# Fórmules

## Contents



## 1 Simmetría

- Delta de Kronecker generalitzada simmetrica és:

$$\delta^{\alpha\beta}_{\mu\nu} = \frac{n(\alpha\beta)}{2}\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu}\right)$$

Con

$$n(\alpha\beta) = \begin{cases} 1 \text{ si } \alpha = \beta \\ 2 \text{ si } \alpha \neq \beta \end{cases}$$

$$n(\alpha\beta) = 2 - \delta_{\alpha\beta}$$

$$\frac{2}{n(\alpha\beta)} = 1 + \delta_{\alpha\beta}$$

- Tipos de sumatorio sobre (ab):

$$\sum_{c,d} \delta^{ab}_{cd}\delta^{cd}_{ef} = \delta^{ab}_{ef} n(ab) \tag{1}$$

$$\sum_{c \leq d} \delta^{ab}_{cd}\delta^{cd}_{ef} = \delta^{ab}_{ef} \tag{2}$$

## 2 Métrica

Consideramos una matriz cuadrada invertible $g$, con componentes $g_{\alpha\beta}$. Definimos $\rho = \sqrt{|\det(g)|}$, y su inversa con $g^{\alpha\mu} g_{\mu\beta} = \delta^{\alpha}_{\beta}$.

- Para cualquier operador diferencial $\delta$:

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\beta\nu} \delta g_{\alpha\beta}$$

$$\delta\rho = \frac{1}{2} \rho g^{\alpha\beta} \delta g_{\alpha\beta}$$

- Si $g$ tiene todas las entradas independientes:

$$\frac{\partial g_{\mu\nu}}{\partial g_{\alpha\beta}} = \delta^{\alpha}_{\mu} \delta^{\beta}_{\nu}$$

$$\frac{\partial g^{\mu\nu}}{\partial g_{\alpha\beta}} = -g^{\mu\alpha} g^{\beta\nu}$$

$$\frac{\partial \rho}{\partial g_{\alpha\beta}} = \frac{1}{2} \rho g^{\alpha\beta}$$

- Si $g$ es simétrica:

$$\frac{\partial g_{\mu\nu}}{\partial g_{\alpha\beta}} = \delta^{\alpha\beta}_{\mu\nu}$$

$$\frac{\partial g^{\mu\nu}}{\partial g_{\alpha\beta}} = -g^{\mu r} g^{s\nu} \delta^{\alpha\beta}_{rs}$$

$$\frac{\partial \rho}{\partial g_{\alpha\beta}} = \frac{1}{2} \rho g^{\mu\nu} \delta^{\alpha\beta}_{\mu\nu} = \frac{1}{2} \rho g^{\alpha\beta} n(\alpha\beta)$$

$$\frac{\partial g_{ab,cd}}{\partial g_{\alpha\beta,\mu\nu}} = \delta^{\alpha\beta}_{ab} \delta^{\mu\nu}_{cd}$$

# 3 Curvatura

Curvatura para una connexión cualquiera, no necesariamente simétrica. El ordre de los subíndeces es importante.

$$R^{\gamma}_{\alpha\mu\beta} = \Gamma^{\gamma}_{\beta\alpha,\mu} - \Gamma^{\gamma}_{\mu\alpha,\beta} + \Gamma^{\gamma}_{\mu\sigma}\Gamma^{\sigma}_{\beta\alpha} - \Gamma^{\gamma}_{\beta\sigma}\Gamma^{\sigma}_{\mu\alpha}$$

Tensores derivados:

- $R_{\lambda\mu,\nu\eta} = g_{\gamma\lambda}R^{\gamma}_{\mu\nu\eta}$
- Tensor de Ricci: $R_{\alpha\beta} = \delta^{\mu}_{\gamma}R^{\gamma}_{\alpha\mu\beta}$
- Curvatura escalar: $R = g^{\mu\eta}g^{\nu\lambda}R_{\lambda\mu\nu\eta} = g^{\mu\eta}R^{\nu}_{\mu\nu\eta} = g^{\mu\eta}R_{\mu\eta}$

Tensor the Ricci para una connexión cualquiera

$$R_{\alpha\beta} = \delta^{\mu}_{\gamma}R^{\gamma}_{\alpha\mu\beta} = \Gamma^{\gamma}_{\beta\alpha,\gamma} - \Gamma^{\gamma}_{\gamma\alpha,\beta} + \Gamma^{\gamma}_{\beta\alpha}\Gamma^{\sigma}_{\sigma\gamma} - \Gamma^{\gamma}_{\beta\sigma}\Gamma^{\sigma}_{\gamma\alpha}$$

Cuando la conexión es la de Levi-Civita de la métrica:

$$R_{\lambda\mu,\nu\eta} = -\frac{1}{2}[g_{\lambda\nu,\mu\eta} - g_{\mu\nu,\eta\lambda} - g_{\lambda\eta,\nu\mu} + g_{\mu\eta,\nu\lambda}] + g_{\tau\sigma}(\Gamma^{\tau}_{\eta\lambda}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\tau}_{\nu\lambda}\Gamma^{\sigma}_{\mu\eta})$$

Otras identidades de la conexión de Levi-Civita.

$$\Gamma^{\tau}_{\nu\lambda} = g^{\tau r}\Gamma_{\nu\lambda r}$$

$$g_{\tau\sigma}(\Gamma^{\tau}_{\eta\lambda}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\tau}_{\nu\lambda}\Gamma^{\sigma}_{\mu\eta}) = g^{rs}(\Gamma_{\eta\lambda r}\Gamma_{\mu\nu s} - \Gamma_{\nu\lambda r}\Gamma_{\mu\eta s})$$

$$\Gamma^{i}_{kj} = \frac{1}{2}g^{is}\left(g_{ks,j} + g_{js,k} - g_{kj,s}\right)$$

$$g_{\lambda\sigma}\frac{\partial}{\partial x^{\mu}}g^{\sigma\nu} = -g^{\sigma\nu}\frac{\partial}{\partial x^{\mu}}g_{\lambda\sigma} = -g^{\sigma\nu}\left(\Gamma^{\eta}_{\mu\lambda}g_{\eta\sigma} + \Gamma^{\eta}_{\mu\sigma}g_{\eta\lambda}\right)$$

$$\frac{\partial}{\partial g_{\alpha\beta,\mu}}\Gamma^{i}_{kj} = \frac{1}{2}\frac{n(\alpha\beta)}{n(kj)}(g^{i\beta}\delta^{\alpha\mu}_{kj} + g^{i\alpha}\delta^{\beta\mu}_{kj} - g^{i\mu}\delta^{\alpha\beta}_{kj})$$

# 4 Hilbert Lagrangian

- Lagrangiana:

$$L = \rho R = K^{\alpha\beta,\mu\nu} g_{\alpha\beta,\mu\nu} + L_0 = S^{\alpha\beta,\mu\nu} g_{\alpha\beta,\mu\nu} + L_0 = n(\mu\nu) L^{(\alpha\beta),(\mu\nu)} g_{(\alpha\beta),(\mu\nu)} + L_0$$

- Coeficiente de las segundas derivadas.

$$K^{\alpha\beta,\mu\nu} = \rho \left( g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu} \right)$$

$$\begin{aligned} \tfrac{\partial K^{\alpha\beta,\mu\nu}}{\partial g_{ab}} = \quad & \rho \delta^{ab}_{rs} \left( \tfrac{1}{2} g^{rs} (g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu}) + g^{\alpha r} (g^{s\beta} g^{\mu\nu} - g^{s\mu} g^{\beta\nu}) \right. \\ & \left. + g^{s\nu} (g^{\alpha\beta} g^{\mu r} - g^{\alpha\mu} g^{\beta r}) \right) = \rho \delta^{ab}_{rs} M^{\alpha\beta,\mu\nu,rs} \end{aligned}$$

Las parejas $(\alpha\beta), (\mu\nu)$ i $(rs)$ estan simetrizadas. Reordenando los índices de distintas formas optenemos distintas expresiones:

$$M^{\alpha\beta,\mu\nu,sr} = \frac{1}{2} g^{rs} (g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu}) - 2 g^{\alpha r} g^{s\mu} g^{\beta\nu} + g^{\alpha r} g^{s\beta} g^{\mu\nu} + g^{s\nu} g^{\alpha\beta} g^{\mu r}$$

$$M^{\alpha\beta,\mu\nu,rs} = \frac{1}{2} g^{rs} (g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu}) + g^{\alpha r} (g^{s\beta} g^{\mu\nu} - g^{s\mu} g^{\beta\nu}) + g^{s\nu} (g^{\alpha\beta} g^{\mu r} - g^{\alpha\mu} g^{\beta r})$$

$$M^{12,36,45} = -\frac{1}{2} g^{12} g^{45} g^{36} + \frac{1}{2} g^{45} g^{13} g^{26} + g^{12} g^{43} g^{56} + g^{36} g^{14} g^{25} - g^{16} g^{43} g^{25} - g^{13} g^{46} g^{25}$$

Simetrizando explícitamente:

$$\begin{aligned} S^{\alpha\beta\mu\nu} &= \tfrac{1}{4} \left( K^{\alpha\beta,\mu\nu} + K^{\alpha\beta,\nu\mu} + K^{\beta\alpha,\mu\nu} + K^{\beta\alpha,\nu\mu} \right) \\ &= \tfrac{1}{2} \left( K^{\alpha\beta,\mu\nu} + K^{\alpha\beta,\nu\mu} \right) = \tfrac{\rho}{2} (g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} - 2 g^{\alpha\beta} g^{\mu\nu}) \end{aligned}$$

- Término sin segundas derivadas:

$$\begin{aligned} L_0 &= \rho g^{\alpha\beta} \{ g^{\gamma\delta} (g_{\delta\mu,\beta} \Gamma^{\mu}_{\alpha\gamma} - g_{\delta\mu,\gamma} \Gamma^{\mu}_{\alpha\beta}) + \Gamma^{\delta}_{\alpha\beta} \Gamma^{\gamma}_{\gamma\delta} - \Gamma^{\delta}_{\alpha\gamma} \Gamma^{\gamma}_{\beta\delta} \} \\ &= \rho g^{\mu\eta} g^{\nu\lambda} g^{rs} (\Gamma_{\eta\lambda r} \Gamma_{\mu\nu s} - \Gamma_{\nu\lambda r} \Gamma_{\mu\eta s}) \end{aligned}$$

$$L_0 = \frac{\rho}{4} g_{12,3} g_{45,6} \Upsilon^{12,3;45,6} = \frac{\rho}{4} \sum_{\substack{r \leq s \\ k \leq l}} g_{rs,3} g_{kl,6} \delta^{rs}_{12} \delta^{kl}_{45} \Upsilon^{12,3;45,6}$$

Simetritzando (12)(45):

$$\Upsilon^{12,3;45,6} = -g^{12}g^{45}g^{36} + 3g^{14}g^{25}g^{36} + 2g^{12}g^{46}g^{35} + 2g^{45}g^{16}g^{32} - 4g^{13}g^{25}g^{46} - 2g^{43}g^{25}g^{16}$$

Propiedades algebraicas:

$$\Upsilon^{12,3;45,6} \neq \Upsilon^{45,3;12,6}; \quad \Upsilon^{12,3;45,6} = \Upsilon^{45,6;12,3}$$

$$\frac{\rho}{4}\Upsilon^{12,3;45,6} = \frac{1}{2}\frac{\partial K^{12,45}}{\partial g_{36}} + \frac{1}{2}g^{25}K^{13,46}$$

Simetritzando (12)(45) i (36) (o (12) $\leftrightarrow$ (45)) (es equivalente si simetritzamos las dos):

$$\Upsilon^{12,3;45,6} = -g^{12}g^{45}g^{36} + 3g^{14}g^{25}g^{36} + 2g^{12}g^{46}g^{35} + 2g^{45}g^{16}g^{32} - 6g^{14}g^{26}g^{35}$$

$$\begin{array}{l}
\frac{\partial \Upsilon^{12,3;45,6}}{\partial g_{78}} = \Upsilon^{12,3;45,6;78} = \delta_{78} \\
g^{17}g^{82}g^{45}g^{36} - 3g^{17}g^{84}g^{25}g^{36} - 2g^{17}g^{28}g^{46}g^{35} - 2g^{47}g^{85}g^{16}g^{32} + 6g^{17}g^{84}g^{26}g^{35} \\
+g^{12}g^{47}g^{85}g^{36} - 3g^{14}g^{27}g^{85}g^{36} - 2g^{12}g^{47}g^{86}g^{35} - 2g^{45}g^{17}g^{86}g^{32} + 6g^{14}g^{27}g^{86}g^{35} \\
+g^{12}g^{45}g^{37}g^{86} - 3g^{14}g^{25}g^{37}g^{86} - 2g^{12}g^{46}g^{37}g^{85} - 2g^{45}g^{16}g^{37}g^{82} + 6g^{14}g^{26}g^{37}g^{85}
\end{array}$$

$$\begin{aligned}
\frac{\partial L_0}{\partial g_{\alpha\beta}} = \quad & \rho\delta^{\alpha\beta}_{12}(\tfrac{1}{2}g^{12}g^{\mu\eta}g^{\nu\lambda}g^{rs} - g^{\mu 1}g^{2\eta}g^{\nu\lambda}g^{rs} \\
& -g^{\mu\nu}g^{\nu 1}g^{2\lambda}g^{rs} - g^{\mu\nu}g^{\nu\lambda}g^{r1}g^{2s})(\Gamma_{\eta\lambda r}\Gamma_{\mu\nu s} - \Gamma_{\nu\lambda r}\Gamma_{\mu\eta s})
\end{aligned}$$

$$\sum_{\alpha\leq\beta} g_{\alpha\beta,\mu}\frac{\partial L_0}{\partial g_{\alpha\beta,\mu}} = 2L_0$$

$$\frac{\partial L_0}{\partial g_{\alpha\beta,\mu}} = \frac{\rho}{2}\sum_{k\leq l} g_{kl,\nu}\delta^{\alpha\beta}_{12}\delta^{kl}_{45}\Upsilon^{12,\mu;45,\nu} \tag{3}$$

- Función característica de segundo orden:

$$\begin{aligned}
L^{\alpha\beta,\mu\nu} \quad & = \tfrac{1}{n(\mu\nu)}\tfrac{\partial L}{\partial g_{\alpha\beta,\mu\nu}} = n(\alpha\beta)S^{\alpha\beta\mu\nu} = \tfrac{n(\alpha\beta)}{2}\rho(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - 2g^{\alpha\beta}g^{\mu\nu}) \\
& = \delta^{\alpha\beta}_{rs}K^{rs,\mu\nu} = \delta^{\mu\nu}_{rs}K^{\alpha\beta,rs}
\end{aligned} .$$

- Función característica de primer orden

$$\begin{aligned}
L^{\alpha\beta,\mu} &=^{?} \frac{\partial L}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3} \frac{1}{n(\mu\nu)} \frac{d}{dx^{\nu}} \left( \frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}} \right) \\
&= \frac{\partial L_0}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3} \frac{d}{dx^{\nu}} L^{\alpha\beta,\mu\nu} \\
&= \rho \sum_{(kl)} \delta_{12}^{\alpha\beta} \delta_{45}^{kl} g_{kl,\nu} N^{12,45,\mu\nu}
\end{aligned}$$

$$N^{12,45,36} = \frac{1}{4} g^{16} g^{23} g^{45} + \frac{1}{4} g^{13} g^{26} g^{45} + \frac{1}{2} g^{14} g^{25} g^{36} - \frac{1}{2} g^{13} g^{24} g^{56} - \frac{1}{2} g^{14} g^{23} g^{56}$$

$$N^{\neg}_{\alpha\beta\mu,abc} = \frac{1}{n(\alpha\beta)} \left( -\frac{1}{6} g_{\alpha a} g_{\beta\mu} g_{bc} - \frac{1}{6} g_{\alpha\mu} g_{\beta a} g_{bc} + \frac{1}{2} g_{\alpha a} g_{\beta b} g_{\mu c} + \frac{1}{12} g_{\alpha c} g_{\beta\mu} g_{ab} + \frac{1}{12} g_{\alpha\mu} g_{\beta c} g_{ab} \right)$$

$$g_{\alpha\beta,\gamma} = (4) \frac{1}{\rho} N^{\neg}_{\alpha\beta\gamma 123} L^{12,3}$$

- Funció característica d'ordre zero.

$$L^{\alpha\beta} = \frac{\partial L}{\partial g_{\alpha\beta}} - \frac{d}{dx^{\mu}} L^{\alpha\beta,\mu} = -\rho \delta_{ij}^{\alpha\beta} \left( R^{ij} - \frac{1}{2} g^{ij} R \right)$$

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = \left( g^{\alpha\mu} g^{\beta\eta} g^{\lambda\nu} - \frac{1}{2} g^{\alpha\beta} g^{\lambda\nu} g^{\mu\eta} \right) R_{\lambda\mu\nu\eta} = T^{\lambda\mu\nu\eta\alpha\beta} R_{\lambda\mu\nu\eta}$$

$$-\rho \delta_{ij}^{\alpha\beta} \left( R^{ij} - \frac{1}{2} g^{ij} R \right) = \frac{\partial L}{\partial g_{\alpha\beta}} - \rho g^{\mu\eta} \frac{\partial g^{\lambda\nu} R_{\lambda\mu\nu\eta}}{\partial g_{\alpha\beta}}$$

$$L^{\alpha\beta} = \frac{\partial L}{\partial g_{\alpha\beta}} - D_{\mu} L^{\alpha\beta,\mu}$$

El que s'ha de provar es:

$$D_{\mu} L^{\alpha\beta,\mu} = \rho g^{\mu\eta} \frac{\partial g^{\lambda\nu} R_{\lambda\mu\nu\eta}}{\partial g_{\alpha\beta}}$$

- Termes con segundas derivadas en las ecuaciones de Euler-Lagrange.

$$D_s D_r \frac{\partial R}{\partial g_{\alpha\beta,rs}} = \rho \sum_{\substack{r\leq s\\ a\leq b}} g_{ab,rs}\delta_{12}^{\alpha\beta}\delta_{45}^{rs}\delta_{36}^{ab}K^{12,45,36} + .....$$

$$D_s \frac{\partial R}{\partial g_{\alpha\beta,s}} = \frac{\rho}{2} \sum_{\substack{r\leq s\\ a\leq b}} g_{ab,rs}\delta_{12}^{\alpha\beta}\delta_{45}^{rs}\delta_{36}^{ab}\Upsilon^{124365} + ....$$

$$\frac{\partial R}{\partial g_{\alpha\beta}} = \rho \sum_{\substack{r\leq s\\ a\leq b}} g_{ab,rs}\delta_{12}^{\alpha\beta}\delta_{45}^{rs}\delta_{36}^{ab}K^{364512} + ...$$

$$K^{364512} = \left(\frac{1}{2}g^{12}(g^{34}g^{65} - g^{36}g^{45}) - 2g^{31}g^{24}g^{65} + g^{31}g^{26}g^{45} + g^{25}g^{36}g^{41}\right)$$

- Otras funciones auxiliares (Hamiltoniano: regularidad, gauge, soluciones...)

$$FL^*H = \sum_{\substack{\alpha\leq\beta\\ \mu\leq\nu}} L^{\alpha\beta,\mu\nu}g_{\alpha\beta,\mu\nu} + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu}g_{\alpha\beta,\mu} - L = \sum_{\substack{\alpha\leq\beta\\ k\leq l}} \rho g_{\alpha\beta,\mu}g_{kl,\nu}\delta_{12}^{\alpha\beta}\delta_{45}^{kl}H^{1245\mu\nu}$$

$$H^{1245\mu\nu} = \frac{1}{4}g^{12}g^{45}g^{\mu\nu} - \frac{1}{4}g^{14}g^{25}g^{\mu\nu} + \frac{1}{2}g^{14}g^{5\mu}g^{2\nu} - \frac{1}{2}g^{12}g^{5\nu}g^{4\mu} \tag{4}$$

$$H =^? \frac{1}{\rho n(cd)n(ab)}p^{abc}p^{def}H_{abcdef}$$

$$U^{\alpha\beta\mu\nu ab} = \partial g_{\alpha\beta}L^{ab,\nu\mu} - \partial g_{ab,\nu}L^{\alpha\beta,\mu} = \rho\delta_{12}^{\alpha\beta}\delta_{45}^{ab}(-\frac{1}{2}\Upsilon^{12,\mu;45,\nu} + M^{12\mu\nu45} + M^{45\nu\mu12})$$

$$-U^{\alpha\beta\mu\nu ab} = H^{\alpha\beta\mu\nu ab} + H^{ab\nu\mu\alpha\beta}$$

$$H^{\alpha\beta\mu\nu ab} = \frac{1}{4}\Upsilon^{12,\mu;45,\nu} - M^{12\mu\nu45}$$

$$M^{12\mu\nu45}+M^{45\nu\mu12} = -g^{12}g^{45}g^{\mu\nu}+\frac{3}{2}g^{12}g^{4\mu}g^{5\nu}+\frac{3}{2}g^{45}g^{1\mu}g^{2\nu}+2g^{14}g^{25}g^{\mu\nu}-2g^{1\nu}g^{4\mu}g^{25}-2g^{1\mu}g^{4\nu}g^{25}$$

(Mejor mirar artículo para la siguiente)

$$U^{\alpha\beta\mu\nu ab} =^{?} \rho\delta_{12}^{\alpha\beta}\delta_{rs}^{ab}(-\frac{1}{2}g^{12}g^{45}g^{\mu\nu}+\frac{1}{2}g^{14}g^{25}g^{\mu\nu}+\frac{1}{2}g^{12}g^{4\mu}g^{5\nu}+\frac{1}{2}g^{45}g^{1\nu}g^{\mu2}-g^{1\nu}g^{4\mu}g^{25})$$