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Title

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Chapter 1

Geometrical background

1.1 Bundles, sections and jets

Field theory studies the dynamics of fields, which are, in the broadest sense, quantities that vary from point to point in space (or spacetime). The fundamental example is a scalar field, which is the assignment of a real value to each point in space (or spacetime). In more precise mathematical terms, if M denotes spacetime (thought of as a manifold with a pseudo-Riemannian metric, as is standard in general relativity), then a scalar field is a smooth map $\phi\colon M \to \mathbb{R}$. However, if we think of a field with vector values, for instance, it is not enough to define it as a map $X\colon M \to \mathbb{R}^4$, at least not if we want our theory to be covariant. Indeed, we think of a vector field as an assignment of a vector at each point in spacetime which is tangent to that point. Thus, the proper definition of a vector field is as a map $X\colon M \to TM$, where TM is the tangent bundle of M, together with the condition $X(p) \in T_pM$, which is just the requirement that $X(p)$ be tangent to p. This condition makes X a *section* of TM. If $\tau_M\colon TM \to M$ is the map which takes a tangent vector to its basepoint, then X being a section of TM is equivalent to $\tau_{TM} \circ X = \mathrm{id}_M$.

This idea points us in the right direction for a more appropriate definition of fields. We introduce the idea of a *fiber bundle*, which is a projection map $E \xrightarrow{\pi} M$, such that for every $p \in M$, $\pi^{-1}(p)$ is diffeomorphic to some fixed manifold F. $\pi^{-1}(p)$ is called the fibre over p, and F is the typical fibre of the bundle. What is happening here is that E is the result of attaching a copy of F at each point of M. If one attempts to draw this, taking M to be 2 dimensional and F 1 dimensional, the result is something that looks much like a hair brush, where the copies of F are the bristles, or fibres, hence the terminology. M is known as the *base space*, E receives the name *total space*.

For a more concrete picture, let's see what this looks like in coordinates. Consider a coordinate system x^μ for M, where μ runs over the dimension of M —which will be 4 if M is spacetime—, and a coordinate system u^α for the typical fibre F, where α runs over the dimension of F. Then points in the total space E can be labeled by (x^μ, u^α). For example, in the tangent bundle, these coordinates label the base point of a vector and its components with respect to a basis (which is generally taken to be basis induced by the coordinates in physics).

1.1.1 Fields as sections of bundles

1.1.2

In classical mechanics, the Lagrangian is defined as a function on the velocity phase space of the system, i.e. the tangent bundle of the configuration space TM, $L\colon TM \to \mathbb{R}$. With the Lagrangian in hand, one defines the action functional as

$$S[\gamma] = \int_a^b L(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t.$$

The precise generalisation of the Lagrangian to classical field theory is a bit more difficult. For one, whatever the Lagrangian is, it has to be something we can integrate over our parameter space, which in general will be a manifold. In the case of general relativity, the parameter space is space-time, for example. This means it has to be related to a top form on the parameter space. The precise language to formulate this idea is the language of jet bundles. In general, a suitable way to think of fields is as sections of some bundle $E \xrightarrow{\pi} M$, where M is the parameter space and the typical fibre of the bundle is the space in which the field takes values. Then,for a first order theory, the Lagrangian $\mathcal{L}$ is a bundle morphism from the first jet bundle $J^1\pi$ to the bundle $\bigwedge^n T^*M$. Then, for a section ϕ of π, $\mathcal{L} \circ j^1\phi$ is a section of $\bigwedge^n T^*M$, i.e., a top form over M. We can then define the action functional as

$$S[\phi] = \int_M \mathcal{L} \circ j^1\phi$$

In field theory, the Lagrangian is no longer just a real-valued function, since it is something we have to integrate over the parameter space, which is spacetime in the context of general relativity. The object we need is going to be a top form. Specifically, if we are studying the dynamics of sections of some bundle $E \xrightarrow{\pi} M$, the Lagrangian density $\mathcal{L}$, or sometimes just the Lagrangian, is a bundle morphism $\mathcal{L}\colon J^1\pi \to \bigwedge^n T^*M$, where n is the dimension of the parameter space M. That $\mathcal{L}$ is a bundle morphism means that for any $j^1_p\phi \in J^1_p\pi$, $\mathcal{L}(j^1_p\phi) \in \wedge^n T^*_p M$, or, more succintly, that $\wedge^n \tau^*_M \circ \mathcal{L} = \pi_1$.

When we introduce contact geometry into the picture one has to enlarge the source space for the Lagrangian, it is now a bundle morphism of the form

$$\mathcal{L}\colon J^1\pi \oplus_M \Lambda^{n-1}T^*M \to \Lambda^n T^*M.$$

The domain is the Whitney sum of the first jet bundle, where the field and its first derivatives live, and of the bundle of $n-1$ forms over M, where the action density lives. In coordinates, the points in $J^1\pi$ can be described as $(x^\mu, u^\alpha, u^\alpha_\mu)$. The points of $\Lambda^{n-1}T^*M$ can be described in coordinates by z^μ. Indeed, $\Lambda^{n-1}T^*M$ and T^*M both are vector bundles of dimension n over M, so they are isomorphic. Indeed, writing $\mathrm{d}^n x$ for $\mathrm{d}^1 x \wedge \cdots \wedge \mathrm{d}^n x$, we define

$$* \,\mathrm{d}x_\mu = (-1)^\mu \frac{\partial}{\partial x^\mu} \lrcorner \,\mathrm{d}^n x$$

where no summation is implied over μ. The $*\,\mathrm{d}x_\mu$ are a basis of the $n-1$ forms. Then we can write for any $z \in \Lambda^{n-1}T^*M$, $z = z^\mu * \mathrm{d}x_\mu$.

For general relativity, we are interested in sections of the bundle $\operatorname{Sym}^2 T^*M$ of symmetric 2-forms over M, specifically the subbundle of nondegenerate symmetric 2-forms, also known as pseudo-Riemannian metrics. The

1.2 Action dependent Lagrangian

The standard formulations of classical mechanics, the Lagrangian and Hamiltonian formalisms, can be used to describe and study a great variety of physical systems. Nevertheless, all of them exhibit *conservative* behaviour. Even the simplest non conservative systems, i.e. dissipative, cannot be described by means of a standard Lagrangian or Hamiltonian. One way of describing non conservative systems by means of a variational principle is through *action-dependent lagrangians*. This theory was originally studied by Herglotz.

Recall that in ordinary classical mechanics, one models particle trajectories as smooth paths in some configuration manifold, M, which generally represents space in Newtonian mechanics and spacetime in relativity, i.e. trajectories are of the form $q\colon [t_0, t_1] \to M$. The Lagrangian L is introduced as a function on the velocity phase space, TM, i.e. $L\colon TM \to \mathbb{R}$, with which the action functional is defined as the integral

$$S[q] = \int_{t_0}^{t_1} L(q^\mu(t), \dot{q}^\mu(t))\,\mathrm{d}t.$$

The Principle of Least Action is the statement that physical trajectories correspond to extrema of the action functional. The condition of being an extremum of the action can be equivalently formulated, by means of the calculus of variations, in terms of the well-known Euler-Lagrange equations.

In this new framework we expand the configuration space with a new variable, and consider paths $(q, z)\colon [t_0, t_1] \to M \times \mathbb{R}$. The new action functional is

$$S[q, z] = z(t_1) - z(t_0). \tag{1.1}$$

Physical trajectories will be those that extremise S *subject to the constraint* $\dot{z}(t) = L(q^\mu(t), \dot{q(t)}, z(t))$. We see that if a path satisfies this constraint then Equation (1.1) becomes

$$S[q, z] = z(t_1) - z(t_0) = \int_{t_0}^{t_1} \dot{z}(t) = \int_{t_0}^{t_1} L(q^\mu(t), \dot{q}^\mu(t), z(t)) \tag{1.2}$$

which makes clear why this is an action dependent Lagrangian: z represents the change in action along the path. And if L is independent of z then Equation (1.2) just reduces to the usual action from Lagrangian mechanics.

Now, it is not obvious how to solve this variational problem, since it is now a constrained optimisation problem. However, using Lagrange multipliers, it can be shown that extrema of this action will satisfy the so-called Herglotz equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\mu} - \frac{\partial L}{\partial q^\mu} - \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}^\mu} = 0 \tag{1.3}$$

These differential equations are the action dependent analog to the Euler-Lagrange equations, and notice that, as expected, if L is independent of z then the Herglotz equations reduce to Euler-Lagrange.

To see that this is useful to describe physical phenomena, consider the following action dependent Lagrangian:

$$L(q, \dot{q}, z) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 - \gamma z.$$

The Herglotz equation for this Lagrangian is

$$m\ddot{q} + m\omega^2 q + \gamma m\dot{q} = 0$$

which one recognises as the equation of motion of a damped harmonic oscillator.