Consider a contact Lagrangian of the form

$$L(\phi, \partial_\mu \phi, z^\mu). \tag{0.1}$$

Given a scalar field $\phi\colon M \to \mathbb{R}$, consider the vector field $z^\mu(\phi)$ which solves the PDE

$$\partial_\mu z^\mu(\phi) = L(\phi, \partial_\mu \phi, z^\mu(\phi)). \tag{0.2}$$

We define the action functional as

$$A(\phi) = \int_M \partial_\mu z^\mu(\phi)(x^\nu)\,\mathrm{d}^4 x = \int_{\partial M} z^\mu(\phi)(x^\nu)\,\mathrm{d}n_\mu. \tag{0.3}$$

We wish to compute the variation of A to determine which is a necessary and sufficient condition ϕ must satisfy in order to be an extremum of A (which will turn out to be the Herglotz equations).

Consider a variation about a field ϕ, i.e. a parameterised family of fields ϕ_ϵ such that $\phi_0 = \phi$ and $\phi_\epsilon|_{\partial M} = \phi|_{\partial M}$, for $\epsilon \in (-\delta, \delta)$. Then ϕ is by definition an extremum of A if and only if

$$\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} A(\phi_\epsilon) = 0 \tag{0.4}$$

for any variation about ϕ. And

$$\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} A(\phi_\epsilon) = \int_{\partial M} \tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} z^\mu(\phi_\epsilon)(x^\nu)\,\mathrm{d}n_\mu. \tag{0.5}$$

We write $\zeta^\mu(\phi_0)$ for $\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\big|_{\epsilon=0} z^\mu(\phi_\epsilon)$ and η for $\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\big|_{\epsilon=0} \phi_\epsilon$. From the definition of z, we have, introducing the shorthand $\Phi_\epsilon(x^\nu) = (\phi_\epsilon(x^\nu), \partial_\mu \phi_\epsilon(x^\nu), z^\mu(\phi_\epsilon)(x^\nu))$,

$$\begin{aligned}
\partial_\mu \zeta^\mu(\phi_0) &= \tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \partial_\mu z^\mu(\phi_\epsilon) && (0.6)\\
&= \tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} L(\Phi_\epsilon) && (0.7)\\
&= \frac{\partial L}{\partial \phi}(\Phi_0)\, \tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \phi_\epsilon + \frac{\partial L}{\partial(\partial_\mu \phi)}(\Phi_0)\, \tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \partial_\mu \phi_\epsilon + \frac{\partial L}{\partial z^\mu}(\Phi_0)\zeta^\mu(\phi_0) && (0.8)\\
&= \frac{\partial L}{\partial \phi}(\Phi_0)\eta + \frac{\partial L}{\partial(\partial_\mu \phi)}(\Phi_0)\partial_\mu \eta + \frac{\partial L}{\partial z^\mu}(\Phi_0)\zeta^\mu(\phi_0) && (0.9)
\end{aligned}$$

This gives a PDE for $\zeta^\mu(\phi_0)$. If $\frac{\partial L}{\partial z^\mu}(\Phi_0)$ is the gradient of some function, i.e. there is a function f such that $\partial_\mu f = \frac{\partial L}{\partial z^\mu}(\Phi_0)$, in which case we say the dependence of L on z is *exact*, then we can solve this equation for $\zeta^\mu(\phi_0)$. Indeed, following the 1-dimensional case, if we try the ansatz

$$\zeta^\mu(\phi_0)(x^\nu) = M^\mu(x^\nu) e^{f(x^\nu)} \tag{0.10}$$

then (writing ζ^μ for $\zeta^\mu(\phi_0)$)

$$\partial_\mu \zeta^\mu(x^\nu) = \partial_\mu f(x^\nu) M^\mu(x^\nu) e^{f(x^\nu)} + \partial_\mu M^\mu(x^\nu) e^{f(x^\nu)} \tag{0.11}$$

so it must be

$$\partial_\mu M^\mu = e^{-f} \left(\frac{\partial L}{\partial \phi}(\Phi_0) \, \tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \phi_\epsilon + \frac{\partial L}{\partial(\partial_\mu \phi)}(\Phi_0) \, \tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \partial_\mu \phi_\epsilon \right). \tag{0.12}$$

Now

$$\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} A(\phi_\epsilon) = \int_{\partial M} \zeta^\mu \, \mathrm{d}n_\mu = \int_{\partial M} e^f M^\mu \, \mathrm{d}n_\mu. \tag{0.13}$$

The field ϕ will be an extremum if and only if this integral vanishes for all variations about ϕ.

PAS MISTERIÓS: equival a veure que s'anuł.la la integral sense l'exponencial.

We rewrite the integral using integration by parts:

$$\int_{\partial M} M^\mu \, \mathrm{d}n_\mu = \int_M \partial_\mu M^\mu \tag{0.14}$$

$$= \int_M e^{-f} \left(\frac{\partial L}{\partial \phi}(\Phi_0)\eta + \frac{\partial L}{\partial(\partial_\mu \phi)}(\Phi_0)\partial_\mu \eta \right) \tag{0.15}$$

$$= \int_M e^{-f} \frac{\partial L}{\partial \phi}(\Phi_0)\eta + \int_{\partial M} e^{-f} \frac{\partial L}{\partial(\partial_\mu \phi)} \eta - \int_M \partial_\mu \left(e^{-f} \frac{\partial L}{\partial(\partial_\mu \phi)} \right) \eta \tag{0.16}$$

The second integral vanishes because η vanishes on ∂M, since $\phi_\epsilon|_{\partial M} = \phi|_{\partial M}$ for all ϵ. Using the product rule for the third integral and recalling that $\partial_\mu f = \frac{\partial L}{\partial z^\mu}(\Phi_0)$ we arrive at

$$\int_{\partial M} M^\mu \, \mathrm{d}n_\mu = \int_M \eta e^{-f} \left(\frac{\partial L}{\partial \phi}(\Phi_0) - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)}(\Phi_0) \right) + \frac{\partial L}{\partial z^\mu} \frac{\partial L}{\partial(\partial_\mu \phi)} \right). \tag{0.17}$$

So, ϕ will be an extremum of the action if and only if this last integral vanishes for any variation. And, if it is to vanish for any variation, it must vanish for any η, thus the whole integrand must vanish. Since e^{-f} is nonzero, this is equivalent to the Herglotz equation,

$$\frac{\partial L}{\partial \phi}(\Phi_0) - \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)}(\Phi_0) \right) + \frac{\partial L}{\partial z^\mu} \frac{\partial L}{\partial(\partial_\mu \phi)} = 0 \tag{0.18}$$