A variational derivation of the field equations of an action-dependent Einstein-Hilbert Lagrangian

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Outline

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Idea: what does an action-dependent version of Einstein gravity look like?

Work done

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UAB

- Writing down the equations of action-dependent GR has two main challenges: it is a field theory and it is a *second order* field theory.

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- An attempt was made in [Laz+18], but we claim the equations found there are not the correct ones. We will use an alternative formulation of the action-dependent theory which allows us to treat second order theories.

Outline

If we allow the Lagrangian to depend on the action, then we have

$$S[q^\mu](t) = \int_a^t L(q^\mu(\tau), \dot{q}^\mu(\tau), S[q^\mu](\tau)) \, \mathrm{d}\tau$$

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So, solve this ODE for any given trajectory and then find which paths are extrema of $S[q^\mu](a) - S[q^\mu](b)$. Not very elegant!

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We want to find the extrema of this functional *subject to the constraint* that z actually be the action.

Lagrange multipliers

Turns out, finding the extrema subject to the constraint is equivalent to finding the extrema of

$$\tilde{S}[q^\mu, z, \lambda] = S[q^\mu, z] - \int_a^b \lambda(t)\left[\dot{z}(t) - L(q^\mu(t), \dot{q}^\mu(t), z(t))\right] \mathrm{d}t$$

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The Euler-Lagrange equations of this functional are

$$\frac{\partial L}{\partial q^\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\mu} + \frac{\partial L}{\partial \dot{q}^\mu}\frac{\partial L}{\partial z} = 0.$$

These are the Herglotz equations.

Example: damped harmonic oscillator

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$$L(q, \dot{q}, z) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 - \gamma z.$$

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The equation of motion is

$$\ddot{q} + \gamma\dot{q} + \omega^2 q = 0,$$

which is the equation of a damped harmonic oscillator.

Generalisation to field theory

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The Herglotz equations for field theory are

$$\frac{\partial L}{\partial \phi^a} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi^a)} + \frac{\partial L}{\partial z^\mu}\frac{\partial L}{\partial(\partial_\mu \phi^a)} = 0.$$

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Action functional	$S[q^\mu, z] = z(b) - z(a)$	$S[\phi^a, z^\mu] = \int_{\partial D} z$
Constraint	$\dot{z} = L(q^\mu, \dot{q}^\mu, z)$	$\mathrm{d}z = \mathcal{L}(\phi^a, \partial_\mu \phi^a, z^\mu)$

Outline

Einstein-Hilbert Lagrangian

Recall the standard Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\textsf{E-H}} = R\omega_g = R\sqrt{g}\,\mathrm{d}^4x$$

where

$$R = g^{ab}R_{ab} = g^{ab}(\partial_m\Gamma^m{}_{ab} - \partial_a\Gamma^m{}_{mb} + \Gamma^m{}_{mn}\Gamma^n{}_{ab} - \Gamma^m{}_{an}\Gamma^n{}_{mb}).$$

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It's second order in g!
The Euler-Lagrange equations of this Lagrangian are the Einstein field equations:

$$R_{ab} - \tfrac{1}{2}g_{ab}R = 0$$

10 second order PDEs.

Action flux coupling

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This means the constraint is

$$\partial_\mu z^\mu = R\sqrt{g} - \theta_\mu z^\mu.$$

Derivation of the modified field equations

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Let's compute the variation of this:

$$\begin{aligned}
\delta\tilde{S}[g_{ab}, z^\nu] &= \int_D \left[(1-\lambda)\delta\partial_\mu z^\mu + \lambda(\delta(R\sqrt{g}) - \theta_\mu \delta z^\mu)\right] \mathrm{d}^4 x \\
&= \int_D (1-\lambda)\partial_\mu \delta z^\mu - \lambda\theta_\mu \delta z^\mu \, \mathrm{d}^4 x + \int_D \lambda\delta(R\sqrt{g}) \, \mathrm{d}^4 x.
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The variations of the action flux and the metric decouple.

Variation of the action flux

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This must vanish for any variation of z^μ, so it must be

$$\partial_\mu \lambda = \lambda\theta_\mu.$$

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&\quad = \int_D \lambda R_{ab}\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x + \int_D \lambda g^{ab}\delta R_{ab}\sqrt{g}\,\mathrm{d}^4x - \int_D \tfrac{1}{2}\lambda g_{ab}R\delta g^{ab}\sqrt{g}\,\mathrm{d}^4x
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If λ weren't there, the second term would vanish and we would get the Einstein field equations. To deal with λ we have to integrate by parts twice.

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&\quad - \int_D (\nabla_n\lambda)(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&= -\int_D \lambda\theta_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x
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&\quad = \int_D \lambda\nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&\quad = \int_D \nabla_n\cancel{\Big(\lambda(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\Big)}\sqrt{g}\,\mathrm{d}^4x \\
&\qquad - \int_D (\nabla_n\lambda)(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
&\quad = -\int_D \lambda\theta_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x
\end{aligned}$$

Variation of the metric

We use

$$g^{ab}\delta R_{ab} = \nabla_n(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})$$

so integrating by parts

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&= \int_D \nabla_n\cancel{\Big(\lambda(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\Big)}\sqrt{g}\,\mathrm{d}^4x \\
&\quad - \int_D \boxed{(\nabla_n\lambda)}(g^{ab}\delta\Gamma^n{}_{ab} - g^{nb}\delta\Gamma^m{}_{mb})\sqrt{g}\,\mathrm{d}^4x \\
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The dissipation tensor

We call the object K_{ab} the *dissipation tensor*.

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which is a $(0,2)$ tensor. This means the equations we have derived are covariant.

Problems with existing equations

In [Laz+17], the derived equations are

$$R_{ab} + \tilde{K}_{ab} - \frac{1}{2} g_{ab}(R + \tilde{K}) = 0$$

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From mechanics, linear action couplings in second order theories lead to terms *quadratic* in the dissipation. K_{ab} is quadratic in θ_μ, but not $\tilde{K}_{ab}$.

Outline

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A variational derivation of the field equations of an action-dependent Einstein-Hilbert Lagrangian

Arnau Mas

Supervised by
Dr Jordi Gaset

Universitat Autònoma de Barcelona

June 27th 2021

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