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# On the multisymplectic formalism for first order field theories

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*Abstract*: The general purpose of this paper is to attempt to clarify the geometrical foundations of first order Lagrangian and Hamiltonian field theories by introducing in a systematic way multisymplectic manifolds, the field theoretical analogues of the symplectic structures used in geometrical mechanics. Much of the confusion surrounding such terms as gauge transformation and symmetry transformation as they are used in the context of Lagrangian theory is thereby eliminated, as we show. We discuss Noether's theorem for general symmetries of Lagrangian and Hamiltonian field theories. The cohomology associated to a group of symmetries of Hamiltonian or Lagrangian field theories is constructed and its relation with the structure of the current algebra is made apparent.

*Keywords*: Multisymplectic structure, jet bundle, first order field theory, multimomentum map, gauge transformation.

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## 1. Introduction

The task of establishing the geometrical foundations of field theory and the variational calculus was not taken seriously until the late sixties. The development of gauge theories as very fundamental models for physical theories has been a highlight in the history of the subject. This history may have been short but it has not been insignificant; an enormous amount of work has been done trying to explore to its deepest roots the geometrical structure of Lagrangian field theories and the calculus of variations [2, 9, 10, 15, 17, 19]. As a result of this work it has emerged as a fundamental principle that fields are sections of a vector bundle $E \xrightarrow{\pi} M$ over some parameter space $M$. It seems to us, however, that there are aspects of the foundations of the classical theory of fields that remain obscure, or are still presented in a misleading way. The

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general purpose of these notes is to discuss some very basic geometric structures lying in the realm of field theory that throw light on some of these points.

It is now generally accepted that the geometrical approach has led to a considerable improvement in our understanding of classical particle mechanics. The fundamental concept of geometrical Hamiltonian mechanics is that of a symplectic structure, which is the axiomatic description of the phase space of a mechanical system [1,3]. It is notable that the geometrical picture of Lagrangian field theories lacks the notion of the corresponding "covariant cotangent bundle" of the bundle $E \to M$. Furthermore, the development of symplectic geometry has been crucial to the construction of a reasonable scheme of quantization for classical mechanical systems, namely geometric quantization [21]; but at present there is no satisfactory scheme of covariant geometric quantization of fields. The role played by momentum maps in a proper formulation of Noether's theorem and reduction theorems in classical mechanics is also widely recognised; this notion has escaped recognition in the ordinary approaches to field theories.

There has been considerable recent interest in the appearence of anomalous terms in the commutation rules for the current algebras of groups of symmetries of some Lagrangian field theories. Although the investigation of these anomalies has been a major topic in quantum field theory for several years [9,12], it has become evident only recently that they are cohomological in origin. On the other hand, it is well-known that in classical mechanics symmetries carry cohomology [20], this cohomology appearing, for example, in the equivariance properties of the momentum map. It is apparent that every effort should be made to clarify the analoguous relations between symmetries and cohomology in Lagrangian field theories. Much of the confusion surrounding these matters comes from the special place that gauge symmetries have in the formulation of physical theories. It is an elementary observation in classical mechanics that contact symmetries are not the only relevant ones in the discussion of the symmetry properties of a given system. A similar observation should be made for field theories. It has become a common belief in field theory that any plausible Lagrangian field theory should be gauge invariant, meaning by this that the Lagrangian of the system is invariant under the full group of bundle automorphisms of the underlying vector bundle. This principle may be appropriate for a Lagrangian field theory modelled on a vector bundle over the physical space-time; but it need not apply in the case of a Lagrangian field theory modelled over a parameter space with different physical meaning. Furthermore, there is no reason for a gauge invariant Lagrangian system not to have other symmetries not necessarily coming from transformations of the underlying vector bundle. Pursuing this idea we should ask ourselves what is the correct formulation of the notion of symmetry and conserved current for more general symmetries than gauge symmetries in Lagrangian field theories.

We feel that all these questions—as well as many others—deserve attention from the perspective of the geometrical foundation of the theory. The first part of the present paper is therefore devoted to a careful discussion of the geometrical constructions appropriate to first order field theories. We emphasise first of all the importance of concepts of affine geometry in these constructions. Starting with a bundle $E \xrightarrow{\pi} M$, $\dim M = m$, one can derive from basic affine geometry the notion of the dual jet bundle of the jet bundle $J_1\pi$. The space $\mathcal{M}\pi$ of affine maps from $J_1\pi$ to $\mathbb{R}$ comes equipped in a natural way with a canonical $m$-form (in very much the same way as $T^*Q$, the cotangent bundle of the manifold $Q$, is equipped with a canonical 1-form). It also possesses a natural fibration over an affine bundle over $E$, that we shall call the dual jet bundle $J_1\pi^*$ of $J_1\pi$. The space $\mathcal{M}\pi$ is the model for the notion of multisymplectic manifolds and $J_1\pi^*$ is the model for covariant phase spaces.

The starting point for our work on these topics was a comparison of the multisymplectic structure for field theories as described by Gotay et al. [11] and the polysymplectic structure defined by Günther [13]. Our proposal for the geometry of first order field theory may be thought of as a synthesis of these two approaches. It has to be admitted that our structure fails to satisfy the first of the criteria for a multidimensional Hamiltonian formalism laid down by Günther, since two spaces are involved, not just one; however, the advantage in terms of clarity is obvious to us, and we hope to our readers too.

One interesting feature of the theory as we develop it is that the Hamiltonian, rather than being a function on $J_1\pi^*$, is a section of the bundle $\mathcal{M}\pi \to J_1\pi^*$. Its use permits us to write down an intrinsic version of the covariant Hamilton equations on $J_1\pi^*$.

We shall develop below structures related to multisymplectic manifolds such as Hamiltonian vector fields and Hamiltonian forms (which play the role of the observables of the theory); and we shall thoroughly discuss group actions on multisymplectic manifolds. Cohomology and momentum maps arise in the same way as they do in symplectic mechanics but with the added feature that higher order cocycles come into the play. We show that, rather than 2-cocycles as is the case in classical mechanics, the cocycles arising from symmetries of a field theory are $(m+1)$-cocycles on the Lie algebra of the group.

The notions of both symmetries and gauge transformations are very natural in the setting of multisymplectic manifolds associated to fiber bundles, but they are very different in nature. Of course when dealing with the very special case of prolongation of bundle automorphisms, both notions coincide and we recover the standard results. The systematic development of cohomology theory for groups of symmetries in field theory leads us to the observation that cocycles appearing in the current algebra commutators are closely related to the gauge-invariance properties of the corresponding Lagrangian. In contrast to the generally accepted belief that anomalous terms appear because of the breaking of gauge invariance, we shall show that such cocycles can appear even for gauge invariant Lagrangians.

The organization of the paper is as follows. The second section of the paper (following this introduction) will be devoted to the description of some basic facts about affine geometry which are needed in the subsequent constructions. Section 3 deals with the main structures associated with a fiber bundle: the multisymplectic manifold $\mathcal{M}\pi$, the dual jet bundle $J_1\pi^*$, Hamiltonians, Hamilton equations, the vertical endomorphism in the first jet bundle $J_1\pi$, Lagrangians, the Cartan form, Euler-Lagrange equations, gauge forms, and the Legendre transformation. Section 4 provides a systematic account of the geometry of group actions on multisymplectic manifolds, and the corresponding cohomology theory. Hamiltonian vector fields, the short exact sequence of Lie algebras associated to any multisymplectic manifold, multisymplectic group actions, momentum maps and their equivariance properties are discussed here as well as the construction of a family of cocycles on the Lie algebra of a given group of transformations with values in the cohomology groups of the multisymplectic manifold. The fifth section is devoted to a discussion of the theory of group actions as described above in the context of Lagrangian and Hamiltonian field theories. Symmetries, current algebras and Noether's theorem are restated along these lines, and the section ends with a detailed account of the relation between symmetries and gauge transformations as well as its cohomological implications. In the sixth and final section there are some examples.

## 2. Basic structures

**2.1. Affine duals.** Let $\mathcal{A}$ be an affine space modelled on a finite dimensional real vector space $\mathcal{V}$. A real-valued function $f$ on $\mathcal{A}$ is an *affine function* if it is an affine map $\mathcal{A} \to \mathbb{R}$, that is to say, if there is an element $\varphi$ of $\mathcal{V}^*$ such that, for all $a_1, a_2 \in \mathcal{A}$,

$$f(a_1) - f(a_2) = \langle a_1 - a_2, \varphi \rangle.$$

An affine function $f$ has the coordinate representation

$$f(a) = f_0 + a^k \varphi_k$$

where $f_0 = f(a_0)$, $a_0$ is the origin of coordinates, $(a^k)$ are the coordinates of $a$, or in other words the components of the vector $a - a_0$ with respect to a given basis for $\mathcal{V}$, and $(\varphi_k)$ are the components of $\varphi$ with respect to the dual basis for $\mathcal{V}^*$. The space $\mathrm{Aff}(\mathcal{A}, \mathbb{R})$ of affine functions on a given affine space $\mathcal{A}$ is itself an affine space, and the constant functions (which are clearly affine) form an affine subspace. (Strictly speaking we are regarding $\mathbb{R}$ here as a 1-dimensional affine space.)

The quotient affine space, that is the set of equivalence classes of affine functions differing by constant functions, will be called the *dual* of $\mathcal{A}$ and denoted by $\mathcal{A}^*$; thus $\mathcal{A}^* = \mathrm{Aff}(\mathcal{A}, \mathbb{R})/\mathbb{R}$ where the $\mathbb{R}$ in the quotient is the space of constant maps. In fact $\mathcal{A}^*$ is isomorphic to $\mathcal{V}^*$ (considered as an affine space), since the affine maps in each equivalence class are precisely those that have the same linear part. With respect to affine coordinates for $\mathcal{A}$ an affine function takes the form $a \mapsto p + p_k a^k$ for certain coefficients $p, p_k$. We may take these for coordinates on $\mathrm{Aff}(\mathcal{A}, \mathbb{R})$. The $p_k$ then serve as coordinates on $\mathcal{A}^*$, in the sense that $(p_k)$ are the coordinates of the equivalence class of the affine function $a \mapsto p_k a^k$.

**2.2. Affine spaces of sections.** Consider a short exact sequence of finite dimensional vector spaces

$$0 \to \mathcal{V} \hookrightarrow \mathcal{U} \xrightarrow{\pi} \mathcal{W} \to 0.$$

Let $\Sigma$ be the set of sections of $\pi$, that is, the linear maps $\sigma\colon \mathcal{W} \to \mathcal{U}$ such that $\pi \circ \sigma = \mathrm{id}_{\mathcal{W}}$. Thus each $\sigma \in \Sigma$ defines a splitting of the exact sequence, and the image of $\sigma$ is a subspace of $\mathcal{U}$ complementary to $\mathcal{V}$. Given any $\sigma \in \Sigma$ and any linear map $\lambda\colon \mathcal{W} \to \mathcal{V}$ we may define a new section $\sigma + \lambda$ by

$$(\sigma + \lambda)(w) = \sigma(w) + \lambda(w).$$

With the action of $\mathrm{Lin}(\mathcal{W}, \mathcal{V})$ (the vector space of linear maps $\mathcal{W} \to \mathcal{V}$) defined in this way $\Sigma$ becomes an affine space modelled on $\mathrm{Lin}(\mathcal{W}, \mathcal{V})$.

Choose a basis $\{e_i\}$ for $\mathcal{W}$ and a basis $\{e_\alpha\}$ for $\mathcal{V}$. Take the corresponding basis $\{e_\alpha \otimes \theta^i\}$ for $\mathrm{L}(\mathcal{W}, \mathcal{V}) \simeq \mathcal{V} \otimes \mathcal{W}^*$, where $\{\theta^i\}$ is the basis for $\mathcal{W}$ dual to $\{e_i\}$. Let $\sigma_0$ be a fixed element of $\Sigma$. Then for any $\sigma \in \Sigma$

$$\sigma = \sigma_0 + (\sigma - \sigma_0) = \sigma_0 + y_i^\alpha e_\alpha \otimes \theta^i$$

for certain coefficients $y_i^\alpha$, which we take for the coordinates of $\sigma$ with respect to $\sigma_0$ as origin and $\{e_\alpha \otimes \theta^i\}$ as basis.

The affine dual $\Sigma^*$ is modelled on the dual of the vector space $\mathrm{Lin}(\mathcal{W}, \mathcal{V})$ or $\mathcal{V} \otimes \mathcal{W}^*$, which is $(\mathcal{V} \otimes \mathcal{W}^*)^* \simeq \mathcal{W} \otimes \mathcal{V}^* \simeq \mathrm{Lin}(\mathcal{V}, \mathcal{W})$. The pairing defining the duality between $\mathrm{Lin}(\mathcal{W}, \mathcal{V})$ and $\mathrm{Lin}(\mathcal{V}, \mathcal{W})$ is given by taking the trace of the composition of an element of each space.

The space of affine functions $\mathrm{Aff}(\Sigma, \mathbb{R})$ has coordinates $p, p_\alpha^i$, where the coordinate form of the corresponding affine function is $p + p_\alpha^i y_i^\alpha$. For coordinates on $\Sigma^*$ we may take the $p_\alpha^i$.

There is a natural realisation of $\mathrm{Aff}(\Sigma, \mathbb{R})$ in terms of $m$-forms on $\mathcal{U}$, where $m = \dim \mathcal{W}$ (and a form in this vector space context is just an alternating multilinear function); after all, an $m$-form on $\mathcal{U}$ is, roughly speaking, a rule for assigning numbers to $m$-dimensional subspaces of $\mathcal{U}$. To make this idea more precise we proceed as follows. First we choose a volume form vol on $\mathcal{W}$. Then for any $m$-form $\omega$ on $\mathcal{U}$ and for any $\sigma \in \Sigma$, $\sigma^*\omega$ is an $m$-form on $\mathcal{W}$ and may therefore be expressed as a multiple of vol:

$$\sigma^*\omega = \hat{\omega}(\sigma)\mathrm{vol}.$$

Then $\hat{\omega}$ is a function on $\Sigma$. This function may be evaluated in terms of a basis $\{e_i\}$ for $\mathcal{W}$:

$$\hat{\omega}(\sigma) = \frac{\omega(\sigma(e_1), \sigma(e_2), \ldots, \sigma(e_m))}{\mathrm{vol}(e_1, e_2, \ldots, e_m)}.$$

It is obviously convenient to choose a basis $\{e_i\}$ for which

$$\mathrm{vol}(e_1, e_2, \ldots, e_m) = 1.$$

We now find the conditions under which $\hat{\omega}$ is an affine function. For any $\lambda \in \mathrm{L}(\mathcal{W}, \mathcal{V})$

$$\hat{\omega}(\sigma + \lambda) - \hat{\omega}(\sigma) = \sum_{j=1}^{m} \omega(\sigma(e_1), \ldots, \lambda(e_j), \ldots, \sigma(e_m)) + \ldots$$

where the unspecified terms contain two or more arguments involving $(\lambda)$.

$$i_{v_1} i_{v_2} \omega = 0 \qquad \text{for all } v_1, v_2 \in \mathcal{V}.$$

In order for $\hat{\omega}$ to be an affine function on $\Sigma$ this expression must be linear in $\lambda$, and so the unspecified terms must all vanish. The necessary and sufficient condition for this is that $\omega$ should give zero whenever two (or more) of its arguments belong to $\mathcal{V}$, or in other words that

$$i_{v_1} i_{v_2} \omega = 0 \qquad \text{for all } v_1, v_2 \in \mathcal{V}.$$

We denote the space of $m$-forms satisfying this condition by $\bigwedge_1^m(\mathcal{U})$.

With respect to a basis $\{e_i\}$ for $\mathcal{W}$ such that $\mathrm{vol}(e_1, e_2, \ldots, e_m) = 1$, a basis $\{e_\alpha\}$ for $\mathcal{V}$, a fixed section $\sigma_0$ of $\pi$, the corresponding basis $\{\sigma_0(e_i), e_\alpha\}$ for $\mathcal{U}$ and the dual basis $\{\theta^i, \theta^\alpha\}$ for $\mathcal{U}^*$ we may write any $\omega \in \bigwedge_1^m(\mathcal{U})$ as

$$\omega = p\,\pi^*\mathrm{vol} + p_\alpha^i \theta^\alpha \wedge \pi^*(i_{e_i}\mathrm{vol})$$

for some coefficients $p, p_\alpha^i$. The corresponding affine function $\hat{\omega}$ is $p + p_\alpha^i y_i^\alpha$. The space of $m$-forms $\bigwedge_1^m(\mathcal{U})$ is isomorphic to $\mathrm{Aff}(\Sigma, \mathbb{R})$. The constant functions correspond to basic forms, that is to say multiples of $\pi^*\mathrm{vol}$; we denote the space of such forms by $\bigwedge_0^m(\mathcal{U})$. Thus $\Sigma^*$ is isomorphic to $\bigwedge_1^m(\mathcal{U}) / \bigwedge_0^m(\mathcal{U})$.

**2.3. The Legendre map.** We consider now arbitrary (that is, not necessarily affine) functions on affine spaces. Let $L$ be a smooth function on an affine space $\mathcal{A}$. Given any point $a_0$ of $\mathcal{A}$ the first order Taylor approximation to $L$ about $a_0$, given by

$$a \mapsto L(a_0) + \langle a - a_0, dL|_{a_0} \rangle,$$

may be regarded as the 'best affine approximation' to $L$ based at $a_0$. (In the expression for the Taylor approximation we have implicitly identified $T_{a_0}^*(\mathcal{A})$ with $\mathcal{V}^*$.) Thus each smooth function $L$ on $\mathcal{A}$ defines a smooth map $\widehat{\mathcal{F}L}: \mathcal{A} \to \mathrm{Aff}(\mathcal{A}, \mathbb{R})$ by

$$\widehat{\mathcal{F}L}(a)(\acute{a}) = L(a) + \langle \acute{a} - a, dL|_a \rangle.$$

Composing this map with the projection $\mathrm{Aff}(\mathcal{A}, \mathbb{R}) \to \mathcal{A}^*$ gives a map

$$\mathcal{F}L: \mathcal{A} \to \mathcal{A}^*$$

which is called the *Legendre map* corresponding to $L$.

In terms of coordinates

$$\widehat{\mathcal{F}L}(a)(\acute{a}) = L(a) + (\acute{a}^k - a^k)\frac{\partial L}{\partial a^k}(a)$$

and so $\widehat{\mathcal{F}L}$ is given by

$$a \mapsto \left( L(a) - a^k \frac{\partial L}{\partial a^k}(a), \frac{\partial L}{\partial a^k}(a) \right).$$

The Legendre map $\mathcal{F}L$ is given by

$$a \mapsto \left(\frac{\partial L}{\partial a^k}(a)\right).$$

If the function $L$ is actually affine, so that

$$L(a_1) - L(a_2) = \langle a_1 - a_2, \varphi\rangle$$

for some $\varphi \in \mathcal{V}^*$, then

$$\widehat{\mathcal{F}L}(a)(\acute{a}) = L(a) + \langle \acute{a} - a, \varphi\rangle = L(\acute{a})$$

so $\widehat{\mathcal{F}L}$ is the constant map $a \mapsto L$, and $\mathcal{F}L$ is the constant map whose value is the set of affine functions differing from $L$ by a constant, that is to say, the equivalence class of affine functions determined by $\varphi$. Conversely, a smooth function on $\mathcal{A}$ whose associated Legendre map is constant is necessarily an affine function.

In the case of an affine space $\Sigma$ of splittings of a short exact sequence of vector spaces the coordinate form of $\widehat{\mathcal{F}L}$ for a smooth function $L$ on $\Sigma$ is

$$\sigma \mapsto \left(L(\sigma) - y_i^\alpha \frac{\partial L}{\partial y_i^\alpha}(\sigma), \frac{\partial L}{\partial y_i^\alpha}(\sigma)\right).$$

If, however, we identify $\mathrm{Aff}(\Sigma, \mathbb{R})$ with $\bigwedge_1^m(\mathcal{U})$ we may think of the construction as associating with each $\sigma \in \Sigma$ an $m$-form $\Theta_L(\sigma) \in \bigwedge_1^m(\mathcal{U})$, namely

$$\begin{aligned}\Theta_L(\sigma) &= \left(L(\sigma) - y_i^\alpha \frac{\partial L}{\partial y_i^\alpha}(\sigma)\right) \pi^*\mathrm{vol} + \frac{\partial L}{\partial y_i^\alpha}(\sigma)\, \theta^\alpha \wedge \pi^*(i_{e_i}\mathrm{vol}) \\ &= L(\sigma)\, \pi^*\mathrm{vol} + \frac{\partial L}{\partial y_i^\alpha}(\sigma)(\theta^\alpha - y_i^\alpha \theta^i) \wedge \pi^*(i_{e_i}\mathrm{vol}).\end{aligned}$$

We can regard this as defining an $m$-form on $\mathcal{U} \times \Sigma$ which is semi-basic with respect to projection onto the first factor. It is the Cartan $m$-form associated with $L$. Note that if $L$ is actually an affine function then $\Theta_L(\sigma)$ is just the $m$-form corresponding to that affine function; in particular it is independent of $\sigma$, and is therefore a basic form (considered as a form on $\mathcal{U} \times \Sigma$).

## 3. Lagrangian and Hamiltonian systems

**3.1. Affine bundles.** A fibre bundle $B \xrightarrow{\pi} M$ is an *affine bundle* if each fibre is an affine space modelled on the corresponding fibre of a vector bundle $V \xrightarrow{\tau} M$ [7,18].

Affine bundles always admit global sections [18].

If $\phi$ and $\phi'$ are two sections of an affine bundle $B \xrightarrow{\pi} M$ then for each $x \in M$ $\phi(x) - \phi'(x)$ is well-defined as an element of $\tau^{-1}(x)$, and so $x \mapsto \phi(x) - \phi'(x)$ defines a section of the vector bundle $V \xrightarrow{\tau} M$ on which $B$ is modelled, which we denote by $\phi - \phi'$.

**3.2. The first jet bundle and its dual.** Let $E \xrightarrow{\pi} M$ be a fibre bundle. At each $y \in E$ we have a short exact sequence of vector spaces

$$0 \to V_yE \hookrightarrow T_yE \xrightarrow{\pi_*} T_{\pi(y)}M \to 0$$

where $V_yE$ is the vertical subspace of $T_yE$, that is, the subspace consisting of vectors tangent to the fibre. Let $\Sigma_y$ be the space of splittings of this sequence. It is an affine space modelled on $\mathrm{Lin}(T_{\pi(y)}M, V_yE)$. The set $\bigcup_y \Sigma_y$ is the total space of an affine bundle over $E$ modelled on the vector bundle $VE \otimes_E \pi^*T^*M$, where $VE$ is the vertical subbundle of $TE$. This affine bundle is the 1-jet bundle of the bundle $E \xrightarrow{\pi} M$; we denote it by $J_1\pi$. The projection $J_1\pi \to E$ is denoted by $\pi_1^0$; and we write $\pi_1$ for the projection $\pi \circ \pi_1^0 \colon J_1\pi \to M$.

By carrying out fibrewise the constructions defined earlier for affine spaces we can define the affine dual bundle $J_1\pi^*$ of $J_1\pi$. The dual $J_1\pi^*$ is an affine bundle over $E$, modelled on $\pi^*TM \otimes_E VE^*$, with projections $\tau_1^0 \colon J_1\pi^* \to E$ and $\tau_1 \colon J_1\pi^* \to M$.

By identifying the space of affine functions on each fibre of $J_1\pi$ with a space of $m$-forms (where $m = \dim M$) we can regard $J_1\pi^*$ as a quotient of a bundle of $m$-forms over $E$. Let $\bigwedge_r^m E$ be the subbundle of the bundle $\bigwedge^m E$ of $m$-forms on $E$ consisting of those $m$-forms which give zero when $r+1$ of their arguments are vertical (in the sense of being sections of $VE$ over $E$); then we have a short exact sequence of affine bundles

$$0 \to \textstyle\bigwedge_0^m E \hookrightarrow \bigwedge_1^m E \to J_1\pi^* \to 0.$$

Here $\bigwedge_0^m E$ is the bundle of semi-basic $m$-forms on $E$. The bundle $\bigwedge_1^m E$ is an affine line bundle (1-dimensional affine bundle) over $J_1\pi^*$.

The bundle $\bigwedge_1^m E$ carries a canonical $m$-form, which may be defined by a generalisation of the definition of the canonical 1-form on the cotangent bundle of a manifold, a generalisation which applies to any bundle of forms of a given degree. Let $\nu \colon \bigwedge_1^m E \to E$ be the projection. Then the canonical $m$-form $\Theta$ is defined by

$$\Theta_w(\xi_1, \xi_2, \ldots, \xi_m) = w(\nu_*\xi_1, \nu_*\xi_2, \ldots, \nu_*\xi_m)$$

where $w \in \bigwedge_1^m E$ and $\xi_i \in T_w \bigwedge_1^m E$; in this definition one takes advantage of the fact that each $w \in \bigwedge_1^m E$ is just an $m$-covector on $T_{\nu(w)}E$.

The closed $(m+1)$-form $d\Theta$ is said to define a *multisymplectic structure* on the bundle $\bigwedge_1^m E$ [11].

An $m$-form on $E$ is a section of $\bigwedge_1^m E$; but it is sometimes convenient to distinguish the map from the geometric object. We write $\tilde{\omega}$ for the section corresponding to the form $\omega$; the two are related through $\Theta$ by

$$\tilde{\omega}^*\Theta = \omega.$$

Given coordinates $(x^i, y^\alpha)$ for $E$ we have coordinates $(x^i, y^\alpha, p, p_\alpha^i)$ on $\bigwedge_1^m E$ adapted to them. We assume given a volume form on $M$, and that the coordinates $(x^i)$ are such that the volume form has the local expression $dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m$. The point $w \in \bigwedge_1^m E$ with coordinates $(x^i, y^\alpha, p, p_\alpha^i)$ is the $m$-covector

$$p\, d^m x + p_\alpha^i\, dy^\alpha \wedge d^{m-1} x_i$$

at the point of $E$ with coordinates $(x^i, y^\alpha)$; here $d^m x$ stands for the volume form $dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m$, and $d^{m-1}x_i = i_{\partial/\partial x^i} d^m x$. With respect to these same coordinates we have the local expression

$$\Theta = p\, d^m x + p_\alpha^i\, dy^\alpha \wedge d^{m-1} x_i$$

for $\Theta$, where $p$ and $p_\alpha^i$ are now to be interpreted as coordinate functions.

To underline the importance of the multisymplectic structure we now change notation, writing $\mathcal{M}\pi$ for the multisymplectic manifold $\bigwedge_1^m E$. We shall denote the projection $\mathcal{M}\pi \to E$ by $\nu$, as before; while the projection $\mathcal{M}\pi \to J_1\pi^*$ will be denoted by $\mu$. Thus $\nu = \tau_1^0 \circ \mu$.

The multisymplectic structure does not pass to the quotient $J_1\pi^*$, and in fact there is no simple canonical object on $J_1\pi^*$ corresponding to $\Theta$. The best one can do is to take $p_\alpha^i dy^\alpha \wedge d^{m-1}x_i$ modulo semi-basic $m$-forms. An alternative is to use the vector-valued 1-form

$$p_\alpha^i\, dy^\alpha \otimes \frac{\partial}{\partial x^i},$$

which is essentially equivalent to the $m$-form via interior product with the volume. This vector-valued 1-form has been called a *polysymplectic structure* [13] on $J_1\pi^*$, and the Hamiltonian theory of fields can be developed in terms of such a structure. However, this approach seems to us to be artificial and unsatisfactory; in particular, the arguments of the vector-valued 1-form have to be restricted to vertical vectors, essentially because it corresponds to an $m$-form which is defined only up to equivalence, or in other words because it is not really a tensorial object.

**3.3. Gauge forms.** As we have seen, each fibre $\nu^{-1}(y)$ of the multisymplectic manifold $\nu\colon \mathcal{M}\pi \to E$ may be identified with the set of affine functions on the fibre of $J_1\pi \to E$ over $y \in E$, that is, functions on $(\pi_1^0)^{-1}(y)$ which are affine in the fibre coordinates. A section of $\mathcal{M}\pi \to E$ may therefore be regarded as an affine function on $J_1\pi$, in other words a function affine on the fibres of $J_1\pi \to E$.

If a volume form vol is chosen on $M$ then with each function $f$ on $J_1\pi$ we may associate a semi-basic $m$-form $f\pi_1{}^*\mathrm{vol}$; and in particular with each affine function we may associate such a semi-basic $m$-form.

In fact the affine structure of $J_1\pi$ allows us to pick out in an invariant way those semi-basic $m$-forms which are affine, in the sense of depending affinely on the fibre coordinates of $J_1\pi \to E$. The set of such forms is a module over $C^\infty E$, which we denote by $\mathrm{Aff}_0^m J_1\pi$. Given a volume form on $M$ we may identify the set of affine functions, and therefore the set of sections of $\mathcal{M}\pi$, with $\mathrm{Aff}_0^m J_1\pi$; this identification is of course dependent on which volume is chosen. In coordinates for which the volume is $d^m x$ the identification is

$$\rho + \rho^i_\alpha y^\alpha_i \Longleftrightarrow (\rho + \rho^i_\alpha y^\alpha_i) d^m x \Longleftrightarrow \rho\, d^m x + \rho^i_\alpha\, dy^\alpha \wedge d^{m-1} x_i$$

where $\rho$ and $\rho^i_\alpha$ are functions of the $x^i$ and $y^\alpha$. For any section $\tilde{\omega}$ of $\mathcal{M}\pi$, with corresponding $m$-form $\omega$ on $E$, we denote by $\hat{\omega}$ the associated affine function on $J_1\pi$, and by $\bar{\omega}$ the associated element of $\mathrm{Aff}^m_0 J_1\pi$ (it being understood that a choice of volume form on $M$ has been made).

Notice that for any section $\sigma$ of $E \to M$, with 1-jet $j_1\sigma$,

$$(j_1\sigma)^* \bar{\omega} = \sigma^* \omega.$$

It follows that if $\omega$ is closed then so is $(j_1\sigma)^*\bar{\omega}$ for every section $\sigma$ of $E$.

An element $\bar{\omega}$ of $\mathrm{Aff}^m_0 J_1\pi$ is called a *gauge form* if $\omega$ is closed. The reason for this terminology will now be explained. Suppose that, in local coordinates,

$$\bar{\omega} = (\rho + \rho^i_\alpha y^\alpha_i) d^m x$$

as before. Then

$$\omega = \rho\, d^m x + \rho^i_\alpha\, dy^\alpha \wedge d^{m-1} x_i,$$

and so

$$d\omega = \frac{\partial \rho}{\partial y^\alpha} dy^\alpha \wedge d^m x + \frac{\partial \rho^j_\alpha}{\partial x^i} dx^i \wedge dy^\alpha \wedge d^{m-1} x_j + \frac{\partial \rho^i_\beta}{\partial y^\alpha} dy^\alpha \wedge dy^\beta \wedge d^m x.$$

Thus $d\omega = 0$ if and only if

$$\frac{\partial \rho^i_\beta}{\partial y^\alpha} = \frac{\partial \rho^i_\alpha}{\partial y^\beta} \qquad \text{and} \qquad \frac{\partial \rho}{\partial y^\alpha} = \frac{\partial \rho^i_\alpha}{\partial x^i}.$$

Thus there are functions $f^i$, defined locally on $E$, and $f$, defined locally on $M$, such that

$$\rho^i_\alpha = \frac{\partial f^i}{\partial y^\alpha} \qquad \text{and} \qquad \rho = \frac{\partial f^i}{\partial x^i} + f.$$

But we can always find functions $g^i$, locally defined on $M$, such that $f = \partial g^i/\partial x^i$, by Poincaré's lemma applied to $f d^m x$; and so without loss of generality we have functions $F^i$, locally defined on $E$, such that

$$\rho^i_\alpha = \frac{\partial F^i}{\partial y^\alpha} \qquad \text{and} \qquad \rho = \frac{\partial F^i}{\partial x^i}.$$

Thus

$$\bar{\omega} = \left( \frac{\partial F^i}{\partial x^i} + \frac{\partial F^i}{\partial y^\alpha} y^i_\alpha \right) d^m x.$$

Then if $\sigma$ is any (local) section of $E$,

$$(j_1\sigma)^*\bar{\omega} = \frac{\partial(F^i \circ \sigma)}{\partial x^i} d^m x$$

or equivalently

$$\hat{\omega} \circ j_1\sigma = \frac{\partial(F^i \circ \sigma)}{\partial x^i}.$$

Thus $\hat{\omega}$ is (locally) a divergence.

The condition that $\omega$ is closed may also be simply expressed in terms of the multisymplectic structure: it is

$$\tilde{\omega}^* d\Theta = 0.$$

**3.4. The vertical endomorphism on the jet bundle.** Suppose given a volume form on $M$, as before. There is a vector-valued $m$-form $S$ on $J_1\pi$, whose values are vertical over $E$, and which is given in coordinates for which the volume form is $d^m x$ by

$$S = ((dy^\alpha - y^\alpha_j dx^j) \wedge d^{m-1}x_i) \otimes \frac{\partial}{\partial y^\alpha_i}.$$

It is called the *vertical endomorphism*, and may be defined intrinsically [16]. We shall consider $S$ as a map $\bigwedge^1 J_1\pi \to \bigwedge^m J_1\pi$.

If $X$ is a vector field on $J_1\pi$ vertical over $M$ then $\mathcal{L}_X S = 0$ if and only if $X$ is the prolongation to $J_1\pi$ of a vector field on $E$. The proof may be given most easily in coordinates, as follows. Suppose that

$$X = \xi^\alpha \frac{\partial}{\partial y^\alpha} + \xi^\alpha_i \frac{\partial}{\partial y^\alpha_i}.$$

We have to show that

$$\frac{\partial \xi^\alpha}{\partial y^\beta_j} = 0 \qquad \text{and} \qquad \xi^\alpha_i = \frac{\partial \xi^\alpha}{\partial x^i} + \frac{\partial \xi^\alpha}{\partial y^\beta} y^\beta_i.$$

Now

$$\begin{aligned}\mathcal{L}_X S &= (d\xi^\alpha - \xi^\alpha_j dx^j) \wedge d^{m-1}x_i \otimes \frac{\partial}{\partial y^\alpha_i} \\ &\quad - (dy^\alpha - y^\alpha_j dx^j) \wedge d^{m-1}x_i \otimes \left( \frac{\partial \xi^\beta}{\partial y^\alpha_i} \frac{\partial}{\partial y^\beta} + \frac{\partial \xi^\beta_k}{\partial y^\alpha_i} \frac{\partial}{\partial y^\beta_k} \right).\end{aligned}$$

Thus $\mathcal{L}_X S = 0$ if and only if

$$\frac{\partial \xi^\alpha}{\partial y^\beta_j} = 0 \qquad \text{and}$$

$$(d\xi^\alpha - \xi^\alpha_j dx^j) \wedge d^{m-1}x_i = \frac{\partial \xi^\alpha_i}{\partial y^\beta_j}(dy^\beta - y^\beta_k dx^k) \wedge d^{m-1}x_j.$$

Using the first of these in the second we obtain

$$\left(\frac{\partial \xi^\alpha}{\partial y^\beta} dy^\beta - \left(\xi^\alpha_j - \frac{\partial \xi^\alpha}{\partial x^j}\right) dx^j\right) \wedge d^{m-1}x_i = \frac{\partial \xi^\alpha_i}{\partial y^\beta_j}(dy^\beta - y^\beta_k dx^k) \wedge d^{m-1}x_j$$

from which the required result follows.

**3.5. Lagrangians and Cartan forms.** A *Lagrangian* is a (smooth) function on $J_1\pi$. The *Cartan m-form* $\Theta_L$ associated with the Lagrangian $L$ is

$$\Theta_L = L\,{\pi_1}^*\mathrm{vol} + S(dL)$$

where vol is the volume form on $M$ with respect to which $S$ is defined. In coordinates for which vol $= d^m x$

$$\begin{aligned}\Theta_L &= L\,d^m x + \frac{\partial L}{\partial y^\alpha_i}(dy^\alpha - y^\alpha_j dx^j) \wedge d^{m-1}x_i \\ &= \left(L - \frac{\partial L}{\partial y^\alpha_i} y^\alpha_i\right) d^m x + \frac{\partial L}{\partial y^\alpha_i} dy^\alpha \wedge d^{m-1}x_i.\end{aligned}$$

We may use a Lagrangian function to define a *Legendre map* $\mathcal{F}L\colon J_1\pi \to J_1\pi^*$ and an *extended Legendre map* $\widehat{\mathcal{F}L}\colon J_1\pi \to \mathcal{M}\pi$, both fibred over the identity of $E$, by applying the affine definitions fibrewise. It is then apparent that the Cartan $m$-form depends on $L$ through $\widehat{\mathcal{F}L}$, and in fact

$$\Theta_L = \widehat{\mathcal{F}L}^*\Theta.$$

A section $\sigma$ of $E \to M$ satisfies the *Euler-Lagrange equations* for the Lagrangian $L$ if

$$(j_1\sigma)^*(i_X d\Theta_L) = 0$$

for all vector fields $X$ on $J_1\pi$. In coordinates

$$\begin{aligned}d\Theta_L &= dL \wedge d^m x + d\left(\frac{\partial L}{\partial y^\alpha_i}\right) \wedge (dy^\alpha - y^\alpha_j dx^j) \wedge d^{m-1}x_i - \frac{\partial L}{\partial y^\alpha_i} dy^\alpha_i \wedge d^m x \\ &= \frac{\partial L}{\partial y^\alpha} dy^\alpha \wedge d^m x + d\left(\frac{\partial L}{\partial y^\alpha_i}\right) \wedge (dy^\alpha - y^\alpha_j dx^j) \wedge d^{m-1}x_i \\ &= (dy^\alpha - y^\alpha_j dx^j) \wedge \left(\frac{\partial L}{\partial y^\alpha} d^m x - d\left(\frac{\partial L}{\partial y^\alpha_i}\right) \wedge d^{m-1}x_i\right).\end{aligned}$$

Take, for a local basis of vector fields,

$$\left\{\frac{\partial}{\partial x^i} + y^\alpha_i \frac{\partial}{\partial y^\alpha},\quad \frac{\partial}{\partial y^\alpha},\quad \frac{\partial}{\partial y^\alpha_i}\right\}.$$

The interior products of $d\Theta_L$ with the first and last of these give multiples of the contact forms $dy^\alpha - y^\alpha_i dx^i$, which automatically vanish when pulled back to $M$ by a jet of a section. Thus the only significant case occurs when $X = \partial/\partial y^\alpha$, which leads to the conditions

$$(j_1\sigma)^*\left(\frac{\partial L}{\partial y^\alpha}d^m x - d\left(\frac{\partial L}{\partial y_i^\alpha}\right)\wedge d^{m-1}x_i\right) = 0$$

which are equivalent to the usual Euler-Lagrange equations for the section $\sigma$.

More generally, note that if $L$ is regular, in the sense that the matrix

$$\frac{\partial^2 L}{\partial y_i^\alpha \partial y_j^\beta}$$

is non-singular, then any section $\Phi$ of $J_1\pi \to M$ which satisfies

$$\Phi^*(i_{\partial/\partial y_i^\alpha} d\Theta_L) = 0$$

is necessarily the 1-jet of a section of $E \to M$. But even then, there is no further condition arising from the choice $X = \partial/\partial x^i + \cdots$.

**3.6. Hamiltonians.** When the Lagrangian function is regular the extended Legendre map is a local embedding of $J_1\pi$ into $\mathcal{M}\pi$ as a codimension 1 submanifold, which is transverse to the fibres of $\mu: \mathcal{M}\pi \to J_1\pi^*$. In favourable cases the image will actually define a section of $\mu$, in which case we shall say that the Lagrangian is *hyperregular*. With this as motivation we define a Hamiltonian on $J_1\pi^*$ to be a section of $\mu$.

We may use a Hamiltonian section $h$ to define an $m$-form on $J_1\pi^*$ by pulling back the canonical $m$-form $\Theta$ from $\mathcal{M}\pi$: we call the form so obtained the *Hamiltonian m-form* associated with $h$ and denote it by $\Theta_h$; thus

$$\Theta_h = h^*\Theta.$$

A section $\phi$ of $J_1\pi^*$ is said to satisfy the *Hamilton equations* for a given Hamiltonian $h$ if

$$\phi^*(i_X d\Theta_h) = 0$$

for all vector fields $X$ on $J_1\pi^*$. In terms of local coordinates $(x^i, y^\alpha, p_\alpha^i)$ for $J_1\pi^*$ and $(x^i, y^\alpha, p, p_\alpha^i)$ for $\mathcal{M}\pi$ the section $h$ may be represented by a local function $H$ in the form

$$p = H(x^i, y^\alpha, p_\alpha^i).$$

Then

$$\Theta_h = H d^m x + p_\alpha^i\, dy^\alpha \wedge d^{m-1}x_i$$

(where, as always, the coordinates on the base are chosen so that vol $= d^m x$). Then

$$i_{\partial/\partial p_\alpha^i} d\Theta_h = \frac{\partial H}{\partial p_\alpha^i} d^m x + dy^\alpha \wedge d^{m-1}x_i$$

$$i_{\partial/\partial y^\alpha} d\Theta_h = \frac{\partial H}{\partial y^\alpha} d^m x - dp_\alpha^i \wedge d^{m-1}x_i$$

from which follow the Hamilton equations

$$\frac{\partial y^\alpha}{\partial x^i} = -\frac{\partial H}{\partial p^i_\alpha} \qquad \text{and} \qquad \frac{\partial p^i_\alpha}{\partial x^i} = \frac{\partial H}{\partial y^\alpha}.$$

The equations obtained by taking $X$ to be $\partial/\partial x^i$ are consequences of these, and simply express the partial derivatives of $H \circ \phi$ as 'total' derivatives of $H$. (These are the equations which are classically written as

$$\frac{dH}{dx^i} = \frac{\partial H}{\partial x^i} + \frac{\partial H}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i} + \frac{\partial H}{\partial p^j_\alpha}\frac{\partial p^j_\alpha}{\partial x^i}.)$$

These Hamilton equations are often described as being covariant. This term must be treated with caution in this context. Clearly, by writing the equations in the invariant form $\phi^*(i_X d\Theta_h) = 0$ we have shown that they are in a sense covariant. However, it is important to remember that the function $H$ is, in general, only locally defined; in other words, there is no true Hamiltonian function, and the local representative $H$ transforms in a non-trivial way under coordinate transformations. When $\mathcal{M}\pi$ is a trivial bundle over $J_1\pi^*$, so that there is a predetermined global section, then the Hamiltonian section may be represented by a global function and no problem arises. This occurs when $E$ is trivial over $M$. In general, however, there is no preferred section of $\mathcal{M}\pi$ over $J_1\pi^*$ to relate the Hamiltonian section to, and in order to write the Hamilton equations in manifestly covariant form one must introduce a connection, as we shall explain below.

For any Lagrangian $L$ we have the following commutative diagram of Legendre maps.

$$\begin{tikzcd}
 & & \mathcal{M}\pi \arrow[d, "\mu"] \\
J_1\pi \arrow[urr, "\widehat{\mathcal{F}L}"] \arrow[rr, "\mathcal{F}L"] \arrow[dr, "\pi_1^0"'] & & J_1\pi^* \arrow[dl, "\tau_1^0"] \\
 & E &
\end{tikzcd}$$

When $L$ is hyperregular $\mathcal{F}L$ is a diffeomorphism and we may therefore define a section $h$ of $\mu$ such that

$$h \circ \mathcal{F}L = \widehat{\mathcal{F}L}.$$

Then

$$\mathcal{F}L^*\Theta_h = \mathcal{F}L^* h^*\Theta = \widehat{\mathcal{F}L}^*\Theta = \Theta_L$$

is the Cartan $m$-form associated with $L$, and a section $\sigma$ of $E \to M$ is a solution of the Euler-Lagrange equations for $L$ if and only if $\phi = \mathcal{F}L \circ j_1\sigma$ is a solution of the

Hamilton equations for $h$. The local Hamiltonian function is given by

$$H = L - y_i^\alpha \frac{\partial L}{\partial y_i^\alpha}$$

in the usual way.

**3.7. Connections.** A connection on a fibre bundle $E \xrightarrow{\pi} M$ is a section $\gamma$ of its 1-jet bundle $\pi_1^0\colon J_1\pi \to E$; that is to say, a choice of complement to the vertical subspace $V_yE$ in $T_yE$ at each point $y \in E$. With the aid of a connection we can define the covariant derivative of any section $\sigma$ of $E \xrightarrow{\pi} M$ as follows. Take the 1-jet $j_1\sigma$ of $\sigma$: it is also a section of the affine bundle $J_1\pi$, and therefore $j_1\sigma - \gamma$ (which measures by how far $\sigma$ fails to be horizontal with respect to $\gamma$) is a section of the underlying vector bundle $VE \otimes_E \pi^*T^*M$. Given any $x \in M$ and any $\xi \in T_xM$ we define an element $\nabla_\xi\sigma$ of $V_{\sigma(x)}E$ by

$$\nabla_\xi \sigma = \langle \xi, j_1\sigma(x) - \gamma(x)\rangle.$$

In coordinates, if $\gamma$ is given by

$$y_i^\alpha = \Gamma_i^\alpha(x^j, y^\beta)$$

then

$$\nabla_\xi \sigma = \xi^i \left( \frac{\partial \sigma}{\partial x^i}(x) - \Gamma_i^\alpha(x, \sigma(x)) \right) \frac{\partial}{\partial y^\alpha}.$$

For any vector field $X$ on M the resulting covariant derivative $\nabla_X\sigma$ is a section of $VE \to M$ along $\sigma$.

A connection on $E$ determines also a section of $\mathcal{M}\pi \to J_1\pi^*$. Each fibre of $\mathcal{M}\pi$ in this fibration is an equivalence class of affine functions on the corresponding fibre of $J_1\pi$; the section of $J_1\pi$ given by the connection picks out a point of each fibre of that bundle; there is a unique member of the equivalence class of affine functions that vanishes at this point, and this defines the section. In terms of coordinates the section of $\mathcal{M}\pi \to J_1\pi^*$ determined by $\gamma$ is given by

$$p = -\Gamma_i^\alpha p_\alpha^i.$$

We may use this section of $\mathcal{M}\pi \to J_1\pi^*$ to pull back the multisymplectic form $\Theta$ to an $m$-form $\Theta_\gamma$ on $J_1\pi^*$. The resulting form is given locally by

$$\Theta_\gamma = -\Gamma_i^\alpha p_\alpha^i\, d^m x + p_\alpha^i\, dy^\alpha \wedge d^{m-1}x_i = p_\alpha^i(dy^\alpha - \Gamma_i^\alpha dx^i) \wedge d^{m-1}x_i.$$

Suppose there is given a Hamiltonian section $h$ of $\mathcal{M} \to J_1\pi^*$. Then since $\mathcal{M}$ is an affine bundle over $J_1\pi^*$ we may take the difference $h - \gamma$, which is a semi-basic $m$-form on $J_1\pi^*$, and may therefore be written as $\tilde{h}\,\tau_1^{0*}\mathrm{vol}$ where $\tilde{h}$ is a globally defined function on $J_1\pi^*$. Then

$$\Theta_h = \Theta_\gamma + \tilde{h}\,\tau_1^{0*}\mathrm{vol}$$

and the Hamilton equations for $h$ may be written

$$\phi^*(i_X(d\Theta_\gamma + d\tilde{h} \wedge \tau_1^{0*}\mathrm{vol})) = 0.$$

In coordinates the equations take the form

$$\frac{\partial y^\alpha}{\partial x^i} - \Gamma_i^\alpha = -\frac{\partial \tilde{h}}{\partial p_\alpha^i} \qquad \text{and} \qquad \frac{\partial p_\alpha^i}{\partial x^i} + p_\beta^j \frac{\partial \Gamma_j^\beta}{\partial y^\alpha} = \frac{\partial \tilde{h}}{\partial y^\alpha}.$$

## 4. Group actions on multisymplectic manifolds

**4.1. Hamiltonian vector fields.** Suppose given a manifold $\mathcal{M}$ together with a closed $(m+1)$-form $\Omega$ defined on it. We shall call such a structure a *multisymplectic manifold.* (Strictly speaking there should be further restrictions on $\Omega$ of an algebraic nature for this term to apply. However, the constructions of this section work for an arbitrary closed $(m+1)$-form, while the next section is concerned only with canonical situations, so these restrictions serve no useful purpose in the present paper.)

A transformation $\phi: \mathcal{M} \to \mathcal{M}$ will be called *canonical* if it preserves the multisymplectic form $\Omega$, that is if $\phi^*\Omega = \Omega$. A vector field $X$ on $\mathcal{M}$ will be called a *locally Hamiltonian* vector field if $X$ is an infinitesimal symmetry of $\Omega$, that is if $\mathcal{L}_X\Omega = 0$, or equivalently if $i_X\Omega$ is closed. It is clear that a vector field $X$ is locally Hamiltonian if and only if its flow consists of (local) canonical transformations. We shall say that a vector field $X$ on $\mathcal{M}$ is *globally Hamiltonian*, or just *Hamiltonian*, if $i_X\Omega = df$, where $f$ is an $(m-1)$-form on $\mathcal{M}$; and we shall say that $f$ is a *Hamiltonian form* on $\mathcal{M}$.

The commutator of two locally Hamiltonian vector fields $X, Y$ is a Hamiltonian vector field, because $i_{[X,Y]}\Omega = d(i_X i_Y \Omega)$, and consequently the set of all Hamiltonian vector fields on $\mathcal{M}$ is an ideal in the Lie subalgebra of all locally Hamiltonian vector fields. The correspondence between Hamiltonian forms $f$ and Hamiltonian vector fields $X$ given by

$$i_X\Omega = df$$

is well defined modulo characteristic vector fields of $\Omega$, because if $X, Y$ are two vector fields such that $i_X\Omega = i_Y\Omega = df$ then clearly $i_{(X-Y)}\Omega = 0$. Furthermore, the characteristic vector fields of $\Omega$ form an ideal in the Lie algebra of Hamiltonian vector fields, since if $X$ is Hamiltonian and $Y$ is characteristic then $i_{[X,Y]}\Omega = d(i_X i_Y \Omega) = 0$. The Lie algebra of Hamiltonian vector fields modulo characteristic vector fields will be denoted by $\mathrm{Ham}(\mathcal{M}, \Omega)$.

Two Hamiltonian forms $f, g$ on $\mathcal{M}$, differing by a closed $(m-1)$-form $h$, correspond to the same class of Hamiltonian vector fields, denoted by $X_f$. In particular, addition of an exact $(m-1)$-form does not change the Hamiltonian vector field associated to a given Hamiltonian form. The set of Hamiltonian forms on $\mathcal{M}$ modulo exact $(m-1)$-forms will be denoted by $\mathcal{H}(\mathcal{M}, \Omega)$, and it is clear that the bracket of two Hamiltonian forms $f, g$ defined by

$$\{f, g\} = i_{X_f} i_{X_g} \Omega$$

induces a Lie algebra structure on $\mathcal{H}(\mathcal{M},\Omega)$, because a straightforward computation shows that

$$\{\{f,g\},h\} + \text{cyclic} = d(i_{X_f} i_{X_g} i_{X_h}\Omega).$$

The preceding remarks shows that we have the following short exact sequence of Lie algebras associated to any multisymplectic manifold $(\mathcal{M},\Omega)$

$$0 \to H^{(m-1)}(\mathcal{M}) \xrightarrow{i} \mathcal{H}(\mathcal{M},\Omega) \xrightarrow{X} \mathrm{Ham}(\mathcal{M},\Omega) \to 0.$$

The map $X$ is defined by $i_{X_f}\Omega = df$ for any $X_f \in X(f)$, and $X$ is actually a Lie algebra homomorphism because

$$d\{f,g\} = d(i_{X_f} i_{X_g}\Omega) = i_{[X_f,X_g]}\Omega.$$

The module $H^{(m-1)}(\mathcal{M})$ should be considered as an abelian Lie algebra.

**4.2. Multimomentum maps.** Let $G$ be a Lie group with Lie algebra $\mathcal{G}$, and let $\phi\colon G \times \mathcal{M} \to \mathcal{M}$ be an action of $G$ on the multisymplectic manifold $\mathcal{M}$ preserving the multisymplectic form $\Omega$. If $\xi_{\mathcal{M}}$ denotes the vector field on $\mathcal{M}$ associated with $\xi \in \mathcal{G}$ by the action, it is clear that $\mathcal{L}_{\xi_{\mathcal{M}}}\Omega = 0$. We will assume that $\phi$ is such that the vector field $\xi_{\mathcal{M}}$ is not only locally Hamiltonian but Hamiltonian, with Hamiltonian form $f_\xi$. The map sending $\xi \in \mathcal{G}$ to $\xi_{\mathcal{M}}$, which will be denoted by $\tilde{\phi}$, is a Lie algebra homomorphism. It defines a class in the second cohomology group of $\mathcal{G}$ with values in $H^{(m-1)}(\mathcal{M})$, as we now show.

$$\begin{tikzcd}
 & & & \mathcal{G} \arrow[dl, "f"'] \arrow[d, "\tilde{\phi}"] & \\
0 \arrow[r] & H^{(m-1)}(\mathcal{M}) \arrow[r, "i"] & \mathcal{H}(\mathcal{M},\Omega) \arrow[r, "X"] & \mathrm{Ham}(\mathcal{M},\Omega) \arrow[r] & 0
\end{tikzcd}$$

Any lifting $f\colon \mathcal{G} \to \mathcal{H}(\mathcal{M},\Omega)$, as shown in the diagram above, defines a 2-cocycle $c$ on $\mathcal{G}$ with values in $H^{(m-1)}(\mathcal{M})$ by means of

$$\{f_\xi, f_\eta\} - f_{[\xi,\eta]} = c(\xi,\eta).$$

Notice that $c(\xi,\eta)$ is a closed $(m-1)$-form; and because $f_\xi$ is defined modulo exact forms we can think of $c(\xi,\eta)$ as an element of $H^2(\mathcal{G}, H^{(m-1)}(\mathcal{M}))$. Conversely, for any class $[c] \in H^2(\mathcal{G}, H^{(m-1)}(\mathcal{M}))$ we have the central extension $\mathcal{G}_c = \mathcal{G} \odot H^{(m-1)}(\mathcal{M})$ with the Lie bracket

$$[(\xi,\alpha),(\eta,\beta)] = ([\xi,\eta], c(\xi,\beta)) \quad \forall \xi,\eta \in \mathcal{G} \quad \forall \alpha,\beta \in H^{m-1}(\mathcal{M})$$

and the lifting $f_c: \mathcal{G}_c \to \mathcal{H}(\mathcal{M},\Omega)$ defined by $f_c(\xi,\alpha) = f_\xi$ which makes the following diagram commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{(m-1)}(\mathcal{M}) & \longrightarrow & \mathcal{G}_c & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & \Vert & & {\scriptstyle f_c}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\phi}} & & \\
0 & \longrightarrow & H^{(m-1)}(\mathcal{M}) & \xrightarrow{i} & \mathcal{H}(\mathcal{M},\Omega) & \xrightarrow{X} & \mathrm{Ham}(\mathcal{M},\Omega) & \longrightarrow & 0.
\end{array}$$

The notion dual to the map $f: \mathcal{G} \to \mathcal{H}(\mathcal{M},\Omega)$ will be called a *covariant multimomentum map*, or simply the *multimomentum map* for the action of $G$ on $\mathcal{M}$. Thus the multimomentum map for the action of $G$ on $\mathcal{M}$ is the map $J: \mathcal{M} \to \mathcal{G}^* \otimes \bigwedge^{m-1}(\mathcal{M})$ such that

$$d\langle \xi, J\rangle = i_{\xi_\mathcal{M}}\Omega$$

for all $\xi \in \mathcal{G}$.

It is clear that for any $g \in G$

$$\begin{aligned} g^* d\langle \xi, J\rangle &= g^*(i_{\xi_\mathcal{M}}\Omega) = i_{(g_*^{-1}\xi_\mathcal{M})}(g^*\Omega) \\ &= i_{(\mathrm{ad}\, g^{-1}\xi)_\mathcal{M}}\Omega = d\langle \mathrm{ad}\, g^{-1}\xi, J\rangle. \end{aligned}$$

Thus for each $\xi \in \mathcal{G}$

$$d(g^*\langle \xi, J\rangle) = d\langle \xi, \mathrm{coad}\, g J\rangle$$

or

$$g^* J = \mathrm{coad}\, g J \qquad \text{modulo a closed } \mathcal{G}^*\text{-valued } (m-1)\text{-form},$$

or

$$g^* J = \mathrm{coad}\, g J + \theta(g) \qquad \text{with } \theta(g) \in \mathcal{G}^* \otimes Z^{m-1}(\mathcal{M}),$$

where $Z^{m-1}(\mathcal{M})$ denotes the set of closed $(m-1)$-forms on $\mathcal{M}$. The following computation shows that $\theta: G \to \mathcal{G}^* \otimes Z^{m-1}(\mathcal{M})$ defines a 1-cocycle on $G$ with values in the $G$-module $\mathcal{G}^* \otimes Z^{m-1}(\mathcal{M})$. Let $g_1, g_2$ be any elements in $G$; then

$$\begin{aligned} \theta(g_1 g_2) &= (g_1 g_2)^* J - \mathrm{coad}(g_1 g_2) J \\ &= g_2^* g_1^* J - \mathrm{coad}(g_1 g_2) J \\ &= g_2^*(\mathrm{coad}\, g_1 J + \theta(g_1)) - \mathrm{coad}(g_1 g_2) J \\ &= \mathrm{coad}\, g_1(g_2^* J) + g_2^*\theta(g_1) - \mathrm{coad}(g_1 g_2) J \\ &= \mathrm{coad}\, g_1 \theta(g_2) + g_2^*\theta(g_1) \end{aligned}$$

where $g^*$ acts on the form part of $J$ and $\mathrm{coad}\, g$ acts on $\mathcal{G}^*$. Obviously the tangent map $T\theta(e)$ to $\theta$ at the unit element $e$ of the group is a map $T\theta(e): \mathcal{G} \to \mathcal{G}^* \otimes Z^{m-1}(\mathcal{M})$ which can be identified with the 2-cocycle $c$ defined previously.

**4.3. Group actions and cohomology.** In the previous section we have described the 2-cocycle $c$ on $\mathcal{G}$ with values in $H^{(m-1)}(\mathcal{M})$. We shall now discuss some aspects of the cocycle $c$ in terms of higher order cocycles of the Lie algebra $\mathcal{G}$.

In order to understand better the significance of this cocycle we shall have to develop some cohomological machinery based in the double complex

$$\bigwedge^{**} = \bigoplus_{p,q\geqslant 0} \bigwedge^{p,q}, \quad \text{where} \quad \bigwedge^{p,q} = \bigwedge^{p}(\mathcal{M}) \otimes \bigwedge^{q}(\mathcal{G}).$$

Here $\bigwedge^p(\mathcal{M})$ denotes the module of ordinary p-forms on the manifold $\mathcal{M}$ and $\bigwedge^q(\mathcal{G})$ denotes the set of alternating multilinear maps on $\mathcal{G}$. We can arrange the modules $\bigwedge^{p,q}$ in a grid, with the index $p$ lying on the $x$-axis and the index $q$ on the $y$-axis. Clearly $\bigwedge^{0,q}$ is the module of functions with values in $\bigwedge^q(\mathcal{G})$ and $\bigwedge^{p,0}$ is the set of ordinary p-forms on $\mathcal{M}$. There are three coboundary operators acting on the double complex $\bigwedge^{**}$: two of them, $d$ and $\partial$, are independent of the action of the group $G$, while the third, $\delta$, depends on the action of $G$ on $\mathcal{M}$. The operator $d\colon \bigwedge^{p,q} \to \bigwedge^{p+1,q}$ is the ordinary exterior differential on the complex of differential forms on $\mathcal{M}$. On the other hand $\partial\colon \bigwedge^{p,q} \to \bigwedge^{p,q+1}$ is the Chevalley coboundary operator in the exterior algebra $\bigwedge^*(\mathcal{G})$ of the Lie algebra $\mathcal{G}$. Finally $\delta\colon \bigwedge^{p,q} \to \bigwedge^{p,q+1}$ is defined by the formula

$$\begin{aligned}\delta\alpha(\xi_1,\ldots,\xi_{q+1}) &= \sum_{i=1}^{q+1}(-1)^{i+1}\mathcal{L}_{\xi_i}\alpha(\xi_1,\ldots,\hat{\xi}_i,\ldots,\xi_{q+1})\\ &\quad+\sum_{i<j}(-1)^{i+j}\alpha([\xi_i,\xi_j],\xi_1,\ldots,\hat{\xi}_i,\ldots,\hat{\xi}_j,\ldots,\xi_{q+1})\end{aligned}$$

for $\alpha \in \bigwedge^{p,q}$, $\xi_1,\ldots,\xi_{q+1} \in \mathcal{G}$, where $\mathcal{L}_{\xi_i}$ means the Lie derivative with respect to the vector field in $\mathcal{M}$ associated to $\xi_i$. It is clear that $\partial$ and $\delta$ both commute with $d$. We shall say that $\alpha$ is a *$q$-cocycle* on $\mathcal{G}$ with values in $\bigwedge^p(\mathcal{M})$ if $\delta\alpha = 0$. It is easy to check the following formulas: for a 1-cocycle $\alpha$

$$\mathcal{L}_\xi\alpha(\zeta) - \mathcal{L}_\zeta\alpha(\xi) - \alpha([\xi,\zeta]) = 0;$$

while for a 2-cocycle $\omega$

$$\begin{aligned}&\mathcal{L}_\xi\omega(\zeta,\eta) - \mathcal{L}_\zeta\omega(\xi,\eta) + \mathcal{L}_\eta\omega(\xi,\zeta)\\ &\qquad - \omega([\xi,\zeta],\eta) \qquad + \omega([\xi,\eta],\zeta) - \omega([\zeta,\eta],\xi) = 0.\end{aligned}$$

Notice that the operator $\delta$ restricted to constant differential forms coincides with $\partial$.

Now let us assume that $(\mathcal{M},\Omega)$ is a multisymplectic manifold and that the Lie group $G$ acts preserving the multisymplectic form $\Omega$. We shall use an adapted version of the tic-tac-toe lemma [4] to construct a family of (local) cocycles on $\mathcal{G}$. Because $\Omega$ is a closed $(m+1)$-form there exists (locally) an $m$-form $\theta^0$ on $\mathcal{M}$ such that $d\theta^0 = \Omega$. Let us define (at least locally) the element $c^1 \in \bigwedge^{m,1}$ by $c^1 = \delta\theta^0$. Then $c^1$ consists of a family of closed $m$-forms on $\mathcal{M}$, because $dc^1 = d\delta\theta^0 = \delta d\theta^0 = \delta\Omega = 0$. Therefore there exists (locally) a family of $(m-1)$-forms $\theta^1 \in \bigwedge^{m-1,1}$ such that $d\theta^1 = c^1$. Let us define $c^2 \in \bigwedge^{m-1,2}$ by $c^2 = \delta\theta^1$. As in the previous discussion, $c^2$ is a family of closed $(m-1)$-forms, and locally there exists a family $\theta^2$ of $(m-2)$-forms such that $d\theta^2 = c^2$. Define $c^3 \in \Omega^{m-2,3}$ as $\delta\theta^2$. Continuing in the same way, after $(m-1)$ steps we will get a family $\theta^{m-1}$ of 1-forms; $c^m = \delta\theta^{m-1}$ will be a family of closed 1-forms, $\theta^m \in \bigwedge^{0,m}$ will satisfy $d\theta^m = c^m$ and eventually we will have found a family of functions $\theta^m$ locally defined on $\mathcal{M}$ with values in $\bigwedge^m(\mathcal{G})$ such that $d\theta^m = c^m$. Defining $c^{m+1} \in \Omega^{0,m+1}$ by $c^{m+1} = \delta\theta^{m+1}$, we find that, as before, $dc^{m+1} = 0$, which means that the $c^{m+1}$ are in fact constants. Furthermore $\delta c^{m+1} = 0$; but $\delta$ coincides with $\partial$ on constant forms. As a consequence of all this we find that $c^{m+1}$ defines an ordinary Chevalley $(m+1)$-cocycle on $\mathcal{G}$.

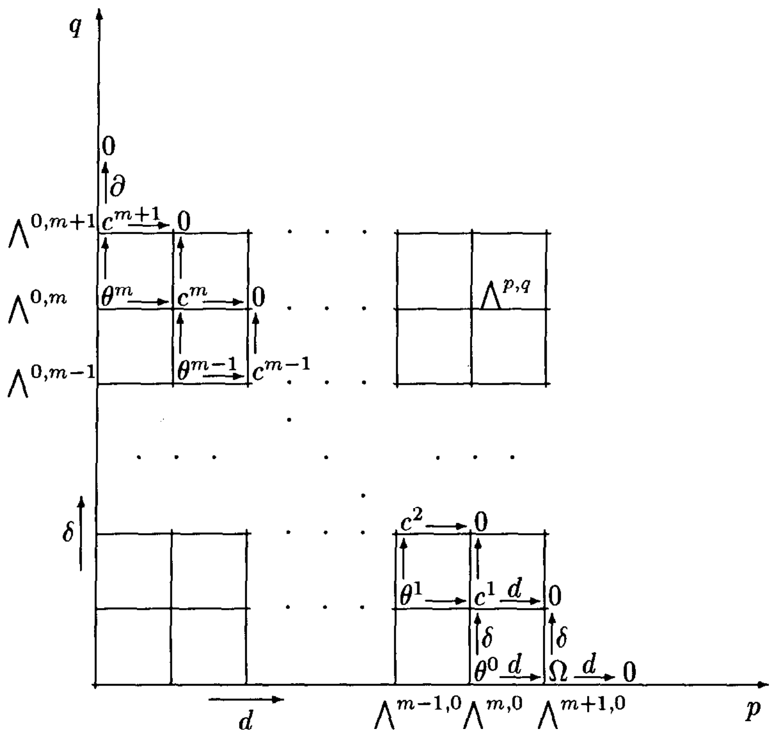


Diagram: The double complex $\bigwedge^{*,*}$

In order to validate this construction we have to study what happens if we choose a different set of $\theta$'s. So let $\tilde{\theta}^0$ be another locally defined $m$-form such that $d\tilde{\theta}^0 = \Omega$. Obviously, there is a closed $m$-form $\beta^0$ such that $\tilde{\theta}^0 = \theta^0 + \beta^0$. Because $\beta^0$ is closed, there is (shrinking the domains if necessary) an $(m-1)$-form $\alpha^0$ such that $d\alpha^0 = \beta^0$. Then

$$\tilde{c}^1 = \delta\tilde{\theta}^0 = \delta\theta^0 + \delta\beta^0 = c^1 + d(\delta\alpha^0)$$

and the cohomology class of $c^1$ does not change. Because $\tilde{c}^1$ is a family of closed $m$-forms there is locally a family of $(m-1)$-forms $\tilde{\theta}^1$ such that $d\tilde{\theta}^1 = \tilde{c}^1$. From the identity above we deduce that

$$\tilde{\theta}^1 = \theta^1 + \delta\alpha^0 + \beta^1$$

where $\beta^1$ is a family of closed $(m-1)$-forms. Repeating the process, we choose a family of $(m-2)$-forms $\alpha^1$ such that $d\alpha^1 = \beta^1$. We find that

$$\tilde{c}^2 = \delta\tilde{\theta}^1 = \delta\theta^1 + \delta\delta\alpha^0 + \delta\beta^1 = c^2 + d(\delta\alpha^1),$$

and so the cohomology class of $c^2$ is unchanged. Continuing this process, we show eventually that $\tilde{c}^{m+1} = c^{m+1} + \delta\beta^m$, where $\beta^m$ is a family of closed 0-forms, that is, constant functions on $\mathcal{M}$. As a consequence, the Chevalley cohomology class of $\tilde{c}^{m+1}$ is the same as the class of $c^{m+1}$.

We shall now consider what happens to the family of cocycles $\{c^1, \ldots, c^{m+1}\}$ when we consider a different action of the group $G$ on the multisymplectic manifold $\mathcal{M}$. Let us suppose that we have two different actions of $G$ on $\mathcal{M}$ preserving the multisymplectic form $\Omega$, and let us denote by $g, \tilde{g}$ the canonical maps corresponding to the same element of the group with respect to the two different multisymplectic actions. We shall say that the two actions are *equivalent* if there exists a canonical diffeomorphism $\psi: \mathcal{M} \to \mathcal{M}$ such that $\psi \circ \tilde{g} = g \circ \psi \;\; \forall g \in G$. In these circumstances it is obvious that $\psi$ induces an equivariant isomorphism between the $G$-modules of $k$-forms on $\mathcal{M}$ corresponding to both actions, and it induces an isomorphism between the cohomology groups of $\mathcal{G}$ with values in the $G$-modules of $k$-forms. In particular this isomorphism becomes the identity on the subgroup $H^{m+1}(\mathcal{G})$ of the group $H^{m+1}(\mathcal{G}, \bigwedge^0(\mathcal{M}))$ leaving the class of $c^{m+1}$ invariant.

We can summarize the above discussion as follows: if $G$ is a Lie group acting in the multisymplectic manifold $\mathcal{M}$ preserving the multisymplectic $m$-form $\Omega$ then there is a correspondence between equivalence classes of multisymplectic group actions and families of classes $\{[c^1], \ldots, [c^{m+1}]\}$ of cocycles of $\mathcal{G}$ with values in the cohomology groups $H^k(\mathcal{M})$ of $\mathcal{M}$; in particular this correspondence assigns a class in $H^{m+1}(\mathcal{G})$ to any multisymplectic group action, namely the $(m+1)$-cohomology class $c^{m+1}$ on $H^{m+1}(\mathcal{G})$.

It is important to note that if any of the families $\theta^k$ involved in the process of defining $c^{m+1}$ vanishes then automatically $c^{m+1} = 0$. In particular, if $\Omega$ is exact, so that $\Omega = d\Theta$, and $\Theta$ is $G$-invariant, then $c^{m+1}$ vanishes. We shall find this situation in the multisymplectic description of Lagrangian field theory, where $\Omega_L = d\Theta_L$. The cohomological properties of the action of any group of covariant canonical transformations preserving $\Omega_L$ will depend exclusively of the invariance properties of the $m$-form $\Theta_L$. This discussion will be continued in the sections to follow.

## 5. Symmetries and gauge transformations

**5.1. Symmetries.** This part of the paper is devoted to a discussion of the notions of symmetry, gauge transformation and Noether's theorem in the setting of first order field theories, using the geometrical background developed in Section 3 (the multisymplectic structure of Lagrangian first order field theories) and the general theory of symmetry for group actions on multisymplectic manifolds described above.

The first thing to do is to specialize the general discussion about group actions on multisymplectic manifolds to the setting of Lagrangian and Hamiltonian field theories. In what follows we shall be dealing with the canonical multisymplectic manifold $(\mathcal{M}\pi, \Omega)$ arising from a bundle $E \xrightarrow{\pi} M$ and the multisymplectic manifolds $(J_1\pi, \Omega_L)$ and $(J_1\pi^*, \Omega_h)$ that we get by selecting a Lagrangian $L$ on $J_1\pi$ or a section $h$ of the bundle $\mathcal{M}\pi \to J_1\pi^*$. We shall call the first of these a *Lagrangian system* and the second a *covariant Hamiltonian system* or simply a *Hamiltonian system.*

A *covariant canonical transformation* for a Lagrangian or a Hamiltonian system will be a canonical transformation, fibred over $M$, of $J_1\pi$ or $J_1\pi^*$ respectively. A *covariant Hamiltonian vector field* $X$ will be a projectable vector field on $J_1\pi$ or $J_1\pi^*$ whose flow consists of covariant canonical transformations: or in other words a vector field $X$, projectable onto $M$, such that the Lie derivative with respect to $X$ of $\Omega_L$ or $\Omega_h$ is zero.

A *symmetry* of the Hamiltonian system $(J_1\pi^*, \Omega_h)$ is a covariant canonical transformation $\Psi$ on $\mathcal{M}\pi$ such that $\Psi$ preserves the section $h$. Notice that $\Psi$ is fibred over $J_1\pi^*$ because it is fibred over $M$, and consequently $\Psi$ induces a transformation $\tilde{\Psi}$ on $J_1\pi^*$ such that $\Psi \circ h = h \circ \tilde{\Psi}$. If $\Psi$ is a symmetry of $(J_1\pi^*, \Omega_h)$ then $\tilde{\Psi}^*\Omega_h = \Omega_h$ and $\tilde{\Psi}$ is a canonical covariant transformation of the multisymplectic manifold $(J_1\pi^*, \Omega_h)$.

Let $\phi$ be a solution of the Hamilton equations for the Hamiltonian system $(J_1\pi^*, \Omega_h)$, so that $\phi^*(i_Y\Omega_h) = 0$ for every vector field $Y$ on $J_1\pi^*$. If $\Psi$ is a symmetry then

$$\begin{aligned}(\tilde{\Psi} \circ \phi)^*(i_Y\Omega_h) &= \phi^*\tilde{\Psi}^*(i_Y\Omega_h)\\ &= \phi^*(i_{\tilde{\Psi}^{-1}{}_*Y}\tilde{\Psi}^*\Omega_h)\\ &= \phi^*(i_{\tilde{\Psi}^{-1}{}_*Y}\Omega_h) = 0;\end{aligned}$$

and $\tilde{\Psi} \circ \phi$ will again be a solution of the Hamilton equations.

We shall define a *symmetry* of a Lagrangian system $(J_1\pi, L)$ in an analogous way as a canonical covariant transformation $\Phi$ of the multisymplectic manifold $(J_1\pi, \Omega_L)$. In general, if $\sigma$ is a solution of the Euler-Lagrange equations, so that $j_1\sigma^*(i_Y\Omega_L) = 0$ for every vector field $Y$ on $J_1\pi$, then clearly $\Phi \circ j_1\sigma$ is again a solution of the Euler-Lagrange equations in the sense that $(\Phi \circ j_1\sigma)^*(i_Y\Omega_L) = 0$ for every $Y$; however, $\Phi \circ j_1\sigma$ is not necessarily a jet extension again. But if $\Phi$ is the first jet prolongation of an automorphism $\phi$ of the bundle $E \to M$, that is, if $\Phi = j_1\phi$, then clearly the transformed solution is again a jet extension of a section of $E$, because $\Phi \circ j_1\sigma = j_1(\phi \circ \sigma)$.

**5.2. Current algebra.** We shall discuss next how a group of symmetries of a Hamiltonian system produces a family of conservation laws satisfying a precise current algebra. Let $G$ be a group of symmetries of the Hamiltonian system $(J_1\pi^*, \Omega_h)$. For any solution $\phi$ of the Hamilton equations and any element $\xi \in \mathcal{G}$ we have

$$0 = \phi^*(i_{\xi_{J_1\pi^*}}\Omega_h) = \phi^*(h^* i_{\xi_{\mathcal{M}\pi}}\Omega) = \phi^*h^*df_\xi = d((h \circ \phi)^* f_\xi).$$

Denoting by $f_\xi[\phi]$ the $(m-1)$-form $(h \circ \phi)^* f_\xi$ on $M$, we have a map $\mathcal{G} \to H^{m-1}(M)$ associating to any $\xi \in \mathcal{G}$ the class of $f_\xi[\phi]$. For any solution $\phi$ the bracket $\{\cdot,\cdot\}$ in $\mathcal{H}(\mathcal{M},\Omega)$ induces a bracket in the currents $f_\xi[\phi]$ in $H^{m-1}(M)$ associated to the elements of the Lie algebra of $G$ by

$$\{f[\phi], g[\phi]\} = (h \circ \phi)^* \{f, g\}.$$

The relation between the bracket $\{\cdot,\cdot\}$ in the set of currents and the Lie algebra bracket $[\cdot,\cdot]$ is obtained by a simple computation: on pulling back by $\phi \circ h$ the commutation relations for $f_\xi$ and $f_\eta$ we have the following commutation rule for the two conservation laws $f_\xi[\phi]$ and $f_\eta[\phi]$

$$\{f_\xi[\phi], f_\eta[\phi]\} = f_{[\xi,\eta]}[\phi] + c^2[\phi](\xi,\eta).$$

Choosing $\xi, \eta$ from the elements of a basis $\{\xi_i\}$ of $\mathcal{G}$, we get

$$\{f_i[\phi], f_j[\phi]\} = c_{ij}^k f_k[\phi] + c_{ij}^2[\phi]$$

where $f_i[\phi] = f_{\xi_i}[\phi]$, $c_{ij}^k$ are the structure constants of $\mathcal{G}$ and $c_{ij}^2[\phi]$ are the components of $c^2[\phi]$ in the basis $\{\xi_i\}$. The term $c_{ij}^2[\phi]$ in the commutation relations above is called the *anomalous term* or also the *Schwinger term* of the theory and as we discussed before is part of a sequence of cocycles obtained from the action of the group of symmetries $G$ on the multisymplectic manifold $\mathcal{M}\pi$.

The above discussion can be rephrased saying that for any infinitesimal generator $\xi$ of a group $G$ of symmetries of the Hamiltonian system $(J_1\pi^*, \Omega_h)$ and any solution $\phi$ of the covariant Hamilton equations there exists a conservation law given by the class of $(m-1)$-forms which contains $f_\xi[\phi]$. The set of conservation laws has an induced Lie algebra structure which is an extension of the Lie algebra $\mathcal{G}$ by $H^{m-1}(M)$ with 2-cocycle $c^2[\phi]$, $c^2[\phi]$ being the pull-back $(h \circ \phi)^* c^2$ of the 2-cocycle $c^2$ defined by the action of the Lie group $G$ on $\mathcal{M}\pi$.

**5.3. Noether's theorem.** Noether's theorem may be stated precisely using the momentum map associated with a group of symmetries of a Lagrangian or Hamiltonian system. As we showed in the last subsection, for any solution $\phi$ of the Hamilton equations we have a map from $\mathcal{G}$ into $H^{m-1}(M)$ sending any infinitesimal generator of the group $\xi$ to a conservation law $f_\xi[\phi]$. In fact we can pull-back the whole momentum map $J\colon \mathcal{M}\pi \to \bigwedge^{m-1} \mathcal{M}\pi \otimes \mathcal{G}^*$ by the section $h \circ \phi$ of $\mathcal{M}\pi \to M$, and on repeating the same argument as before we find that

$$d(h \circ \phi)^* J) = 0.$$

The element $(h \circ \phi)^* J \in H^{m-1}(M) \otimes \mathcal{G}^*$ will be denoted by $J[\phi]$; Noether's theorem simply states that $J[\phi]$ is a $\mathcal{G}^*$-valued conserved current for any solution $\phi$ of Hamilton's equations.

The same statement holds when we consider Lagrangian systems $(J_1\pi, \Omega_L)$. We can compute the form of the conservation laws by noticing that, because $\mathcal{L}_\xi \Omega_L = 0$, we have

$$\mathcal{L}_\xi \Theta_L = \beta(\xi)$$

with $\beta(\xi)$ a closed $m$-form. (We shall shortly identify $\beta$ as the 1-cocycle associated with the action of $G$ on $J_1\pi$). This implies that

$$i_{\xi_{J_1\pi}} \Omega_L + d(i_{\xi_{J_1\pi}} \Theta_L) = \beta(\xi);$$

assuming that $\beta(\xi)$ is exact, so that $\beta(\xi) = dh_\xi$ where $h_\xi$ is an $(m-1)$-form on $J_1\pi$, we find that the class of the Hamiltonian form associated to $\xi$ is given by

$$f_\xi = -\langle \xi_{J_1\pi}, \Theta_L \rangle + h_\xi.$$

In general we cannot be more specific about the form of the conserved currents $f_\xi[j_1\sigma]$ for a solution $\sigma$ of the Euler-Lagrange equations, but if the group of symmetries is in fact acting on $J_1\pi$ by prolongation of an action of $G$ on $E \to M$ by bundle automorphisms, we can compute the form of $J[j_1\sigma]$ explicitly in terms of the Lagrangian $L$. As we shall show below, for such a symmetry group $\beta(\xi)$ is actually a gauge form and $h_\xi$ is a $(m-1)$-form on $E$. Moreover, for any $\alpha \in J_1\pi$

$$\begin{aligned}
\langle \xi_{J_1\pi}, \Theta_L \rangle(\alpha) &= \langle \xi_{J_1\pi}, \widehat{\mathcal{F}L}^* \Theta \rangle(\alpha) \\
&= \langle \widehat{\mathcal{F}L}_* \xi_{J_1\pi}, \Theta \rangle (\widehat{\mathcal{F}L}(\alpha) \\
&= \langle \xi_{J_1\pi^*}, \Theta \rangle (\widehat{\mathcal{F}L}(\alpha)) \\
&= \langle \xi_E, \widehat{\mathcal{F}L}(\alpha) \rangle.
\end{aligned}$$

This means that

$$f_\xi(\alpha) = \langle \xi_E, \widehat{\mathcal{F}L}(\alpha) \rangle + h_\xi(\alpha).$$

**5.4. Gauge transformations.** We have defined gauge forms as closed affine semi-basic $m$-forms on $J_1\pi$. Notice that if $\lambda = L\pi_1{}^*\mathrm{vol}$ is a gauge form then $\Omega_\lambda$, the Cartan $(m+1)$-form defined by $\lambda$, vanishes identically. (It will be convenient in this subsection to label the Cartan form by the Lagrangian density rather than the function.) This is because when $L$ is affine $\widehat{\mathcal{F}L} = L$. Moreover, $\widehat{\mathcal{F}L}^* \Theta = \tilde{\lambda}^* \Theta$, and thus $\Omega_\lambda = d\Theta_\lambda = d\lambda$, which vanishes because $\lambda$ is closed. Conversely, if $\lambda = L\pi_1{}^*\mathrm{vol}$ is a density for which $\Omega_\lambda = 0$ $L$ must be affine, since

$$i_{\partial/\partial y_i^\alpha} \Omega_\lambda = -\frac{\partial^2 L}{\partial y_i^\alpha \partial y_j^\beta}(dy^\beta - y_k^\beta dx^k) \wedge d^{m-1}x_j,$$

as follows from the calculation in Section 3.5; and so

$$\frac{\partial^2 L}{\partial y_i^\alpha \partial y_j^\beta} = 0.$$

Thus $\Omega_\lambda = d\lambda = 0$ as before, and $\lambda$ must be a gauge form. Thus the kernel of the Cartan operator $\lambda \mapsto \Omega_\lambda$ is just the space of gauge forms.

Incidentally, to say that the Cartan $(m+1)$-form of $\lambda$ vanishes is not the same as to say that the Euler-Lagrange equations of $\lambda$ are null. This is because there are non-vanishing Cartan forms for which $(j^1\phi)^*\Omega_\lambda = 0$ for every 1-jet of a section of $\pi$. This matter has been discussed by Hojman [14].

A transformation of $J_1\pi$ which changes the Lagrangian density $\lambda$ by a gauge form $\gamma$ will be called a *gauge transformation.* Thus a gauge transformation $\Psi$ is such that

$$\Psi^*\lambda = \lambda + \gamma.$$

Traditionally, for the discussion in the preceding subsection, gauge transformations have been identified with symmetries of the Lagrangian system, but it is not always the case that a gauge transformation is a symmetry. The reason for this apparent paradox lies in the fact that the construction of the Cartan form uses the geometrical structure of $J_1\pi$ through the vertical endomorphism $S$. It is clear that if $\Psi$ is a gauge transformation of the Lagrangian density $\lambda$, so that $\Psi^*\lambda = \lambda + \gamma$, then

$$\Omega_{\Psi^*\lambda} = \Omega_\lambda + \Omega_\gamma = \Omega_\lambda.$$

But in general it is not true that $\Psi^*\Omega_\lambda = \Omega_{\Psi^*\lambda}$. In fact

$$\Psi^*\Theta_\lambda = \Psi^*(\lambda + S(d\lambda)) = \lambda + \gamma + \Psi^*(S(d\lambda))$$

which in general does not differ from $\Theta_{\Psi^*\lambda}$ by a closed form.

There is a distinguished subset of gauge transformations, those that preserve the vertical endomorphism $S$, that is, those transformations $\Psi$ such that $\Psi_*S = S\Psi_*$. They will be called in what follows natural gauge transformations. For a natural gauge transformation $\Psi$, we have (as a consequence of the computation above)

$$\begin{aligned}\Psi^*\Theta_\lambda &= \lambda + \gamma + \Psi^*(S(d\lambda)) \\ &= \lambda + S\Psi^*(d\lambda) + \gamma + S\Psi^*(d\gamma) \\ &= \Theta_\lambda + \Theta_\gamma\end{aligned}$$

and it immediately follows from this computation that for natural gauge transformations $\Psi^*\Omega_\lambda = \Omega_\lambda$. If $\sigma$ is a solution of the Euler-Lagrange equations, clearly

$$\begin{aligned}(\Psi \circ j_1\sigma)^*(i_X\Omega_\lambda) &= (j_1\sigma)^*\Psi^*(i_X\Omega_\lambda) \\ &= (j_1\sigma)^*(i_{X'}\Psi^*\Omega_\lambda) \\ &= (j_1\sigma)^*(i_{X'}\Omega_\lambda) = 0,\end{aligned}$$

where $X' = {\Psi_*}^{-1}X$, and consequently $\Psi$ is a symmetry of the Lagrangian system. Thus we have proved that natural gauge transformations of the Lagrangian density $\lambda$ are symmetries of the Lagrangian system $(J_1\pi, \Omega_\lambda)$.

Let $\psi: E \to E$ be a fibred diffeomorphism, fibred over the identity, and let $\Psi = j_1\psi$ be its prolongation to $J_1\pi$. Then for any section $\sigma$ of $E \xrightarrow{\pi} M$, $\psi \circ \sigma$ is a section of $E \xrightarrow{\pi} M$ again, and $\Psi \circ j_1\sigma = j_1(\psi \circ \sigma)$ is a 1-jet section of $J_1\pi \to M$. Moreover, for any vector field $X$ vertical over $E$ (over $M$), $\Psi_* X$ is also vertical over $E$ (over $M$). Then it is clear that $\Psi_* S = S \circ \Psi_*$, and bundle automorphisms of $E \to M$ are the best candidates for natural gauge transformations. For this reason gauge transformations that are prolongations to $J_1\pi$ of bundle automorphism are natural gauge transformations. In fact, it follows from the result of Section 3.4 that any natural gauge transformation of $J_1\pi$ which is fibred over the identity of $M$ and which lies in a one-parameter group must be of this form. Moreover, this is the situation encountered in most physical systems. The group of bundle automorphisms has itself been called the group of gauge transformations without any reference to any Lagrangian density, under the assumption that any reasonable and meaningful Lagrangian should possess this group as its group of natural gauge transformations.

**5.5. Cohomology and gauge transformations.** Let $G$ be a group of symmetries of the Lagrangian system $(J_1\pi, \Omega_\lambda)$. As we know from the general discussion about symmetries and cohomology, this group has an associated family of cocycles $\{c^1, c^2, \ldots, c^{m+1}\}$. The first element of this family, $c^1$, is obtained by looking at the infinitesimal variation of $\Theta_\lambda$ under the group $G$, that is

$$c^1(\xi) = \mathcal{L}_\xi \Theta_\lambda$$

where $\mathcal{L}_\xi$ represents the Lie derivative of $\Theta_\lambda$ along the vector field on $J_1\pi$ associated with $\xi \in \mathcal{G}$. Let $g_t = \exp t\xi$ be the one-parameter group of transformations defined by $\xi$; then

$$g_t^* \Theta_\lambda = \Theta_\lambda + \alpha_t$$

where $\alpha_t$ is a one-parameter family of closed $m$-forms on $J_1\pi$. Clearly, it holds $c^1(\xi) = \mathrm{d}/\mathrm{d}t|_{t=0}\, \alpha_t$. Notice that $\alpha_t$ need not to be a family of gauge forms.

On the other hand, if $G$ were at the same time a group of gauge transformations of $(J_1\pi, \Omega_\lambda)$, then

$$g_t^* \lambda = \lambda + \bar{\gamma}_t$$

where $\bar{\gamma}_t$ is a family of gauge forms on $J_1\pi$. We shall obtain a 1-cocycle on $\mathcal{G}$ with values in gauge forms by means of

$$\mathcal{L}_\xi \lambda = \bar{c}^1(\xi)$$

where $\bar{c}^1(\xi) = \mathrm{d}/\mathrm{d}t|_{t=0}\, \bar{\gamma}_t$; this is clearly a gauge form for all $\xi$. Notice that, as a consequence of the discussion in the previous subsection, the cocycles $c^1$ and $\bar{c}^1$ need not be the same unless $G$ is a group of natural gauge transformations. If this is the case, from the computations above

$$g_t^* \Theta_\lambda = \Theta_\lambda + \pi_1^{0*} \gamma_t$$

and we find that

$$c^1(\xi) = \pi_1^{0*}\bar{c}^1(\xi).$$

In this case $c^1(\xi)$ is a family of closed $m$-forms on $E$. We can find (at least locally) a family $\theta^1(\xi)$ of $(m-1)$-forms on $E$ such that $d\theta^1(\xi) = c^1(\xi)$ and then define $c^2(\xi,\eta) = \delta\theta^1(\xi,\eta)$. We have shown that $c^2$ is precisely the cocycle that appears in the commutation relations for the currents $f_\xi$ associated with the infinitesimal generators of the group $G$, and this discussion gives us the precise relation occurring between the invariance properties of Lagrangian densities and the anomalous terms appearing in the commutators of the corresponding conserved quantities. We can conclude that for groups of natural gauge transformations of the Lagrangian system $(J_1\pi, \Omega_\lambda)$ there is a unique family of cocycles $\{c^1, c^2, \ldots, c^{m+1}\}$ with values in closed forms in $E$ (notice that for general symmetries the cocycles take their values in closed forms on $J_1\pi$).

It is remarkable that anomalous terms (2-cocyles) in the commutation relations of the current algebra of the group of symmetries of the Lagrangian could occur even for natural gauge transformations, in contrast to the usual belief that anomalous terms arise because of the breaking of the gauge invariance of the theory. Of course the Lagrangian density has to be gauge covariant with respect to the group $G$ in order to provide the 1-cocycle $c^1$ starting the sequence $\{c^1, c^2, \ldots, c^{m+1}\}$. In spite of this the ordinary situation in physics for the occurrence of anomalous terms in the commutation relations for the currents happens for groups of transformations that are not groups of gauge transformations of the Lagrangian. We could apply in a very similar way the cohomology theory developed above, starting with the 1-cocycle $C^1(\xi) = \delta\Omega_\lambda(\xi)$, and proceeding in exactly the same way we would obtain a family of cocycles $\{C^1, C^2, \ldots, C^{m+2}\}$. Some of the implications of this situation have been analysed in [5].

## 6. Some examples

**6.1. Classical mechanics.** One of the supposed advantages of the formalism described in this paper is that it applies equally to classical analytical mechanics of particles and to field theories. We shall therefore briefly indicate precisely how classical mechanics fits into the picture.

The main point to recognise is that it is the time-dependent version of classical mechanics that is relevant. This corresponds to $m = 1$, the base manifold $M$ being R, representing the time axis; $E = Q \times \mathbb{R}$ is a trivial bundle over R, $Q$ being the configuration space of the system; a section of $E$ is just a curve in $Q$. Then $J_1\pi = TQ \times \mathbb{R}$, the vertical endomorphism is a vector-valued 1-form, namely the one described in [6], and the Cartan form is the usual one for time-dependent Lagrangian mechanics. Furthermore $\mathcal{M}\pi = T^*(Q \times \mathbb{R})$ and the multisymplectic form $\Omega$ is the canonical symplectic form on $T^*(Q \times \mathbb{R})$. Since the bundle $E$ is a product $\mathcal{M}$ is trivial over $J_1\pi^*$; the latter manifold can be identified with $T^*Q \times \mathbb{R}$, and a Hamiltonian section defines a Hamiltonian function in an invariant way. The Hamilton equations are equivalent to the specification of the characteristic line-element field of the 2-form $\omega - dH \wedge dt$, where $\omega$

is the standard symplectic 2-form on $T^*Q$ (regarded as a form on $T^*Q \times \mathbb{R}$). In fact if we regard the trivialisation of $E \to \mathbb{R}$ as defining a connection then the corresponding section of $\mathcal{M}\pi \to J_1\pi^*$ is just given by $p = 0$, and the form $d\Theta_\gamma$ on $J_1\pi^* = T^*Q \times \mathbb{R}$ obtained by pulling back the multisymplectic form $d\Theta$ is just $\omega$.

The short exact sequence of Lie algebras derived in Section 4.1 becomes in this case

$$0 \to H^0(Q) \xrightarrow{i} \mathcal{F}(T^*(Q \times \mathbb{R})) \xrightarrow{\sigma} \mathrm{Ham}(T^*(Q \times \mathbb{R}), \Omega) \to 0.$$

In ordinary classical mechanics the cohomology class we get by carrying out the construction described in Section 4.2 is precisely the symplectic 2-cocycle defined by the symplectic action of the group $G$; this can be seen directly from the formula $\{f_\xi, f_\eta\} - f_{[\xi,\eta]} = c^2(\xi, \eta)$. This symplectic cocycle is in fact well defined even if the action is not Hamiltonian (in the sense that there do not exist globally defined Hamiltonians for the elements of the Lie algebra of $G$).

**6.2. Scalar fields.** The geometrical structure for scalar field theories is, in an obvious sense, dual to that for particle mechanics: it is based on the opposite fibration of $Q \times \mathbb{R}$, namely $Q \times \mathbb{R} \to Q$. Then $J_1\pi = T^*Q \times \mathbb{R}$, $\mathcal{M}\pi = T(Q \times \mathbb{R})$, and $J_1\pi^* = TQ \times \mathbb{R}$. The Hamilton equations may be written

$$\frac{\partial y}{\partial x^i} = -\frac{\partial H}{\partial p^i} \qquad \text{and} \qquad \frac{\partial p^i}{\partial x^i} = \frac{\partial H}{\partial y}.$$

**6.3. The multisymplectic structure associated with a distribution.** We have concentrated on the multisymplectic structure associated with a fibre bundle $E \xrightarrow{\pi} M$. In fact the basic ideas behind this construction, namely the definition of $J_1\pi$ in terms of sections of the short exact sequence of tangent spaces arising from the fibration, and the identification of the affine duals of the fibres of $J_1\pi \to E$ with quotient spaces of covectors, work in a rather more general context.

Let $E$ be a differential manifold with a distribution $\mathcal{D}$, that is to say a choice of vector subspace $\mathcal{D}_y \subset T_yE$ at each point $y \in E$, of constant dimension over $E$ and varying smoothly in the appropriate sense; in other words, $\mathcal{D}$ is a vector subbundle of $TE$. We do not assume that $\mathcal{D}$ is integrable (in the sense of Frobenius). Then at each point $y \in E$ we have a short exact sequence of vector spaces

$$0 \to \mathcal{D}_y \hookrightarrow T_yE \to T_yE/\mathcal{D}_y \to 0.$$

Let $\Sigma_y$ be the space of splittings of this sequence: it is an affine space modelled on $\mathrm{Lin}(T_yE/\mathcal{D}_y, \mathcal{D}_y)$. The collection of all such splittings, $\bigcup_y \Sigma_y$, is the total space of an affine bundle over $E$ modelled on the vector bundle $\mathcal{D} \otimes TE/\mathcal{D}$; we call this affine bundle the *virtual* 1-*jet bundle* of the distribution $\mathcal{D}$ and denote it by $\pi_\mathcal{D}\colon J_1\mathcal{D} \to E$. The use of the term 'virtual' is supposed to indicate the similarity between $J_1\mathcal{D}$ and a genuine 1-jet bundle while recognising that there is no underlying 'space-time' manifold $M$ in this case; perhaps it is the space-time that should be thought of as virtual.

We may introduce coordinates on the fibres of $J_1\mathcal{D}$ as follows. Suppose that $\mathcal{D}$ has codimension $m$. Let $\{\theta^i\}$, $i = 1, 2, \ldots, m$, be a local basis for the characterising 1-forms for $\mathcal{D}$, so that $\langle \mathcal{D}, \theta^i \rangle = 0$. Extend $\{\theta^i\}$ to a local basis of 1-forms for $E$ by the addition of 1-forms $\{\theta^\alpha\}$, $\alpha = m+1, m+2, \ldots, \dim E$. Then any complement to $\mathcal{D}_y$ at any point $y \in E$ may be uniquely specified as the subspace of $T_y E$ simultaneously annihilated by 1-forms $\{\theta^\alpha + y_i^\alpha \theta^i\}$ for certain coefficients $y_i^\alpha$; and these coefficients may be used as fibre coordinates. Notice that this definition essentially proves the local triviality of $J_1 \mathcal{D}$, by relating it to the existence of local bases of 1-forms with appropriate properties.

The construction of analogues of $\mathcal{M}\pi$ and $J_1\pi^*$ proceeds in a straightforward way. In particular we may identify the space of affine functions on a fibre of $J_1\mathcal{D}$ with a space of covectors on $E$, and are thereby led to consider the subbundle $\mathcal{MD}$ of $\bigwedge^m E$ consisting of $m$-covectors $w$ at all points $y \in E$ which satisfy

$$i_{v_1} i_{v_2} w = 0 \text{ for all } v_1, v_2 \in \mathcal{D}_y.$$

Such covectors take the form

$$p\,\theta^1 \wedge \theta^2 \wedge \ldots \wedge \theta^m + \sum_{i=1}^{m} p_\alpha^i\, \theta^\alpha \wedge \theta^1 \wedge \ldots \widehat{\theta^i} \ldots \wedge \theta^m$$

when expressed in terms of the basis of 1-forms described above (where the overhat indicates omission of the corresponding 1-form); thus $(p, p_i^\alpha)$ will serve as fibre coordinates on $\mathcal{MD}$. Furthermore, $\mathcal{MD}$ is a multisymplectic manifold with multisymplectic form defined exactly as in the case of a 1-jet bundle.

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