A variational derivation of the field equations of an action-dependent Einstein-Hilbert Lagrangian

Arnau Mas

Supervised by
Dr Jordi Gaset

Universitat Autònoma de Barcelona
June 27th 2021

Universitat Autònoma de Barcelona

Outline

Outline

Background

Background

- Modified theories of gravity: $f(R)$, Brans-Dicke, etc. to answer questions from cosmology

- Modified theories of gravity: $f(R)$, Brans-Dicke, etc. to answer questions from cosmology
- Action-dependent Lagrangian mechanics: variational formulation of dissipative systems, applied in many other fields (thermodynamics, control theory).

Background

- Modified theories of gravity: $f(R)$, Brans-Dicke, etc. to answer questions from cosmology
- Action-dependent Lagrangian mechanics: variational formulation of dissipative systems, applied in many other fields (thermodynamics, control theory).

Idea: what does an action-dependent version of Einstein gravity look like?

Work done

- Writing down the equations of action-dependent GR has two main challenges: it is a field theory and it is a *second order* field theory.

Work done

- Writing down the equations of action-dependent GR has two main challenges: it is a field theory and it is a *second order* field theory.
- An attempt was made in [Laz+18], but we claim the equations found there are not the correct ones. We will use an alternative formulation of the action-dependent theory which allows us to treat second order theories.

Outline

If we allow the Lagrangian to depend on the action, then we have

$$S[q^\mu](t) = \int_a^t L(q^\mu(\tau), \dot{q}^\mu(\tau), S[q^\mu](\tau))\,\mathrm{d}\tau$$

Herglotz problem: implicit version

If we allow the Lagrangian to depend on the action, then we have

$$S[q^\mu](t) = \int_a^t L(q^\mu(\tau), \dot{q}^\mu(\tau), S[q^\mu](\tau))\, \mathrm{d}\tau$$

and differentiating

$$\dot{S}[q^\mu](t) = L(q^\mu(t), \dot{q}^\mu(t), S[q^\mu](t)).$$

If we allow the Lagrangian to depend on the action, then we have

$$S[q^\mu](t) = \int_a^t L(q^\mu(\tau), \dot{q}^\mu(\tau), S[q^\mu](\tau))\,\mathrm{d}\tau$$

and differentiating

$$\dot{S}[q^\mu](t) = L(q^\mu(t), \dot{q}^\mu(t), S[q^\mu](t)).$$

So, solve this ODE for any given trajectory and then find which paths are extrema of $S[q^\mu](a) - S[q^\mu](b)$.

Herglotz problem: implicit version

If we allow the Lagrangian to depend on the action, then we have

$$S[q^\mu](t) = \int_a^t L(q^\mu(\tau), \dot{q}^\mu(\tau), S[q^\mu](\tau))\,\mathrm{d}\tau$$

and differentiating

$$\dot{S}[q^\mu](t) = L(q^\mu(t), \dot{q}^\mu(t), S[q^\mu](t)).$$

So, solve this ODE for any given trajectory and then find which paths are extrema of $S[q^\mu](a) - S[q^\mu](b)$. Not very elegant!

Herglotz problem: constrained version

Consider expanded trajectories q^μ, z, so z tracks *some quantity* along the path.

Consider expanded trajectories q^μ, z, so z tracks *some quantity* along the path. If we want z to be the change in action then we require

$$\dot{z}(t) = L(q^\mu(t), \dot{q}^\mu(t), z(t)).$$

Herglotz problem: constrained version

Consider expanded trajectories q^μ, z, so z tracks *some quantity* along the path. If we want z to be the change in action then we require

$$\dot{z}(t) = L(q^\mu(t), \dot{q}^\mu(t), z(t)).$$

So the action functional becomes

$$S[q^\mu, z] = z(b) - z(a).$$

Herglotz problem: constrained version

Consider expanded trajectories q^μ, z, so z tracks *some quantity* along the path. If we want z to be the change in action then we require

$$\dot{z}(t) = L(q^\mu(t), \dot{q}^\mu(t), z(t)).$$

So the action functional becomes

$$S[q^\mu, z] = z(b) - z(a).$$

We want to find the extrema of this functional

Herglotz problem: constrained version

Consider expanded trajectories q^μ, z, so z tracks *some quantity* along the path. If we want z to be the change in action then we require

$$\dot{z}(t) = L(q^\mu(t), \dot{q}^\mu(t), z(t)).$$

So the action functional becomes

$$S[q^\mu, z] = z(b) - z(a).$$

We want to find the extrema of this functional *subject to the constraint* that z actually be the action.

Lagrange multipliers

Turns out, finding the extrema subject to the constraint is equivalent to finding the extrema of

$$\tilde{S}[q^\mu, z, \lambda] = S[q^\mu, z] - \int_a^b \lambda(t)\left[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\right] \mathrm{d}t$$

Turns out, finding the extrema subject to the constraint is equivalent to finding the extrema of

$$\begin{aligned}\tilde{S}[q^\mu, z, \lambda] &= S[q^\mu, z] - \int_a^b \lambda(t)\big[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\big]\,\mathrm{d}t \\ &= \int_a^b \dot{z}(t)\,\mathrm{d}t - \int_a^b \lambda(t)\big[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\big]\,\mathrm{d}t\end{aligned}$$

Lagrange multipliers

Turns out, finding the extrema subject to the constraint is equivalent to finding the extrema of

$$\begin{aligned}\tilde{S}[q^\mu, z, \lambda] &= S[q^\mu, z] - \int_a^b \lambda(t)\left[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\right] \mathrm{d}t \\ &= \int_a^b \dot{z}(t)\,\mathrm{d}t - \int_a^b \lambda(t)\left[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\right] \mathrm{d}t \\ &= \int_a^b (1 - \lambda(t))\dot{z}(t) + \lambda(t) L(q^\mu(t), \dot{q}^\mu(t), z(t))\,\mathrm{d}t.\end{aligned}$$

Lagrange multipliers

Turns out, finding the extrema subject to the constraint is equivalent to finding the extrema of

$$\begin{aligned}\tilde{S}[q^\mu, z, \lambda] &= S[q^\mu, z] - \int_a^b \lambda(t)\left[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\right] \mathrm{d}t \\ &= \int_a^b \dot{z}(t)\,\mathrm{d}t - \int_a^b \lambda(t)\left[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\right] \mathrm{d}t \\ &= \int_a^b (1-\lambda(t))\dot{z}(t) + \lambda(t) L(q^\mu(t), \dot{q}^\mu(t), z(t))\,\mathrm{d}t.\end{aligned}$$

Lagrange multipliers

Turns out, finding the extrema subject to the constraint is equivalent to finding the extrema of

$$\begin{aligned}\tilde{S}[q^\mu, z, \lambda] &= S[q^\mu, z] - \int_a^b \lambda(t)\big[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\big]\,\mathrm{d}t \\ &= \int_a^b \dot{z}(t)\,\mathrm{d}t - \int_a^b \lambda(t)\big[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\big]\,\mathrm{d}t \\ &= \int_a^b (1-\lambda(t))\dot{z}(t) + \lambda(t)L(q^\mu(t), \dot{q}^\mu(t), z(t))\,\mathrm{d}t.\end{aligned}$$

The Euler-Lagrange equations of this functional are

$$\frac{\partial L}{\partial q^\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\mu} + \frac{\partial L}{\partial \dot{q}^\mu}\frac{\partial L}{\partial z} = 0.$$

These are the Herglotz equations.

The harmonic oscillator with linear action coupling is

The harmonic oscillator with linear action coupling is

$$L(q,\dot{q},z) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 - \gamma z.$$

Example: damped harmonic oscillator

The harmonic oscillator with linear action coupling is

$$L(q, \dot{q}, z) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 - \gamma z.$$

The equation of motion is

The harmonic oscillator with linear action coupling is

$$L(q,\dot{q},z)=\tfrac{1}{2}m\dot{q}^2-\tfrac{1}{2}m\omega^2q^2-\gamma z.$$

The equation of motion is

$$\ddot{q}+\gamma\dot{q}+\omega^2q=0,$$

which is the equation of a damped harmonic oscillator.

Generalisation to field theory

There is a subtlety when thinking of action-dependent Lagrangians for field theory. We must replace z by a 3-form $z = z^\mu \, \mathrm{d}x_\mu$ which we call the action flux.

Generalisation to field theory

There is a subtlety when thinking of action-dependent Lagrangians for field theory. We must replace z by a 3-form $z = z^\mu \, \mathrm{d}x_\mu$ which we call the action flux. The action flux does *not* transform like a vector.

Generalisation to field theory

There is a subtlety when thinking of action-dependent Lagrangians for field theory. We must replace z by a 3-form $z = z^\mu \, \mathrm{d}x_\mu$ which we call the action flux. The action flux does *not* transform like a vector.
The Herglotz equations for field theory are

$$\frac{\partial L}{\partial \phi^a} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi^a)} + \frac{\partial L}{\partial z^\mu} \frac{\partial L}{\partial(\partial_\mu \phi^a)} = 0.$$

	Mechanics	Field theory

	Mechanics	Field theory
Action flux	$z(t)$	$z^\mu(x^\nu)\,\mathrm{d}x_\mu$

Generalisation to field theory

	Mechanics	Field theory
Action flux	$z(t)$	$z^\mu(x^\nu)\,\mathrm{d}x_\mu$
Action functional	$S[q^\mu, z] = z(b) - z(a)$	$S[\phi^a, z^\mu] = \int_{\partial D} z$

Generalisation to field theory

	Mechanics	Field theory
Action flux	$z(t)$	$z^\mu(x^\nu)\,\mathrm{d}x_\mu$
Action functional	$S[q^\mu, z] = z(b) - z(a)$	$S[\phi^a, z^\mu] = \int_{\partial D} z$
Constraint	$\dot{z} = L(q^\mu, \dot{q}^\mu, z)$	$\mathrm{d}z = \mathcal{L}(\phi^a, \partial_\mu \phi^a, z^\mu)$

Outline

Recall the standard Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\textsf{E-H}} = R\omega_g = R\sqrt{g}\,\mathrm{d}^4x$$

where

$$R = g^{ab}R_{ab} = g^{ab}(\partial_m\Gamma^m{}_{ab} - \partial_a\Gamma^m{}_{mb} + \Gamma^m{}_{mn}\Gamma^n{}_{ab} - \Gamma^m{}_{an}\Gamma^n{}_{mb}).$$

Einstein-Hilbert Lagrangian

Recall the standard Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\textsf{E-H}} = R\omega_g = R\sqrt{g}\,\mathrm{d}^4x$$

where

$$R = g^{ab}R_{ab} = g^{ab}(\partial_m\Gamma^m{}_{ab} - \partial_a\Gamma^m{}_{mb} + \Gamma^m{}_{mn}\Gamma^n{}_{ab} - \Gamma^m{}_{an}\Gamma^n{}_{mb}).$$

It's second order in g!

Einstein-Hilbert Lagrangian

Recall the standard Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\text{E-H}} = R\omega_g = R\sqrt{g}\,\mathrm{d}^4x$$

where

$$R = g^{ab}R_{ab} = g^{ab}(\partial_m\Gamma^m{}_{ab} - \partial_a\Gamma^m{}_{mb} + \Gamma^m{}_{mn}\Gamma^n{}_{ab} - \Gamma^m{}_{an}\Gamma^n{}_{mb}).$$

It's second order in g!

The Euler-Lagrange equations of this Lagrangian are the Einstein field equations:

$$R_{ab} - \tfrac{1}{2}g_{ab}R = 0$$

10 second order PDEs.

Action flux coupling

How do we couple the action-flux to the Lagrangian?

Action flux coupling

How do we couple the action-flux to the Lagrangian? Simplest is linear coupling,

$$\mathcal{L}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, z^{\mu}) = R\omega_g - \theta \wedge z.$$

Action flux coupling

How do we couple the action-flux to the Lagrangian? Simplest is linear coupling,

$$\mathcal{L}_{\text{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, z^\mu) = R\omega_g - \theta \wedge z.$$

θ must be a covector, $\theta = \theta_\mu \, \mathrm{d}x^\mu$, the *dissipation form*.

Action flux coupling

How do we couple the action-flux to the Lagrangian? Simplest is linear coupling,

$$\mathcal{L}_{\textsf{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, z^\mu) = R\omega_g - \theta \wedge z.$$

θ must be a covector, $\theta = \theta_\mu \, \mathrm{d}x^\mu$, the *dissipation form*.
In coordinates

$$\theta \wedge z = \theta_\mu z^\mu \, \mathrm{d}^4 x.$$

How do we couple the action-flux to the Lagrangian? Simplest is linear coupling,

$$\mathcal{L}_{\textsf{E-H}}(g_{ab}, g_{ab,\mu}, g_{ab,\mu\nu}, z^\mu) = R\omega_g - \theta \wedge z.$$

θ must be a covector, $\theta = \theta_\mu \, \mathrm{d}x^\mu$, the *dissipation form*.
In coordinates

$$\theta \wedge z = \theta_\mu z^\mu \, \mathrm{d}^4 x.$$

This means the constraint is

$$\partial_\mu z^\mu = R\sqrt{g} - \theta_\mu z^\mu.$$

Derivation of the modified field equations

Let's sketch the derivation of the equations.

Let's sketch the derivation of the equations. Write out the expanded action

$$\tilde{S}[g_{ab}, z^\nu] = \int_D \left[(1-\lambda)\partial_\mu z^\mu + \lambda(R\sqrt{g} - \theta_\mu z^\mu)\right] \mathrm{d}^4 x.$$

Derivation of the modified field equations

Let's sketch the derivation of the equations. Write out the expanded action

$$\tilde{S}[g_{ab}, z^\nu] = \int_D \left[(1-\lambda)\partial_\mu z^\mu + \lambda(R\sqrt{g} - \theta_\mu z^\mu)\right] \mathrm{d}^4x.$$

Let's compute the variation of this:

$$\begin{aligned}
\delta\tilde{S}[g_{ab}, z^\nu] &= \int_D \left[(1-\lambda)\delta\partial_\mu z^\mu + \lambda(\delta(R\sqrt{g}) - \theta_\mu \delta z^\mu)\right] \mathrm{d}^4x \\
&= \int_D (1-\lambda)\partial_\mu \delta z^\mu - \lambda\theta_\mu \delta z^\mu \,\mathrm{d}^4x + \int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x.
\end{aligned}$$

Let's sketch the derivation of the equations. Write out the expanded action

$$\tilde{S}[g_{ab}, z^\nu] = \int_D \left[(1-\lambda)\partial_\mu z^\mu + \lambda(R\sqrt{g} - \theta_\mu z^\mu)\right] \mathrm{d}^4x.$$

Let's compute the variation of this:

$$\begin{aligned}\delta\tilde{S}[g_{ab}, z^\nu] &= \int_D \left[(1-\lambda)\delta\partial_\mu z^\mu + \lambda(\delta(R\sqrt{g}) - \theta_\mu \delta z^\mu)\right] \mathrm{d}^4x \\ &= \int_D (1-\lambda)\partial_\mu \delta z^\mu - \lambda\theta_\mu \delta z^\mu \,\mathrm{d}^4x + \int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x.\end{aligned}$$

Derivation of the modified field equations

Let's sketch the derivation of the equations. Write out the expanded action

$$\tilde{S}[g_{ab}, z^\nu] = \int_D \left[(1-\lambda)\partial_\mu z^\mu + \lambda(R\sqrt{g} - \theta_\mu z^\mu)\right] \mathrm{d}^4 x.$$

Let's compute the variation of this:

$$\begin{aligned}\delta\tilde{S}[g_{ab}, z^\nu] &= \int_D \left[(1-\lambda)\delta\partial_\mu z^\mu + \lambda(\delta(R\sqrt{g}) - \theta_\mu \delta z^\mu)\right] \mathrm{d}^4 x \\ &= \int_D (1-\lambda)\partial_\mu \delta z^\mu - \lambda\theta_\mu \delta z^\mu \, \mathrm{d}^4 x + \int_D \lambda\delta(R\sqrt{g})\, \mathrm{d}^4 x.\end{aligned}$$

Let's sketch the derivation of the equations. Write out the expanded action

$$\tilde{S}[g_{ab}, z^\nu] = \int_D \left[(1-\lambda)\partial_\mu z^\mu + \lambda(R\sqrt{g} - \theta_\mu z^\mu)\right] \mathrm{d}^4x.$$

Let's compute the variation of this:

$$\begin{aligned}\delta\tilde{S}[g_{ab}, z^\nu] &= \int_D \left[(1-\lambda)\delta\partial_\mu z^\mu + \lambda(\delta(R\sqrt{g}) - \theta_\mu \delta z^\mu)\right] \mathrm{d}^4x \\ &= \int_D (1-\lambda)\partial_\mu \delta z^\mu - \lambda\theta_\mu \delta z^\mu \,\mathrm{d}^4x + \int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x.\end{aligned}$$

Derivation of the modified field equations

Let's sketch the derivation of the equations. Write out the expanded action

$$\tilde{S}[g_{ab}, z^\nu] = \int_D \left[(1-\lambda)\partial_\mu z^\mu + \lambda(R\sqrt{g} - \theta_\mu z^\mu)\right] \mathrm{d}^4 x.$$

Let's compute the variation of this:

$$\begin{aligned}\delta\tilde{S}[g_{ab}, z^\nu] &= \int_D \left[(1-\lambda)\delta\partial_\mu z^\mu + \lambda(\delta(R\sqrt{g}) - \theta_\mu \delta z^\mu)\right] \mathrm{d}^4 x \\ &= \int_D (1-\lambda)\partial_\mu \delta z^\mu - \lambda\theta_\mu \delta z^\mu \, \mathrm{d}^4 x + \int_D \lambda\delta(R\sqrt{g})\, \mathrm{d}^4 x.\end{aligned}$$

The variations of the action flux and the metric decouple.

Integrating by parts in the variation of the action flux,

Variation of the action flux

Integrating by parts in the variation of the action flux,

$$\int_D (1-\lambda)\partial_\mu \delta z^\mu - \lambda \theta_\mu \delta z^\mu \, \mathrm{d}^4 x$$

Variation of the action flux

Integrating by parts in the variation of the action flux,

$$
\begin{aligned}
&\int_D (1-\lambda)\partial_\mu \delta z^\mu - \lambda\theta_\mu \delta z^\mu \, \mathrm{d}^4 x \\
&\quad = \int_D \partial_\mu\big((1-\lambda)\delta z^\mu\big) \, \mathrm{d}^4 x + \int_D (\partial_\mu \lambda - \lambda\theta_\mu)\delta z^\mu \, \mathrm{d}^4 x
\end{aligned}
$$

Variation of the action flux

Integrating by parts in the variation of the action flux,

$$\int_D (1-\lambda)\partial_\mu \delta z^\mu - \lambda\theta_\mu \delta z^\mu \, \mathrm{d}^4x$$
$$= \cancel{\int_D \partial_\mu((1-\lambda)\delta z^\mu)\, \mathrm{d}^4x} + \int_D (\partial_\mu\lambda - \lambda\theta_\mu)\delta z^\mu \, \mathrm{d}^4x$$

Variation of the action flux

Integrating by parts in the variation of the action flux,

$$\int_D (1-\lambda)\partial_\mu \delta z^\mu - \lambda\theta_\mu \delta z^\mu \,\mathrm{d}^4x$$
$$= \cancel{\int_D \partial_\mu((1-\lambda)\delta z^\mu)\,\mathrm{d}^4x} + \int_D (\partial_\mu\lambda - \lambda\theta_\mu)\delta z^\mu \,\mathrm{d}^4x$$

This must vanish for any variation of z^μ, so it must be

$$\partial_\mu\lambda = \lambda\theta_\mu.$$

Variation of the metric

The variation of the second integral term gives three terms

Variation of the metric

The variation of the second integral term gives three terms

$$\delta(R\sqrt{g}) = \delta(g^{ab}R_{ab}\sqrt{g}) = \delta g^{ab}R_{ab}\sqrt{g} + g_{ab}\delta R^{ab}\sqrt{g} + g^{ab}R_{ab}\delta\sqrt{g}$$

The variation of the second integral term gives three terms

$$\delta(R\sqrt{g}) = \delta(g^{ab}R_{ab}\sqrt{g}) = \delta g^{ab}R_{ab}\sqrt{g} + g_{ab}\delta R^{ab}\sqrt{g} + g^{ab}R_{ab}\delta\sqrt{g}$$

So

$$\int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x =$$

Variation of the metric

The variation of the second integral term gives three terms

$$\delta(R\sqrt{g}) = \delta(g^{ab}R_{ab}\sqrt{g}) = \delta g^{ab}R_{ab}\sqrt{g} + g_{ab}\delta R^{ab}\sqrt{g} + g^{ab}R_{ab}\delta\sqrt{g}$$

So

$$\int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x =$$
$$= \int_D \lambda\delta g^{ab}R_{ab}\sqrt{g}\,\mathrm{d}^4x + \int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x + \int_D \lambda R\delta\sqrt{g}\,\mathrm{d}^4x$$

The variation of the second integral term gives three terms

$$\delta(R\sqrt{g}) = \delta(g^{ab}R_{ab}\sqrt{g}) = \delta g^{ab}R_{ab}\sqrt{g} + g_{ab}\delta R^{ab}\sqrt{g} + g^{ab}R_{ab}\delta\sqrt{g}$$

So

$$\begin{aligned}
&\int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x = \\
&\quad = \int_D \lambda\delta g^{ab}R_{ab}\sqrt{g}\,\mathrm{d}^4x + \int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x + \int_D \lambda R\delta\sqrt{g}\,\mathrm{d}^4x \\
&\quad = \int_D \lambda R_{ab}\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x + \int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x - \int_D \tfrac{1}{2}\lambda g_{ab}R\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x
\end{aligned}$$

Variation of the metric

$$\int_D \lambda \delta(R\sqrt{g})\,\mathrm{d}^4x =$$

$$= \int_D \lambda R_{ab}\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x + \int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x - \int_D \tfrac{1}{2}\lambda g_{ab}R\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x$$

Variation of the metric

$$\int_D \lambda\delta(R\sqrt{g})\,\mathrm{d}^4x =$$

$$= \boxed{\int_D \lambda R_{ab}\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x} + \int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x - \boxed{\int_D \tfrac{1}{2}\lambda g_{ab}R\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x}$$

If λ weren't there, the second term would vanish and we would get the Einstein field equations. To deal with λ we have to integrate by parts twice.

Variation of the metric

We use

$$g^{ab}\delta R_{ab} = \nabla_n (g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})$$

Variation of the metric

We use

$$g^{ab}\delta R_{ab} = \nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})$$

so integrating by parts

$$\int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x =$$

Variation of the metric

We use

$$g^{ab}\delta R_{ab} = \nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})$$

so integrating by parts

$$\begin{aligned}&\int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x = \\ &\quad = \int_D \lambda\nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x\end{aligned}$$

We use

$$g^{ab}\delta R_{ab} = \nabla_n (g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})$$

so integrating by parts

$$\begin{aligned}
&\int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x = \\
&\quad = \int_D \lambda\nabla_n (g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&\quad = \int_D \nabla_n \left(\lambda(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\right)\sqrt{g}\,\mathrm{d}^4x
\end{aligned}$$

Variation of the metric

We use

$$g^{ab}\delta R_{ab} = \nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})$$

so integrating by parts

$$\begin{aligned}
&\int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x = \\
&\quad = \int_D \lambda\nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&\quad = \int_D \nabla_n\left(\lambda(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\right)\sqrt{g}\,\mathrm{d}^4x \\
&\qquad - \int_D (\nabla_n\lambda)(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x
\end{aligned}$$

We use

$$g^{ab}\delta R_{ab} = \nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})$$

so integrating by parts

$$\begin{aligned}
&\int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x = \\
&\quad = \int_D \lambda\nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&\quad = \int_D \nabla_n\left(\lambda(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\right)\sqrt{g}\,\mathrm{d}^4x \\
&\qquad - \int_D (\nabla_n\lambda)(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&\quad = -\int_D \lambda\theta_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x
\end{aligned}$$

Variation of the metric

We use

$$g^{ab}\delta R_{ab} = \nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})$$

so integrating by parts

$$\begin{aligned}
&\int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x = \\
&\quad = \int_D \lambda\nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&\quad = \int_D \cancel{\nabla_n\Big(\lambda(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\Big)}\sqrt{g}\,\mathrm{d}^4x \\
&\qquad - \int_D (\nabla_n\lambda)(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&\quad = -\int_D \lambda\theta_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x
\end{aligned}$$

Variation of the metric

We use

$$g^{ab}\delta R_{ab} = \nabla_n (g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})$$

so integrating by parts

$$\begin{aligned}
&\int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x = \\
&\quad = \int_D \lambda\nabla_n (g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&\quad = \int_D \nabla_n \cancel{\Big(\lambda (g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\Big)}\sqrt{g}\,\mathrm{d}^4x \\
&\qquad - \int_D (\nabla_n\lambda)(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&\quad = -\int_D \lambda\theta_n (g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x
\end{aligned}$$

We use

$$\delta\Gamma^a{}_{bc} = \frac{1}{2} g^{am} (\nabla_c \delta g_{bm} + \nabla_b \delta g_{mc} - \nabla_m \delta g_{bc}).$$

We use

$$\delta\Gamma^a{}_{bc} = \frac{1}{2} g^{am}(\nabla_c \delta g_{bm} + \nabla_b \delta g_{mc} - \nabla_m \delta g_{bc}).$$

so $g^{ab}\delta R_{ab}$ splits into 6 terms.

Variation of the metric

We use

$$\delta\Gamma^a{}_{bc} = \tfrac{1}{2} g^{am}(\nabla_c \delta g_{bm} + \nabla_b \delta g_{mc} - \nabla_m \delta g_{bc}).$$

so $g^{ab}\delta R_{ab}$ splits into 6 terms. We will go through one in detail

$$-\tfrac{1}{2}\int_D \lambda\theta_n g^{ab} g^{nk} \nabla_b \delta g_{ak} \sqrt{g}\, \mathrm{d}^4 x$$

Variation of the metric

We use

$$\delta\Gamma^a{}_{bc} = \tfrac{1}{2} g^{am}(\nabla_c \delta g_{bm} + \nabla_b \delta g_{mc} - \nabla_m \delta g_{bc}).$$

so $g^{ab}\delta R_{ab}$ splits into 6 terms. We will go through one in detail

$$\begin{aligned} &-\tfrac{1}{2}\int_D \lambda\theta_n g^{ab} g^{nk} \nabla_b \delta g_{ak} \sqrt{g}\,\mathrm{d}^4 x \\ &\quad = -\tfrac{1}{2}\int_D \lambda\theta_n \nabla_b (g^{ab} g^{nk} \delta g_{ak}) \sqrt{g}\,\mathrm{d}^4 x \end{aligned}$$

Variation of the metric

We use

$$\delta\Gamma^{a}{}_{bc} = \tfrac{1}{2} g^{am}(\nabla_c \delta g_{bm} + \nabla_b \delta g_{mc} - \nabla_m \delta g_{bc}).$$

so $g^{ab}\delta R_{ab}$ splits into 6 terms. We will go through one in detail

$$\begin{aligned}
&-\tfrac{1}{2}\int_D \lambda\theta_n g^{ab} g^{nk} \nabla_b \delta g_{ak} \sqrt{g}\,\mathrm{d}^4 x \\
&\quad= -\tfrac{1}{2}\int_D \lambda\theta_n \nabla_b (g^{ab} g^{nk} \delta g_{ak}) \sqrt{g}\,\mathrm{d}^4 x \\
&\quad= -\tfrac{1}{2}\int_D \nabla_b (\lambda\theta_n g^{ab} g^{nk} \delta g_{ak}) \sqrt{g}\,\mathrm{d}^4 x + \tfrac{1}{2}\int_D \nabla_b (\lambda\theta_n)(g^{ab} g^{nk} \delta g_{ak}) \sqrt{g}\,\mathrm{d}^4 x
\end{aligned}$$

Variation of the metric

We use

$$\delta\Gamma^a{}_{bc} = \tfrac{1}{2} g^{am}(\nabla_c \delta g_{bm} + \nabla_b \delta g_{mc} - \nabla_m \delta g_{bc}).$$

so $g^{ab}\delta R_{ab}$ splits into 6 terms. We will go through one in detail

$$\begin{aligned}
&-\tfrac{1}{2}\int_D \lambda\theta_n g^{ab} g^{nk} \nabla_b \delta g_{ak} \sqrt{g}\,\mathrm{d}^4x \\
&\quad = -\tfrac{1}{2}\int_D \lambda\theta_n \nabla_b (g^{ab} g^{nk} \delta g_{ak}) \sqrt{g}\,\mathrm{d}^4x \\
&\quad = -\tfrac{1}{2}\int_D \nabla_b (\lambda\theta_n g^{ab} g^{nk} \delta g_{ak}) \sqrt{g}\,\mathrm{d}^4x + \tfrac{1}{2}\int_D \nabla_b(\lambda\theta_n)(g^{ab} g^{nk} \delta g_{ak}) \sqrt{g}\,\mathrm{d}^4x \\
&\quad = -\int_D \lambda(\theta_b\theta_n + \nabla_b\theta_n)\delta g^{bn} \sqrt{g}\,\mathrm{d}^4x.
\end{aligned}$$

Variation of the metric

We use

$$\delta\Gamma^a{}_{bc} = \tfrac{1}{2} g^{am}(\nabla_c \delta g_{bm} + \nabla_b \delta g_{mc} - \nabla_m \delta g_{bc}).$$

so $g^{ab}\delta R_{ab}$ splits into 6 terms. We will go through one in detail

$$\begin{aligned}
&-\tfrac{1}{2}\int_D \lambda\theta_n g^{ab} g^{nk} \nabla_b \delta g_{ak} \sqrt{g}\,\mathrm{d}^4x \\
&\quad = -\tfrac{1}{2}\int_D \lambda\theta_n \nabla_b (g^{ab} g^{nk} \delta g_{ak}) \sqrt{g}\,\mathrm{d}^4x \\
&\quad = \cancel{-\tfrac{1}{2}\int_D \nabla_b (\lambda\theta_n g^{ab} g^{nk} \delta g_{ak}) \sqrt{g}\,\mathrm{d}^4x} + \tfrac{1}{2}\int_D \nabla_b(\lambda\theta_n)(g^{ab} g^{nk} \delta g_{ak}) \sqrt{g}\,\mathrm{d}^4x \\
&\quad = -\int_D \lambda(\theta_b\theta_n + \nabla_b\theta_n)\delta g^{bn} \sqrt{g}\,\mathrm{d}^4x.
\end{aligned}$$

Final equations

After some calculation, one finds

Final equations

After some calculation, one finds

$$\delta\tilde{S}[g_{ab}, z^\mu] = \int_D (\partial_\mu\lambda - \lambda\theta_\mu)\delta z^\mu \, \mathrm{d}^4x + \int_D \lambda(R_{ab} - \tfrac{1}{2}Rg_{ab} - K_{ab} + Kg_{ab})\delta g^{ab}\sqrt{g}\, \mathrm{d}^4x$$

with $K_{ab} = \theta_a\theta_b + \nabla_{(a}\theta_{b)}$.

Final equations

After some calculation, one finds

$$\delta\tilde{S}[g_{ab}, z^\mu] = \int_D (\partial_\mu\lambda - \lambda\theta_\mu)\delta z^\mu \,\mathrm{d}^4x \\ + \int_D \lambda(R_{ab} - \tfrac{1}{2}Rg_{ab} - K_{ab} + K g_{ab})\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x$$

with $K_{ab} = \theta_a\theta_b + \nabla_{(a}\theta_{b)}$.
The stationary configurations will be those for which this vanishes for any variation, so we arrive at

Final equations

After some calculation, one finds

$$\delta\tilde{S}[g_{ab}, z^\mu] = \int_D (\partial_\mu\lambda - \lambda\theta_\mu)\delta z^\mu \,\mathrm{d}^4x + \int_D \lambda(R_{ab} - \tfrac{1}{2}Rg_{ab} - K_{ab} + Kg_{ab})\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x$$

with $K_{ab} = \theta_a\theta_b + \nabla_{(a}\theta_{b)}$.
The stationary configurations will be those for which this vanishes for any variation, so we arrive at

$$R_{ab} - \tfrac{1}{2}Rg_{ab} - K_{ab} + Kg_{ab} = 0.$$

Final equations

After some calculation, one finds

$$\delta\tilde{S}[g_{ab}, z^\mu] = \int_D (\partial_\mu\lambda - \lambda\theta_\mu)\delta z^\mu \, \mathrm{d}^4x + \int_D \lambda(R_{ab} - \tfrac{1}{2}Rg_{ab} - K_{ab} + Kg_{ab})\delta g^{ab}\sqrt{g}\, \mathrm{d}^4x$$

with $K_{ab} = \theta_a\theta_b + \nabla_{(a}\theta_{b)}$.
The stationary configurations will be those for which this vanishes for any variation, so we arrive at

$$\boxed{R_{ab} - \tfrac{1}{2}Rg_{ab}} - \boxed{K_{ab} + Kg_{ab} = 0}.$$

Outline

The dissipation tensor

We call the object K_{ab} the *dissipation tensor*.

The dissipation tensor

We call the object K_{ab} the *dissipation tensor*. Coordinate free we can write

$$K(X,Y) = (\theta \otimes \theta)(X,Y) + \frac{1}{2}\left[(\nabla_X \theta)Y + (\nabla_Y \theta)X\right]$$

which is a $(0,2)$ tensor.

The dissipation tensor

We call the object K_{ab} the *dissipation tensor*. Coordinate free we can write

$$K(X,Y) = (\theta \otimes \theta)(X,Y) + \frac{1}{2}\left[(\nabla_X \theta)Y + (\nabla_Y \theta)X\right]$$

which is a $(0,2)$ tensor. This means the equations we have derived are covariant.

Problems with existing equations

In [Laz+17], the derived equations are

$$R_{ab} + \tilde{K}_{ab} - \frac{1}{2} g_{ab}(R + \tilde{K}) = 0$$

where

$$\tilde{K}_{ab} = \theta_m \Gamma^m{}_{ab} - \frac{1}{2}\left(\theta_a \Gamma^m{}_{mb} + \theta_b \Gamma^m{}_{am}\right).$$

Problems with existing equations

In [Laz+17], the derived equations are

$$R_{ab} + \tilde{K}_{ab} - \tfrac{1}{2} g_{ab}(R + \tilde{K}) = 0$$

where

$$\tilde{K}_{ab} = \theta_m \Gamma^m{}_{ab} - \tfrac{1}{2}\left(\theta_a \Gamma^m{}_{mb} + \theta_b \Gamma^m{}_{am}\right).$$

$\tilde{K}_{ab}$ is not a tensor so they are *not* covariant!

Problems with existing equations

In [Laz+17], the derived equations are

$$R_{ab} + \tilde{K}_{ab} - \frac{1}{2} g_{ab}(R + \tilde{K}) = 0$$

where

$$\tilde{K}_{ab} = \theta_m \Gamma^m{}_{ab} - \frac{1}{2}\left(\theta_a \Gamma^m{}_{mb} + \theta_b \Gamma^m{}_{am}\right).$$

$\tilde{K}_{ab}$ is not a tensor so they are *not* covariant!
From mechanics, linear action couplings in second order theories lead to terms *quadratic* in the dissipation. K_{ab} is quadratic in θ_μ, but not $\tilde{K}_{ab}$.

Outline

Conclusions

As a recap

Conclusions

As a recap

- The Herglotz problem can be turned into an unconstrained optimisation problem.

Conclusions

As a recap

- The Herglotz problem can be turned into an unconstrained optimisation problem.
- Because we can write down an action, we can take its variation and deal with second order theories.

Conclusions

As a recap

- The Herglotz problem can be turned into an unconstrained optimisation problem.
- Because we can write down an action, we can take its variation and deal with second order theories.
- We have derived the field equations for the Einstein-Hilbert Lagrangian with a linear action flux coupling.

Conclusions

As a recap

- The Herglotz problem can be turned into an unconstrained optimisation problem.
- Because we can write down an action, we can take its variation and deal with second order theories.
- We have derived the field equations for the Einstein-Hilbert Lagrangian with a linear action flux coupling.
- The result are a set of equations which are *covariant* and with the expected structure.

Conclusions

As a recap

- The Herglotz problem can be turned into an unconstrained optimisation problem.
- Because we can write down an action, we can take its variation and deal with second order theories.
- We have derived the field equations for the Einstein-Hilbert Lagrangian with a linear action flux coupling.
- The result are a set of equations which are *covariant* and with the expected structure.

Further work

- What do the solutions look like?

Further work

- What do the solutions look like?
- What happens when we add matter terms?

Further work

- What do the solutions look like?
- What happens when we add matter terms?
- What kind of cosmology do we get?

Further work

- What do the solutions look like?
- What happens when we add matter terms?
- What kind of cosmology do we get?
- What happens with different action flux couplings?

Further work

UAB

- What do the solutions look like?
- What happens when we add matter terms?
- What kind of cosmology do we get?
- What happens with different action flux couplings?

A variational derivation of the field equations of an action-dependent Einstein-Hilbert Lagrangian

Arnau Mas

Supervised by
Dr Jordi Gaset

Universitat Autònoma de Barcelona
June 27th 2021