$$

Without going through the details again, the other integral in Equation (2.17) can be brought to the form

$$\begin{aligned}\tfrac{1}{2}\int_D \lambda\theta_n\nabla_k(g^{ab}g^{nk}\delta g_{ab})\sqrt{g}\,\mathrm{d}^4x &= -\tfrac{1}{2}\int_D \nabla_k(\lambda\theta_n)g^{ab}g^{nk}\delta g_{ab}\sqrt{g}\,\mathrm{d}^4x \\ &= \tfrac{1}{2}\int_D \lambda(\theta_k\theta_n + \nabla_k\theta_n)g^{ab}g^{nk}g_{ma}g_{lb}\delta g^{ml}\sqrt{g}\,\mathrm{d}^4x \\ &= \tfrac{1}{2}\int_D \lambda g^{nk}(\theta_k\theta_n + \nabla_k\theta_n)g_{ml}\delta g^{ml}\sqrt{g}\,\mathrm{d}^4x\end{aligned}$$

There is still another integral we need to evaluate, namely

$$\int_D(\partial_n\lambda)g^{nb}\delta\Gamma^m{}_{mb}\sqrt{g}\,\mathrm{d}^4x = \tfrac{1}{2}\int_D \lambda\theta_n g^{nb}g^{mk}(\nabla_b\delta g_{mk} + \nabla_m\delta g_{kb} - \nabla_k\delta g_{mb})\sqrt{g}\,\mathrm{d}^4x. \tag{2.18}$$

Because m and k are symmetrised, the second and third terms cancel, leaving us with

$$\begin{aligned}\tfrac{1}{2}\int_D \lambda\theta_n g^{nb}g^{mk}\nabla_b\delta g_{mk}\sqrt{g}\,\mathrm{d}^4x &= -\tfrac{1}{2}\int_D \nabla_b(\lambda\theta_n)g^{nb}g^{mk}\delta g_{mk}\sqrt{g}\,\mathrm{d}^4x \qquad (2.19)\\ &= \tfrac{1}{2}\int_D \lambda(\theta_b\theta_n + \nabla_b\theta_n)g^{nb}g^{mk}g_{am}g_{lk}\delta g^{al}\sqrt{g}\,\mathrm{d}^4x \qquad (2.20)\\ &= \tfrac{1}{2}\int_D \lambda g^{nb}(\theta_b\theta_n + \nabla_b\theta_n)g_{al}\delta g^{al}\sqrt{g}\,\mathrm{d}^4x. \qquad (2.21)\end{aligned}$$

We have calculated all the integrals we need. Before we put them all together, let us make the following observation:

$$\nabla_a\theta_b = \partial_a\theta_b - \Gamma^m{}_{ab}\theta_m = \partial_b\theta_a - \Gamma^m{}_{ba}\theta_m = \nabla_b\theta_a$$

which uses the fact that θ must be closed. Therefore we can define the follwoing symmetric

(0,2) tensor

$$K_{ab} = \theta_a\theta_b + \nabla_{(a}\theta_{b)}. \tag{2.22}$$

So, after liberal relabeling of indices, we find that Equation (2.10) becomes

$$\delta\tilde{S}[g_{ab}, z^\mu] = \int_D (\partial_\mu\lambda - \lambda\theta_\mu)\delta z^\mu \, \mathrm{d}^4x + \int_D \lambda(R_{ab} - \tfrac{1}{2}Rg_{ab} - K_{ab} + Kg_{ab})\delta g^{ab}\sqrt{g}\, \mathrm{d}^4x. \tag{2.23}$$

Applying the fundamental theorem of the calculus of variations, the action will be stationary if and only if the integrands of both terms vanish. From the first integral we get Equation (2.11), which we have already used. And from the second one we get the modified Einstein field equations

$$R_{ab} - \tfrac{1}{2}Rg_{ab} - K_{ab} + Kg_{ab} = 0 \tag{2.24}$$

with K_{ab} defined as in Equation (2.22) and $K = g^{mn}K_{mn}$ its trace.

Note that K is indeed a tensor. One could see this by showing that it transforms like one, or alternatively by observing that K_{ab} are the components of $\theta \otimes \theta + \nabla\theta$, which is certainly a tensor.

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