

# **Stochastic models, estimation, and control**

**VOLUME 2**

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To Kristen and Keryn . . .  
the "other women" in my life

# Preface

As was true of Volume 1, the purpose of this book is twofold. First, it attempts to develop a thorough understanding of the fundamental concepts incorporated in stochastic processes, estimation, and control. Second, and of equal importance, it provides experience and insights into applying the theory to realistic practical problems. Basically, it investigates the theory and derives from it the tools required to reach the ultimate objective of systematically generating effective designs for estimators and stochastic controllers for operational implementation.

Perhaps most importantly, the entire text follows the basic principles of Volume 1 and concentrates on presenting material in the most lucid, best motivated, and most easily grasped manner. It is oriented toward an engineer or an engineering student, and it is intended both to be a textbook from which a reader can *learn* about estimation and stochastic control and to provide a good reference source for those who are deeply immersed in these areas. As a result, considerable effort is expended to provide graphical representations, physical interpretations and justifications, geometrical insights, and practical implications of important concepts, as well as precise and mathematically rigorous development of ideas. With an eye to practicality and eventual implementation of algorithms in a digital computer, emphasis is maintained on the case of continuous-time dynamic systems with sampled-data measurements available; nevertheless, corresponding results for discrete-time dynamics or for continuous-time measurements are also presented. These algorithms are developed in detail, to the point where the various design trade-offs and performance evaluations involved in achieving an efficient, practical configuration can be understood. Many examples and problems are used throughout the text to aid comprehension of important concepts. Furthermore, there is an extensive set of references in each chapter to allow pursuit of ideas in the open literature once an understanding of both theoretical concepts and practical implementation issues has been established through the text.

This volume builds upon the foundations set in Volume 1. The seven chapters of that volume yielded linear stochastic system models driven by white Gaussian noises and the optimal Kalman filter based upon models of that form. In this volume, Chapters 8–10 extend these ideas to consider optimal smoothing in addition to filtering, compensation of linear model inadequacies while exploiting the basic insights of linear filtering (including an initial study of the important extended Kalman filter algorithm), and adaptive estimation based upon linear models in which uncertain parameters are embedded. Subsequently, Chapter 11 properly develops nonlinear stochastic system models, which then form the basis for the design of practical nonlinear estimation algorithms in Chapter 12.

This book forms a self-contained set with Volume 1, and together with Volume 3 on stochastic control, can provide a fundamental source for studying stochastic models, estimation, and control. In fact, they are an outgrowth of a three-quarter sequence of graduate courses taught at the Air Force Institute of Technology; and thus the text and problems have received thorough class testing. Students had previously taken a basic course in applied probability theory, and many had also taken a first control theory course, linear algebra, and linear system theory; but the required aspects of these disciplines have also been developed in Volume 1. The reader is assumed to have been exposed to advanced calculus, differential equations, and some vector and matrix analysis on an engineering level. Any more advanced mathematical concepts are developed within the text itself, requiring only a willingness on the part of the reader to deal with new means of conceiving a problem and its solution. Although the mathematics becomes relatively sophisticated at times, efforts are made to motivate the need for, and to stress the underlying basis of, this sophistication.

The author wishes to express his gratitude to the many students who have contributed significantly to the writing of this book through their feedback to me—in the form of suggestions, questions, encouragement, and their own personal growth. I regard it as one of God's many blessings that I have had the privilege to interact with these individuals and to contribute to their growth. The stimulation of technical discussions and association with Professors Michael Athans, John Deyst, Nils Sandell, Wallace Vander Velde, William Widnall, and Alan Willsky of the Massachusetts Institute of Technology, Professor David Kleinman of the University of Connecticut, and Professors Jurgen Gobien, James Negro, J. B. Peterson, and Stanley Robinson of the Air Force Institute of Technology has also had a profound effect on this work. I deeply appreciate the continued support provided by Dr. Robert Fontana, Chairman of the Department of Electrical Engineering at AFIT, and the painstaking care with which many of my associates have reviewed the manuscript. Finally, I wish to thank my wife, Beverly, and my children, Kristen and Keryn, without whose constant love and support this effort could not have been fruitful.

# Notation

## Vectors, Matrices

*Scalars* are denoted by upper or lower case letters in italic type.

*Vectors* are denoted by lower case letters in boldface type, as the vector  $\mathbf{x}$  made up of components  $x_i$ .

*Matrices* are denoted by upper case letters in boldface type, as the matrix  $\mathbf{A}$  made up of elements  $A_{ij}$  (ith row, jth column).

## Random Vectors (Stochastic Processes), Realizations (Samples), and Dummy Variables

*Random vectors* are set in boldface sans serif type, as  $\mathbf{x}(\cdot)$  or frequently just as  $\mathbf{x}$  made up of scalar components  $x_i$ ;  $\mathbf{x}(\cdot)$  is a mapping from the sample space  $\Omega$  into real Euclidean  $n$ -space  $R^n$ : for each  $\omega_k \in \Omega$ ,  $\mathbf{x}(\omega_k) \in R^n$ .

*Realizations* of the random vector are set in boldface roman type, as  $\mathbf{x}$ :  $\mathbf{x}(\omega_k) = \mathbf{x}$ .

*Dummy variables* (for arguments of density or distribution functions, integrations, etc.) are denoted by the equivalent Greek letter, such as  $\xi$  being associated with  $\mathbf{x}$ : e.g., the density function  $f_{\mathbf{x}}(\xi)$ . The correspondences are  $(\mathbf{x}, \xi)$ ,  $(\mathbf{y}, p)$ ,  $(\mathbf{z}, \zeta)$ ,  $(\mathbf{Z}, \mathcal{Z})$ .

*Stochastic processes* are set in boldface sans serif type, just as random vectors are. The  $n$ -vector stochastic process  $\mathbf{x}(\cdot, \cdot)$  is a mapping from the product space  $T \times \Omega$  into  $R^n$ , where  $T$  is some time set of interest: for each  $t_j \in T$  and  $\omega_k \in \Omega$ ,  $\mathbf{x}(t_j, \omega_k) \in R^n$ . Moreover, for each  $t_j \in T$ ,  $\mathbf{x}(t_j, \cdot)$  is a random vector, and for each  $\omega_k \in \Omega$ ,  $\mathbf{x}(\cdot, \omega_k)$  can be thought of as a particular time function and is called a *sample* out of the process. In analogy with random vector realizations, such samples are set in boldface roman type:  $\mathbf{x}(\cdot, \omega_k) = \mathbf{x}(\cdot)$  and  $\mathbf{x}(t_j, \omega_k) = \mathbf{x}(t_j)$ . Often the second argument of a stochastic process is suppressed:  $\mathbf{x}(t, \cdot)$  is often written as  $\mathbf{x}(t)$ , and this stochastic process evaluated at time  $t$  is to be distinguished from a process sample  $\mathbf{x}(t)$  at that same time.

*Subscripts*

a:	augmented
b:	backward running
c:	continuous-time
d:	discrete-time
f:	final time; or filter (shaping filter)

n:	nominal
ss:	steady state
t:	truth model
0:	initial time

*Superscripts*

<sup>T</sup> :	transpose (matrix)
<sup>*</sup> :	complex conjugate transpose; or transformed coordinates
<sup>-1</sup> :	inverse (matrix)

<sup>#</sup> :	pseudoinverse
<sup>^</sup> :	estimate
<sup>-</sup> :	Fourier transform; or steady state solution

*Matrix and Vector Relationships*

$\mathbf{A} > \mathbf{0}$ :  $\mathbf{A}$  is positive definite.

$\mathbf{A} \geq \mathbf{0}$ :  $\mathbf{A}$  is positive semidefinite.

$\mathbf{x} \leq \mathbf{a}$ : componentwise,  $x_1 \leq a_1, x_2 \leq a_2, \dots$ , and  $x_n \leq a_n$ .

*Commonly Used Abbreviations and Symbols*

$E\{\cdot\}$	expectation	w.p.1	with probability of one
$E\{\cdot \cdot\}$	conditional expectation	$ \cdot $	determinant of
$\exp(\cdot)$	exponential	$\ \cdot\ $	norm of
lim.	limit	$\sqrt{\cdot}$	matrix square root of
l.i.m.	limit in mean (square)	$\in$	(see Volume 1)
$\ln(\cdot)$	natural log	$\subset$	element of
m.s.	mean square	{·}	subset of
max.	maximum	$\{\cdot\}$	set of; such as
min.	minimum	$\{\mathbf{x} \in \mathbf{X} : \mathbf{x} \leq \mathbf{a}\}$	$\{\mathbf{x} \in \mathbf{X} : \mathbf{x} \leq \mathbf{a}\}$ , i.e., the set
$R^n$	Euclidean $n$ -space		of $\mathbf{x} \in \mathbf{X}$ such that
$\text{sgn}(\cdot)$	signum (sign of)		$x_i \leq a_i$ for all $i$
$\text{tr}(\cdot)$	trace		

*List of symbols and pages where they are defined or first used*

$\mathbf{A}$	14; 78; 137	$\mathbf{A}_{\text{TS}}$	220; 225
$\mathbf{A}_{\text{GS}}$	222; 226	$\mathbf{a}$	75

$\mathcal{A}_i$	97	$\mathbf{K}_{\text{GS}}$	222; 226
$\mathcal{A}_i^N$	98	$\mathbf{K}_{\text{SL}}$	244
$\mathbf{B}$	41; 160	$\mathbf{K}_{\text{TS}}$	220; 225
$\mathbf{B}_d$	2	$\mathbf{K}_{\text{ss}}$	57
$\hat{\mathbf{b}}_m(t_i^-)$	227	$L$	35, 75
$\hat{\mathbf{b}}_m(t)$	249	$\mathbf{m}_x$	162; 199
$\hat{\mathbf{b}}_m(t_i^-)$	224; 225	$m$	40
$\hat{\mathbf{b}}_p(t)$	249	$\mathbf{n}_{\text{GP}}$	164
$\hat{\mathbf{b}}_p(t/t_{i-1})$	224; 226	$\mathbf{n}_p$	163
$\mathbf{C}$	25	$n$	39
$c_i$	237	$\mathbf{P} = \mathbf{P}(t/t)$	58; 248; 249
$\mathbf{dl}$	181	$\mathbf{P}(t_i^-)$	5; 9; 45; 216
$\mathbf{dx}$	39; 161; 181	$\mathbf{P}(t_i^+)$	5; 9; 44; 216
$\mathbf{dp}$	39; 161	$\mathbf{P}(t/t_{i-1})$	44; 215
$\mathbf{d}\beta_m$	57; 163; 246	$\mathbf{P}(t_i/t_j)$	6; 10
$\mathbf{d}\psi$	186	$\mathbf{P}_N$	37
$E\{\cdot\}$	26; 39	$\mathbf{P}_b^{-1}(t_i^-)$	3; 6; 10
$E\{\cdot \cdot\}$	1	$\mathbf{P}_b^{-1}(t_i^+)$	3; 6; 9
$\mathbf{e}$	26; 98; 137	$\mathbf{P}_{xx}$	162
$\mathbf{F}$	20; 41; 45; 160	$\mathbf{P}_{xx}(t)$	162; 199
$F_x$	162	$\mathbf{P}_{xx}(t, t + \tau)$	162
$F_{x y}$	162; 167	$\mathbf{P}_0$	2; 9; 70
$f$	39	$P(A)$	163
$f_x$	161; 175	$\mathbf{p}(t_i)$	168
$f_x(\xi, t   \mathbf{p}, t')$	174	$p$	75
$f_{x,y}$	161	$p_k(t_i)$	130; 131
$f_{x y}$	35; 161; 174	$\mathbf{Q}$	39
$\mathcal{F}(t)$	244	$\mathbf{Q}_d$	2; 70
$\mathcal{F}$	176	$\mathbf{R}$	2; 40; 160
$\mathbf{G}$	39; 160	$\mathbf{R}_c$	58; 160
$\mathbf{G}[\mathbf{x}(t), t]$	166; 181	$\mathbf{R}_{i\text{-aged}}$	29
$\mathbf{G}_d$	2	$\mathbf{r}$	137
$\mathbf{G}_d[\mathbf{x}(t_i), t_i]$	238	$\mathbf{r}_i$	80
$\mathbf{H}$	2; 41; 44; 160	$r$	39
$H_i$	239	$r_{xy}$	28
$\mathbf{h}$	40	$\mathbf{s}^1$	82
$\mathcal{H}(t_i)$	244	$s$	44
$\mathbf{l}$	161; 175; 178; 181	$\mathbf{T}$	114; 169
$\mathbf{I}$	6	$T$	xiii; 161
$\mathbf{l}$	36	$t_f$	4
$\mathbf{J}$	81	$t_i$	1
$J$	243	$\mathbf{u}$	2; 39
$\mathbf{K}$	6; 44; 58; 59	$\mathbf{v}$	2; 40

$\mathbf{v}_c$	57; 246	$\mathbf{z}(t)$	57; 160
$\mathbf{W}$	8	$\mathbf{z}(t_i)$	1; 40; 160
$\mathbf{w}$	39; 160	$\mathbf{z}_i$	1
$\mathbf{w}_d$	2	$\mathbf{\hat{p}}$	39; 160
$\mathbf{x}$	1; 39; 160	$\mathbf{\hat{p}}_m$	57; 163; 246
$\hat{\mathbf{x}}(t)$	58; 248; 249	$\mathbf{\hat{p}}'$	176; 192
$\hat{\mathbf{x}}(t_i^-)$	1; 9; 45; 216	$\delta \mathbf{x}$	41
$\hat{\mathbf{x}}(t_i^+)$	1; 9; 44; 216	$\delta \mathbf{u}$	41
$\hat{\mathbf{x}}(t/t_{i-1})$	44; 215	$\delta \mathbf{z}$	41
$\hat{\mathbf{x}}(t_i/t_j)$	1; 10	$\varepsilon$	32
$\hat{\mathbf{x}}_{\text{MAP}}$	242	$\lambda$	163
$\hat{\mathbf{x}}_N$	35; 36	$v$	256
$\hat{\mathbf{x}}_b(t_i^-)$	2	$\Sigma$	242
$\hat{\mathbf{x}}_b(t_i^+)$	6	$\sigma$	55; 67
$\hat{\mathbf{x}}_k$	131	$\sigma^2$	28
$\hat{\mathbf{x}}_0$	9; 39; 70	$\Phi$	2; 161
$\hat{\mathbf{y}}_b(t_i^-)$	6; 10	$\phi$	238
$\hat{\mathbf{y}}_b(t_i^+)$	6; 9	$\phi_x$	197
$\mathbf{z}$	1; 213	$\bar{\psi}_{xx}(\omega)$	63
$\mathbf{z}(t_i, t_j)$	2	$\psi$	186
$\mathbf{z}_N$	80	$\Omega$	xiii
$\mathbf{z}_i$	1	$\omega$	1; 63

# CHAPTER 8

## Optimal smoothing

### 8.1 INTRODUCTION

In the previous chapters, we have considered linear system models and optimal filtering, the optimal estimation of the state at time  $t_i$ ,  $\mathbf{x}(t_i, \omega_j) = \mathbf{x}(t_i)$ , based upon knowledge of all measurements taken up to time  $t_i$ :

$$\mathbf{z}(t_1, \omega_j) = \mathbf{z}_1, \quad \mathbf{z}(t_2, \omega_j) = \mathbf{z}_2, \quad \dots, \quad \mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$$

or equivalently,  $\mathbf{Z}(t_i, \omega_j) = \mathbf{Z}_i$ . We have actually considered optimal prediction as well in attempting to estimate  $\mathbf{x}(t_i)$  based on knowledge of  $\mathbf{Z}(t_{i-1}, \omega_j) = \mathbf{Z}_{i-1}$ . Under our assumptions, the optimal estimate of  $\mathbf{x}(t_i)$ , based on knowledge of available measurement information, has been the conditional expectation of  $\mathbf{x}(t_i)$ , conditioned on that information:

$$\hat{\mathbf{x}}(t_i^+) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_i, \omega_j) = \mathbf{Z}_i\} \quad (8-1)$$

$$\hat{\mathbf{x}}(t_i^-) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_{i-1}, \omega_j) = \mathbf{Z}_{i-1}\} \quad (8-2)$$

In fact, these values were shown to be optimal with respect to many different criteria.

The Kalman filter, or square root implementation of the same estimator, provides the best estimate of  $\mathbf{x}(t_i)$  based on all measurements through time  $t_i$  in a recursive manner, and it is thus ideally suited to real-time computations. However, if one were willing (or able) to wait until after time  $t_i$  to generate an optimal estimate of  $\mathbf{x}(t_i)$ , then a *better* estimate than the  $\hat{\mathbf{x}}(t_i^+)$  provided by the Kalman filter could be produced in most cases [6, 7, 9, 23, 28]. The additional information contained in the measurements taken after time  $t_i$  can be exploited to provide this improvement in estimation accuracy. The *optimal smoothed estimate* [36] (again under many criteria) is

$$\hat{\mathbf{x}}(t_i/t_j) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_j, \omega_k) = \mathbf{Z}_j\}, \quad j > i \quad (8-3)$$

and the subject of optimal smoothing is concerned with developing efficient, practical algorithms for calculating this estimate.

Section 8.2 formulates the smoothing problem and presents a conceptual approach to smoothing of combining the outputs of a filter running forward from initial time  $t_0$  to the current time  $t_i$ , and a separate filter running backward from terminal time  $t_f$  to  $t_i$ . Three useful classes of smoothing problems, characterized by the manner in which  $t_i$  and  $t_j$  can vary in (8-3), are presented in Section 8.3, and then discussed individually in the ensuing three sections.

## 8.2 BASIC STRUCTURE

Explicit equations for various forms of optimal smoothers are generally quite complicated. However, the basic smoothing concept and underlying structure can be discerned readily by dividing the estimation problem into two parts, one involving the past and present measurements and the other based on future measurements alone, and combining the results.

Consider a discrete-time model (possibly “equivalent discrete”):

$$\mathbf{x}(t_{i+1}) = \Phi(t_{i+1}, t_i)\mathbf{x}(t_i) + \mathbf{B}_d(t_i)\mathbf{u}(t_i) + \mathbf{G}_d(t_i)\mathbf{w}_d(t_i) \quad (8-4)$$

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \quad (8-5)$$

with the usual assumptions on  $\mathbf{x}(t_0)$ ,  $\mathbf{w}_d(\cdot, \cdot)$ , and  $\mathbf{v}(\cdot, \cdot)$ : Gaussian and independent of each other, initial conditions with mean  $\hat{\mathbf{x}}_0$  and covariance  $\mathbf{P}_0$ , white and zero-mean processes of strengths  $\mathbf{Q}_d(t_i)$  and  $\mathbf{R}(t_i)$ , respectively, for all times of interest. Now assume we are trying to estimate  $\mathbf{x}(t_i)$  from measurement data through time  $t_j$ , with  $j > i$ . Put all of the measurements up through time  $t_i$  into a single composite vector  $\mathbf{Z}(t_i)$ , or perhaps more explicitly  $\mathbf{Z}(t_1, t_i)$ , denoting the fact that its partitions are  $\mathbf{z}(t_1)$ ,  $\mathbf{z}(t_2)$ , ...,  $\mathbf{z}(t_i)$ . Similarly, put all “future” measurements,  $\mathbf{z}(t_{i+1})$ ,  $\mathbf{z}(t_{i+2})$ , ...,  $\mathbf{z}(t_j)$ , into a single composite vector  $\mathbf{Z}(t_{i+1}, t_j)$ . Conceptually, a three-part procedure can now be employed to estimate  $\mathbf{x}(t_i)$ :

(1) Calculate

$$\hat{\mathbf{x}}(t_i^+) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_1, t_i) = \mathbf{Z}_{1,i}\} \quad (8-6)$$

by means of a filter running forward in time from time  $t_0$  to time  $t_i$ . A priori information about  $\mathbf{x}(t_0)$  is used to initialize this filter.

(2) Independently, calculate

$$\hat{\mathbf{x}}_b(t_i^-) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_{i+1}, t_j) = \mathbf{Z}_{(i+1), j}\} \quad (8-7)$$

by means of a filter that is run backwards in time from time  $t_j$  to time  $t_{i+1}$ , plus a one-step “prediction” backward to time  $t_i$ . The notation  $\hat{\mathbf{x}}_b(t_i^-)$  is meant to denote the estimate of  $\mathbf{x}(t_i)$  provided by the backward-running filter (thus the subscript b), just before the measurement at time  $t_i$  is incorporated (thus the minus superscript on  $t_i^-$ ). Note that time  $t_i^-$  is to the right of  $t_i^+$  on

a real-time scale for the backward filter, as shown in Fig. 8.1, since minus and plus denote before and after measurement incorporation, respectively. The “initial” condition for the backward-running filter is established by viewing  $\mathbf{x}(t_j)$  as a random vector about which you have no a priori statistical information, i.e.,  $\mathbf{P}_b^{-1}(t_j^-) = \mathbf{0}$ . Thus, an inverse-covariance formulation is appropriate for the backward filter. (This will be developed further in Section 8.4.)

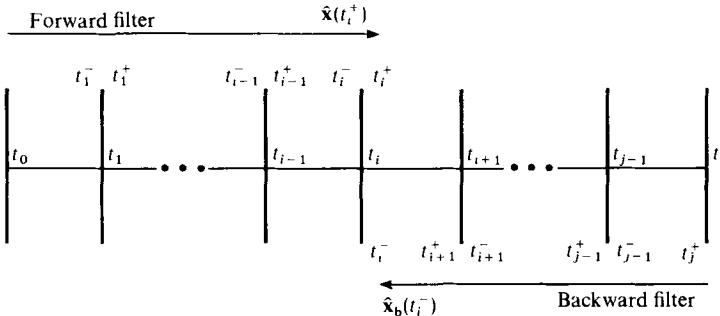


FIG. 8.1 Forward and backward filter operation.

(3) The smoothed estimate of  $\mathbf{x}(t_i)$ ,  $\hat{\mathbf{x}}(t_i/t_j)$  as defined in (8-3), is generated by optimally combining the value of  $\hat{\mathbf{x}}(t_i^+)$  from the forward filter (incorporating initial condition information about  $\mathbf{x}(t_0)$  and measurement information from  $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_i$ ) and  $\hat{\mathbf{x}}_b(t_i^-)$  from the backward filter (incorporating measurement information from  $\mathbf{z}_{i+1}, \mathbf{z}_{i+2}, \dots, \mathbf{z}_j$ ). This combination is accomplished by viewing  $\hat{\mathbf{x}}(t_i^+)$  and  $\hat{\mathbf{x}}_b(t_i^-)$  as two separate “observations” of  $\mathbf{x}(t_i)$  and assigning relative weighting according to the confidence you have in the precision of each, indicated by  $\mathbf{P}(t_i^+)$  and  $\mathbf{P}_b(t_i^-)$ , respectively. Another way of thinking of this process is to consider the backward filter output  $\hat{\mathbf{x}}_b(t_i^-)$  as providing an additional “measurement” with which to update the forward filter. Note that we choose to process  $\mathbf{z}(t_i, \omega_i) = \mathbf{z}_i$  in the forward filter; we could just as easily have chosen to process it in the backward filter instead, as long as this data does not enter into both filters and thus be counted twice in the smoothed estimate.

### 8.3 THREE CLASSES OF SMOOTHING PROBLEMS

There are many different classes of smoothing problems, each being determined by the manner in which the time parameters  $t_i$  and  $t_j$  are allowed to vary in the desired smoothed estimate  $\hat{\mathbf{x}}(t_i/t_j)$ . However, there are three classes of particular interest because of their applicability to realistic problems, namely, fixed-interval, fixed-point, and fixed-lag smoothing problems [10, 23, 24, 27, 32, 36].

To describe *fixed-interval smoothing*, let an experiment (system operation, mission, etc.) be conducted, and let measurement data be collected over the interval from initial time  $t_0$  to final time  $t_f$ ,  $[t_0, t_f]$ . After all of the data has been collected, it is desired to obtain the optimal estimate of  $\mathbf{x}(t_i)$  for all time  $t_i \in [t_0, t_f]$ , based on all measurements taken in the interval. Offline computations are thus inherently involved in generating the optimal fixed-interval smoothed estimate,

$$\begin{aligned}\hat{\mathbf{x}}(t_i/t_f) &= E\{\mathbf{x}(t_i)|\mathbf{Z}(t_f) = \mathbf{Z}_f\} \\ t_i &= t_0, t_1, \dots, t_f; \quad t_f = \text{fixed final time}\end{aligned}\quad (8-8)$$

Figure 8.2a represents these calculations schematically. This estimation technique is used for post-experiment data reduction to obtain refined state estimates

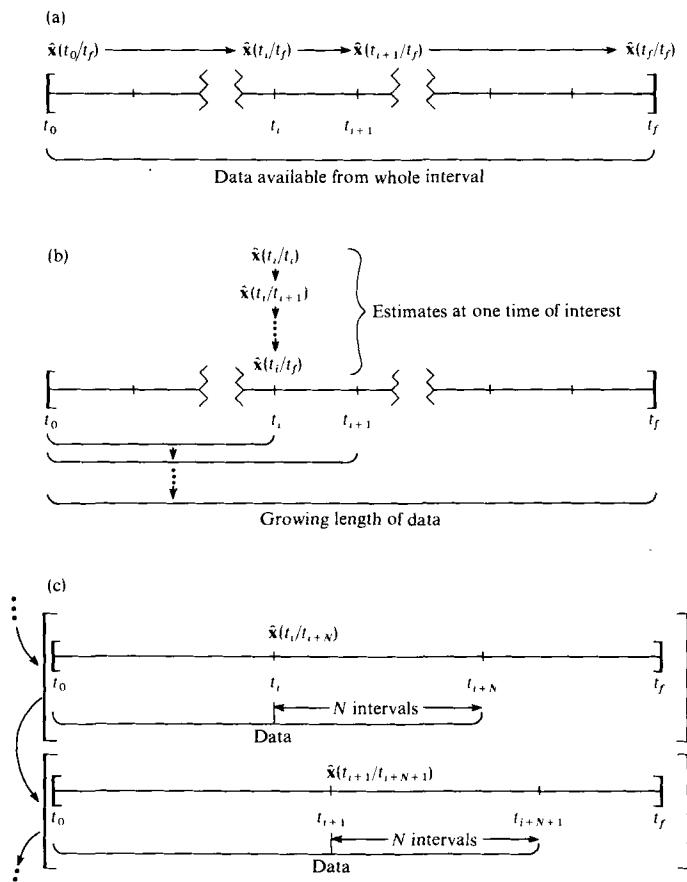


FIG. 8.2 Three types of smoothers: (a) fixed-interval smoothing, (b) fixed-point smoothing, (c) fixed-lag smoothing.

of better quality than that provided by online filters. It is also possible to use fixed-interval smoothing to estimate values of control inputs as well as states, to assess whether the “deterministic” controls were actually of commanded magnitude. A specific example would be post-flight analysis of a missile, generating smoothed estimates of both trajectory parameters (states) and thrust actually produced by the rocket motors for that flight (controls).

To consider *fixed-point smoothing*, let there be a certain point (or points) in time at which the value of the system state is considered critical. For example, conditions at engine burnout time are critical to rocket booster problems. Thus, one would desire an estimate of  $\mathbf{x}(t_i)$  for fixed  $t_i$ , conditioned on more and more data as measurements become available in real time:

$$\hat{\mathbf{x}}(t_i/t_j) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_j) = \mathbf{Z}_j\} \quad (8-9)$$

$t_i$  fixed;  $t_j = t_i, t_{i+1}, \dots, t_f$

This is the optimal fixed-point smoothed estimate, as depicted in Fig. 8.2b.

Finally, let measurements be taken, but assume that it is admissible for your application to generate an optimal estimate of  $\mathbf{x}(t_i)$ , not at time  $t_i$ , but at time  $t_{i+N}$ , where  $N$  is a fixed integer. Thus, to estimate  $\mathbf{x}(t_i)$ , you have available not only the measurements

$$\mathbf{z}(t_1, \omega_k) = \mathbf{z}_1, \quad \mathbf{z}(t_2, \omega_k) = \mathbf{z}_2, \quad \dots, \quad \mathbf{z}(t_i, \omega_k) = \mathbf{z}_i$$

but also the  $N$  additional measurements

$$\mathbf{z}(t_{i+1}, \omega_k) = \mathbf{z}_{i+1}, \quad \dots, \quad \mathbf{z}(t_{i+N}, \omega_k) = \mathbf{z}_{i+N}$$

and you are willing to delay the computation of the estimate of  $\mathbf{x}(t_i)$  until  $t_{i+N}$  to take advantage of the additional information in these  $N$  measurements. We wish to generate the optimal *fixed-lag smoothed estimate*,

$$\hat{\mathbf{x}}(t_i/t_{i+N}) = E\{\mathbf{x}(t_i) | \mathbf{Z}(t_{i+N}) = \mathbf{Z}_{i+N}\} \quad (8-10)$$

$t_i = t_0, t_1, \dots, t_{f-N}; \quad N = \text{fixed integer}$

Such an estimator is depicted in Fig. 8.2c and is particularly applicable to communications and telemetry data reduction.

## 8.4 FIXED-INTERVAL SMOOTHING

To develop the fixed-interval smoother, we shall exploit the work of Fraser [6, 7], who first showed it to be just a suitable combination of two optimal filters. Let the *forward filter* recursively produce a state estimate  $\hat{\mathbf{x}}(t_k^-)$  and error covariance  $\mathbf{P}(t_k^-)$  before incorporation of measurement  $\mathbf{z}_k$ , and  $\hat{\mathbf{x}}(t_k^+)$  and  $\mathbf{P}(t_k^+)$  after incorporation, for  $k = 1, 2, \dots, i$ . Notationally, let  $\hat{\mathbf{x}}_b(t_k^-)$  and  $\mathbf{P}_b(t_k^-)$  denote the state estimate and error covariance before incorporating measurement  $\mathbf{z}_k$  into the *backward filter*, and let  $\hat{\mathbf{x}}_b(t_k^+)$  and  $\mathbf{P}_b(t_k^+)$  be analogous

quantities after incorporation. Because the backward filter is of inverse covariance formulation [17], it will actually incorporate  $\mathbf{z}_k$  to generate  $\mathbf{P}_b^{-1}(t_k^+)$  and  $\hat{\mathbf{y}}_b(t_k^+) = \mathbf{P}_b^{-1}(t_k^+) \hat{\mathbf{x}}_b(t_k^+)$ , and then propagate backward in time to form  $\mathbf{P}_b^{-1}(t_{k-1}^-)$  and  $\hat{\mathbf{y}}_b(t_{k-1}^-) = \mathbf{P}_b^{-1}(t_{k-1}^-) \hat{\mathbf{x}}_b(t_{k-1}^-)$ , for  $k = f, f-1, \dots, i+1$ . Thus, we have the situation depicted in Fig. 8.1, with  $t_j = t_f$ . Subsequently, the optimal fixed-interval smoothed estimate and associated error covariance can be evaluated totally in terms of quantities available from the two filters as [6, 7, 34, 35]:

$$\hat{\mathbf{x}}(t_i/t_f) = \mathbf{P}(t_i/t_f) [\mathbf{P}^{-1}(t_i^+) \hat{\mathbf{x}}(t_i^+) + \mathbf{P}_b^{-1}(t_i^-) \hat{\mathbf{x}}_b(t_i^-)] \quad (8-11a)$$

$$= \mathbf{P}(t_i/t_f) [\mathbf{P}^{-1}(t_i^+) \hat{\mathbf{x}}(t_i^+) + \hat{\mathbf{y}}_b(t_i^-)] \quad (8-11b)$$

$$\mathbf{P}^{-1}(t_i/t_f) = \mathbf{P}^{-1}(t_i^+) + \mathbf{P}_b^{-1}(t_i^-) \quad (8-12)$$

There are a number of ways of deriving (8-11) and (8-12), and a demonstration by analogy to previous filter results is presented here because of its simplicity. In Chapter 5, we started from an estimate  $\hat{\mathbf{x}}(t_i^-)$  and associated error covariance  $\mathbf{P}(t_i^-)$  just before the measurement  $\mathbf{z}(t_i, \omega_k) = \mathbf{z}_i$  was incorporated, and derived the update equations as

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i) [\mathbf{z}_i - \mathbf{H}(t_i) \hat{\mathbf{x}}(t_i^-)] \quad (8-13a)$$

$$= [\mathbf{I} - \mathbf{K}(t_i) \mathbf{H}(t_i)] \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i) \mathbf{z}_i \quad (8-13b)$$

where the gain  $\mathbf{K}(t_i)$  is given by

$$\mathbf{K}(t_i) = \mathbf{P}(t_i^-) \mathbf{H}^T(t_i) [\mathbf{H}(t_i) \mathbf{P}(t_i^-) \mathbf{H}^T(t_i) + \mathbf{R}(t_i)]^{-1} \quad (8-14a)$$

$$= \mathbf{P}(t_i^+) \mathbf{H}^T(t_i) \mathbf{R}(t_i)^{-1} \quad (8-14b)$$

and the covariance update as

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{K}(t_i) \mathbf{H}(t_i) \mathbf{P}(t_i^-) \quad (8-15a)$$

$$= [\mathbf{I} - \mathbf{K}(t_i) \mathbf{H}(t_i)] \mathbf{P}(t_i^-) \quad (8-15b)$$

or, equivalently,

$$\mathbf{P}^{-1}(t_i^+) = \mathbf{P}^{-1}(t_i^-) + \mathbf{H}^T(t_i) \mathbf{R}^{-1}(t_i) \mathbf{H}(t_i) \quad (8-16)$$

Because it will be useful to the ensuing derivation, (8-15b) can be rearranged as

$$[\mathbf{I} - \mathbf{K}(t_i) \mathbf{H}(t_i)] = \mathbf{P}(t_i^+) \mathbf{P}^{-1}(t_i^-) \quad (8-17)$$

Substituting this and (8-14b) into (8-13b) yields an equivalent expression for  $\hat{\mathbf{x}}(t_i^+)$  as:

$$\begin{aligned} \hat{\mathbf{x}}(t_i^+) &= [\mathbf{P}(t_i^+) \mathbf{P}^{-1}(t_i^-)] \hat{\mathbf{x}}(t_i^-) + \mathbf{P}(t_i^+) \mathbf{H}^T(t_i) \mathbf{R}^{-1}(t_i) \mathbf{z}_i \\ &= \mathbf{P}(t_i^+) [\mathbf{P}^{-1}(t_i^-) \hat{\mathbf{x}}(t_i^-) + \mathbf{H}^T(t_i) \mathbf{R}^{-1}(t_i) \mathbf{z}_i] \end{aligned} \quad (8-18)$$

Now consider the analogous problem of having an estimate  $\hat{\mathbf{x}}(t_i^+)$  and associated  $\mathbf{P}(t_i^+)$ , based on all measurements up to and including  $\mathbf{z}_i$ , and now it is desired to update that estimate with the “measurement”  $\hat{\mathbf{x}}_b(t_i^-)$  with an

associated “measurement error” covariance matrix  $\mathbf{P}_b(t_i^-)$ . Since  $\hat{\mathbf{x}}_b(t_i^-)$  would represent a “measurement” of the entire state, the  $\mathbf{H}(t_i)$  for this update would be the identity matrix. When the update is performed, the resulting state estimate will be the optimal estimate of  $\mathbf{x}(t_i)$  based on  $\mathbf{Z}_i$  and  $\hat{\mathbf{x}}_b(t_i^-)$ , and since  $\hat{\mathbf{x}}_b(t_i^-)$  embodies all the information about  $\mathbf{x}(t_i)$  based (solely) on the data  $\mathbf{z}_{i+1}, \mathbf{z}_{i+2}, \dots, \mathbf{z}_f$ , this is then the optimal estimate of  $\mathbf{x}(t_i)$  based on  $\mathbf{Z}_f$ , or  $\hat{\mathbf{x}}(t_i/t_f)$ . The associated estimate error covariance after update would be  $\mathbf{P}(t_i/t_f)$ . Thus, for this analogy, the replacements shown in Table 8.1 can be made.

TABLE 8.1

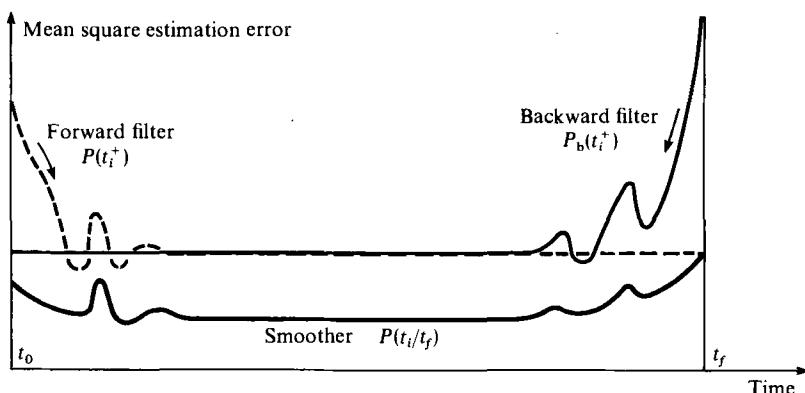
Original filter update	Smoother relation	Original filter update	Smoother relation
$\hat{\mathbf{x}}(t_i^-)$	$\hat{\mathbf{x}}(t_i^+)$	$\mathbf{H}(t_i)$	$\mathbf{I}$
$\hat{\mathbf{x}}(t_i^+)$	$\hat{\mathbf{x}}(t_i/t_f)$	$\mathbf{R}(t_i)$	$\mathbf{P}_b(t_i^-)$
$\mathbf{z}_i$	$\hat{\mathbf{x}}_b(t_i^-)$	$\mathbf{P}(t_i^-)$	$\mathbf{P}(t_i^+)$
		$\mathbf{P}(t_i^+)$	$\mathbf{P}(t_i/t_f)$

Using these replacements, (8-18) and (8-16) become equal to (8-11) and (8-12), respectively, as desired.

Equation (8-12) indicates that

$$\mathbf{P}(t_i/t_f) \leq \mathbf{P}(t_i^+) \quad (8-19)$$

i.e., that the smoothed estimate is at least as good as the filtered estimate for all time. Generally, it will in fact be better except at the terminal time  $t_f$ , as shown graphically in Fig. 8.3 for a typical case. The conditions on the smoother at time  $t_f$ , and their implications for the “initial conditions” on the backward filter, warrant further attention.

FIG. 8.3  $\mathbf{P}(t_i^+), \mathbf{P}_b(t_i^+), \mathbf{P}(t_i/t_f)$  for typical estimation problem.

At terminal time  $t_f$ , the optimal smoothed estimate is

$$\hat{\mathbf{x}}(t_f/t_f) \triangleq E\{\mathbf{x}(t_f) | \mathbf{Z}(t_f) = \mathbf{Z}_f\} \quad (8-20)$$

By definition, this must be equivalent to the forward filter estimate  $\hat{\mathbf{x}}(t_f^+)$ , since

$$\hat{\mathbf{x}}(t_f^+) \triangleq E\{\mathbf{x}(t_f) | \mathbf{Z}(t_f) = \mathbf{Z}_f\} \quad (8-21)$$

Similarly, it must be true that

$$\mathbf{P}(t_f/t_f) = \mathbf{P}(t_f^+) \quad (8-22)$$

Since (8-12) is true for all time, let  $t_i = t_f$  and obtain

$$\mathbf{P}^{-1}(t_f/t_f) = \mathbf{P}^{-1}(t_f^+) + \mathbf{P}_b^{-1}(t_f^-) \quad (8-23)$$

Combining (8-22) and (8-23) yields the starting condition for the backward filter:

$$\mathbf{P}_b^{-1}(t_f^-) = \mathbf{0} \quad (8-24)$$

The boundary condition on  $\hat{\mathbf{x}}_b$ ,  $\hat{\mathbf{x}}_b(t_f^-)$ , is as yet totally unknown. However, it is finite, so

$$\hat{\mathbf{y}}_b(t_f^-) = \mathbf{P}_b^{-1}(t_f^-)\hat{\mathbf{x}}_b(t_f^-) = \mathbf{0} \quad (8-25)$$

Thus, the backward filter is formulated as an inverse covariance filter, using the starting conditions (8-24) and (8-25). The forward filter may be formulated either as a conventional Kalman filter or an inverse covariance form optimal estimator, the former being more prevalent.

Once  $\hat{\mathbf{x}}(t_i^+)$  and  $\hat{\mathbf{x}}_b(t_i^-)$  have been calculated in the forward and backward filters, respectively, (8-11) and (8-12) can be used to combine these results. However, it would be computationally more efficient to process  $\mathbf{P}(t_i/t_f)$  directly, instead of using (8-12) to generate  $\mathbf{P}^{-1}(t_i/t_f)$  and inverting for use in (8-11). Algebraic manipulation of (8-12) yields this desirable form as

$$\begin{aligned} \mathbf{P}(t_i/t_f) &= [\mathbf{I} - \mathbf{W}(t_i)\mathbf{P}_b^{-1}(t_i^-)]\mathbf{P}(t_i^+)[\mathbf{I} - \mathbf{W}(t_i)\mathbf{P}_b^{-1}(t_i^-)]^T \\ &\quad + \mathbf{W}(t_i)\mathbf{P}_b^{-1}(t_i^-)\mathbf{W}^T(t_i) \end{aligned} \quad (8-26)$$

where

$$\mathbf{W}(t_i) = \mathbf{P}(t_i^+)[\mathbf{I} + \mathbf{P}(t_i^+)\mathbf{P}_b^{-1}(t_i^-)]^{-1T} \quad (8-27)$$

The smoothed estimate relation (8-11) is in a useful computational form if the forward filter is of inverse covariance form, since  $\mathbf{P}^{-1}(t_i^+)\hat{\mathbf{x}}(t_i^+)$  would be available directly as  $\hat{\mathbf{y}}(t_i^+)$  and then

$$\hat{\mathbf{x}}(t_i/t_f) = \mathbf{P}(t_i/t_f)[\hat{\mathbf{y}}(t_i^+) + \hat{\mathbf{y}}_b(t_i^-)] \quad (8-28)$$

However, for the more typical case in which the forward filter is of conventional covariance form, a relation that does not require an inversion of the  $n$ -by- $n$   $\mathbf{P}(t_i^+)$  matrix would be preferable. To achieve such a form, premultiply (8-12)

by  $\mathbf{P}(t_i^+)$  to obtain

$$\mathbf{P}(t_i^+) \mathbf{P}^{-1}(t_i/t_f) = \mathbf{I} + \mathbf{P}(t_i^+) \mathbf{P}_b^{-1}(t_i^-)$$

Inverting this expression yields

$$\mathbf{P}(t_i/t_f) \mathbf{P}^{-1}(t_i^+) = [\mathbf{I} + \mathbf{P}(t_i^+) \mathbf{P}_b^{-1}(t_i^-)]^{-1} \quad (8-29)$$

Finally, substitution of (8-29) into (8-11b) produces

$$\hat{\mathbf{x}}(t_i/t_f) = [\mathbf{I} + \mathbf{P}(t_i^+) \mathbf{P}_b^{-1}(t_i^-)]^{-1} \hat{\mathbf{x}}(t_i^+) + \mathbf{P}(t_i/t_f) \hat{\mathbf{y}}_b(t_i^-) \quad (8-30)$$

Although (8-30) requires computation of  $[\mathbf{I} + \mathbf{P}(t_i^+) \mathbf{P}_b^{-1}(t_i^-)]^{-1}$ , this is the same inverse used in (8-27), so we have in fact reduced the number of inversions required.

Let us now *summarize the fixed-interval smoother*. First a conventional *forward filter* is initialized by

$$\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0 \quad (8-31a)$$

$$\mathbf{P}(t_0) = \mathbf{P}_0 \quad (8-31b)$$

Time propagations between sample times are processed by

$$\hat{\mathbf{x}}(t_{k+1}^-) = \Phi(t_{k+1}, t_k) \hat{\mathbf{x}}(t_k^+) + \mathbf{B}_d(t_k) \mathbf{u}(t_k) \quad (8-32a)$$

$$\mathbf{P}(t_{k+1}^-) = \Phi(t_{k+1}, t_k) \mathbf{P}(t_k^+) \Phi^T(t_{k+1}, t_k) + \mathbf{G}_d(t_k) \mathbf{Q}_d(t_k) \mathbf{G}_d^T(t_k) \quad (8-32b)$$

Measurement updates are given by

$$\mathbf{K}(t_k) = \mathbf{P}(t_k^-) \mathbf{H}^T(t_k) [\mathbf{H}(t_k) \mathbf{P}(t_k^-) \mathbf{H}^T(t_k) + \mathbf{R}(t_k)]^{-1} \quad (8-33a)$$

$$\hat{\mathbf{x}}(t_k^+) = \hat{\mathbf{x}}(t_k^-) + \mathbf{K}(t_k) [\mathbf{z}_k - \mathbf{H}(t_k) \hat{\mathbf{x}}(t_k^-)] \quad (8-33b)$$

$$\mathbf{P}(t_k^+) = \mathbf{P}(t_k^-) - \mathbf{K}(t_k) \mathbf{H}(t_k) \mathbf{P}(t_k^-) \quad (8-33c)$$

Equations (8-32) and (8-33) are applied iteratively for  $k = 1, 2, \dots, i$  to produce  $\hat{\mathbf{x}}(t_i^+)$ . Note that (8-33c) could be replaced by the Joseph form if desired. Square root forms are also possible, but offline processing is usually done on computers with long enough wordlength that numerics do not dictate such implementation.

The inverse-covariance *backward filter* is initialized at terminal time  $t_f$  through (note the use of  $t_f^-$ , since a measurement update to incorporate  $\mathbf{z}_f$  precedes the first backward time propagation):

$$\hat{\mathbf{y}}_b(t_f^-) = \mathbf{0} \quad (8-34a)$$

$$\mathbf{P}_b^{-1}(t_f^-) = \mathbf{0} \quad (8-34b)$$

Measurement updates are generated by

$$\hat{\mathbf{y}}_b(t_k^+) = \hat{\mathbf{y}}_b(t_k^-) + \mathbf{H}^T(t_k) \mathbf{R}^{-1}(t_k) \mathbf{z}_k \quad (8-35a)$$

$$\mathbf{P}_b^{-1}(t_k^+) = \mathbf{P}_b^{-1}(t_k^-) + \mathbf{H}^T(t_k) \mathbf{R}^{-1}(t_k) \mathbf{H}(t_k) \quad (8-35b)$$

The estimate is propagated backward in time to the preceding measurement sample time via [6, 7, 12, 15, 33–35]:

$$\begin{aligned} \mathbf{J}(t_k) &= \mathbf{P}_b^{-1}(t_k^+) \mathbf{G}_d(t_{k-1}) [\mathbf{G}_d^T(t_{k-1}) \mathbf{P}_b^{-1}(t_k^+) \mathbf{G}_d(t_{k-1}) \\ &\quad + \mathbf{Q}_d^{-1}(t_{k-1})]^{-1} \end{aligned} \quad (8-36a)$$

$$\mathbf{L}(t_k) = \mathbf{I} - \mathbf{J}(t_k) \mathbf{G}_d^T(t_{k-1}) \quad (8-36b)$$

$$\hat{\mathbf{y}}_b(t_{k-1}^-) = \Phi^T(t_k, t_{k-1}) \mathbf{L}(t_k) [\hat{\mathbf{y}}_b(t_k^+) - \mathbf{P}_b^{-1}(t_k^+) \mathbf{B}_d(t_{k-1}) \mathbf{u}(t_{k-1})] \quad (8-36c)$$

$$\begin{aligned} \mathbf{P}_b^{-1}(t_{k-1}^-) &= \Phi^T(t_k, t_{k-1}) \{ \mathbf{L}(t_k) \mathbf{P}_b^{-1}(t_k^+) \mathbf{L}^T(t_k) \\ &\quad + \mathbf{J}(t_k) \mathbf{Q}_d^{-1}(t_{k-1}) \mathbf{J}^T(t_k) \} \Phi(t_k, t_{k-1}) \end{aligned} \quad (8-36d)$$

Note the time indices on the state transition matrices: these are indeed proper for backward propagation of adjoint system relations. Equations (8-35) and (8-36) are applied recursively for  $k = f, (f-1), \dots, (i+1)$  to generate  $\hat{\mathbf{y}}_b(t_i^-)$ .

At time  $t_i$ , the *smoothed estimate* is calculated by combining  $\hat{\mathbf{x}}(t_i^+)$ ,  $\mathbf{P}(t_i^+)$ ,  $\hat{\mathbf{y}}_b(t_i^-)$  and  $\mathbf{P}_b^{-1}(t_i^-)$  through

$$\mathbf{X}(t_i) = [\mathbf{I} + \mathbf{P}(t_i^+) \mathbf{P}_b^{-1}(t_i^-)]^{-1} \quad (8-37a)$$

$$\mathbf{W}(t_i) = \mathbf{P}(t_i^+) \mathbf{X}^T(t_i) \quad (8-37b)$$

$$\mathbf{Y}(t_i) = \mathbf{I} - \mathbf{W}(t_i) \mathbf{P}_b^{-1}(t_i^-) \quad (8-37c)$$

$$\mathbf{P}(t_i/t_f) = \mathbf{Y}(t_i) \mathbf{P}(t_i^+) \mathbf{Y}^T(t_i) + \mathbf{W}(t_i) \mathbf{P}_b^{-1}(t_i^-) \mathbf{W}^T(t_i) \quad (8-37d)$$

$$\hat{\mathbf{x}}(t_i/t_f) = \mathbf{X}(t_i) \hat{\mathbf{x}}(t_i^+) + \mathbf{P}(t_i/t_f) \hat{\mathbf{y}}_b(t_i^-) \quad (8-37e)$$

One effective computational procedure is to compute and store values of  $\mathbf{P}_b^{-1}(t_k^-)$  and  $\hat{\mathbf{y}}_b(t_k^-)$  from running the backward filter for  $k = f, (f-1), \dots, 1$ . Then the forward filter is run across the interval, simultaneously generating the values of  $\hat{\mathbf{x}}(t_i/t_f)$  and  $\mathbf{P}(t_i/t_f)$  from (8-37) by using the values stored during the backward filter operation.

In some post-experimental data reduction applications, such as trajectory analyses, it is desirable to determine an estimate of the control inputs based on measurement data, to discern whether actual control inputs were equal to nominally commanded values. Let the system dynamics be described by

$$\mathbf{x}(t_{i+1}) = \Phi(t_{i+1}, t_i) \mathbf{x}(t_i) + \mathbf{G}_d(t_i) [\mathbf{u}(t_i) + \mathbf{w}_d(t_i)] \quad (8-38)$$

where  $\mathbf{u}(t_i)$  is the nominal value of the control input at time  $t_i$ . This is the original problem model (8-4) with the assumption that  $\mathbf{B}_d(t_i) \equiv \mathbf{G}_d(t_i)$  for all times of interest (this is rather nonrestrictive, since zero entries can be added to  $\mathbf{u}(t_i)$  or  $\mathbf{w}_d(t_i)$  as required for compatibility). A smoothed estimate of the actual control applied at time  $t_i$  based on all measurement data  $\hat{\mathbf{u}}(t_i/t_f)$  can then be determined, along with an associated covariance for  $[\mathbf{u}(t_i) + \mathbf{w}_d(t_i)]$ ,  $\mathbf{U}(t_i/t_f)$ . The two sets of filter relations are altered only to set  $\mathbf{B}_d(t_k) \equiv \mathbf{G}_d(t_k)$ , and then the following

computations are added to smoother relations (8-37):

$$\mathbf{V}(t_i) = \mathbf{Q}_d(t_i) \mathbf{G}_d^T(t_i) \boldsymbol{\Phi}^T(t_i, t_{i+1}) \quad (8-39a)$$

$$\mathbf{K}_u(t_i) = \mathbf{V}(t_i) \mathbf{X}^T(t_i) \quad (8-39b)$$

$$\hat{\mathbf{u}}(t_i/t_f) = \mathbf{u}(t_i) + \mathbf{K}_u(t_i) [\hat{\mathbf{x}}_b(t_i^-) - \mathbf{P}_b^{-1}(t_i^-) \hat{\mathbf{x}}(t_i^+)] \quad (8-39c)$$

$$\mathbf{U}(t_i/t_f) = \mathbf{Q}_d(t_i) - \mathbf{V}(t_i) \mathbf{P}_b^{-1}(t_i^-) \mathbf{K}_u^T(t_i) \quad (8-39d)$$

The question naturally arises whether or not smoothing is worth the additional burden beyond that of filtering for a particular application. The *smoothability* criterion as defined by Fraser [6] is meant to judge whether smoothing will provide a state estimate which is superior to that obtained by the simpler means of extrapolating the final forward filter state estimate backward in time through repeated usage of (8-32a) arranged as

$$\hat{\mathbf{x}}'(t_k) = \boldsymbol{\Phi}(t_k, t_{k+1}) [\hat{\mathbf{x}}'(t_{k+1}) - \mathbf{B}_d(t_k) \mathbf{u}(t_k)] \quad (8-40a)$$

starting from

$$\hat{\mathbf{x}}'(t_f) = \hat{\mathbf{x}}(t_f^+) \quad (8-40b)$$

By this definition, those components of the state which are controllable with respect to the dynamic driving noise are smoothable. Thus, one should not consider smoothing in those cases for which an adequate model is a linear system with no driving noise, i.e.,  $\mathbf{Q}_d(t_i) \equiv \mathbf{0}$ . Moreover, the larger the elements (eigenvalues) of the stochastic controllability matrix, the greater the relative benefit of smoothing instead of filtering.

**EXAMPLE 8.1** Consider application [6] of the discrete-time smoother to an oscillatory system for which an adequate model is a lightly damped, constant coefficient, second order stochastic differential equation; for instance, a mass, spring, and dashpot described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

where  $\omega_n$  is the undamped natural frequency of the system and  $\zeta$  is the damping ratio, with numerical values

$$\omega_n = 6 \text{ rad/sec} \cong 1 \text{ rev/sec}, \quad \zeta = 0.16$$

Scalar measurements of  $x_1(t_i)$  are taken every 0.1 sec, modeled as

$$\mathbf{z}(t_i) = [1 \ 0] \mathbf{x}(t_i) + \mathbf{v}(t_i)$$

Initial condition information about  $\mathbf{x}(t_0)$  is that it can be modeled as a Gaussian random variable with mean and covariance

$$\hat{\mathbf{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{P}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let an equivalent discrete-time model be generated, and let  $Q_d = 10^{-2}$  and  $R = 10^{-4}$ .

Figures 8.4 and 8.5 portray the diagonal elements (on log scale) of the resulting forward filter  $\mathbf{P}(t_i^+)$  and smoother  $\mathbf{P}(t_i/t_f)$  for a time interval of 10 sec. Note that the smoother estimate is never worse than that of the filter, this being the consequence of (8-12), as discussed earlier.

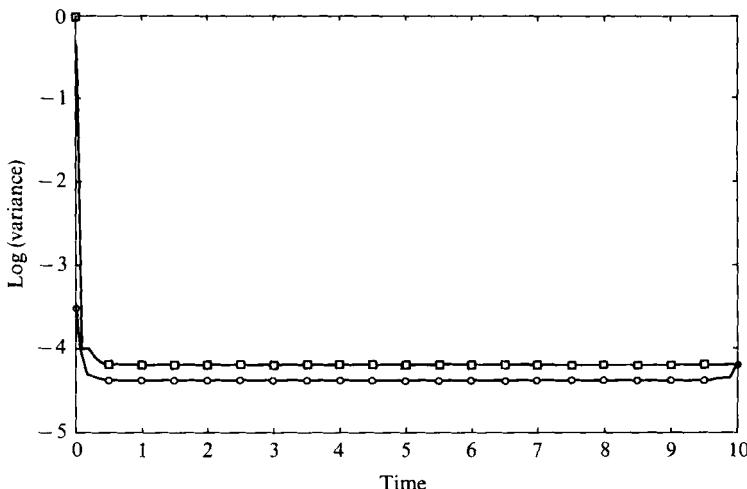


FIG. 8.4 One-one element of covariance matrices,  $R = 10^{-4}$ ,  $Q_d = 10^{-2}$ :  $\square$ , forward filter;  $\circ$ , smoother. From Fraser [6].

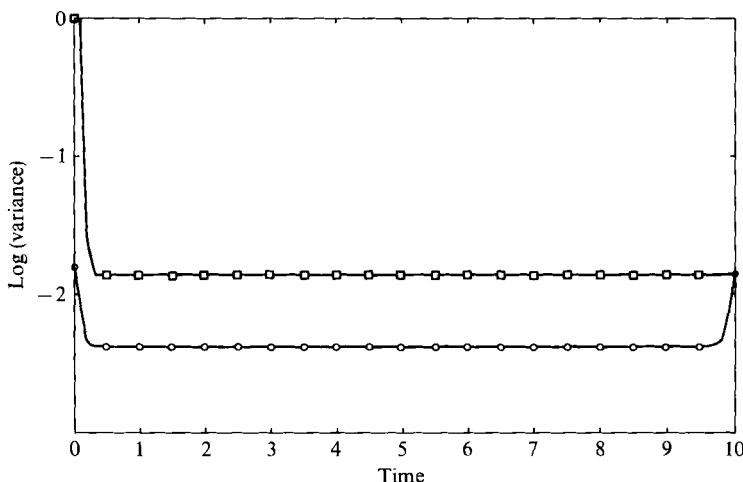


FIG. 8.5 Two-two element of covariance matrices,  $R = 10^{-4}$ ,  $Q_d = 10^{-2}$ :  $\square$ , forward filter;  $\circ$ , smoother. From Fraser [6].

Also note that there are indeed two transients for the smoother: the transient in the beginning being due to the forward filter and the one at the end due to the backward filter. For this case, both filters were essentially in steady state for most of the time interval of interest, and thus so was the smoother.

The improvement of smoothing over filtering is noticeably more significant for estimating  $x_2$  than for  $x_1$ . This is true because the  $x_2$  state is more strongly smoothable:  $x_2(\cdot, \cdot)$  is driven directly by the dynamic driving noise  $w(\cdot, \cdot)$ , while  $x_1(\cdot, \cdot)$  is separated from  $w(\cdot, \cdot)$  by an integration. ■

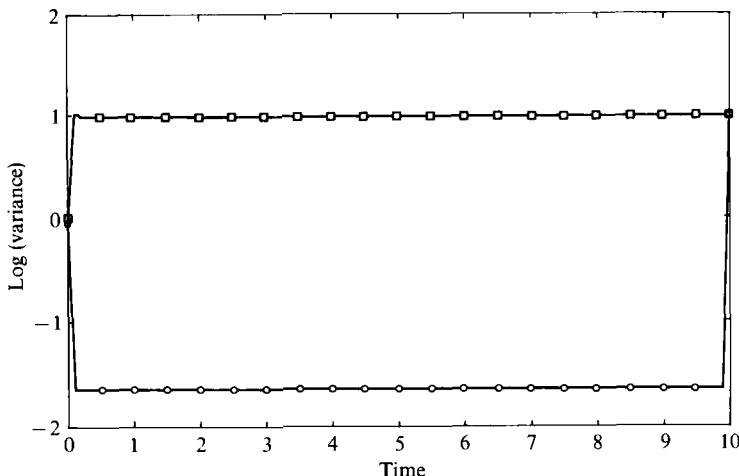


FIG. 8.6 Two–two element of covariance matrices,  $R = 10^{-4}$ ,  $Q_d = 10$ :  $\square$ , forward filter;  $\circ$ , smoother. From Fraser [6].

EXAMPLE 8.2 Consider the same problem as in Example 8.1, but now investigate the effect of varying  $Q_d$ . For Fig. 8.6, a plot of  $\log P_{22}$  for the forward filter and smoother,  $Q_d$  was increased from  $10^{-2}$  to 10. As  $Q_d$  is made larger,  $x_2$  becomes “more smoothable,” and the improvement of smoothing over filtering becomes larger, as seen by comparing this plot to Fig. 8.5.

For Fig. 8.7,  $Q_d$  was set at zero, and there is virtually no separation between the smoother and filter variances. For the case of no dynamic driving noise, the state vector is not “smoothable,” and the smoother result is merely the backward extrapolation of the forward filter final covariance. ■

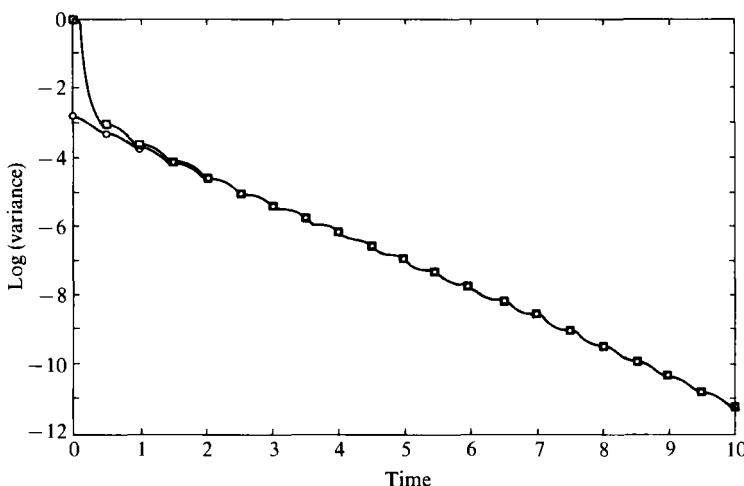


FIG. 8.7 Two–two element of covariance matrices,  $R = 10^{-4}$ ,  $Q_d = 0$ :  $\square$ , forward filter;  $\circ$ , smoother. From Fraser [6].

**EXAMPLE 8.3** The previous examples, cases in which smoothing significantly improves the filter estimate, as shown in Figs. 8.5 and 8.6, are characterized by both the filter and smoother quickly reaching steady state operation. For these cases, the  $Q_d/R$  ratio is large: the uncertainty in the state propagation is large compared to relatively accurate measurements, so the new estimate is much more heavily dependent upon the new measurement than prior estimates; this combined with noise stationarity caused the filter to reach steady state operation quickly.

The steady state values attained by the filter and smoother are independent of the  $\mathbf{P}_0$  matrix. The filter steady state is reached after the transient effects due to  $\mathbf{P}_0$  have settled out, and the backward filter is independent of  $\mathbf{P}_0$  by definition. Since  $\mathbf{P}(t_i/t_f)$  is a function of the two filter covariances, the smoother steady state is also independent of  $\mathbf{P}_0$ .

Figure 8.8 exemplifies a case with a long transient compared to the time interval of interest: the same problem as in the previous examples, but with  $Q_d = 10^{-4}$  and  $R = 10$ . Note the lack of significant improvement by smoothing. Also note that the variance oscillations occur at a frequency twice that of the system characteristic frequency, typical of estimation problems with oscillatory systems. ■

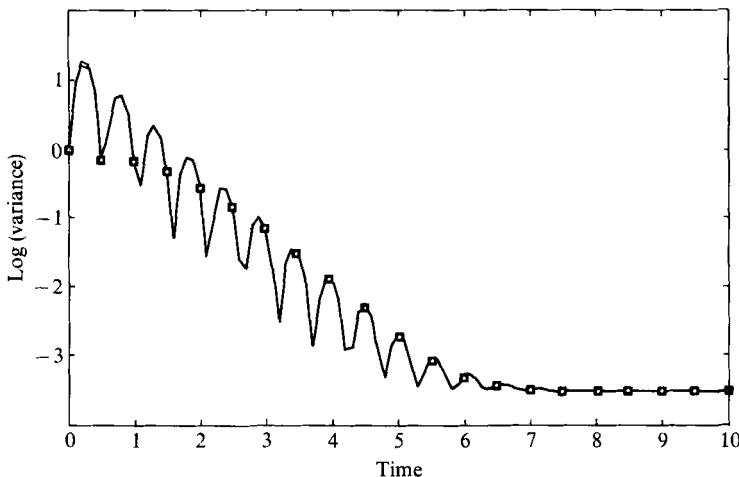


FIG. 8.8 Two-two element of covariance matrices,  $R = 10$ ,  $Q_d = 10^{-4}$ : □, forward filter; ○, smoother. From Fraser [6].

The smoother algorithm of (8-31)–(8-37) can be shown to be equivalent to the optimal fixed-interval smoother obtained by Meditch [20, 21, 23, 24]. Computationally, this alternate method entails performing the forward filter computations and storing  $\hat{\mathbf{x}}(t_i^-)$ ,  $\mathbf{P}(t_i^-)$ ,  $\hat{\mathbf{x}}(t_i^+)$ , and  $\mathbf{P}(t_i^+)$  for all time over the interval  $[t_0, t_f]$ . Then, starting from the boundary condition

$$\hat{\mathbf{x}}(t_f/t_f) = \hat{\mathbf{x}}(t_f^+) \quad (8-41)$$

the smoothed estimate is generated backward in time via

$$\hat{\mathbf{x}}(t_i/t_f) = \hat{\mathbf{x}}(t_i^+) + \mathbf{A}(t_i)[\hat{\mathbf{x}}(t_{i+1}/t_f) - \hat{\mathbf{x}}(t_{i+1}^-)] \quad (8-42)$$

where the “smoothing estimator gain matrix”  $\mathbf{A}(t_i)$  is given by

$$\mathbf{A}(t_i) = \mathbf{P}(t_i^+) \Phi^T(t_{i+1}, t_i) \mathbf{P}^{-1}(t_{i+1}^-) \quad (8-43)$$

Again, the indices in  $\Phi^T$  in this expression are correct as shown, appropriate to the state transition matrix for propagating adjoint system quantities backward in time. Also note that an  $n$ -by- $n$  inversion is required for each recursion of (8-43). The covariance of the zero-mean Gaussian estimation error  $[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i/t_f)]$ , as given in (8-37d), can be generated backwards from

$$\mathbf{P}(t_f/t_f) = \mathbf{P}(t_f^+) \quad (8-44)$$

using the recursion

$$\mathbf{P}(t_i/t_f) = \mathbf{P}(t_i^+) + \mathbf{A}(t_i)[\mathbf{P}(t_{i+1}/t_f) - \mathbf{P}(t_{i+1}^-)]\mathbf{A}^T(t_i) \quad (8-45)$$

For the special case of  $\mathbf{Q}_d(t_i) \equiv \mathbf{0}$ ,  $\mathbf{A}(t_i)$  becomes  $\Phi(t_i, t_{i+1})$  and

$$\hat{\mathbf{x}}(t_i/t_f) = \Phi(t_i, t_f)\hat{\mathbf{x}}(t_f^+) - \sum_{k=i}^{f-1} \Phi(t_i, t_{k+1})\mathbf{B}_d(t_k)\mathbf{u}(t_k) \quad (8-46a)$$

$$\mathbf{P}(t_i/t_f) = \Phi(t_i, t_f)\mathbf{P}(t_f^+)\Phi^T(t_i, t_f) \quad (8-46b)$$

as would be predicted in view of the smoothability criterion and (8-40).

## 8.5 FIXED-POINT SMOOTHING

Meditch [10, 19–21, 23, 24] has shown that the optimal fixed-point smoother can be expressed in the following form. Starting from the initial condition

$$\hat{\mathbf{x}}(t_i/t_i) = \hat{\mathbf{x}}(t_i^+) \quad (8-47)$$

with  $\hat{\mathbf{x}}(t_i^+)$  obtained from a concurrently running Kalman filter, the relation

$$\hat{\mathbf{x}}(t_i/t_j) = \hat{\mathbf{x}}(t_i/t_{j-1}) + \mathbf{W}(t_j)[\hat{\mathbf{x}}(t_j^+) - \hat{\mathbf{x}}(t_j^-)] \quad (8-48)$$

is solved for fixed  $t_i$ , letting  $t_j = t_{i+1}, t_{i+2}, \dots, t_f$ , with the gain matrix  $\mathbf{W}(t_j)$  evaluated recursively as

$$\mathbf{W}(t_j) \triangleq \prod_{k=i}^{j-1} \mathbf{A}(t_k) = \mathbf{W}(t_{j-1})\mathbf{A}(t_{j-1}) \quad (8-49a)$$

$$\mathbf{A}(t_k) \triangleq \mathbf{P}(t_k^+)\Phi^T(t_{k+1}, t_k)\mathbf{P}^{-1}(t_{k+1}^-) \quad (8-49b)$$

The values for  $\hat{\mathbf{x}}(t_j^+)$  and  $\hat{\mathbf{x}}(t_j^-)$  in this expression are obtained from the Kalman filter iterating forward in time. In fact,

$$[\hat{\mathbf{x}}(t_j^+) - \hat{\mathbf{x}}(t_j^-)] = \mathbf{K}(t_j)[\mathbf{z}_j - \mathbf{H}(t_j)\hat{\mathbf{x}}(t_j^-)] \quad (8-50)$$

so (8-48) can be interpreted as the means of “reflecting back” the information embodied in the filter residual at each time  $t_j$  about the state value at time  $t_i$ . Moreover, the vector in (8-50) is directly available from the filter. The error committed by this estimator,  $[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i/t_j)]$ , is Gaussian and zero mean for  $j = i, i+1, \dots, f$ , with covariance:

$$\mathbf{P}(t_i/t_j) = \mathbf{P}(t_i/t_{j-1}) + \mathbf{W}(t_j)[\mathbf{P}(t_j^+) - \mathbf{P}(t_j^-)]\mathbf{W}^T(t_j) \quad (8-51a)$$

$$= \mathbf{P}(t_i/t_{j-1}) - \mathbf{W}(t_j)\mathbf{K}(t_j)\mathbf{H}(t_j)\mathbf{P}(t_j^-)\mathbf{W}^T(t_j) \quad (8-51b)$$

solved forward for  $t_j = t_{i+1}, t_{i+2}, \dots, t_f$  from the initial condition

$$\mathbf{P}(t_i/t_i) = \mathbf{P}(t_i^+) \quad (8-52)$$

To avoid the inversion of  $\mathbf{P}(t_{j+1}^-)$  as required for each recursion of (8-49b), the equivalent form due to Fraser (derived in a fashion analogous to that of Section 8.4) can be used [6, 7]. This algorithm instead requires knowledge of  $\mathbf{R}^{-1}(t_k)$ , which can be precomputed for all  $k$ . As in the Meditch smoother, a Kalman filter is an integral part of the estimator. Letting  $t_i$  be fixed,  $\hat{\mathbf{x}}(t_i/t_j)$  is calculated iteratively for  $t_j = t_{i+1}, t_{i+2}, \dots, t_f$  from

$$\hat{\mathbf{x}}(t_i/t_j) = \hat{\mathbf{x}}(t_i/t_{j-1}) + \mathbf{W}(t_j)\mathbf{H}^T(t_j)\mathbf{R}^{-1}(t_j)[\mathbf{z}_j - \mathbf{H}(t_j)\hat{\mathbf{x}}(t_j^-)] \quad (8-53)$$

using (8-47) as an initial condition, with  $\mathbf{W}(t_j)$  generated by means of the recursion

$$\mathbf{S}(t_j) = \mathbf{H}^T(t_j)\mathbf{R}^{-1}(t_j)\mathbf{H}(t_j) \quad (8-54a)$$

$$\mathbf{W}(t_j) = \mathbf{W}(t_{j-1})\Phi^T(t_j, t_{j-1})[\mathbf{I} - \mathbf{S}(t_j)\mathbf{P}(t_j^+)] \quad (8-54b)$$

starting from

$$\mathbf{W}(t_i) = \mathbf{P}(t_i^+) \quad (8-54c)$$

In (8-53), the bracketed term is the residual that is directly available from the filter, and does not require separate computation. The error covariance can be computed in this form as

$$\mathbf{P}(t_i/t_j) = \mathbf{P}(t_i/t_{j-1}) - \mathbf{W}(t_j)[\mathbf{S}(t_j)\mathbf{P}(t_j^-)\mathbf{S}(t_j) + \mathbf{S}(t_j)]\mathbf{W}^T(t_j) \quad (8-55)$$

for  $j = i+1, i+2, \dots, f$ , with initial condition given by (8-52).

## 8.6 FIXED-LAG SMOOTHING

The optimal fixed-lag smoother for an  $N$ -step time lag can be generated from the relation [10, 21, 23, 24]

$$\begin{aligned} \hat{\mathbf{x}}(t_{i+1}/t_{i+N+1}) &= \Phi(t_{i+1}, t_i)\hat{\mathbf{x}}(t_i/t_{i+N}) \\ &+ \mathbf{C}(t_{i+N+1})\mathbf{K}(t_{i+N+1})[\mathbf{z}_{i+N+1} - \mathbf{H}(t_{i+N+1})\hat{\mathbf{x}}(t_{i+N+1}^-)] \\ &+ \mathbf{U}(t_{i+1})[\hat{\mathbf{x}}(t_i/t_{i+N}) - \hat{\mathbf{x}}(t_i^+)] \end{aligned} \quad (8-56)$$

where the  $n$ -by- $n$  gain matrix  $\mathbf{C}(t_{i+N+1})$  is given by

$$\mathbf{C}(t_{i+N+1}) = \prod_{k=i+1}^{i+N} \mathbf{A}(t_k) = \mathbf{A}^{-1}(t_i)\mathbf{C}(t_{i+N})\mathbf{A}(t_{i+N}) \quad (8-57a)$$

where  $\mathbf{A}(t_k)$  is defined in (8-49b), and the  $n$ -by- $n$   $\mathbf{U}(t_{i+1})$  is given by

$$\mathbf{U}(t_{i+1}) = \mathbf{G}_d(t_i)\mathbf{Q}_d(t_i)\mathbf{G}_d^T(t_i)\Phi^T(t_i, t_{i+1})\mathbf{P}^{-1}(t_i^+) \quad (8-57b)$$

In (8-56), the first term on the right hand side indicates the time propagation of the previous smoothed estimate, the second term is the correction due to

incorporating the new information available from measurement at time  $t_{i+N+1}$ , and the third term is the correction due to dynamic noise smoothing beyond the capabilities of the Kalman filter. The entire quantity  $\{\mathbf{K}(t_{i+N+1})[\mathbf{z}_{i+N+1} - \mathbf{H}(t_{i+N+1})\hat{\mathbf{x}}(t_{i+N+1}^-)]\}$  is available from the Kalman filter which is run simultaneously with the smoother algorithm.

The smoothed estimate is calculated by (8-56) for  $t_i = t_0, t_1, \dots, t_{f-N}$  from the initial condition  $\hat{\mathbf{x}}(t_0/t_N)$ . Thus, in order to initialize the fixed-lag smoother, an optimal fixed-point smoother as described in the previous section must be iterated  $N$  times to start from  $\hat{\mathbf{x}}(t_0/t_0) = \hat{\mathbf{x}}_0$  and compute  $\hat{\mathbf{x}}(t_0/t_1), \hat{\mathbf{x}}(t_0/t_2), \dots, \hat{\mathbf{x}}(t_0/t_N)$  in succession.

The covariance of the zero-mean error committed by this estimator can be computed by

$$\begin{aligned}\mathbf{P}(t_{i+1}/t_{i+N+1}) &= \mathbf{P}(t_{i+1}^-) - \mathbf{C}(t_{i+N+1})\mathbf{K}(t_{i+N+1})\mathbf{H}(t_{i+N+1})\mathbf{P}(t_{i+N+1}^-)\mathbf{C}^T(t_{i+N+1}) \\ &\quad - \mathbf{A}^{-1}(t_i)[\mathbf{P}(t_i^+) - \mathbf{P}(t_i/t_{i+N})]\mathbf{A}^{-1}(t_i)^T\end{aligned}\quad (8-58)$$

for  $i = 0, 1, \dots, (f - N - 1)$ , starting from the initial condition  $\mathbf{P}(t_0/t_N)$ , which is also obtained as the output of  $N$  initial iterations of the optimal fixed-point smoother. The computational and storage burden of this algorithm is seen to be considerably greater than that of an optimal filter for the same problem (which is inherently part of the smoother structure itself). Consequently, the performance benefit of incorporating  $N$  additional measurements into each estimate must warrant the added burden and  $N$ -step delay of availability of the state estimate, before this estimator would become preferable to a more straightforward filter algorithm.

## 8.7 SUMMARY

This chapter has generated algorithms for the optimal smoothed estimate of the state at time  $t_i$ , based on measurement data through time  $t_j$ , where  $j > i$ ; namely,  $\hat{\mathbf{x}}(t_i/t_j) = E\{\mathbf{x}(t_i)|\mathbf{Z}(t_j) = \mathbf{Z}_j\}$ . Conceptually, the smoothing problem can be decomposed into two filtering problems: one using initial conditions and the “past” history of data  $\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_i\}$  and the other incorporating only “future” measurements  $\{\mathbf{z}_{i+1}, \mathbf{z}_{i+2}, \dots, \mathbf{z}_j\}$ , with the smoothed estimate being the optimal combination of these two filter outputs. Figure 8.2 portrayed the three useful types of smoothers: the fixed-interval smoother described by (8-31)–(8-37), the fixed-point smoother given by (8-52)–(8-55), and the fixed-lag smoother specified by (8-56)–(8-58). If it is acceptable to make an estimate of  $\mathbf{x}(t_i)$  at some time later than  $t_i$  itself, and if the additional computational and storage requirements are not prohibitive, a smoother can provide estimates superior to those of a filter. The degree of improvement can be assessed by comparing  $\mathbf{P}(t_i/t_j)$  to  $\mathbf{P}(t_i^+)$  from a filter, so that a tradeoff decision can be made rationally.

Optimal smoothers can also be developed analogously for the cases of nonwhite measurement corruption noise [3, 25, 26] or continuous-time measurements [18–20, 23], and extensions of the concepts in this chapter can be used to generate nonlinear smoothers [1, 2, 4, 5, 8, 13, 22, 29, 34, 35]. Innovations approaches [10, 11, 14–16], maximum likelihood methods [30, 31], least squares [10, 11, 14], and dynamic programming [5] can also be applied to generate smoothing algorithms.

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## PROBLEMS

**8.1** Consider the example used previously in Chapter 5 of Volume 1 of a gyro on test, described by the scalar dynamics equation

$$\dot{x}(t) = -x(t) + w(t)$$

where  $x(\cdot, \cdot)$  is gyro drift rate and  $w(\cdot, \cdot)$  is white Gaussian noise of zero mean and variance kernel  $Q\delta(\tau)$ , where  $Q = 2 \text{ deg}^2/\text{hr}$ . The output is sampled every 0.25 hr, and this output is described as

$$z(t_i) = x(t_i) + v(t_i)$$

where  $v(\cdot, \cdot)$  is a white Gaussian noise of zero mean and with power given by  $R = 0.5 \text{ deg}^2/\text{hr}^2$ . For this problem, we assumed initial conditions of  $\hat{x}_0 = 0$ ,  $P_0 = 1 \text{ deg}^2/\text{hr}^2$ . The state transition matrix was found to be  $\Phi(t_{i+1}, t_i) = 0.78$ .

The optimal error covariance of the optimal filter for a two-hour interval of data sampling was found to be as shown in Table 8.P1.

TABLE 8.P1

$t_i$ (hr)	$P(t_i^-)$ ( $\text{deg}^2/\text{hr}^2$ )	$P(t_i^+)$ ( $\text{deg}^2/\text{hr}^2$ )
0		1.0
0.25	1.0	0.33
0.50	0.59	0.27
0.75	0.55	0.26
1.00	0.54	0.25
1.25	0.54	0.25
1.50	0.54	0.25
1.75	0.54	0.25
2.00	0.54	0.25

Now calculate the optimal fixed-interval smoother results for the same data. Write out the state estimate and covariance relations for the backward filter and smoother, and actually compute the smoother error covariance for the two-hour interval.

**8.2** Repeat Problem 8.1, but use the alternate fixed-interval smoother results of (8-41)–(8-46). Demonstrate the equivalence of these two means of generating the smoothed estimate.

**8.3** Reconsider Problem 8.1, but now assume that the state value at  $t = 0.75$  hr is of specific interest. Generate the fixed-point smoother result to estimate the gyro drift rate  $x$  at that time, based on measurements taken sequentially over the entire two-hour interval, using (8-47)–(8-52).

**8.4** Repeat the previous problem, using the alternate fixed-point smoother results, (8-53)–(8-55), and demonstrate the equivalence of these two alternate computations of the smoothed estimate.

**8.5** Reconsider Problem 8.1, but now compute the fixed-lag smoother result for a 3-step time lag. Note the need of the fixed-point smoother for initialization over the first three measurements. Compare these results to the previous Kalman filter estimate.

**8.6** Repeat the preceding problem, but for a 1-step time lag. Compare performance to both the 3-step fixed-lag smoother and the Kalman filter.

**8.7** We have attained continuous-time filter equations as

$$\begin{aligned} d\hat{\mathbf{x}}/dt &= \mathbf{F}\hat{\mathbf{x}} + \mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}[\mathbf{z} - \mathbf{H}\hat{\mathbf{x}}] \\ d\mathbf{P}/dt &= \mathbf{FP} + \mathbf{PF}^T + \mathbf{GQG}^T - \mathbf{PH}^T\mathbf{R}^{-1}\mathbf{HP} \end{aligned}$$

Now derive the continuous-time fixed-interval optimal smoother relations.

(a) One means of deriving this result is to express the sampled-data results of (8-31)–(8-37) in the limit as the sample period goes to zero. In Chapter 5 of Volume 1, such a limiting operation was used to derive the filter equations above. Analogously take the limit of the backward filter equations (8-34)–(8-36), and combine these results with those of the forward filter via the analog of (8-37), to produce the smoother for continuous-time measurements.

(b) An alternate derivation of the continuous-measurement fixed-interval smoother has also been suggested. Since the backward filter propagates the estimates backwards in time, let  $\tau = t_f - t$ , and then write the system description,

$$\begin{aligned} d\mathbf{x}(t)/dt &= \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{w}(t) \\ \mathbf{z}(t) &= \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}(t) \end{aligned}$$

in terms of the variable  $\tau$ . By so doing, the equations for the backward filter in terms of  $\hat{x}_b$  and  $P_b$  can be written as an analog to the forward filter, with  $(-\mathbf{F})$  replacing  $\mathbf{F}$  and  $(-\mathbf{G})$  replacing  $\mathbf{G}$ . Knowing  $dP_b/d\tau$ , an expression can be obtained for  $dP_b^{-1}/d\tau$ , using

$$dP_b^{-1}/d\tau = -P_b^{-1}(dP_b/d\tau)P_b^{-1}$$

Writing  $\hat{y}_b(\tau) = P_b^{-1}(\tau)\hat{x}_b(\tau)$  and differentiating yields a “state” equation for  $\hat{y}_b(\tau)$ . Starting conditions can be specified as for the discrete-time case, and finally the smoother relationships derived from the forward and backward filters.

One must be careful about misinterpreting these results. Proper characterization of properties of reverse-time stochastic processes must be considered, as accomplished in [12, 15, 33–35].

- 8.8** Consider the system and measurement described by the scalar equations

$$\dot{x} = w, \quad z = x + v$$

where  $w$  and  $v$  are zero-mean white Gaussian noises, independent of each other, and of strength  $Q$  and  $R$ , respectively. Show that, in steady state operation, the optimal smoothed estimate of  $x$  is just the average of the forward and backward filter estimates.

**8.9** Generate a numerically precise form of fixed-interval smoother by writing the forward filter (8-31)–(8-33) in square root form (see Chapter 7 of Volume 1) and the backward filter (8-34)–(8-36) in square root inverse covariance form, and combining the results according to (8-37) written in factored form.

**8.10** The concept of *adjoint differential and difference equations* appears in this chapter and will reappear later in optimal control problems.

Given a homogeneous linear differential equation  $\dot{x}(t) = \mathbf{F}(t)x(t)$ , the associated “adjoint” differential equation is the differential equation for the  $n$ -vector  $p(t)$  such that the inner product of  $p(t)$  with  $x(t)$  is constant for all time:

$$x(t)^T p(t) = \text{const}$$

- (a) Take the derivative of this expression to show that the adjoint equation associated with  $\dot{x}(t) = \mathbf{F}(t)x(t)$  is

$$\dot{p}(t) = -\mathbf{F}^T(t)p(t)$$

- (b) If  $\Phi_x(t, t_0)$  is the state transition matrix associated with  $\mathbf{F}(t)$  and  $\Phi_p(t, t_0)$  is the state transition matrix associated with  $[-\mathbf{F}^T(t)]$ , then show that

$$\Phi_p(t, t_0) = \Phi_x^T(t_0, t) = [\Phi_x^T(t, t_0)]^{-1}$$

To do this, show that  $[\Phi_p^T(t, t_0)\Phi_x(t, t_0)]$  and  $\mathbf{I}$  satisfy the same differential equation and initial condition.

- (c) Show that, as a function of its second argument,  $\Phi_x(t, \tau)$  must satisfy

$$\partial[\Phi_x(t, \tau)]/\partial\tau = -\Phi_x(t, \tau)\mathbf{F}(\tau)$$

or, in other words,

$$\partial[\Phi_x^T(t, \tau)]/\partial\tau = [-\mathbf{F}(t)^T]\Phi_x^T(t, \tau)$$

- (d) For discrete-time models, consider the relationship

$$x(t_{i+1})^T p(t_{i+1}) = x(t_i)^T p(t_i)$$

and the homogeneous state difference equation

$$x(t_{i+1}) = \Phi(t_{i+1}, t_i)x(t_i)$$

to develop the associated adjoint difference equation.

(e) Adjoints can be understood in a more general linear operator sense as well. Let  $V$  and  $W$  be inner product spaces, i.e., linear vector spaces with inner products defined on them as  $\langle \cdot, \cdot \rangle_V$  and  $\langle \cdot, \cdot \rangle_W$ , respectively. Let  $\mathcal{A}$  be a linear operator (transformation, mapping, function, etc.) from  $V$  into  $W$ , i.e.,  $\mathcal{A}$  transforms any  $v \in V$  into a  $w \in W$  in a manner that can be represented by a simple matrix multiplication,  $w = Av$ . If there exists a transformation  $\mathcal{A}^*$  from  $W$  into  $V$  such that, for all  $v \in V$  and  $w \in W$ ,

$$\langle \mathcal{A}v, w \rangle_W = \langle v, \mathcal{A}^*w \rangle_V$$

then  $\mathcal{A}^*$  is called the adjoint of  $\mathcal{A}$ . Whenever  $\mathcal{A}^*$  exists, it is unique, linear, and has the property that if  $(\mathcal{A}^*)^*$  exists,  $(\mathcal{A}^*)^* = \mathcal{A}$ . For the previous part of this problem,  $V$  and  $W$  are both a finite dimensional Euclidean space  $R^n$ , and an appropriate inner product is

$$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle_{R^n} = \mathbf{x}_1^T \mathbf{x}_2$$

and  $\mathcal{A}^*$  is guaranteed to exist. For part (a), let  $\mathcal{A}\mathbf{x}(t) = \dot{\mathbf{x}}(t)$  and find  $\mathcal{A}^*$  such that

$$\langle \mathcal{A}\mathbf{x}(t), \mathbf{p}(t) \rangle_{R^n} = \langle \mathbf{x}(t), \mathcal{A}^*\mathbf{p}(t) \rangle_{R^n}$$

For part (d), let  $\mathcal{A}\mathbf{x}(t_i) = \mathbf{x}(t_{i+1})$  and find  $\mathcal{A}^*$  such that

$$\langle \mathcal{A}\mathbf{x}(t_i), \mathbf{p}(t_{i+1}) \rangle_{R^n} = \langle \mathbf{x}(t_i), \mathcal{A}^*\mathbf{p}(t_{i+1}) \rangle_{R^n}$$

# CHAPTER 9

## Compensation of linear model inadequacies

### 9.1 INTRODUCTION

Up to this point, we have assumed that linear models for the system dynamics and measurement relations are adequate for developing optimal estimators. No model is perfect, and it is especially true that a linear model is the result of either purposeful approximation and simplification or lack of knowledge about the system being modeled. Thus, although a linear model might depict the predominant aspects of a problem, it is always an erroneous model to some degree. Unfortunately, when an estimator is developed on the basis of such an erroneous model, it can “learn the wrong state too well [22]” when operated for a long enough period of time, especially if the strengths of noises in the assumed model are small. If  $\mathbf{Q}_d(t_i)$ , the strength of the dynamic driving noise in the model, has small eigenvalues for all time, the filter-computed error covariance matrix and gain matrix are correspondingly “small.” Thus, the filter’s estimate is highly dependent upon the output of its internal model and not affected dramatically by measurements coming in from the “real world.” Too heavy a reliance on the internal model can cause the state estimate and the “true state” values in the “real world” to diverge. In the extreme case, the filter gains are zeroed and the state estimator is totally divorced from the real world, simultaneously indicating through its computed error covariance matrix that ultimate confidence should be placed in the accuracy of its computed estimate.

There is always some discrepancy between the performance indication propagated by the filter (the computed  $\mathbf{P}(t_i^-)$  and  $\mathbf{P}(t_i^+)$ ) and the actual performance achieved in realistic applications, because the model embodied in the filter cannot be exactly correct. Such discrepancy is termed *divergence* [12–16, 18, 19, 22, 26, 39, 41, 47, 50–53, 58, 60]. In some instances, often denoted as “apparent divergence [14],” the true estimation errors are larger in magnitude than those indicated by the filter-propagated error covariance matrix, but the

true magnitudes remain bounded. Better filter performance could be achieved through better “tuning” of the filter algorithm, or some other form of model compensation, or through improved numerical precision. “True divergence [14]” is a more critical case in which the filter-computed error covariance remains bounded while the true estimation errors grow unbounded: the filter not only provides inadequate estimates, but is totally unaware of the existence of a problem. True divergence can be caused either by unmodeled or mismodeled effects, or by numerical instabilities. Numerical problems were treated in Chapter 7 (Volume 1), and here we address the problem of model inadequacy. Note that, in order for true divergence to be due to modeling problems, the true system must exhibit a behavior of certain quantities growing without bound (such as position error in an inertial navigation system), since the estimator algorithm itself will be stable if its internal model is stochastically observable and controllable—a nonrestrictive assumption.

This chapter seeks to compensate for the inadequacies of an assumed linear model, thereby exploiting linear estimation concepts and insights as much as possible, rather than to dispose of them entirely in preference of full-scale nonlinear estimation. Compensation techniques that are described in the following sections are:

- (1) addition of pseudonoise to the assumed model and artificial lower bounding of error covariance matrix elements,
- (2) limiting of effective filter memory and overweighting most recent data,
- (3) finite memory filtering,
- (4) linearized Kalman filtering,
- (5) extended Kalman filtering.

The first of these methods entails telling the filter that it should decrease its confidence in its own model. The second and third are concerned with the case in which a linear model is indeed adequate, but only for a limited length of time propagation. Finally, the last two attempt to exploit linear models and methods in the case in which a substantially better depiction of the true system would be in the form of a nonlinear model.

The special case of state estimation in the face of uncertain parameters in a linear dynamics model and linear measurement model, or in the statistical description of noises entering the system model, will be reserved for the following chapter. Adaptive estimators will also be discussed in general in that chapter. Full-scale nonlinear models and estimation will be developed in detail in ensuing chapters.

## 9.2 PSEUDONOISE ADDITION AND ARTIFICIAL LOWER BOUNDING OF $\mathbf{P}$

*Addition of pseudonoise* to the dynamics model, as by increasing the elements of  $\mathbf{Q}_d(t_i)$  for all time, was previously discussed in the context of tuning a simplified, reduced order Kalman filter [30]. Essentially, the dominant (linear)

aspects of the dynamics are included in the model, and one accounts for the many neglected effects by introducing additional uncertainty into the model. By adding such fictitious noise, or pseudonoise, to the dynamics model, one “tells” the filter that it cannot neglect the incompleteness or inadequacy of the representation that its internal model provides of the true system.

**EXAMPLE 9.1** In Sections 4.11 and 6.5 (Volume 1), alternative models of a bias of  $b(t) = 0$  and  $b(t) = w(t)$ , with  $w(\cdot, \cdot)$  a zero-mean white Gaussian noise, were discussed and resulting filter performance depicted. The latter model can be interpreted as the original  $b(t) = 0$ , but with pseudonoise added to reflect a conviction that this basic model is not totally adequate.

Without the pseudonoise addition, the filter would compute a variance of the error in estimating bias that converged to zero. Simultaneously, the filter gain on that state channel would be zeroed, thereby precluding use of further measurement data to maintain a viable estimate. Consequently, if the “bias” parameter were in fact slowly varying, or if it underwent a sudden change of value (failure, etc.), the true estimation error might grow to a nonnegligible magnitude. This problem context will be repeated in examples throughout this chapter. ■

Pseudonoise can similarly be added to the measurement relations to reflect uncertainty in the adequacy of the assumed measurement model as well.

**EXAMPLE 9.2** Suppose that an adequate model for a measurement of some state  $x_k$  were

$$z(t_i) = x_k(t_i) + b(t_i) + n(t_i) + v(t_i)$$

where  $b(\cdot, \cdot)$  is a bias,  $n(\cdot, \cdot)$  is a time-correlated noise, and  $v(\cdot, \cdot)$  is a white Gaussian noise of strength  $R(t_i)$  for all  $t_i$ . If a filter were to be based on a reduced-order model that eliminated  $b$  and  $n$  as state variables, the measurement model would become

$$z(t_i) = x_k(t_i) + v(t_i)$$

The strength  $R(t_i)$  should be increased to account for these neglected effects. ■

Many approaches have been suggested as “systematic” means of adding pseudonoises [12, 14, 22, 52]. However, most are basically ad hoc procedures that require iterative adjustment (guided by physical insights where possible) until the filter performance observed in a performance sensitivity analysis (see Chapter 6, Volume 1) is acceptable.

A somewhat more formal result is provided by the concept of a *minimum variance reduced order (MVRO) estimator* [4, 20, 62]. Suppose that a truth model description of a system were

$$\mathbf{x}_t(t_{i+1}) = \Phi_t(t_{i+1}, t_i)\mathbf{x}_t(t_i) + \mathbf{w}_{dt}(t_i) \quad (9-1a)$$

$$\mathbf{z}_t(t_i) = \mathbf{H}_t(t_i)\mathbf{x}_t(t_i) + \mathbf{v}_t(t_i) \quad (9-1b)$$

where  $\mathbf{x}_t$  is of dimension  $n_t$ . Assume we want to develop an  $n$ -dimensional filter in terms of states

$$\mathbf{x}(t_i) = \mathbf{C}_t(t_i)\mathbf{x}_t(t_i) \quad (9-2)$$

(in terms of the notation of Chapter 6, the desired quantities are the entire filter state, so that  $\mathbf{y}(t_i) = \mathbf{x}(t_i)\mathbf{C}(t_i) = \mathbf{I}$ ). An estimator is chosen to be of the

form

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i)[\mathbf{z}_t(t_i) - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)] \quad (9-3a)$$

$$\hat{\mathbf{x}}(t_{i+1}^-) = \Phi(t_{i+1}, t_i)\hat{\mathbf{x}}(t_i^+) \quad (9-3b)$$

where  $\mathbf{K}$ ,  $\mathbf{H}$ , and  $\Phi$  are to be chosen so as to minimize the mean square error function  $\text{tr}[E\{\mathbf{e}_t(t_i^+)\mathbf{e}_t^T(t_i^+)\}]$  for all sample times  $t_i$ , where the error  $\mathbf{e}_t(t_i^+)$  is given by

$$\mathbf{e}_t(t_i^+) = \hat{\mathbf{x}}(t_i^+) - \mathbf{C}_t(t_i)\mathbf{x}_t(t_i) \quad (9-4)$$

Generation of  $E\{\mathbf{e}_t(t_i^+)\mathbf{e}_t^T(t_i^+)\}$  by a set of expressions as in the covariance analyses of Chapter 6 and differentiation with respect to  $\mathbf{K}$ ,  $\mathbf{H}$ , and  $\Phi$  then yield the general MVRO estimator. Constrained MVRO estimators, in which  $\mathbf{H}$  and  $\Phi$  are selected by the designer and optimization is performed only with respect to  $\mathbf{K}$ , or in which additional assumptions are imposed to simplify the resulting estimator form, have also been applied in practice.

**EXAMPLE 9.3** A constrained MVRO estimator of practical importance is one in which  $\mathbf{C}_t$  is of the form  $[\mathbf{I} : \mathbf{0}]$ , such that the estimator state forms the first  $n$  variables in the truth model state. Thus, the truth model becomes

$$\begin{bmatrix} \mathbf{x}_{t1}(t_{i+1}) \\ \mathbf{x}_{t2}(t_{i+1}) \end{bmatrix} = \begin{bmatrix} \Phi_{t11}(t_{i+1}, t_i) & | & \Phi_{t12}(t_{i+1}, t_i) \\ \Phi_{t21}(t_{i+1}, t_i) & | & \Phi_{t22}(t_{i+1}, t_i) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t1}(t_i) \\ \mathbf{x}_{t2}(t_i) \end{bmatrix} + \begin{bmatrix} \mathbf{w}_{dt1}(t_i) \\ \mathbf{w}_{dt2}(t_i) \end{bmatrix}$$

$$\mathbf{z}_t(t_i) = [\mathbf{H}_{t1}(t_i) : \mathbf{H}_{t2}(t_i)] \begin{bmatrix} \mathbf{x}_{t1}(t_i) \\ \mathbf{x}_{t2}(t_i) \end{bmatrix} + \mathbf{v}_t(t_i)$$

and the estimator provides estimates of  $\mathbf{x}_{t1}$ . A further constraint is imposed that impulsive feedback correction of all estimated variables is employed. Under these assumptions, a covariance analysis can be performed using the  $n_t$ -dimensional  $[(\mathbf{x}_{t1} - \hat{\mathbf{x}})^T : \mathbf{x}_{t2}^T]^T$  instead of the  $(n_t + n)$ -dimensional vector  $[\mathbf{x}_t^T : \hat{\mathbf{x}}^T]^T$  as in Chapter 6, and the optimal choice of  $\mathbf{H}$  and  $\Phi$  in the filter are  $\mathbf{H}_{t1}$  and  $\Phi_{t11}$ , respectively. The vector  $[(\mathbf{x}_{t1} - \hat{\mathbf{x}})^T : \mathbf{x}_{t2}^T]^T$  satisfies the same dynamics as  $[\mathbf{x}_{t1}^T : \mathbf{x}_{t2}^T]^T$  and updates according to

$$\begin{bmatrix} (\mathbf{x}_{t1} - \hat{\mathbf{x}})(t_i^+) \\ \mathbf{x}_{t2}(t_i^+) \end{bmatrix} = \begin{bmatrix} (\mathbf{I} - \mathbf{K}\mathbf{H}_{t1}) & | & \mathbf{K}\mathbf{H}_{t2} \\ \mathbf{0} & | & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{x}_{t1} - \hat{\mathbf{x}})(t_i^-) \\ \mathbf{x}_{t2}(t_i^-) \end{bmatrix} + \begin{bmatrix} -\mathbf{K}\mathbf{v}_t \\ \mathbf{0} \end{bmatrix}$$

Letting  $\mathbf{P}_t$  be the associated covariance matrix, partitioned as

$$\mathbf{P}_t = \begin{bmatrix} \mathbf{P}_{t111} & | & \mathbf{P}_{t112} \\ \mathbf{P}_{t112}^T & | & \mathbf{P}_{t222} \end{bmatrix}$$

the MVRO estimator gain, found via  $\partial\{\text{tr}\mathbf{P}_{t111}^+\}/\partial\mathbf{K} = \mathbf{0}$ , is

$$\mathbf{K} = [\mathbf{I} : \mathbf{0}]\mathbf{P}_t^{-1}\mathbf{H}_t^T\{\mathbf{H}_t\mathbf{P}_t^{-1}\mathbf{H}_t^T + \mathbf{R}_t\}^{-1}$$

$$\begin{aligned} &= [\mathbf{P}_{t111}^{-1} : \mathbf{P}_{t112}^{-1}] \begin{bmatrix} \mathbf{H}_{t1}^T \\ \mathbf{H}_{t2}^T \end{bmatrix} \left\{ [\mathbf{H}_{t1} : \mathbf{H}_{t2}] \begin{bmatrix} \mathbf{P}_{t111}^{-1} & | & \mathbf{P}_{t112}^{-1} \\ \mathbf{P}_{t112}^{-1} & | & \mathbf{P}_{t222}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{H}_{t1}^T \\ \mathbf{H}_{t2}^T \end{bmatrix} + \mathbf{R}_t \right\}^{-1} \\ &= \mathbf{P}_{t111}^{-1}\mathbf{H}_{t1}^T\{\mathbf{H}_{t1}\mathbf{P}_{t111}^{-1}\mathbf{H}_{t1}^T + \mathbf{R}_t + \mathbf{R}'\}^{-1} + \mathbf{K}' \end{aligned}$$

where

$$\begin{aligned}\mathbf{R}' &= \mathbf{H}_{t1} \mathbf{P}_{t12}^{-} \mathbf{H}_{t2}^T + \mathbf{H}_{t2} \mathbf{P}_{t12}^{-T} \mathbf{H}_{t1}^T + \mathbf{H}_{t2} \mathbf{P}_{t22}^{-} \mathbf{H}_{t2}^T \\ \mathbf{K}' &= \mathbf{P}_{t12}^{-} \mathbf{H}_{t2}^T \{ \mathbf{H}_{t1} \mathbf{P}_{t11}^{-} \mathbf{H}_{t1}^T + \mathbf{R}_t + \mathbf{R}' \}^{-1}\end{aligned}$$

From the dynamics equation for  $[(\mathbf{x}_{t1} - \hat{\mathbf{x}})^T : \mathbf{x}_t^T]^T$ , the covariance  $\mathbf{P}_t$  propagates according to

$$\mathbf{P}_t(t_{i+1}) = \Phi_{t(i+1, t_i)} \mathbf{P}_t(t_i^+) \Phi_{t(i+1, t_i)}^T + \mathbf{Q}_{dt}(t_i)$$

from which the upper left partition can be extracted as

$$\begin{aligned}\mathbf{P}_{t11}(t_{i+1}^-) &= \Phi_{t11}(t_{i+1}, t_i) \mathbf{P}_{t11}(t_i^+) \Phi_{t11}^T(t_{i+1}, t_i) + \mathbf{Q}_{dt11} + \mathbf{Q}_d' \\ \mathbf{Q}_d' &= \Phi_{t11} \mathbf{P}_{t12}^+ \Phi_{t12}^T + \Phi_{t12} \mathbf{P}_{t12}^{+T} \Phi_{t11}^T + \Phi_{t12} \mathbf{P}_{t22}^T \Phi_{t12}^T\end{aligned}$$

Such an MVRO estimator design was conducted in the design of the reduced-order navigation Kalman filter (in U-D covariance factorization form) for the Offensive Avionics System (OAS) intended for Air Force operational use in the 1980s [62]. ■

As seen in the preceding example, if  $\mathbf{C}_t$  in (9-2) is of the form  $[\mathbf{I} : \mathbf{0}]$ , then the MVRO estimator has a structure of the usual Kalman filter state equations and covariance update equations, but with modified covariance propagation and gain relations as

$$\mathbf{K}(t_i) = \mathbf{P}(t_i^-) \mathbf{H}^T(t_i) [\mathbf{H}(t_i) \mathbf{P}(t_i^-) \mathbf{H}^T(t_i) + \mathbf{R}_t(t_i) + \mathbf{R}'(t_i)]^{-1} + \mathbf{K}'(t_i) \quad (9-5a)$$

$$\mathbf{P}(t_{i+1}^-, t_i) = \Phi(t_{i+1}) \mathbf{P}(t_i^+) \Phi^T(t_{i+1}, t_i) + [\mathbf{C}_t \mathbf{Q}_{dt}(t_i) \mathbf{C}_t^T] + \mathbf{Q}_d'(t_i) \quad (9-5b)$$

where the structure of  $\mathbf{R}'(t_i)$  and  $\mathbf{Q}_d'(t_i)$  to describe appropriate *pseudonoise addition* and the structure of  $\mathbf{K}'(t_i)$  to delineate additional appropriate *filter gain compensation* are completely defined by the MVRO estimator solution. These results provide a design goal for performance of a compensated filter of given reduced-order dimension and state selection, a more realistic design goal than that provided by the full-order Kalman filter based upon the truth model. Since  $\mathbf{P}_{t12}$  and  $\mathbf{P}_{t22}$  are not computed explicitly by the online filter, the MVRO estimator result can be used either for implementations employing precomputed gains or for insights into the structure of online approximations to  $\mathbf{R}'$ ,  $\mathbf{Q}_d'$ , and  $\mathbf{K}'$ .

Thus, complete compensation for unmodeled errors is seen to require both pseudonoise addition and additional gain compensation. The structure of an added gain compensation will be seen again in Section 9.3, where the Schmidt epsilon technique is discussed.

A technique closely related to pseudonoise addition is the *artificial lower bounding of computed error covariance matrix elements*. If the calculated value of  $\mathbf{P}(t_i^+)$  embodies very small entries (or eigenvalues) then this method artificially increases these entries before time propagation rather than artificially increase the elements of  $\mathbf{Q}_d(t_i)$  for the propagation itself as above. The overall effect, though, is essentially the same: subsequent values of  $\mathbf{P}(t_j^-)$ ,  $\mathbf{P}(t_j^+)$ , and  $\mathbf{K}(t_j)$  are “larger,” and the filter places less relative weighting upon its own internal model. One means of implementing this technique is to preselect the

$n$  minimum admissible values for the diagonal terms of  $\mathbf{P}^+$ , denoted as  $\sigma_{1-\min}^2, \sigma_{2-\min}^2, \dots, \sigma_{n-\min}^2$ , and also a maximum correlation coefficient  $r_{\max}$ . In online filter operation, the standard calculation of  $\mathbf{P}(t_i^+)$  is performed, followed immediately by

$$P_{jj}^*(t_i^+) = \max\{P_{jj}(t_i^+), \sigma_{j-\min}^2\} \quad (9-6a)$$

for  $j = 1, 2, \dots, n$ . In other words, if the computed value falls below  $\sigma_{j-\min}^2$ , it is artificially increased to the minimum admissible value. The off-diagonal terms of  $\mathbf{P}^*(t_i^+)$  are then generated from

$$M_{jk} = r_{\max}^2 P_{jj}^*(t_i^+) P_{kk}^*(t_i^+) \quad (9-6b)$$

$$P_{jk}^*(t_i^+) = \begin{cases} P_{jk}(t_i) & \text{if } |P_{jk}(t_i^+)| < M_{jk}(t_i) \\ \text{sgn}[P_{jk}(t_i^+)] \sqrt{M_{jk}(t_i)} & \text{otherwise} \end{cases} \quad (9-6c)$$

The  $\mathbf{P}^*(t_i^+)$  matrix is then used in place of  $\mathbf{P}(t_i^+)$  in the filter. Although this does prevent diagonal terms from going too small (or negative, due to numerics) and the correlations from becoming too large (yielding potential singularity problems), this is totally an ad hoc procedure, and  $\mathbf{P}(t_i^+)$  generally does not provide any form of optimal approximation to the true estimation error covariance.

It should be noted that both of these methods are used to circumvent numerical difficulties in a Kalman filter as well as to account for modeling shortcomings.

### 9.3 LIMITING EFFECTIVE FILTER MEMORY AND OVERWEIGHTING MOST RECENT DATA

Suppose we believe that a given linear system is adequate over certain lengths of time, but we have low confidence that the same model accurately portrays propagations over long time intervals. For instance, a linear perturbation model might be adequate for a satellite orbit determination problem while the satellite is in view and telemetry data is being processed continuously, but the same model might be inadequate for propagating between intermittent data and will probably be grossly unsatisfactory for propagating between the time the satellite goes below the horizon and when it reappears on the opposite horizon. Conceptually, we would like to eliminate the effect of older data from a current state estimate if that data is thought to be no longer meaningful, due to the erroneous system model degrading the information in measurements from the distant past.

One means of limiting effective memory length is called *exponential age-weighting of data* [2, 9, 14, 17, 22, 36, 37, 45, 46, 49, 59, 61], as developed originally by Fagin [9]. Basically, as one moves along in time, the assumed strengths of measurement corruption noises for prior measurements are arti-

ficially increased before their influence is brought to bear on the current state estimate. Suppose you are at time  $t_i$ , and a conventional Kalman filter algorithm would provide a state estimate based on a sequence of measurement noise strengths  $\mathbf{R}(t_1), \mathbf{R}(t_2), \dots, \mathbf{R}(t_i)$ . Let  $s$  be some real number greater than one, and generate a new sequence of assumed noise strengths using

$$\mathbf{R}_{i\text{-aged}}(t_j) = s^{(i-j)} \mathbf{R}(t_j) \quad (9-7)$$

for  $j = 1, 2, \dots, i$ . It is convenient to think of the factor  $s$  as

$$s = e^{\Delta t / T_a} \quad (9-8)$$

where  $\Delta t$  is the (constant) measurement sample period ( $t_{i+1} - t_i$ ), and  $T_a$  is the exponential age-weighting time constant: the smaller the value of  $T_a$ , the faster the prior data will be “aged,” and thus “forgotten” by the resulting filter.

Figure 9.1 portrays the result of this technique applied to the case of stationary noise statistics,  $\mathbf{R}(t_j) = \mathbf{R}$  for all  $t_j$ . At time  $t_i$ , the modified noise strengths

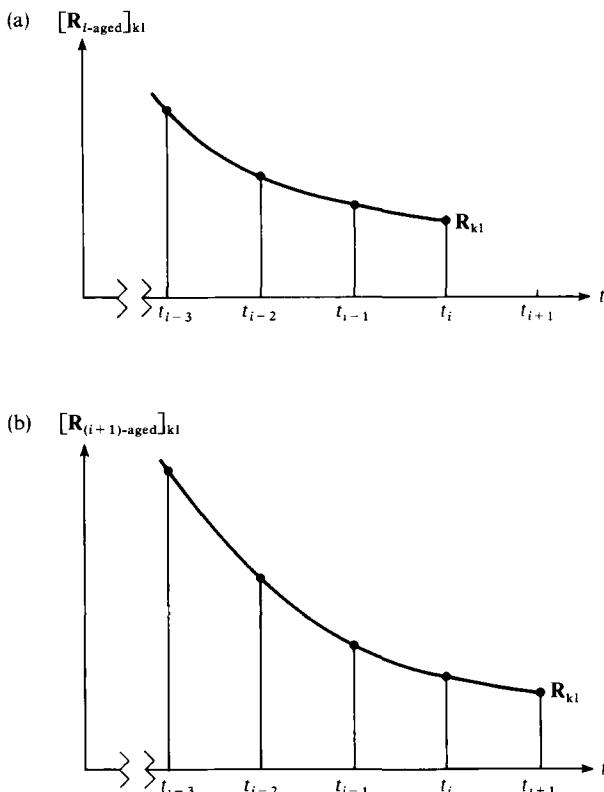


FIG. 9.1 Computed  $\mathbf{R}_{i\text{-aged}}(t_j)$  and  $\mathbf{R}_{(i+1)\text{-aged}}(t_j)$  values. (a) At time  $t_i$ . (b) At time  $t_{i+1}$ .

would be

$$\mathbf{R}_{i\text{-aged}}(t_j) = e^{(i-j)\Delta t/T_a} \mathbf{R} \quad (9-9a)$$

for  $j = 1, 2, \dots, i$ , as in plot (a). The estimate of  $\mathbf{x}(t_i)$  would then be generated, conditioned on the measurement history  $\mathbf{Z}_i$  using this description of corruptive noises. At the next measurement update time  $t_{i+1}$ , a new computation of noise strengths would yield

$$\mathbf{R}_{(i+1)\text{-aged}}(t_j) = e^{(i+1-j)\Delta t/T_a} \mathbf{R} \quad (9-9b)$$

for  $j = 1, 2, \dots, (i+1)$ . The estimate of  $\mathbf{x}(t_{i+1})$  would then be based upon the same measurement history  $\mathbf{Z}_i$  plus the additional  $\mathbf{z}_{i+1}$ , but with a totally different assumed history of noise strengths.

Such a concept might at first seem to preclude a simple iterative algorithm. However, if the optimal estimates of  $\mathbf{x}(t_i)$  and  $\mathbf{x}(t_{i+1})$  are generated independently and compared, it can be shown that a recursion almost identical to the conventional Kalman filter can be used to implement the desired estimator. For propagating between measurement times, the computations are

$$\hat{\mathbf{x}}(t_{i+1}^-) = \Phi(t_{i+1}, t_i) \hat{\mathbf{x}}(t_i^+) + \mathbf{B}_d(t_i) \mathbf{u}(t_i) \quad (9-10a)$$

$$\bar{\mathbf{P}}(t_{i+1}^-) = s\Phi(t_{i+1}, t_i) \bar{\mathbf{P}}(t_i^+) \Phi^T(t_{i+1}, t_i) + \mathbf{G}_d(t_i) \mathbf{Q}_d(t_i) \mathbf{G}_d^T(t_i) \quad (9-10b)$$

and for measurement updating,

$$\mathbf{K}(t_i) = \bar{\mathbf{P}}(t_i^-) \mathbf{H}^T(t_i) [\mathbf{H}(t_i) \bar{\mathbf{P}}(t_i^-) \mathbf{H}^T(t_i) + \mathbf{R}(t_i)]^{-1} \quad (9-11a)$$

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i) [\mathbf{z}_i - \mathbf{H}(t_i) \hat{\mathbf{x}}(t_i^-)] \quad (9-11b)$$

$$\bar{\mathbf{P}}(t_i^+) = \bar{\mathbf{P}}(t_i^-) - \mathbf{K}(t_i) \mathbf{H}(t_i) \bar{\mathbf{P}}(t_i^-) \quad (9-11c)$$

These relations are identical in form to those of a conventional Kalman filter, except for the age-weighting factor in the time propagation equation (9-10b). However, the notation  $\bar{\mathbf{P}}$  instead of  $\mathbf{P}$  has been adopted purposely, because it can be shown that  $\bar{\mathbf{P}}$  in these relations is not in general the estimation error covariance matrix.

**EXAMPLE 9.4** Recall the bias estimation problem previously considered in Example 9.1. Assume that discrete-time measurements of some unknown constant (bias) are available in the form of

$$\mathbf{z}(t_i) = b + \mathbf{v}(t_i)$$

where  $b$  is the bias value and  $\mathbf{v}(\cdot, \cdot)$  is a zero-mean white Gaussian noise of strength  $R$ . Propose a bias model as

$$\dot{b}(t) = 0$$

where  $b(t_0)$  is assumed to be a Gaussian random variable with mean  $b_0$  and variance  $P_{b0}$ . Thus, for using (9-10) and (9-11), we identify

$$\Phi = 1, \quad \mathbf{B}_d = 0, \quad \mathbf{H} = 1, \quad \mathbf{G}_d \mathbf{Q}_d \mathbf{G}_d^T = 0, \quad \mathbf{R} = R$$

In the previous example, the gain of a conventional filter based on such a model was shown to converge to a steady state value of zero. Here we have, from (9-10) and (9-11),

$$\begin{aligned}\bar{P}(t_{i+1}^-) &= s\bar{P}(t_i^+) \\ \bar{P}(t_{i+1}^+) &= \bar{P}(t_i^-) - \frac{\bar{P}(t_i^-)^2}{\bar{P}(t_i^-) + R} \\ &= s\bar{P}(t_i^+) - \frac{s^2\bar{P}^2(t_i^+)}{s\bar{P}(t_i^+) + R} \\ &= \frac{s\bar{P}(t_i^+)R}{s\bar{P}(t_i^+) + R}\end{aligned}$$

For steady state performance, we solve  $\bar{P}(t_{i+1}^+) = \bar{P}(t_i^+)$ , which yields

$$\begin{aligned}R\bar{P} + s\bar{P}^2 &= s\bar{P}R \rightarrow s\bar{P}^2 + R\bar{P}(1-s) = 0 \\ \rightarrow \bar{P} &= R(1-1/s)\end{aligned}$$

Thus, the steady state gain is

$$K_{ss} = \frac{s\bar{P}}{s\bar{P} + R} = \frac{sR(1-1/s)}{sR(1-1/s) + R} = 1 - \frac{1}{s}$$

For any  $s > 1$ , i.e., for any finite  $T_a$  in (9-8),  $K_{ss}$  is greater than zero, as desired. ■

Other forms of “fading memory” filters have also been derived [37, 59] that are algebraically equivalent to (9-10) and (9-11). Moreover, matrix weighting factors have been considered in addition to scalar factors in order to provide different rates of fading for different filter channels [2, 46].

A similar compensation method is the overweighting of the single most recent measurement, again in an effort to discount the effect of previous data information being propagated through an erroneous dynamics model. By overweighting the current measurement (or, underweighting prior data relatively), the values of  $\mathbf{P}$  and  $\mathbf{K}$  calculated by the filter are prevented from becoming unrealistically small. One means of accomplishing this objective for the case of scalar measurements is the *Schmidt epsilon technique* [14, 22, 26, 54, 55], developed to resolve some of the difficulties encountered in designing the Kalman filter for the C-5A aircraft navigation system.

Let us propose a modification to the state update equation as

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i)[z_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)] + \epsilon' \widehat{\Delta\mathbf{x}}(t_i^+) \quad (9-12)$$

where  $\widehat{\Delta\mathbf{x}}(t_i^+)$  is the estimate of  $[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^-)]$  based only on the measurement  $\mathbf{z}(t_i, \omega_j) = z_i$  and  $\epsilon'$  is a scalar scale factor to be determined. The least squares estimate of  $[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^-)]$  based only upon the scalar measurement residual  $[z_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]$  can be shown to be (with  $R^{-1}(t_i)$  chosen as the weighting factor in the cost function to be minimized):

$$\widehat{\Delta\mathbf{x}}(t_i^+) = [\mathbf{H}^T(t_i)R^{-1}(t_i)\mathbf{H}(t_i)]^\# \mathbf{H}^T(t_i)R^{-1}(t_i)[z_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)] \quad (9-13)$$

where  $\#$  denotes the Penrose pseudoinverse [43, 44]. Note that the  $n$ -by- $n$  matrix  $[\mathbf{H}^T(t_i)R^{-1}(t_i)\mathbf{H}(t_i)]$  is of rank one since  $\mathbf{H}(t_i)$  is 1-by- $n$ , and thus it does

not have an ordinary inverse. Moreover, in this case,

$$\begin{aligned} [\mathbf{H}^T(t_i)R^{-1}(t_i)\mathbf{H}(t_i)]^\# &= \mathbf{H}^\#(t_i)R^{-1\#}(t_i)\mathbf{H}^{\#T}(t_i) \\ &= \mathbf{H}^\#(t_i)R(t_i)\mathbf{H}^{\#T}(t_i) \end{aligned} \quad (9-14a)$$

where

$$\mathbf{H}^\#(t_i) = \frac{\mathbf{H}^T(t_i)}{\mathbf{H}(t_i)\mathbf{H}^T(t_i)} \quad (9-14b)$$

since then the 1-by- $n$   $\mathbf{H}(t_i)$  times the  $n$ -by-1  $\mathbf{H}^\#(t_i)$  yields the 1-by-1 identity, one, as would be desired of an “inverse” for  $\mathbf{H}(t_i)$ . Putting (9-14) into (9-13) produces

$$\widehat{\Delta \mathbf{x}}(t_i^+) = \frac{\mathbf{H}^T(t_i)}{\mathbf{H}(t_i)\mathbf{H}^T(t_i)} [z_i - \mathbf{H}(t_i)\widehat{\mathbf{x}}(t_i^-)] \quad (9-15)$$

Note that the vector measurement case does not admit the simple forms of (9-14b) and (9-15) because  $[\mathbf{H}(t_i)\mathbf{H}^T(t_i)]$  would no longer be a scalar.

Now substitute (9-15) into (9-12), defining the Schmidt  $\varepsilon$  parameter for convenience, as

$$\varepsilon = \varepsilon' \frac{\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + R(t_i)}{R(t_i)}$$

to write the modified update relation as

$$\widehat{\mathbf{x}}(t_i^+) = \widehat{\mathbf{x}}(t_i^-) + \left[ \frac{\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \varepsilon[R(t_i)\mathbf{H}^T(t_i)/\mathbf{H}(t_i)\mathbf{H}^T(t_i)]}{\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + R(t_i)} \right] [z_i - \mathbf{H}(t_i)\widehat{\mathbf{x}}(t_i^-)] \quad (9-16)$$

Thus, the effective gain of the modified filter is

$$\mathbf{K}_{\text{modified}}(t_i) = \mathbf{K}_{\text{Kalman}}(t_i) + \mathbf{K}_{\text{overweight}}(t_i) \quad (9-17)$$

where  $\mathbf{K}_{\text{overweight}}(t_i)$  is proportional to  $\varepsilon$ . The valid range of  $\varepsilon$  is from 0 to 1. When  $\varepsilon = 0$ , the conventional Kalman filter results. On the other hand, when  $\varepsilon = 1$ , premultiplying (9-16) by  $\mathbf{H}(t_i)$  yields

$$\mathbf{H}(t_i)\widehat{\mathbf{x}}(t_i^+) = \mathbf{H}(t_i)\widehat{\mathbf{x}}(t_i^-) + [1][z_i - \mathbf{H}(t_i)\widehat{\mathbf{x}}(t_i^-)] = z_i$$

In other words, the estimate  $\widehat{\mathbf{x}}(t_i^+)$  is computed such that  $\mathbf{H}(t_i)\widehat{\mathbf{x}}(t_i^+)$  is forced to agree exactly with the measurement data. The appropriate value for  $\varepsilon$  for a given application must be determined through a performance analysis. It can be shown that the corresponding covariance of the estimation error is

$$\mathbf{P}(t_i^+)_{\text{modified}} = \mathbf{P}(t_i^+)_{\text{Kalman}} + \frac{\varepsilon^2 R^2(t_i) \mathbf{H}^T(t_i) \mathbf{H}(t_i)}{[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + R(t_i)][\mathbf{H}(t_i)\mathbf{H}^T(t_i)]^2} \quad (9-18)$$

**EXAMPLE 9.5** Returning to the bias estimation problem of Example 9.1, the steady state gain of the filter would not converge to zero, but to

$$K_{ss} = 0 + \varepsilon[(R \cdot 1)/(1 \cdot 1)]/(1 \cdot 0 \cdot 1 + R) = \varepsilon \quad \blacksquare$$

## 9.4 FINITE MEMORY FILTERING

All estimators discussed up to this point are of the growing memory type:  $\hat{\mathbf{x}}(t_i^+)$  is an estimate of  $\mathbf{x}(t_i)$  based on the initial condition information and all measurements to time  $t_i$ ,  $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_i$ . As time progresses, the estimator “remembers” an ever-growing number of measurements. However, if a system model were thought to be valid over a time interval equal to  $N$  sample periods but inadequate for longer durations, it would be inappropriate to have such a growing memory. For example, slowly varying parameters might be modeled as “essentially constant over the most recent  $N$  sample periods,” whereas modeling them as constants for all time would mismodel true behavior. Thus, one might seek an estimate of  $\mathbf{x}(t_i)$  that is based only upon the most recent  $N$  measurements.

**EXAMPLE 9.6** Suppose a parameter of interest is modeled as a constant bias,

$$\mathbf{x}(t) = k = \text{const} \quad \text{for all } t$$

and it is desired to estimate its value from discrete-time measurements (with sample period  $\Delta t$ ), modeled as the parameter plus zero-mean white Gaussian noise:

$$\mathbf{z}(t_i) = \mathbf{x}(t_i) + \mathbf{v}(t_i)$$

One reasonable estimator would be a simple averager:

$$\hat{x}(t_i) = \frac{1}{i} \sum_{j=1}^i z_j$$

Now let the true parameter be best modeled as a bias plus a ramp,

$$x_i(t) = k + at$$

In this case, the average estimation error would be

$$E\{x_i(t_i) - \hat{x}(t_i)\} = E\{[k + at_i] - \frac{1}{i} \sum_{j=1}^i [k + at_j + v(t_j)]\} = \frac{1}{2}a\Delta t(i-1)$$

This is a quantity which grows without bound.

If, instead, the estimator averaged only the most recent  $N$  data points,

$$\hat{x}_N(t_i) = \begin{cases} \frac{1}{i} \sum_{j=1}^i z_j & i \leq N \\ \frac{1}{N} \sum_{j=i-N+1}^i z_j & i > N \end{cases}$$

then the mean error due to the erroneous model would be

$$E\{x_i(t_i) - \hat{x}_N(t_i)\} = \begin{cases} \frac{1}{2}a\Delta t(i-1) & i \leq N \\ \frac{1}{2}a\Delta t(N-1) & i > N \end{cases}$$

As shown in Fig. 9.2, this does not grow unbounded in time. ■

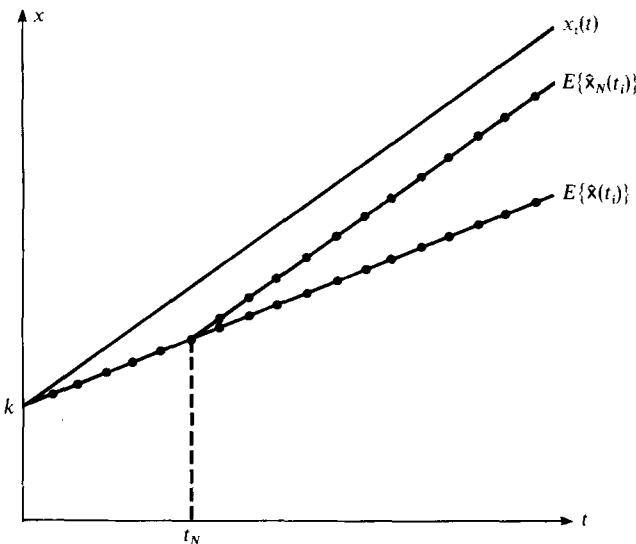


FIG. 9.2 True and estimated state values for Example 9.3.

This section develops finite memory filters, which provide estimates of the form

$$E(\mathbf{x}(t_i) | \mathbf{z}(t_{i-N+1}, \omega_j) = \mathbf{z}_{i-N+1}, \mathbf{z}(t_{i-N+2}, \omega_j) = \mathbf{z}_{i-N+2}, \dots, \mathbf{z}(t_i, \omega_j) = \mathbf{z}_i)$$

where  $N$  is some fixed integer chosen for a particular application [21, 22, 56, 57]. Other names for filters of this type are “limited memory filters,” and “sliding arc” or “sliding window” filters, since they involve a state estimate based on a sliding arc of the  $N$  most recent measurements, denoted as  $\hat{x}_N(t_i^+)$ .

As in the case of growing length filters, steady state operation is possible if the assumed model entails a time-invariant system and stationary noise statistics. However, a finite memory filter can exhibit instability problems that are not encountered in the corresponding growing memory filter (the system model must now be stochastically observable and controllable with respect to any time interval defined by  $N$  consecutive sample times, i.e., of length equal to  $(N-1)$  sample periods, for the filter to be stable) [22, 57].

A filter of this form can always be implemented by maintaining storage of the  $N$  most recent measurement values, and performing an  $N$ -step recursion ( $N$  updates,  $N-1$  propagations) of an inverse covariance filter (to allow algo-

rithm startup with no assumed “a priori” information about the state) each time the real system progresses one sample period. However, this can place a severe, if not prohibitive, computational load upon the computer. Consequently, an equivalent one-step recursion would be extremely desirable, in the general form [57] of

$$\hat{\mathbf{x}}_N(t_i^+) = \mathbf{C}_1(t_i)\hat{\mathbf{x}}_N(t_{i-1}^+) + \mathbf{C}_2(t_i)\mathbf{z}_i - \mathbf{C}_3(t_i)\mathbf{z}_{i-N} \quad (9-19)$$

To derive such a result, assume  $\hat{\mathbf{x}}_N(t_{i-1}^+)$  is known; note that it is based on  $\mathbf{z}_{i-N}, \mathbf{z}_{i-N+1}, \dots, \mathbf{z}_{i-1}$ . Now calculate  $\hat{\mathbf{x}}_{N+1}(t_i^+)$ , based on the  $N$  measurements just delineated plus  $\mathbf{z}_i$ , by means of a time propagation and measurement update, expressing the result in terms of  $\hat{\mathbf{x}}_N(t_{i-1}^+)$  and  $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$ . Second, assume  $\hat{\mathbf{x}}_N(t_i^+)$  is known, and calculate  $\hat{\mathbf{x}}_{N+1}(t_i^+)$  in terms of  $\hat{\mathbf{x}}_N(t_i^+)$  and  $\mathbf{z}(t_{i-N}, \omega_j) = \mathbf{z}_{i-N}$ . Finally, equate the two expressions for  $\hat{\mathbf{x}}_{N+1}(t_i^+)$  to obtain a recursive relationship of the form given by (9-19).

By looking at the inverse covariance form of the filter, this concept can be interpreted in an appealing manner. As one steps from time  $t_{i-1}$  to time  $t_i$ , the information in  $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$  is added to the propagated state estimate and the information in  $\mathbf{z}(t_{i-N}, \omega_j) = \mathbf{z}_{i-N}$  is subtracted from it, recalling that  $\mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i)$  is the “information” embodied in the current measurement. Thus, even though (9-19) is recursive, it is still necessary to store the most recent  $N$  measurements for processing, since knowledge of  $\mathbf{z}_{i-N}$  is required at time  $t_i$ .

Although (9-19) seems to be an attractive formulation, useful recursion relations of this structure have not been generated for the exact solution to the general case. In fact, the calculations of  $\mathbf{C}_1$ ,  $\mathbf{C}_2$ , and  $\mathbf{C}_3$  are so complex that the straightforward  $N$ -step recursion is preferable. However, in the particular case of no dynamic noise ( $\mathbf{Q}_d(t_i) \equiv \mathbf{0}$  for all  $t_i$ ), such a form can be derived, and these have found sufficient applications to warrant further study.

One form of the finite memory filter for  $\mathbf{Q}_d(t_i) \equiv \mathbf{0}$  can be derived [56, 57] as the maximum likelihood estimate that maximizes the likelihood function

$$L(\xi, \mathcal{L}_{i, i-N+1}) = \ln\{f_{\mathbf{z}(t_i), \dots, \mathbf{z}(t_{i-N+1})|\mathbf{x}(t_i)}(\xi_i, \dots, \xi_{i-N+1} | \xi)\} \quad (9-20)$$

This is, in fact, an appropriate choice of likelihood function to obtain an estimate of  $\mathbf{x}(t_i)$  based only on information contained in the most recent  $N$  measurements, with no “a priori” information about  $\mathbf{x}(t_{i-N+1})$ . For algebraic simplicity, the deterministic input will be neglected here, but then added later to the final result. By repeated application of Bayes’ rule, the density function in (9-20) can be expressed as

$$\begin{aligned} f_{\mathbf{z}(t_i), \dots, \mathbf{z}(t_{i-N+1})|\mathbf{x}(t_i)} &= f_{\mathbf{z}(t_i)|\mathbf{z}(t_{i-1}), \dots, \mathbf{z}(t_{i-N+1}), \mathbf{x}(t_i)} f_{\mathbf{z}(t_{i-1}), \dots, \mathbf{z}(t_{i-N+1})|\mathbf{x}(t_i)} \\ &\vdots \\ &= \prod_{j=i-N+1}^i f_{\mathbf{z}(t_j)|\mathbf{z}(t_{j-1}), \dots, \mathbf{z}(t_{i-N+1}), \mathbf{x}(t_i)} \end{aligned} \quad (9-21)$$

where the term evaluated for  $j = i - N + 1$  is  $f_{z(t_{i-N+1})|x(t_i)}(\zeta_{i-N+1} | \xi)$ . For the case of  $\mathbf{Q}_d(t_i) \equiv \mathbf{0}$ , conditioning on the realization of  $\mathbf{x}(t_i)$  completely determines the densities appearing in (9-21), independent of  $\mathbf{z}(t_{j-1}), \dots, \mathbf{z}(t_{i-N+1})$ :

$$\begin{aligned} & f_{z(t_j)|z(t_{j-1}), \dots, z(t_{i-N+1}), x(t_i)}(\zeta_j | \zeta_{j-1}, \dots, \zeta_{i-N+1}, \xi) \\ &= f_{z(t_j)|x(t_i)}(\zeta_j | \xi) \\ &= \frac{1}{(2\pi)^{m/2} |\mathbf{R}(t_j)|^{1/2}} \exp\left\{-\frac{1}{2} [\zeta_j - \mathbf{H}(t_j)\Phi(t_j, t_i)\xi]^T \mathbf{R}^{-1}(t_j) \right. \\ &\quad \times \left. [\zeta_j - \mathbf{H}(t_j)\Phi(t_j, t_i)\xi]\right\} \end{aligned} \quad (9-22)$$

Therefore, the likelihood function in (9-20) can be written as

$$\begin{aligned} L(\xi, \mathcal{L}_{i, i-N+1}) &= -\frac{Nm}{2} \ln(2\pi) - \frac{1}{2} \sum_{j=i-N+1}^i \ln |\mathbf{R}(t_j)| \\ &\quad - \frac{1}{2} \sum_{j=i-N+1}^i [\zeta_j - \mathbf{H}(t_j)\Phi(t_j, t_i)\xi]^T \mathbf{R}^{-1}(t_j) \\ &\quad \times [\zeta_j - \mathbf{H}(t_j)\Phi(t_j, t_i)\xi] \end{aligned} \quad (9-23)$$

The maximum likelihood estimate is generated as the solution to

$$\frac{\partial L(\xi, \mathcal{L}_{i, i-N+1})}{\partial \xi} \Bigg|_{\xi=\hat{\mathbf{x}}_N(t_i^+)} = \mathbf{0}^T \quad (9-24)$$

This yields

$$\hat{\mathbf{x}}_N(t_i^+) = \mathcal{I}^{-1}(t_i, t_{i-N+1}) \left[ \sum_{j=i-N+1}^i \Phi^T(t_j, t_i) \mathbf{H}^T(t_j) \mathbf{R}^{-1}(t_j) \zeta_j \right] \quad (9-25)$$

where  $\zeta_j$  is a realized value of  $\mathbf{z}(t_j)$  and  $\mathcal{I}(t_i, t_{i-N+1})$  is the  $N$ -step information matrix, defined as

$$\mathcal{I}(t_i, t_{i-N+1}) = \sum_{j=i-N+1}^i \Phi^T(t_j, t_i) \mathbf{H}^T(t_j) \mathbf{R}^{-1}(t_j) \mathbf{H}(t_j) \Phi(t_j, t_i) \quad (9-26)$$

This  $N$ -step information matrix is a measure of the certainty of the state estimate due only to the  $N$  most recent measurement values, analogous to the growing length information matrix  $\mathcal{I}(t_i, t_1)$  discussed in Section 5.7 (Volume 1). To be assured of the existence of  $\mathcal{I}^{-1}(t_i, t_{i-N+1})$  in (9-25),  $\mathcal{I}(t_i, t_{i-N+1})$  must be of full rank for all  $t_i$ : unlike  $\mathcal{I}(t_i, t_1)$ , the rank of  $\mathcal{I}(t_i, t_{i-N+1})$  is not necessarily a nondecreasing function of  $t_i$ . For the first  $N$  sample times,  $\mathcal{I}(t_i, t_{i-N+1})$  is replaced by  $\mathcal{I}(t_i, t_1)$ , and in general there is some minimum number of sample times greater than one before  $\mathcal{I}(t_i, t_1)$  is of rank  $n$ , since there are usually more states than measurements ( $n > m$ ).

A recursion can be developed for the  $N$ -step information matrix, for  $i > N$ , by writing  $\mathcal{J}(t_i, t_{i-N+1})$  and  $\mathcal{J}(t_{i-1}, t_{i-N})$  through (9-26) and equating like terms, to yield

$$\begin{aligned}\mathcal{J}(t_i, t_{i-N+1}) &= \Phi^T(t_{i-1}, t_i)\mathcal{J}(t_{i-1}, t_{i-N})\Phi(t_{i-1}, t_i) + \mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i) \\ &\quad - \Phi^T(t_{i-N}, t_i)\mathbf{H}^T(t_{i-N})\mathbf{R}^{-1}(t_{i-N})\mathbf{H}(t_{i-N})\Phi(t_{i-N}, t_i)\end{aligned}\quad (9-27)$$

Thus, to obtain  $\mathcal{J}(t_i, t_{i-N+1})$  from the one-step propagated  $\mathcal{J}(t_{i-1}, t_{i-N})$ , add the information obtained from  $\mathbf{z}(t_i)$  and subtract the information due to  $\mathbf{z}(t_{i-N})$ . Similar algebraic manipulation of (9-25) yields a recursive state estimate relation as

$$\begin{aligned}\hat{\mathbf{x}}_N(t_i^+) &= \Phi(t_i, t_{i-1})\hat{\mathbf{x}}_N(t_{i-1}^+) + \mathcal{J}^{-1}(t_i, t_{i-N+1}) \\ &\quad \times \{\mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)[\mathbf{z}_i - \mathbf{H}(t_i)\Phi(t_i, t_{i-1})\hat{\mathbf{x}}_N(t_{i-1}^+)] \\ &\quad - \Phi^T(t_{i-N}, t_i)\mathbf{H}^T(t_{i-N})\mathbf{R}^{-1}(t_{i-N}) \\ &\quad \times [\mathbf{z}_{i-N} - \mathbf{H}(t_{i-N})\Phi(t_{i-N}, t_{i-1})\hat{\mathbf{x}}_N(t_{i-1}^+)]\}\end{aligned}\quad (9-28)$$

In (9-28),  $\{\mathbf{H}(t_i)\Phi(t_i, t_{i-1})\hat{\mathbf{x}}_N(t_{i-1}^+)\}$  is the optimal prediction of  $\mathbf{z}_i$  before that measurement is taken, as propagated from  $\hat{\mathbf{x}}_N(t_{i-1}^+)$ , and  $\{\mathbf{H}(t_{i-N})\Phi(t_{i-N}, t_{i-1})\hat{\mathbf{x}}_N(t_{i-1}^+)\}$  is similarly the best estimate (“backward prediction”) of  $\mathbf{z}_{i-N}$  as generated from  $\hat{\mathbf{x}}_N(t_{i-1}^+)$ . The two bracketed terms in (9-28) are thus residuals at time  $t_i$  and  $t_{i-N}$  as propagated from the best state estimate at time  $t_{i-1}$ . Allowing deterministic control inputs extends (9-28) to

$$\hat{\mathbf{x}}_N(t_i^-) = \Phi(t_i, t_{i-1})\hat{\mathbf{x}}_N(t_{i-1}^+) + \mathbf{B}_d(t_{i-1})\mathbf{u}(t_{i-1}) \quad (9-29a)$$

$$\hat{\mathbf{x}}_N(t_{i-N}/t_{i-1}) = \Phi(t_{i-N}, t_{i-1})\hat{\mathbf{x}}_N(t_{i-1}^+) - \sum_{j=i-N+1}^{i-1} \Phi(t_{i-N}, t_j)\mathbf{B}_d(t_{j-1})\mathbf{u}(t_{j-1}) \quad (9-29b)$$

$$\begin{aligned}\hat{\mathbf{x}}_N(t_i^+) &= \hat{\mathbf{x}}_N(t_i^-) + \mathcal{J}^{-1}(t_i, t_{i-N+1})\{\mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}_N(t_i^-)] \\ &\quad - \Phi^T(t_{i-N}, t_i)\mathbf{H}^T(t_{i-N})\mathbf{R}^{-1}(t_{i-N})[\mathbf{z}_{i-N} - \mathbf{H}(t_{i-N})\hat{\mathbf{x}}_N(t_{i-N}/t_{i-1})]\}\end{aligned}\quad (9-29c)$$

Thus, besides requiring storage of the most recent  $N$  measurement values, the  $\mathbf{H}$ ,  $\mathbf{R}^{-1}$ ,  $\Phi$ ,  $\mathbf{B}_d$ , and  $\mathbf{u}$  values must also be stored and an  $n$ -by- $n$  matrix inversion performed each iteration, a significant computer burden.

Another formulation [21, 22] of the finite memory filter for  $\mathbf{Q}_d(t_i) \equiv \mathbf{0}$  is given by

$$\hat{\mathbf{x}}_N(t_i^+) = \mathbf{P}_N(t_i^+)[\mathbf{P}^{-1}(t_i^+)\hat{\mathbf{x}}(t_i^+) - \mathbf{P}^{-1}(t_i/t_{i-N})\hat{\mathbf{x}}(t_i/t_{i-N})] \quad (9-30a)$$

$$\mathbf{P}_N^{-1}(t_i^+) = \mathbf{P}^{-1}(t_i^+) - \mathbf{P}^{-1}(t_i/t_{i-N}) \quad (9-30b)$$

where  $\hat{\mathbf{x}}(t_i^+)$  and  $\mathbf{P}(t_i^+)$  are the state estimate and error covariance of a conventional growing-memory filter, and  $\hat{\mathbf{x}}(t_i/t_{i-N})$  and  $\mathbf{P}(t_i/t_{i-N})$  are the optimal

prediction of  $\mathbf{x}(t_i)$  based on the measurements up through  $\mathbf{z}(t_{i-N}, \omega_j) = \mathbf{z}_{i-N}$  and covariance of the error  $[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i/t_{i-N})]$ , given by

$$\hat{\mathbf{x}}(t_i/t_{i-N}) = \Phi(t_i, t_{i-N})\hat{\mathbf{x}}(t_{i-N}^+) + \sum_{j=i-N+1}^i \Phi(t_i, t_j)\mathbf{B}_d(t_{j-1})\mathbf{u}(t_{j-1}) \quad (9-31a)$$

$$\mathbf{P}(t_i/t_{i-N}) = \Phi(t_i, t_{i-N})\mathbf{P}(t_{i-N}^+)\Phi^T(t_i, t_{i-N}) \quad (9-31b)$$

For (9-31), the values of  $\hat{\mathbf{x}}(t_{i-N}^+)$  and  $\mathbf{P}(t_{i-N}^+)$  could be obtained either from storage of these values from the conventional filter or without storage from a second conventional filter running simultaneously. This result is expressed as the weighted difference of two growing-memory filter estimates, the structure being readily interpreted in a manner analogous to that used for the smoother results, (8-11) and (8-12), of the previous chapter. Unfortunately, such a structure embodies two growing-length memory filters, each of these being the very set of erroneous and potentially divergent calculations we seek to avoid through finite memory filtering.

Nevertheless, this form provides insights into a practical approximate method [21, 22] that substantially reduces the computational and storage burden of finite memory filters for the case of  $\mathbf{Q}_d(t_i) \equiv \mathbf{0}$ . In this method, old data is eliminated in batches of  $N$ , so that the memory length varies between  $N$  and  $2N$  measurements. The procedure starts by iterating a conventional Kalman filter algorithm  $N$  times to generate  $\hat{\mathbf{x}}(t_N^+)$  and  $\mathbf{P}(t_N^+)$ , and these two values are put into storage. Then the conventional filter is run to time  $t_{2N}$ , producing  $\hat{\mathbf{x}}(t_{2N}^+)$  and  $\mathbf{P}(t_{2N}^+)$  based upon the first  $2N$  measurements. Using the quantities previously stored, the prediction  $\hat{\mathbf{x}}(t_{2N}/t_N)$  and associated covariance  $\mathbf{P}(t_{2N}/t_N)$  are computed as

$$\hat{\mathbf{x}}(t_{2N}/t_N) = \Phi(t_{2N}, t_N)\hat{\mathbf{x}}(t_N^+) + \sum_{j=N+1}^{2N} \Phi(t_{2N}, t_j)\mathbf{B}_d(t_{j-1})\mathbf{u}(t_{j-1}) \quad (9-32a)$$

$$\mathbf{P}(t_{2N}/t_N) = \Phi(t_{2N}, t_N)\mathbf{P}(t_N^+)\Phi^T(t_{2N}, t_N) \quad (9-32b)$$

Then  $\hat{\mathbf{x}}_N(t_{2N}^+)$  and  $\mathbf{P}_N(t_{2N}^+)$  are calculated from (9-20) as

$$\mathbf{P}_N^{-1}(t_{2N}^+) = \mathbf{P}^{-1}(t_{2N}^+) - \mathbf{P}^{-1}(t_{2N}/t_N) \quad (9-33a)$$

$$\hat{\mathbf{x}}_N(t_{2N}^+) = \mathbf{P}_N(t_{2N}^+)[\mathbf{P}^{-1}(t_{2N}^+)\hat{\mathbf{x}}(t_{2N}^+) - \mathbf{P}^{-1}(t_{2N}/t_N)\hat{\mathbf{x}}(t_{2N}/t_N)] \quad (9-33b)$$

Now  $\hat{\mathbf{x}}_N(t_{2N}^+)$  and  $\mathbf{P}_N(t_{2N}^+)$  are put into storage for future use and also used as initial conditions for a conventional Kalman filter algorithm, which is run out to time  $t_{3N}$ . At this time, the prediction  $\hat{\mathbf{x}}(t_{3N}/t_{2N})$  and covariance  $\mathbf{P}(t_{3N}/t_{2N})$  are computed analogously to (9-32), replacing  $\hat{\mathbf{x}}(t_N^+)$  by  $\hat{\mathbf{x}}_N(t_{2N}^+)$  and  $\mathbf{P}(t_N^+)$  by  $\mathbf{P}_N(t_{2N}^+)$ . These results are combined analogously to (9-33) to approximate  $\hat{\mathbf{x}}_N(t_{3N}^+)$  and  $\mathbf{P}_N(t_{3N}^+)$ . The recursion continues in this fashion. Estimates are in fact available at every measurement sample time, not just every  $N$  points, and

these estimates will be based upon  $(N), (N + 1), \dots, (2N - 1), (N), (N + 1), \dots$  measurements iteratively. Such an approximate filter has been applied with success to orbital determination problems and other applications [21, 22].

## 9.5 LINEARIZED AND EXTENDED KALMAN FILTERS

Consider an estimation problem involving a continuous-time system with discrete-time measurements. Unlike cases discussed previously, assume that a linear model does not provide a valid description of the problem, i.e., that nonlinearities in the deterministic portion of the state dynamics and measurement models are not negligible. Let the system state be well modeled as satisfying the nonlinear stochastic differential equation

$$\mathbf{d}\mathbf{x}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] dt + \mathbf{G}(t) d\beta(t) \quad (9-34a)$$

where  $\mathbf{f}(\cdot, \cdot, \cdot)$  is a known  $n$ -vector of functions of three arguments,  $\mathbf{u}(\cdot)$  is an  $r$ -vector of deterministic control input functions, and  $\beta(\cdot, \cdot)$  is Brownian motion with diffusion  $\mathbf{Q}(t)$  for all  $t \in T$ . Equation (9-34a) is a relation for the stochastic differential  $\mathbf{d}\mathbf{x}(\cdot, \cdot)$  in the sense that integrating the right hand side of the equation yields the manner of time evolution of the  $\mathbf{x}(\cdot, \cdot)$  process. It can also be written in white noise notation as

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] + \mathbf{G}(t)\mathbf{w}(t) \quad (9-34b)$$

where  $\mathbf{w}(\cdot, \cdot)$  is a zero-mean white Gaussian noise process with covariance kernel

$$E\{\mathbf{w}(t)\mathbf{w}^T(t + \tau)\} = \mathbf{Q}(t)\delta(\tau) \quad (9-35)$$

In analogy to Section 2.3 (Volume 1),  $\mathbf{f}(\cdot, \cdot, \cdot)$  is assumed Lipschitz in its first argument, continuous in its second argument (and  $\mathbf{u}(\cdot)$  is assumed piecewise continuous), and piecewise continuous in its third argument. More general nonlinear stochastic differential equations will be developed in detail in Chapter 11, at which time sufficient conditions for the existence of a solution will be discussed in greater detail. At this point, we view (9-34) as a modest generalization of models considered previously, replacing  $[\mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)]$  by  $\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]$ . Note that the dynamic driving noise,  $\beta(\cdot, \cdot)$  in (9-34a) or  $\mathbf{w}(\cdot, \cdot)$  in (9-34b), is still assumed to enter in a linear additive fashion. This is representative of a wide class of practical applications and warrants special consideration, especially since the insights from linear estimation will be directly exploitable.

The initial condition  $\mathbf{x}(t_0)$  for (9-34) is assumed to be a Gaussian random  $n$ -vector with mean  $\hat{\mathbf{x}}_0$  and covariance  $\mathbf{P}_0$ . As is true of the dynamic noise, the Gaussian assumption is motivated in part by the desire to exploit linear estimation theory; the  $\mathbf{x}(\cdot, \cdot)$  process described by (9-34) will not in general be Gaussian.

Let the discrete-time measurements be modeled in general as a known nonlinear function of the state plus linearly additive noise corruption for all  $t_i \in T$ , as

$$\mathbf{z}(t_i) = \mathbf{h}[\mathbf{x}(t_i), t_i] + \mathbf{v}(t_i) \quad (9-36)$$

where  $\mathbf{h}[\cdot, \cdot]$  is a known  $m$ -vector of functions of state and time, and  $\mathbf{v}(\cdot, \cdot)$  is a white Gaussian noise sequence of mean zero and covariance kernel

$$E\{\mathbf{v}(t_i)\mathbf{v}^T(t_j)\} = \begin{cases} \mathbf{R}(t_i) & t_i = t_j \\ \mathbf{0} & t_i \neq t_j \end{cases} \quad (9-37)$$

Again, this is a modest increase in complexity over previous models, replacing  $[\mathbf{H}(t_i)\mathbf{x}(t_i)]$  by  $\mathbf{h}[\mathbf{x}(t_i), t_i]$ . In fact, there are many applications in which either  $\mathbf{f}$  or  $\mathbf{h}$  in the adequate system model is in fact a linear function.

Given such a system model, it is desired to generate an “optimal” state estimate. This will be viewed later in the context of estimation with general nonlinear models, but here we seek to apply the previously developed linear estimation results directly. To do so, assume that we can generate a nominal (reference) state trajectory,  $\mathbf{x}_n(\cdot)$ , for all time  $t \in T$ , starting from the initial condition  $\mathbf{x}_n(t_0) = \mathbf{x}_{n0}$  and satisfying the deterministic differential equation:

$$\dot{\mathbf{x}}_n(t) = \mathbf{f}[\mathbf{x}_n(t), \mathbf{u}(t), t] \quad (9-38)$$

Here  $\mathbf{f}(\cdot, \cdot, \cdot)$  and  $\mathbf{u}(\cdot)$  are identical to those in (9-34). In fact, we could assume that  $\mathbf{x}_n(\cdot)$  were driven by some nominal deterministic control function  $\mathbf{u}_n(\cdot)$  different from  $\mathbf{u}(\cdot)$ , but we will confine our attention to  $\mathbf{u}_n(t) \triangleq \mathbf{u}(t)$  for all  $t \in T$  to maintain adequacy of linear perturbation techniques as much as possible. Associated with such a nominal state trajectory would be the sequence of nominal measurements,

$$\mathbf{z}_n(t_i) = \mathbf{h}[\mathbf{x}_n(t_i), t_i] \quad (9-39)$$

where  $\mathbf{h}[\cdot, \cdot]$  is as given by (9-36).

Now consider the perturbation of the state from the assumed nominal trajectory:  $[\mathbf{x}(t) - \mathbf{x}_n(t)]$  for all  $t \in T$ . This is a stochastic process satisfying

$$[\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_n(t)] = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] - \mathbf{f}[\mathbf{x}_n(t), \mathbf{u}(t), t] + \mathbf{G}(t)\mathbf{w}(t) \quad (9-40)$$

i.e., a nonlinear stochastic differential equation written in white noise notation, with  $\mathbf{x}_n(t)$  given for all  $t \in T$  as the solution to (9-38). Equation (9-40) can be expressed as a series by expanding about  $\mathbf{x}_n(t)$ :

$$[\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_n(t)] = \left. \frac{\partial \mathbf{f}[\mathbf{x}, \mathbf{u}(t), t]}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_n(t)} [\mathbf{x}(t) - \mathbf{x}_n(t)] + \text{h.o.t.} + \mathbf{G}(t)\mathbf{w}(t) \quad (9-41)$$

where the zero-order term in the Taylor series for  $\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]$  has been canceled by the second term in (9-40) and where “h.o.t.” are terms in powers of

$[\mathbf{x}(t) - \mathbf{x}_n(t)]$  greater than one. A first order approximation to this equation, called the variational or perturbation equation, can be written as

$$\dot{\delta\mathbf{x}}(t) = \mathbf{F}[t; \mathbf{x}_n(t)] \delta\mathbf{x}(t) + \mathbf{G}(t)\mathbf{w}(t) \quad (9-42)$$

where  $\delta\mathbf{x}(\cdot, \cdot)$  is a first order approximation to the process  $[\mathbf{x}(\cdot, \cdot) - \mathbf{x}_n(\cdot, \cdot)]$ , and  $\mathbf{F}[t; \mathbf{x}_n(t)]$  is the  $n$ -by- $n$  matrix of partial derivatives of  $\mathbf{f}$  with respect to its first argument, evaluated along the nominal trajectory:

$$\mathbf{F}[t; \mathbf{x}_n(t)] \triangleq \left. \frac{\partial \mathbf{f}[\mathbf{x}, \mathbf{u}(t), t]}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_n(t)} \quad (9-43)$$

The solution to Eq. (9-42) is a viable approximation to the solution of (9-40) as long as the deviations from the nominal state trajectory are small enough for the higher order terms in (9-41) to be negligible (the strength of  $\mathbf{w}(\cdot, \cdot)$  might be increased somewhat to account for these terms). The appropriate initial condition would thus be to model  $\delta\mathbf{x}(t_0)$  as a Gaussian random variable with mean  $[\hat{\mathbf{x}}_0 - \mathbf{x}_{n0}]$  and covariance  $\mathbf{P}_0$ ;  $\hat{\mathbf{x}}_0$  and  $\mathbf{x}_{n0}$  are usually equal, so  $E\{\delta\mathbf{x}(t_0)\}$  is typically  $\mathbf{0}$ . Note that, had we allowed  $\mathbf{u}(t)$  to assume a value different than  $\mathbf{u}_n(t)$ , with  $\delta\mathbf{u}(t) \triangleq [\mathbf{u}(t) - \mathbf{u}_n(t)]$ , (9-42) could be expanded to

$$\dot{\delta\mathbf{x}}(t) = \mathbf{F}[t; \mathbf{x}_n(t), \mathbf{u}_n(t)] \delta\mathbf{x}(t) + \mathbf{B}[t; \mathbf{x}_n(t), \mathbf{u}_n(t)] \delta\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t) \quad (9-44)$$

with  $\mathbf{B}[t; \mathbf{x}_n(t), \mathbf{u}_n(t)]$  as  $\partial\mathbf{f}/\partial\mathbf{u}$  evaluated along the nominal. However, letting  $\delta\mathbf{u}(t)$  be nonzero compromises the validity of small perturbation approximations unnecessarily, and we shall therefore assume  $\delta\mathbf{u}(t) \equiv \mathbf{0}$  for all  $t \in T$ , as mentioned before.

In a similar manner, we can consider the measurement perturbations for each time  $t_i$ , through (9-36) and (9-39), as

$$[\mathbf{z}(t_i) - \mathbf{z}_n(t_i)] = \mathbf{h}[\mathbf{x}(t_i), t_i] - \mathbf{h}[\mathbf{x}_n(t_i), t_i] + \mathbf{v}(t_i) \quad (9-45)$$

A linearization of this relation yields the perturbation measurement model of

$$\delta\mathbf{z}(t_i) = \mathbf{H}[t_i; \mathbf{x}_n(t_i)] \delta\mathbf{x}(t_i) + \mathbf{v}(t_i) \quad (9-46)$$

where  $\mathbf{H}[t_i; \mathbf{x}_n(t_i)]$  for each  $t_i$  is the  $m$ -by- $n$  matrix of partial derivatives of  $\mathbf{h}$  with respect to its first argument, evaluated along the nominal trajectory:

$$\mathbf{H}[t_i; \mathbf{x}_n(t_i)] \triangleq \left. \frac{\partial \mathbf{h}[\mathbf{x}, t_i]}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_n(t_i)} \quad (9-47)$$

Thus,  $\delta\mathbf{z}(\cdot, \cdot)$  is an approximation to the difference process  $[\mathbf{z}(\cdot, \cdot) - \mathbf{z}_n(\cdot, \cdot)]$  that is valid to first order. A realization of the actual measurement process model,  $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$ , will be different from  $\mathbf{z}_n(t_i)$  because of two effects, the noise corruption  $\mathbf{v}(t_i, \omega_j)$  and the fact that  $\mathbf{h}[\mathbf{x}(t_i, \omega_j), t_i]$  is not in general equal to  $\mathbf{h}[\mathbf{x}_n(t_i), t_i]$ . Additionally,  $\delta\mathbf{z}(t_i, \omega_j)$  will differ from  $[\mathbf{z}(t_i, \omega_j) - \mathbf{z}_n(t_i)]$  because of the approximations inherent in (9-46).

Now one can consider applying linear filtering theory to the system model prescribed by (9-42) and (9-46). Using the a priori nominal trajectory  $\mathbf{x}_n(\cdot)$ , i.e., the (precomputable) solution to (9-38), the required matrices  $\mathbf{F}[t; \mathbf{x}_n(t)]$  and  $\mathbf{H}[t_i; \mathbf{x}_n(t_i)]$  can be evaluated, provided the derivatives involved exist. The input “measurement” for this filter at time  $t_i$  is in fact the difference value  $[\mathbf{z}(t_i, \omega_j) - \mathbf{z}_n(t_i)]$ . The output of such a filter would be the optimal estimate of  $\delta\mathbf{x}(t)$  for all  $t \in T$ , denoted as  $\widehat{\delta\mathbf{x}}(t)$ , and this could be added to the nominal value  $\mathbf{x}_n(t)$  to establish an estimate of the total state:

$$\hat{\mathbf{x}}(t) \triangleq \mathbf{x}_n(t) + \widehat{\delta\mathbf{x}}(t) \quad (9-48)$$

This form of estimator is called the *linearized Kalman filter*, or *perturbation Kalman filter*. It is computationally advantageous compared to an “optimal” nonlinear filter, but it can suffer from large magnitude errors if the “true” and nominal trajectories differ significantly. In fact, the feedforward error state space Kalman filters of Chapter 6 (Volume 1) were examples of this configuration.

The basic idea of the *extended Kalman filter* is to *relinearize* about each estimate  $\hat{\mathbf{x}}(t_i^+)$  once it has been computed. As soon as a new state estimate is made, a new and better reference state trajectory is incorporated into the estimation process. In this manner, one enhances the validity of the assumption that deviations from the reference (nominal) trajectory are small enough to allow linear perturbation techniques to be employed with adequate results. The extended Kalman filter will now be derived using the insights from the linearized Kalman filter, and then the actual algorithm for implementation will be summarized and applied to realistic problems.

First consider the linear filter used to estimate  $\delta\mathbf{x}(\cdot)$  in the development of the linearized Kalman filter above. If we have just processed the measurement  $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$  to obtain  $\hat{\mathbf{x}}(t_i^+)$ , let us relinearize about the trajectory starting from that value instead of  $\mathbf{x}_n(t_i)$ . Let  $\mathbf{x}_n(t/t_i)$  denote the solution to the nominal trajectory differential equation but starting from the “initial” condition of  $\hat{\mathbf{x}}(t_i^+)$ , i.e., the solution over the interval  $[t, t_{i+1}]$  of

$$\dot{\mathbf{x}}_n(t/t_i) = \mathbf{f}[\mathbf{x}_n(t/t_i), \mathbf{u}(t), t] \quad (9-49a)$$

$$\mathbf{x}_n(t_i/t_i) = \hat{\mathbf{x}}(t_i^+) \quad (9-49b)$$

At time  $t_i$ , just after measurement incorporation and relinearization, the best estimate of  $\delta\mathbf{x}(t_i)$  is (because of the relinearization process):

$$\widehat{\delta\mathbf{x}}(t_i^+) = \mathbf{0} \quad (9-50)$$

To propagate the state perturbation estimate to the next sample time  $t_{i+1}$ , we would employ the relinearized evaluation of  $\mathbf{F}, \mathbf{F}[t; \mathbf{x}_n(t/t_i)]$ . If we let  $\widehat{\delta\mathbf{x}}(t/t_i)$  denote the estimate of  $\delta\mathbf{x}(t)$  based on the measurements through  $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$ ,

for  $t$  in the interval  $[t_i, t_{i+1})$ , then it is the solution to

$$\dot{\widehat{\mathbf{x}}}(t/t_i) = \mathbf{F}[t; \mathbf{x}_n(t/t_i)] \widehat{\delta\mathbf{x}}(t/t_i) \quad (9-51a)$$

subject to the initial condition

$$\widehat{\delta\mathbf{x}}(t_i/t_i) = \widehat{\delta\mathbf{x}}(t_i^+) = \mathbf{0} \quad (9-51b)$$

Thus, it can be seen that  $\widehat{\delta\mathbf{x}}(t/t_i)$  is identically zero over the entire interval  $[t_i, t_{i+1})$ , and so

$$\widehat{\delta\mathbf{x}}(t_{i+1}^-) = \widehat{\delta\mathbf{x}}(t_{i+1}/t_i) = \mathbf{0} \quad (9-52)$$

The measurement to be presented to the relinearized filter at time  $t_{i+1}$  is  $[\mathbf{z}(t_{i+1}, \omega_j) - \mathbf{z}_n(t_{i+1}/t_i)]$ , where

$$\mathbf{z}_n(t_{i+1}/t_i) \triangleq \mathbf{h}[\mathbf{x}_n(t_{i+1}/t_i), t_{i+1}] \quad (9-53)$$

Consequently, the measurement update for the relinearized filter is

$$\widehat{\delta\mathbf{x}}(t_{i+1}^+) = \widehat{\delta\mathbf{x}}(t_{i+1}^-) + \mathbf{K}(t_{i+1}) [\{\mathbf{z}_{i+1} - \mathbf{z}_n(t_{i+1}/t_i)\} - \mathbf{H}(t_{i+1}) \widehat{\delta\mathbf{x}}(t_{i+1}^-)]$$

In view of (9-52) and (9-53), this becomes

$$\widehat{\delta\mathbf{x}}(t_{i+1}^+) = \mathbf{K}(t_{i+1}) \{\mathbf{z}_{i+1} - \mathbf{h}[\mathbf{x}_n(t_{i+1}/t_i), t_{i+1}]\} \quad (9-54)$$

where  $\mathbf{K}(t_{i+1})$  is computed using  $\mathbf{P}(t_{i+1}^-)$  and  $\mathbf{H}(t_{i+1})$  evaluated along the most recent “nominal,”  $\mathbf{x}_n(t/t_i)$  for  $t \in [t_i, t_{i+1})$ .

To achieve the final form of the extended Kalman filter, consider combining the most recent “nominal”  $\mathbf{x}_n(t/t_i)$  with the state perturbation estimate  $\widehat{\delta\mathbf{x}}(t/t_i)$  to generate an estimate of the full state. Assuming that an adequate model of  $\mathbf{x}(t)$  for all  $t$  in  $[t_i, t_{i+1})$  is

$$\mathbf{x}(t) = \mathbf{x}_n(t/t_i) + \widehat{\delta\mathbf{x}}(t) \quad (9-55)$$

we define the optimal estimate of  $\mathbf{x}(t)$  in this interval as

$$\hat{\mathbf{x}}(t/t_i) \triangleq \mathbf{x}_n(t/t_i) + \widehat{\delta\mathbf{x}}(t/t_i) \quad (9-56)$$

Since  $\widehat{\delta\mathbf{x}}(t/t_i)$  is zero over the entire interval between measurement times  $t_i$  and  $t_{i+1}$ , the best estimate of the total state over this interval is obtained as the solution to

$$\dot{\hat{\mathbf{x}}}(t/t_i) = \mathbf{f}[\hat{\mathbf{x}}(t/t_i), \mathbf{u}(t), t] \quad (9-57a)$$

starting from the initial condition

$$\hat{\mathbf{x}}(t_i/t_i) = \hat{\mathbf{x}}(t_i^+) \quad (9-57b)$$

Note that this estimate propagation (prediction) employs the *nonlinear* system model dynamic equations. To incorporate the measurement at time  $t_{i+1}$ , we

again invoke the assumed adequacy of (9-55) to write

$$\begin{aligned}\hat{\mathbf{x}}(t_{i+1}^+) &= \hat{\mathbf{x}}(t_{i+1}/t_i) + \widehat{\delta\mathbf{x}}(t_{i+1}^+) \\ &= \hat{\mathbf{x}}(t_{i+1}/t_i) + \mathbf{K}(t_{i+1})\{\mathbf{z}_{i+1} - \mathbf{h}[\hat{\mathbf{x}}(t_{i+1}/t_i), t_{i+1}]\}\end{aligned}\quad (9-58)$$

where  $\hat{\mathbf{x}}(t_{i+1}/t_i)$  is obtained from the solution of (9-57).

The *extended Kalman filter* will now be summarized [5, 6, 22, 26, 53]. Let the system of interest be described by the dynamics model

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] + \mathbf{G}(t)\mathbf{w}(t) \quad (9-59)$$

where  $\mathbf{x}(t_0)$  is modeled as a Gaussian random  $n$ -vector with mean  $\hat{\mathbf{x}}_0$  and covariance  $\mathbf{P}_0$ ,  $\mathbf{u}(\cdot)$  is an  $r$ -vector of known input functions, and  $\mathbf{w}(\cdot, \cdot)$  is a zero-mean white Gaussian  $s$ -vector process, independent of  $\mathbf{x}(t_0)$  and of strength  $\mathbf{Q}(t)$  for all time  $t \in T$ . Let the available discrete-time measurements be modeled as the  $m$ -vector process  $\mathbf{z}(\cdot, \cdot)$ : for each  $t_i \in T$ ,

$$\mathbf{z}(t_i) = \mathbf{h}[\mathbf{x}(t_i), t_i] + \mathbf{v}(t_i) \quad (9-60)$$

where  $\mathbf{v}(\cdot, \cdot)$  is a zero-mean white Gaussian  $m$ -vector process, independent of  $\mathbf{x}(t_0)$  and  $\mathbf{w}(\cdot, \cdot)$ , and of strength  $\mathbf{R}(t_i)$  for all  $t_i$  of interest.

The extended Kalman filter *measurement update* incorporates the measurement  $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$  by means of

$$\mathbf{K}(t_i) = \mathbf{P}(t_i^-)\mathbf{H}^T[t_i; \hat{\mathbf{x}}(t_i^-)]\{\mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)]\mathbf{P}(t_i^-)\mathbf{H}^T[t_i; \hat{\mathbf{x}}(t_i^-)] + \mathbf{R}(t_i)\}^{-1} \quad (9-61)$$

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i)\{\mathbf{z}_i - \mathbf{h}[\hat{\mathbf{x}}(t_i^-), t_i]\} \quad (9-62)$$

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{K}(t_i)\mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)]\mathbf{P}(t_i^-) \quad (9-63a)$$

$$\begin{aligned}&= \{\mathbf{I} - \mathbf{K}(t_i)\mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)]\}\mathbf{P}(t_i^-)\{\mathbf{I} - \mathbf{K}(t_i)\mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)]\}^T \\ &\quad + \mathbf{K}(t_i)\mathbf{R}(t_i)\mathbf{K}^T(t_i)\end{aligned}\quad (9-63b)$$

where  $\mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)]$  is defined as the  $m$ -by- $n$  partial derivative matrix:

$$\mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)] \triangleq \left. \frac{\partial \mathbf{h}[\mathbf{x}, t_i]}{\partial \mathbf{x}} \right|_{\mathbf{x}=\hat{\mathbf{x}}(t_i^-)} \quad (9-64)$$

The estimate is *propagated* forward to the next sample time  $t_{i+1}$  by integrating

$$\dot{\hat{\mathbf{x}}}(t/t_i) = \mathbf{f}[\hat{\mathbf{x}}(t/t_i), \mathbf{u}(t), t] \quad (9-65)$$

$$\dot{\mathbf{P}}(t/t_i) = \mathbf{F}[t; \hat{\mathbf{x}}(t/t_i)]\mathbf{P}(t/t_i) + \mathbf{P}(t/t_i)\mathbf{F}^T[t; \hat{\mathbf{x}}(t/t_i)] + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t) \quad (9-66)$$

from time  $t_i$  to  $t_{i+1}$ , using the initial conditions provided by (9-62) and (9-63):

$$\hat{\mathbf{x}}(t_i/t_i) = \hat{\mathbf{x}}(t_i^+) \quad (9-67a)$$

$$\mathbf{P}(t_i/t_i) = \mathbf{P}(t_i^+) \quad (9-67b)$$

(For the first interval, from  $t_0$  to  $t_1$ , the initial conditions would be  $\hat{\mathbf{x}}_0$  and  $\mathbf{P}_0$ , respectively.) In (9-66),  $\mathbf{F}[t; \hat{\mathbf{x}}(t/t_i)]$  is the  $n$ -by- $n$  partial derivative matrix:

$$\mathbf{F}[t; \hat{\mathbf{x}}(t/t_i)] \triangleq \left. \frac{\partial \mathbf{f}[\mathbf{x}, \mathbf{u}(t), t]}{\partial \mathbf{x}} \right|_{\mathbf{x} = \hat{\mathbf{x}}(t/t_i)} \quad (9-68)$$

for all  $t$  in the interval  $[t_i, t_{i+1})$ . Upon integrating (9-65) and (9-66) to the next sample time,  $\hat{\mathbf{x}}(t_{i+1}^-)$  and  $\mathbf{P}(t_{i+1}^-)$  are defined as

$$\hat{\mathbf{x}}(t_{i+1}^-) = \hat{\mathbf{x}}(t_{i+1}/t_i) \quad (9-69a)$$

$$\mathbf{P}(t_{i+1}^-) = \mathbf{P}(t_{i+1}/t_i) \quad (9-69b)$$

for use in the next measurement update.

**EXAMPLE 9.7** One means of numerically integrating (9-65) and (9-66) from one sample time to the next would be to divide the interval  $[t_i, t_{i+1})$  into  $N$  equal subintervals and apply a first order Euler integration technique to each subinterval. For example, let  $N = 2$ , and let each subinterval be  $\Delta t$  seconds long. First  $\hat{\mathbf{x}}(t_i/t_i)$  and  $t_i$  are substituted into  $\mathbf{f}$  in (9-65) to evaluate  $\dot{\hat{\mathbf{x}}}(t_i/t_i)$ , with which  $\hat{\mathbf{x}}(t_i + \Delta t/t_i)$  is computed as ( $\triangleq$ )

$$\hat{\mathbf{x}}(t_i + \Delta t/t_i) \triangleq \hat{\mathbf{x}}(t_i/t_i) + [\dot{\hat{\mathbf{x}}}(t_i/t_i)] \Delta t$$

Similarly,  $\mathbf{F}[t_i; \hat{\mathbf{x}}(t_i/t_i)]$  is computed and then (9-66) evaluated to yield  $\dot{\mathbf{P}}(t_i/t_i)$ , from which is calculated

$$\mathbf{P}(t_i + \Delta t/t_i) \triangleq \mathbf{P}(t_i/t_i) + [\dot{\mathbf{P}}(t_i/t_i)] \Delta t$$

The second subinterval propagation follows similarly. The computed  $\hat{\mathbf{x}}(t_i + \Delta t/t_i)$  and values of  $\mathbf{u}(t_i + \Delta t)$  and  $(t_i + \Delta t)$  are used to evaluate  $\mathbf{f}$  in (9-65), generating an approximate  $\dot{\hat{\mathbf{x}}}(t_i + \Delta t/t_i)$ , and then  $\hat{\mathbf{x}}(t_{i+1}/t_i)$  is calculated from

$$\hat{\mathbf{x}}(t_{i+1}/t_i) \triangleq \hat{\mathbf{x}}(t_i + 2\Delta t/t_i) \triangleq \hat{\mathbf{x}}(t_i + \Delta t/t_i) + [\dot{\hat{\mathbf{x}}}(t_i + \Delta t/t_i)] \Delta t$$

Once  $\mathbf{F}[t_i + \Delta t; \hat{\mathbf{x}}(t_i + \Delta t/t_i)]$  is computed, (9-66) can yield  $\dot{\mathbf{P}}(t_i + \Delta t/t_i)$ , and then

$$\mathbf{P}(t_{i+1}/t_i) \triangleq \mathbf{P}(t_i + 2\Delta t/t_i) \triangleq \mathbf{P}(t_i + \Delta t/t_i) + [\dot{\mathbf{P}}(t_i + \Delta t/t_i)] \Delta t$$

Integration accuracy is improved if the value assumed by the derivatives at the midpoints of the subintervals are used in place of their values at the beginning of each subinterval. Consider the first subinterval. In (9-65),  $\hat{\mathbf{x}}(t_i + 0.5\Delta t/t_i)$  would not be available, but  $\mathbf{u}(t_i + 0.5\Delta t)$  or an approximation to it of  $\frac{1}{2}\{\mathbf{u}(t_i) + \mathbf{u}(t_i + \Delta t)\}$  is, and so  $\mathbf{f}[\hat{\mathbf{x}}(t_i/t_i), \mathbf{u}(t_i), t_i]$  could be replaced by  $\mathbf{f}[\hat{\mathbf{x}}(t_i/t_i), \mathbf{u}(t_i + 0.5\Delta t), (t_i + 0.5\Delta t)]$ . Once (9-65) is integrated forward, both  $\hat{\mathbf{x}}(t_i/t_i)$  and  $\hat{\mathbf{x}}(t_i + \Delta t/t_i)$  are available for use in (9-66), so  $\mathbf{F}[t_i; \hat{\mathbf{x}}(t_i/t_i)]$  could be supplanted by  $\mathbf{F}[(t_i + 0.5\Delta t); \frac{1}{2}\{\hat{\mathbf{x}}(t_i/t_i) + \hat{\mathbf{x}}(t_i + \Delta t/t_i)\}]$ . Furthermore,  $[\mathbf{G}(t_i)\mathbf{Q}(t_i)\mathbf{G}^T(t_i)]$  could be replaced by its value, or averaged approximation, at the time  $(t_i + 0.5\Delta t)$ .

On the other extreme, if computation time is critical, (9-66) can be approximated by evaluating the partial derivative matrix only once over the entire sample period, effectively replacing  $\mathbf{F}[t; \hat{\mathbf{x}}(t/t_i)]$  by  $\mathbf{F}[t_i; \hat{\mathbf{x}}(t_i^+)]$  in (9-66). A somewhat better approximation for time-varying systems would replace  $\mathbf{F}[t; \hat{\mathbf{x}}(t/t_i)]$  by  $\mathbf{F}[t; \hat{\mathbf{x}}(t_i^+)]$ . ■

The time propagation relations can be written equivalently as

$$\hat{\mathbf{x}}(t_{i+1}^-) = \hat{\mathbf{x}}(t_i^+) + \int_{t_i}^{t_{i+1}} \mathbf{f}[\hat{\mathbf{x}}(t/t_i), \mathbf{u}(t), t] dt \quad (9-70)$$

$$\begin{aligned} \mathbf{P}(t_{i+1}^-) &= \Phi[t_{i+1}, t_i; \hat{\mathbf{x}}(\tau/t_i)] \mathbf{P}(t_i^+) \Phi^T[t_{i+1}, t_i; \hat{\mathbf{x}}(\tau/t_i)] \\ &\quad + \int_{t_i}^{t_{i+1}} \Phi[t_{i+1}, t; \hat{\mathbf{x}}(\tau/t_i)] \mathbf{G}(t) \mathbf{Q}(t) \mathbf{G}^T(t) \Phi^T[t_{i+1}, t; \hat{\mathbf{x}}(\tau/t_i)] dt \end{aligned} \quad (9-71)$$

In (9-71),  $\Phi[t_{i+1}, t; \hat{\mathbf{x}}(\tau/t_i)]$  denotes the state transition matrix associated with  $\mathbf{F}[\tau; \hat{\mathbf{x}}(\tau/t_i)]$  for all  $\tau \in [t_i, t_{i+1}]$ .

Note that  $\mathbf{F}$  (and thus  $\Phi$ ),  $\mathbf{H}$ ,  $\mathbf{K}$ , and  $\mathbf{P}$  are evaluated by knowing the most recent estimate of the nominal (reference) state trajectory. In contradistinction to the conventional and linearized Kalman filters, the equations for propagating and updating the estimation error covariance matrix are *coupled* to the state estimate relations. Therefore the covariance and gain matrices *cannot* be pre-computed without knowledge of the state estimates and thus of the actual measurement values.

**EXAMPLE 9.8** Recall the satellite in planar orbit discussed in Examples 2.7 and 2.8 of Volume 1. A reasonably good model of satellite motion in terms of range  $r(t)$  from the earth center and angle  $\theta(t)$  relative to a fixed coordinate system can be expressed in terms of  $x_1 = r$ ,  $x_2 = \dot{r}$ ,  $x_3 = \theta$ , and  $x_4 = \dot{\theta}$  as  $\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] + \mathbf{G}(t)\mathbf{w}(t)$  by adding acceleration-level pseudonoise to the model in Example 2.7:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_1(t)x_4^2(t) - \frac{G}{x_1^2(t)} + u_r(t) \\ x_4(t) \\ -\frac{2}{x_1(t)}x_4(t)x_2(t) + \frac{1}{x_1(t)}u_\theta(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

Assume that discrete-time measurements of both range and angle are available, as

$$\begin{bmatrix} z_1(t_i) \\ z_2(t_i) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \\ x_3(t_i) \\ x_4(t_i) \end{bmatrix} + \begin{bmatrix} v_1(t_i) \\ v_2(t_i) \end{bmatrix}$$

Thus, for this problem,  $\mathbf{h}[\mathbf{x}(t_i), t_i] = \mathbf{H}\mathbf{x}(t_i)$ . Based on range and angle measurements from a ground tracking station and the models above, it is desired to generate a state estimate over a given time interval.

A linearized Kalman filter could be based upon an assumed nominal of a circular orbit at radius  $r_0$ :

$$\mathbf{x}_n(t) = [r_0, 0, \omega t, \omega]^T$$

with  $u_r(t) = u_\theta(t) = 0$  and  $G = r_0^{-3}\omega^2$ . The perturbation state estimate propagation would be based upon (9-42) with

$$\mathbf{F}[t; \mathbf{x}_n(t)] \triangleq \left. \frac{\partial \mathbf{f}[\mathbf{x}, \mathbf{u}(t), t]}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}_n(t)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2r_0\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega/r_0 & 0 & 0 \end{bmatrix}$$

and the update based on (9-46) with  $\mathbf{H}$  as given above. Figure 9.3a depicts a “true satellite” trajectory as an ellipse, with sampled data measurements and position estimates generated via  $\hat{x}_1(t_i^+) = \hat{r}(t_i^+)$  and  $\hat{x}_3(t_i^+) = \hat{\theta}(t_i^+)$  from a particular run of this filter. Note that the assumption of small deviations from the assumed nominal is subject to serious doubt, and filter performance suffers.

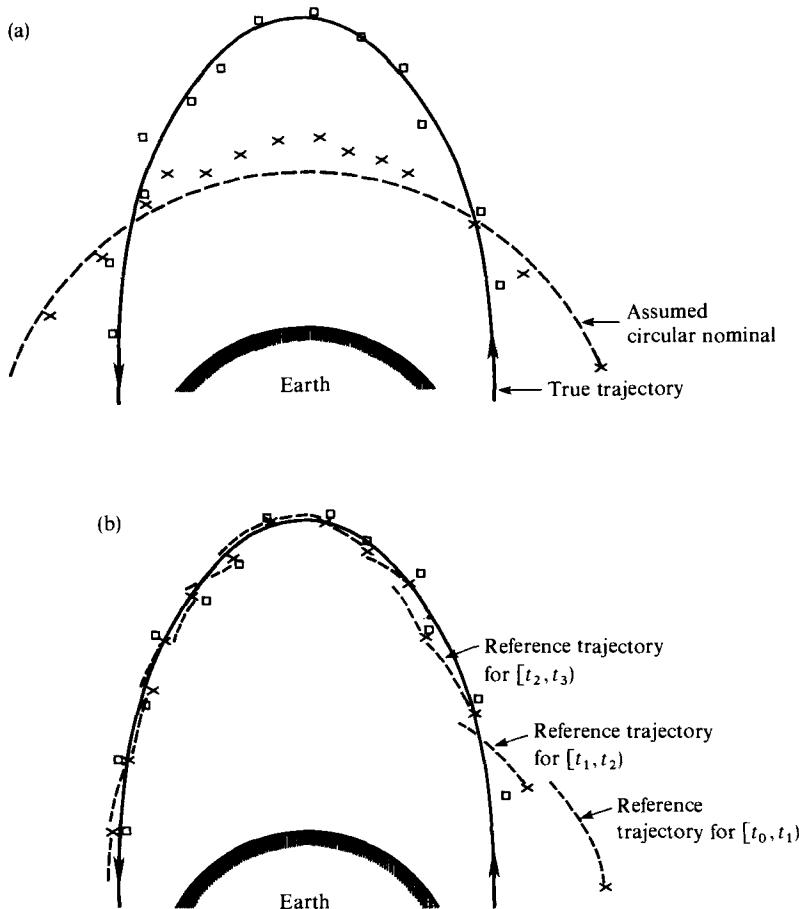


FIG. 9.3 (a) Linearized and (b) extended Kalman filters applied to orbit determination.  $\square$ , sampled data measurement information;  $x$ , position estimate from filter ( $\hat{x}(t_i^+)$ ).

On the other hand, an extended Kalman filter for this problem would be given by (9-61)–(9-69), with  $F[t; \hat{x}(t/t_i)]$  in (9-66) given by

$$F[t; \hat{x}(t/t_i)] \triangleq \left. \frac{\partial f[\mathbf{x}, \mathbf{u}(t), t]}{\partial \mathbf{x}} \right|_{\mathbf{x}=\hat{x}(t/t_i)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \left( x_4^2 + \frac{2G}{x_1^3} \right) & 0 & 0 & 2x_1x_4 \\ 0 & 0 & 0 & 1 \\ \frac{2x_2x_4}{x_1^2} & -\frac{2x_4}{x_1} & 0 & -\frac{2x_2}{x_1} \end{bmatrix}_{\mathbf{x}=\hat{x}(t/t_i)}$$

where  $\hat{\mathbf{x}}(t/t_i)$  for  $t \in [t_i, t_{i+1})$  represents a new reference trajectory as a conic arc over each sample period. For the same measurement realizations as considered for the linearized filter, Fig. 9.3b depicts the position estimates and successive reference trajectories generated by an extended Kalman filter. Due to relinearizations, small perturbation assumptions are more valid and performance is better than that achieved in Fig. 9.3a. Unlike the linearized Kalman filter, if a second run of this “experiment” were conducted, yielding a different set of measurement realizations, an entirely different sequence of reference trajectories would result. ■

Not only is it impossible to precompute covariance and gain time histories because of coupling of second moments to state estimates and measurement realizations, a priori covariance performance (sensitivity) analyses as described in Chapter 6 (Volume 1) are also precluded as a complete analysis method for the extended Kalman filter. Instead, its performance characteristics must be portrayed through a Monte Carlo analysis [38]. Nevertheless, covariance analyses are exploited to some degree in the design of extended Kalman filters. If a nominal state trajectory is chosen (corresponding to either “typical” or “worst case” situations), then a covariance analysis of a linearized Kalman filter operating over this nominal affords an approximate analysis of an extended Kalman filter with small deviations from this same nominal. In other words, it is assumed that the “true”  $\mathbf{x}(t)$  is well approximated by some trajectory  $\mathbf{x}_n(t)$  for all  $t$ , but the filter does not really know what these values are in actual operation, so it uses its best estimate  $\hat{\mathbf{x}}(t/t_i)$  to evaluate the partial derivative matrices  $\mathbf{F}$  and  $\mathbf{H}$ . However, an a priori covariance analysis can be performed by evaluating these partial derivatives based on  $\mathbf{x}_n(t)$  for all  $t$  instead. In this manner, an initial filter tuning and local perturbation analysis of the extended filter can be performed efficiently. Subsequently, a Monte Carlo analysis is conducted to investigate filter performance thoroughly, as especially the ability of the filter to recover from a condition in which an estimate  $\hat{\mathbf{x}}(t/t_i)$  is significantly different from the “true”  $\mathbf{x}(t)$  value at some  $t \in T$ .

One particular application of extended Kalman filtering is the estimation of state variables in a system well described by a linear model, but in which are embedded some uncertain parameters in the  $\mathbf{F}$  (or  $\Phi$ ),  $\mathbf{B}$ , and/or  $\mathbf{H}$  matrices [28, 63]. These uncertain parameters can be treated as additional state variables, resulting in a nonlinear model since these new states are multiplied by the original state variables in the model. There are other approaches to the problem of simultaneous estimation of states and parameters in a linear model structure, and the next chapter will study this problem extensively, but the extended Kalman filter has been shown to be a viable candidate for a means of solution.

**EXAMPLE 9.9** This example investigates thrust-vector controlling for spacecraft that can undergo significant bending. The angular orientation of a gimballed rocket engine nozzle (relative to the spacecraft body) is commanded not only to achieve desired guidance corrections, but also to damp out vehicle bending modes, during the thrusting period.

Consider a pitch plane description of the docked configuration of the Apollo Command and Service Module (CSM) with the Lunar Module (LM). This is a vehicle for which bending is especially important because of the narrow docking tunnel between the two modules, as shown in Fig. 9.4.

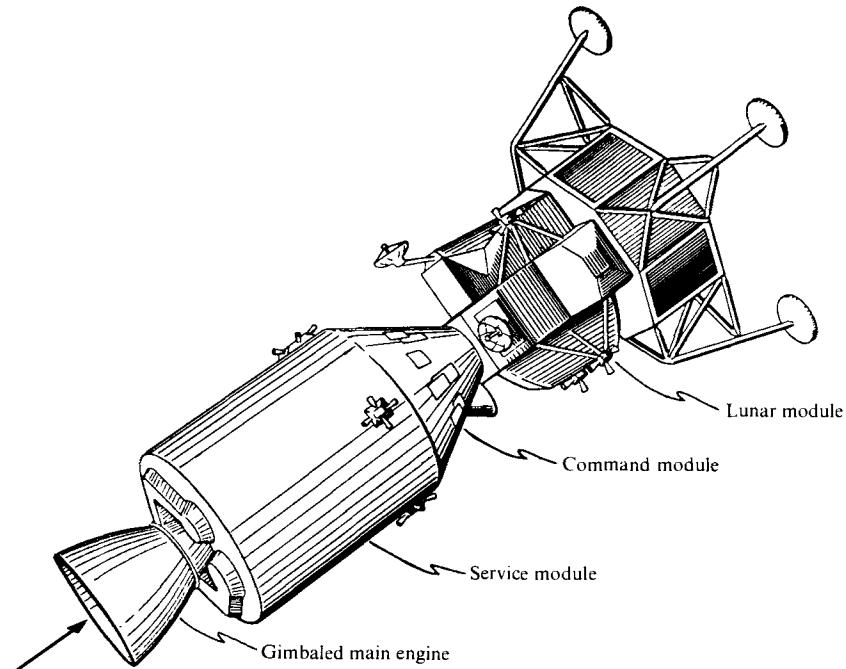


FIG. 9.4 The Apollo spacecraft: docked configuration of CSM and LM. From Widnall [64], with permission of MIT Press.

The basic vehicle dynamics can be described by

$$\frac{d}{dt} \begin{bmatrix} \omega(t) \\ \theta(t) \\ v_b(t) \\ q(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0.0815 & 1.13 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_b^2 & -11 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} \omega(t) \\ \theta(t) \\ v_b(t) \\ q(t) \\ \delta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix} \delta_{com}(t) + \begin{bmatrix} 1.13 \\ 0 \\ -11 \\ 0 \\ 0 \end{bmatrix} w(t)$$

where  $\omega$  and  $\theta$  are rigid body motion angular velocity and attitude (relative to an inertial reference direction),  $v_b$  and  $q$  are the velocity and position of the generalized bending coordinate,  $\delta$  is the angle of the gimbled engine nozzle relative to the vehicle centerline,  $\delta_{com}$  is the commanded gimbal angle, and  $w$  models the vibrational disturbance due to the rocket engine. Figure 9.5 presents a block diagram portrayal of the system model. Notice that the bending mode is modeled as an undamped second order system; true damping ratio is on the order of 0.005 and so it has been neglected for this problem. Later we will return to this problem to consider design of a feedback controller to provide  $\delta_{com}(\cdot)$  simply as some input command signal. The engine gimbal servo system is modeled to consider design of a feedback controller to provide  $\delta_{com}(\cdot)$ , but here we concentrate on the estimation problem and consider  $\delta_{com}(\cdot)$  simply as some input command signal. The engine gimbal servo system is modeled as a first order lag driven by  $\delta_{com}$ : this state is stochastically uncontrollable, and so should not enter the vector of states to be estimated; instead,  $\delta(t)$  can be treated as a known input, with an auxiliary equation of

$$\dot{\delta}(t) = -10\delta(t) + 10\delta_{com}(t); \quad \delta(t_0) = 0$$

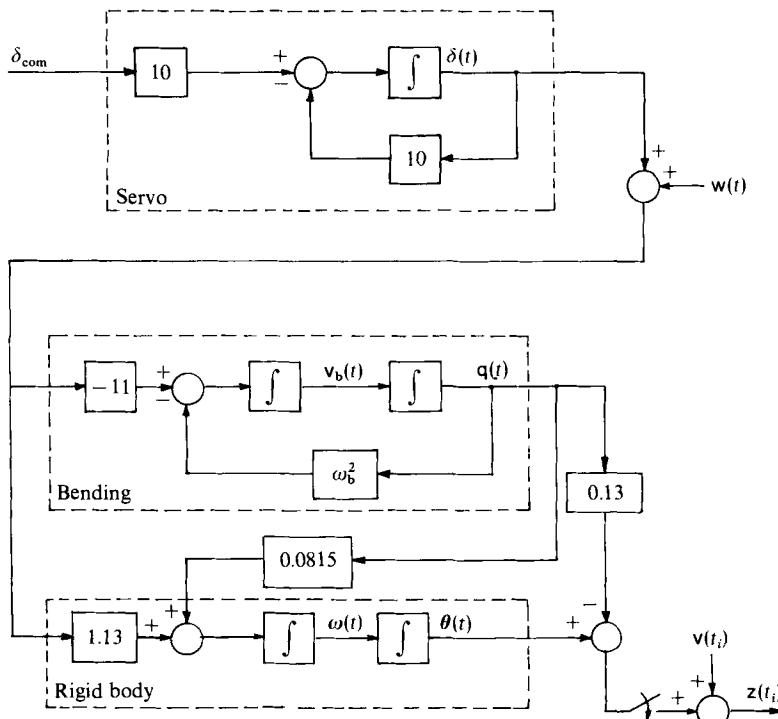


FIG. 9.5 System model for thrust vector control problem.

to determine its value once  $\delta_{\text{com}}(\cdot)$  is known. The dynamic driving noise  $w(\cdot, \cdot)$  enters the equations as a zero-mean white Gaussian noise random thrust vector angle, with power spectral density value of  $0.0004 \text{ rad}^2/\text{Hz}$ ; such a disturbance can be expected to cause a lateral velocity of about 2 ft/sec in 100 sec of engine burn time.

In the system dynamics model,  $\omega_b^2$  is the undamped natural frequency of the bending mode, a slowly varying parameter whose value is known only with some uncertainty (in fact, initial tests in the Apollo program revealed values substantially different from those predicted analytically). Use of a wrong assumed value not only degrades performance of an ensuing feedback controller design, but an error of only a few rad/sec can cause controller-induced instability: the applied control would actually increase the bending mode amplitude rather than damp it out.

The system model is rather simple. A more realistic model would describe the full scale problem by coupling the fundamental yaw plane rigid body and bending modes, the torsional bending mode, fuel slosh and center-of-gravity shift effects. Higher order modes could be neglected for this application. However, attention is confined to the five states described above for illustrative purposes.

A sampled measurement is available every 0.1 sec from the inertial measurement unit (IMU) of angular orientation of the vehicle at the IMU location with respect to inertial space, modeled as

$$z(t_i) = \theta(t_i) - 0.13q(t_i) + v(t_i)$$

where  $\theta(t_i)$  is the rigid body angular attitude, the second term is the generalized bending coordinate  $q(t_i)$  multiplied by the slope of the bending mode at the IMU station per unit displacement of  $q(t_i)$ , and  $v(\cdot, \cdot)$  is a zero-mean white Gaussian noise used to model readout quantization errors and other uncertainties. If quantization error is assumed to be uniformly distributed over the quanti-

zation interval of 0.0002 rad associated with the analog-to-digital conversion, then the variance of this error is  $\frac{1}{12}(0.0002)^2 \text{ rad}^2$ . Since this is the major contribution to measurement uncertainty,  $R(t_i)$  is set to this value for all  $t_i$  (it could have been increased slightly to account for neglected effects).

An optimal estimate of the state variables  $\omega$ ,  $\theta$ ,  $v_b$ , and  $q$  and the uncertain parameter  $\omega_b^2$  could be provided by an extended Kalman filter by treating  $[\omega_b^2]$  as an additional state. This "state" could be modeled as a random bias plus pseudonoise (i.e., a random walk) as

$$d[\omega_b^2(t)]/dt = n(t)$$

where  $n(\cdot, \cdot)$  is a white Gaussian noise of appropriate strength. Thus, the dynamics model for filter development is

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \omega(t) \\ \theta(t) \\ v_b(t) \\ q(t) \\ [\omega_b^2(t)] \end{bmatrix} &= \begin{bmatrix} 0.0815q(t) + 1.13\delta(t) \\ \omega(t) \\ -[\omega_b^2(t)][q(t)] - 11\delta(t) \\ v_b(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 1.13 & 0 \\ 0 & 0 \\ -11 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w(t) \\ n(t) \end{bmatrix} \\ \dot{\mathbf{x}}(t) &= \mathbf{f}[\mathbf{x}(t), u(t)] + \mathbf{G} \quad \mathbf{w}(t) \end{aligned}$$

where  $\delta(t)$  is generated deterministically from commanded control  $\delta_{\text{com}}(t)$  by the auxiliary equation

$$\dot{\delta}(t) = -10\delta(t) + 10\delta_{\text{com}}(t); \quad \delta(t_0) = 0$$

Note the nonlinearity in  $f_3[\mathbf{x}(t), u(t)]$ . The measurement model is

$$\begin{aligned} z(t_i) &= [0 \quad 1 \quad 0 \quad -0.13 \quad 0] \mathbf{x}(t_i) + v(t_i) \\ (z(t_i)) &= \mathbf{H} \quad \mathbf{x}(t_i) + v(t_i) \end{aligned}$$

Figure 9.6 presents a representative time history (i.e., from one run out of a Monte Carlo analysis) of the error in the filter estimate of  $[\omega_b^2]$ . For this simulation, the "true" value of  $[\omega_b^2]$  was set

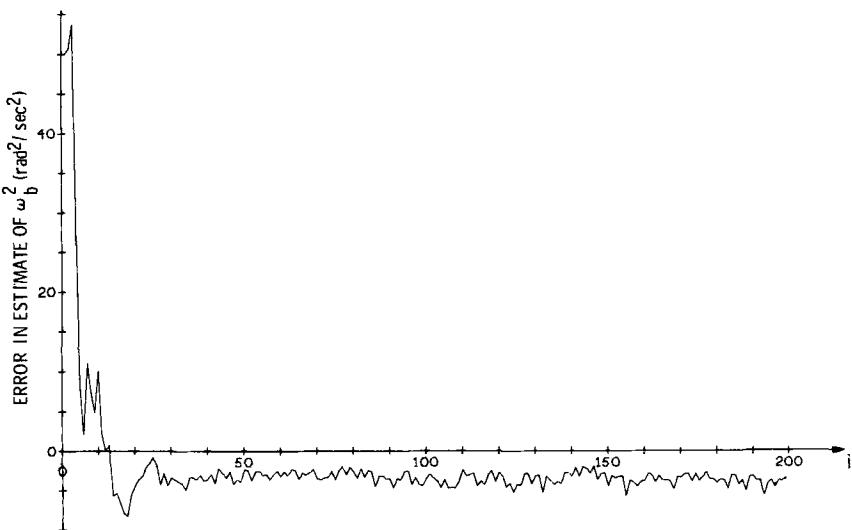


FIG. 9.6 Parameter estimate error produced by extended Kalman filter.

equal to  $150 \text{ rad}^2/\text{sec}^2$ , while the filter model assumed  $[\omega_b^2(t_0)]$  to be described as a Gaussian random variable with mean  $100 \text{ rad}^2/\text{sec}^2$  and variance  $1000 \text{ rad}^4/\text{sec}^4$ . This choice of variance, and the selection of  $200 \text{ rad}^4/\text{sec}^5$  for the strength of the white noise  $n(\cdot, \cdot)$ , are empirically determined "good" values achieved through filter tuning. Thus the filter exhibits an ability to estimate states and parameters simultaneously; the corresponding state estimation precision far exceeds that of a conventional Kalman filter that erroneously assumes  $[\omega_b^2] = 100 \text{ rad}^2/\text{sec}^2$ .

The estimator performance is strongly dependent upon the assumed second order statistics on  $[\omega_b^2(t_0)]$  and  $n(\cdot, \cdot)$ , larger numerical values causing greater oscillations in the steady state performance, and lower ones causing slower transients. The latter case is depicted in Fig. 9.7, for which the initial variance on  $\omega_b^2$  was reduced from  $1000$  to  $10 \text{ rad}^4/\text{sec}^4$  and the strength of  $n(\cdot, \cdot)$  reduced from  $200$  to  $2 \text{ rad}^4/\text{sec}^5$ . ■

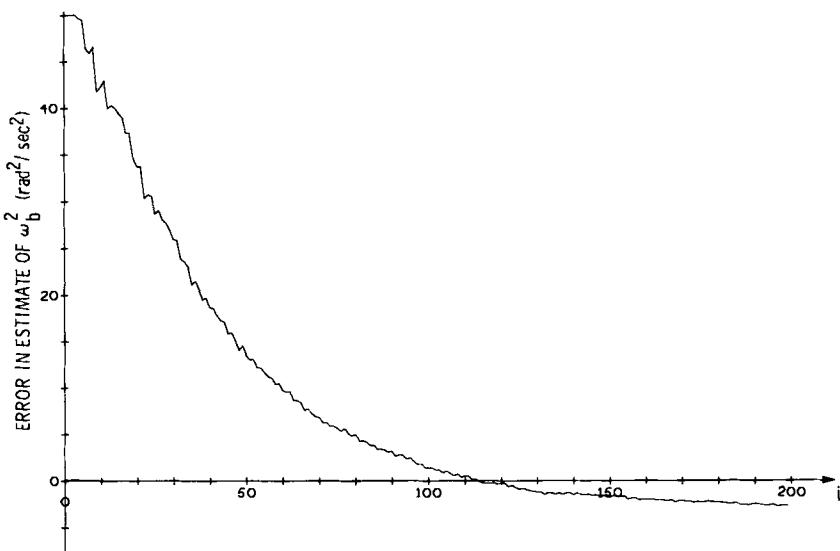


FIG. 9.7 Parameter estimate error when variances for parameter model are reduced.

The preceding example revealed a bias error in the state estimate produced by an extended Kalman filter. This is very characteristic of extended Kalman filters, and it is due to the neglected higher order effects inherent in trying to exploit linear perturbation concepts. The more pronounced the nonlinearities are in a given application, the more seriously one can expect performance to be degraded by this effect [8, 22]. This can be compensated to some degree by "tuning" [30] the filter such that its internally computed error variances match the true mean squared errors (variances plus squares of the mean errors) as indicated in a Monte Carlo performance analysis [38]. This will be discussed more extensively in Chapter 12, and alternative means of compensating for such bias errors (called "bias correction terms") will be developed at that time.

Extended Kalman filters have been used in many practical applications [1, 3, 7, 10, 11, 23–27, 29, 31–35, 39, 40, 42, 48, 53, 60, 65]. The following example presents one such implementation.

**EXAMPLE 9.10** In this example [31, 35], an extended Kalman filter is designed to track a long-range object, using outputs from a forward looking infrared (FLIR) sensor as measurements. The FLIR sensor generates outputs of an array of infrared detectors as they are mechanically scanned through a limited field of view, with digitized outputs corresponding to the average intensity over each picture element (pixel; assumed square, of dimension defined by the optics and physical size of the detectors) in a complete picture image (frame of data). A frame of data is available each 1/30 sec, and an 8-by-8 pixel array out of the complete frame, called a “tracking window,” is provided to the filter as a measurement (i.e., 64 intensity values).

Long-range targets appear as point sources of infrared radiation. Due to the physics of wave propagation and optics, the resulting radiation intensity pattern (called glint, and assumed to be time invariant), can be modeled as a two-dimensional Gaussian function over the FLIR image plane, centered at  $[x_{\text{peak}}(t), y_{\text{peak}}(t)]$  relative to the center of the 8-by-8 pixel array, and having circular equal-intensity contours:

$$I_{\text{target}}(x, y, t) = I_{\max} \exp\left\{(-1/2\sigma_g^2)([x - x_{\text{peak}}(t)]^2 + [y - y_{\text{peak}}(t)]^2)\right\}$$

where  $I_{\max}$  is the peak intensity value (assumed known) and  $\sigma_g$  is the dispersion of the Gaussian glint function. Figure 9.8 portrays this graphically.

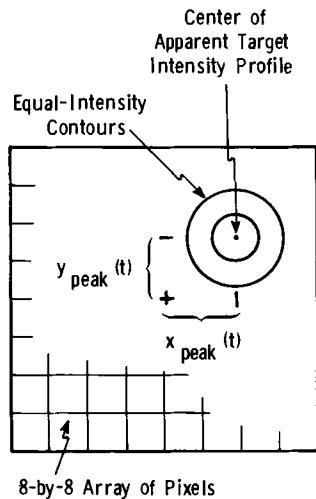


FIG. 9.8 Apparent target intensity pattern on image plane. From [31], © 1980 IEEE.

The apparent location of the target is actually the sum of effects due to true target dynamics, atmospheric disturbances (jitter), and vibration:

$$x_{\text{peak}}(t) = x_{\text{dyn}}(t) + x_{\text{atm}}(t) + x_{\text{vib}}(t)$$

and similarly for  $y_{\text{peak}}(t)$ . We want to estimate  $x_{\text{dyn}}$  and  $y_{\text{dyn}}$  separately from other effects, based on real-time noise-corrupted FLIR sensor data.

To provide a simple, generally applicable target dynamics model that accounts for time-correlated behavior of realistic targets, an independent zero-mean first order Gauss-Markov model in each direction was chosen:

$$\begin{aligned}\dot{x}_{\text{dyn}}(t) &= -(1/\tau_d)x_{\text{dyn}}(t) + w_{x\text{dyn}}(t) \\ E\{w_{x\text{dyn}}(t)w_{x\text{dyn}}(t + \tau)\} &= 2\sigma_d^2/\tau_d \delta(\tau)\end{aligned}$$

By suitable choice of  $x_{\text{dyn}}$  variance  $\sigma_d^2$  and correlation time  $\tau_d$ , amplitude and rate-of-change characteristics of a variety of targets as seen in the image plane can be well represented.

Through spectral analysis of the atmospheric jitter phenomenon, it was found that  $x_{\text{atm}}$  and  $y_{\text{atm}}$  can each be adequately modeled as the output of a third order shaping filter, described by

$$G(s) = K\omega_2^2 / [(s + \omega_1)(s + \omega_2^2)]$$

driven by unit strength white Gaussian noise. Because  $\omega_2 \gg \omega_1$  and the significant frequencies of the problem were well below  $\omega_2$ , this was approximated by

$$G'(s) = K\omega_1 / (s + \omega_1)$$

in the eventual filter design with very small degradation in performance. Because the tracker under consideration was ground based, vibration effects were considered negligible compared to other effects.

Thus, the dynamics model for the filter is a four-state ( $\mathbf{x} = [x_{\text{dyn}}, y_{\text{dyn}}, x_{\text{atm}}, y_{\text{atm}}]^T$ ) linear time-invariant model. This yields very simple propagations:

$$\begin{aligned}\hat{\mathbf{x}}(t_{i+1}^-) &= \Phi \hat{\mathbf{x}}(t_i^+) \\ \mathbf{P}(t_{i+1}^-) &= \Phi \mathbf{P}(t_i^+) \Phi^T + \mathbf{Q}_d\end{aligned}$$

with constant, precomputed  $\Phi$  and  $\mathbf{Q}_d = \int \Phi \mathbf{Q} \mathbf{Q}^T \Phi^T d\tau$ .

The target intensity pattern is corrupted by background noise (clutter, etc.) and inherent FLIR errors (thermal noise, dark current, etc.) before being available in the measurements. Letting  $z_{jk}(t_i)$  denote the measurement at time  $t_i$  of the average intensity over the pixel in the  $j$ th row and  $k$ th column of the 8-by-8 array,

$$z_{jk}(t_i) = \frac{1}{A_p} \iint_{\substack{\text{region of} \\ \text{jkth pixel}}} I_{\text{target}}(x, y, t_i) dx dy + v_{jk}(t_i)$$

where  $A_p$  is the area of one pixel, and  $v_{jk}(t_i)$  models the combined intensity effects of background and FLIR errors on the  $jk$ th pixel. Arraying 64 such scalar equations into a single measurement vector yields a model of the form

$$\mathbf{z}(t_i) = \mathbf{h}[\mathbf{x}(t_i), t_i] + \mathbf{v}(t_i)$$

Note that spatial correlation of the background and FLIR errors are readily represented by the off-diagonal terms of the  $\mathbf{R}$  matrix associated with  $\mathbf{v}$ . In the actual filter implementation, the  $\mathbf{h}$  function was approximated by replacing the two-dimensional integral terms by the value of  $I_{\text{target}}$  evaluated at the center of the  $jk$ th pixel.

Due to the large number of measurements, the usual measurement update given by (9-61)–(9-63) was computationally inefficient, (9-61) requiring the inversion of a 64-by-64 matrix. To circumvent this problem, the algebraically equivalent form

$$\begin{aligned}\mathbf{I}(t_i^-) &= [\mathbf{P}(t_i^-)]^{-1} \\ \mathbf{I}(t_i^+) &= \mathbf{I}(t_i^-) + \mathbf{H}^T(t_i) \mathbf{R}^{-1}(t_i) \mathbf{H}(t_i) \\ \mathbf{P}(t_i^+) &= [\mathbf{I}(t_i^+)]^{-1} \\ \mathbf{K}(t_i) &= \mathbf{P}(t_i^+) \mathbf{H}^T(t_i) \mathbf{R}^{-1}(t_i)\end{aligned}$$

was employed. This form only requires two 4-by-4 matrix inversions online;  $\mathbf{R}^{-1}$  is constant and is generated once offline.

The performance capabilities of this filter were evaluated and compared to those of a previously proposed correlation tracker by means of a Monte Carlo analysis (involving 20 sample simulations for each run). Figures 9.9 and 9.10 portray the sample mean error  $\pm 1$  standard deviation committed by the filter in estimating the target horizontal location, and the corresponding correlation tracker

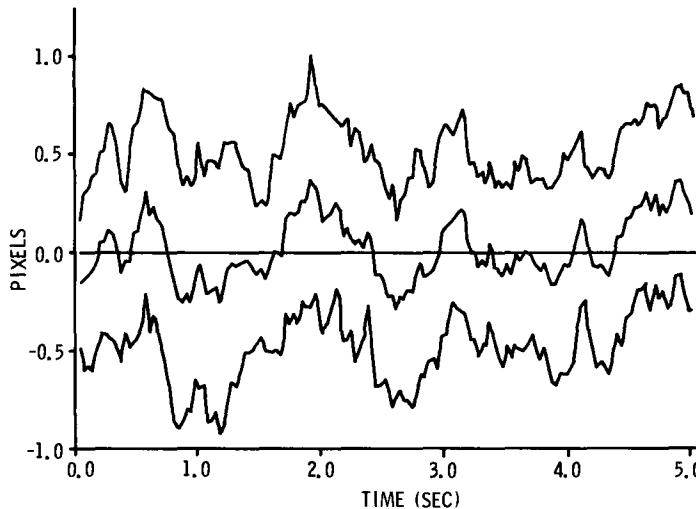


FIG. 9.9  $\hat{x}_{\text{dyn}}$  mean error  $\pm 1\sigma$ . S/N = 10,  $\sigma_g = 3$  pixels,  $\sigma_d = \sigma_a = 1$  pixel,  $\tau_d = 1$  sec. From [31], © 1980 IEEE.

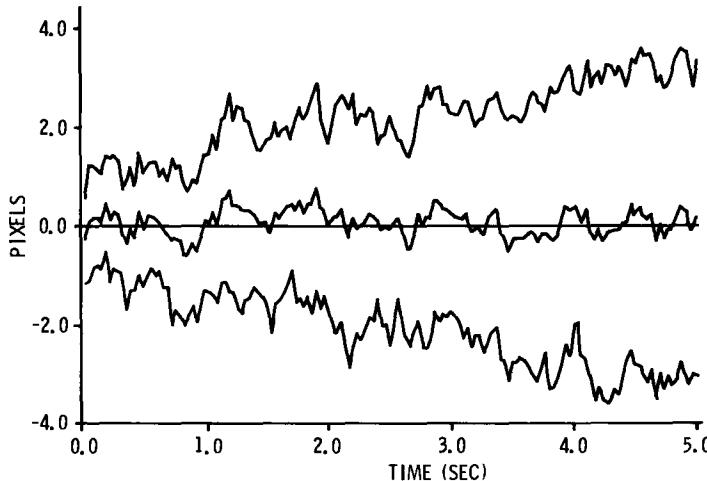


FIG. 9.10 Correlator x tracking error mean  $\pm 1\sigma$ . S/N = 10,  $\sigma_g = 3$  pixels,  $\sigma_d = \sigma_a = 1$  pixel,  $\tau_d = 1$  sec. From [31], © 1980 IEEE.

error, for a particular representative case. Note that signal-to-noise ratio S/N is defined here as

$$S/N = \frac{I_{\max}}{[1\sigma \text{ value of (background + FLIR) noise}]} \quad (1)$$

$\sigma_g$  specifies the intensity pattern spot size on the FLIR image plane relative to pixel size,  $\sigma_d$  and  $\tau_d$  describe target dynamics, and  $\sigma_a$  is the root mean square atmospheric jitter.

Tables 9.1 and 9.2 depict the performance of the two tracking algorithms at the end of 5 sec as a function of realistic S/N (20, 10, and 1) and  $\sigma_g$  values (3 pixels and 1 pixel). These results are for

TABLE 9.1

*Mean Error and  $1\sigma$  Error Comparisons with  $\sigma_g = 3$  Pixels*

S/N	Correlation tracker		Extended Kalman filter	
	Mean error (pixels)	$1\sigma$ error (pixels)	Mean error (pixels)	$1\sigma$ error (pixels)
20	0.5	1.5	0.0	0.2
10	3.0	3.0	0.0	0.2
1	15.0	30.0	0.0	0.8

TABLE 9.2

*Mean Error and  $1\sigma$  Error Comparisons with  $\sigma_g = 1$  Pixel*

S/N	Correlation tracker		Extended Kalman filter	
	Mean error (pixels)	$1\sigma$ error (pixels)	Mean error (pixels)	$1\sigma$ error (pixels)
20	7.0	8.0	0.0	0.2
10	8.0	10.0	0.0	0.2
1	15.0	30.0	0.0	0.8

$\tau_d = 1$  sec and averaged over realistic cases of  $\sigma_d/\sigma_a = 5, 1$ , and  $0.2$ . The extended Kalman filter consistently outperforms the correlation tracker. Lowering S/N to 1 degrades the filter performance slightly and decreasing spot size to 1 pixel (i.e.,  $\sigma_g = 1$  pixel in Table 9.2) has no noticeable effect, whereas both of these variations seriously affect the correlation tracker.

The filter exploits knowledge unused by the tracker—size, shape, and motion characteristics of target, and atmospheric jitter spectral description—to yield the enhanced performance. Because the correlation tracker makes no specific assumptions about the target, it may be a more robust algorithm: able to perform well in the face of large variations in the true characteristics of the system. Nevertheless, the filter is not overly sensitive to small errors in its assumed model of the tracking task. For instance, with  $S/N = 20$ ,  $\sigma_g = 3$  pixels,  $\sigma_d/\sigma_a = 5$ , and  $\tau_d = 1$  sec, the filter's mean error remained at 0, and  $1\sigma$  error only increased from 0.2 to 0.3 pixels when its assumed  $\sigma_g$  was changed from 3 pixels to 1. ■

The preceding example considered vector measurement updates. Note that, unlike the linear or linearized Kalman filters, the extended Kalman filter can generate a different state estimate if measurements are incorporated recursively at a sample time instead. Consider (9-61)–(9-65) with  $\mathbf{R}(t_i)$  diagonal. If  $\mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)]$  is computed and decomposed into  $m$  rows  $\mathbf{H}_1, \mathbf{H}_2, \dots, \mathbf{H}_m$ , then vector updating and  $n$  iterative scalar updates yield the same value of  $\hat{\mathbf{x}}(t_i^+)$ . However, if each new  $\hat{\mathbf{x}}(t_i^+)$  generated by incorporating a single scalar measurement is used to evaluate the ensuing  $\mathbf{H}_k$ , to provide a better point about which to

linearize  $\mathbf{h}$ , as

$$\mathbf{H}_k = \left. \frac{\partial \mathbf{h}_k[\mathbf{x}, t_i]}{\partial \mathbf{x}} \right|_{\mathbf{x} = \hat{\mathbf{x}}(t_i^+)} \quad \begin{array}{l} \text{incorporating } (k-1) \\ \text{components of } \mathbf{z}(t_i) \end{array} \quad (9-72)$$

then the two forms of updating will differ in their results.

Of practical importance because of significantly reduced online computation are *precomputed-gain extended Kalman filters*. They provide state estimates according to (9-62) and (9-65), but with gain histories precomputed as in the linearized Kalman filter, using a precomputed nominal state trajectory. Note that this in fact differs from the linearized filter, which uses the same  $\mathbf{K}(t_i)$  time history, but generates state estimates according to

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i)\{\mathbf{z}_i - \mathbf{h}[\mathbf{x}_n(t_i), t_i] - \mathbf{H}(t_i; \mathbf{x}_n(t_i))[\hat{\mathbf{x}}(t_i^-) - \mathbf{x}_n(t_i)]\} \quad (9-73)$$

$$\dot{\hat{\mathbf{x}}}(t/t_i) = \mathbf{f}[\mathbf{x}_n(t_i), t_i] + \mathbf{F}(t; \mathbf{x}_n(t))[\hat{\mathbf{x}}(t/t_i) - \mathbf{x}_n(t)] \quad (9-74)$$

Within this class of estimators, the *constant-gain extended Kalman filter* (CGEKF) [50, 51] is especially significant. For a *time-invariant* nonlinear system description in which the nominal is an *equilibrium* state trajectory,  $\mathbf{x}_n = \text{const}$ , the linear perturbation equations are themselves time invariant, and the  $\mathbf{K}(t_i)$  associated with the linearized filter can reach a steady state value,  $\mathbf{K}_{ss}$ . Then the constant-gain extended Kalman filter is defined by

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}_{ss}\{\mathbf{z}_i - \mathbf{h}[\hat{\mathbf{x}}(t_i^-)]\} \quad (9-75)$$

$$\dot{\hat{\mathbf{x}}}(t/t_i) = \mathbf{f}[\hat{\mathbf{x}}(t/t_i), \mathbf{u}(t)] \quad (9-76)$$

Such a filter form will be useful when incorporated into a control system that constantly drives the system state toward the equilibrium condition, thereby maintaining the adequacy of models based on linearizations about that pre-computed value. More will be said about this form in Chapter 15 (Volume 3) on stochastic controllers for nonlinear systems.

*Continuous-time continuous-measurement extended Kalman filters* can be based upon state dynamics and measurement models of

$$d\mathbf{x}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] dt + \mathbf{G}(t) d\beta(t) \quad (9-77a)$$

$$d\mathbf{y}(t) = \mathbf{h}[\mathbf{x}(t), t] dt + d\beta_m(t) \quad (9-77b)$$

often written in the less rigorous form

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] + \mathbf{G}(t)\mathbf{w}(t) \quad (9-78a)$$

$$\mathbf{z}(t) = \mathbf{h}[\mathbf{x}(t), t] + \mathbf{v}_c(t) \quad (9-78b)$$

by formally letting  $\mathbf{w} = d\beta/dt$ ,  $\mathbf{v}_c = d\beta_m/dt$ ,  $\mathbf{z} = d\mathbf{y}/dt$ , where  $\mathbf{w}(\cdot, \cdot)$  and  $\mathbf{v}_c(\cdot, \cdot)$  are zero-mean white (Gaussian) noises that are uncorrelated with each other

with statistics (using subscript c to denote continuous-time)

$$E\{\mathbf{w}(t)\mathbf{w}^T(t + \tau)\} = \mathbf{Q}(t)\delta(\tau) \quad (9-79a)$$

$$E\{\mathbf{v}_c(t)\mathbf{v}_c^T(t + \tau)\} = \mathbf{R}_c(t)\delta(\tau) \quad (9-79b)$$

The estimator equation is given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}[\hat{\mathbf{x}}(t), \mathbf{u}(t), t] + \mathbf{K}(t)\{\mathbf{z}(t) - \mathbf{h}[\hat{\mathbf{x}}(t), t]\} \quad (9-80)$$

where the gain  $\mathbf{K}(t)$  is given by

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^T[t; \hat{\mathbf{x}}(t)]\mathbf{R}_c^{-1}(t) \quad (9-81)$$

and  $\mathbf{P}(t)$  satisfies

$$\begin{aligned} \dot{\mathbf{P}}(t) &= \mathbf{F}[t; \hat{\mathbf{x}}(t)]\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T[t; \hat{\mathbf{x}}(t)] + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t) \\ &\quad - \mathbf{P}(t)\mathbf{H}^T[t; \hat{\mathbf{x}}(t)]\mathbf{R}_c^{-1}(t)\mathbf{H}[t; \hat{\mathbf{x}}(t)]\mathbf{P}(t) \end{aligned} \quad (9-82)$$

where  $\mathbf{F}[t; \hat{\mathbf{x}}(t)]$  and  $\mathbf{H}[t; \hat{\mathbf{x}}(t)]$  are the partials of  $\mathbf{f}$  and  $\mathbf{h}$ , respectively, with respect to  $\mathbf{x}$ , evaluated at  $\mathbf{x} = \hat{\mathbf{x}}(t)$ . Precomputed and constant gain forms of this filter can also be applied in practice [50, 51].

*Nonlinear full-order observers* can be defined of the form of (9-62) and (9-65), or (9-80) for continuous measurements, with the gain  $\mathbf{K}(t_i)$  or  $\mathbf{K}(t)$  chosen by other means than used in the extended Kalman filter. For constant-gain observers, *pole placement* or more extensive *eigenvalue/eigenvector assignment techniques* can be applied to the system of perturbation equations about an equilibrium solution to achieve desirable observer characteristics (see Chapter 14, Section 4, in Volume 3). Another means of establishing gain sequences in the discrete-measurement form is through *stochastic approximations* [14], yielding estimators that are not optimal with respect to statistical characteristics but that possess desirable, well-defined convergence properties, even in the face of large parameter uncertainties.

In certain applications, local iterations (over a single sample period) on nominal trajectory redefinition and subsequent relinearization may be warranted for performance improvement and practical computationally. The resulting *iterated extended Kalman filter* and *iterated linearized filter-smoother* are capable of providing better performance than the basic extended Kalman filter, especially in the face of significant nonlinearities, because of the improved reference trajectories incorporated into the estimates [8, 22, 52].

Recall the extended Kalman filter algorithm, (9-61)–(9-69). The fundamental idea of the iterated filter is that, once  $\hat{\mathbf{x}}(t_i^+)$  is generated by using (9-62), then this value would serve as a better state estimate than  $\hat{\mathbf{x}}(t_i^-)$  for evaluating  $\mathbf{h}$  and  $\mathbf{H}$  in the measurement update relations. Then the state estimate after measurement incorporation could be recomputed, iteratively if desired. Thus, (9-61)–(9-62) would be replaced by setting  $\hat{\mathbf{x}}_0$  equal to  $\hat{\mathbf{x}}(t_i^-)$  and performing an

iteration on

$$\mathbf{K}(t_i) = \mathbf{P}(t_i^-) \mathbf{H}^T(t_i; \hat{\mathbf{x}}_k) [\mathbf{H}(t_i; \hat{\mathbf{x}}_k) \mathbf{P}(t_i^-) \mathbf{H}^T(t_i; \hat{\mathbf{x}}_k) + \mathbf{R}(t_i)]^{-1} \quad (9-83)$$

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i) \{ \mathbf{z}_i - \mathbf{h}(\hat{\mathbf{x}}_k, t_i) - \mathbf{H}(t_i, \hat{\mathbf{x}}_k) [\hat{\mathbf{x}}(t_i^-) - \hat{\mathbf{x}}_k] \} \quad (9-84)$$

for  $k = 0, 1, \dots, N-1$ , and then setting  $\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}_N$  (the iteration could be stopped when consecutive values  $\hat{\mathbf{x}}_k$  and  $\hat{\mathbf{x}}_{k+1}$  differ by less than a preselected amount). Note that  $\hat{\mathbf{x}}_1$  is just the  $\hat{\mathbf{x}}(t_i^+)$  estimate provided by a simple extended Kalman filter.

The algorithm just described addresses the problem of significant nonlinearities by reevaluating  $\mathbf{h}$  and  $\mathbf{H}$  to achieve a better  $\hat{\mathbf{x}}(t_i^+)$ . This will also improve estimation over future intervals because of improved succeeding reference trajectories. It is also possible to improve the reference trajectory used for propagation over  $[t_{i-1}, t_i]$  once the measurement  $\mathbf{z}_i$  is taken, by applying smoothing techniques backward to time  $t_{i-1}$ . Incorporating such a local iteration into the extended Kalman filter structure yields what is termed the iterated linearized filter-smoother [22], but because of its limited applicability due to significant computational burden, the algorithm is not presented in explicit detail.

## 9.6 SUMMARY

This chapter presented means of extending linear models, techniques, and insights further into the problem of state estimation. Fundamentally, these methods entailed (1) decreasing the filter's confidence in the adequacy of the linear model within its structure, (2) telling the filter to discount or totally ignore older data because of the cumulative errors resulting from an erroneous linear model for time propagations, and (3) attempting to incorporate nonlinearities into the model while still exploiting linear estimation techniques. The next chapter investigates estimation based upon a linear system model with uncertain parameters within its structure. Succeeding chapters will then address proper development of general nonlinear stochastic models, and state estimation based upon such extended models.

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## PROBLEMS

**9.1** Recall the example of a gyro on test, as described in Problem 8.1, and the Kalman filter developed to estimate the gyro drift rate  $x(t)$ . Assume that adequate performance is not being achieved, and you suspect that this is caused by modeling inadequacies. We want to explore some of the compensation techniques of this chapter on filter operation. As discussed in Example 5.4 of Volume 1, "tracking" properties of the filter are indicated by the gain  $K(t_i)$ : if  $K(t_i)$  is approximately one,  $\hat{x}(t_i^+)$  is approximately equal to the most recent measurement  $z_i$ ; if  $K(t_i)$  is very small, then  $\hat{x}(t_i^+)$  is not "tracking" the measurements closely, but rather is heavily weighting the output of its own internal system model.

(a) Generate the  $K(t_i)$  time history corresponding to the two-hour interval as depicted in Problem 8.1 and Example 5.4.

(b) Demonstrate the effect of adding "pseudonoise" by increasing  $Q$  from  $2 \text{ deg}^2/\text{hr}$  to  $4 \text{ deg}^2/\text{hr}$ , and computing the time histories of  $P(t_i^-)$ ,  $P(t_i^+)$ , and  $K(t_i)$  values.

(c) As an alternative, consider putting a lower bound of 0.35 on  $P(t_i^+)$ , and generate the corresponding time histories.

(d) Repeat for the case of using age-weighting instead, with an age-weighting time constant  $T_a$  of 1 hr. Repeat with  $T_a = 0.5$  hr.

(e) Apply the Schmidt  $\varepsilon$  technique to this problem, with  $\varepsilon = 0.25$ . Repeat with  $\varepsilon = 0.5, 0.75$ , and 1.0.

(f) Consider development of a one-state MVRO estimator for this problem, assuming that a better model ("truth model") for the gyro would modify

$$\dot{x}(t) = -x(t) + w(t)$$

with  $w(\cdot, \cdot)$  having a power spectral density value of  $2 \text{ deg}^2/\text{hr}$  over all frequencies, into the model

$$\dot{x}(t) = -x(t) + n(t)$$

where  $n(\cdot, \cdot)$  is a zero-mean first-order Gauss-Markov process with power spectral density value of about 2 at the low frequencies but band-limited with a break frequency of 3 rad/sec:

$$\bar{\psi}_{nn}(\omega) = 18/(\omega^2 + 9)$$

or

$$E\{n(t)n(t + \tau)\} = 6 \exp\{-3|\tau|\}$$

Relate this result to the previous modifications via pseudonoise addition and filter gain compensation.

**9.2** Assume that a certain sensor acts as a low-pass filter in providing an output voltage signal  $s$ , in response to a variation in the quantity being measured, denoted as  $u$ . Due to the effects of parameter variations in the sensor and associated electronics, a slowly drifting bias  $b$  is also introduced into the output. Furthermore, instrument noise and quantization error affect the digital readout from the sensor package, this being effectively modeled by white noise  $v$  of zero mean and strength  $R$ . Thus, a model of the sensor would be as shown in Fig. 9.P1. With the sample period chosen as  $\Delta t$  sec, the slowly drifting bias is essentially a constant for 3 sample periods, but such a model becomes invalid for 4 or more periods.

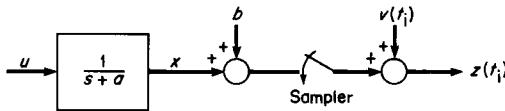


FIG. 9.P1

(a) Write the equations for the appropriate finite memory filter to estimate  $x$ , a "smoothed" version of the input  $u$ .

(b) Generate the filter equations to estimate  $x$  using

- (1) a pseudonoise addition,
- (2) the Schmidt  $\varepsilon$  technique, and
- (3) the Fagin age-weighting technique

as ad hoc procedures to maintain a valid estimate of  $b$  and  $x$ ; explain how the value of the "design" parameters,  $Q$ ,  $\varepsilon$ , and  $\tau$ , respectively, might be established.

**9.3** Derive the age-weighting filter results (9-10) and (9-11) by the means described in the text prior to these relations.

**9.4** Show that the age-weighted or fading memory filter described by (9-10) and (9-11) can also be expressed in the following convenient manner if  $\mathbf{Q}_d = 0$ : For measurement update at time  $t_i$ , either

$$\mathbf{K}(t_i) = \bar{\mathbf{P}}'(t_i^-) \mathbf{H}^T(t_i) [\mathbf{H}(t_i) \bar{\mathbf{P}}'(t_i^-) \mathbf{H}^T(t_i) + \{\mathbf{R}(t_i)/s\}]^{-1}$$

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i) [\mathbf{z}_i - \mathbf{H}(t_i) \hat{\mathbf{x}}(t_i^-)]$$

$$\bar{\mathbf{P}}'(t_i^+) = s \{\bar{\mathbf{P}}'(t_i^-) - \mathbf{K}(t_i) \mathbf{H}(t_i) \bar{\mathbf{P}}'(t_i^-)\}$$

or

$$\begin{aligned}\bar{\mathbf{P}}'(t_i^+) &= [\{s\bar{\mathbf{P}}'(t_i^-)\}^{-1} + \mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i)\mathbf{H}(t_i)]^{-1} \\ \mathbf{K}(t_i) &= \bar{\mathbf{P}}'(t_i^+)\mathbf{H}^T(t_i)\mathbf{R}^{-1}(t_i) \\ \hat{\mathbf{x}}(t_i^+) &= \hat{\mathbf{x}}(t_i^-) + \mathbf{K}(t_i)[\mathbf{z}_i - \mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)]\end{aligned}$$

and, for time propagation to the next sample time,

$$\begin{aligned}\hat{\mathbf{x}}(t_{i+1}^-) &= \Phi(t_{i+1}, t_i)\hat{\mathbf{x}}(t_i^+) + \mathbf{B}_d(t_i)\mathbf{u}(t_i) \\ \bar{\mathbf{P}}'(t_{i+1}^-) &= \Phi(t_{i+1}, t_i)\bar{\mathbf{P}}'(t_i^+)\Phi^T(t_{i+1}, t_i)\end{aligned}$$

- 9.5** (a) Generate the finite memory filter for the problem of Examples 9.1, 9.4, and 9.5:

$$\dot{\mathbf{b}}(t) = 0, \quad \mathbf{z}(t_i) = \mathbf{b}(t_i) + \mathbf{v}(t_i)$$

Using (9-25)–(9-29), develop finite memory filters for  $N = 2, 5$ , and  $10$ . Compare these to the results of the text example.

(b) Apply the approximate filter discussed at the end of Section 9.4 to this problem (see (9-32), (9-33), etc.), for  $N$  of  $2, 5$ , and  $10$ .

- (c) Show how the finite memory filters above change if the dynamics model is altered to

$$\dot{\mathbf{b}}(t) = -[0.1]\mathbf{b}(t)$$

**9.6** Derive the recursion relation (9-27) by writing out  $\mathcal{J}(t_i, t_{i-N+1})$  and  $\mathcal{J}(t_{i-1}, t_{i-N})$  according to (9-26) and equating like terms.

**9.7** (a) Write the explicit relations required for the linearized Kalman filter for Example 9.8. Which of these can be precomputed?

(b) Compare these to the detailed relations for implementing the extended Kalman filter for the same problem. Which of these can be precomputed?

(c) Now generate the precomputed-gain extended Kalman filter for this problem. Compare the details of this filter to the two previous algorithms.

(d) How would filter performance change if only range or only angle measurements were available? Consider observability of the linearized model (as linearized about a nominal circular orbit) for each case. What would this indicate about the linearized filter performance? What about the filters of parts (b) and (c) above?

(e) Show how the three types of filters in (a), (b), and (c) change if the available measurements are in the form of pressure and a direction cosine,

$$\begin{aligned}\mathbf{z}_1'(t_i) &= p_0 \exp\{-\gamma \mathbf{x}_1(t_i)\} + \mathbf{v}_1'(t_i) \\ \mathbf{z}_2'(t_i) &= \cos\{\mathbf{x}_3(t_i)\} + \mathbf{v}_2'(t_i)\end{aligned}$$

where  $p_0$  is sea level pressure and  $\gamma$  is a known decay factor, instead of simple range and angle as in Example 9.8.

**9.8** Consider a measurement update of the extended Kalman filter of Example 9.8. Explicitly write out the equations to incorporate the range and angle measurements as a two-dimensional vector update, and compare these to the iterative scalar updating with these two measurements. Note the difference of the two results.

**9.9** (a) Write out the explicit equations for the extended Kalman filter for the thrust-vector control problem of Example 9.9. Note that  $\mathbf{w}(t)$  does not affect the servo:  $\delta(t_0)$  is known exactly and angle commands  $\delta_{com}$  are the only servo input; what implications does this have on the dimension of the state vector in the filter? In actual practice,  $\delta_{com}$  is constant over the interval between

measurement updates, and  $\delta_{\text{com}}(t_i)$  for use from  $t_i$  to  $t_{i+1}$  is computed as

$$\delta_{\text{com}}(t_i) = -\mathbf{G}_c^*(t_i; t_f, \hat{\omega}_b^2)\hat{\mathbf{x}}(t_i^+)$$

where  $\mathbf{G}_c^*$  is the optimal controller gain matrix, propagated backwards from final time  $t_f$  using the best estimate of  $\omega_b^2$ , and  $\hat{\mathbf{x}}(t_i^+)$  is the optimal estimate of the five original states in the linear system model. This will be developed in Chapters 13–15 in Volume 3. However, for this problem consider  $\delta_{\text{com}}$  as a deterministic input.

(b) Now generate the precomputed-gain extended Kalman filter for this same problem. Be aware of implications of the assumed equilibrium solution that is used for gain evaluation. Consider a constant-gain extended Kalman filter. How does this compare to the precomputed-gain filter, and why might it be a suitable replacement for the precomputed-gain filter in this application?

**9.10** Assume that you want to design a filter to separate a signal  $s(t)$  from a signal-plus-noise input, modeled as

$$i(t) = s(t) + n(t)$$

where the signal and noise are assumed stationary, zero-mean, and uncorrelated with each other, with power spectral density descriptions, respectively, of

$$\bar{\psi}_{ss}(\omega) = \frac{\frac{5}{12}}{\omega^2 + 4}$$

$$\bar{\psi}_{nn}(\omega) = \frac{\frac{7}{12}}{\omega^2 + 16}$$

(a) To do so, assume that you take sampled-data measurements every 0.2 sec, of the form

$$z(t_i) = s(t_i) + n(t_i) + v(t_i)$$

where  $v(\cdot, \cdot)$  is zero-mean white Gaussian discrete-time noise of variance 0.01, modeling the effect of errors induced by analog-to-digital conversion with a finite wordlength (see Problem 6.2 of Volume 1). Generate the appropriate Kalman filter to estimate  $s(t)$ .

(b) Now assume that filter performance has been considered inadequate in actual operation. You suspect some unmodeled effects, and so you conduct some further analysis of the signal generator. The initial model has been a zero-mean white Gaussian noise  $w(\cdot, \cdot)$ , of strength  $\frac{5}{12}$ ; i.e.,

$$E\{w(t)w(t + \tau)\} = \frac{5}{12}\delta(t)$$

driving a linear equation

$$\dot{x}(t) = -2x(t) + w(t)$$

with an output of

$$s(t) = x(t)$$

After some analysis, a refined model is produced as

$$\begin{aligned}\dot{x}(t) &= -2x(t) + 0.05x^2(t) + w(t) \\ s(t) &= x(t) - 0.02x^2(t)\end{aligned}$$

Design the linearized and extended Kalman filters for this application, explicitly writing out all required vectors and matrices.

(c) Generate the precomputed-gain and constant-gain extended Kalman filters for this problem, and compare to the results in (b).

(d) Describe how one could account for these nonlinearities in ways other than using an extended Kalman filter.

**9.11** Reconsider Problem 9.10, but now in the context of continuous-time measurements being available. Assume measurements are available as

$$\mathbf{z}(t) = \mathbf{s}(t) + \mathbf{n}(t)$$

where  $\bar{\psi}_{ss}(\omega)$  is as given in Problem 9.10, but

$$\bar{\psi}_{nn}(\omega) = \frac{\frac{7}{12}}{\omega^2 + 16} + 0.1$$

- (a) Generate the appropriate continuous-measurement Kalman filter to estimate  $s(t)$ .
- (b) Consider the model modification given in part (b) of Problem 9.10, and design the linearized and extended Kalman filters for this continuous-measurement case.
- (c) As in the previous problem, generate the precomputed-gain and constant-gain extended Kalman filters for this problem, and compare these to the results of part (b).

**9.12** (a) Explicitly write out the extended Kalman filter equations for Example 9.10.  
 (b) Assume that a controller can point the center of the field of view toward any commanded location in one sample period, without error. Thus, at time  $t_i$ , we can propagate the estimate to obtain  $\hat{\mathbf{x}}(t_{i+1})$ , and the first two components,  $\hat{x}_{dyn}(t_{i+1})$  and  $\hat{y}_{dyn}(t_{i+1})$ , can be used as pointing commands. How should the filter's computed  $\hat{\mathbf{x}}(t_{i+1})$  be modified if this control action is taken? How does this affect the next residuals? Compare to "open-loop" tracking in which the center of the field of view continues to point in one direction for all time.

(c) Show how the filter changes when a dynamics model appropriate for some airborne targets,

$$\begin{aligned}\dot{x}_{dyn}(t) &= v_x(t) \\ \dot{v}_x(t) &= a_x(t) \\ \ddot{a}_x(t) &= (-1/T_a)a_x(t) + w_x(t)\end{aligned}$$

with  $w_x(\cdot, \cdot)$  zero-mean white Gaussian noise with  $E\{\mathbf{w}_x(t)\mathbf{w}_x(t+\tau)\} = Q_x \delta(\tau)$ , and similarly for the  $y$  direction, replaces the benign target dynamics model of Example 9.10. Note that  $a_x(\cdot, \cdot)$  is a first order Gauss-Markov process, with autocorrelation kernel of

$$E\{\mathbf{a}(t)\mathbf{a}(t+\tau)\} = \sigma_{ax}^2 \exp\{-|\tau|/T_a\}$$

and that  $\sigma_{ax}^2$  and  $T_a$  (and corresponding values for the  $y$  axis) can be selected to match acceleration characteristics of many airborne vehicles.

(d) Repeat part (c), but now use the *nonlinear* model for acceleration that assumes the target will exhibit constant turn-rate trajectories (with pseudonoise added to account for model inadequacies, etc.),

$$\ddot{\mathbf{a}}(t) = -\omega^2 \mathbf{v}(t) + \mathbf{w}(t)$$

where

$$\omega = |\mathbf{v}(t) \times \mathbf{a}(t)| / |\mathbf{v}(t)|^2$$

and  $|\cdot|$  denotes magnitude, and  $\times$  denotes cross-product. Be particularly careful in evaluation of  $\partial f_i / \partial \mathbf{x}$ : recall that, by definition of partial derivatives,  $\partial f_i(x_1, x_2, \dots, x_j, \dots, x_n) / \partial x_j$  is to be evaluated under the assumption that all other  $x$  components besides  $x_j$  are to be held constant.

**9.13** This is a computer problem that assumes access to a Monte Carlo filter performance evaluation algorithm such as [38], as described in Section 6.8 and Problem 7.14 of Volume 1 for the linear model case; extensions to the nonlinear truth model (and filter) case will be discussed in

Chapter 11 (and Chapter 12). Consider a “truth model” description for Example 9.8 for the case of  $u_r = u_t = 0$  and let  $G = 1$ , so that one possible nominal solution to the original nonlinear equations would be the circular orbit with  $r_0 = 1$  radius unit,  $\omega = 1 \text{ rad}/(\text{time unit})$ , and  $\theta(t) = \omega t$ . Note that the radial velocity is zero and tangential velocity is  $r_0\omega = 1 \text{ radius unit}/(\text{time unit})$  for this orbit.

If an impulsive velocity change  $\Delta v$  is applied tangentially at  $t = 0$ , then a “Hohmann transfer” ellipse results as in Fig. 9.P2. Here  $r_0$  is both the radius of the circular orbit and the perigee of the elliptical orbit, while  $r_1$  is the apogee of the ellipse;  $v_c$  is the velocity magnitude for the circular orbit (i.e.,  $r_0\omega$ ) and  $[v_c + \Delta v]$  is the velocity at  $r_0$  for the Hohmann ellipse. The values of  $v_c$ ,  $\Delta v$ ,  $r_0$ , and  $r_1$  can be related by

$$(\Delta v/v_c)^2 = 3 - [2r_0/(r_0 + r_1)] - 2\sqrt{2r_1/(r_0 + r_1)}$$

Consider a highly elliptical trajectory in which  $r_1 = 10r_0$ . This yields  $\Delta v \cong 0.3484v_c$ , and thereby the appropriate initial condition on  $\theta$  at  $t = 0$  for the elliptical orbit.

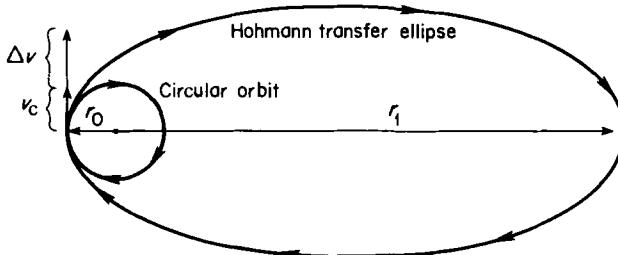


FIG. 9.P2

Assume measurements of  $\theta$  only, corrupted by white measurement noise of variance  $R$ , every  $\Delta t = 0.5$  time units. Design both linearized and extended Kalman filters based on the model given in Example 9.8, with  $u_r \equiv u_t \equiv 0$  and  $w_1(\cdot, \cdot)$  and  $w_2(\cdot, \cdot)$  assumed to be zero-mean white Gaussian noises, independent of each other, and of strengths  $Q_1$  and  $Q_2$ , respectively. Initially assume the nominal circular orbit for both filters.

Compare the performance of linearized and extended Kalman filters over an interval of 5 time units, letting the filter-assumed  $R$  and truth model  $R$  agree, for the cases of

- (a) truth model = circular orbit,  $R = 0.01$ ,  $Q_1 = Q_2 = 10^{-6}$ ,
- (b) truth model = circular orbit,  $R = 0.0001$ ,  $Q_1 = Q_2 = 10^{-6}$ ,
- (c) truth model = ellipse,  $R = 0.01$ ,  $Q_1 = Q_2 = 10^{-6}$ ,
- (d) truth model = ellipse,  $R = 0.0001$ ,  $Q_1 = Q_2 = 10^{-6}$ ,
- (e) truth model = ellipse,  $R = 0.01$ ,  $Q_1$  and  $Q_2$  tuned for best performance,
- (f) truth model = ellipse,  $R = 0.0001$ ,  $Q_1$  and  $Q_2$  tuned for best performance.

Specifically look at the mean  $\pm 1\sigma$  (standard deviation) time histories of errors between true and estimated variables.

# CHAPTER 10

## Parameter uncertainties and adaptive estimation

### 10.1 INTRODUCTION

For many applications of interest, various techniques can be used to produce an adequate system description in the form of a linear system model driven by known inputs and white Gaussian noises, with which it is possible to develop an optimal state estimator and/or controller. However, the optimality of these devices is dependent upon complete knowledge of the parameters that define the best model for system dynamics, output relations, and statistical description of uncertainties. In any practical application, these quantities are known only with some uncertainty, and the performance degradation that results from improperly assigned values can be severe.

The extent of the model uncertainty and the sensitivity of filter performance to such uncertainty vary substantially from one problem to another. For example, in spacecraft thrust vector control problems, the physics involved are well understood and can lead to a very accurate linear system model, but certain key parameters, as bending mode natural frequency, are typically known imprecisely. An optimal filter or filter/controller combination is sensitively tuned to this parameter, and even a small deviation in its value can cause inadequate, or even unstable, control. On the other hand, the model is less well defined for the dynamics involved in many process control applications. Distillation towers would be a prime example: a wide range of complex dynamics occur within each tower, but practical experience has shown that adequate control is possible by assuming an approximate model of a second order system plus a time delay. Although the towers are qualitatively similar, the appropriate parameters for the linear model differ for each particular tower, and vary slowly in time as well. A priori information about these parameters would typically be in the form of only a range of physically admissible values.

Thus, in order to improve the quality of the state estimates, it would be desirable in many instances to estimate a number of uncertain parameters in the dynamics or measurement model simultaneously in an online fashion. This is often termed combined *state estimation and system identification* [6, 13, 28, 31, 35, 55, 58, 63, 82, 84, 88, 89, 91, 112, 115, 116, 117]. A number of methods have been suggested for developing such capacity, such as using an extended Kalman filter to solve the nonlinear estimation problem that results from treating the parameters as additional state variables. However, these techniques usually depend upon a priori parameter statistical information (difficult to provide with confidence in many practical situations), or require a difficult interpretation of physical knowledge about the parameters (i.e., that they are more slowly varying than the states) into the specification of appropriate noise strengths to drive the dynamics model. One objective of this chapter is to provide a feasible state and parameter estimator that (1) does not require complete a priori parameter statistics, but can utilize any such information that is available, (2) allows the engineer to use his knowledge that the parameters are slowly varying (if at all) in a direct and physically meaningful manner, and (3) provides, or yields to approximations that provide, both online capability and adequate performance.

Previous chapters demonstrated the effect of improper filter tuning, i.e., improper selection of parameters to describe noise statistics in a problem formulation, upon the precision of state estimation. One would like to readjust the assumed noise strengths in the filter's internal model, based upon information obtained in real time from the measurements becoming available, so that the filter is continually "tuned" as well as possible. Such an algorithm is often termed an *adaptive or self-tuning estimation* algorithm, and this class of algorithms will also be investigated in this chapter [5, 6, 59, 88, 91, 103, 141]. Actually, this differs from a state estimation/system identification algorithm only in which parameters are assumed to be uncertain enough to require estimation, and all such algorithms could be properly termed adaptive estimators, or simultaneous estimators of states and uncertain parameters. Because of the interrelation of concepts involved in problem formulation and solution for these two areas, it is fruitful to consider them together [62, 72].

In all cases, the key to adaptation will be the residuals of the state estimator. Since these are the differences between actual measurements and best measurement predictions based upon the filter's internal model, consistent mismatch indicates erroneous model formulation, and particular characteristics of the mismatch can be exploited to perform the needed adaptation.

Section 10.2 formulates the basic problem of parameter uncertainties and adaptive estimation. The next four sections treat uncertainties in  $\Phi$  and  $\mathbf{B}_d$ , using maximum likelihood techniques to develop the full scale estimator, evaluating its performance capabilities, and attaining online applicability without severely degrading performance. Similarly, Section 10.7 develops the

maximum likelihood estimator for the case of uncertainties being confined to  $\mathbf{Q}_d$  and/or  $\mathbf{R}$ . In subsequent sections, other solution methods are developed: Bayesian and multiple model filtering algorithms, correlation methods, and covariance matching techniques.

## 10.2 PROBLEM FORMULATION

Assume that the system of interest can be described by means of the linear stochastic difference equation

$$\mathbf{x}(t_{i+1}) = \Phi(t_{i+1}, t_i)\mathbf{x}(t_i) + \mathbf{B}_d(t_i)\mathbf{u}(t_i) + \mathbf{G}_d(t_i)\mathbf{w}_d(t_i) \quad (10-1)$$

from which are available discrete-time measurements modeled by the linear relation

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \quad (10-2)$$

It is assumed that  $\mathbf{w}_d(\cdot, \cdot)$  and  $\mathbf{v}(\cdot, \cdot)$  are independent, zero-mean, white Gaussian noise processes with covariance kernels

$$E\{\mathbf{w}_d(t_i)\mathbf{w}_d^T(t_j)\} = \mathbf{Q}_d(t_i)\delta_{ij} \quad (10-3)$$

$$E\{\mathbf{v}(t_i)\mathbf{v}^T(t_j)\} = \mathbf{R}(t_i)\delta_{ij} \quad (10-4)$$

where  $\mathbf{Q}_d(t_i)$  is positive semidefinite and  $\mathbf{R}(t_i)$  is positive definite for all  $t_i$ . The initial condition on the state is known only with some uncertainty in general, and  $\mathbf{x}(t_0)$  is described by means of a Gaussian random vector assumed independent of  $\mathbf{w}_d(\cdot, \cdot)$  and  $\mathbf{v}(\cdot, \cdot)$ , with known mean and covariance:

$$E\{\mathbf{x}(t_0)\} = \hat{\mathbf{x}}_0 \quad (10-5a)$$

$$E\{[\mathbf{x}(t_0) - \hat{\mathbf{x}}_0][\mathbf{x}(t_0) - \hat{\mathbf{x}}_0]^T\} = \mathbf{P}_0 \quad (10-5b)$$

where  $\mathbf{P}_0$  is positive semidefinite.

In a system identification problem context, some parameters defining the state transition matrix  $\Phi(t_{i+1}, t_i)$ , input matrix  $\mathbf{B}_d(t_i)$ , or measurement matrix  $\mathbf{H}(t_i)$  time histories are not determined completely. These parameters might be explicit elements of these matrices, but need not be. For instance, (10-1) would most naturally arise as an equivalent discrete-time model to represent a continuous-time system with sampled-data measurements; then a single uncertain parameter in the  $\mathbf{F}(t)$  matrix of the continuous-time description will generally affect many elements of both  $\Phi(t_{i+1}, t_i)$  and  $\mathbf{B}_d(t_i)$ . Uncertainties in  $\mathbf{H}(t_i)$  can be deemphasized for two reasons: (1) generally  $\mathbf{H}(t_i)$  is known more precisely than  $\Phi(t_{i+1}, t_i)$  or  $\mathbf{B}_d(t_i)$  in practical problems and (2) uncertainties in  $\mathbf{H}$  often cannot be distinguished from uncertainties in  $\Phi$  or  $\mathbf{B}_d$  on the basis of observing time histories of  $\mathbf{z}(t_i)$  and  $\mathbf{u}(t_i)$  values, so one can choose a state space description that avoids uncertain parameters in  $\mathbf{H}$ . For example, in the simple

case of a scalar state system model with no control inputs or driving noise, the  $i$ th measurement is  $[H(t_i)\Phi(t_i, t_0)x_0 + v(t_i)]$ , from which an estimate can be made of the product  $H(t_i)\Phi(t_i, t_0)$ , but not of both  $H(t_i)$  and  $\Phi(t_i, t_0)$  separately. Note that uncertainties in  $\mathbf{G}_d(t_i)$  are not included, but this case can be treated equivalently as uncertain parameters in  $\mathbf{Q}_d(t_i)$ .

In an adaptive estimator context, uncertain parameters are considered to exist in the  $\mathbf{Q}_d$  and/or  $\mathbf{R}$  matrices. These are to be estimated simultaneously with the states in a filter algorithm.

The designer often lacks sufficient information to develop valid or complete statistical or probability density function models for these parameters. In fact, he might be able to provide at best a range of possible values and a most probable value for each parameter by examining the physics of the problem. These parameters are distinguished from the state variables in that they will vary significantly more slowly than the states, and may in fact be time invariant. It is this characterization of uncertain parameters that an effective estimation algorithm should exploit in a direct manner.

The objective is to improve the quality of the state estimation (and possibly of a stochastic controller as well, to be discussed later) by simultaneously estimating some of the uncertain parameters in the model structure. From a computational feasibility standpoint, it will be important to perform a sensitivity analysis [82, 83, 85, 100, 101] a priori to identify the parameters that are crucial to estimate, rather than to estimate all whose values are at all uncertain. Moreover, it may be advantageous computationally to attempt to choose the best parameter value from a set of discrete values rather than to consider a continuous range of possible values. Such practical aspects will be considered once the full-scale solution to the problem has been described in detail.

Of the many possible means of formulating this problem, the maximum likelihood technique has been chosen for development in this chapter [65, 82, 108, 115, 121, 147]. This choice was motivated by many considerations. First, some properties of a general maximum likelihood estimator make it especially attractive [25, 107, 121, 137, 145]. The following characteristics have been proven by Cramér [25] for the case of independent, identically distributed measurements, and the corresponding generalizations to the context of the current problem will be developed subsequently. If an efficient estimate exists (i.e., if there exists an unbiased estimate with finite covariance such that no other unbiased estimate has a lower covariance), it can always be found through maximum likelihood methods. Further, if an efficient estimate exists, the likelihood equation will have a unique solution that equals the efficient estimate [127]. Under rather general conditions, the likelihood equation has a solution that converges in probability to the true value of the variables as the number of sample elements grows without bound; i.e., it is consistent [10–12, 109, 139]. This solution is an asymptotically Gaussian and asymptotically efficient

estimate. Kerr [68] further asserts these additional properties. If any single sufficient statistic for the estimated variable exists, the maximum likelihood estimate will be sufficient, and under very general conditions, it will be at least asymptotically sufficient and unbiased. Even though the estimate will generally be biased for small samples, it will provide the unique minimum attainable variance estimate under the existence of sufficient statistics, attaining the Cramér–Rao lower bound [137] if this is possible. Without sufficient statistics, this optimal behavior cannot be proven for small samples, but it will still be asymptotically optimal and usually a good small-sample estimator.

With regard to the bias, maximum likelihood estimates tend to have the true value of the estimated variable near the center of their distributions, so that the bias is often negligible. Levin [75] further states that if the measurement noise variance is small with respect to the actual system output, then the bias in the estimate of the pulse transfer function parameters of the system model will be negligible compared to the standard deviation of the estimate. He also demonstrates that, in the absence of the Gaussian assumption on the noises, the maximum likelihood estimate will at least provide a generalized least squares fit to data.

Besides these desirable properties, other considerations tend to favor maximum likelihood techniques as well. Once a particular conditional density has been propagated, the various possible estimates, as the mode, median, or mean, will often be very close, especially for unimodal densities concentrated about their mean. Therefore, it is logical to choose the estimate that is easiest to compute or approximate. Schweppe [119–121] and others have noted that estimates based on the peak value of the density often have this characteristic. Furthermore, the mean and median estimate computations are made complex by truncating the density, as by imposing an admissible range of parameter values, whereas the maximum likelihood estimate is not affected at all. In his work toward simultaneously estimating the system state and noise statistics, Abramson [1] chose the maximum likelihood method partially because of severe analytical difficulties in a minimum variance formulation. Especially since one objective of this chapter is to provide online algorithms, the rationale of using the estimator which yields the simplest implementation is important, so the use of maximum likelihood techniques is further substantiated.

A potentially simpler technique might be the method of weighted least squares [52, 54, 143]. With the “proper” choice of weighting factors, this method can derive the Kalman filter and other powerful estimation results. However, the appropriate choice of the weighting factors is usually based upon considerable hindsight gained from comparison to other estimation techniques. The primary conceptual disadvantage to this method is that it does not attempt to propagate statistical or probabilistic information in time, and the knowledge of certain conditional densities provides a complete, rational basis of estimation. Although the available statistical information can often be incorporated

into the weighting factors, the estimation technique itself does not reveal how to do so.

Bayesian estimation is conceptually satisfying in that it propagates the conditional density of the variables to be estimated, conditioned upon the values assumed by the sample elements that are actually observed. However, this requires an a priori specification of a parameter density function, and sufficient statistical information to define such a density adequately is quite often lacking in real applications. A maximum likelihood formulation need not suffer from this drawback. The classical likelihood function is the conditional density of the measurements, conditioned upon the value of the uncertain parameters, and the estimation is then a form of hypothesis testing to find the values that maximize the probability of the events (the measurements) that have actually occurred. However, there are also more general forms of likelihood functions [82, 120], in the form of appropriately defined conditional densities, which can exploit as much, or as little, of the a priori statistical information as is available. In this respect, the maximum likelihood approach is more generally applicable than Bayesian methods. Similarly, Kashyap [65] has asserted that maximum likelihood methods are more generally applicable than least squares, the instrumental variable method, stochastic approximation, or other methods commonly suggested for system identification purposes [2, 51, 71, 114].

Perhaps the single greatest disadvantage to maximum likelihood estimation is the lack of theoretical knowledge about the behavior of the estimates for small sample sizes. Much is known about the asymptotic behavior as the number of samples becomes infinite, however, and a considerable amount of practical experience with the method instills confidence in its viability.

To exploit the maximum likelihood method, one must choose an appropriate likelihood function. This in turn requires a mathematical model for the parameters. An “optimal” estimation of the parameters should exploit the knowledge that their values will be more consistent from measurement sample time to sample time than will any of the states or noises associated with the problem. An adequate and readily applied model is that the parameters are *essentially constant over any given interval of  $N$  sample periods* (where  $N$  is chosen by the designer) [29, 82, 121]. That is, at a given time  $t_i$ , the parameters are modeled as remaining constant over the samples  $(i - N + 1)$  to  $i$ . At the next sample time  $t_{i+1}$ , the parameters are again assumed to be constants, though possibly of different magnitude, over the samples  $(i - N + 2)$  to  $(i + 1)$ . A polynomial of any fixed order could be used to model parameters over (longer) fixed-duration intervals, but use of a constant can usually be justified as the best a priori approximation, the assumed form that is easiest to work with as a designer (as, to set the appropriate value of  $N$ ), and the form that yields the least complex estimation equations (of distinct importance for online applicability).

Note that such a model does not require the designer to make assumptions about the form of a priori probability densities to describe the parameters, a

difficult task leading to rather arbitrary choices, often with erroneously biasing effects. Moreover, there is no contorted stochastic description of the parameters, such as the output of an integrator driven by white Gaussian noise of strength chosen in an iterative manner so as to yield reasonable results. Instead, the designer must determine an adequate value of  $N$  for the “essentially constant over  $N$  sample periods” model from the physics of the particular problem at hand: a more straightforward application of his physical knowledge. Factors involved in the choice of  $N$  are fully discussed later in Section 10.6.

Based upon this model for the parameters, one is naturally led to considering a fixed-length memory type of estimator for the parameters. Conceptually, one can “remember” only the most recent  $N$  data points and ask, what constant values for the parameters will fit these  $N$  samples best in a maximum likelihood sense? By proper choice of likelihood function, such a parameter estimate can be combined with a state estimate that depends either on the most recent  $N$  samples of data or on all data taken since the initial time. The choice of the likelihood function for eventual implementation will depend upon tractability of the resulting equations, physical sensibility of the likelihood function, and overall performance of the estimator.

This formulation will provide an online estimator that would remain sensitive to any slow parameter variations, whereas a growing-length estimator employing a constant-parameter model would become less sensitive to parameter changes occurring later in the interval of interest than those occurring earlier. On the other hand, an estimator with  $N > 1$  will not be overly sensitive to single points of bad data, as would be the case if the parameter estimate were based upon only the current single measurement.

Apparent drawbacks of the fixed-length memory formulation of parameter estimation would be the necessity to “remember”  $N$  sets of measurement data at all times and the foreseeable use of “batch processing” of the data collected over an  $N$ -period interval each time a parameter estimate is made. With regard to objectionable memory requirements, as the system passes from time  $t_{i-1}$  to  $t_i$ , the information provided by the  $(i - N)$ th set of data would be removed, and the data from the  $i$ th measurement could be overwritten into these locations of an inexpensive form of memory. Batch processing will not be required online: because the parameters are slowly varying, they can be estimated less frequently than the state variables (for instance, only every  $N$  sample periods, at least after an initial transient period of the first  $N$  sample times).

### 10.3 UNCERTAINTIES IN $\Phi$ AND $\mathbf{B}_d$ : LIKELIHOOD EQUATIONS

Assume that an adequate system representation has been generated in the form of Eqs. (10-1)–(10-5), but that  $p$  uncertain parameters have been identified in the state transition matrix ( $\Phi$ ) and deterministic input matrix ( $\mathbf{B}_d$ ) time

histories. Let these parameters be arrayed as the components of a  $p$ -dimensional vector, denoted as  $\mathbf{a}$ . Further assume that the following matrices (required for parameter estimation) can be evaluated (or approximated) for all times of interest:  $\partial\Phi(t_{i+1}, t_i)/\partial a_k$  and  $\partial\mathbf{B}_d(t_i)/\partial a_k$  for  $k = 1, 2, \dots, p$ .

First one must ask if simultaneous state and parameter estimation is possible to accomplish. One cannot really attempt to estimate the “true” values of states and parameters in a system: there are no  $n$ -dimensional processes in nature, only processes that can be modeled adequately with an  $n$ -state representation. Thus, one attempts to find the values of states and parameters that, when substituted into the assumed model structure, yield a model output behavior that best duplicates the actual system performance in some respect. The ability to perform the estimation will therefore be determined by *conditions upon the mathematical model* employed as an adequate representation. If the estimation is impossible with the originally formulated problem, one may need to incorporate different measurements, additional measurements, or a modified system model in order to satisfy these conditions.

Basically, of the system modes (quasi-static perhaps) that a particular parameter affects, at least one must be (1) observable, (2) excited by the initial conditions or controllable with respect to the points of entry of the dynamic noise  $\mathbf{w}_d(\cdot, \cdot)$  or those of the deterministic input (assumed not identically zero), and (3) such that  $\Phi$  and  $\mathbf{B}_d$  do not assume identical values over a range of parameter values or for a number of discrete parameter values (as in an aliasing phenomenon). Refined theoretical “identifiability” criteria have been established in the literature [10, 11, 27, 36, 82, 102, 124, 127–129, 132–134], but they are generally difficult to verify. The generalized ambiguity function concept of Section 10.5 will also indicate whether the estimation of particular parameters is feasible (in a practical manner), and it will further predict the accuracy with which each can be estimated. Thus, adequacy of performance would be evaluated practically through an ambiguity function analysis and simulation results.

Now let us develop the likelihood equations to be solved to generate the state and parameter estimates. For any specified likelihood function  $L[\theta(t_i), \mathcal{X}_i]$ , where  $\theta(t_i)$  is the vector of variables to be estimated and  $\mathcal{X}_i$  is the set of realized values of the measurements to be used as data, the objective of maximum likelihood estimation is to find that value of  $\theta^*(t_i)$  that maximizes  $L[\theta(t_i), \mathcal{X}_i]$  as a function of  $\theta(t_i)$ . When  $L[\theta(t_i), \mathcal{X}_i]$  is differentiable with respect to  $\theta(t_i)$  and the maximum actually lies within the admissible range of parameter values, this can be obtained by solving the likelihood equation:

$$\frac{\partial L[\theta(t_i), \mathcal{X}_i]}{\partial \theta(t_i)} \Bigg|_{\theta(t_i) = \theta^*(t_i)} = \mathbf{0}^T \quad (10-6)$$

Table 10.1 presents the likelihood functions for which results have been obtained [82]. Those in the left numbered column yield growing-length memory algorithms, whereas the right numbered column entries yield fixed-length

TABLE 10.1

*Candidate Likelihood Functions*

(1) $\ln f_{\mathbf{x}(t_i), \mathbf{a} \mathbf{z}(t_i)}(\xi, \alpha   \mathcal{L}_i)$	(1) $\ln f_{\mathbf{x}(t_i), \mathbf{a} \mathbf{z}_N(t_i)}(\xi, \alpha   \mathcal{L}_{i, i-N+1})$
(2) $\ln f_{\mathbf{z}(t_i) \mathbf{x}(t_i), \mathbf{a}}(\mathcal{L}_i   \xi, \alpha)$	(2a) $\ln f_{\mathbf{z}_N(t_i) \mathbf{x}(t_i), \mathbf{a}}(\mathcal{L}_{i, i-N+1}   \xi, \alpha)$
	(2b) $\ln f_{\mathbf{z}_N(t_i) \mathbf{z}(t_{i-N}), \mathbf{x}(t_i), \mathbf{a}}(\mathcal{L}_{i, i-N+1}   \mathcal{L}_{i-N}, \xi, \alpha)$
(3) $\ln f_{\mathbf{x}(t_i) \mathbf{z}(t_i), \mathbf{a}}(\xi   \mathcal{L}_i, \alpha)$	(3) $\ln f_{\mathbf{x}(t_i) \mathbf{z}_N(t_i), \mathbf{a}}(\xi   \mathcal{L}_{i, i-N+1}, \alpha)$
(4) $\ln f_{\mathbf{x}(t_i), \mathbf{z}(t_i) \mathbf{a}}(\xi, \mathcal{L}_i   \alpha)$	(4a) $\ln f_{\mathbf{x}(t_i), \mathbf{z}_N(t_i) \mathbf{a}}(\xi, \mathcal{L}_{i, i-N+1}   \alpha)$
	(4b) $\ln f_{\mathbf{x}(t_i), \mathbf{z}_N(t_i) \mathbf{z}(t_{i-N}), \mathbf{a}}(\xi, \mathcal{L}_{i, i-N+1}   \mathcal{L}_{i-N}, \alpha)$

$\mathbf{x}(t_i)$  = state vector at time  $t_i$   
 $\mathbf{a}$  = parameter vector  
 $\mathbf{z}(t_i)$  = measurement history  $\mathbf{z}(t_1), \mathbf{z}(t_2), \dots, \mathbf{z}(t_i)$   
 $\mathbf{z}_N(t_i)$  = most recent  $N$  measurements  $\mathbf{z}(t_{i-N+1}), \mathbf{z}(t_{i-N+2}), \dots, \mathbf{z}(t_i)$   
 $\mathbf{z}(t_{i-N})$  = measurement history  $\mathbf{z}(t_1), \mathbf{z}(t_2), \dots, \mathbf{z}(t_{i-N})$

memory parameter estimators with either growing-length or fixed-length state estimators. Consequently, the functions on the left also serve to generate starting procedures for the corresponding functions on the right before  $N$  samples have been accumulated.

Of these prospective likelihood functions, the fourth entry is probably the best choice for most applications, in that it exploits all a priori information typically available and can yield an effective and computationally feasible estimator. To see this, let us compare the various entries.

The first entry in Table 10.1 would be the most “logical” choice from a Bayesian viewpoint, but it requires specification of a priori parameter statistics (unlike the others since  $\mathbf{a}$  appears to the right of the conditioning sign in the other entries). Such statistics are usually difficult, if not impossible, to assess, and  $\mathbf{a}$  is assumed to be of specific form (Gaussian, uniform, etc.) solely to admit solutions to the problem formulated. In the case of a uniform density between two boundary values, the same results can be achieved more readily by using the fourth entry in the table and limiting the estimates to the specified range.

The second entry in Table 10.1 is the “classical” likelihood function for generating maximum likelihood estimates. However, Bayes’ rule (applied to entries in the left column of Table 10.1 for sake of argument) yields

$$f_{\mathbf{z}(t_i)|\mathbf{x}(t_i), \mathbf{a}} f_{\mathbf{x}(t_i)|\mathbf{a}} = f_{\mathbf{x}(t_i), \mathbf{z}(t_i)|\mathbf{a}} \quad (10-7)$$

In other words, the information contained in the second entry is also contained in the fourth entry, but the latter’s dependence upon the propagation of the a priori state statistics, given by  $f_{\mathbf{x}(t_i)|\mathbf{a}}(\xi | \alpha)$ , has been removed. In most practical problems, initial state statistics are both available and useful for estimation performance improvement.

Bayes' rule can also be used to depict the difference between the third and fourth entries of Table 10.1:

$$f_{\mathbf{x}(t_i)|\mathbf{z}(t_i), \alpha} f_{\mathbf{z}(t_i)|\alpha} = f_{\mathbf{x}(t_i), \mathbf{z}(t_i)|\alpha} \quad (10-8)$$

Because  $f_{\mathbf{z}(t_i)|\alpha}(\mathcal{Z}_i|\alpha)$  is not an explicit function of the state values, the state estimators found by using either the third or fourth entry in (10-6) will be of the same algorithmic form. However, the third entry does not include the very terms that are highly sensitive to parameter values and thereby yield good parameter estimates.

Thus, the fourth entry is the preferable form. In the right column of the table, entry (4a) will yield a fixed-length memory estimator of both states and parameters, whereas (4b) will combine a growing-length memory state estimator with a fixed-length memory parameter estimator. This latter form will be the most useful, from the viewpoints of both performance and computational burden. Consequently, attention will be concentrated on this form and the corresponding entry in the left column of Table 10.1 for the associated startup procedure.

First, consider  $\ln f_{\mathbf{x}(t_i), \mathbf{z}(t_i)|\alpha}(\xi, \mathcal{Z}_i|\alpha)$ . Bayes' rule can be applied repeatedly to obtain

$$\begin{aligned} f_{\mathbf{x}(t_i), \mathbf{z}(t_i)|\alpha} &= f_{\mathbf{x}(t_i)|\mathbf{z}(t_i), \alpha} f_{\mathbf{z}(t_i)|\alpha} \\ &= f_{\mathbf{x}(t_i)|\mathbf{z}(t_i), \alpha} f_{\mathbf{z}(t_i)|\mathbf{z}(t_{i-1}), \alpha} f_{\mathbf{z}(t_{i-1})|\alpha} \\ &\vdots \\ &= f_{\mathbf{x}(t_i)|\mathbf{z}(t_i), \alpha} \prod_{j=1}^i f_{\mathbf{z}(t_j)|\mathbf{z}(t_{j-1}), \alpha} \end{aligned} \quad (10-9)$$

where, since the first measurement occurs at time  $t_1$ , the term in the product for  $j = 1$  is  $f_{\mathbf{z}(t_1)|\alpha}(\zeta_1|\alpha)$ . Each of the separate densities in (10-9) can be written out explicitly as a Gaussian density:

$$\begin{aligned} f_{\mathbf{x}(t_i)|\mathbf{z}(t_i), \alpha}(\xi | \mathcal{Z}_i, \alpha) &= \frac{1}{(2\pi)^{n/2} |\mathbf{P}(t_i^+)|^{1/2}} \exp\{\cdot\} \\ \{\cdot\} &= \left\{ -\frac{1}{2} [\xi - \hat{\mathbf{x}}(t_i^+)]^T \mathbf{P}(t_i^+)^{-1} [\xi - \hat{\mathbf{x}}(t_i^+)] \right\} \end{aligned} \quad (10-10)$$

where  $\hat{\mathbf{x}}(t_i^+)$  and  $\mathbf{P}(t_i^+)$  are implicitly dependent upon the particular parameter values upon which the density is conditioned, and

$$\begin{aligned} f_{\mathbf{z}(t_j)|\mathbf{z}(t_{j-1}), \alpha}(\zeta_j | \mathcal{Z}_{j-1}, \alpha) &= \frac{1}{(2\pi)^{m/2} |\mathbf{A}(t_j)|^{1/2}} \exp\{\cdot\} \\ \{\cdot\} &= \left\{ -\frac{1}{2} [\zeta_j - \mathbf{H}(t_j)\hat{\mathbf{x}}(t_j^-)]^T \mathbf{A}(t_j)^{-1} [\zeta_j - \mathbf{H}(t_j)\hat{\mathbf{x}}(t_j^-)] \right\} \end{aligned} \quad (10-11)$$

where

$$\mathbf{A}(t_j) = \mathbf{H}(t_j)\mathbf{P}(t_j^-)\mathbf{H}^T(t_j) + \mathbf{R}(t_j) \quad (10-12)$$

where again  $\hat{\mathbf{x}}(t_j^-)$ ,  $\mathbf{P}(t_j^-)$ , and  $\mathbf{A}(t_j)$  are implicitly functions of the given parameter values. By substituting these expressions into (10-9), the likelihood function can be written as

$$\begin{aligned} & \ln f_{\mathbf{x}(t_i), \mathbf{z}(t_i)|\mathbf{a}}(\xi, \mathcal{L}_i | \alpha) \\ &= -\frac{n+im}{2} \ln(2\pi) - \frac{1}{2} \ln(|\mathbf{P}(t_i^+)|) - \frac{1}{2} \sum_{j=1}^i \ln(|\mathbf{A}(t_j)|) \\ &\quad - \frac{1}{2} [\xi - \hat{\mathbf{x}}(t_i^+)]^T \mathbf{P}(t_i^+)^{-1} [\xi - \hat{\mathbf{x}}(t_i^+)] \\ &\quad - \frac{1}{2} \sum_{j=1}^i [\zeta_j - \mathbf{H}(t_j)\hat{\mathbf{x}}(t_j^-)]^T \mathbf{A}(t_j)^{-1} [\zeta_j - \mathbf{H}(t_j)\hat{\mathbf{x}}(t_j^-)] \end{aligned} \quad (10-13)$$

The desired likelihood equations are found through Eq. (10-6), which can be written as the simultaneous solution to

$$\frac{\partial}{\partial \xi} [\ln f_{\mathbf{x}(t_i), \mathbf{z}(t_i)|\mathbf{a}}(\xi, \mathcal{L}_i | \alpha)] \Bigg|_{\substack{\xi = \mathbf{x}^*(t_i) \\ \alpha = \mathbf{a}^*(t_i)}} = \mathbf{0}^T \quad (10-14a)$$

$$\frac{\partial}{\partial \alpha} [\ln f_{\mathbf{x}(t_i), \mathbf{z}(t_i)|\mathbf{a}}(\xi, \mathcal{L}_i | \alpha)] \Bigg|_{\substack{\xi = \mathbf{x}^*(t_i) \\ \alpha = \mathbf{a}^*(t_i)}} = \mathbf{0}^T \quad (10-14b)$$

Using (10-13), the first of these becomes

$$-[\xi - \hat{\mathbf{x}}(t_i^+)]^T \mathbf{P}(t_i^+)^{-1} \Bigg|_{\substack{\xi = \mathbf{x}^*(t_i) \\ \alpha = \mathbf{a}^*(t_i)}} = \mathbf{0}^T$$

for which the solution is

$$\mathbf{x}^*(t_i) = \hat{\mathbf{x}}(t_i^+) \Bigg|_{\alpha = \mathbf{a}^*(t_i)} \quad (10-15)$$

In other words, the maximum likelihood estimate of the system state at time  $t_i$  is provided by the Kalman filter algorithm of Chapter 5, with the estimate of  $\mathbf{a}^*(t_i)$  replacing the nominal parameter values for the required propagations.

Now consider the partial derivative of the likelihood function (10-13) with respect to the  $p$ -dimensional vector of parameters, as in (10-14b). Typical forms to appear in this evaluation are the partials of a log of a determinant and of a quadratic form involving an inverse matrix; such partials can be expressed

as [82]:

$$\frac{\partial \ln|\mathbf{X}|}{\partial \alpha_k} = \frac{\partial \ln|\mathbf{X}|}{\partial |\mathbf{X}|} \frac{\partial |\mathbf{X}|}{\partial \alpha_k} = \frac{1}{|\mathbf{X}|} \frac{\partial |\mathbf{X}|}{\partial \alpha_k} = \text{tr} \left\{ \mathbf{X}^{-1} \frac{\partial \mathbf{X}}{\partial \alpha_k} \right\} \quad (10-16a)$$

$$\frac{\partial \mathbf{X}^{-1}}{\partial \alpha_k} = -\mathbf{X}^{-1} \frac{\partial \mathbf{X}}{\partial \alpha_k} \mathbf{X}^{-1} \quad (10-16b)$$

where  $\mathbf{X}$  is a square matrix and  $\alpha_k$  is the  $k$ th component of  $\alpha$ . Using these relations, the desired partial derivatives of (10-13) become

$$\begin{aligned} & -2 \frac{\partial}{\partial \alpha_k} \{ \ln f_{\mathbf{x}(t_i), \mathbf{z}(t_i)|\mathbf{a}}(\xi, \mathcal{L}_i | \alpha) \} \\ &= \text{tr} \left\{ \mathbf{P}(t_i^+)^{-1} \frac{\partial \mathbf{P}(t_i^+)}{\partial \alpha_k} \right\} \\ & - 2 \frac{\partial \hat{\mathbf{x}}(t_i^+)^T}{\partial \alpha_k} \mathbf{P}(t_i^+)^{-1} [\xi - \hat{\mathbf{x}}(t_i^+)] \\ & - [\xi - \hat{\mathbf{x}}(t_i^+)]^T \mathbf{P}(t_i^+)^{-1} \frac{\partial \mathbf{P}(t_i^+)}{\partial \alpha_k} \mathbf{P}(t_i^+)^{-1} [\xi - \hat{\mathbf{x}}(t_i^+)] \\ & + \sum_{j=1}^i \text{tr} \left\{ \mathbf{A}(t_j)^{-1} \frac{\partial \mathbf{A}(t_j)}{\partial \alpha_k} \right\} \\ & - 2 \sum_{j=1}^i \frac{\partial \hat{\mathbf{x}}(t_j^-)^T}{\partial \alpha_k} \mathbf{H}(t_j)^T \mathbf{A}(t_j)^{-1} [\zeta_j - \mathbf{H}(t_j) \hat{\mathbf{x}}(t_j^-)] \\ & - \sum_{j=1}^i [\zeta_j - \mathbf{H}(t_j) \hat{\mathbf{x}}(t_j^-)]^T \mathbf{A}(t_j)^{-1} \frac{\partial \mathbf{A}(t_j)}{\partial \alpha_k} \mathbf{A}(t_j)^{-1} [\zeta_j - \mathbf{H}(t_j) \hat{\mathbf{x}}(t_j^-)] \end{aligned} \quad (10-17)$$

From Eq. (10-15),  $\xi$  is simultaneously being set equal to  $\hat{\mathbf{x}}(t_i^+)|_{\alpha=\mathbf{a}^*(t_i)}$ , so  $[\xi - \hat{\mathbf{x}}(t_i^+)] = \mathbf{0}$  in (10-17). Since  $\mathbf{f}^T \mathbf{g} = \text{tr}\{\mathbf{f}\mathbf{g}^T\} = \text{tr}\{\mathbf{g}\mathbf{f}^T\}$  for general vectors  $\mathbf{f}$  and  $\mathbf{g}$ , the likelihood equation for each parameter can be written as (for  $k = 1, 2, \dots, p$ ):

$$\begin{aligned} & \text{tr} \left\{ \mathbf{P}(t_i^+)^{-1} \frac{\partial \mathbf{P}(t_i^+)}{\partial \alpha_k} \right\} - 2 \sum_{j=1}^i \frac{\partial \hat{\mathbf{x}}(t_j^-)^T}{\partial \alpha_k} \mathbf{H}(t_j)^T \mathbf{A}(t_j)^{-1} [\mathbf{z}_j - \mathbf{H}(t_j) \hat{\mathbf{x}}(t_j^-)] \\ & + \sum_{j=1}^i \text{tr} \left\{ [\mathbf{A}(t_j)^{-1} - \mathbf{A}(t_j)^{-1} [\mathbf{z}_j - \mathbf{H}(t_j) \hat{\mathbf{x}}(t_j^-)] [\mathbf{z}_j - \mathbf{H}(t_j) \hat{\mathbf{x}}(t_j^-)]^T \mathbf{A}(t_j)^{-1}] \right. \\ & \times \left. \frac{\partial \mathbf{A}(t_j)}{\partial \alpha_k} \right\}_{\alpha=\mathbf{a}^*(t_i)} = 0 \end{aligned} \quad (10-18)$$

Unfortunately, there is *no general closed form solution* to this equation, and so an iterative procedure will have to be employed to determine  $\mathbf{a}^*(t_i)$ . The solution

to the  $p$  equations of the form of (10-18) and the result of (10-15) yield the simultaneous maximum likelihood estimate of states and parameters.

To provide a fixed-length memory parameter estimator, consider the likelihood function  $\ln f_{\mathbf{x}(t_i), \mathbf{z}_N(t_i)|\mathbf{z}(t_{i-N}), \mathbf{a}}(\xi, \mathcal{L}_{i, i-N+1}|\mathcal{L}_{i-N}, \alpha)$ , where  $\mathbf{z}_N(t_i)$  is an  $Nm$ -dimensional random vector composed of the  $N$  most recent measurements,  $\mathbf{z}(t_{i-N+1}), \mathbf{z}(t_{i-N+2}), \dots, \mathbf{z}(t_i)$ , a particular realization of which would be denoted as  $\mathbf{Z}_{i, i-N+1}$ , and the corresponding dummy vector as  $\mathcal{L}_{i, i-N+1}$ . By Bayes' rule,

$$\begin{aligned} f_{\mathbf{x}(t_i), \mathbf{z}_N(t_i)|\mathbf{z}(t_{i-N}), \mathbf{a}} &= f_{\mathbf{x}(t_i)|\mathbf{z}_N(t_i), \mathbf{z}(t_{i-N}), \mathbf{a}} f_{\mathbf{z}_N(t_i)|\mathbf{z}(t_{i-N}), \mathbf{a}} \\ &= f_{\mathbf{x}(t_i)|\mathbf{z}_N(t_i), \mathbf{z}(t_{i-N}), \mathbf{a}} \frac{f_{\mathbf{z}_N(t_i), \mathbf{z}(t_{i-N})|\mathbf{a}}}{f_{\mathbf{z}(t_{i-N})|\mathbf{a}}} \\ &= f_{\mathbf{x}(t_i)|\mathbf{z}(t_i), \mathbf{a}} \frac{f_{\mathbf{z}(t_i)|\mathbf{a}}}{f_{\mathbf{z}(t_{i-N})|\mathbf{a}}} \\ &= f_{\mathbf{x}(t_i)|\mathbf{z}(t_i), \mathbf{a}} \prod_{j=i-N+1}^i f_{\mathbf{z}(t_j)|\mathbf{z}(t_{j-1}), \mathbf{a}} \end{aligned} \quad (10-19)$$

This is identical in form to (10-9) except that the lower limit on the product terms is  $(i - N + 1)$  instead of 1. By an analogous derivation, it can be concluded that the *desired estimator is given by (10-15) and (10-18) with the lower limits on the summations changed to  $(i - N + 1)$* :

$$\mathbf{x}^*(t_i) = \hat{\mathbf{x}}(t_i^+) \Big|_{\mathbf{a}=\mathbf{a}^*(t_i)} \quad (10-20)$$

with the  $p$  components  $a_k^*(t_i)$  satisfying the likelihood equations

$$\begin{aligned} \text{tr} \left\{ \mathbf{P}(t_i^+)^{-1} \frac{\partial \mathbf{P}(t_i^+)}{\partial \alpha_k} \right\} - 2 \sum_{j=i-N+1}^i \frac{\partial \hat{\mathbf{x}}(t_j^-)^T}{\partial \alpha_k} \mathbf{H}(t_j)^T \mathbf{A}(t_j)^{-1} \mathbf{r}_j \\ + \sum_{j=i-N+1}^i \text{tr} \left\{ [\mathbf{A}(t_j)^{-1} - \mathbf{A}(t_j)^{-1} \mathbf{r}_j \mathbf{r}_j^T \mathbf{A}(t_j)^{-1}] \frac{\partial \mathbf{A}(t_j)}{\partial \alpha_k} \right\} \Big|_{\mathbf{a}=\mathbf{a}^*(t_i)} = 0 \end{aligned} \quad (10-21)$$

for  $k = 1, 2, \dots, p$ , where

$$\mathbf{A}(t_j) = \mathbf{H}(t_j) \mathbf{P}(t_j^-) \mathbf{H}(t_j)^T + \mathbf{R}(t_j) \quad (10-22a)$$

$$\mathbf{r}_j = \mathbf{z}_j - \mathbf{H}(t_j) \hat{\mathbf{x}}(t_j^-) \quad (10-22b)$$

#### 10.4 UNCERTAINTIES IN $\Phi$ AND $\mathbf{B}_d$ : FULL-SCALE ESTIMATOR

The previous section yielded likelihood equations, (10-15) and (10-18) or (10-20) and (10-21), the solution to which will provide the desired estimates of system states and parameters. Since these equations do not have a general closed form solution, it is necessary to consider iterative techniques.

The general form of the likelihood equations is given by Eq. (10-6), where  $\theta(t_i)$  is composed of an  $n$ -dimensional partition of states and a  $p$ -dimensional partition of parameters. If this equation were to be solved iteratively by the Newton–Raphson method [57, 64], the new estimate of  $\theta(t_i)$ , denoted as  $\hat{\theta}^*(t_i)$ , could be expressed in terms of a trial solution or previous best estimate, denoted as  $\hat{\theta}_*(t_i)$ , as (see Problem 10.10):

$$\hat{\theta}^*(t_i) = \hat{\theta}_*(t_i) - \left[ \frac{\partial^2 L[\hat{\theta}_*(t_i), \mathbf{Z}_i]}{\partial \theta^2} \right]^{-1} \frac{\partial L[\hat{\theta}_*(t_i), \mathbf{Z}_i]}{\partial \theta} \quad (10-23)$$

where the notation  $\{\partial L[\hat{\theta}_*(t_i), \mathbf{Z}_i]/\partial \theta\}^\top$  is meant to depict the vector quantity  $\{\partial L[\theta, \mathbf{Z}_i]/\partial \theta\}^\top$  evaluated at the point  $\theta = \hat{\theta}_*(t_i)$ : the *gradient* or *score vector*. To use this algorithm, the second derivative matrix, or Hessian,  $\partial^2 L[\hat{\theta}_*(t_i), \mathbf{Z}_i]/\partial \theta^2$  must be of full rank. Local iterations of (10-23) would be possible conceptually, but would not be practical for online applications. Unfortunately, even if only one iteration of (10-23) were to be processed at each time step, the computation and inversion of the required Hessian would put a prohibitive load on an online computer [39, 43].

Rao [107] has suggested an approximation called “scoring” which simplifies the computations substantially while maintaining accuracy over large samples. The approximation made is that

$$\frac{\partial^2 L[\hat{\theta}_*(t_i), \mathbf{Z}_i]}{\partial \theta^2} \cong -\mathbf{J}[t_i, \hat{\theta}_*(t_i)] \quad (10-24)$$

where the matrix

$$\mathbf{J}[t_i, \hat{\theta}_*(t_i)] \triangleq E \left\{ \frac{\partial L[\theta, \mathbf{Z}(t_i)]}{\partial \theta} \middle| \theta = \hat{\theta}_*(t_i) \right\} \quad (10-25)$$

is termed the *conditional information matrix* by Rao [107]. It can be shown that, in fact [82],

$$-\mathbf{J}[t_i, \hat{\theta}_*(t_i)] = E \left\{ \frac{\partial^2 L[\theta, \mathbf{Z}(t_i)]}{\partial \theta^2} \middle| \theta = \hat{\theta}_*(t_i) \right\} \quad (10-26)$$

so that the approximation being made is that the second derivative  $\partial^2 L/\partial \theta^2[\hat{\theta}_*(t_i), \mathbf{Z}_i]$ , for a particular realization of  $\mathbf{Z}(t_i, \omega_k) = \mathbf{Z}_i$ , can be adequately represented by its ensemble average over all possible measurement time histories. The desirability of such an approximation is that it requires generation of only first order information instead of first and second order, as required in using (10-23) directly. Thus, the first order correction to  $\hat{\theta}_*(t_i)$  becomes

$$\hat{\theta}^*(t_i) = \hat{\theta}_*(t_i) + \mathbf{J}[t_i, \hat{\theta}_*(t_i)]^{-1} \frac{\partial L[\hat{\theta}_*(t_i), \mathbf{Z}_i]}{\partial \theta} \quad (10-27)$$

Various other iterative techniques have been investigated, but “scoring” has been found to be a superior algorithm. The error committed by the approximation (10-24) is of order  $1/N$  for large  $N$ , and for large  $N$  its convergence rate approaches that of the Newton–Raphson method. In general, it converges less rapidly than a Newton–Raphson algorithm near the solution, but it converges from a larger region than Newton–Raphson and has very substantial computational advantages over this method. On the other hand, it converges more rapidly than a conventional gradient algorithm, requiring somewhat more computation to do so. Conjugate gradient methods might also be employed, but the literature reports certain undesirable numerical characteristics of this algorithm for the parameter estimation application.

There are some disadvantages to the “scoring” method, but these can be countered in practical usage. First, initial evaluations of  $\mathbf{J}[t_i, \hat{\theta}_*(t_i)]$  yield a matrix with small magnitude entries (and not a good approximation to the negative Hessian), so that its inverse has very large entries. Thus, a precomputed  $\mathbf{J}^{-1}$  (the value to which  $\mathbf{J}^{-1}$  might converge in simulation studies for an average value of  $\boldsymbol{\theta}$ , for instance) could be used in the recursion (10-27) for the first few sample periods, until  $\mathbf{J}[t_i, \hat{\theta}_*(t_i)]$  has grown to a reasonable “magnitude.” Another disadvantage might be the need to calculate the value and inverse of  $\mathbf{J}[t_i, \hat{\theta}_*(t_i)]$  repeatedly. However, since  $\mathbf{J}[t_i, \hat{\theta}_*(t_i)]$  typically will not change substantially after an initial transient, one could retain a fixed computed value for  $\mathbf{J}^{-1}$  after a certain state, periodically verifying whether the corresponding  $\mathbf{J}$  is still “close” to a currently computed  $\mathbf{J}[t_i, \hat{\theta}_*(t_i)]$ . The effect of these ad hoc approaches will be depicted through examples later.

Based on the previous discussion, the recursions to yield a full-scale solution to the state and parameter estimation problem will now be developed. This is not intended to provide online capability, but rather to establish the best performance achievable from maximum likelihood methods. Section 10.6 will develop means of furnishing online applicability based on insights gained from the full-scale estimator, and the performance and computational loading of the resulting estimators can then be compared to the benchmark of the full-scale estimator.

To implement the scoring iteration for equations of the form (10-21), it is necessary to generate the score  $\{\partial L[\hat{\mathbf{x}}_*(t_i), \hat{\mathbf{a}}_*(t_i), \mathbf{Z}_i] / \partial \mathbf{a}\}^T$  and the conditional information matrix  $\mathbf{J}[t_i, \hat{\mathbf{x}}_*(t_i), \hat{\mathbf{a}}_*(t_i)]$ . The score is a  $p$ -dimensional vector whose components are of the form of Eq. (10-21) times  $-\frac{1}{2}$ , but in which the terms are evaluated with the parameter estimate  $\hat{\mathbf{a}}_*(t_i)$ , rather than the actual but unknown maximum likelihood estimate  $\mathbf{a}^*(t_i)$ . To evaluate the required terms, it is convenient to decompose the  $N$ -step score into the sum of the  $N$  most recent single-measurement scores,  $\mathbf{s}^1[\mathbf{Z}_i, \hat{\mathbf{a}}_*(t_i)]$ , and a final term denoted as  $\gamma[\mathbf{Z}_i, \hat{\mathbf{a}}_*(t_i)]$ :

$$\frac{\partial L}{\partial a_k} [\hat{\mathbf{x}}_*(t_i), \hat{\mathbf{a}}_*(t_i), \mathbf{Z}_i] = \gamma_k [\mathbf{Z}_i, \hat{\mathbf{a}}_*(t_i)] + \sum_{j=i-N+1}^i s_k^{-1} [\mathbf{Z}_j, \hat{\mathbf{a}}_*(t_i)] \quad (10-28)$$

where

$$\begin{aligned} s_k^{-1}[\mathbf{Z}_j, \mathbf{a}] &= \frac{\partial \hat{\mathbf{x}}(t_j^-)^T}{\partial a_k} \mathbf{H}(t_j)^T \mathbf{A}(t_j)^{-1} \mathbf{r}_j \\ &\quad - \frac{1}{2} \text{tr} \left\{ [\mathbf{A}(t_j)^{-1} - \mathbf{A}(t_j)^{-1} \mathbf{r}_j \mathbf{r}_j^T \mathbf{A}(t_j)^{-1}] \frac{\partial \mathbf{A}(t_j)}{\partial a_k} \right\} \end{aligned} \quad (10-29a)$$

$$\gamma_k[\mathbf{Z}_i, \mathbf{a}] = -\frac{1}{2} \text{tr} \left\{ \mathbf{P}(t_i^+)^{-1} \frac{\partial \mathbf{P}(t_i^+)}{\partial a_k} \right\} \quad (10-29b)$$

The notation is meant to depict that, although  $s_k^{-1}[\mathbf{Z}_j, \mathbf{a}]$  explicitly contains only the residual at time  $t_j$ , this single-measurement score is in fact a function of the entire measurement history  $\mathbf{Z}_j$ .

With regard to the conditional information matrix, it is possible to obtain a closed-form expression for the case of the parameter value assuming its true (but unknown) value  $\mathbf{a}_t$  [82, 145]. It can be shown that the  $k$ th component of  $\mathbf{J}[t_i, \hat{\mathbf{x}}_*(t_i), \mathbf{a}_t]$  can be decomposed into the sum of the  $N$  most recent single-sample terms and one additional term [82]:

$$\begin{aligned} J_{kl}[t_i, \hat{\mathbf{x}}_*(t_i), \mathbf{a}_t] &= E\{\gamma_k[\mathbf{Z}(t_i), \mathbf{a}] \gamma_l[\mathbf{Z}(t_i), \mathbf{a}] | \mathbf{a} = \mathbf{a}_t\} \\ &\quad + \sum_{j=i-N+1}^i E\{s_k^{-1}[\mathbf{Z}(t_j), \mathbf{a}] s_l^{-1}[\mathbf{Z}(t_j), \mathbf{a}] | \mathbf{a} = \mathbf{a}_t\} \end{aligned} \quad (10-30)$$

where

$$\begin{aligned} E\{s_k^{-1}[\mathbf{Z}(t_j), \mathbf{a}] s_l^{-1}[\mathbf{Z}(t_j), \mathbf{a}] | \mathbf{a} = \mathbf{a}_t\} \\ = \frac{1}{2} \text{tr} \left[ \mathbf{A}(t_j)^{-1} \frac{\partial \mathbf{A}(t_j)}{\partial a_k} \mathbf{A}(t_j)^{-1} \frac{\partial \mathbf{A}(t_j)}{\partial a_l} \right. \\ \left. + 2\mathbf{A}(t_j)^{-1} \mathbf{H}(t_j) E \left\{ \frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} \frac{\partial \hat{\mathbf{x}}(t_j^-)^T}{\partial a_l} \middle| \mathbf{a} = \mathbf{a}_t \right\} \mathbf{H}(t_j)^T \right] \end{aligned} \quad (10-31a)$$

$$\begin{aligned} E\{\gamma_k[\mathbf{Z}(t_i), \mathbf{a}] \gamma_l[\mathbf{Z}(t_i), \mathbf{a}] | \mathbf{a} = \mathbf{a}_t\} \\ = \frac{1}{2} \text{tr} \left[ \mathbf{P}(t_i^+)^{-1} \frac{\partial \mathbf{P}(t_i^+)}{\partial a_k} \mathbf{P}(t_i^+)^{-1} \frac{\partial \mathbf{P}(t_i^+)}{\partial a_l} \right. \\ \left. + 2\mathbf{P}(t_i^+)^{-1} E \left\{ \frac{\partial \hat{\mathbf{x}}(t_i^+)}{\partial a_k} \frac{\partial \hat{\mathbf{x}}(t_i^+)^T}{\partial a_l} \middle| \mathbf{a} = \mathbf{a}_t \right\} \right] \end{aligned} \quad (10-31b)$$

The equations above are exact: no approximations were used to derive them. The approximation to be made is that the same expressions can be used for parameter values other than  $\mathbf{a}_t$ .

The parameter estimator seeks the value of the parameter vector, constant over the most recent  $N$  sample times, that best matches the system model

to the measurements taken from the real system. Thus, if the estimate is being made at time  $t_i$ , then  $\hat{\mathbf{x}}(t_{i-N}^+)$  and  $\mathbf{P}(t_{i-N}^+)$  are considered to be immutable; thus, the initial conditions for the  $N$ -step recursions are

$$\hat{\mathbf{x}}(t_{i-N}^+) = \text{previously computed} \quad (10-32a)$$

$$\mathbf{P}(t_{i-N}^+) = \text{previously computed} \quad (10-32b)$$

$$\frac{\partial \hat{\mathbf{x}}(t_{i-N}^+)}{\partial a_k} = \mathbf{0} \quad (\text{for all } k) \quad (10-32c)$$

$$\frac{\partial \mathbf{P}(t_{i-N}^+)}{\partial a_k} = \mathbf{0} \quad (\text{for all } k) \quad (10-32d)$$

$$E \left\{ \frac{\partial \hat{\mathbf{x}}(t_{i-N}^+)}{\partial a_k} \frac{\partial \hat{\mathbf{x}}(t_{i-N}^+)^T}{\partial a_l} \middle| \mathbf{a} = \hat{\mathbf{a}}_*(t_i) \right\} = \mathbf{0} \quad (\text{for all } k \text{ and } l) \quad (10-32e)$$

The computational process will now be described. Let the current time be  $t_i$ , and let the parameter estimate most recently computed (at the previous sample time  $t_{i-1}$ , assuming a new parameter estimate is generated at every time step) be denoted as  $\hat{\mathbf{a}}_*(t_i)$ . Using  $\hat{\mathbf{a}}_*(t_i)$  and starting from the initial conditions in (10-32), perform an  $N$ -step recursion of

- (1)  $\hat{\mathbf{x}}(t_{j-1}^+) \rightarrow \hat{\mathbf{x}}(t_j^-) \rightarrow \hat{\mathbf{x}}(t_j^+)$ .
- (2)  $\mathbf{P}(t_{j-1}^+) \rightarrow \mathbf{P}(t_j^-) \rightarrow \mathbf{P}(t_j^+)$ .
- (3) Compute  $\mathbf{s}^1[\mathbf{Z}_j, \hat{\mathbf{a}}_*(t_i)]$  and add to a running sum.
- (4) Compute  $E\{\mathbf{s}^1[\mathbf{Z}(t_j), \mathbf{a}] \mathbf{s}^{1T}[\mathbf{Z}(t_j), \mathbf{a}] \mid \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\}$  and add to a running sum.

At the end of the  $N$ -step recursion,  $\gamma[\mathbf{Z}_i, \hat{\mathbf{a}}_*(t_i)]$  is computed and added to the score running sum to form  $\{\partial L[\hat{\mathbf{x}}_*(t_i), \hat{\mathbf{a}}_*(t_i), \mathbf{Z}_i] / \partial \mathbf{a}\}^T$ , and similarly  $E\{\gamma[\mathbf{Z}(t_i), \mathbf{a}] \gamma^T[\mathbf{Z}(t_i), \mathbf{a}] \mid \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\}$  is computed and added to the conditional information matrix running sum to generate  $\mathbf{J}[t_i, \hat{\mathbf{x}}_*(t_i), \hat{\mathbf{a}}_*(t_i)]$ . Finally, a new parameter estimate is computed by means of

$$\hat{\mathbf{a}}^*(t_i) = \hat{\mathbf{a}}_*(t_i) + \mathbf{J}[t_i, \hat{\mathbf{x}}_*(t_i), \hat{\mathbf{a}}_*(t_i)]^{-1} \{\partial L[\hat{\mathbf{x}}_*(t_i), \hat{\mathbf{a}}_*(t_i), \mathbf{Z}_i] / \partial \mathbf{a}\}^T \quad (10-33)$$

Thus are generated a state estimate  $\hat{\mathbf{x}}(t_i^+)$ , evaluated using  $\hat{\mathbf{a}}_*(t_i)$ , and a new parameter estimate  $\hat{\mathbf{a}}^*(t_i)$ . Local iterations are possible to improve convergence, but they would require substantial computation. (These are often considered for offline identification.)

The recursions will now be described in detail. To propagate from just after measurement incorporation at time  $t_{j-1}$  to just before measurement incorporation at time  $t_j$ , for  $j = (i - N + 1), (i - N + 2), \dots, i$ , the relations (implicitly based on  $\hat{\mathbf{a}}_*(t_i)$ ) are in three categories. First, the *state related equations* are:

$$\hat{\mathbf{x}}(t_j^-) = \Phi(t_j, t_{j-1}) \hat{\mathbf{x}}(t_{j-1}^+) + \mathbf{B}_d(t_{j-1}) \mathbf{u}(t_{j-1}) \quad (10-34)$$

$$\mathbf{P}(t_j^-) = \Phi(t_j, t_{j-1}) \mathbf{P}(t_{j-1}^+) \Phi^T(t_j, t_{j-1}) + \mathbf{G}_d(t_{j-1}) \mathbf{Q}_d(t_{j-1}) \mathbf{G}_d^T(t_{j-1}) \quad (10-35)$$

$$\mathbf{A}(t_j) = \mathbf{H}(t_j)[\mathbf{P}(t_j^-)\mathbf{H}^T(t_j)] + \mathbf{R}(t_j) \quad (10-36)$$

$$\mathbf{K}(t_j) = [\mathbf{P}(t_j^-)\mathbf{H}^T(t_j)]\mathbf{A}^{-1}(t_j) \quad (10-37)$$

These are the usual Kalman filter equations, but the  $\Phi$  and  $\mathbf{B}_d$  matrices are evaluated using the most recent parameter estimate,  $\hat{\mathbf{a}}_*(t_i)$ .

The *score equations* necessary for time propagation are given by the  $p$  sets of “sensitivity system” equations

$$\begin{aligned} \frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} &= \Phi(t_j, t_{j-1}) \frac{\partial \hat{\mathbf{x}}(t_{j-1}^+)}{\partial a_k} + \frac{\partial \Phi(t_j, t_{j-1})}{\partial a_k} \hat{\mathbf{x}}(t_{j-1}^+) \\ &\quad + \frac{\partial \mathbf{B}_d(t_{j-1})}{\partial a_k} \mathbf{u}(t_{j-1}) \end{aligned} \quad (10-38)$$

$$\begin{aligned} \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} &= \frac{\partial \Phi(t_j, t_{j-1})}{\partial a_k} \mathbf{P}(t_{j-1}^+) \Phi^T(t_j, t_{j-1}) + \Phi(t_j, t_{j-1}) \mathbf{P}(t_{j-1}^+) \frac{\partial \Phi^T(t_j, t_{j-1})}{\partial a_k} \\ &\quad + \Phi(t_j, t_{j-1}) \frac{\partial \mathbf{P}(t_{j-1}^+)}{\partial a_k} \Phi^T(t_j, t_{j-1}) \end{aligned} \quad (10-39)$$

$$\frac{\partial \mathbf{A}(t_j)}{\partial a_k} = \mathbf{H}(t_j) \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} \mathbf{H}^T(t_j) \quad (10-40)$$

for  $k = 1, 2, \dots, p$ . Note that  $\partial \Phi(t_j, t_{j-1})/\partial a_k$  and  $\partial \mathbf{B}_d(t_{j-1})/\partial a_k$  must be known for all  $k$ . Note also that  $\partial \mathbf{u}(t_{j-1})/\partial a_k = \mathbf{0}$  because, when at time  $t_i$ , control inputs through time  $t_{i-1}$  have already been applied and cannot be changed, even for feedback control.

*Conditional information matrix computations* for propagating forward in time are (removing time arguments equal to  $t_{j-1}$  and shortening “ $\mathbf{a} = \hat{\mathbf{a}}_*(t_i)$ ” to “ $\hat{\mathbf{a}}_*$ ” in the conditioning):

$$\begin{aligned} E\left\{\frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} \frac{\partial \hat{\mathbf{x}}^T(t_j^-)}{\partial a_l} \middle| \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\right\} &= \Phi E\left\{\frac{\partial \hat{\mathbf{x}}^+}{\partial a_k} \frac{\partial \hat{\mathbf{x}}^{+T}}{\partial a_l} \middle| \hat{\mathbf{a}}_*\right\} \Phi^T + \frac{\partial \Phi}{\partial a_k} E\{\hat{\mathbf{x}}^+ \hat{\mathbf{x}}^{+T} | \hat{\mathbf{a}}_*\} \frac{\partial \Phi^T}{\partial a_l} \\ &\quad + \Phi E\left\{\frac{\partial \hat{\mathbf{x}}^+}{\partial a_k} \hat{\mathbf{x}}^{+T} \middle| \hat{\mathbf{a}}_*\right\} \frac{\partial \Phi^T}{\partial a_l} + \frac{\partial \Phi}{\partial a_k} E\left\{\hat{\mathbf{x}}^+ \frac{\partial \hat{\mathbf{x}}^{+T}}{\partial a_l} \middle| \hat{\mathbf{a}}_*\right\} \Phi^T \\ &\quad + \frac{\partial \mathbf{B}_d}{\partial a_k} E\{\mathbf{u}\mathbf{u}^T | \hat{\mathbf{a}}_*\} \frac{\partial \mathbf{B}_d^T}{\partial a_l} \\ &\quad + \frac{\partial \Phi}{\partial a_k} E\{\hat{\mathbf{x}}^+ \mathbf{u}^T | \hat{\mathbf{a}}_*\} \frac{\partial \mathbf{B}_d^T}{\partial a_l} + \frac{\partial \mathbf{B}_d}{\partial a_k} E\{\mathbf{u} \hat{\mathbf{x}}^{+T} | \hat{\mathbf{a}}_*\} \frac{\partial \Phi^T}{\partial a_l} \\ &\quad + \Phi E\left\{\frac{\partial \hat{\mathbf{x}}^+}{\partial a_k} \mathbf{u}^T \middle| \hat{\mathbf{a}}_*\right\} \frac{\partial \mathbf{B}_d^T}{\partial a_l} + \frac{\partial \mathbf{B}_d}{\partial a_k} E\left\{\mathbf{u} \frac{\partial \hat{\mathbf{x}}^{+T}}{\partial a_l} \middle| \hat{\mathbf{a}}_*\right\} \Phi^T \end{aligned} \quad (10-41)$$

$$\begin{aligned} E\{\hat{\mathbf{x}}(t_j^-)\hat{\mathbf{x}}^T(t_j^-) \mid \hat{\mathbf{a}} = \hat{\mathbf{a}}_*(t_i)\} &= \Phi E\{\hat{\mathbf{x}}^+ \hat{\mathbf{x}}^{+\top} \mid \hat{\mathbf{a}}_*\} \Phi^T + \mathbf{B}_d E\{\mathbf{u}\mathbf{u}^T \mid \hat{\mathbf{a}}_*\} \mathbf{B}_d^T \\ &\quad + \Phi E\{\hat{\mathbf{x}}^+ \mathbf{u}^T \mid \hat{\mathbf{a}}_*\} \mathbf{B}_d^T + \mathbf{B}_d E\{\mathbf{u}\hat{\mathbf{x}}^{+\top} \mid \hat{\mathbf{a}}_*\} \Phi^T \end{aligned} \quad (10-42)$$

$$\begin{aligned} E\left\{\frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} \hat{\mathbf{x}}^T(t_j^-) \mid \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\right\} &= \Phi E\left\{\frac{\partial \hat{\mathbf{x}}^+}{\partial a_k} \hat{\mathbf{x}}^{+\top} \mid \hat{\mathbf{a}}_*\right\} \Phi^T + \frac{\partial \Phi}{\partial a_k} E\{\hat{\mathbf{x}}^+ \hat{\mathbf{x}}^{+\top} \mid \hat{\mathbf{a}}_*\} \Phi^T \\ &\quad + \Phi E\left\{\frac{\partial \hat{\mathbf{x}}^+}{\partial a_k} \mathbf{u}^T \mid \hat{\mathbf{a}}_*\right\} \mathbf{B}_d^T + \frac{\partial \mathbf{B}_d}{\partial a_k} E\{\mathbf{u}\mathbf{u}^T \mid \hat{\mathbf{a}}_*\} \mathbf{B}_d^T \\ &\quad + \frac{\partial \Phi}{\partial a_k} E\{\hat{\mathbf{x}}^+ \mathbf{u}^T \mid \hat{\mathbf{a}}_*\} \mathbf{B}_d^T + \frac{\partial \mathbf{B}_d}{\partial a_k} E\{\mathbf{u}\hat{\mathbf{x}}^{+\top} \mid \hat{\mathbf{a}}_*\} \Phi^T \end{aligned} \quad (10-43)$$

$$\begin{aligned} E\{s_k^{-1}[\mathbf{Z}(t_j), \mathbf{a}] s_l^{-1}[\mathbf{Z}(t_j), \mathbf{a}] \mid \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\} \\ = \frac{1}{2} \text{tr} \left[ \mathbf{A}^{-1}(t_j) \frac{\partial \mathbf{A}(t_j)}{\partial a_k} \mathbf{A}^{-1}(t_j) \frac{\partial \mathbf{A}(t_j)}{\partial a_l} \right. \\ \left. + 2\mathbf{A}^{-1}(t_j) \mathbf{H}(t_j) E\left\{\frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} \frac{\partial \hat{\mathbf{x}}^T(t_j^-)}{\partial a_l} \mid \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\right\} \mathbf{H}^T(t_j) \right] \end{aligned} \quad (10-44)$$

Equations (10-42) and (10-43) are needed to define certain terms in (10-41), and once (10-43) is computed, both it and its transpose will be used. The initial conditions at  $t_{i-N}$  on (10-41) and (10-43) are  $\mathbf{0}$ , and  $E\{\hat{\mathbf{x}}(t_{i-N}^+) \hat{\mathbf{x}}^T(t_{i-N}^+) \mid \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\}$  is approximated by  $E\{\hat{\mathbf{x}}(t_{i-N}^+) \hat{\mathbf{x}}^T(t_{i-N}^+) \mid \mathbf{a} = \hat{\mathbf{a}}_*(t_{i-1})\}$  as generated at time  $t_{i-1}$ . The expectations involving the control are evaluated in one of two ways. If the  $\mathbf{u}(\cdot)$  time history is completely precomputed (zero, for instance), then  $E\{\mathbf{u}(\cdot)^T \mid \hat{\mathbf{a}}_*\}$  becomes  $\mathbf{u}E\{(\cdot)^T \mid \hat{\mathbf{a}}_*\}$ , and thus the following recursions must be processed to evaluate the required terms:

$$E\{\hat{\mathbf{x}}(t_j^+) \mid \hat{\mathbf{a}}_*\} = \Phi E\{\hat{\mathbf{x}}^+ \mid \hat{\mathbf{a}}_*\} + \mathbf{B}_d \mathbf{u} \quad (10-45a)$$

$$\begin{aligned} E\left\{\frac{\partial \hat{\mathbf{x}}(t_j^+)}{\partial a_k} \mid \hat{\mathbf{a}}_*\right\} &= [\mathbf{I} - \mathbf{K}(t_j) \mathbf{H}(t_j)] \\ &\quad \times \left[ \Phi E\left\{\frac{\partial \hat{\mathbf{x}}^+}{\partial a_k} \mid \hat{\mathbf{a}}_*\right\} + \frac{\partial \Phi}{\partial a_k} E\{\hat{\mathbf{x}}^+ \mid \hat{\mathbf{a}}_*\} + \frac{\partial \mathbf{B}_d}{\partial a_k} \mathbf{u} \right] \end{aligned} \quad (10-45b)$$

On the other hand, if the control were computed in feedback form as  $\mathbf{u}(t_j) = -\mathbf{G}_c(t_j) \hat{\mathbf{x}}(t_j^+)$ , then the expectations of the form  $E\{\mathbf{u}(\cdot)^T \mid \hat{\mathbf{a}}_*\}$  become  $-\mathbf{G}_c E\{\hat{\mathbf{x}}^+(\cdot)^T \mid \hat{\mathbf{a}}_*\}$  and  $E\{\mathbf{u}\mathbf{u}^T \mid \hat{\mathbf{a}}_*\}$  becomes  $\mathbf{G}_c E\{\hat{\mathbf{x}}^+ \hat{\mathbf{x}}^{+\top} \mid \hat{\mathbf{a}}_*\} \mathbf{G}_c^T$ , for which the recursions have already been evaluated.

To incorporate the measurement at time  $t_j$ , the state relations are:

$$\mathbf{r}_j = \mathbf{z}_j - \mathbf{H}(t_j) \hat{\mathbf{x}}(t_j^-) \quad (10-46)$$

$$\mathbf{D}(t_j) = \mathbf{I} - \mathbf{K}(t_j) \mathbf{H}(t_j) \quad (10-47)$$

$$\hat{\mathbf{x}}(t_j^+) = \hat{\mathbf{x}}(t_j^-) + \mathbf{K}(t_j) \mathbf{r}_j \quad (10-48)$$

$$\mathbf{P}(t_j^+) = \mathbf{P}(t_j^-) - \mathbf{K}(t_j) \mathbf{H}(t_j) \mathbf{P}(t_j^-) \quad (10-49a)$$

$$= \mathbf{D}(t_j) \mathbf{P}(t_j^-) \mathbf{D}^T(t_j) + \mathbf{K}(t_j) \mathbf{R}(t_j) \mathbf{K}^T(t_j) \quad (10-49b)$$

After the first time through these relations, the values of  $\hat{\mathbf{x}}(t_{i-N+1}^+)$  and  $\mathbf{P}(t_{i-N+1}^+)$  are stored in memory to serve as the initial conditions at the next sample time,  $t_{i+1}$ .

The score equations to be processed during a measurement update are

$$\mathbf{n}_j = \mathbf{A}^{-1}(t_j) \mathbf{r}_j \quad (10-50)$$

$$\mathbf{C}(t_j) = \mathbf{A}^{-1}(t_j) - \mathbf{n}_j \mathbf{n}_j^T \quad (10-51)$$

and, for  $k = 1, 2, \dots, p$ ,

$$s_k^{-1}[\mathbf{Z}_j, \hat{\mathbf{a}}_*(t_i)] = \frac{\partial \hat{\mathbf{x}}^T(t_j^-)}{\partial a_k} [\mathbf{H}^T(t_j) \mathbf{n}_j] - \frac{1}{2} \text{tr} \left\{ \mathbf{C}(t_j) \frac{\partial \mathbf{A}(t_j)}{\partial a_k} \right\} \quad (10-52)$$

$$\frac{\partial \hat{\mathbf{x}}(t_j^+)}{\partial a_k} = \mathbf{D}(t_j) \left\{ \frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} + \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} [\mathbf{H}^T(t_j) \mathbf{n}_j] \right\} \quad (10-53)$$

$$\frac{\partial \mathbf{P}(t_j^+)}{\partial a_k} = \mathbf{D}(t_j) \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} \mathbf{D}^T(t_j) \quad (10-54)$$

It should be noted that, despite its simple appearance, Eq. (10-54) does account for the variation of  $\mathbf{K}(t_j)$  with respect to  $a_k$ . This can be verified by taking the partial of (10-49b) with respect to  $a_k$ , expressing  $\partial \mathbf{K}(t_j)/\partial a_k$  in terms of  $\partial \mathbf{P}(t_j^-)/\partial a_k$ , and collecting like terms. The  $p$  scalar results of the computations in (10-52) are added to the running sums for generating the score vector.

The conditional information matrix relations are

$$\begin{aligned} E & \left\{ \frac{\partial \hat{\mathbf{x}}(t_j^+)}{\partial a_k} \frac{\partial \hat{\mathbf{x}}^T(t_j^+)}{\partial a_l} \middle| \mathbf{a} = \hat{\mathbf{a}}_*(t_i) \right\} \\ &= \mathbf{D}(t_j) E \left\{ \frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} \frac{\partial \hat{\mathbf{x}}^T(t_j^-)}{\partial a_l} \middle| \mathbf{a} = \hat{\mathbf{a}}_*(t_i) \right\} \mathbf{D}^T(t_j) \\ &+ \mathbf{D}(t_j) \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} \mathbf{H}^T(t_j) \mathbf{A}^{-1}(t_j) \mathbf{H}(t_j) \frac{\partial \mathbf{P}(t_j^-)}{\partial a_l} \mathbf{D}^T(t_j) \end{aligned} \quad (10-55)$$

$$E\{\hat{\mathbf{x}}(t_j^+) \hat{\mathbf{x}}^T(t_j^+) | \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\} = E\{\hat{\mathbf{x}}(t_j^-) \hat{\mathbf{x}}^T(t_j^-) | \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\} + \mathbf{K}(t_j) \mathbf{A}(t_j) \mathbf{K}^T(t_j) \quad (10-56)$$

$$\begin{aligned} & E\left\{ \frac{\partial \hat{\mathbf{x}}(t_j^+)}{\partial a_k} \hat{\mathbf{x}}^T(t_j^+) | \mathbf{a} = \hat{\mathbf{a}}_*(t_i) \right\} \\ &= \mathbf{D}(t_j) \left[ E\left\{ \frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} \hat{\mathbf{x}}^T(t_j^-) | \mathbf{a} = \hat{\mathbf{a}}_*(t_i) \right\} + \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} \mathbf{H}^T(t_j) \mathbf{K}^T(t_j) \right] \quad (10-57) \end{aligned}$$

Note that the propagation and update equations simplify substantially for a parameter  $a_k$  confined totally to  $\mathbf{B}_d$ . Not only is  $\partial \Phi / \partial a_k = \mathbf{0}$ , but so are all second order statistics terms of the form  $\partial \mathbf{P}^- / \partial a_k$ ,  $\partial \mathbf{P}^+ / \partial a_k$ , and  $\partial \mathbf{A} / \partial a_k$ .

At the end of the  $N$ -step recursion,  $\mathbf{P}(t_i^+)$ ,  $\partial \hat{\mathbf{x}}(t_i^+)/\partial a_k$ ,  $\partial \mathbf{P}(t_i^+)/\partial a_k$ , and  $E\{[\partial \hat{\mathbf{x}}(t_i^+)/\partial a_k][\partial \hat{\mathbf{x}}(t_i^+)/\partial a_l]^T | \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\}$  will have been evaluated for all values of  $k$  and  $l$ . These are precisely the terms required to form the final components of summations to yield the score vector and conditional information matrix according to (10-29b) and (10-31b). These are computed and added to the appropriate running sums to generate  $\{\partial L[\hat{\mathbf{x}}_*(t_i), \hat{\mathbf{a}}_*(t_i), \mathbf{Z}_i]/\partial \mathbf{a}\}^T$  and  $\mathbf{J}[t_i, \hat{\mathbf{x}}_*(t_i), \hat{\mathbf{a}}_*(t_i)]$  via (10-28) and (10-30). Subsequently, Eq. (10-33) is processed to yield the new parameter estimate  $\hat{\mathbf{a}}^*(t_i)$ .

**EXAMPLE 10.1** To appreciate the magnitude of the computations required by the full-scale estimator, consider a hypothetical problem involving the estimation of a five-dimensional state, two parameters influencing the state transition matrix  $\Phi$ , and one parameter affecting  $\mathbf{B}_d$ . The control vector is two dimensional, computed in feedback fashion using the current state estimate (the controller gains are assumed to be precomputed). In addition, the state is driven by a scalar dynamic noise, and two-dimensional measurements are made at each sample time. Finally, it has been found that the parameters can be modeled adequately as constants over an interval of ten sample times.

Table 10.2 portrays the number of multiplications, additions, subtractions, and matrix inversions to be processed in progressing a single sample period with the state and parameter estimation. The totals are separated into the evaluations necessary to propagate and update the state  $\mathbf{x}$ , score  $\mathbf{s}$ , and conditional information matrix  $\mathbf{J}$  and finally into the additional computations to achieve a new parameter estimate. (General results as functions of dimensions  $n$ ,  $m$ ,  $r$ , and  $s$  are in [82].) If local iterations are to be considered, the numbers in the table correspond to the number of computations in each local iteration.

TABLE 10.2  
*Full-Scale Solution of Hypothetical Problem*

Term	Multiplications	Additions	Subtractions	Inversions
$\mathbf{x}$	8800	6638	270	10 (2 $\times$ 2)
$\mathbf{s}$	25742	20618	60	1 (5 $\times$ 5)
$\mathbf{J}$	120437	89765	0	0
$\mathbf{a}^*$	9	9	0	1 (3 $\times$ 3)

It can be seen from this table that, even for the moderate state dimension of this problem, the computational load is great. In fact, the number of multiplications and additions grows in proportion to the cube of the state dimension  $n$ , and the burden on the computer thus becomes overwhelming as  $n$  increases. ■

The inordinately large number of calculations required for the conditional information matrix, as evidenced in the preceding example, motivates search for a means of approximating its value in a simple fashion. Offline precomputation is a possibility, but discussion of this method is deferred until Section 10.6. Another technique would be to employ the approximation [82]

$$E \left\{ \frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} \frac{\partial \hat{\mathbf{x}}^T(t_j^-)}{\partial a_l} \middle| \mathbf{a} = \hat{\mathbf{a}}_*(t_i) \right\} \cong \frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} \frac{\partial \hat{\mathbf{x}}^T(t_j^-)}{\partial a_l} \quad (10-58)$$

This states that the expectation of the matrix  $\{[\partial \hat{\mathbf{x}}(t_j^-)/\partial a_k][\partial \hat{\mathbf{x}}^T(t_j^-)/\partial a_l]\}$  over all possible noise sequences can be adequately represented by the value it would attain due to the particular sequence that is assumed to have generated the measurement data. A heuristic justification for (10-58) is that the scoring approximation to the Newton-Raphson method removed the dependence of the Hessian (second derivative) matrix upon the particular sequence of data taken, and this dependence is now regained in a different, but related, manner. From a practical standpoint, this should reduce sensitivity of the estimates to incorrectly assumed values for the noise statistics.

By incorporating Eq. (10-58) into (10-44), the component terms of the conditional information matrix become

$$\begin{aligned} E\{s_k^{-1}[\mathbf{Z}(t_j), \mathbf{a}]s_l^{-1}[\mathbf{Z}(t_j), \mathbf{a}] \mid \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\} &\cong \frac{1}{2} \text{tr} \left[ \mathbf{A}^{-1}(t_j) \frac{\partial \mathbf{A}(t_j)}{\partial a_k} \mathbf{A}^{-1}(t_j) \frac{\partial \mathbf{A}(t_j)}{\partial a_l} \right] \\ &+ \frac{\partial \hat{\mathbf{x}}^T(t_j^-)}{\partial a_k} \mathbf{H}^T(t_j) \mathbf{A}^{-1}(t_j) \mathbf{H}(t_j) \frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_l} \end{aligned} \quad (10-59)$$

Similarly, (10-31b) becomes

$$\begin{aligned} E\{\gamma_k[\mathbf{Z}(t_i), \mathbf{a}]\gamma_l[\mathbf{Z}(t_i), \mathbf{a}] \mid \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\} &\cong \frac{1}{2} \text{tr} \left[ \mathbf{P}^{-1}(t_i^+) \frac{\partial \mathbf{P}(t_i^+)}{\partial a_k} \mathbf{P}^{-1}(t_i^+) \frac{\partial \mathbf{P}(t_i^+)}{\partial a_l} \right] \\ &+ \frac{\partial \hat{\mathbf{x}}^T(t_i^+)}{\partial a_k} \mathbf{P}^{-1}(t_i^+) \frac{\partial \hat{\mathbf{x}}(t_i^+)}{\partial a_l} \end{aligned} \quad (10-60)$$

From these relations can be seen the tremendous benefit of using the proposed approximation:  $\mathbf{J}$  can be evaluated using only the propagation and update relations necessary for the state and score computations. Equation (10-59) replaces (10-41)–(10-45) and (10-55)–(10-57), and Eq. (10-60) replaces (10-31b).

**EXAMPLE 10.2** For the hypothetical problem posed in Example 10.1, the approximation just described reduces the required number of multiplications from 120,437 to 1598, and the number of additions from 89,765 to 1117. Of these, only 790 multiplications and 464 additions are required for the 10-term sum of components given by (10-59), the remainder being for the final term given by (10-60); this is caused by the difference between the state and measurement dimensions. ■

The approximate solution method just described yields vast reductions in required computation. Nevertheless, investigations [82] have shown the corresponding accuracy and rate of convergence to be exceptionally high.

**EXAMPLE 10.3** Consider a system modeled by the discrete-time representation (with sample period of 0.1 sec)

$$\begin{bmatrix} x_1(t_{i+1}) \\ x_2(t_{i+1}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_d(t_i)$$

$$z(t_i) = [1 \quad 0] \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix} + v(t_i)$$

Let  $w_d(\cdot, \cdot)$  and  $v(\cdot, \cdot)$  be zero-mean white Gaussian noises with constant variances  $Q_d = 10/3$  and  $R = 0.1$ , and let the a priori statistics on Gaussian  $x(t_0)$  be

$$\hat{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$

For the purpose of simulation, let the true system parameters be  $a_1 = -1$  and  $a_0 = 0.8$ , resulting in a damped second order mode (the discrete-time system eigenvalues are given by  $\lambda = -\frac{1}{2}[a_1 \pm \sqrt{a_1^2 - 4a_0}] \cong 0.5 \pm j0.74$ , each of magnitude less than unity; thus the mode is stable). Moreover, let the true initial state value be  $x_{t_0} = [10 \quad 10]^T$ .

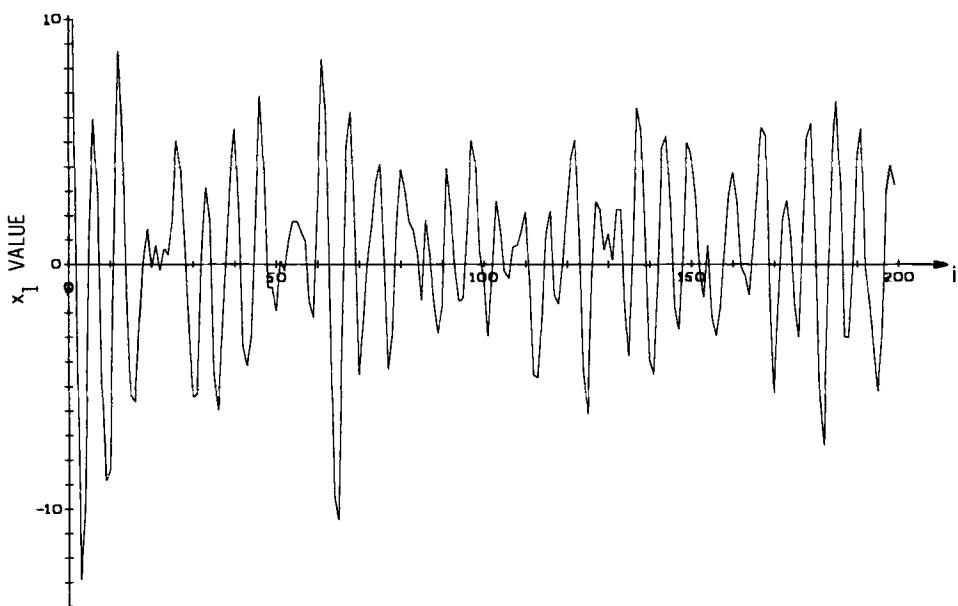


FIG. 10.1 Typical  $x_1$  state trajectory.

For the system described, a typical state trajectory for  $x_1$  versus sample time  $i$  for 200 sample times (i.e., 20 sec) is portrayed in Fig. 10.1. The corresponding  $x_2$  sample trajectory is identical but displaced one time-step ahead. Since the measurement is a rather accurate indication of  $x_1(t_i)$ , the errors in a state estimate  $\hat{x}_1(t_i)$  will be rather small regardless of the parameter estimate, whereas the errors in  $\hat{x}_2(t_i)$ , a prediction of the next value of  $x_1$ , will be more dependent upon the accuracy of the assumed system model. This is borne out in Fig. 10.2: plots (a) and (b) depict the errors in the estimates  $\hat{x}_1(t_i)$  and  $\hat{x}_2(t_i)$  from a single run of a Kalman filter with a correctly evaluated system model ( $a_1 = -1$ ,  $a_0 = 0.8$ ), whereas (c) and (d) plot the same errors for a Kalman filter that assumes

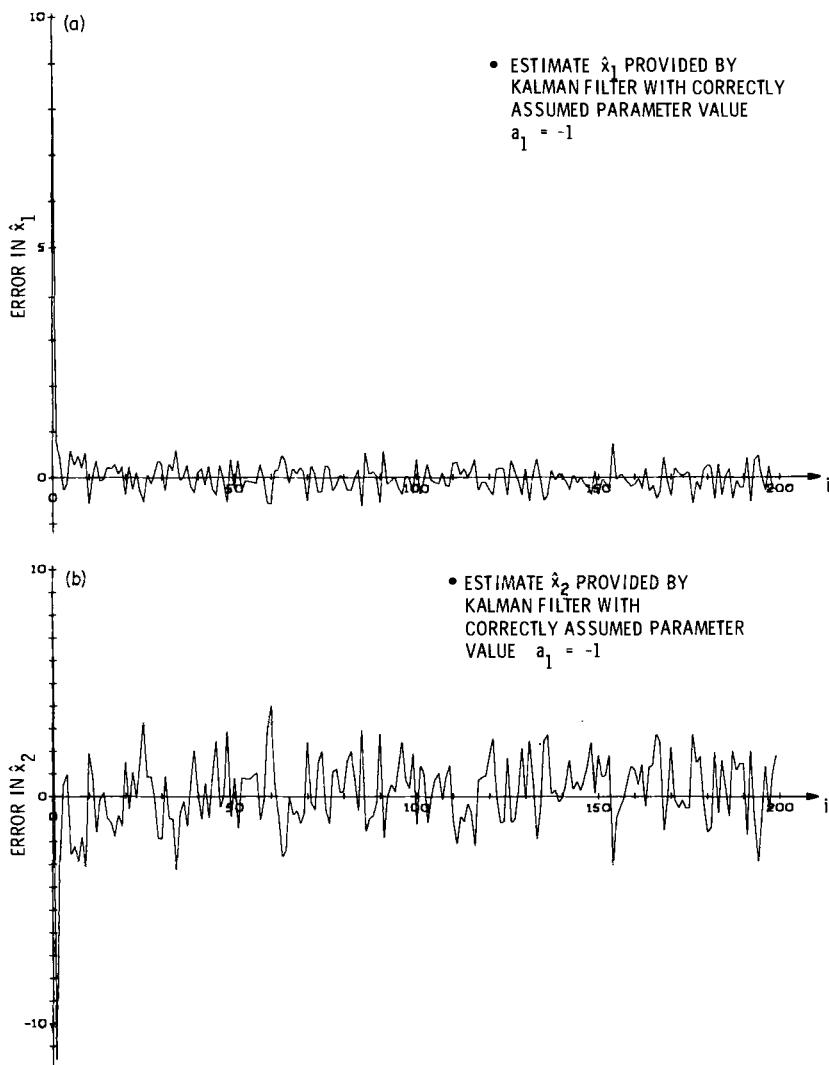


FIG. 10.2 (a) Error in  $\hat{x}_1$  for assumed parameter value of  $-1$ . (b) Error in  $\hat{x}_2$  for assumed parameter value of  $-1$ .

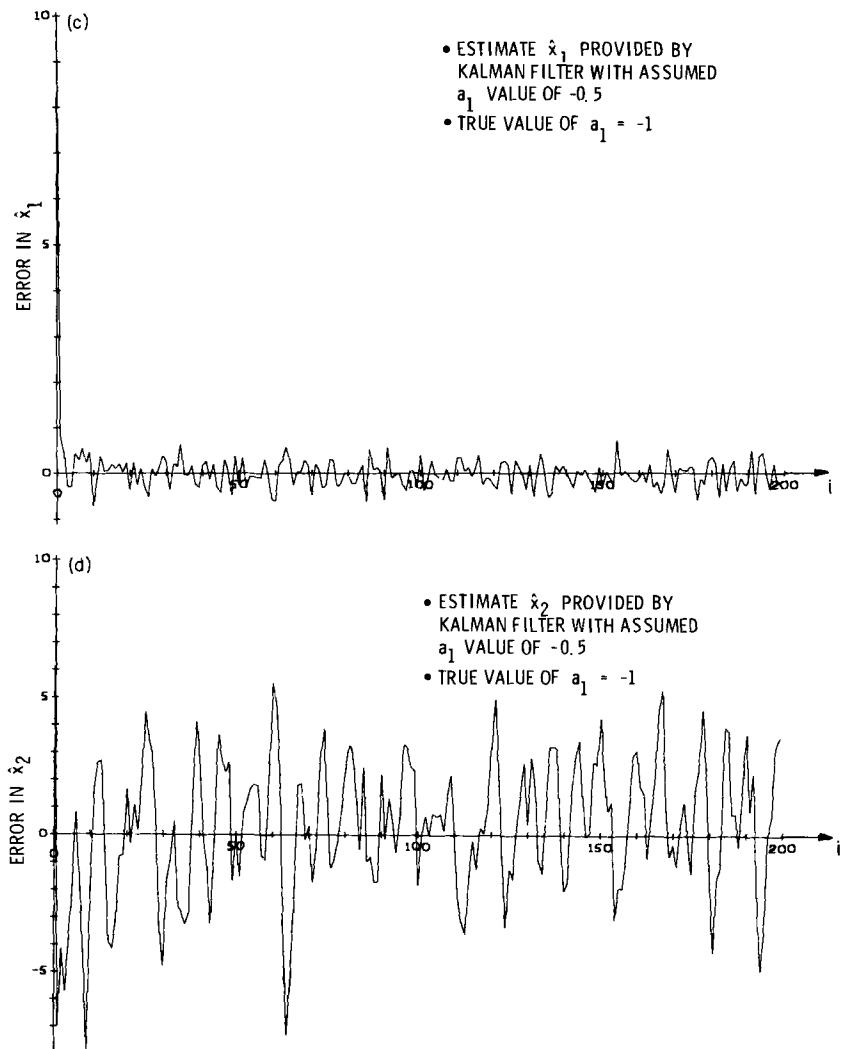


FIG. 10.2 (c) Error in  $\hat{x}_1$  for assumed parameter value of  $-0.5$ . (d) Error in  $\hat{x}_2$  for assumed parameter value of  $-0.5$ .

$a_1 = -0.5$ . Thus, plots (a) and (b) represent the best state estimate performance attainable from an estimator of both states and parameters, a performance level approached as the parameter estimate converges to the true value.

The state and parameter estimator using the approximation of (10-58)–(10-60) was implemented to estimate  $x_1$ ,  $x_2$ , and  $a_1$ , with the initial guess of  $a_1$  set at  $-0.5$ . Figure 10.3 presents typical parameter estimate error trajectories, plot (a) pertaining to an estimator with the number ( $N$ ) of samples in the fixed-length memory parameter estimator set equal to 10, and plot (b) corresponding to  $N = 30$ .

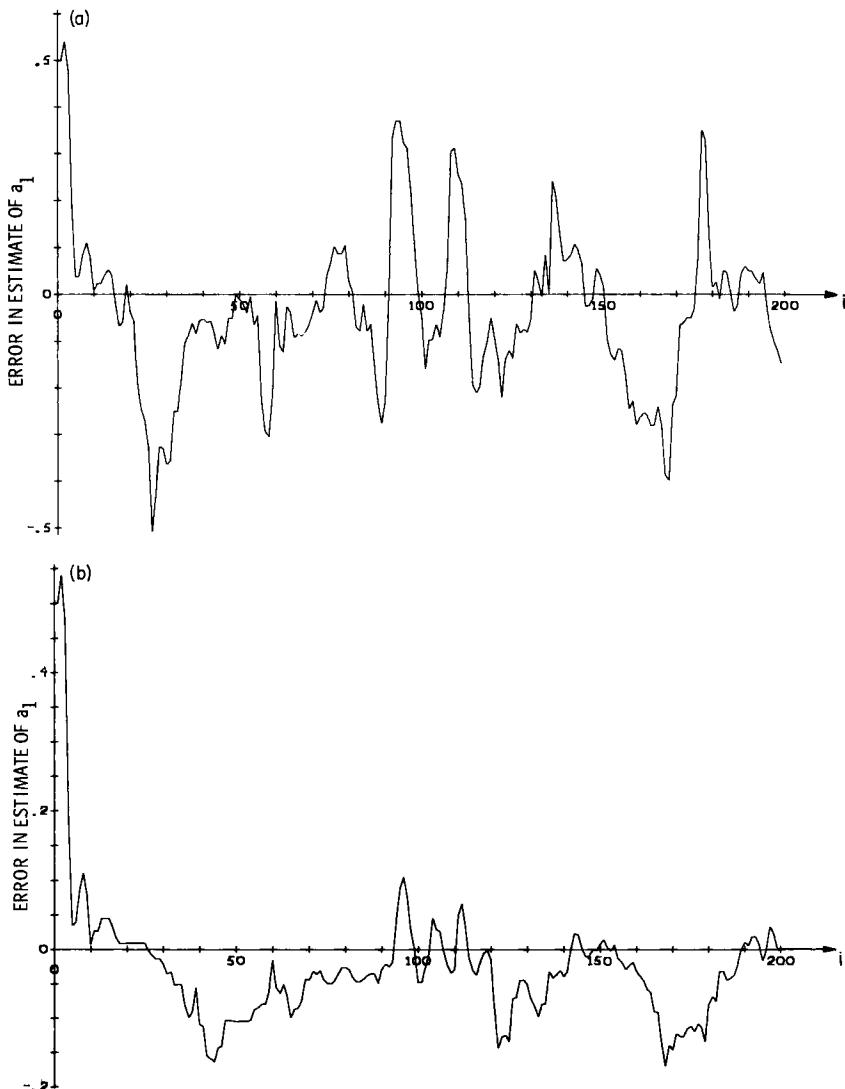
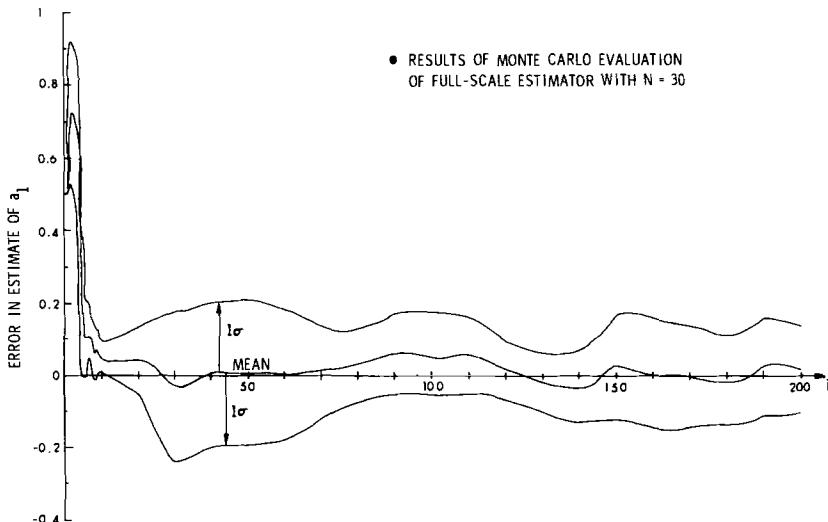
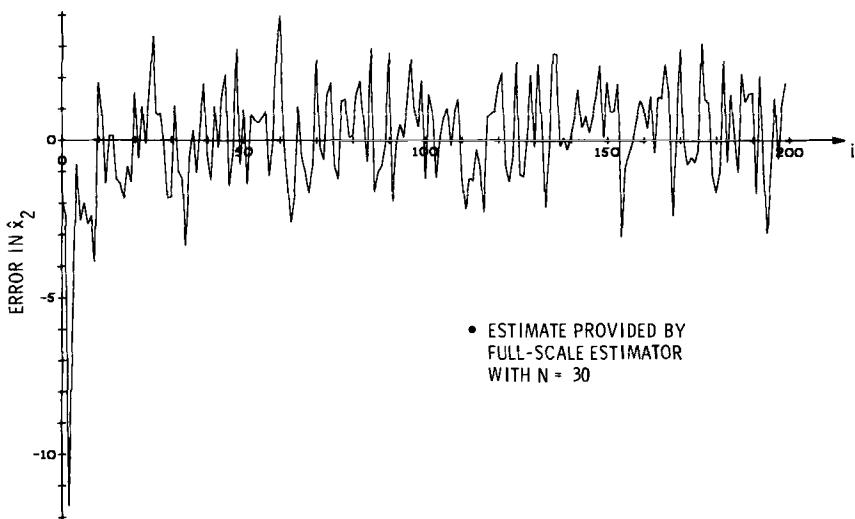


FIG. 10.3 (a) Parameter estimate error trajectory; full-scale estimator with  $N = 10$ . (b) Parameter estimate error trajectory; full-scale estimator with  $N = 30$ .

The larger  $N$  is seen to be very effective in reducing the magnitude of the error fluctuations, approximately halving the average magnitudes.

The results of a Monte Carlo evaluation, using five simulations differing from each other only in the particular realizations of  $w_d(\cdot, \cdot)$  and  $v(\cdot, \cdot)$  employed, are portrayed in Fig. 10.4 as a plot of mean  $\pm 1$  standard deviation values for all time. The initial transient behavior is undesirable in that the average parameter error grows rapidly for a few samples before converging to a small

FIG. 10.4 Mean  $\pm 1\sigma$  values of full-scale parameter estimate errors.FIG. 10.5 Error in  $\hat{x}_2$  when parameter is estimated simultaneously.

"steady state" value. The error grows because of the particular state initial conditions (not changed for the Monte Carlo runs), and its rate of change is large partially due to the large magnitude of  $J^{-1}$  for the first few samples. Practical experience has shown that this undesirable behavior can be removed entirely by delaying the first parameter estimate until the average transient settles out, in this case until  $i = 5$ , or  $t_i = 0.5$  sec.

For all cases discussed, the state estimate error trajectories do in fact converge to the behavior of the Kalman filter with correctly evaluated parameters. This is readily seen by comparing Fig. 10.5,

a plot of the error in  $\hat{x}_2(t_i)$  generated by the estimator that yielded the parameter estimate error in Fig. 10.3b, to Fig. 10.2b. ■

**EXAMPLE 10.4** Consider the same problem as in the previous example, but now let both  $a_1$  and  $a_0$  be estimated simultaneously with the states. The filter initially assumed erroneous parameter values of  $a_1 = -0.5$  and  $a_0 = 0.3$ , with true values maintained at  $-1$  and  $0.8$ , respectively.

Figure 10.6a presents the error in the corresponding estimate of  $a_1$ , and Fig. 10.6b pertains to  $a_0$ . The only difference between this simulation and that portrayed by Fig. 10.3b is the addition of

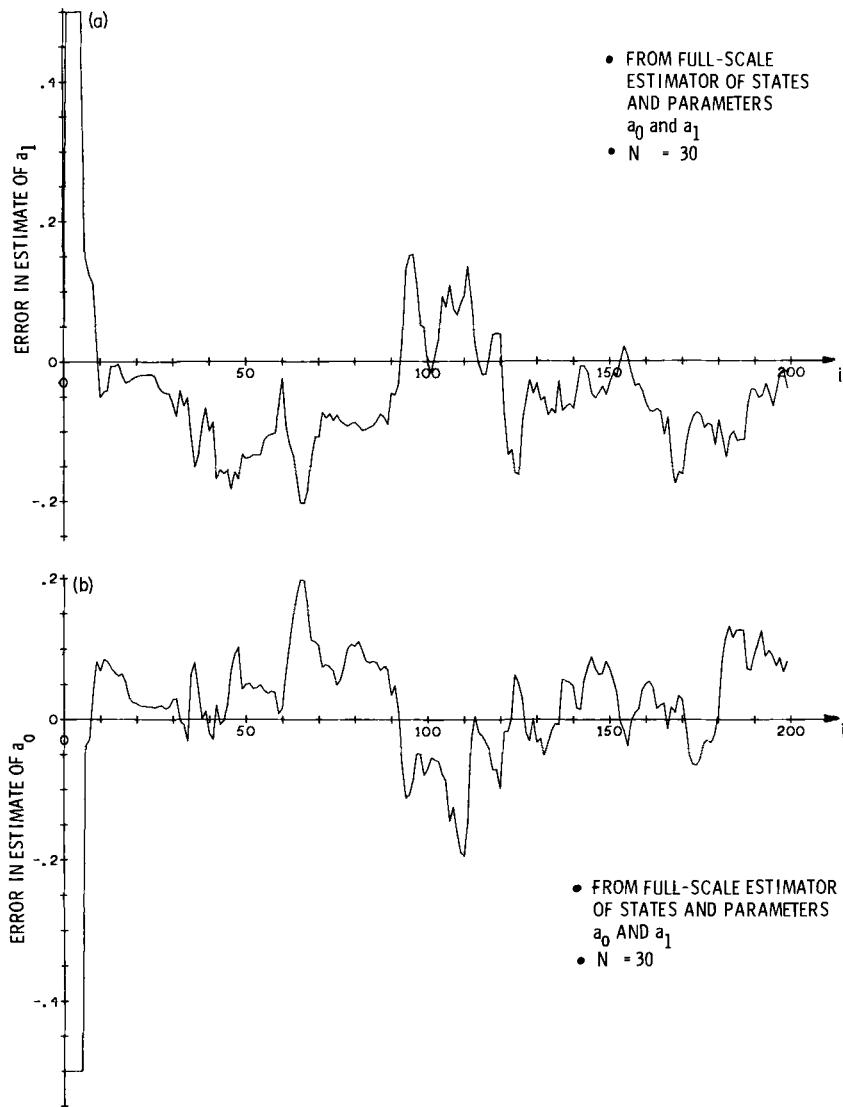


FIG. 10.6 (a) Error in estimate of the parameter  $a_1$ . (b) Error in estimate of the parameter  $a_0$ .

the  $a_0$  estimate and the delay of the first parameter estimate to reduce the magnitude of initial transients: identical random numbers were used to simulate the noise inputs for both cases. Therefore, Fig. 10.6a is directly comparable to Fig. 10.3b, and the ability to perform an accurate estimate is seen to be maintained in the face of additional parameter uncertainty.

Direct comparison of the two plots in Fig. 10.6 reveals that the increased error magnitude for  $a_1$  corresponds directly to an error of opposite sign in estimating  $a_0$ . This is to be expected since the parameter estimator is basically fitting the assumed system model to the acquired data. As a result, the errors in the state estimates for the two cases are virtually indistinguishable. ■

## 10.5 UNCERTAINTIES IN $\Phi$ AND $B_d$ : PERFORMANCE ANALYSIS

To be able to use this estimation technique with confidence, the user must be able to predict the performance achievable from the algorithm. This section provides two distinct performance analyses to satisfy this need: characterization of asymptotic properties and ambiguity function portrayal of estimation precision.

*Asymptotic properties* refer to behavior exhibited by the estimator as the number of measurements processed grows without bound. Besides being of theoretical importance for growing-length memory estimators, these properties also delineate trends of estimator behavior as time progresses, for sufficiently large  $N$  associated with the fixed-length memory parameter estimator. Under a set of very nonrestrictive “regularity” conditions [82, 107, 145], the state and parameter estimator described in the previous section yields a parameter estimator  $\hat{\mathbf{a}}^*(t_i)$  that is:

- (1) *consistent*: it converges in probability to the true (but unknown) value  $\mathbf{a}_t$  as the number of sample elements processed grows without bound (theoretically, as  $N \rightarrow \infty$ );
- (2) *asymptotically unbiased*;
- (3) *asymptotically Gaussian distributed with mean  $\mathbf{a}_t$  and covariance  $J[t_i, \mathbf{a}_t]^{-1}$* ;
- (4) *asymptotically efficient*: in the limit as the number of sample elements grows without bound, the estimator  $\hat{\mathbf{a}}^*(t_i)$  is unbiased, has finite error covariance, and there is no other unbiased estimator whose error covariance is smaller than that of  $\hat{\mathbf{a}}^*(t_i)$ .

Furthermore, since the parameter estimate does converge to the true value  $\mathbf{a}_t$ , the behavior of the corresponding state estimator converges to that of the Kalman filter that uses the true value  $\mathbf{a}_t$  for the parameters. The proofs of these claims [82], not included herein, are conditioned upon the regularity conditions mentioned previously and the assumption that the model of the parameters as constant over the interval of interest (i.e., over all time) is valid. Thus, to be assured of estimator behavior well described by this theoretical prediction, one should seek as large a value of  $N$  for the fixed-length memory parameter

estimator as allowed by the physics of the problem and by computer capacity.

Considerable information about performance to be expected from the estimator can be obtained through the concept of a *generalized ambiguity function* [82, 121], the average value of the likelihood function upon which the estimator is based. Let  $L[\theta(t_i), \mathcal{L}_i]$  denote that likelihood function, and then the ambiguity function  $\mathcal{A}_i(\cdot, \cdot)$  is defined as the scalar function such that

$$\mathcal{A}_i(\theta, \theta_t) \triangleq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} L[\theta, \mathcal{L}_i] f_{\mathbf{z}(t_i)|\theta(t_i)}(\mathcal{L}_i | \theta_t) d\mathcal{L}_i \quad (10-61)$$

where  $\theta$  is some value of the estimated variables at time  $t_i$  and  $\theta_t$  is the true, but unknown, value of these quantities. For a given value of  $\theta_t$ , this function of  $\theta$  provides both a global and local performance analysis. If it has multiple peaks, it indicates that individual likelihood functions realized by a particular set of measurement data may well have multiple peaks, which can cause convergence to local maxima rather than global, or cause failure of any convergence at all. In fact, the name “ambiguity function” is derived from the fact that multiple peaks tend to cause ambiguities. Furthermore, the curvature of the ambiguity function in the immediate vicinity of its peak value at  $\theta_t$  (various chosen values of  $\theta_t$  would be investigated in practice) conveys the preciseness with which a maximum likelihood estimate can be discerned. In fact, this curvature can be inversely related to the Cramér–Rao lower bound [121, 137] on the estimate error covariance matrix.

To be more specific, if  $\hat{\mathbf{a}}$  is an estimator of the scalar parameter  $a$ , whose true but unknown value is  $a_t$ , its bias error is

$$b(a_t, t_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\hat{a} - a_t) f_{\mathbf{z}(t_i)|\mathbf{a}}(\mathcal{L}_i | a_t) d\mathcal{L}_i \quad (10-62)$$

The Cramér–Rao lower bound on maximum likelihood estimate error variance can then be expressed as

$$E\{[\hat{a} - a_t]^2\} \geq \frac{\left[ 1 + \left. \frac{\partial b}{\partial a} \right|_{a=a_t} \right]^2}{\left. - \frac{\partial^2}{\partial a^2} \mathcal{A}_i(a, a_t) \right|_{a=a_t}} \quad (10-63)$$

For a vector parameter  $\mathbf{a}$ , the unbiased version of (10-63) would be

$$E\{[\hat{\mathbf{a}} - \mathbf{a}_t][\hat{\mathbf{a}} - \mathbf{a}_t]^T\} \geq \left[ \left. - \frac{\partial^2}{\partial \mathbf{a}^2} \mathcal{A}_i(\mathbf{a}, \mathbf{a}_t) \right|_{\mathbf{a}=\mathbf{a}_t} \right]^{-1} \quad (10-64)$$

Although biases are generally present in maximum likelihood estimates, they are often ignored in performance analyses for expediency. The covariance lower bound so derived serves the purpose in design as the highest attainable goal in performance, the standard of comparison for all practical, suboptimal implementations.

An analogous development can be made for an  $N$ -step *generalized ambiguity function* defined as

$$\mathcal{A}_i^N(\mathbf{a}, \mathbf{a}_t) \triangleq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} L[\mathbf{a}, \mathcal{L}_{i, i-N+1}] f_{\mathbf{z}_{N(t_i)}|\mathbf{a}(t_i)}(\mathcal{L}_{i, i-N+1} | \mathbf{a}_t) d\mathcal{L}_{i, i-N+1} \quad (10-65)$$

This would be instrumental in evaluating the fixed-length memory parameter estimator of basic interest to us. It can be used as a design tool, able to delineate sensitivity of estimation accuracy to

- (1) various sizes of the fixed-length memory ( $N$ ),
- (2) form of the assumed system model (particular state variables chosen, state dimension, etc.),
- (3) types of measurements taken (the structure of the  $\mathbf{H}(t_i)$  matrices) and their precision (the  $\mathbf{R}(t_i)$  time history),
- (4) the magnitude and uncertainty in the initial conditions,
- (5) the dynamic driving noise intensities,
- (6) control inputs,
- (7) use of alternate likelihood functions,
- (8) removal of certain terms from the likelihood equations to reduce computational burden.

Thus, the ambiguity function can be an invaluable tool to ensure adequate performance of a state and parameter estimator.

It can be shown [82] that the ambiguity function value  $\mathcal{A}_i^N(\mathbf{a}, \mathbf{a}_t)$  for any  $\mathbf{a}$  and  $\mathbf{a}_t$  can be evaluated entirely from the output of a state estimator sensitivity analysis as described in Section 6.8 of Volume 1. Let the “truth model” be identical to the model upon which a Kalman filter is based, except that the parameters are evaluated as  $\mathbf{a}_t$  in the former, and as  $\mathbf{a}$  in the latter. Let the estimation error  $\mathbf{e}_t(\cdot, \cdot)$  be defined as the difference between the entire state estimate  $\hat{\mathbf{x}}(\cdot, \cdot)$  and the truth model state  $\mathbf{x}_t(\cdot, \cdot)$ . Furthermore, let the covariances of this estimation error before and after measurement incorporation be denoted in accordance with Eq. (6-104) as  $\mathbf{P}_e(t_j^-; \mathbf{a}_t, \mathbf{a})$  and  $\mathbf{P}_e(t_j^+; \mathbf{a}_t, \mathbf{a})$ , respectively, where the notation is meant to reflect the dependence of these values on the particular choices of  $\mathbf{a}_t$  and  $\mathbf{a}$ . Then  $\mathcal{A}_i^N(\mathbf{a}, \mathbf{a}_t)$  can be evaluated as

$$\begin{aligned} \mathcal{A}_i^N(\mathbf{a}, \mathbf{a}_t) = & \sum_{j=i-N+1}^i \left[ \frac{1}{2}m \ln(2\pi) - \frac{1}{2} \ln[|\mathbf{A}(t_j; \mathbf{a})|] \right. \\ & - \frac{1}{2} \text{tr}\{\mathbf{A}^{-1}(t_j; \mathbf{a}) [\mathbf{H}(t_j)\mathbf{P}_e(t_j^-; \mathbf{a}_t, \mathbf{a})\mathbf{H}^T(t_j) + \mathbf{R}(t_j)]\} \Big] \\ & + \left[ -\frac{1}{2}n \ln(2\pi) - \frac{1}{2} \ln[|\mathbf{P}(t_i^+; \mathbf{a})|] - \frac{1}{2} \text{tr}\{\mathbf{P}^{-1}(t_i^+; \mathbf{a})\mathbf{P}_e(t_i^+; \mathbf{a}_t, \mathbf{a})\} \right] \end{aligned} \quad (10-66)$$

The separation of terms in the above expression is intended to display the effect of the individual terms in Eq. (10-19) on the  $N$ -step ambiguity function.

The last trace term warrants special attention: this corresponds to the likelihood equation terms that contain the vector  $[\xi - \hat{x}(t_i^+)]$ , all of which vanish, so the effect of this trace term on the ambiguity function (and Cramér–Rao lower bound) should be ignored.

This result is especially convenient. One would normally perform an error sensitivity analysis of a standard Kalman filter for a range of  $a$  values for each chosen  $a_t$ , to determine whether simultaneous parameter estimation were warranted and, if so, which parameters might be the most critical to estimate. By so doing, one would also be providing much of the data required to generate an ambiguity function performance analysis of the parameter estimation itself.

**EXAMPLE 10.5** Consider the application introduced in Example 10.3. Figure 10.7 presents the ambiguity function for  $i = 50$  and  $N = 30$  plotted as a function of the parameter  $(-a_1)$ , with sign reversal to correspond to the entry in the transition matrix. The true value,  $(-a_{11})$ , is 1.0. As seen from the figure, the ambiguity function is unimodal, with sufficient curvature at the peak value to predict successful parameter estimation.

By calculating the ambiguity function in similar fashion for other values of  $N$ , Eq. (10-63) can be used to establish the Cramér–Rao lower bound on an unbiased parameter estimate error variance as a function of  $N$ , as presented in Fig. 10.8. Since this is a lower bound on achievable performance, it specifies a minimum  $N$  required to yield a desired accuracy in parameter estimation for particular values of  $\mathbf{R}$ ,  $\mathbf{Q}_d$ , and  $\mathbf{P}_0$ . Moreover, it indicates that, for small  $N$ , enlarging the fixed-length memory can substantially improve precision, but for  $N$  above approximately 30, this length must be enlarged appreciably to effect a moderate reduction in error variance. This would be invaluable knowledge to aid determination of the  $N$  to be used online.

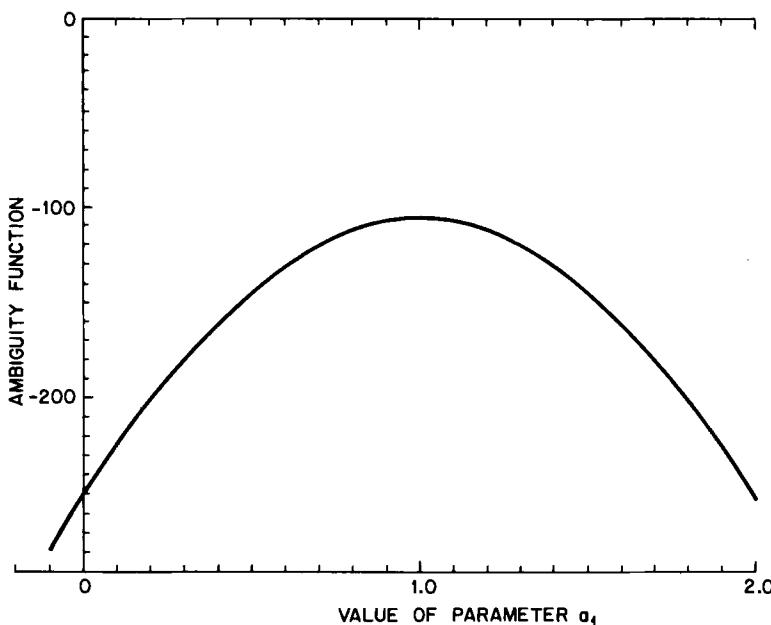


FIG. 10.7 Ambiguity function for  $i = 50$  and  $N = 30$ .

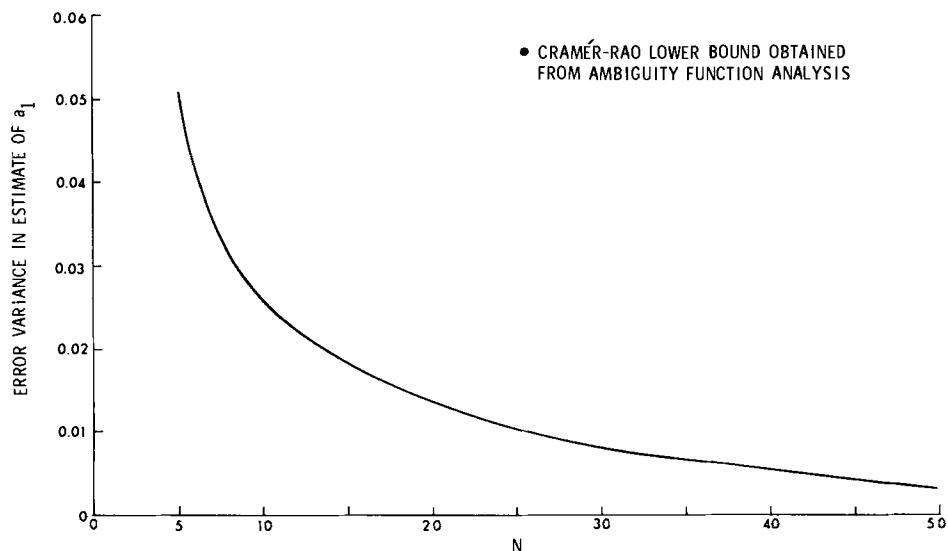


FIG. 10.8 Lower bound on parameter estimate error variance vs  $N$  at  $i = 50$ .

Return to Fig. 10.3: As predicted by the ambiguity function analysis, the larger interval ( $N = 30$  versus  $N = 10$ ) is very effective in reducing the magnitude of the error fluctuations. From Fig. 10.8, the lower bounds on the  $1\sigma$  values are about 0.16 for  $N = 10$  and 0.09 for  $N = 30$ , almost a halving of the standard deviation. This is corroborated by the plots in Fig. 10.3. Moreover, the  $1\sigma$  value in Fig. 10.4 has converged to the close vicinity of the lower bound provided by the ambiguity function analysis. ■

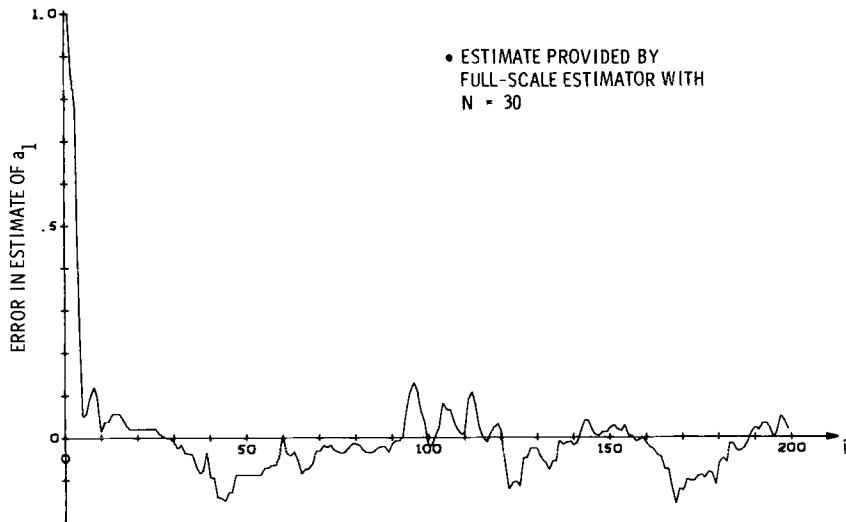


FIG. 10.9 Parameter estimate error for Example 10.6.

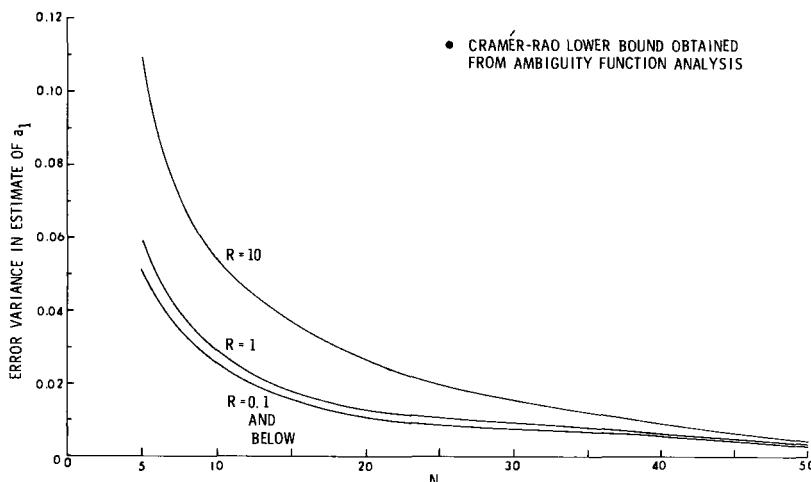


FIG. 10.10 Lower bound on parameter estimate error variance vs  $N$  at  $i = 50$ .

**EXAMPLE 10.6** Figure 10.9 is a typical plot of parameter estimate error for the case of  $N = 30$ ,  $R$  changed from 0.1 to 0.05, and  $\hat{a}_1(t_0)$  changed from  $-0.5$  to 0. This reveals that an ability to acquire a good parameter estimate is not strongly dependent upon the initial error. Over a large range of initial values of  $\hat{a}_1(t_0)$ , the estimator was able to converge to the immediate neighborhood of the true value within five sample periods.

Comparing this plot to Fig. 10.3b, the effect of reducing the measurement noise variance  $R$  is not very apparent in the post-transient, or “tracking,” phase. This and other runs confirm the trends discernible from Fig. 10.10, a plot of the Cramér–Rao bound as a function of  $N$ , at  $i = 50$ , for various values of  $R$ . Only for  $R$  greater than 0.1 does variation of measurement precision have a significant effect on parameter estimate precision. ■

## 10.6 UNCERTAINTIES IN $\Phi$ AND $\mathbf{B}_d$ : ATTAINING ONLINE APPLICABILITY

The full-scale solution algorithm of Section 10.4 requires a substantial amount of computation after each new measurement is incorporated. Consequently, this section seeks to attain online applicability while maintaining adequate estimation accuracy. One means of providing such applicability is through *approximations* designed to extract the essence of the full-scale solution: simplified iterative solution procedures, inclusion of only the most significant terms in the estimator likelihood equations, and precomputation and curve-fitting of various needed quantities [82]. *Methods not inherently involving approximations* will also be explored: use of advantageous state space representations, exploitation of symmetry, and modification of measurement incorporation [82].

Before detailing these aspects, it is appropriate to mention some general considerations that will prevail because of the time constraint inherent in

online applications. First of all, parameter estimation should be restricted to the *least number of parameters that will provide acceptable state estimation precision and overall system performance*. Furthermore, the *slowly varying nature* of parameters can be exploited, such as by estimating the parameters less frequently than the states. An algorithm that calculates a state estimate immediately upon receiving a measurement, using a previous parameter estimate, and then performs parameter estimation “in the background,” is superior in the online context to one that requires an iteration of the parameter calculations before it makes its (mathematically superior) state estimate available. Moreover, if a parameter estimate were processed only every  $N$  sample periods, only the running sums for  $\mathbf{s}$  and  $\mathbf{J}$  need be stored, and the memory requirements are vastly reduced.

Choice of the fixed-length memory size  $N$  affects online feasibility as well. Factors that cause one to seek small  $N$  are: (1) the limit imposed by the validity of the model that the system parameters are “essentially constant” over  $N$  sample periods, (2) the limit imposed by the capacity of the computer to accomplish the required calculations in an allotted portion of a sample period, (3) memory requirements, and (4) the necessity of a small interval to maintain adequate parameter tracking, since retention of large amounts of “old” data can seriously affect initial convergence and also cause significant lag in response to true parameter variations. On the other hand, reasons for making  $N$  large include: (1) large  $N$  yields less estimator susceptibility to errors due to single “bad” measured data points and reduces high frequency oscillation of the estimates, (2) larger  $N$  values allow greater confidence that the actual estimator will exhibit behavior well described by the theoretical asymptotic properties, (3) the Cramér–Rao lower bound is inversely related to  $N$  and in fact establishes a minimum admissible  $N$  for possibly providing adequate performance, and (4) certain approximations such as (10-24) and (10-58) become more valid with increasing  $N$ . A tradeoff must be analyzed, and the value of  $N$  that is most compatible with these diverse factors chosen as the actual size for implementation.

Some simplified iterative solution procedures have already been described, such as using (10-58)–(10-60) to evaluate  $\mathbf{J}[t_i, \hat{\mathbf{x}}_*(t_i), \hat{\mathbf{a}}_*(t_i)]$ . To avoid numerical difficulties of inverting a matrix with very small eigenvalues and large condition number, another approximation was also suggested: using a precomputed  $\mathbf{J}^{-1}$  for a number of initial sample periods or deferring the first parameter estimate until a prespecified number of samples has been taken, to ensure adequate size of the computed  $\mathbf{J}$  entries.

Even these approximations do not yield online applicability, however. Every sample period, we are still required to regenerate the state estimates, residuals, and score and  $\mathbf{J}$  component terms of the parameter vector estimate over an  $N$ -step interval, precluding real-time computation in most practical problems. Since the parameters can be assumed to be essentially constant over  $N$  periods,

then once a good parameter estimate is made, it should not vary significantly in  $N$  steps. Therefore, for some applications, it is adequate to estimate the parameters only every  $N$  sample periods (at least after an initial acquisition transient), with no reevaluation of terms over that interval. Let a parameter estimate  $\hat{\mathbf{a}}^*(t_j)$  be made, and use it for the next  $N$  sample periods to propagate and update the state, score, and conditional information matrix relations (using (10-59) and (10-60) to simplify  $\mathbf{J}$  computations). At each sample time  $t_i$  in this interval, running sum registers with contents denoted as  $\tilde{\mathbf{s}}(t_i)$  and  $\tilde{\mathbf{J}}(t_i)$  (zeroed at time  $t_j$ ) are updated according to

$$\tilde{\mathbf{s}}(t_i) = \tilde{\mathbf{s}}(t_{i-1}) + \mathbf{s}^1[\mathbf{Z}_i, \hat{\mathbf{a}}^*(t_j)] \quad (10-67a)$$

$$\tilde{\mathbf{J}}(t_i) = \tilde{\mathbf{J}}(t_{i-1}) + \mathbf{J}^1[\mathbf{Z}_i, \hat{\mathbf{a}}^*(t_j)] \quad (10-67b)$$

with  $\mathbf{s}^1$  and  $\mathbf{J}^1$  computed as in (10-52) and (10-59), respectively. At the end of the  $N$  steps, (10-29b) and (10-60) are computed and the results added to the running sums to generate  $\mathbf{s}[\mathbf{Z}_{j+N}, \hat{\mathbf{a}}^*(t_j)]$  and  $\mathbf{J}[\mathbf{Z}_{j+N}, \hat{\mathbf{a}}^*(t_j)]$ , and finally the new parameter estimate is generated by means of

$$\hat{\mathbf{a}}^*(t_{j+N}) = \hat{\mathbf{a}}^*(t_j) + \mathbf{J}[\mathbf{Z}_{j+N}, \hat{\mathbf{a}}^*(t_j)]^{-1}\mathbf{s}[\mathbf{Z}_{j+N}, \hat{\mathbf{a}}^*(t_j)] \quad (10-68)$$

Advantages of this simplified iterative solution procedure are low computation and storage load and more rapid availability of the state estimate at each sample time (desirable, and critical if the estimator is to be used in conjunction with a feedback controller). However, a parameter estimate is produced only every  $N$  samples, yielding slower initial convergence to a good estimate and an inherent lag in estimating a parameter change. These effects can be alleviated by changing the size of the interval from  $N$  to  $N/I$  sample periods, with  $I$  a small integer by which  $N$  is divisible. This effectively reduces the lag in responding to measurement information that the parameter values are different than previously estimated. Then, to incorporate the expectation that the parameters will be essentially constant over  $N$  steps, the  $I$  most recent parameter estimates can be averaged.

**EXAMPLE 10.7** Recall the hypothetical problem discussed in Examples 10.1 and 10.2. Table 10.3 reveals the reduction in computations afforded by this online conceptualization. These numbers pertain to a sample period in which a parameter estimate is made—the case in which the most

TABLE 10.3

*Online Conceptualization for Hypothetical Problem*

Term	Multiplications	Additions	Subtractions	Inversions
$\mathbf{x}$	889	671	27	1 ( $2 \times 2$ )
$\mathbf{s}$	2801	2249	6	1 ( $5 \times 5$ )
$\mathbf{J}$	887	694	0	0
$\hat{\mathbf{a}}^*$	9	9	0	1 ( $3 \times 3$ )

TABLE 10.4  
*Online Conceptualization—No Parameter Update*

Term	Multiplications	Additions	Subtractions	Inversions
x	889	671	27	1 ( $2 \times 2$ )
s	2549	2041	6	0
J	79	47	0	0
a*	0	0	0	0

calculations must be performed within a sample period. Note that the approximation for J discussed in Example 10.2 is employed.

For the case of not updating the parameter estimate, the required calculations are as in Table 10.4. The differences between this and the previous table are due mostly to the removal of the final components of the score and conditional information matrix given by Eqs. (10-29b) and (10-60), requiring a substantial amount of the J computations because  $n$  is much greater than  $m$ . ■

**EXAMPLE 10.8** Recall Example 10.3 and the full-scale estimator results portrayed in Figs. 10.3 and 10.4. To provide a direct basis of comparison, identical system simulations (with the same noise samples) were used to analyze the performance of the online estimator just described.

Figure 10.11a depicts the parameter estimate error committed by the online algorithm that estimates the parameter every ten samples (with  $N = 10$ ). For the first ten sample periods, a parameter estimate is made every period (without any recalculation of  $s^1$  or  $J^1$  values) in order to improve the initial transient characteristics. Tests conducted without the initially increased estimate frequency required about 3 parameter estimates, or 30 sample periods, for the parameter value to be acquired: on the average, five times as long as required for the decay of the transient in Fig. 10.11. Since the additional computations are minimal, this procedure is an attractive enhancement to performance.

Full-scale estimator and online estimator capabilities are very comparable, as seen by comparing Figs. 10.11a and 10.3a. This is true, despite a reduction from 12.95 to 1.10 sec of IBM 360/75 computer time to perform the estimation over 200 sample periods (not including simulation and program setup times). The associated Kalman state estimator required 0.55 sec.

Since an appropriate value of  $N$  has been determined as equal to 30, the online estimator could be generated with  $N = 30$  (after an initial 10 samples of more rapid estimation, to avoid a 3-iteration transient equal to 90 sample periods in this case). Or, the estimator with  $N = 10$  can be used, averaging each new parameter estimate with the two previous ones; the original unaveraged estimate is stored for future averaging, while the averaged value is used as the actual current parameter estimate. Figure 10.11b presents the results of this procedure. Here, averaging was started at  $i = 40$  to yield a graph scale identical to that of Fig. 10.11a, facilitating a comparison of the post-transient error magnitudes; averaging two estimates at  $i = 20$  and three at  $i = 30$  would be recommended for actual use. Averaging is not used in the first ten sample periods, since this would degrade initial acquisition capabilities. The accuracy of the estimates following  $i = 40$  in Fig. 10.11b strongly resembles that attained by the full-scale estimator with  $N = 30$ , as depicted in Fig. 10.3b. For this case, the reduction in computer time is even more pronounced, from 38.98 to 1.10 sec of IBM 360/75 time.

This method also decreases the lag time in responding to parameter variations from that obtained by simply setting  $N$  to 30. If a good estimate is achieved at sample instant  $i$  and the "true" parameter value begins to vary slowly, the latter technique will not respond until sample instant  $(i + 30)$ , whereas the former would start to respond at  $(i + 10)$ . Moreover, the former would make three parameter estimate iterations by instant  $(i + 30)$ , with more valid  $s^1$  and  $J^1$  values for instants

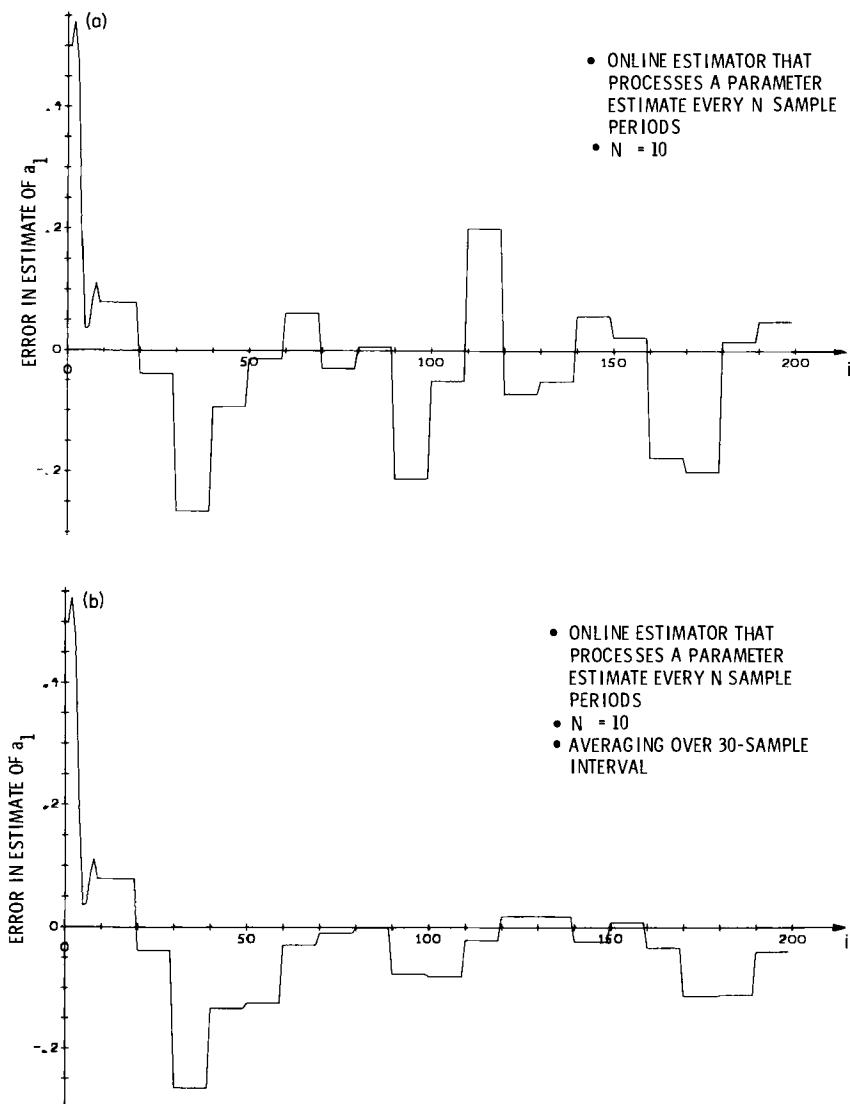
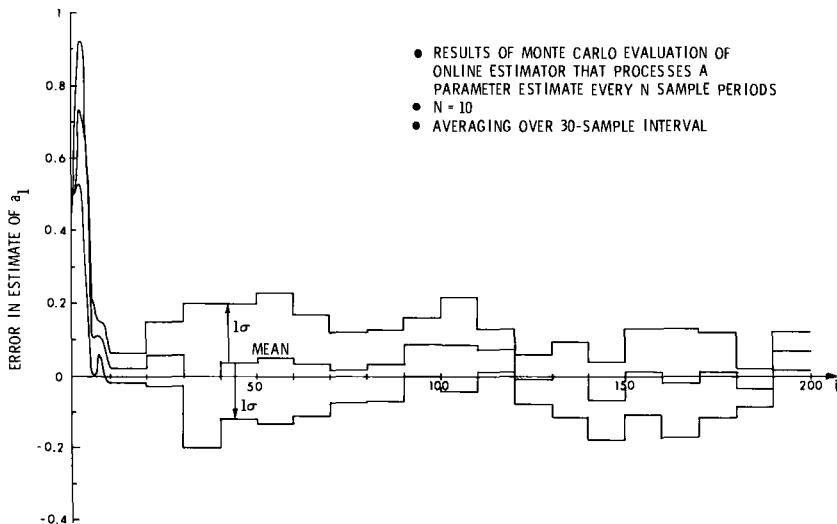


FIG. 10.11 (a) Parameter estimate error; online estimator with  $N = 10$ . (b) Parameter estimate error; online estimator with  $N = 10$  plus averaging.

$(i + 11)$  to  $(i + 30)$ , and thus would generally converge to a better estimate at instant  $(i + 30)$  than the estimator with  $N = 30$ .

Figure 10.12 presents the mean  $\pm 1\sigma$  time histories obtained from a Monte Carlo analysis of the online estimator with  $N = 10$  and averaging over the three most recent estimates, identical to the simulations used to generate Fig. 10.4. Comparison of these plots verifies the fact that the online technique attains performance almost identical to that of the more complex algorithm. ■

FIG. 10.12 Mean  $\pm 1\sigma$  values of online parameter estimate errors.

If it is desired to evaluate a parameter estimate more often, a somewhat different online approximate technique can be used. Whether or not a new parameter estimate is to be made at time  $t_i$ , the one-step terms  $s^1[Z_i, \hat{a}^*(t_j)]$  and  $J^1[Z_i, \hat{a}^*(t_j)]$  are evaluated as real time progresses from  $t_{i-1}$  to  $t_i$ , where  $t_j$  is the most recent time that a parameter estimate was made. Also, the new running sums for  $s$  and  $J$  are computed by adding these newest  $s^1$  and  $J^1$  values and subtracting off the oldest, corresponding to time  $t_{i-N}$ :

$$\tilde{s}(t_i) = \tilde{s}(t_{i-1}) + s^1[Z_i, \hat{a}^*(t_j)] - s^1[Z_{i-N}, \hat{a}^*(t_k)] \quad (10-69a)$$

$$\tilde{J}(t_i) = \tilde{J}(t_{i-1}) + J^1[Z_i, \hat{a}^*(t_j)] - J^1[Z_{i-N}, \hat{a}^*(t_k)] \quad (10-69b)$$

where  $\hat{a}^*(t_k)$  was the available estimate of  $a$  when  $s$  and  $J$  were computed at time  $t_{i-N}$ . Note that this technique thus requires additional computer memory to store the  $N$  most recent  $s^1$  and  $J^1$  evaluations. Whenever a new parameter estimate is desired, the results of calculating (10-29b) and (10-60) are added to the appropriate running sums to produce  $s[Z_i, \hat{a}^*(t_j)]$  and  $J[Z_i, \hat{a}^*(t_j)]$ , and  $\hat{a}^*(t_i)$  is computed in a manner analogous to Eq. (10-68). As in the previous form, using an interval of less than  $N$  periods can be used to improve transient and tracking characteristics. Subsequently, the estimates over an  $N$ -period interval can be averaged to incorporate the constant-over- $N$ -step model into the parameter estimate and remove high frequency fluctuations.

**EXAMPLE 10.9** For the hypothetical problem discussed in Examples 10.1, 10.2, and 10.7, the only change in Tables 10.3 and 10.4 are the incorporation of three more subtractions for  $s$  and six more subtractions for  $J$ , as seen by comparing Eqs. (10-69) and (10-67). ■

**EXAMPLE 10.10** Recall the problem in Examples 10.3 and 10.8. Although an appropriate  $N$  was found to be 30, the lack of term regeneration requires a smaller interval for adequate performance, and a 5-step size was chosen. Averaging over a 30-sample interval was then incorporated.

Figure 10.13a plots the parameter estimate error from such an estimator that generates a new parameter value every sample period, produced by the same noise sequences as generated Figs. 10.3 and 10.11. Averaging was started at the initial time, causing the initial transient to settle out somewhat more slowly; this aspect can be removed by starting the averaging later. As seen from the

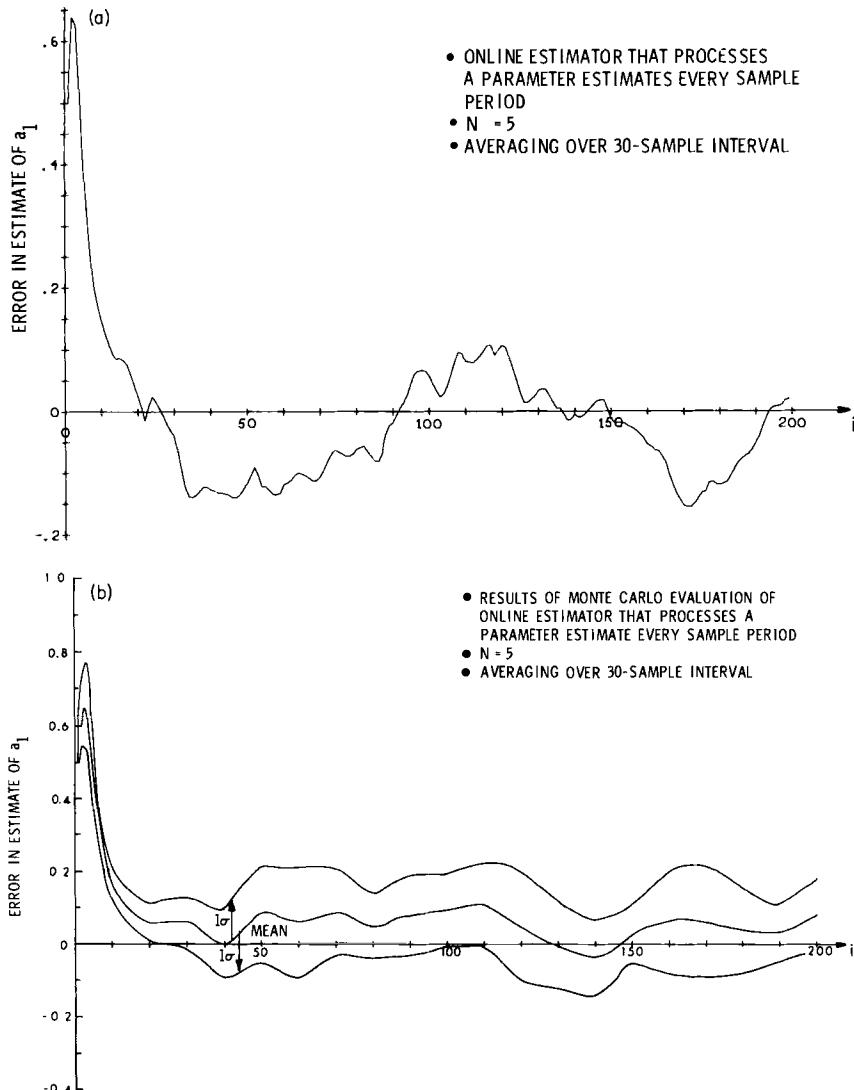


FIG. 10.13 (a) Parameter estimate error; online estimator with  $N = 5$  plus averaging. (b) Mean  $\pm 1\sigma$  values of online parameter estimate errors.

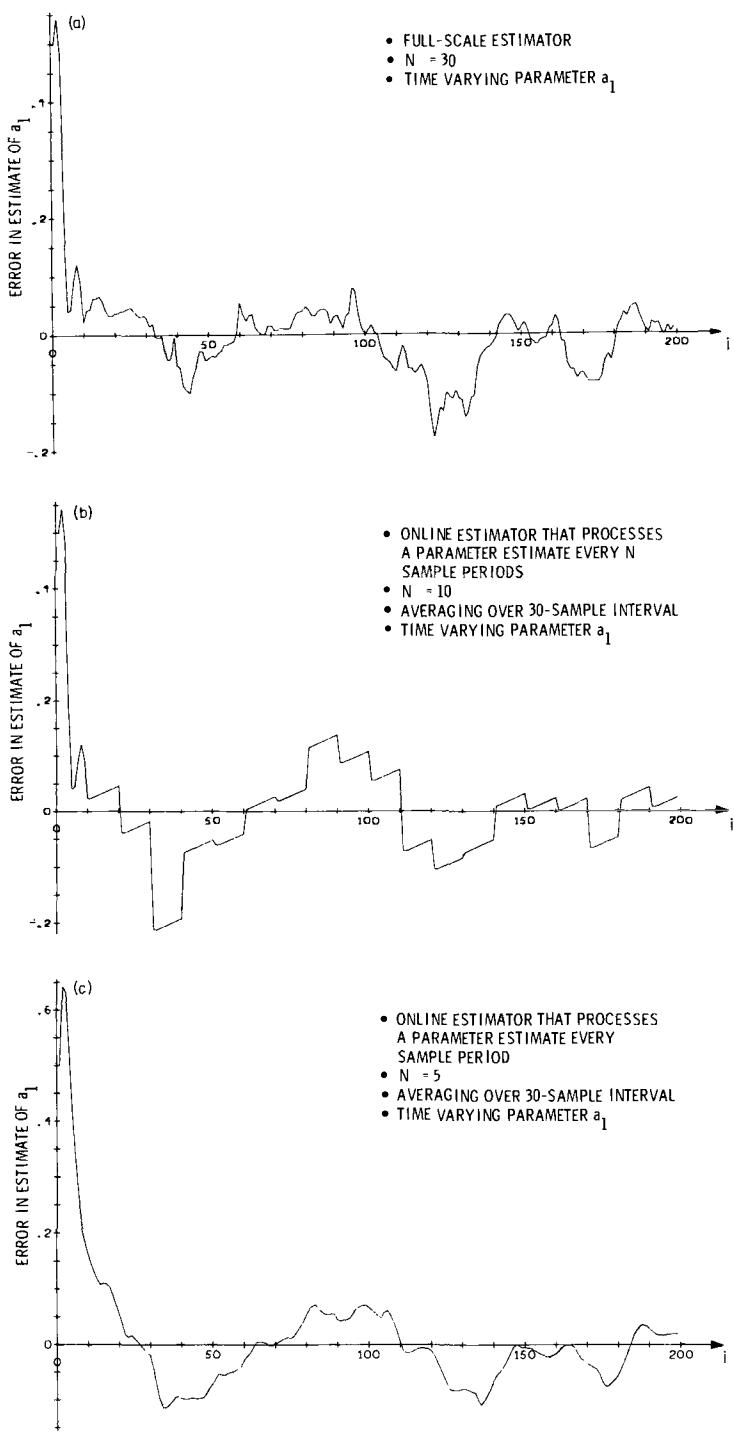


FIG. 10.14 Parameter estimate error with variable true parameter. (a) Full-scale estimator, (b) and (c) online estimators.

figure, the tracking behavior of this online method is comparable to that of the full-scale estimator. The Monte Carlo results shown in Fig. 10.13b (based on the same noise simulation runs as for Fig. 10.4) further substantiate this fact.

Very similar performance is achievable, with less memory load, by including only every second or third estimate in the averaging, or by making a parameter estimate only every second or third sample period. The undesirable initial deviation can be removed by delaying the time of the first parameter estimate by 5 sample periods. ■

**EXAMPLE 10.11** The ability of the estimators to track a slowly time-varying parameter value was examined by letting the “true” parameter in the previous problem double its magnitude (linearly) in the 200 sample period interval. Figures 10.14a, b, and c portray the parameter estimate errors committed by the full-scale and two online estimators, respectively. Figure 10.14b displays the slope of the “true” parameter growth, since the simulation computed the error every period while this estimate was made only every ten samples after  $i = 10$ . Thus, although the parameter actually changed by 0.075 every 30-step interval, i.e., 15% of its original value, all three estimators were able to track its value very adequately with a model of the parameter as constant over 30 periods. ■

To enhance online applicability, not only can the iterative solution procedure be approximated, but the *likelihood equations themselves can be approximated* as well. Analyses based on ambiguity functions and simulations can indicate the relative sensitivities of individual terms in the likelihood equations to parameter values, and the less sensitive terms can be neglected. Experience has shown that, of the possible implementations of this form, the most successful is the *inclusion of only weighted least squares type of terms*. Not only does this remove a considerable amount of computation, but it generally provides estimates of the same quality as attained with the more complex algorithm. In fact, certain terms removed in this manner can be shown to contribute a bias to the parameter estimate, further motivating their removal, provided that the remaining terms can effect an adequate estimate.

Basically, if Gaussian models are involved, the maximum likelihood method involves maximizing

$$L(\boldsymbol{\theta}, \mathbf{Z}_i) = \ln[(2\pi)^{-n/2} |\mathbf{Y}|^{-1/2} \exp\{-\frac{1}{2}\boldsymbol{\xi}^T \mathbf{Y}^{-1} \boldsymbol{\xi}\}] \quad (10-70)$$

as a function of  $\boldsymbol{\theta}$ , where both  $\boldsymbol{\xi}$  and  $\mathbf{Y}$  are functions of  $\boldsymbol{\theta}$ , yielding the  $p$  simultaneous equations ( $k = 1, 2, \dots, p$ ):

$$\frac{\partial \boldsymbol{\xi}^T}{\partial \theta_k} \mathbf{Y}^{-1} \boldsymbol{\xi} + \frac{1}{2} \text{tr} \left\{ \mathbf{Y}^{-1} \frac{\partial \mathbf{Y}}{\partial \theta_k} \right\} - \frac{1}{2} \boldsymbol{\xi}^T \mathbf{Y}^{-1} \frac{\partial \mathbf{Y}}{\partial \theta_k} \mathbf{Y}^{-1} \boldsymbol{\xi} \Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}^*} = 0 \quad (10-71)$$

If the dependence of  $\mathbf{Y}$  on  $\boldsymbol{\theta}$  is neglected, the likelihood equations reduce to

$$\frac{\partial \boldsymbol{\xi}^T}{\partial \theta_k} \mathbf{Y}^{-1} \boldsymbol{\xi} \Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}^*} = 0 \quad (10-72)$$

which are also the weighted least squares relations for maximizing  $(-\frac{1}{2}\boldsymbol{\xi}^T \mathbf{Y}^{-1} \boldsymbol{\xi})$ , or minimizing  $(\frac{1}{2}\boldsymbol{\xi}^T \mathbf{Y}^{-1} \boldsymbol{\xi})$ , by appropriate choice of  $\boldsymbol{\theta}$ . In Eq. (10-71), the second term does not depend on the actual sequence of measurement values, and thus

at best it contributes nothing to a valid means of estimating parameters, and at worst it contributes a bias or destabilization. The third term is not as sensitive to parameter variations as the first, and if it can be neglected, then there is no need to generate expressions for  $\partial \mathbf{Y} / \partial \theta_k$  in the estimator algorithm.

The implementation equations for this approximation are vastly simplified. To propagate between sample times, the state equations are as given by (10-34)–(10-37). However, the partials of the covariance matrices with respect to each parameter  $a_k$  are now neglected, and only (10-38) remains of the score relations, and the conditional information matrix relations reduce to the single computation (see (10-59)):

$$\begin{aligned} E\{s_k^{-1}[\mathbf{Z}(t_i), \mathbf{a}] s_i^{-1}[\mathbf{Z}(t_i), \mathbf{a}]\} | \mathbf{a} = \hat{\mathbf{a}}_*(t_i)\} \\ \cong \frac{\partial \hat{\mathbf{x}}^T(t_i^-)}{\partial a_k} \mathbf{H}^T(t_i) \mathbf{A}^{-1}(t_i) \mathbf{H}(t_i) \frac{\partial \hat{\mathbf{x}}(t_i^-)}{\partial a_l} \end{aligned} \quad (10-73)$$

For measurement updates, the state relations remain unchanged from (10-46) to (10-49), but the score relations reduce to (10-50) and

$$s_k^{-1}[\mathbf{Z}_i, \hat{\mathbf{a}}_*(t_i)] = \frac{\partial \hat{\mathbf{x}}^T(t_i^-)}{\partial a_k} \mathbf{H}^T(t_i) \mathbf{n}_i \quad (10-74)$$

$$\frac{\partial \hat{\mathbf{x}}(t_i^+)}{\partial a_k} = \mathbf{D}(t_i) \frac{\partial \hat{\mathbf{x}}(t_i^-)}{\partial a_k} \quad (10-75)$$

and no  $\mathbf{J}$  computations are necessary. Moreover, the final terms added to the score running sums corresponding to (10-29b) go to zero, and the associated  $\mathbf{J}$  term given by (10-60) need not be added either.

**EXAMPLE 10.12** Table 10.5 depicts the required computations for the hypothetical problem described earlier, using the online estimator that provides a parameter estimate more frequently than every  $N$  samples. These results are directly comparable to those of Example 10.9, and the reductions are seen to be substantial. If no new parameter estimate is provided, only the last row of the table is affected. ■

TABLE 10.5

*Online Method Using Weighted Least Squares Terms Only*

Term	Multiplications	Additions	Subtractions	Inversions
$\mathbf{x}$	889	671	27	1 ( $2 \times 2$ )
$\mathbf{s}$	275	189	3	0
$\mathbf{J}$	36	24	6	0
$\mathbf{a}^*$	9	9	0	1 ( $3 \times 3$ )

**EXAMPLE 10.13** Figure 10.15 portrays the performance of an online estimator identical to that described in Example 10.10, except that only the weighted least squares type terms are retained. Plot (a) is the parameter estimate error for the same representative run as shown in Fig. 10.13a,

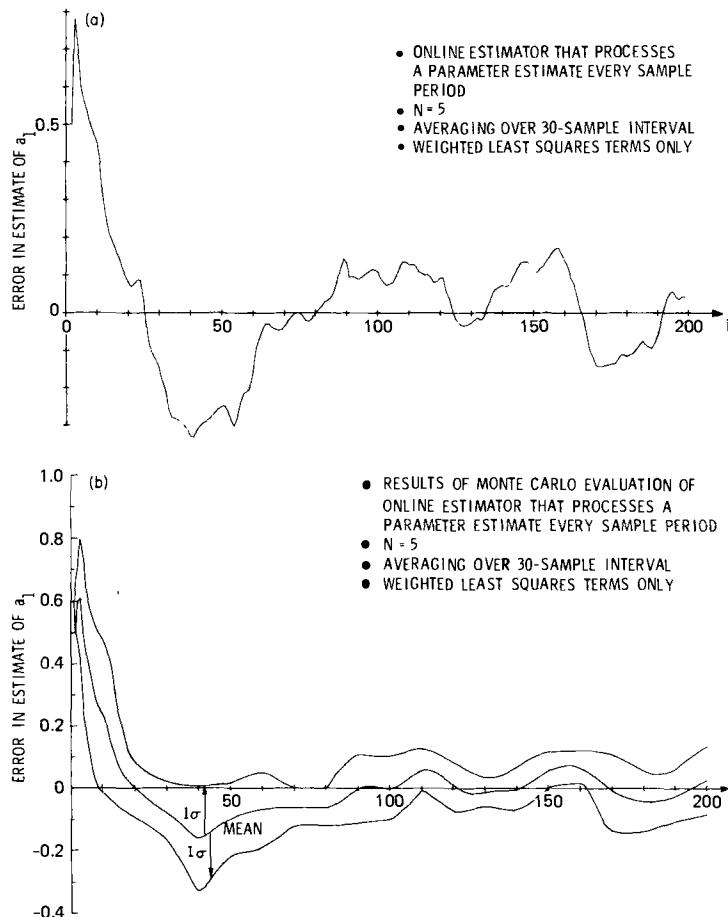


FIG. 10.15 (a) Parameter estimate error for online estimator using only WLS terms. (b) Mean  $\pm 1\sigma$  values of online parameter estimate errors; WLS terms only.

and plot (b) presents the Monte Carlo simulation results comparable to Fig. 10.13b. Comparison of these plots reveals that the  $1\sigma$  bands are wider in Fig. 10.15b for the initial stages, but become narrower than those in Fig. 10.13b as time progresses beyond the initial transient period. More importantly, though, the bias is seen to be decreased in Fig. 10.15b.

This performance is gained with a significant reduction in computer time. IBM 360/75 time for estimator implementation over 200 samples was reduced from 6.40 to 2.68 sec, the major difference attributed to the removal of the propagation of  $\partial \mathbf{P} / \partial a$ . ■

In order to improve the efficiency of the estimator algorithms, it would be advantageous to *precompute and store* various terms for online use, rather than calculate all quantities in real time. *Curve-fitting* of simple functions, such as piecewise-linear functions, to precomputed time histories can be used to

minimize storage requirements and maintain feasibility. Quantities could be precomputed as a function of the parameter values  $\mathbf{a}$ , and then these required values could be evaluated using either the most recent estimate of  $\mathbf{a}$  or a nominal  $\mathbf{a}$  value that provides reasonable performance over a range of possible parameter values.

First of all,  $\mathbf{J}^{-1}$  typically reaches a “steady state” value very quickly, and needs updating infrequently, if at all, after an initial transient. In fact, there is considerable advantage to using such a precomputed  $\mathbf{J}^{-1}$  before such time as an online computed  $\mathbf{J}^{-1}$  would complete the initial transient stage, as discussed previously. By using a precomputed  $\mathbf{J}^{-1}$  initially, the estimator would be employing a form of weighted gradient iteration instead of scoring, with better initial convergence properties for the parameter estimate.

Matrix propagations that can be precomputed parametrically as a function of  $\mathbf{a}$  would be the  $\mathbf{P}$  sequence (and related  $\mathbf{A}$  and  $\mathbf{K}$ ), the  $\partial\mathbf{P}/\partial a_k$  sequence and associated  $\partial\mathbf{A}/\partial a_k$  for all  $p$  values of  $k$ , and finally the controller gains if feedback control is used. (Controller design, and specifically assumed certainty equivalence controllers as applicable here, will be discussed in Volume 3.) These sequences could be evaluated over an interval of interest using a number of fixed values of the parameters. Simple but adequate approximating functions would be curve-fitted to express these sequence values as a function of  $\mathbf{a}$ .

**EXAMPLE 10.14** Table 10.6 presents the calculations required for the hypothetical problem, using the online estimator with frequent parameter updating, but with precomputed  $\mathbf{J}^{-1}$  and matrix propagations as described above. These results are directly comparable to those of Example 10.9.

TABLE 10.6  
*Online Method; Precomputed Matrices*

Term	Multiplications	Additions	Subtractions	Inversions
$\mathbf{x}$	69	56	2	0
$\mathbf{s}$	309	229	9	0
$\mathbf{J}$	0	0	0	0
$\mathbf{a}^*$	9	9	0	0

TABLE 10.7  
*Online Method with WLS Terms Only; Precomputed Matrices*

Term	Multiplications	Additions	Subtractions	Inversions
$\mathbf{x}$	69	56	2	0
$\mathbf{s}$	275	189	9	0
$\mathbf{J}$	0	0	0	0
$\mathbf{a}^*$	9	9	0	0

Similarly, Table 10.7 presents the corresponding results for the case of including only weighted least squares type terms. Comparison to Table 10.5 of Example 10.12 reveals the marked improvement afforded by precomputation.

Note that computations required to evaluate the functions of  $a$  for the curve-fitted approximations are not included in these two tables. ■

**EXAMPLE 10.15** If the Monte Carlo simulations of Example 10.13 are examined, the scalar  $J^{-1}$  is seen to reach a value of approximately 0.05 after the initial transient period. Figure 10.16 plots the parameter estimate error when  $J^{-1}$  is set equal to 0.05 for all time in the estimator whose performance was given by Fig. 10.15a (with the same noise simulations used). The most notable change is the improved transient behavior: the initial rise in error is less, it decreases to 0.05 in about half the time, and yet overshoots zero by less. In certain other simulations that previously required a delay in making the first parameter estimate, the use of precomputed  $J^{-1}$  yields adequate performance without the delay, since the initial values of  $J^{-1}$  are no longer disproportionately large. Substantiating results were obtained with both the other online estimator and the full-scale version.

Precomputed state estimator gains as functions of the parameter value also proved successful for this problem. Steady state gains were attained to four significant figures in four sample periods over the entire range of parameter values, so only steady state gains were used. Least squares curve-fitting techniques yielded an approximation of

$$\begin{aligned} k_1 &= 0.971392 - 0.000035a_1 + 0.000680a_1^2 \\ k_2 &= -0.9505a_1 \end{aligned}$$

that commits an error of, at worst, one part in  $10^4$  over a range of  $a_1$  from -2 to 0. Estimation performance remained essentially the same, simulation results agreeing to two significant figures by the end of the second sample period, and being virtually indistinguishable thereafter. ■

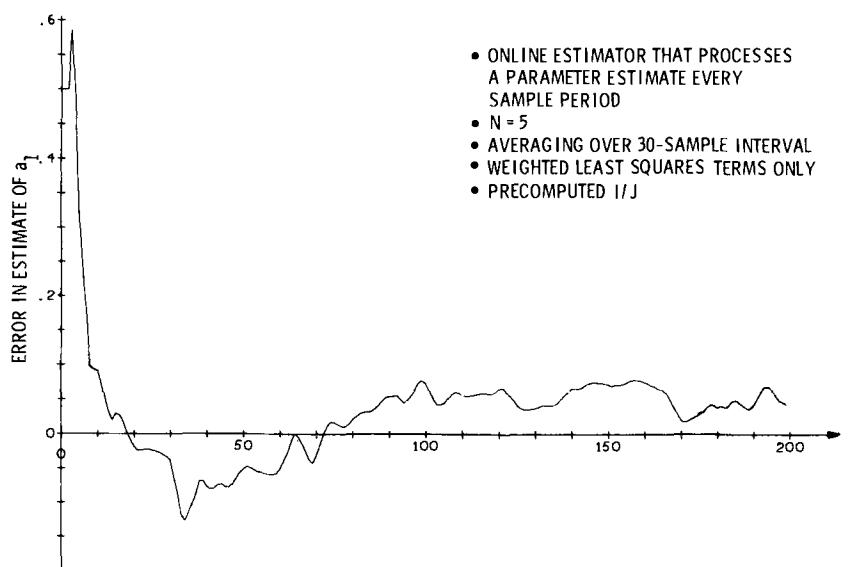


FIG. 10.16 Parameter estimate error for estimator using precomputed  $1/J$ .

One method of enhancing online feasibility that does not inherently involve approximations is the use of *advantageous state space representations*. Since a single system input–output relation can be represented by an infinite number of state vector models, it is beneficial to choose a particular model that embodies the greatest number of zero matrix elements, thereby requiring the fewest computations in final implementation. If a system can be modeled as

$$\mathbf{x}(t_{i+1}) = \Phi(t_{i+1}, t_i)\mathbf{x}(t_i) + \mathbf{B}_d(t_i)\mathbf{u}(t_i) + \mathbf{G}_d(t_i)\mathbf{w}_d(t_i) \quad (10-76a)$$

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \quad (10-76b)$$

then it is possible to define a new state vector through use of an invertible transformation matrix  $\mathbf{T}(t_i)$  as [19]

$$\mathbf{x}(t_i) = \mathbf{T}(t_i)\mathbf{x}^*(t_i) \quad (10-77a)$$

$$\mathbf{x}^*(t_i) = \mathbf{T}^{-1}(t_i)\mathbf{x}(t_i) \quad (10-77b)$$

and obtain a model with an input–output relation identical to that of (10-76) in the form of

$$\mathbf{x}^*(t_{i+1}) = \Phi^*(t_{i+1}, t_i)\mathbf{x}^*(t_i) + \mathbf{B}_d^*(t_i)\mathbf{u}(t_i) + \mathbf{G}_d^*(t_i)\mathbf{w}_d(t_i) \quad (10-78a)$$

$$\mathbf{z}(t_i) = \mathbf{H}^*(t_i)\mathbf{x}^*(t_i) + \mathbf{v}(t_i) \quad (10-78b)$$

where the system matrices are generated by

$$\Phi^*(t_{i+1}, t_i) = \mathbf{T}^{-1}(t_{i+1})\Phi(t_{i+1}, t_i)\mathbf{T}(t_i) \quad (10-79a)$$

$$\mathbf{B}_d^*(t_i) = \mathbf{T}^{-1}(t_{i+1})\mathbf{B}_d(t_i) \quad (10-79b)$$

$$\mathbf{G}_d^*(t_i) = \mathbf{T}^{-1}(t_{i+1})\mathbf{G}_d(t_i) \quad (10-79c)$$

$$\mathbf{H}^*(t_i) = \mathbf{H}(t_i)\mathbf{T}(t_i) \quad (10-79d)$$

In the case of scalar measurements, a particularly convenient form [82, 144] is given by the discrete-time counterpart of standard observable phase variable form (this form will also be exploited later in controller design as well). With scalar measurements,  $\mathbf{H}(t_i)$  becomes a 1-by- $n$  matrix, or vector transpose, so Eq. (10-76b) can be written as

$$\mathbf{z}(t_i) = \mathbf{h}^T(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \quad (10-80)$$

If the original system model is completely observable, then an invertible transformation matrix can be generated explicitly through

$$\mathbf{T}^{-1}(t_i) = \begin{bmatrix} \mathbf{h}^T(t_i) \\ \mathbf{h}^T(t_{i+1})\Phi(t_{i+1}, t_i) \\ \vdots \\ \mathbf{h}^T(t_{i+n-1})\Phi(t_{i+n-1}, t_i) \end{bmatrix} \quad (10-81)$$

that will generate a new system state transition matrix as

$$\Phi^*(t_{i+1}, t_i) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \phi^{*\top}(t_{i+1}, t_i) \end{bmatrix} \quad (10-82)$$

where  $\mathbf{0}$  is  $(n - 1)$ -by-1,  $\mathbf{I}$  is an  $(n - 1)$ -by- $(n - 1)$  identity matrix, and  $\phi^{*\top}(t_{i+1}, t_i)$  is given by

$$\phi^{*\top}(t_{i+1}, t_i) = \mathbf{h}^T(t_{i+n})\Phi(t_{i+n}, t_i)\mathbf{T}(t_i) \quad (10-83)$$

Moreover,  $\mathbf{B}_d^*(t_i)$  and  $\mathbf{G}_d^*(t_i)$  are given by (10-79b) and (10-79c), and the new measurement matrix becomes

$$\mathbf{h}^{*\top}(t_i) = [1 \quad 0 \quad \cdots \quad 0] \quad (10-84)$$

From (10-84) it is clear that there are no uncertain parameters in  $\mathbf{h}^{*\top}(t_i)$ , as assumed earlier.

For the case of vector measurements, a row of the measurement matrix  $\mathbf{H}(t_i)$ , with respect to which the system model is completely observable, is denoted as  $\mathbf{h}^T(t_i)$ , and the same procedure is followed. The remaining portion of the new measurement matrix is obtained from (10-79d). If no such row of  $\mathbf{H}(t_i)$  can be found, a modified procedure can be used [82, 144]. In general, vector measurements will yield a  $\mathbf{H}^*(t_i)$  that is no longer devoid of uncertainties. The estimator form can be expanded to admit uncertain parameters in  $\mathbf{H}$  to address this case, or an equivalent update using  $m$  scalar updates can be generated, iteratively applying the previous results (if complete observability is provided by each scalar measurement).

If the system model is time invariant, then either the standard observable form above or the discrete-time modified Jordan canonical form [19], analogous to that discussed in Chapter 2 [83], can be used to advantage. The mode separation provided by the latter form can significantly reduce the amount of computation required to evaluate  $\partial\hat{\mathbf{x}}/\partial a_k$  and  $\partial\mathbf{P}/\partial a_k$  recursions, since each parameter will enter into only one or a few system modes. Moreover, this form generally allows a shorter computer wordlength to achieve the same precision in system representation as that attained by other state variable forms.

**EXAMPLE 10.16** Table 10.8 presents the required number of calculations for the hypothetical problem, using canonical variables and the online estimator formulation including  $\mathbf{H}^*(t_i)$  uncertainties. Table 10.9 relates the corresponding values when only weighted least square terms are used, and Table 10.10 employs precomputations to reduce the loading further. For all of these tables, it is assumed that there are three first order modes and one second order, that one uncertain parameter affects the second order mode, another a first order mode, and the last is an uncertain parameter in  $\mathbf{B}_d$  that affects a first order mode. The numbers cited do not include the computation of required functional evaluations for the elements of the system matrices, which would further motivate use of canonical variables. The numbers in parentheses are the portion of the totals due to allowing uncertain parameters in  $\mathbf{H}^*$ : for scalar measurements, there would be no such contribution. ■

TABLE 10.8  
*Online Method; Canonical Form*

Term	Multiplications	Additions	Subtractions	Inversions
x	606	473	27	1 ( $2 \times 2$ )
s	1394 (115)	1136 (95)	69 (60)	1 ( $5 \times 5$ )
J	887	698 (4)	6	0
a*	9	9	0	1 ( $3 \times 3$ )

TABLE 10.9  
*Online Method with WLS Terms Only; Canonical Form*

Term	Multiplications	Additions	Subtractions	Inversions
x	606	473	27	1 ( $2 \times 2$ )
s	162 (25)	137 (11)	13 (10)	0
J	36	28 (4)	6	0
a*	9	9	0	1 ( $3 \times 3$ )

TABLE 10.10  
*Online Method; WLS Terms, Precomputed Matrices, Canonical Form*

Term	Multiplications	Additions	Subtractions	Inversions
x	46	43	2	0
s	162 (25)	137 (11)	13 (10)	0
J	0	0	0	0
a*	9	9	0	0

*Exploiting symmetry* can enhance not only storage and computation requirements by computing only nonredundant elements of symmetric matrices, but also numerical precision through implementation of square root forms. *Modification of measurement incorporation*, iteratively updating the partitions of  $\mathbf{z}_i$  as discussed previously, can also increase the online applicability of the estimator.

**EXAMPLE 10.17** Recall the thrust vector control problem of Example 9.9, in which an extended Kalman filter was used to estimate both the states and the uncertain parameter ( $\omega_b^2$ ). Figure 10.17a presents a representative parameter estimate error trajectory as generated by the full-scale estimator with  $N = 30$ . As in previous examples, the initial behavior characteristics are improved by delaying the first estimate, allowing the computed  $J^{-1}$  to attain a reasonable magnitude before using it in the estimation: Fig. 10.17b depicts the result of postponing the first parameter estimate until the sixth sample instant.

The corresponding state estimation accuracy is enhanced considerably, as seen by comparing the errors in the estimates of generalized bending coordinate velocity with and without simultaneous

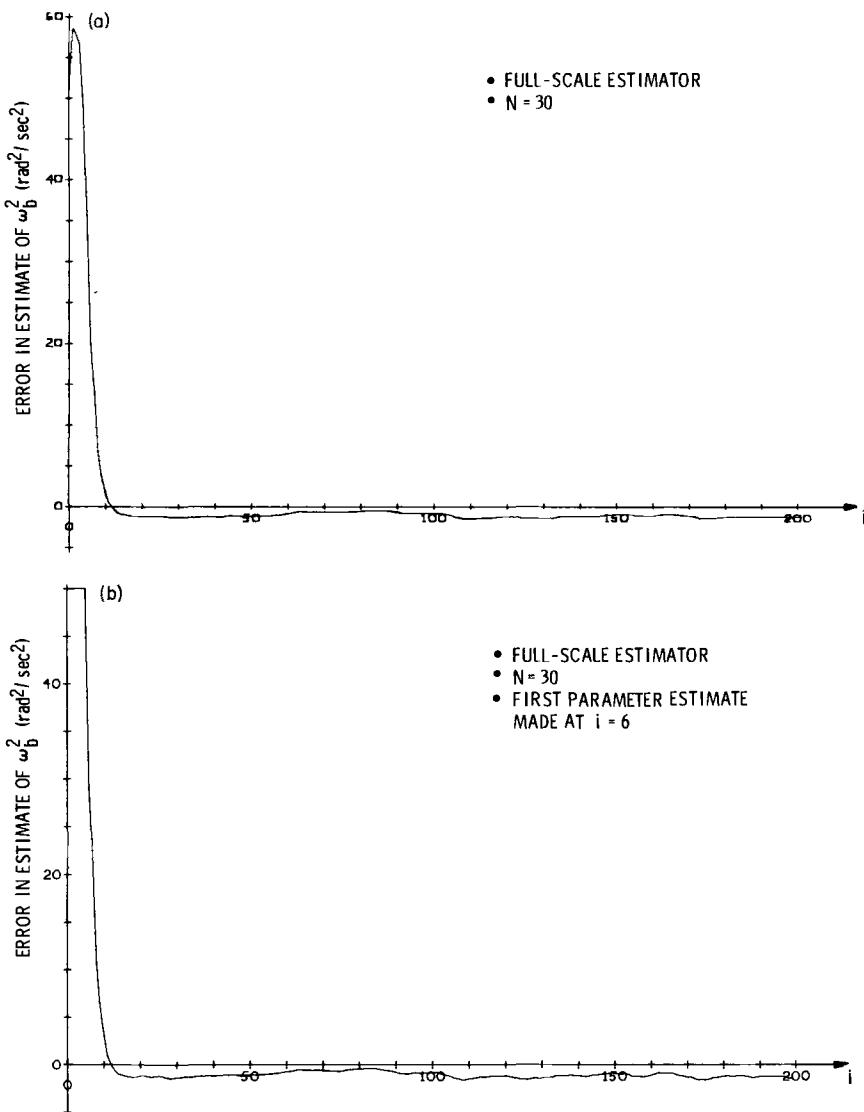


FIG. 10.17 (a) Full-scale parameter estimate error for  $N = 30$ . (b) Full-scale parameter estimate error when first estimate is delayed.

parameter estimation, as in Fig. 10.18. The other state estimates display similar improvement, and results are consistent when mean  $\pm 1\sigma$  (standard deviation) plots from a Monte Carlo analysis are studied [82]. In fact, because of the precision of the parameter estimate, the state trajectories associated with the full-scale estimator are essentially equivalent to the output of a Kalman filter tuned to the correct value of  $\omega_b^2$ .

The online estimator that processes a parameter estimate every  $N$  sample periods ( $N = 10$ ), combined with averaging over the three most recent estimates, performed well, as seen in the

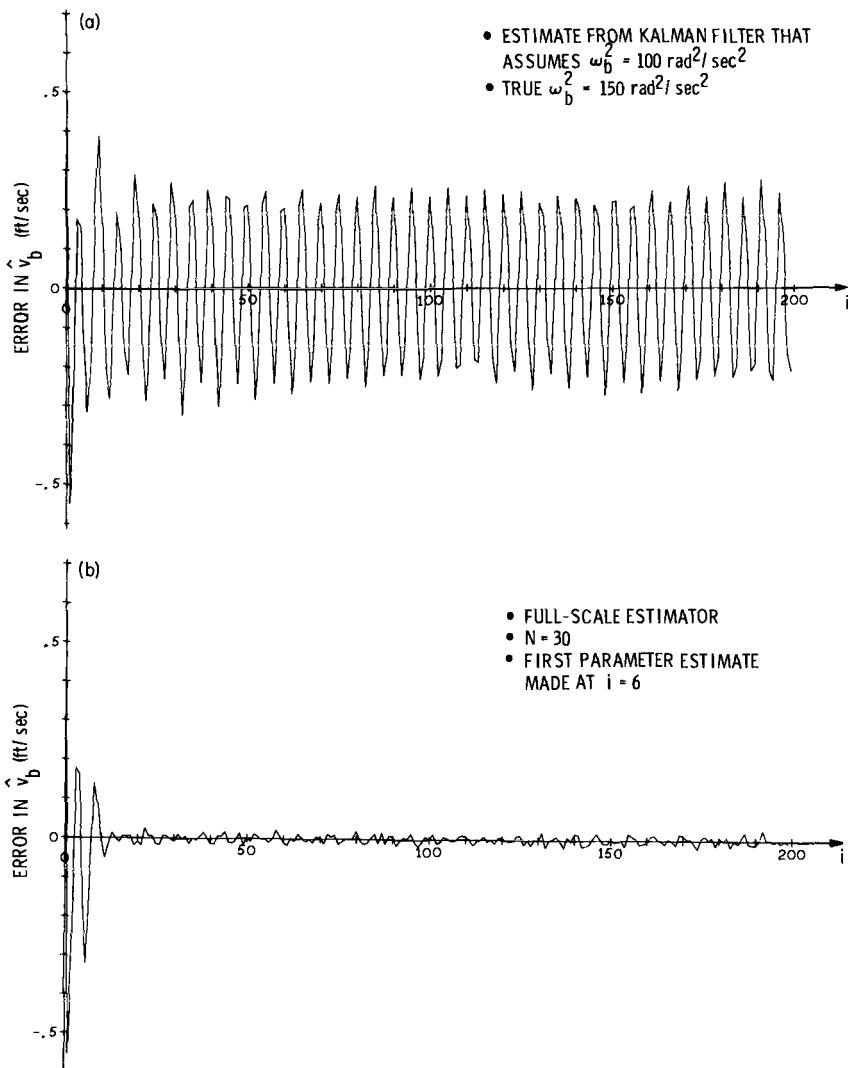


FIG. 10.18 (a)  $\hat{v}_b$  error when parameter is not estimated. (b)  $\hat{v}_b$  error when parameter is estimated.

typical parameter estimate error trajectory (generated by the same noise simulations as for Fig. 10.17) in Fig. 10.19a. As in the previous examples, by estimating the parameter more often for the first few instants, convergence to a good value was very rapid, followed by estimation accuracy that equals that of the full-scale estimator [82].

The error trajectory of Fig. 10.19b was generated by the same sequence of noise inputs, but the estimator itself differed in two respects. First, it incorporated only weighted least squares type terms, and the influence of dropping the other terms is minimal. Second, the initial parameter

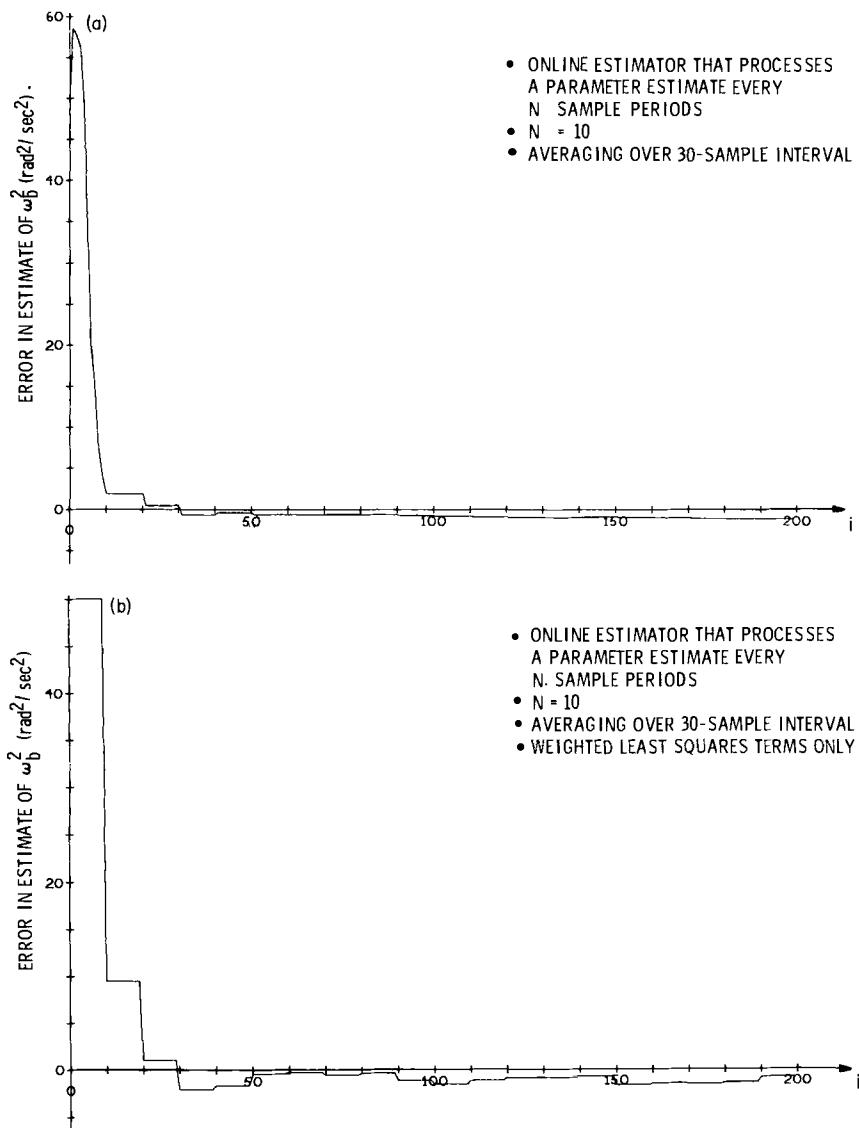


FIG. 10.19 (a) Online parameter estimate error. (b) Online parameter estimate errors; WLS terms only.

estimation frequency was not increased, and the resulting slower transient response is characteristic; if this response is satisfactory, this form might be preferable because of the reduced computational load.

The online procedure that produces a new parameter estimate every period also nearly duplicated the performance of the full-scale technique, similarly with considerably less computational expense. ■

## 10.7 UNCERTAINTIES IN $\mathbf{Q}_d$ AND $\mathbf{R}$

The concept of the previous sections can be applied directly to the problem of state estimation in the face of uncertain parameters in the dynamic noise covariance  $\mathbf{Q}_d$  or the measurement noise covariance  $\mathbf{R}$  [3, 5, 6, 16, 18, 48, 49, 70, 87, 88, 91, 126, 141]. In fact, the solution to (10-15) and (10-18) will again provide the maximum likelihood estimate of states and parameters assuming constant parameters for all time, whereas the solution to (10-20) and (10-21) will provide the estimates exploiting the constant-over- $N$ -steps model for parameters. The only difference in algorithm form lies in the evaluation of the partial derivatives in these relations [1].

In detail, the modifications to the full-scale estimator equations (10-28)–(10-57) are as follows. The score time propagation equations become

$$\frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} = \Phi(t_j, t_{j-1}) \frac{\partial \hat{\mathbf{x}}(t_{j-1}^+)}{\partial a_k} \quad (10-38')$$

$$\begin{aligned} \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} &= \Phi(t_j, t_{j-1}) \frac{\partial \mathbf{P}(t_{j-1}^+)}{\partial a_k} \Phi^T(t_j, t_{j-1}) \\ &\quad + \mathbf{G}_d(t_{j-1}) \frac{\partial \mathbf{Q}_d(t_{j-1})}{\partial a_k} \mathbf{G}_d^T(t_{j-1}) \end{aligned} \quad (10-39')$$

$$\frac{\partial \mathbf{A}(t_j)}{\partial a_k} = \mathbf{H}(t_j) \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} \mathbf{H}^T(t_j) + \frac{\partial \mathbf{R}(t_j)}{\partial a_k} \quad (10-40a')$$

$$\frac{\partial \mathbf{E}(t_j)}{\partial a_k} = \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} \mathbf{H}^T(t_j) - \mathbf{K}(t_j) \frac{\partial \mathbf{A}(t_j)}{\partial a_k} \quad (10-40b')$$

where the additional term in (10-40b') is defined for computational efficiency and is related to the partial of the filter gain  $\mathbf{K}$  by  $[\partial \mathbf{K} / \partial a_k] = [\partial \mathbf{E} / \partial a_k] \mathbf{A}^{-1}$ . Of Eqs. (10-41)–(10-43), only the first term in (10-41) remains; similarly, (10-45) is not required. The modifications to the score measurement update relations are

$$\frac{\partial \hat{\mathbf{x}}(t_j^+)}{\partial a_k} = \mathbf{D}(t_j) \frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} + \frac{\partial \mathbf{E}(t_j)}{\partial a_k} \mathbf{n}_j \quad (10-53')$$

$$\frac{\partial \mathbf{P}(t_j^+)}{\partial a_k} = \mathbf{D}(t_j) \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} \mathbf{D}^T(t_j) + \mathbf{K}(t_j) \frac{\partial \mathbf{R}(t_j)}{\partial a_k} \mathbf{K}^T(t_j) \quad (10-54')$$

For the conditional information matrix, (10-55) becomes

$$\begin{aligned} E \left\{ \frac{\partial \hat{\mathbf{x}}(t_j^+)}{\partial a_k} \frac{\partial \hat{\mathbf{x}}^T(t_j^+)}{\partial a_l} \middle| \mathbf{a} = \hat{\mathbf{a}}_*(t_i) \right\} \\ = \mathbf{D}(t_j) E \left\{ \frac{\partial \hat{\mathbf{x}}(t_j^-)}{\partial a_k} \frac{\partial \hat{\mathbf{x}}^T(t_j^-)}{\partial a_l} \middle| \mathbf{a} = \hat{\mathbf{a}}_*(t_i) \right\} \mathbf{D}^T(t_j) + \frac{\partial \mathbf{E}(t_j)}{\partial a_k} \mathbf{A}^{-1}(t_j) \frac{\partial \mathbf{E}(t_j)}{\partial a_l} \end{aligned} \quad (10-55')$$

and (10-56) and (10-57) are no longer required.

As previously, the full-scale estimator is not feasible for online computations, so means of simplifying the algorithm are required. This is even more true in the present case, since the number of parameters to be estimated, and thus the amount of computation, is often greater in a practical self-tuning filter context than in a state estimation and system identification application (except “black box” identification). Even if  $\mathbf{Q}_d$  and  $\mathbf{R}$  are assumed to be diagonal to limit the number of uncertain parameters, this generally yields more parameters than encountered in many identification adaptations.

The two online conceptualizations of the previous section can be exploited to enhance feasibility, as can the use of precomputed (or infrequently recomputed)  $\mathbf{J}^{-1}$ . Also the idea of retaining only the most significant terms is fruitful to pursue, but the dominant terms in this case differ from those in Section 10.6.

Recall the reasoning associated with Eqs. (10-70)–(10-72). Now, rather than neglect the dependence of  $\mathbf{Y}$  on  $\boldsymbol{\theta}$ , it is appropriate to neglect the dependence of  $\xi$  on  $\boldsymbol{\theta}$ . This is equivalent to ignoring the second term in Eq. (10-21), the explicit likelihood equation for this problem, to yield

$$\sum_{j=i-N+1}^i \text{tr} \left\{ [\mathbf{A}(t_j)^{-1} - \mathbf{A}(t_j)^{-1} \mathbf{r}_j \mathbf{r}_j^T \mathbf{A}(t_j)^{-1}] \frac{\partial \mathbf{A}(t_j)}{\partial a_k} \right\} + \text{tr} \left\{ \mathbf{P}(t_i^+)^{-1} \frac{\partial \mathbf{P}(t_i^+)}{\partial a_k} \right\} \Big|_{\mathbf{a}=\hat{\mathbf{a}}^*(t_i)} = 0 \quad (10-85a)$$

Moreover, the last term in this expression is often dominated by the first, especially for reasonably sized  $N$ , so it too is usually neglected. Note that this is equivalent to removing  $\ln f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i), \mathbf{a}}(\xi | \mathcal{L}_i, \mathbf{a})$  from the likelihood function when seeking the estimate of parameters:

$$\sum_{j=i-N+1}^i \text{tr} \left\{ [\mathbf{A}(t_j)^{-1} - \mathbf{A}(t_j)^{-1} \mathbf{r}_j \mathbf{r}_j^T \mathbf{A}(t_j)^{-1}] \frac{\partial \mathbf{A}(t_j)}{\partial a_k} \right\} \Big|_{\mathbf{a}=\hat{\mathbf{a}}^*(t_i)} = 0 \quad (10-85b)$$

For either of the “pseudo” likelihood equation forms given in (10-85), the computational burden is reduced considerably. There is no longer any need to compute  $\partial \hat{\mathbf{x}}^- / \partial a_k$  or  $E\{[\partial \hat{\mathbf{x}} / \partial a_k][\partial \hat{\mathbf{x}} / \partial a_l]^T | \mathbf{a} = \hat{\mathbf{a}}^*(t_i)\}$ , thereby removing (10-38'), (10-40b'), (10-41), (10-53'), and (10-55'), and simplifying (10-44) and (10-52) to

$$E\{s_k^{-1}[\mathbf{Z}(t_j), \mathbf{a}] s_l^{-1}[\mathbf{Z}(t_j), \mathbf{a}] | \mathbf{a} = \hat{\mathbf{a}}^*(t_i)\} = \frac{1}{2} \text{tr} \left\{ \mathbf{A}^{-1}(t_j) \frac{\partial \mathbf{A}(t_j)}{\partial a_k} \mathbf{A}^{-1}(t_j) \frac{\partial \mathbf{A}(t_j)}{\partial a_l} \right\} \quad (10-44')$$

$$s_k^{-1}[\mathbf{Z}_j, \hat{\mathbf{a}}^*(t_i)] = -\frac{1}{2} \text{tr} \left\{ \mathbf{C}(t_j) \frac{\partial \mathbf{A}(t_j)}{\partial a_k} \right\} \quad (10-52')$$

It should be noted that numerical results [1] have indicated the conditional information matrix based upon (10-44') to exhibit generally smaller eigenvalues than that based on (10-44), with resultant increased numerical difficulty when its inverse is computed.

A further approximation can be made to obtain explicit estimates of  $\mathbf{R}$  and  $\mathbf{Q}_d$ . When these approximations are made, the existence of independent and unique solutions for the uncertain parameters and convergence properties are subject to question. If  $\mathbf{R}$  or  $\mathbf{Q}_d$  is to be estimated separately, a reasonable solution is usually achievable. Simultaneous estimation of parameters from both  $\mathbf{Q}_d$  and  $\mathbf{R}$  is not as well behaved. It is generally true, for all algorithms and not just those based upon the maximum likelihood concept, that  $\mathbf{R}$  parameter estimates are more precise than  $\mathbf{Q}_d$  parameter estimates.

Using (10-40a'), the "pseudo" likelihood equation (10-85b) can be written as

$$\sum_{j=i-N+1}^i \text{tr} \left\{ [\mathbf{A}^{-1}(t_j) - \mathbf{A}^{-1}(t_j)\mathbf{r}_j\mathbf{r}_j^T\mathbf{A}^{-1}(t_j)]\mathbf{H}(t_j) \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} \mathbf{H}^T(t_j) \right\} = 0 \quad (10-86a)$$

for  $a_k$  an unknown parameter in  $\mathbf{Q}_d$ , and

$$\begin{aligned} & \sum_{j=i-N+1}^i \left( \text{tr} \left\{ [\mathbf{A}^{-1}(t_j) - \mathbf{A}^{-1}(t_j)\mathbf{r}_j\mathbf{r}_j^T\mathbf{A}^{-1}(t_j)]\mathbf{H}(t_j) \frac{\partial \mathbf{P}(t_j^-)}{\partial a_k} \mathbf{H}^T(t_j) \right\} \right. \\ & \left. + [\mathbf{A}^{-1}(t_j) - \mathbf{A}^{-1}(t_j)\mathbf{r}_j\mathbf{r}_j^T\mathbf{A}^{-1}(t_j)]_{kk} \right) = 0 \end{aligned} \quad (10-86b)$$

for  $a_k$  a diagonal element of  $\mathbf{R}$ ,  $a_k = R_{kk}$ . To obtain an explicit relation for  $\mathbf{R}$ , assume  $\mathbf{Q}_d$  to be known completely. If, as in many applications,

$$\mathbf{H}(t_j) \frac{\partial \mathbf{P}(t_j^-)}{\partial R_{kk}} \mathbf{H}^T(t_j) \ll \mathbf{I}$$

then (10-86b) can be approximated by its second term alone:

$$\begin{aligned} 0 &= \sum_{j=i-N+1}^i [\mathbf{A}^{-1}(t_j) - \mathbf{A}^{-1}(t_j)\mathbf{r}_j\mathbf{r}_j^T\mathbf{A}^{-1}(t_j)]_{kk} \\ &= \sum_{j=i-N+1}^i [\mathbf{A}^{-1}(t_j)\{\mathbf{A}(t_j) - \mathbf{r}_j\mathbf{r}_j^T\}\mathbf{A}^{-1}(t_j)]_{kk} \\ &= \sum_{j=i-N+1}^i [\mathbf{A}^{-1}(t_j)\{\mathbf{H}(t_j)\mathbf{P}(t_j^-)\mathbf{H}^T(t_j) + \mathbf{R}(t_j) - \mathbf{r}_j\mathbf{r}_j^T\}\mathbf{A}^{-1}(t_j)]_{kk} \end{aligned} \quad (10-87)$$

which would be satisfied for all  $k$  if the term in braces, { }, is zero. If the estimation process is essentially time invariant over the most recent  $N$  steps, i.e.,  $\mathbf{A}^{-1}(t_j) \cong \text{const}$  over these steps, then an estimate of  $\mathbf{R}(t_i)$  can be defined as

$$\hat{\mathbf{R}}(t_i) = \frac{1}{N} \sum_{j=i-N+1}^i [\mathbf{r}_j\mathbf{r}_j^T - \mathbf{H}(t_j)\mathbf{P}(t_j^-)\mathbf{H}^T(t_j)] \quad (10-88a)$$

$$\cong \left[ \frac{1}{N} \sum_{j=i-N+1}^i \mathbf{r}_j\mathbf{r}_j^T \right] - \mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) \quad (10-88b)$$

where the  $\mathbf{r}_j$  and  $\mathbf{P}(t_j^-)$  are computed on the basis of previous parameter estimates, and the term in brackets is recognized as an ergodic approximation to the covariance of the zero-mean residuals,  $\mathbf{A}(t_i) = \mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{R}(t_i)$ . Even for reasonably large  $N$ , these expressions can lead to an estimate of  $\mathbf{R}(t_i)$  that is not positive definite. An alternative, better conditioned estimate can be generated by noting that

$$\mathbf{A}^{-1}(t_j)\mathbf{r}_j = \mathbf{R}^{-1}(t_j)[\mathbf{z}_j - \mathbf{H}(t_j)\hat{\mathbf{x}}(t_j^+)] \quad (10-89)$$

so that a development analogous to (10-87) yields

$$\begin{aligned} \hat{\mathbf{R}}(t_i) &= \frac{1}{N} \sum_{j=i-N+1}^i \{[\mathbf{z}_j - \mathbf{H}(t_j)\hat{\mathbf{x}}(t_j^+)][\mathbf{z}_j - \mathbf{H}(t_j)\hat{\mathbf{x}}(t_j^+)]^T \\ &\quad + \mathbf{H}(t_j)\mathbf{P}(t_j^+)\mathbf{H}^T(t_j)\} \end{aligned} \quad (10-90)$$

Although (10-90) is superior to (10-88) numerically [1], it requires a greater amount of computation, since neither  $[\mathbf{z}_j - \mathbf{H}(t_j)\hat{\mathbf{x}}(t_j^+)]$  nor  $[\mathbf{H}(t_j)\mathbf{P}(t_j^+)\mathbf{H}^T(t_j)]$  is already calculated in the estimation algorithm.

Obtaining an explicit estimate of  $\mathbf{Q}_d$  is less straightforward, and a unique estimate may well not exist without additional constraints. The partial derivative in (10-86a) is evaluated by means of (10-39'), which can be approximated by neglecting the first term. Again assuming that the estimation process is essentially in steady state over the most recent  $N$  steps, (10-86a) can be transformed into a set of equations that are satisfied if

$$\sum_{j=i-N+1}^i [\Phi(t_j, t_{j-1})\mathbf{P}(t_{j-1}^+)\Phi^T(t_j, t_{j-1}) + \mathbf{G}_d(t_{j-1})\mathbf{Q}_d(t_{j-1})\mathbf{G}_d^T(t_{j-1}) \\ - \mathbf{P}(t_j^+) - \Delta\mathbf{x}_j\Delta\mathbf{x}_j^T] = \mathbf{0} \quad (10-91)$$

where

$$\Delta\mathbf{x}_j = \hat{\mathbf{x}}(t_j^+) - \hat{\mathbf{x}}(t_j^-) = \mathbf{K}(t_j)\mathbf{r}_j \quad (10-92)$$

If  $\mathbf{G}_d(t_{j-1})$  is invertible for all  $j$  (as especially  $\mathbf{G}_d(t_{j-1}) \triangleq \mathbf{I}$ ), then an estimate of  $\mathbf{Q}_d(t_i)$  can be defined as

$$\hat{\mathbf{Q}}_d(t_i) = \frac{1}{N} \sum_{j=i-N+1}^i \{\mathbf{G}_d^{-1}(t_{j-1})[\Delta\mathbf{x}_j\Delta\mathbf{x}_j^T + \mathbf{P}(t_j^+) \\ - \Phi(t_j, t_{j-1})\mathbf{P}(t_{j-1}^+)\Phi^T(t_j, t_{j-1})]\mathbf{G}_d^{-1}(t_{j-1})^T\} \quad (10-93)$$

A physical interpretation of this result is developed in Problem 10.17 [61, 86]. If  $\mathbf{G}_d(t_{j-1})$  is not invertible, then  $\mathbf{G}_d^{-1}(t_{j-1})$  in (10-93) can be replaced by the pseudoinverse calculated as

$$\mathbf{G}_d^*(t_{j-1}) = [\mathbf{G}_d(t_{j-1})^T\mathbf{G}_d(t_{j-1})]^{-1}\mathbf{G}_d(t_{j-1})^T \quad (10-94)$$

It should be noted that (10-88) or (10-90) is an approximate solution for  $\hat{\mathbf{R}}$  assuming  $\mathbf{Q}_d$  to be known, and (10-93) is a solution for  $\hat{\mathbf{Q}}_d$  assuming  $\mathbf{R}$

known. The two solutions can be used, with caution, to estimate both matrices simultaneously, knowing that the resulting estimates are not independent. This dependency can cause biased estimates that do not distinguish between errors in  $\hat{\mathbf{R}}$  and  $\hat{\mathbf{Q}}_d$ .

EXAMPLE 10.18 Consider a second order time-invariant system with

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$$

where the undamped natural frequency  $\omega_n$  is 0.1 rad/sec and the damping ratio  $\zeta$  is 0.05. Further, let  $\mathbf{P}_0 = 10\mathbf{I}$ , and let measurement samples be taken from the system every second for 200 sec.

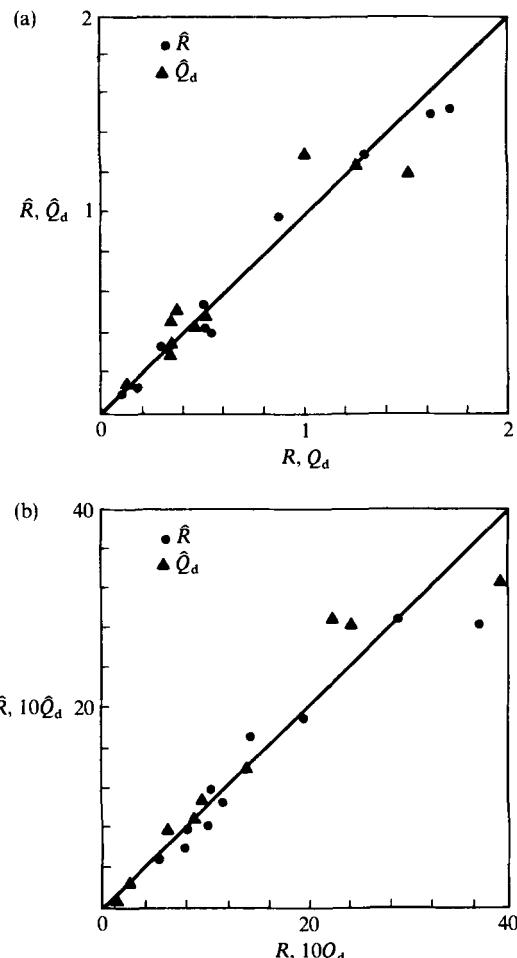


FIG. 10.20 (a) Estimating scalar  $R$  and  $Q_d$ . (b) Estimating scalar  $R$  and  $Q_d$ ; larger true values. From Abramson [1].

This example will indicate the performance for an estimator with  $N$  chosen appropriately: for  $N = 25$  and above this precision is essentially constant, whereas smaller  $N$  yields increasing error magnitudes.

First consider a single input, single output problem [1] in which the equivalent discrete-time model is described by  $\mathbf{G}_d^T = [0 \ 1]$  and  $\mathbf{H} = [1 \ 0]$ . Figure 10.20a depicts the results of ten different simulations, in which true values of  $Q_d$  and  $R$  were each varied between 0.1 and 2, while the estimator assumed initial conditions of  $\hat{Q}_d = 0.5$  and  $\hat{R} = 1.0$  in all cases. If there were no estimation error, the estimates of  $R$  and  $Q_d$  would lie along the diagonal  $\hat{R} = R$  and  $\hat{Q}_d = Q_d$ . The dispersion about this line is a measure of the estimation error. Figure 10.20b portrays the same results, but letting  $Q_d$  vary from 0 to 4 and  $R$  from 5 to 35, with  $\hat{Q}_d = 1$  and  $\hat{R} = 10$  initially. Both plots show good agreement between true and estimated values.

Use of a precomputed  $\mathbf{J}^{-1}$ , based upon the initially assumed parameter values, yielded results almost identical to that depicted in Fig. 10.20. This is shown in Table 10.11, which presents the mean value (averaged over the ten simulations corresponding to each plot of Fig. 10.20) of the true parameter  $a_t$ , and the mean and root mean squared values of the error ( $\hat{a} - a_t$ ). In fact, for the cases run, precomputation of  $\mathbf{J}^{-1}$  improved estimation performance.

If  $\mathbf{G}_d$  and  $\mathbf{H}$  are both changed to 2-by-2 identity matrices, and the a priori values of  $\hat{Q}_d$  and  $\hat{R}$  are

$$\hat{\mathbf{Q}}_d = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \quad \hat{\mathbf{R}} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

then Fig. 10.21 plots estimation performance data similar to that of Fig. 10.20, again for a 10-run simulation. Increasing the number of quantities to be estimated did not degrade the estimator performance, but substantially more computation is required to estimate four parameters instead of two. Table 10.12 also reveals that a precomputed  $\mathbf{J}^{-1}$  is again useful to employ.

The estimator based upon the "pseudo" likelihood equation (10-85b) exhibited severe numerical problems, due to the near singularity of the computed  $\mathbf{J}$  matrix. When a gradient algorithm was used, however, estimates that agreed well with the solution to the full likelihood equations were achieved.

The explicit suboptimal estimates given by (10-90) and (10-93) performed well when the a priori assumed parameter values (for the state estimation) were close to the true values. However, this performance degraded more significantly than that of the full-scale estimator as the a priori values were made less accurate. Figure 10.22a portrays the full-scale estimator results for a 10-run

TABLE 10.11  
*Estimation of Scalar  $Q_d$  and  $R$*

		Case	Mean parameter value	Mean error	RMS error
Fig. 10.20a	$Q_d$	Full-scale	0.619	0.022	0.143
	$Q_d$	Precomputed $\mathbf{J}^{-1}$	0.619	0.017	0.153
	$R$	Full-scale	0.756	-0.056	0.115
	$R$	Precomputed $\mathbf{J}^{-1}$	0.756	-0.052	0.110
Fig. 10.20b	$Q_d$	Full-scale	1.329	0.010	0.523
	$Q_d$	Precomputed $\mathbf{J}^{-1}$	1.329	0.008	0.339
	$R$	Full-scale	15.599	-1.250	3.535
	$R$	Precomputed $\mathbf{J}^{-1}$	15.599	-1.056	3.186

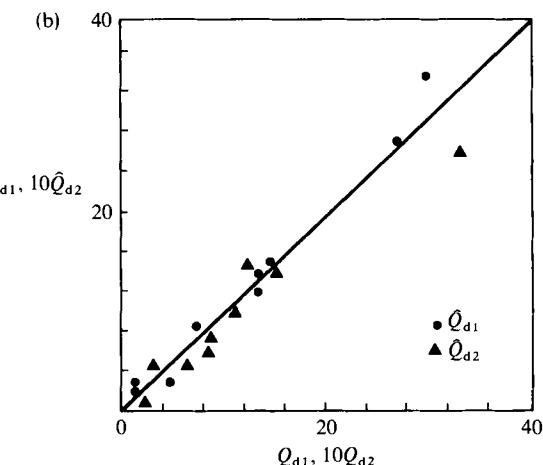
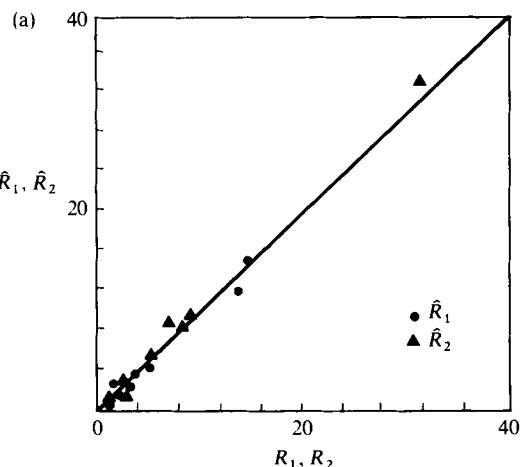


FIG. 10.21 (a) Estimating two diagonal terms of  $\mathbf{R}$ . (b) Estimating two diagonal terms of  $\mathbf{Q}_d$ . From Abramson [1].

TABLE 10.12  
Estimation of 2-by-2 Diagonal  $Q_d$  and  $R$

	Case	Mean parameter value	Mean error	RMS error
$Q_{d1}$	Full-scale	11.198	0.678	2.075
	Precomputed $\mathbf{J}^{-1}$	11.198	0.841	2.216
$Q_{d2}$ <sup>a</sup>	Full-scale	1.896	0.082	0.786
	Precomputed $\mathbf{J}^{-1}$	1.896	0.177	0.776
$R_1$ <sup>a</sup>	Full-scale	9.167	0.191	1.245
	Precomputed $\mathbf{J}^{-1}$	9.167	0.552	1.639
$R_2$	Full-scale	11.388	0.062	1.752
	Precomputed $\mathbf{J}^{-1}$	11.388	0.257	1.753

<sup>a</sup> Correspond to  $Q_d$  and  $R$  of Table 10.11, for the case of Fig. 10.20b.

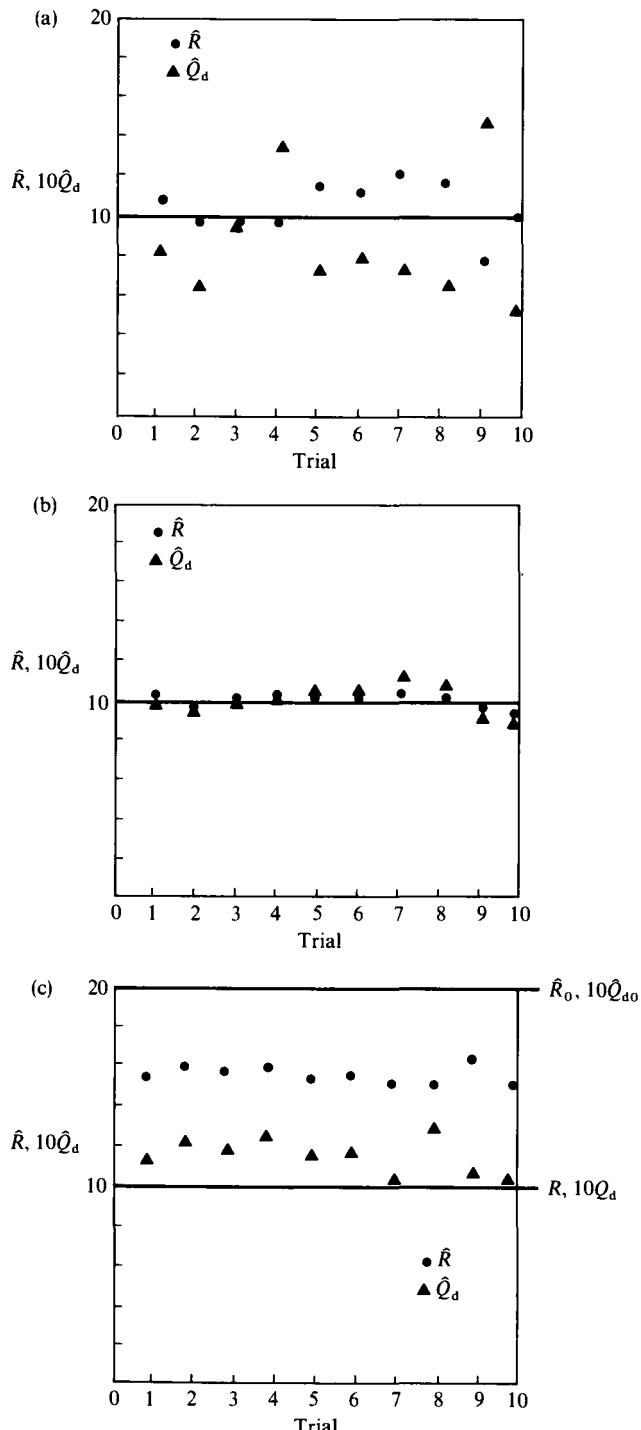


FIG. 10.22 (a) Full-scale estimator. (b) Explicit suboptimal estimator; good a priori parameter values. (c) Explicit suboptimal estimator; poor a priori parameter values. From Abramson [1].

TABLE 10.13

*Full-Scale vs Explicit Suboptimal Estimators*

Case		True parameter value	Mean error	RMS error
Full-scale estimator	$Q_d$	1	-0.140	0.326
	$R$	10	0.443	1.404
Explicit suboptimal good a priori values	$Q_d$	1	0.029	0.077
	$R$	10	0.154	0.373
Explicit suboptimal poor a priori values	$Q_d$	1	0.141	0.160
	$R$	10	5.585	5.601

simulation involving the single input-single output form of this example, letting the true value of  $Q_d$  and  $R$  be 1 and 10, respectively, for all cases. Initial values of  $\hat{Q}_d$  and  $\hat{R}$  were set equal to these values as well. Figure 10.22b is the corresponding result for the explicit suboptimal estimates. When the initial values of  $\hat{Q}_d$  and  $\hat{R}$  are changed to 2 and 20, respectively, the full-scale estimator produces results essentially identical to Fig. 10.22a, whereas the explicit suboptimal estimates are biased significantly towards these erroneous initial conditions, as seen in Fig. 10.22c. Table 10.13 summarizes the 10-run average of these plot values. ■

Another adaptive suboptimal filter can be proposed by recalling that the objective of the adaptation is to improve the state estimation performance. As long as the  $nm$  elements of the adaptively set filter gain  $\mathbf{K}$  are appropriate, the "accuracy" in estimating  $Q_d$  and  $R$ , or the uniqueness of these solutions, is of secondary importance. Again under the assumption of steady state performance over the most recent  $N$  sample times, a unique estimate of  $\mathbf{K}$  and  $\mathbf{A}$  can be obtained even if a unique estimate of  $Q_d$  cannot. The likelihood equations are again written as in (10-21), but ignoring the first term as negligible. If the matrix  $\mathbf{A}$  is chosen as one of the parameters to be estimated, then for  $a_k = \mathbf{A}$ , the third term in (10-21) dominates the second, and an expansion as used in (10-87) can be used to generate an estimate of  $\mathbf{A}$  as

$$\hat{\mathbf{A}}(t_i) = \frac{1}{N} \sum_{j=i-N+1}^i \mathbf{r}_j \mathbf{r}_j^T \quad (10-95)$$

as interpreted physically below (10-88). If uncertain parameters in  $\mathbf{K}$  are to be estimated simultaneously, (10-95) causes the third term in (10-21) to reduce to zero, yielding

$$\sum_{j=i-N+1}^i \frac{\partial \hat{\mathbf{x}}(t_j^-)^T}{\partial a_k} \mathbf{H}^T(t_j) \mathbf{A}(t_j)^{-1} \mathbf{r}_j = 0 \quad (10-96)$$

as the equations to be solved iteratively for  $a_k$  as elements of  $\mathbf{K}$ , along with (10-95). Once  $\hat{\mathbf{A}}$  and  $\hat{\mathbf{K}}$  are achieved, estimates of  $\mathbf{R}$  and  $Q_d$  could be obtained,

if desired, by solving the steady state equations for  $\mathbf{A}$ ,  $\mathbf{K}$ ,  $\mathbf{P}^-$ , and  $\mathbf{P}^+$  (as given by (10-35)–(10-37) and (10-49) for  $\mathbf{Q}_d$  and  $\mathbf{R}$  as [88, 91]

$$\hat{\mathbf{P}}^- = \hat{\mathbf{K}}\hat{\mathbf{A}}(\mathbf{H}^T)^\# \quad (10-97a)$$

$$\hat{\mathbf{P}}^+ = \hat{\mathbf{P}}^- - \hat{\mathbf{K}}\mathbf{H}\hat{\mathbf{P}}^- \quad (10-97b)$$

$$\hat{\mathbf{R}} = \hat{\mathbf{A}} - \mathbf{H}\hat{\mathbf{K}}\hat{\mathbf{A}} \quad (10-97c)$$

$$\hat{\mathbf{Q}}_d = \mathbf{G}_d^\# [\hat{\mathbf{P}}^- - \Phi\hat{\mathbf{P}}^+\Phi^T]\mathbf{G}_d^\# \quad (10-97d)$$

## 10.8 BAYESIAN AND MULTIPLE MODEL FILTERING ALGORITHMS

Let  $\mathbf{a}$  denote the vector of uncertain parameters in a given model, allowing them to affect any or all of  $\Phi$ ,  $\mathbf{B}_d$ ,  $\mathbf{H}$ ,  $\mathbf{Q}_d$ , and  $\mathbf{R}$ . The purpose of Bayesian estimation is to compute the conditional density function:

$$f_{\mathbf{x}(t_i), \mathbf{a}|\mathbf{Z}(t_i)}(\xi, \alpha | \mathbf{Z}_i) = f_{\mathbf{x}(t_i)|\mathbf{a}, \mathbf{Z}(t_i)}(\xi | \alpha, \mathbf{Z}_i) f_{\mathbf{a}|\mathbf{Z}(t_i)}(\alpha | \mathbf{Z}_i) \quad (10-98)$$

Under the assumed model form of (10-1)–(10-5), the first density on the right hand side of this expression would be Gaussian, with mean  $\hat{\mathbf{x}}(t_i^-)$  and covariance  $\mathbf{P}(t_i^-)$  as computed by a Kalman filter, for each given value of the parameter vector  $\mathbf{a}$ .

First assume that  $\mathbf{a}$  can assume any value in a continuous range  $A \subset \mathbb{R}^p$ . Then the second term in (10-98) can be expressed as

$$\begin{aligned} f_{\mathbf{a}|\mathbf{Z}(t_i)}(\alpha | \mathcal{Z}_i) &= f_{\mathbf{a}|\mathbf{z}(t_i), \mathbf{z}(t_{i-1})}(\alpha | \zeta_i, \mathcal{Z}_{i-1}) \\ &= \frac{f_{\mathbf{a}, \mathbf{z}(t_i)|\mathbf{z}(t_{i-1})}(\alpha, \zeta_i | \mathcal{Z}_{i-1})}{f_{\mathbf{z}(t_i)|\mathbf{z}(t_{i-1})}(\zeta_i | \mathcal{Z}_{i-1})} \\ &= \frac{f_{\mathbf{z}(t_i)|\mathbf{a}, \mathbf{z}(t_{i-1})}(\zeta_i | \alpha, \mathcal{Z}_{i-1}) f_{\mathbf{a}|\mathbf{z}(t_{i-1})}(\alpha | \mathcal{Z}_{i-1})}{\int_A f_{\mathbf{z}(t_i)|\mathbf{a}, \mathbf{z}(t_{i-1})}(\zeta_i | \alpha, \mathcal{Z}_{i-1}) f_{\mathbf{a}|\mathbf{z}(t_{i-1})}(\alpha | \mathcal{Z}_{i-1}) d\alpha} \end{aligned} \quad (10-99)$$

Since  $f_{\mathbf{z}(t_i)|\mathbf{a}, \mathbf{z}(t_{i-1})}(\zeta_i | \alpha, \mathcal{Z}_{i-1})$  is Gaussian, with mean  $\mathbf{H}(t_i)\hat{\mathbf{x}}(t_i^-)$  and covariance  $[\mathbf{H}(t_i)\mathbf{P}(t_i^-)\mathbf{H}^T(t_i) + \mathbf{R}(t_i)]$ , for each value of the parameter vector, (10-99) could conceptually be solved recursively, starting from an a priori density  $f_a(\alpha)$ . A state estimate would then be generated as the conditional mean

$$\begin{aligned} E\{\mathbf{x}(t_i) | \mathbf{Z}(t_i) = \mathbf{Z}_i\} &= \int_{-\infty}^{\infty} \xi f_{\mathbf{x}(t_i)|\mathbf{Z}(t_i)}(\xi | \mathbf{Z}_i) d\xi \\ &= \int_{-\infty}^{\infty} \xi \left[ \int_A f_{\mathbf{x}(t_i), \mathbf{a}|\mathbf{Z}(t_i)}(\xi, \alpha | \mathbf{Z}_i) d\alpha \right] d\xi \\ &= \int_{-\infty}^{\infty} \xi \left[ \int_A f_{\mathbf{x}(t_i)|\mathbf{a}, \mathbf{Z}(t_i)}(\xi | \alpha, \mathbf{Z}_i) f_{\mathbf{a}|\mathbf{Z}(t_i)}(\alpha | \mathbf{Z}_i) d\alpha \right] d\xi \\ &= \int_A \left[ \int_{-\infty}^{\infty} \xi f_{\mathbf{x}(t_i)|\mathbf{a}, \mathbf{Z}(t_i)}(\xi | \alpha, \mathbf{Z}_i) d\xi \right] f_{\mathbf{a}|\mathbf{Z}(t_i)}(\alpha | \mathbf{Z}_i) d\alpha \end{aligned} \quad (10-100)$$

which is obtained by use of marginal densities, Bayes' rule, and interchange of order of integration. Note that the expression in brackets on the last line of (10-100) is  $\hat{\mathbf{x}}(t_i^+)$  as produced by a Kalman filter for a particular parameter vector value. Unfortunately, the integrations involved in (10-99) and (10-100) make this estimate computationally infeasible for online usage.

To enhance feasibility, let the parameter vector instead assume only a finite number of values. This finite set might be the result of discretizing a continuous parameter space: selecting a set of values  $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_K\}$  that are dispersed throughout the region of reasonable parameter vector values. Or, a problem of interest might naturally be described by discrete parameter values, as for sensor failure detection in which each  $\mathbf{a}_k$  would correspond to a particular configuration of some sensors failed and the others operational.

Assuming that the parameter vector in fact assumes one of the values  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_K$ , one can seek an algorithm to produce the true conditional mean and covariance of the state simultaneously with identification of the "true" parameter value [80]. However, if a continuous parameter space has been discretized, the "true" parameter value will not be identical to one of the  $\mathbf{a}_k$ 's, but "near" to one [8, 22, 23, 97]. With a sufficiently fine discretization, the approximate solution afforded by such an algorithm will often provide adequate performance. Optimum discretization of a continuous parameter space is a subject of current research.

Conceptually, associated with each  $\mathbf{a}_k$  is a different system model of the form given by (10-1)–(10-5). Thus,  $\mathbf{a}$  is considered a discrete random variable, and each realization  $\mathbf{a}_k$  corresponds to a particular model being selected by nature as the best representation of a given system. For Bayesian estimation, an a priori density function must be specified for  $\mathbf{a}$ . Letting  $p_k(t_0)$  be the probability that  $\mathbf{a}$  assumes the value  $\mathbf{a}_k$  at time  $t_0$ , this density is

$$f_{\mathbf{a}}(\boldsymbol{\alpha}) = \sum_{k=1}^K p_k(t_0) \delta(\boldsymbol{\alpha} - \mathbf{a}_k) \quad (10-101)$$

Note that the  $p_k(t_0)$  values must be such that

$$p_k(t_0) \geq 0 \quad \text{for all } k \quad (10-102a)$$

$$\sum_{k=1}^K p_k(t_0) = 1 \quad (10-102b)$$

and that their values reflect one's best guess about which particular models are most likely to be correct. For example, if all models are equally likely, then  $p_k(t_0) = 1/K$  for all  $k$ .

Now define the hypothesis conditional probability  $p_k(t_i)$  as

$$p_k(t_i) \triangleq \text{prob}\{\mathbf{a} = \mathbf{a}_k | \mathbf{Z}(t_i) = \mathbf{Z}_i\} \quad (10-103)$$

which also satisfies relations analogous to (10-102). It is desired to obtain recursive relations for both  $p_k$  (for  $k = 1, 2, \dots, K$ ) and the conditional mean and covariance of the state, given the measurement history. In a development totally analogous to (10-99) and (10-100), we can write, for  $k = 1, 2, \dots, K$ ,

$$p_k(t_i) = \frac{f_{\mathbf{z}(t_i)|\mathbf{a}, \mathbf{z}(t_{i-1})}(\mathbf{z}_i | \mathbf{a}_k, \mathbf{Z}_{i-1}) p_k(t_{i-1})}{\sum_{j=1}^K f_{\mathbf{z}(t_i)|\mathbf{a}, \mathbf{z}(t_{i-1})}(\mathbf{z}_i | \mathbf{a}_j, \mathbf{Z}_{i-1}) p_j(t_{i-1})} \quad (10-104)$$

$$\begin{aligned} \hat{\mathbf{x}}(t_i^+) &= E\{\mathbf{x}(t_i) | \mathbf{Z}(t_i) = \mathbf{Z}_i\} \\ &= \int_{-\infty}^{\infty} \xi \left[ \sum_{k=1}^K f_{\mathbf{x}(t_i)|\mathbf{a}, \mathbf{z}(t_i)}(\xi | \mathbf{a}_k, \mathbf{Z}_i) p_k(t_i) \right] d\xi \\ &= \sum_{k=1}^K \hat{\mathbf{x}}_k(t_i^+) p_k(t_i) \end{aligned} \quad (10-105)$$

where  $\hat{\mathbf{x}}_k(t_i^+)$  is the state estimate produced by a Kalman filter based on the assumption that the parameter vector equals  $\mathbf{a}_k$ . Thus, the overall state estimate is the probabilistically weighted average of the state estimates generated by each of  $K$  separate Kalman filters, using the hypothesis conditional probabilities  $p_k(t_i)$  as the appropriate weighting factors. The conditional covariance of  $\mathbf{x}(t_i)$  is

$$\begin{aligned} \mathbf{P}(t_i^+) &= E\{[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^+)][\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^+)]^T | \mathbf{Z}(t_i) = \mathbf{Z}_i\} \\ &= \int_{-\infty}^{\infty} [\xi - \hat{\mathbf{x}}(t_i^+)] [\xi - \hat{\mathbf{x}}(t_i^+)]^T f_{\mathbf{x}(t_i)|\mathbf{z}(t_i)}(\xi | \mathbf{Z}_i) d\xi \\ &= \sum_{k=1}^K p_k(t_i) \left\{ \int_{-\infty}^{\infty} [\xi - \hat{\mathbf{x}}(t_i^+)] [\xi - \hat{\mathbf{x}}(t_i^+)]^T f_{\mathbf{x}(t_i)|\mathbf{a}, \mathbf{z}(t_i)}(\xi | \mathbf{a}_k, \mathbf{Z}_i) d\xi \right\} \\ &= \sum_{k=1}^K p_k(t_i) \{ \mathbf{P}_k(t_i^+) + [\hat{\mathbf{x}}_k(t_i^+) - \hat{\mathbf{x}}(t_i^+)] [\hat{\mathbf{x}}_k(t_i^+) - \hat{\mathbf{x}}(t_i^+)]^T \} \end{aligned} \quad (10-106)$$

where  $\mathbf{P}_k(t_i^+)$  is the state error covariance computed by the Kalman filter based upon  $\mathbf{a}_k$ . From (10-106) it can be seen that  $\mathbf{P}(t_i^+)$  cannot be precomputed, since it involves  $p_k(t_i)$ ,  $\hat{\mathbf{x}}_k(t_i^+)$ , and  $\hat{\mathbf{x}}(t_i^+)$ , all of which require knowledge of the measurement history.

The adaptive filter that results from this development is known as the *multiple model filtering algorithm* [7–9, 11, 12, 14, 15, 22–24, 30, 50, 53, 69, 73, 80, 87, 97, 98, 125, 131, 146], and its structure is depicted in Figure 10.23. It is composed of a bank of  $K$  separate Kalman filters, each based on a particular value  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_K$  of the parameter vector. When the measurement  $\mathbf{z}_i$  becomes available at sample time  $t_i$ , the residuals  $\mathbf{r}_1(t_i), \mathbf{r}_2(t_i), \dots, \mathbf{r}_K(t_i)$  are generated in the  $K$  filters, and passed on for processing by the hypothesis conditional probability computation, an implementation of (10-104). Specifically, each

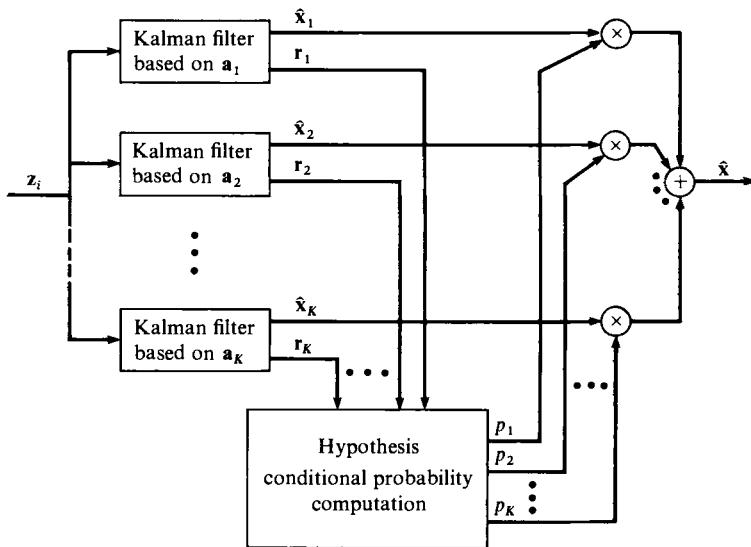


FIG. 10.23 Multiple model filtering algorithm.

$f_{\mathbf{z}(t_i) | \mathbf{a}, \mathbf{z}(t_{i-1})}(\mathbf{z}_i | \mathbf{a}_k, \mathbf{Z}_{i-1})$  can be evaluated as

$$f_{\mathbf{z}(t_i) | \mathbf{a}, \mathbf{z}(t_{i-1})}(\mathbf{z}_i | \mathbf{a}_k, \mathbf{Z}_{i-1}) = \frac{1}{(2\pi)^{m/2} |\mathbf{A}_k(t_i)|^{1/2}} \exp\{-\cdot\}$$

$$\{\cdot\} = \left\{ -\frac{1}{2} \mathbf{r}_k^T(t_i) \mathbf{A}_k^{-1}(t_i) \mathbf{r}_k(t_i) \right\} \quad (10-107)$$

where  $\mathbf{A}_k(t_i)$  is generated in the  $k$ th Kalman filter as

$$\mathbf{A}_k(t_i) = \mathbf{H}_k(t_i) \mathbf{P}_k(t_i^-) \mathbf{H}_k^T(t_i) + \mathbf{R}_k(t_i) \quad (10-108)$$

These  $K$  evaluations, along with memory of the previous  $p_k(t_{i-1})$  values, allow computation of the current hypothesis conditional probabilities  $p_1(t_i), p_2(t_i), \dots, p_K(t_i)$  according to (10-104). The  $p_k(t_i)$  values, in turn, are used as weighting coefficients to generate  $\hat{\mathbf{x}}(t_i^+)$  according to (10-105).

If it is desired to produce an estimate of the parameter vector itself, the conditional mean of  $\mathbf{a}$  at time  $t_i$  is

$$\begin{aligned} \hat{\mathbf{a}}(t_i) &\triangleq E\{\mathbf{a}(t_i) | \mathbf{Z}(t_i) = \mathbf{Z}_i\} = \int_{-\infty}^{\infty} \alpha f_{\mathbf{a} | \mathbf{Z}(t_i)}(\alpha | \mathbf{Z}_i) d\alpha \\ &= \int_{-\infty}^{\infty} \alpha \left[ \sum_{k=1}^K p_k(t_i) \delta(\alpha - \mathbf{a}_k) \right] d\alpha \\ &= \sum_{k=1}^K \mathbf{a}_k p_k(t_i) \end{aligned} \quad (10-109)$$

An indication of the precision of this estimate would be given by the conditional covariance of  $\mathbf{a}(t_i)$ ,

$$E\{\mathbf{a} - \hat{\mathbf{a}}(t_i)\}[\mathbf{a} - \hat{\mathbf{a}}(t_i)]^T | \mathbf{Z}(t_i) = \mathbf{Z}_i = \sum_{k=1}^K [\mathbf{a}_k - \hat{\mathbf{a}}(t_i)][\mathbf{a}_k - \hat{\mathbf{a}}(t_i)]^T p_k(t_i) \quad (10-110)$$

Note that neither these calculations nor (10-106) need to be processed to obtain  $\hat{\mathbf{x}}(t_i^+)$  however.

Heuristically, one would expect that the residuals of the Kalman filter based upon the “correct” model will be consistently smaller than the residuals of the other mismatched filters. If this is true, then Eqs. (10-104) and (10-107) will cause the “correct” probability  $p_k(t_i)$ , i.e., the one whose index is associated with the “correct” filter model, to increase, while causing the others to decrease. The performance of this algorithm is dependent upon a significant difference between the residual characteristics in the “correct” and the “mismatched model” filters. In fact, if the residuals instead are consistently of the same magnitude, then Eqs. (10-104) and (10-107) result in the growth of the  $p_k$  associated with the filter with the smallest value of  $|\mathbf{A}_k|$ . The  $|\mathbf{A}_k|$  values are independent not only of the residuals, but also of the “correctness” of the  $K$  models, and so such a result would be totally erroneous. It is therefore important not to add too much dynamics pseudonoise during tuning, since this tends to mask differences between good and bad models. Unfortunately, no rigorous general proofs are available concerning the asymptotic properties of the hypothesis conditional probabilities. Partial results do indicate convergence of  $p_k$  to unity for the filter based on the true system model, or highest  $p_k$  being associated with the “closest to true” system model in the discretized parameter case [8, 22, 23, 97].

**EXAMPLE 10.19** Consider the single axis motion of a vehicle affected by control force  $u$  and drag  $a$ . Letting  $x_1$  denote vehicle position, and  $x_2$  velocity, a continuous-time model of its motion is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Let a radar be located at the origin, and take measurements every  $\Delta t = 0.1$  sec:

$$\mathbf{z}(t_i) = [1 \ 0] \mathbf{x}(t_i) + \mathbf{v}(t_i)$$

with  $R(t_i) \equiv 1$ . The equivalent discrete-time model of the system is

$$\begin{bmatrix} x_1(t_{i+1}) \\ x_2(t_{i+1}) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{a}(1 - e^{-a\Delta t}) \\ 0 & e^{-a\Delta t} \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{a}(1 - e^{-a\Delta t}) \end{bmatrix} u(t_i) + \begin{bmatrix} w_{d1}(t_i) \\ w_{d2}(t_i) \end{bmatrix}$$

where pseudonoise  $\mathbf{w}_d(\cdot, \cdot)$  of strength  $\mathbf{Q}_d$  is added to reflect uncertainty in the model. The initial states are  $x_1(t_0) = 100$  and  $x_2(t_0) = 50$ . A multiple model filter is to be used to estimate  $x_1$  and  $x_2$ , and to identify  $a$ , where  $a$  is assumed to take on only values  $a_1 = 0$ ,  $a_2 = 0.5$ , or  $a_3 = 1$ . Note that, for those values, the discrete-time model elements are as shown in the following tabulation:

Parameter value	$\Phi_{d12}, B_{d2}$	$\Phi_{d22}$
$a_1 = 0$	0.1000	1.000
$a_2 = 0.5$	0.0975	0.951
$a_3 = 1$	0.0950	0.900

Three cases were run, holding the parameter  $a$  constant in the “true” system, equal to one of the three possible values in each case. For these cases, the filters were initialized with  $\hat{x}(t_0)$  set to the correct  $x(t_0)$  values,  $P_0 = I$ ,  $u \equiv 0$ ,  $Q_d = 0$ , and the initial hypothesis probabilities uniformly distributed:  $p_k(t_0) = \frac{1}{3}$  for  $k = 1, 2, 3$ . Figure 10.24 presents the time histories of the hypothesis conditional probabilities for these cases: the “true” system is always identified in 10 sample periods or less. This illustrates the ability of the algorithm to identify constant parameters with rapid, well-behaved convergence. ■

EXAMPLE 10.20 Consider the same application as in the preceding example, but let the true parameter value undergo jump changes such that it equals  $a_1 = 0$  for  $t \in [0, 2)$ ,  $a_2 = 0.5$  for

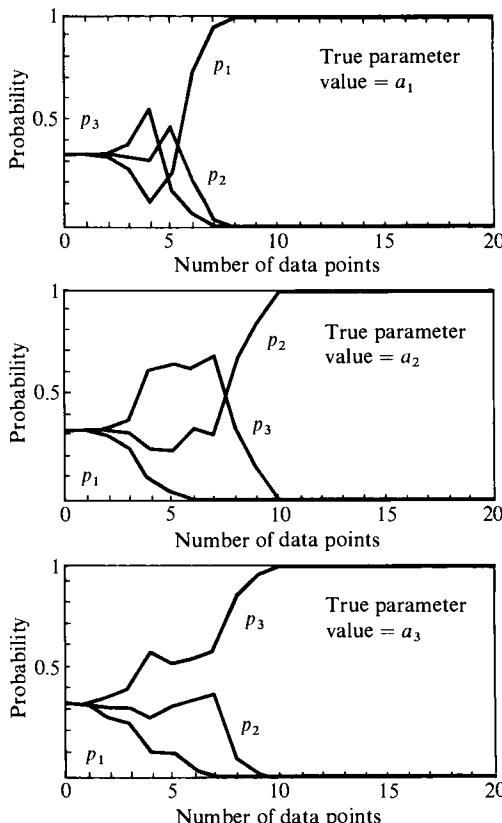


FIG. 10.24 Hypothesis conditional probabilities for Example 10.19. From Athans and Chang [8].

$t \in [2, 4]$ , and  $a_3 = 1$  for  $t \in [4, 6]$ . Further assume that the control force  $u$  is equal to 50 for all time, and known to the estimator.

To allow the estimator to adapt to the changing parameter value, two ad hoc compensations are introduced. First, the hypothesis conditional probabilities are artificially bounded below by a small number (0.0005) to prevent any of them from converging to zero, which would make it very difficult for them to change significantly in response to a subsequent change in true parameter value. Second, pseudonoise of appropriate strength ( $\mathbf{Q}_d = \mathbf{I}$ ) is added to the filter models. With no such noise addition, once the true parameter is identified, the algorithm can become locked to a single filter output, and mismatched filter estimates can drift significantly from true state values. When the parameter value changes and one of these "mismatched" filters becomes the "correct" one, a very long transient is required to achieve proper identification. The pseudonoise addition causes each Kalman filter to generate state estimates sufficiently close to the true states to allow adaptation to parameter changes.

Figure 10.25 portrays the parameter estimate provided by the algorithm as modified above. It demonstrates the parameter tracking ability provided by the ad hoc modifications. The control input  $u$  is also of significance here, in that it provides a persistent excitation of the true system to enhance identifiability. ■

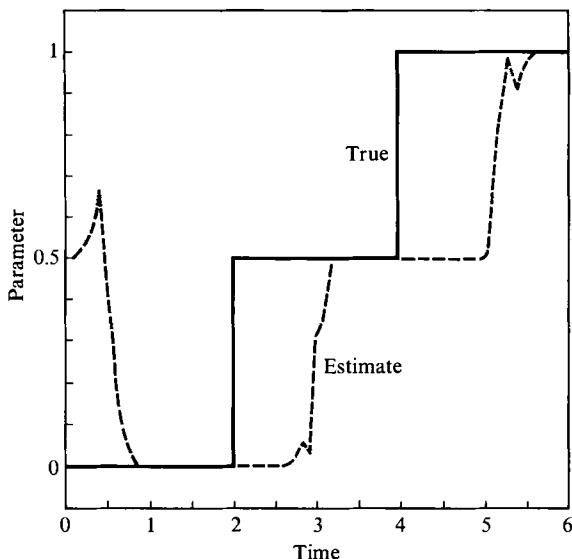


FIG. 10.25 Parameter estimates for Example 10.20. From Athans and Chang [8].

The preceding example treated time-varying parameters by adding pseudonoise to the system model and artificially bounding probability descriptions, two compensation techniques discussed in Chapter 9. A "constant-over- $N$ -steps" model of parameters could also be proposed, but an efficient algorithm based upon such a model has yet to be generated. As an alternative, one could explicitly model possible parameter variations. In this case, the optimum state estimate is the weighted sum of estimates generated by filters matched to all

possible parameter *histories* [8, 22, 23]. If the parameter vector can assume any of  $K$  values at each sample time, then  $K^t$  elemental estimators are required at time  $t$ : such exponential growth in memory requirements is impractical. However, if the parameter vector is additionally assumed to be Markov, i.e., the present value depends only on the previous single value, then the number of elemental estimators remains at  $K^2$ . This is certainly more feasible, but for reasonable levels of discretization,  $K^2$  may well be prohibitive compared to  $K$ , and thus the estimator with ad hoc modifications as in Example 10.20 might well be the preferable means of tracking time-varying parameters. This is especially true since, as stated before, parameters are expected to vary significantly more slowly than states, and not exhibit consistently dynamic behavior from one sample time to the next. Another ad hoc modification for tracking slowly varying parameters, allowing for both a large range of possible values and sufficiently fine discretization, is to incorporate a dynamically redefinable (or “moving”) bank of filters instead of a statically fixed set of bank members.

The previous example also indicated the usefulness of certain forms of inputs to aid the identifiability of systems. The subject of *optimum design of inputs* [4, 40, 45, 78, 79, 93, 94, 104, 130, 136, 142, 148] for system identification and other objectives will be pursued in subsequent chapters on controller synthesis (in Volume 3).

## 10.9 CORRELATION METHODS FOR SELF-TUNING: RESIDUAL “WHITENING”

Correlation methods have been used classically for estimation in time series analysis, deriving equations to relate system parameters to an observed autocorrelation function and then solving these to obtain parameter estimates [17, 60, 88, 90, 91]. Such techniques are most applicable to time-invariant system descriptions and stationary noises. Time-varying, or constant-over- $N$ -step extensions are possible, but this section will assume system time invariance and noise stationarity.

Although output autocorrelations have been exploited to some degree, estimates based upon autocorrelation of a filter’s residuals are more efficient because the residual sequence is less correlated than the output sequence. As shown previously, residuals form a white Gaussian sequence in a “truly optimal” filter, one based upon a complete and perfectly tuned model. However, a sub-optimal or mistuned filter will exhibit a time-correlated residual sequence. Thus, an adaptive technique might be developed to adjust the uncertain parameters in  $\mathbf{Q}_d$  and  $\mathbf{R}$  so as to “whiten” the residuals of the state estimator.

One such algorithm [88, 91] performs a correlation test on observed residuals to determine statistically whether adaptation is required. If so, and the number of unknown elements in  $\mathbf{Q}_d$  is less than  $nm$  (the number of states times the number of measurements), then asymptotically Gaussian, unbiased, and con-

sistent estimates of  $\mathbf{Q}_d$  and  $\mathbf{R}$  are generated. If there are more than  $nm$  parameters in  $\mathbf{Q}_d$ , the steady state gain of the filter is estimated without an explicit estimate of  $\mathbf{Q}_d$ .

The basic concepts and algorithm will now be developed. Assuming that a Kalman filter embodies a constant gain  $\mathbf{K}_s$  not necessarily equal to the optimal steady state gain, an expression can be developed [88] for the autocorrelation of the stationary Gaussian residual process  $\mathbf{r}(\cdot, \cdot)$  by writing

$$\begin{aligned}\mathbf{r}(t_i) &= \mathbf{z}(t_i) - \mathbf{H}\hat{\mathbf{x}}(t_i^-) = \mathbf{H}[\mathbf{x}(t_i) - \hat{\mathbf{x}}(t_i^-)] + \mathbf{v}(t_i) \\ &= \mathbf{H}\mathbf{e}(t_i^-) + \mathbf{v}(t_i)\end{aligned}\quad (10-111)$$

Thus, the desired autocorrelation becomes

$$\begin{aligned}\mathbf{A}_k &\triangleq E\{\mathbf{r}(t_i)\mathbf{r}^T(t_{i-k})\} \\ &= \begin{cases} \mathbf{H}E\{\mathbf{e}(t_i^-)\mathbf{e}^T(t_{i-k})\}\mathbf{H}^T + \mathbf{H}E\{\mathbf{e}(t_i)\mathbf{v}^T(t_{i-k})\} & k > 0 \\ \mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R} & k = 0 \end{cases} \quad (10-112a) \\ &\quad (10-112b)\end{aligned}$$

where  $\mathbf{P}^-$  is the steady state error covariance matrix found as the solution to [83, Problem 6.7]

$$\mathbf{P}^- = \Phi[(\mathbf{I} - \mathbf{K}_s\mathbf{H})\mathbf{P}^-(\mathbf{I} - \mathbf{K}_s\mathbf{H})^T + \mathbf{K}_s\mathbf{R}\mathbf{K}_s^T]\Phi^T + \mathbf{G}_d\mathbf{Q}_d\mathbf{G}_d^T \quad (10-113)$$

Note the use of suboptimal  $\mathbf{K}_s$ , but optimal  $\mathbf{Q}_d$  and  $\mathbf{R}$  in the two preceding expressions. Equation (10-112a) can be expanded by noting that  $\mathbf{e}(t_i^-)$  satisfies

$$\mathbf{e}(t_i^-) = \Phi(\mathbf{I} - \mathbf{K}_s\mathbf{H})\mathbf{e}(t_{i-1}^-) - \Phi\mathbf{K}_s\mathbf{v}(t_{i-1}) + \mathbf{G}_d\mathbf{w}_d(t_{i-1}) \quad (10-114)$$

By iterating this expression  $k$  times, it can be shown that

$$\begin{aligned}E\{\mathbf{e}(t_i^-)\mathbf{e}^T(t_{i-k})\} &= [\Phi(\mathbf{I} - \mathbf{K}_s\mathbf{H})]^k\mathbf{P}^- \\ E\{\mathbf{e}(t_i^-)\mathbf{v}^T(t_{i-k})\} &= -[\Phi(\mathbf{I} - \mathbf{K}_s\mathbf{H})]^{k-1}\Phi\mathbf{K}_s\mathbf{R}\end{aligned}$$

which, when substituted into (10-112a), yields

$$\mathbf{A}_k = \begin{cases} \mathbf{H}[\Phi(\mathbf{I} - \mathbf{K}_s\mathbf{H})]^{k-1}\Phi[\mathbf{P}^-\mathbf{H}^T - \mathbf{K}_s\mathbf{A}_0] & k > 0 \\ \mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R} & k = 0 \end{cases} \quad (10-115a) \quad (10-115b)$$

Equations (10-115) and (10-113) describe the desired residual autocorrelation completely, since  $\mathbf{A}_{-k} = \mathbf{A}_k$ . Note that for an optimal filter in which  $\mathbf{K}_s$  is set equal to  $\{\mathbf{P}^-\mathbf{H}^T(\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{R})^{-1}\}$ ,  $\mathbf{A}_k \equiv 0$  for  $k \neq 0$ .

In online usage, the value of  $\mathbf{P}^-$  is unknown, since optimal  $\mathbf{Q}_d$  and  $\mathbf{R}$  values are needed for its evaluation. Basically, (10-115a) can be written for  $k = 1, 2, \dots, n$  in order to solve for  $\mathbf{P}^-\mathbf{H}^T$ , in terms of the computed sample autocorrelations (based on the same ergodic assumptions as (10-95))

$$\hat{\mathbf{A}}_k = \frac{1}{N} \sum_{i=k+1}^N \mathbf{r}_i \mathbf{r}_{i-k}^T \quad (10-116)$$

Although this is a biased estimate of  $\mathbf{A}_k$  (dividing by  $N - k - 1$  instead of  $N$  removes the bias), it is preferable since it yields less mean squared error than the corresponding unbiased estimate [88]. Once  $\widehat{\mathbf{P}^{-}\mathbf{H}^T}$  is obtained, this can be used to obtain asymptotically unbiased and consistent estimates of  $\mathbf{R}$  and  $\mathbf{K}$ , and of  $\mathbf{Q}_d$  if desired as well. Setting up (10-115a) for  $k = 1, 2, \dots, n$  and rearranging yields

$$\widehat{\mathbf{P}^{-}\mathbf{H}^T} = (\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T \begin{bmatrix} \widehat{\mathbf{A}}_1 + \mathbf{H}\Phi\mathbf{K}_s\widehat{\mathbf{A}}_0 \\ \widehat{\mathbf{A}}_2 + \mathbf{H}\Phi\mathbf{K}_s\widehat{\mathbf{A}}_1 + \mathbf{H}\Phi^2\mathbf{K}_s\widehat{\mathbf{A}}_0 \\ \vdots \\ \widehat{\mathbf{A}}_n + \mathbf{H}\Phi\mathbf{K}_s\widehat{\mathbf{A}}_{n-1} + \cdots + \mathbf{H}\Phi^n\mathbf{K}_s\widehat{\mathbf{A}}_0 \end{bmatrix} \quad (10-117)$$

where  $\mathbf{M}$  is the  $nm$ -by- $n$  product of the observability matrix and the nonsingular transition matrix  $\Phi$ :

$$\mathbf{M} \triangleq \begin{bmatrix} \mathbf{H} \\ \mathbf{H}\Phi \\ \vdots \\ \mathbf{H}\Phi^{n-1} \end{bmatrix} \Phi \quad (10-118)$$

Since  $\mathbf{M}$  is of rank  $n$ ,  $[(\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T]$  in (10-117) is an appropriate choice of pseudoinverse of  $\mathbf{M}$ . Now  $\mathbf{R}$  can be estimated as

$$\widehat{\mathbf{R}} = \widehat{\mathbf{A}}_0 - \widehat{\mathbf{H}\mathbf{P}^{-}\mathbf{H}^T} \quad (10-119)$$

A better filter gain than  $\mathbf{K}_s$  is evaluated by manipulation of  $\mathbf{P}^-$ . Equation (10-113) expresses the true error covariance  $\mathbf{P}^-$  associated with the suboptimal filter, whereas  $\mathbf{P}_*^-$  associated with an optimally tuned filter would satisfy

$$\mathbf{P}_*^- = \Phi(\mathbf{P}_*^- - \mathbf{K}_*\mathbf{H}\mathbf{P}_*^-)\Phi^T + \mathbf{G}_d\mathbf{Q}_d\mathbf{G}_d^T \quad (10-120)$$

Differencing (10-113) and (10-120) yields an equation for  $[\mathbf{P}_*^- - \mathbf{P}^-] \triangleq \delta\mathbf{P}^-$ :

$$\begin{aligned} \delta\mathbf{P}^- &= \Phi[\delta\mathbf{P}^- - (\mathbf{P}^-\mathbf{H}^T + \delta\mathbf{P}^-\mathbf{H}^T)(\mathbf{A}_0 + \mathbf{H}\delta\mathbf{P}^-\mathbf{H}^T)^{-1}(\mathbf{H}\mathbf{P}^- + \mathbf{H}\delta\mathbf{P}^-) \\ &\quad + \mathbf{K}_*\mathbf{H}\mathbf{P}^- + \mathbf{P}^-\mathbf{H}^T\mathbf{K}_s^T - \mathbf{K}_s\mathbf{A}_0\mathbf{K}_s^T]\Phi^T \end{aligned} \quad (10-121)$$

This is solved for  $\delta\mathbf{P}^-$  using the computed  $\widehat{\mathbf{A}}_0$  and  $\widehat{\mathbf{P}^{-}\mathbf{H}^T}$  from (10-116) and (10-117), yielding  $\widehat{\delta\mathbf{P}^-}$ . Finally, an estimate of the optimal gain is produced as

$$\begin{aligned} \widehat{\mathbf{K}}_* &\triangleq \widehat{\mathbf{P}_*^-\mathbf{H}^T[\mathbf{H}\mathbf{P}_*^-\mathbf{H}^T + \mathbf{R}]^{-1}} \\ &\triangleq \widehat{[(\mathbf{P}^- + \delta\mathbf{P}^-)\mathbf{H}^T][\mathbf{H}\mathbf{P}^-\mathbf{H}^T + \mathbf{H}\delta\mathbf{P}^-\mathbf{H}^T + \mathbf{R}]^{-1}} \\ &\simeq [\widehat{\mathbf{P}^-\mathbf{H}^T} + \widehat{\delta\mathbf{P}^-\mathbf{H}^T}][\widehat{\mathbf{A}}_0 + \widehat{\mathbf{H}\delta\mathbf{P}^-\mathbf{H}^T}]^{-1} \end{aligned} \quad (10-122)$$

For non-real-time applications, local iterations of (10-116)–(10-122) on the same set of  $N$  measurements  $\{\mathbf{z}_i\}$  could be used to improve these estimates.

With each iteration, the residual sequence would become increasingly more white, with resulting better estimates. Two proposed methods of estimating  $\mathbf{G}_d \mathbf{Q} \mathbf{G}_d^T$ , if desired, are to use (10-120) directly with  $\hat{\mathbf{K}}_*$  given by (10-122) and  $\mathbf{P}_*$  given either by  $[\mathbf{P}^- + \delta \mathbf{P}]$  using (10-113) and (10-121), or by

$$\widehat{\mathbf{P}_*} = \hat{\mathbf{K}}_* \hat{\mathbf{A}}_0 (\mathbf{H}^T)^* = \hat{\mathbf{K}}_* \hat{\mathbf{A}}_0 (\mathbf{H} \mathbf{H}^T)^{-1} \mathbf{H} \quad (10-123)$$

which is obtained by solving  $\mathbf{K}_* = \mathbf{P}_*^- \mathbf{H}^T \mathbf{A}_0^{-1}$  for  $\mathbf{P}_*^-$ , as in (10-97) [88].

**EXAMPLE 10.21** The above correlation technique is applied to a position- and velocity-aided inertial navigation problem in one direction. The inertial system errors are modeled by a damped Schuler loop (position and velocity error states) forced by an exponentially time-correlated stationary input to depict gyro drift. Both measurements are corrupted by exponentially time-correlated as well as white noises.

The augmented system description therefore has five states, and the equivalent discrete-time model for an iteration period of 0.1 sec is:

$$\begin{aligned} \mathbf{x}(t_{i+1}) &= \begin{bmatrix} 0.75 & -1.74 & -0.3 & 0 & -0.15 \\ 0.09 & 0.91 & -0.0015 & 0 & -0.008 \\ 0 & 0 & 0.95 & 0 & 0 \\ 0 & 0 & 0 & 0.55 & 0 \\ 0 & 0 & 0 & 0 & 0.905 \end{bmatrix} \mathbf{x}(t_i) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 24.6 & 0 & 0 \\ 0 & 0.835 & 0 \\ 0 & 0 & 1.83 \end{bmatrix} \mathbf{w}_d(t_i) \\ \mathbf{z}(t_i) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t_i) + \mathbf{v}(t_i) \end{aligned}$$

with

$$\mathbf{Q}_d = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

The actual values of  $q_1$ ,  $q_2$ ,  $q_3$ ,  $r_1$ , and  $r_2$  are unity, but they are assumed unknown, with the best a priori guesses as  $\hat{q}_1 = 0.25$ ,  $\hat{q}_2 = 0.5$ ,  $\hat{q}_3 = 0.75$ ,  $\hat{r}_1 = 0.4$ , and  $\hat{r}_2 = 0.6$ . It is desired to estimate these values adaptively based on measurements  $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_N$ , where for this case  $N$  is set at a large value, 950.

The nonwhiteness of the residuals of the mistuned filter can be demonstrated in the following manner. It can be shown that the diagonal term  $[\hat{\mathbf{A}}_k]_{ii}$  in (10-116) is an asymptotically Gaussian estimator of  $[\mathbf{A}_k]_{ii}$  with variance converging to  $[\hat{\mathbf{A}}_0]_{ii}^2/N$ . Thus, the 95%-confidence limits on  $[\hat{\mathbf{A}}_k]_{ii}$  are  $\pm(1.96/N^{1/2})[\hat{\mathbf{A}}_0]_{ii}$ . One can look at a set of values  $[\hat{\mathbf{A}}_k]_{ii}$  for  $k > 0$  and check the number of times they lie outside this confidence band. If this number is less than 5% of the total, the residual sequence can be considered white. In view of (10-112), note that  $A_k = 0$  for all  $k > 0$  for a true white noise and that one would desire  $A_k$  magnitudes to lie within an envelope that decays (perhaps exponentially) with  $k$  in order to call the noise "essentially white"; however, since autocorrelation estimates for larger  $k$  are based on fewer data points ( $N - k$ ), there is more uncertainty in these estimated values, and so comparison to a fixed confidence band for all  $k$  is reasonable. (Other statistical tests are available [17, 38, 60, 96, 140], some being more appropriate for small  $N$ .) For this case, the test was applied to  $[\hat{\mathbf{A}}_k]_{ii}$  for  $k = 1, 2, \dots, 40$ , and four evaluations (10% of the total) exceeded the confidence limits, and so the hypothesis that the residuals are white can be rejected.

The results of estimating the uncertain parameters repeatedly on the same batch of data are shown in Table 10.14. It is seen that most of the adaptation is accomplished during the first iteration.

TABLE 10.14

*Estimates of  $\mathbf{Q}_d$  and  $\mathbf{R}$  Based on 950 Measurements*

Number of iterations	$\hat{q}_1$	$\hat{q}_2$	$\hat{q}_3$	$\hat{r}_1$	$\hat{r}_2$	Likelihood function $L(\hat{\mathbf{Q}}_d, \hat{\mathbf{R}}, \mathbf{Z})$	Percentage of $\hat{A}_k$ values outside 95%-confidence band	
							Measurement 1	Measurement 2
0	0.25	0.5	0.75	0.4	0.6	-7.008	10	10
1	0.73	1.31	0.867	1.444	0.776	-6.514	2.5	5
2	0.87	1.39	0.797	1.537	0.767	-6.511	2.5	5
3	0.91	1.40	0.776	1.565	0.765	-6.510	2.5	5
4	0.92	1.41	0.770	1.573	0.765	-6.509	2.5	5
Check case	1.0	1.0	1.0	1.0	1.0	-6.507	2.5	5

Further iterations do not increase the likelihood function  $L(\mathbf{Q}_d, \mathbf{R}, \mathbf{Z})$  much, where

$$L(\mathbf{Q}_d, \mathbf{R}, \mathbf{Z}) = -\frac{1}{2N} \sum_{i=1}^N \mathbf{r}_i^T (\mathbf{H}\mathbf{P}^{-1}\mathbf{H}^T + \mathbf{R})^{-1} \mathbf{r}_i - \frac{1}{2} \ln |\mathbf{H}\mathbf{P}^{-1}\mathbf{H}^T + \mathbf{R}| - \frac{m}{2} \ln(2\pi)$$

even though the changes in  $\mathbf{Q}_d$  and  $\mathbf{R}$  are significant. Moreover, a check case was run with true values of  $\mathbf{Q}_d$  and  $\mathbf{R}$  in the filter, and the likelihood function is seen to be very close to that achieved

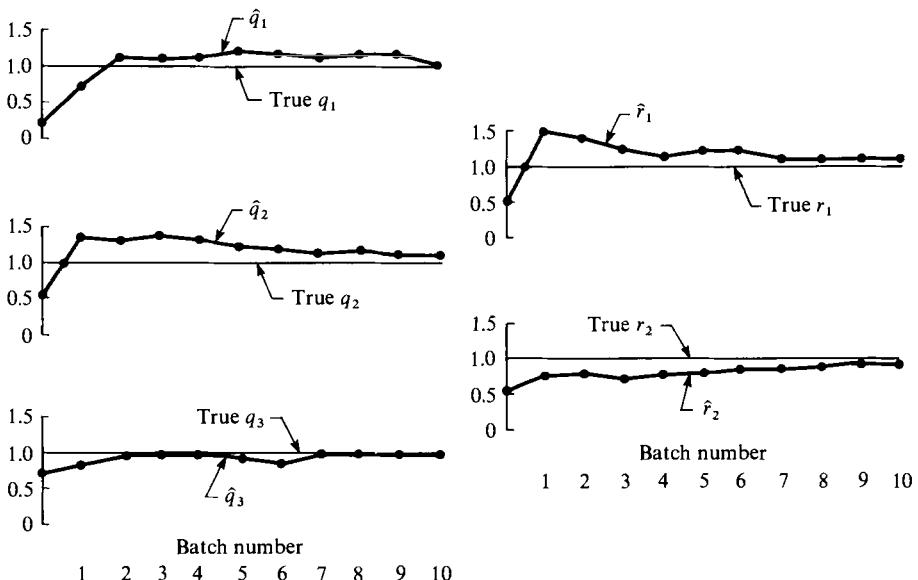


FIG. 10.26 Online adaptation. From Mehra [88], © 1970 IEEE.

in the first iteration. This indicates that the obtained estimates are quite close to the maximum likelihood estimates and that the ambiguity function has very low curvature.

This table reflects the performance of  $\mathbf{Q}_d$  estimates achieved by manipulating Eq. (10-113) by repeated back substitution of  $\mathbf{P}^-$  and selection of a linearly independent subset of equations. However, including more equations and performing a least squares fit or using one of the methods described in the text above provide substantially the same performance. Furthermore, essentially the same values are achieved when different initial estimates of  $\mathbf{Q}_d$  and  $\mathbf{R}$  are employed.

After adaptation,  $[\hat{\mathbf{A}}_k]_{11}$  and  $[\hat{\mathbf{A}}_k]_{22}$  were compared to the 95% confidence limits for  $k = 1, 2, \dots, 40$ . As seen in Table 10.14, only 2.5% (i.e., one point) of the  $[\hat{\mathbf{A}}_k]_{11}$  values and 5% of the  $[\hat{\mathbf{A}}_k]_{22}$  points exceeded these limits, supporting the hypothesis that the residuals are white.

Figure 10.26 [88] demonstrates the performance of this estimator in online usage, generating new estimates of  $\mathbf{Q}_d$  and  $\mathbf{R}$  on ten successive batches of  $N$  measurements ( $N = 950$ ). Simple averaging is performed on the estimated values: if  $\hat{\mathbf{a}}_N(k)$  is the parameter estimate based on the  $k$ th batch of  $N$  measurements, then the parameter estimate  $\hat{\mathbf{a}}(k)$  produced by the algorithm after processing  $k$  batches is

$$\hat{\mathbf{a}}(k) = \begin{cases} \hat{\mathbf{a}}_N(k) & k = 1 \\ \hat{\mathbf{a}}(k-1) + (1/k)[\hat{\mathbf{a}}_N(k) - \hat{\mathbf{a}}(k-1)] & k = 2, 3, \dots \end{cases}$$

Note the asymptotic convergence of the estimates towards their true values in this figure. ■

## 10.10 COVARIANCE MATCHING AND OTHER TECHNIQUES

Covariance matching [91, 111, 122, 123] entails comparing online estimates of residual covariances and their theoretical values as computed by the state filter, and modifying the tuning of  $\mathbf{Q}_d$  and/or  $\mathbf{R}$  until the two agree. This modification can often be confined to  $\mathbf{Q}_d$  alone, since the uncertainty in the appropriate noise strength is usually greater for dynamics noise than measurement corruption noise.

**EXAMPLE 10.22** Consider tracking a vehicle moving along a single axis from a radar located at the origin of the coordinate system. One possible model to use in a tracking filter would be

$$\begin{aligned} \mathbf{x}(t_{i+1}) &= \mathbf{x}(t_i) + [\Delta t]\mathbf{u}(t_i) \\ \mathbf{u}(t_{i+1}) &= \mathbf{u}(t_i) + [\Delta t]\mathbf{w}_d(t_i) \end{aligned}$$

where  $\mathbf{x}(t_i)$  is target position at time  $t_i$ ,  $\mathbf{u}(t_i)$  is velocity,  $\mathbf{w}_d(t_i)$  is a white Gaussian noise meant to model the vehicle acceleration, and  $\Delta t$  is the sample period at which the radar generates noise-corrupted position measurements.

If the target is nonmaneuvering, or essentially maintaining constant velocity, then a very small value of  $Q_d$  will provide good estimation accuracy. However, if the target initiates dynamic maneuvering, this  $Q_d$  value will not allow the state estimate to track the true oscillations well. An increased  $Q_d$ , or perhaps even a better acceleration model than a simple white noise, is required to provide precision tracking. On the other hand, these modifications may severely corrupt the tracking performance during more benign segments of the vehicle trajectory.

Throughout the range of these scenarios, a single measurement model might be used:

$$\mathbf{z}(t_i) = \mathbf{x}(t_i) + \mathbf{v}(t_i)$$

with a single  $R$  to represent the radar precision regardless of vehicle maneuvering. ■

Assume that  $\mathbf{R}$  is known. Then the empirically generated estimate of residual covariance,

$$\hat{\mathbf{A}} = \frac{1}{N} \sum_{j=i-N+1}^i \mathbf{r}_j \mathbf{r}_j^T \quad (10-124)$$

with  $N$  chosen to provide statistical smoothing, can be compared to its theoretical value  $[\mathbf{H}(t_i) \mathbf{P}(t_i^-) \mathbf{H}^T(t_i) + \mathbf{R}(t_i)]$  from the Kalman filter. For instance, if  $\hat{\mathbf{A}}$  exceeds the filter-derived value (on the basis of eigenvalues, diagonal terms, norm, etc.), then  $\mathbf{Q}_d$  should be increased. One means of obtaining a better  $\mathbf{Q}_d$  would be to manipulate

$$\mathbf{A}(t_i) = \mathbf{H}\mathbf{P}(t_i^-)\mathbf{H}^T + \mathbf{R} = \mathbf{H}[\Phi\mathbf{P}(t_{i-1}^+)\Phi^T + \mathbf{G}_d\mathbf{Q}_d\mathbf{G}_d^T]\mathbf{H}^T + \mathbf{R}$$

into

$$\mathbf{H}\mathbf{G}_d\mathbf{Q}_d\mathbf{G}_d^T\mathbf{H}^T = \mathbf{A}(t_i) - \mathbf{H}\Phi\mathbf{P}(t_{i-1}^+)\Phi^T\mathbf{H}^T - \mathbf{R} \quad (10-125)$$

Even if  $\hat{\mathbf{A}}(t_i)$  and the  $\mathbf{P}(t_{i-1}^+)$  erroneously calculated by the filter were used to evaluate the right hand side of this equation, a unique solution for  $\mathbf{Q}_d$  or  $[\mathbf{G}_d \ \mathbf{Q}_d \ \mathbf{G}_d^T]$  generally does not exist because  $\mathbf{H}$  is seldom of rank  $n$ . If the number of unknowns in  $\mathbf{Q}_d$  is restricted, a unique solution can be generated. Otherwise, pseudoinverses can be used. However, the convergence properties of this method are generally subject to question.

If instead  $\mathbf{Q}_d$  is assumed known, then  $\mathbf{R}$  can be estimated as

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{j=i-N+1}^i \mathbf{r}_j \mathbf{r}_j^T - \mathbf{H}\mathbf{P}(t_i^-)\mathbf{H}^T \quad (10-126)$$

Notice that this and the  $\mathbf{Q}_d$  result above are identical to the explicit suboptimal solutions for the maximum likelihood estimates, as given in Section 10.7.

There are numerous related ad hoc adaptation techniques. For instance, one might model  $\mathbf{Q}_d$  as

$$\mathbf{Q}_d = \mathbf{Q}_{do} + a[\Delta\mathbf{Q}_d] \quad (10-127)$$

where  $\mathbf{Q}_{do}$  is a nominal value,  $\Delta\mathbf{Q}_d$  is a perturbation to that nominal, and  $a$  is a scalar parameter to be adjusted for best agreement between empirical and theoretical residual covariances. Further, the parameter might be constrained to assume only discrete values, resulting in a multiple model filtering algorithm similar in structure to that of Section 10.8, but in which the state estimate is generated from the one filter with the best residual characteristics, rather than as a weighted sum of  $K$  filter outputs.

In many proposed algorithms, a statistical test is performed on the observed residuals to determine if and when to undergo adaptation. Optimal tests can be developed from statistical decision theory [121, 137], but more practical

suboptimal tests are usually implemented, such as determining if a certain percentage of the  $N$  most recent residuals have surpassed computed  $2\sigma$  or  $3\sigma$  bounds, or determining if the current residual is sufficiently beyond a given threshold while simultaneously the previous  $N$  residuals have been of the same sign [49, 66, 67, 91]. However, such a test introduces a time delay in adapting to changes in the true system. If this is not tolerable, reprocessing of the  $N$  most recent measurements (stored for this eventuality) can be used to enhance transient estimation performance once the need to adapt is declared.

**EXAMPLE 10.23** One proposed filter for the tracking problem, as discussed in Example 10.22, is to provide two different values of  $Q_d$ , or perhaps two different acceleration models altogether, one for the nonmaneuvering case and one for the more dynamic case. When in the nominal nonmaneuvering mode, the filter declares the target to be maneuvering if  $N$  (two or more) consecutive residuals are of the same sign and outside the computed  $3\sigma$  level [26, 49, 66, 67, 87, 99, 131].

However, the target will have been maneuvering for some time before the actual declaration, and the state estimates may well have diverged enough due to the use of the “erroneous” no-maneuver model to cause recovery problems. Therefore, the most recent  $N$  raw measurements (the number of residuals used for the maneuver decision) are retrieved from storage and reprocessed using the heavier weighting of the gains generated by the “maneuver”  $Q_d$ . Thus, the residual behavior is used to discern the inadequacy of the internal model, and then there is a backsliding of the computation “time” to the point at which the model is first thought to become inadequate. All measurements from that time forward are reprocessed, until the algorithm returns to real-time operation.

Of course, multiple model adaptive filtering provides a viable alternative method for this case. ■

## 10.11 SUMMARY

This chapter has considered the online simultaneous estimation of states and uncertain parameters. The basic motivation for the adaptive parameter estimation is to improve the filter’s internal model, with regard to both structure ( $\Phi$ ,  $\mathbf{B}_d$ , and  $\mathbf{H}$ ) and assumed noise strengths ( $\mathbf{Q}_d$  and  $\mathbf{R}$ ), thereby enhancing state estimation performance. Offline methods and algorithms devoted to parameter identification without state estimation have not been discussed in detail, but the substantial amount of literature in these areas [5, 6, 20, 21, 32–34, 37, 41, 42, 44–47, 56, 63, 74, 76–78, 81, 92, 95, 105, 106, 110, 113, 118, 130, 135, 138] can be understood in terms of the concepts presented here. Moreover, attention has been confined to applications in which linear models suffice, although the same basic methods can be used for the more general case.

In many applications, the robustness of the Kalman filter or other estimation algorithms yields adequate state estimation precision despite parameter uncertainties. It may be sufficient to add “pseudonoise” to the filter model to cause a heavier weighting of real-world measurements, thereby decreasing sensitivity to erroneously assumed parameter values. However, the contributions of the parameter errors to model uncertainty often cannot be treated adequately as “equivalent” white noise effects. Sensitivity analyses can be used

to indicate whether simultaneous parameter estimation is required and, if so, which parameters should be included to improve performance the most.

If uncertain parameters in  $\Phi$ ,  $B_d$ , or  $H$  are to be estimated, one can treat these as additional state variables, modeled as random constants, and formulate an extended Kalman filter to handle the resulting nonlinear state estimation problem. Although this is attractive computationally, such solutions often exhibit objectionable bias and convergence properties. Maximum likelihood techniques provide perhaps the best estimation performance, but are burdensome. This chapter revealed the use of a constant-over- $N$ -steps parameter model, approximations, and efficiency improving techniques to derive algorithms with both online practicality and desirable performance. Discretization of the possible parameter values and Bayesian techniques produced the multiple model filtering algorithm, which may become particularly attractive with the increasing use of microprocessors and distributed computation in computer architecture.

If uncertainties lie in  $Q_d$  or  $R$ , maximum likelihood techniques and multiple model filtering algorithms can similarly be exploited. More ad hoc methods, such as residual "whitening" and covariance matching, are also available, requiring less computation and storage but generally providing inferior performance.

Throughout this chapter, the residuals in the state estimator have played a key role. It is this difference between measurement data and their model-predicted values that allows any model adaptation to take place. The various estimators of this chapter have differed primarily in the means of exploiting the residuals' characteristics in order to provide simultaneous estimation of uncertain model parameters.

The topic of designing control inputs to aid the parameter identification was introduced in this chapter. However, detailed treatment of this aspect will be postponed until Chapter 15 (Volume 3).

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## PROBLEMS

**10.1** Consider the simple problem formulation of (10-1)–(10-5) in which all variables are *scalar*. Let scalar  $\Phi$  and  $B_d$  be uncertain parameters to estimate simultaneously with the scalar state  $x$ .

- (a) Write out the likelihood equations, (10-20) and (10-21), in explicit detail.
- (b) Generate the full-scale estimator equations (10-32)–(10-57) for this special case.
- (c) Demonstrate the simplification due to using the approximation given by (10-58)–(10-60).
- (d) Develop the explicit evaluation of the ambiguity function given by (10-66) by assuming a *scalar* truth model (so model order reduction effects do not come into play).
- (e) Generate the online estimator equations based upon (10-67) and (10-68); compare these to the online estimator based on (10-69).
- (f) Demonstrate the simplification due to including only weighted least squares type terms, as discussed in relation to (10-73)–(10-75). Compare the number of computations to that required when all likelihood equation terms are included.
- (g) What portion of these estimator equations can be precomputed and stored, perhaps as a function of assumed parameter values?
- (h) Assume that possible  $\Phi$  values are adequately represented by three discrete values,  $\Phi_1$ ,  $\Phi_2$ , and  $\Phi_3$ , while  $B_d$  is adequately represented by two values,  $B_{d1}$  and  $B_{d2}$ . Develop the multiple model filtering algorithm for this problem description.

**10.2** Write out the equations in explicit detail for the simultaneous state estimation and system identification algorithms of

- (a) Example 10.3.
- (b) Example 10.4.
- (c) Example 10.8.
- (d) Example 10.10.
- (e) Example 10.13.
- (f) Example 10.15.
- (g) Example 10.19.

**10.3** As in Problem 10.1, consider the simple problem formulation of (10-1)–(10-5) in which all variables are *scalar*. Now consider either  $Q_d$  or  $R$  being uncertain parameters requiring simultaneous estimation along with the state.

- (a) Write the full-scale estimator equations (10-32)–(10-57), as modified by the results of the initial part of Section 10.7, for the case of estimating  $x$  and  $Q_d$ .
- (b) Explicitly show the simplification due to using (10-85).
- (c) Establish the closed-form approximation (10-93) for this problem.
- (d) Assume that possible  $Q_d$  values are adequately represented by three discrete values,  $Q_{d1}$ ,  $Q_{d2}$ , and  $Q_{d3}$ . Develop the multiple model filtering algorithm for this problem.
- (e) Repeat (a) for the case of estimating  $x$  and  $R$ .

- (f) Repeat (b) for the case of estimating  $\mathbf{x}$  and  $R$ .
- (g) Establish the closed-form approximations (10-88) and (10-90) for this problem, and compare them.
- (h) Assume that possible  $R$  values are well represented by three discrete values,  $R_1$ ,  $R_2$ , and  $R_3$ . Develop the multiple model filtering algorithm for these models.
- (i) Develop the multiple model filtering algorithm for uncertain  $Q_d$  and  $R$ , discretized as in (d) and (h).
- (j) Apply (10-95)–(10-97) to this problem, for cases of uncertainties in  $Q_d$ , or  $R$ , or both.
- (k) Develop the self-tuning adaptive filter for this problem, based on correlation methods, as described in Section 10.9. Consider cases of  $Q_d$  uncertain,  $R$  uncertain, and both  $Q_d$  and  $R$  uncertain.
- (l) Develop the self-tuning adaptive filter for this problem, based on covariance matching techniques, as described in Section 10.10. Consider cases of  $Q_d$  uncertain,  $R$  uncertain, and both  $Q_d$  and  $R$  uncertain.

**10.4** Write out the equations in explicit detail for the self-tuning adaptive estimators of

- (a) Example 10.18, single input–single output case, with and without precomputed  $\mathbf{J}^{-1}$ .
- (b) Example 10.18, two input–two output case, with and without precomputed  $\mathbf{J}^{-1}$ .
- (c) Example 10.18, scalar  $Q_d$  estimated via (10-93) and scalar  $R$  via either (10-88) or (10-90).
- (d) Example 10.18, 2-by-2  $Q_d$  estimated via (10-93) and 2-by-2  $\mathbf{R}$  via either (10-88) or (10-90).
- (e) Example 10.21.
- (f) Example 10.22 by covariance matching.
- (g) Example 10.22 by multiple model filtering.

**10.5** (a) Consider an alternative to the development of (10-15) and (10-18) that would be useful for the case of  $\mathbf{Q}_d \equiv \mathbf{0}$ . Instead of using (10-9), show that Bayes' rule can also produce

$$f_{\mathbf{x}(t_i), \mathbf{z}(t_i)|\mathbf{a}} = f_{\mathbf{x}(t_i)|\mathbf{a}} \prod_{j=1}^i f_{\mathbf{z}(t_j)|\mathbf{z}(t_{j-1}), \mathbf{x}(t_j), \mathbf{a}}$$

and that each term in this product can be expressed *conveniently* in Gaussian form if  $\mathbf{Q}_d \equiv \mathbf{0}$ . Generate these forms explicitly and establish the appropriate likelihood equations to be solved simultaneously for  $\mathbf{x}^*(t_i)$  and  $\mathbf{a}^*(t_i)$ .

(b) Repeat for the case of fixed-length memory parameter estimation and fixed-length memory state estimation by showing Bayes' rule can produce

$$f_{\mathbf{x}(t_i), \mathbf{z}_N(t_i)|\mathbf{a}} = f_{\mathbf{x}(t_i)|\mathbf{a}} \prod_{j=i-N+1}^i f_{\mathbf{z}(t_j)|\mathbf{x}(t_i), \mathbf{a}}$$

**10.6** (a) Develop the maximum likelihood estimator equations as in Section 10.3 for entries labeled (1) in Table 10.1.

- (b) Repeat for entries labeled (2).
- (c) Repeat for entries labeled (3).

**10.7** Derive the finite-memory maximum likelihood state and parameter estimator based upon the likelihood function  $\ln f_{\mathbf{x}(t_i), \mathbf{z}_N(t_i)|\mathbf{a}}(\xi, \mathcal{Z}_N | \mathbf{a})$ . Show that the estimator is of the same form as in Section 10.4, except that the initial conditions  $\tilde{\mathbf{x}}(t_{i-N}^+)$ ,  $\mathbf{P}(t_{i-N}^+)$ , and  $E\{\tilde{\mathbf{x}}(t_{i-N}^+) \tilde{\mathbf{x}}^T(t_{i-N}^+)\}$  are replaced by  $\tilde{\mathbf{x}}(t_{i-N})$ ,  $\tilde{\mathbf{P}}(t_{i-N})$ , and  $\tilde{\mathbf{x}}(t_{i-N}) \tilde{\mathbf{x}}^T(t_{i-N})$ , respectively, where  $\tilde{\mathbf{x}}(t_{i-N})$  is generated from a one-step propagation from the previous value  $\tilde{\mathbf{x}}(t_{i-N-1})$ .

$$\tilde{\mathbf{x}}(t_{i-N}) = \Phi(t_{i-N}, t_{i-N-1}; \hat{\mathbf{a}}_*(t_i)) \tilde{\mathbf{x}}(t_{i-N-1}) + \mathbf{B}_d(t_{i-N-1}; \hat{\mathbf{a}}_*(t_i)) \mathbf{u}(t_{i-N-1})$$

for  $i \geq (N + 1)$ , starting from the initial condition  $\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$ , and similarly  $\tilde{\mathbf{P}}(t_{i-N})$  is obtained from

$$\begin{aligned}\tilde{\mathbf{P}}(t_{i-N}) &= \Phi(t_{i-N}, t_{i-N-1}; \hat{\mathbf{a}}_*(t_i))\tilde{\mathbf{P}}(t_{i-N-1})\Phi^T(t_{i-N}, t_{i-N-1}; \hat{\mathbf{a}}_*(t_i)) \\ &\quad + \mathbf{G}_d(t_{i-N-1})\mathbf{Q}_d(t_{i-N-1})\mathbf{G}_d^T(t_{i-N-1})\end{aligned}$$

for  $i \geq (N + 1)$ , with initial condition  $\tilde{\mathbf{P}}(t_0) = \mathbf{P}_0$ .

**10.8** Consider the special case of a single input-single output linear dynamic system

$$\mathbf{y}(t_{i+1}) = \phi_0 \mathbf{y}(t_i) + \cdots + \phi_{n-1} \mathbf{y}(t_{i-n+1}) + bu(t_i) + w_d(t_i)$$

where  $\phi_0, \dots, \phi_{n-1}$ , and  $b$  are uncertain,  $u$  is a control input, and  $w_d(\cdot, \cdot)$  is zero-mean white Gaussian discrete-time noise of constant variance  $Q_d$ . Note that this can be expressed in state space model form as

$$\begin{aligned}[\mathbf{x}(t_{i+1})] &= \begin{bmatrix} \mathbf{0} \\ \phi^T \end{bmatrix} [\mathbf{x}(t_i)] + \begin{bmatrix} \mathbf{0} \\ b \end{bmatrix} u(t_i) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} w_d(t_i) \\ \mathbf{z}(t_i) &= [\mathbf{0}^T \mid 1] \mathbf{x}(t_i)\end{aligned}$$

by letting

$$\begin{aligned}\mathbf{x}^T(t_i) &\triangleq [y(t_{i-n+1}) \cdots y(t_{i-1}) \ y(t_i)] \\ \phi^T &\triangleq [\phi_{n-1} \cdots \phi_1 \ \phi_0]\end{aligned}$$

(a) Generate the likelihood equations to be solved to form the *parameter estimator* based upon a likelihood function of

$$\ln f_{\mathbf{Z}_n(t_i)|\mathbf{Z}(t_{i-N}), \mathbf{s}} = \sum_{j=i-N+1}^i \ln f_{\mathbf{z}(t_j)|\mathbf{z}(t_{j-1}), \mathbf{s}}$$

that would be appropriate for parameters modeled as essentially constant over  $N$  steps. Show that each term in the preceding summation is Gaussian, with mean  $[\phi^T \mathbf{Z}_n(t_{j-1}) + bu(t_{j-1})]$  and variance  $Q_d$ , where  $\mathbf{Z}_n(t_{j-1})$  is the vector of the most recent  $n$  measurements at time  $t_{j-1}$ , with components  $z(t_{j-n})$  to  $z(t_{j-1})$ ; then take the appropriate partials with respect to  $\phi$  and  $b$  to yield the desired results.

(b) Show that these equations *can* be solved in closed form as

$$\begin{bmatrix} \phi^* \\ b^* \end{bmatrix} = \begin{bmatrix} \Lambda & \lambda \\ \lambda^T & \lambda_0 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{j=i-N+1}^i \mathbf{Z}_n(t_{j-1}) z(t_j) \\ \sum_{j=i-N+1}^i u(t_{j-1}) z(t_j) \end{bmatrix}$$

where

$$\begin{aligned}\Lambda &= \sum_{j=i-N+1}^i \mathbf{Z}_n(t_{j-1}) \mathbf{Z}_n^T(t_{j-1}) \\ \lambda &= \sum_{j=i-N+1}^i \mathbf{Z}_n(t_{j-1}) u(t_{j-1}) \\ \lambda_0 &= \sum_{j=i-N+1}^i u^2(t_{j-1})\end{aligned}$$

(c) Show how this result extends to the case of time-varying  $Q_d(t_i)$ .

**10.9** In deriving the expressions given by (10-31) for evaluating the scalar elements of the conditional information matrix in (10-30), it is necessary to evaluate the form  $E\{[\mathbf{y}^T \mathbf{W}_1 \mathbf{y}] [\mathbf{y}^T \mathbf{W}_2 \mathbf{y}]\}$  where  $\mathbf{y}$  is a zero-mean Gaussian random vector with covariance  $\mathbf{Y}$ , and where  $\mathbf{W}_1$  and  $\mathbf{W}_2$  are

arbitrary weighting matrices. This is a useful result in and of itself, so we want to show that

$$E\{[\mathbf{y}^T \mathbf{W}_1 \mathbf{y}] [\mathbf{y}^T \mathbf{W}_2 \mathbf{y}]\} = \text{tr}\{\mathbf{W}_1 \mathbf{Y}\} \text{tr}\{\mathbf{W}_2 \mathbf{Y}\} + 2 \text{tr}\{\mathbf{W}_1 \mathbf{Y} \mathbf{W}_2 \mathbf{Y}\}$$

To show this, first show that

$$\mathbf{y} = \sqrt{\mathbf{Y}} \mathbf{x}$$

where  $\mathbf{x}$  is a zero-mean Gaussian random vector with covariance  $\mathbf{I}$ . Then show

$$\begin{aligned} E\{[\mathbf{y}^T \mathbf{W}_1 \mathbf{y}] [\mathbf{y}^T \mathbf{W}_2 \mathbf{y}]\} &= E\{[\mathbf{x}^T \mathbf{A}_1 \mathbf{x}] [\mathbf{x}^T \mathbf{A}_2 \mathbf{x}]\} \\ &= \sum_{i=1}^n \sum_{j=1}^n \lambda_{1i} \lambda_{2j} E\{\mathbf{x}_i^2 \mathbf{x}_j^2\} \end{aligned}$$

with  $\mathbf{A}_1$  and  $\mathbf{A}_2$  appropriately defined, and  $\lambda_{1i}$  and  $\lambda_{2j}$  being the eigenvalues of  $\mathbf{A}_1$  and  $\mathbf{A}_2$ , respectively, for  $i$  and  $j$  both taking on values from 1 to  $n$ . Demonstrate the validity of these equalities, and use them to derive the final result.

**10.10** Consider maximizing a likelihood function  $L[\boldsymbol{\theta}, \mathbf{Z}]$  as a function of  $\boldsymbol{\theta}$  and for given  $\mathbf{Z}$ , by numerical means. The solution point is then defined by

$$\partial L[\boldsymbol{\theta}, \mathbf{Z}] / \partial \boldsymbol{\theta} = \mathbf{0}^T$$

(a) Assume you have a trial point  $\boldsymbol{\theta}_k$  and the gradient evaluated at that point is nonzero:

$$\partial L[\boldsymbol{\theta}, \mathbf{Z}] / \partial \boldsymbol{\theta}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_k} \neq \mathbf{0}^T$$

The question is, for a given length of step to the next iterate value  $\boldsymbol{\theta}_{k+1}$ , in which direction should we step so as to make the greatest change in  $L$ ? Define this step as

$$\Delta \boldsymbol{\theta}_k = \boldsymbol{\theta}_{k+1} - \boldsymbol{\theta}_k$$

Then, under the constraint that the length of the step is  $s$ , i.e.,  $\Delta \boldsymbol{\theta}_k^T \Delta \boldsymbol{\theta}_k = s^2$ , show that the best step to take is along the gradient, or “steepest ascent,” direction, yielding an iteration

$$\begin{aligned} \boldsymbol{\theta}_{k+1} &= \boldsymbol{\theta}_k + \Delta \boldsymbol{\theta}_k \\ &= \boldsymbol{\theta}_k + \left\{ \frac{s}{\sqrt{(\partial L / \partial \boldsymbol{\theta})|_{\boldsymbol{\theta}_k} (\partial L / \partial \boldsymbol{\theta})|_{\boldsymbol{\theta}_k}^T}} \right\} \frac{\partial L}{\partial \boldsymbol{\theta}}|_{\boldsymbol{\theta}_k}^T \end{aligned}$$

where the term in braces is viewed as a scalar step-size control parameter, yielding a step of length  $s$  in the gradient direction. The easiest way to show this is by using Lagrange multipliers: maximize the value of

$$\{L[\boldsymbol{\theta}_{k+1}, \mathbf{Z}] - L[\boldsymbol{\theta}_k, \mathbf{Z}]\} + v\{\Delta \boldsymbol{\theta}_k^T \Delta \boldsymbol{\theta}_k - s^2\}$$

with respect to  $\boldsymbol{\theta}_{k+1}$  and  $v$ . It is useful to consider a Taylor series for  $L[\boldsymbol{\theta}_{k+1}, \mathbf{Z}]$  carried to first order:

$$L[\boldsymbol{\theta}_{k+1}, \mathbf{Z}] \cong L[\boldsymbol{\theta}_k, \mathbf{Z}] + \left[ \frac{\partial L}{\partial \boldsymbol{\theta}} \Big|_{\boldsymbol{\theta}_k} \right] \Delta \boldsymbol{\theta}_k$$

to derive this *gradient optimization algorithm*.

(b) Consider the case of scalar  $\theta$ . We want to find the value of  $\theta$  that yields the stationary point in  $L$ , i.e., where

$$\partial L[\theta, \mathbf{Z}] / \partial \theta|_{\theta^*} = 0$$

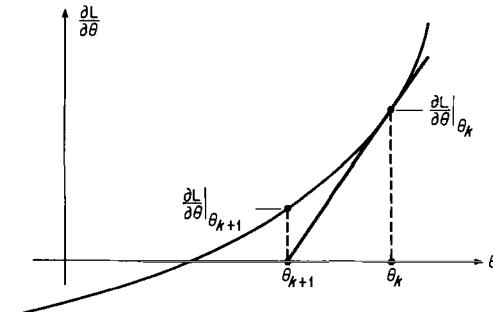


FIG. 10.P1 Basic Newton-Raphson concept.

Thus, if we plot  $\partial L / \partial \theta$  versus  $\theta$  as in the Fig. 10.P1, we are searching for the zero crossing of  $\partial L / \partial \theta$ . Suppose you start at some value  $\theta_k$ . One means of efficiently iterating towards the zero-crossing is to approximate the gradient  $\partial L / \partial \theta$  by the local tangent at  $\theta_k$ , and find the value of  $\theta$  where this local tangent crosses zero and call that point  $\theta_{k+1}$ . Show this yields the iteration

$$\theta_{k+1} = \theta_k - \left[ \frac{\partial^2 L}{\partial \theta^2} \Big|_{\theta_k} \right]^{-1} \left[ \frac{\partial L}{\partial \theta} \Big|_{\theta_k} \right]$$

known as the *Newton-Raphson method*.

(c) Repeat part (b) for the case of vector  $\theta$ , and show that the result generalizes to the form depicted in (10-23).

(d) Note the similarities and differences of the forms of the gradient and Newton-Raphson iterations just derived.

### 10.11 Demonstrate the validity of (10-26) by writing

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{z}(t_i)|\mathbf{a}}(\mathbf{z}| \mathbf{a}) d\mathbf{z} = 1$$

so that

$$\frac{\partial^2 \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{z}(t_i)|\mathbf{a}}(\mathbf{z}| \mathbf{a}) d\mathbf{z} \right\}}{\partial a_k \partial a_l} = 0$$

Combining this with the fact that

$$f = \exp\{\ln f\} = \exp\{L\}$$

yields the desired result.

**10.12** Develop the count for the number of multiplications, additions, subtractions, and inversions for state  $\hat{\mathbf{x}}$ , score  $\mathbf{s}$ , conditional information matrix  $\mathbf{J}$ , and new parameter estimate  $\hat{\mathbf{a}}^*$  as depicted in Examples 10.1, 10.2, 10.7, 10.9, 10.12, 10.14, and 10.16, but written as functions of  $n$  (dimension of  $\mathbf{x}$ ),  $r$  (dimension of  $\mathbf{u}$ ),  $s$  (dimension of  $\mathbf{w}$ ),  $m$  (dimension of  $\mathbf{z}$ ), and  $p$  (dimension of  $\mathbf{a}$ ). (Let  $p = p_A + p_B$ , where  $p_B$  is the number of parameters whose effects are confined to  $\mathbf{B}_d$  only, whereas  $p_A$  corresponds to parameters that affect both  $\Phi$  and  $\mathbf{B}_d$  in general: express the result in terms of  $p_A$  and  $p_B$  separately.)

**10.13** (a) By applying the transformation of (10-77) to the model equations (10-76), demonstrate the validity of (10-78) and (10-79).

(b) Show the validity of (10-81)–(10-84) by substitution into (10-76)–(10-79).

**10.14** Consider the estimator equations developed in Sections 10.3 and 10.4 for the case of uncertain parameters in  $\Phi$  and  $\mathbf{B}_d$ . As discussed in the beginning of Section 10.2, often we need not consider uncertainties in  $\mathbf{H}$ . But in some cases, especially when transformations of state variables are considered (see the discussion following (10-84)), we must consider such uncertain parameters. Generalize the results of Sections 10.3 and 10.4 to allow  $\mathbf{H}$  to be affected by components of the parameter vector  $\mathbf{a}$ . Specifically detail the modifications to the full-scale estimator equations (10-32)–(10-57) as a result of this extension.

**10.15** Derive relation (10-54) for  $\partial \mathbf{P}(t_j^+)/\partial a_k$  for use in maximum likelihood estimation of states and parameters in  $\Phi$  and  $\mathbf{B}_d$  according to the method discussed just below that equation in the text.

**10.16** Demonstrate that  $(1/N) \sum_j \mathbf{r}_j \mathbf{r}_j^T$  is an ergodic approximation to  $[\mathbf{H}(t_i) \mathbf{P}(t_i^-) \mathbf{H}^T(t_i) + \mathbf{R}(t_i)]$ , as claimed below (10-88).

**10.17** (a) This problem develops a physical interpretation of the closed-form approximation for  $\hat{\mathbf{Q}}_d(t_i)$  given by (10-93). Recall the covariance update and propagation equations for a Kalman filter,

$$\begin{aligned}\mathbf{P}(t_j^+) &= \mathbf{P}(t_j^-) - \mathbf{K}(t_j) \mathbf{H}(t_j) \mathbf{P}(t_j^-) \\ \mathbf{P}(t_{j+1}^-) &= \Phi(t_{j+1}, t_j) \mathbf{P}(t_j^+) \Phi^T(t_{j+1}, t_j) + \mathbf{G}_d(t_j) \mathbf{Q}_d(t_j) \mathbf{G}_d^T(t_j)\end{aligned}$$

By noting that  $E\{\Delta \mathbf{x}_j \Delta \mathbf{x}_j^T\} = \mathbf{K}(t_j) \mathbf{H}(t_j) \mathbf{P}(t_j^-)$  and then approximating the ensemble average by a temporal average over the most recent  $N$  samples of data, derive (10-93) as a physically reasonable approximation.

(b) To reduce storage requirements, a fading memory approximation to the finite memory result above is sometimes implemented in actual applications, i.e.,

$$\hat{\mathbf{Q}}_d(t_i) = k \hat{\mathbf{Q}}_d(t_{i-1}) + [1 - k] \hat{\mathbf{Q}}_{d1}(t_i)$$

where  $\hat{\mathbf{Q}}_{d1}(t_i)$  is a single term from the summation in (10-93) and  $k \geq 0.8$  typically. Show how  $k \in (0, 0.8)$  versus  $k \in (0.8, 1.0)$  would affect error variance and responsiveness of this estimate, and relate this to  $N$  of (10-93).

**10.18** Explicitly derive relations (10-104) and (10-105) for the multiple model filtering algorithm by the method described in the text.

**10.19** (a) Derive (10-112).

(b) Iterate (10-114)  $k$  times and demonstrate the validity of the two expectation relations below it.

(c) Generate the expression given by (10-115).

(d) Explicitly write out (10-115a) for  $k = 1, 2, \dots, n$  and rearrange to derive (10-117).

(e) Take the difference of (10-113) and (10-120) to yield (10-121), noting that

$$\mathbf{K}_* = \mathbf{P}_*^- \mathbf{H}^T (\mathbf{H} \mathbf{P}_*^- \mathbf{H}^T + \mathbf{R})^{-1} = \mathbf{P}_*^- \mathbf{H}^T (\mathbf{A}_0 + \mathbf{H} \delta \mathbf{P}^- \mathbf{H}^T)^{-1}$$

(f) Explicitly generate the two optional methods for providing an estimate of  $\mathbf{G}_d \mathbf{Q}_d \mathbf{G}_d^T$ , as outlined below (10-122) in the text.

**10.20** Assume that you are trying to achieve the “best” linear time-invariant system model to represent the input–output characteristics of an unknown system. This problem explores the issue of identifiability of the system model.

(a) Let each system input point in turn be individually driven by wideband noise with stationary statistics of known power spectral density characteristics, and let each system output be analyzed to yield the power spectral density of each in response to a particular input. Assume perfect measurements. Describe fully how this information might be used to generate a linear frequency domain model of the system. Interpret the idea of identifiability for this application.

(b) Assume that some basic modeling has been completed and that the dimension  $n$  of the state vector  $\mathbf{x}$  to model the system has been established. Consider a free dynamic system modeled as

$$\mathbf{x}(t_{i+1}) = \Phi \mathbf{x}(t_i)$$

where  $\mathbf{x}(t_0)$  is known and all states can be observed. Using the first  $n$  values of  $\mathbf{x}(t_i)$  from a specific initial condition  $\mathbf{x}_0$ , develop a necessary condition for  $\Phi$  to be determined completely, and interpret this physically.

(c) If only a single measurement were available for part (b),

$$z(t_i) = \mathbf{h}^T \mathbf{x}(t_i)$$

with  $\mathbf{h}^T$  constant. It is desired to "identify" the equivalent system

$$\mathbf{x}^*(t_{i+1}) = \Phi^* \mathbf{x}^*(t_i), \quad z(t_i) = \mathbf{h}^{*T} \mathbf{x}^*(t_i)$$

where  $\Phi^*$  and  $\mathbf{h}^{*T}$  are of the form

$$\Phi^* = \begin{bmatrix} 0 & 1 & & & 0 \\ & \ddots & & & \\ & & \ddots & & \\ 0 & & & 0 & 1 \\ \hline \phi_1 & \phi_2 & & & \phi_n \end{bmatrix}, \quad \mathbf{h}^{*T} = [1 \ 0 \ \cdots \ 0]$$

By considering the first  $2n$  measurements, the  $n$  unknown elements of  $\Phi^*$  can be evaluated under certain conditions. What are these conditions?

(d) Now consider a general system description as follows. Let  $\mathbf{x}$  be a state function that maps  $(\Theta \times U \times I)$  into  $R^n$ , where  $\Theta$  is the (Hilbert) space of uncertain parameters  $\theta$ ,  $U$  is the space of admissible controls, and  $I$  is a time interval in  $R^1$ . Let  $Y$  be the (Hilbert) space of output functions, with elements  $\mathbf{y}(\cdot) \in Y$  defined by

$$\mathbf{y}(t) = \mathbf{h}[\mathbf{x}, \theta, \mathbf{u}, t] + \mathbf{v}_e(t)$$

for all  $t$ , where  $\mathbf{v}_e(t)$  is an additive noise (*eventually* to be described as a sample from a white zero-mean random process with covariance kernel  $E\{\mathbf{v}_e(t)\mathbf{v}_e^T(t+\tau)\} = \mathbf{R}_e\delta(\tau)$ ). Since we have no dynamics driving noise, a complete state solution exists for any nominal value of  $\theta$ , once  $\mathbf{u}$  and  $\mathbf{x}(t_0)$  are specified. Thus, we can write  $\mathbf{y}(t) = \mathbf{h}[\theta, t] + \mathbf{v}_e(t)$ . The local identifiability of this system model has been related to being able to solve uniquely the least squares problem (neglecting  $\mathbf{v}_e(t)$  for the moment)

$$\min_{\delta\theta} \|\delta\mathbf{y} - \mathcal{H}\delta\theta\|^2 = \min_{\delta\theta} \langle [\delta\mathbf{y} - \mathcal{H}\delta\theta], [\delta\mathbf{y} - \mathcal{H}\delta\theta] \rangle_Y$$

where  $\mathcal{H}$  is the output sensitivity operator, which can be thought of as the partial derivative of  $\mathbf{h}$  with respect to  $\theta$  evaluated at the nominal value of the parameters,  $\theta_0$ . (It is also the "Frechet differential" of  $\mathbf{h}$  at  $\theta_0$ .)  $\langle \cdot, \cdot \rangle_Y$  is the inner product defined on the space for establishing "length" of the vectors  $\mathbf{y}(\cdot) \in Y$ . Thus, if a unique least squares solution  $\delta\theta$  can be found, i.e., if the "best" value of the small variation of  $\theta$  from  $\theta_0$  can be determined uniquely from the small variation of  $\mathbf{y}$  from the value  $\mathbf{y}_0$  that it would assume if  $\theta_0$  were the true parameter value, then the system is said to be locally identifiable. Under what conditions does such a unique solution exist?

To gain insights, consider this as a generalization to the unweighted least squares problem discussed at the end of Section 3.11 in Volume 1. There, outputs were elements of  $Y = R^m$ , with inner product defined as

$$\langle \mathbf{y}_1, \mathbf{y}_2 \rangle_{R^m} = \mathbf{y}_1^T \mathbf{y}_2$$

for all  $\mathbf{y}_1, \mathbf{y}_2 \in R^m$ . Here let us assume that  $Y$  is the space of square integrable functions that map the interval  $[t_0, t_f]$  into  $R^m$  (sometimes denoted as  $L_2([t_0, t_f], R^m)$ ), with associated inner product as

$$\langle \mathbf{y}_1(\cdot), \mathbf{y}_2(\cdot) \rangle_Y = \int_{t_0}^{t_f} \mathbf{y}_1(t)^T \mathbf{y}_2(t) dt$$

Moreover, simple premultiplication by a matrix  $\mathbf{H}$  and then by its transpose in the previous case, as in writing  $\mathbf{H}^T \mathbf{H} \hat{\mathbf{x}}^+ = \mathbf{H}^T \mathbf{y}$  in order to solve for  $\hat{\mathbf{x}}^+$  as  $[\mathbf{H}^T \mathbf{H}]^{-1} \mathbf{H}^T \mathbf{y}$ , are replaced by sequential applications of the operators  $\mathcal{H}$  and then its adjoint  $\mathcal{H}^*$ , to form the operator  $\mathcal{H}^* \mathcal{H}$ . Letting  $\Theta$  be  $R^p$  and recalling Problem 8.10e, we can define the adjoint  $\mathcal{H}^*$  as the mapping from  $Y = L_2([t_0, t_f], R^m)$  into  $\Theta = R^p$  that satisfies

$$\langle \mathcal{H} \delta\theta, \delta\mathbf{y} \rangle_Y = \langle \delta\theta, \mathcal{H}^* \delta\mathbf{y} \rangle_{R^p}$$

Explicitly show the form of the adjoint mapping  $\mathcal{H}^*$ , and use it to express the conditions under which the system model is locally identifiable.

(e) Now consider the generalized, or weighted, least squares problem,

$$\min_{\delta\theta} \|\delta\mathbf{y} - \mathcal{H} \delta\theta\|_{R_c^{-1}}^2 = \min_{\delta\theta} \langle [\delta\mathbf{y} - \mathcal{H} \delta\theta], [\delta\mathbf{y} - \mathcal{H} \delta\theta] \rangle_{R_c^{-1}}$$

where  $\mathcal{R}_c$  is a positive self-adjoint operator from the space  $Y$  back into  $Y$ , such that  $\mathcal{R}_c^{-1}$  exists and is self-adjoint. Thus the generalized inner product can be appropriately defined for any  $\mathbf{y}_1(\cdot)$ ,  $\mathbf{y}_2(\cdot) \in Y$  by

$$\langle \mathbf{y}_1, \mathbf{y}_2 \rangle_{R_c^{-1}} \triangleq \langle \mathbf{y}_1, \mathcal{R}_c^{-1} \mathbf{y}_2 \rangle_Y = \langle \mathcal{R}_c^{-1} \mathbf{y}_1, \mathbf{y}_2 \rangle_Y$$

due to the self-adjoint property. Obtain the solution to this generalized least squares problem.

The positive, self-adjoint operator  $(\mathcal{H}^* \mathcal{R}_c^{-1} \mathcal{H})$  that is inherent in this solution is a mapping from  $\Theta$  into  $\Theta$ , and it provides information about the quality of the estimate  $\delta\theta$ . Show that if  $\Theta$  is Euclidean  $p$ -space  $R^p$  and  $Y$  is  $L_2([t_0, t_f], R^m)$ , then this operator is representable as a  $p$ -by- $p$  matrix, recognized to be the *Fisher information matrix* at the point  $\theta_0$ . What are the conditions for local identifiability of the system model, in terms of this matrix?

# CHAPTER 11

## Nonlinear stochastic system models

### 11.1 INTRODUCTION

Suppose we are given a physical system that can be subjected to our own known controls and to inputs beyond our control, and that it can provide certain noise-corrupted measurements to us. Then the objective of a mathematical model associated with this real system is to generate an adequate, tractable representation of the behavior of all system outputs of interest. As stated earlier in Volume 1, adequacy and tractability are subjective and are functions of the intended use of the model. Our desire to develop estimators and controllers causes us to impose a certain structure upon our models. It has already been seen that linear system models driven by white Gaussian noise are not only descriptive of many observed phenomena, but also yield a basis of useful estimator algorithms. Controller design based on such a model is also tractable and useful practically, and this will be developed subsequently.

Nevertheless, there is a large and important class of problems for which linear stochastic system models are not adequate. In this chapter we consider extensions of this model to admit nonlinearities explicitly. Because we wish to base estimators and controllers upon the extended model, we will be motivated to preserve the Markov nature of the state stochastic process that had previously been obtained with linear dynamic system models. This motivation and the appropriate model structure are developed more fully in Section 11.2, and the fundamental characteristics of Markov processes are presented in Section 11.3. In order to develop nonlinear stochastic differential equations properly, stochastic integrals and differentials are first defined and characterized in Section 11.4. Then the subsequent section investigates the associated stochastic differential equations. Since the solution processes for these equations are Markov, their characteristics are described completely by transition probability densities,

and the time propagation of these important densities is the subject of the last section. The stochastic process models of this chapter will be exploited by nonlinear filter and controller derivations in the sequel.

## 11.2 EXTENSIONS OF LINEAR SYSTEM MODELING

Recall the results of the linear system modeling efforts of Chapter 4 (Volume 1). There we attempted to exploit a model of the form of linear state dynamics driven only by known inputs and white Gaussian noise

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t) \quad (11-1)$$

starting from a Gaussian  $\mathbf{x}(t_0)$  with known mean  $\hat{\mathbf{x}}_0$  and covariance  $\mathbf{P}_0$ , along with a linear measurement corrupted by additive white Gaussian noise of either the discrete-time form

$$\mathbf{z}(t_i) = \mathbf{H}(t_i)\mathbf{x}(t_i) + \mathbf{v}(t_i) \quad (11-2a)$$

or of the continuous-time form

$$\mathbf{z}(t) = \mathbf{H}(t)\mathbf{x}(t) + \mathbf{v}_c(t) \quad (11-2b)$$

The noise processes in (11-1) and (11-2) were assumed independent of  $\mathbf{x}(t_0)$  and (at least initially) independent of each other, with mean of zero and correlation kernels given by

$$E\{\mathbf{w}(t)\mathbf{w}^T(t + \tau)\} = \mathbf{Q}(t)\delta(\tau) \quad (11-3)$$

and either

$$E\{\mathbf{v}(t_i)\mathbf{v}^T(t_j)\} = \mathbf{R}(t_i)\delta_{ij} \quad (11-4a)$$

or

$$E\{\mathbf{v}_c(t)\mathbf{v}_c^T(t + \tau)\} = \mathbf{R}_c(t)\delta(\tau) \quad (11-4b)$$

White noise inputs were justified on the basis that desired time correlation properties of physically observed phenomena could be produced sufficiently well by passing white Gaussian noise through a linear shaping filter, which could then be augmented to the basic state model to yield an overall model of the form above.

As intuitive and physically motivated as this model seemed, it was not suitable because the solution to (11-1) could not be developed properly using ordinary Riemann integrals. Linear stochastic differential equations and their solutions were properly developed through (Wiener) stochastic integrals and Brownian motion. Brownian motion (or the Wiener process) is a zero-mean vector process  $\beta(\cdot, \cdot)$  that has independent Gaussian increments with

$$E\{[\beta(t_2) - \beta(t_1)][\beta(t_2) - \beta(t_1)]^T\} = \int_{t_1}^{t_2} \mathbf{Q}(t) dt \quad (11-5)$$

that was shown to be continuous but nondifferentiable (with probability one or “almost surely,” and in the mean square sense). Its hypothetical derivative would be  $\mathbf{w}(\cdot, \cdot)$  that appears in (11-1). A (*Wiener*) *stochastic integral* [10, 43, 44, 71]

$$\mathbf{l}(t, \cdot) = \int_{t_0}^t \mathbf{A}(\tau) d\beta(\tau, \cdot) \quad (11-6)$$

could then be defined for known (nonrandom)  $\mathbf{A}(\cdot)$  by means of a mean square limit

$$\begin{aligned} \mathbf{l}(t, \cdot) &= \text{l.i.m.}_{N \rightarrow \infty} \int_{t_0}^t \mathbf{A}_N(\tau) d\beta(\tau, \cdot) \\ &= \text{l.i.m.}_{N \rightarrow \infty} \sum_{i=0}^{N-1} \mathbf{A}_N(t_i) [\beta(t_{i+1}, \cdot) - \beta(t_i, \cdot)] \end{aligned} \quad (11-7)$$

where  $N$  is the number of time cuts made in  $[t_0, t]$  and  $\mathbf{A}_N(\tau) = \mathbf{A}(t_i)$  for all  $\tau \in [t_i, t_{i+1}]$ . Viewed as a stochastic process, this stochastic integral is itself a Brownian motion process with rescaled diffusion: Gaussian, zero-mean, with

$$E\{\mathbf{l}(t_2) - \mathbf{l}(t_1)\}[\mathbf{l}(t_2) - \mathbf{l}(t_1)]^T = \int_{t_1}^{t_2} \mathbf{A}(\tau) \mathbf{Q}(\tau) \mathbf{A}^T(\tau) d\tau \quad (11-8)$$

*Stochastic differentials* are properly defined as functions which, when integrated over appropriate limits, yield the associated stochastic integrals: given (11-6), the stochastic differential  $d\mathbf{l}(t, \cdot)$  of  $\mathbf{l}(t, \cdot)$  is

$$d\mathbf{l}(t, \cdot) = \mathbf{A}(t) d\beta(t, \cdot) \quad (11-9)$$

Thus, the properly written linear stochastic differential equation directly related to (11-1) is

$$d\mathbf{x}(t) = \mathbf{F}(t)\mathbf{x}(t) dt + \mathbf{B}(t)\mathbf{u}(t) dt + \mathbf{G}(t) d\beta(t) \quad (11-10)$$

where  $\beta(\cdot, \cdot)$  is of diffusion strength  $\mathbf{Q}(t)$  for all  $t$  of interest, as given by (11-5) or equivalently as

$$E\{d\beta(t) d\beta^T(t)\} = \mathbf{Q}(t) dt \quad (11-11)$$

The solution to (11-10) is the stochastic process  $\mathbf{x}(\cdot, \cdot)$  given by

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^t \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau) d\tau + \int_{t_0}^t \Phi(t, \tau)\mathbf{G}(\tau) d\beta(\tau) \quad (11-12)$$

with  $\Phi$  the state transition matrix associated with  $\mathbf{F}$ . To characterize a stochastic process completely would generally entail specification of the joint probability density (or distribution if the density cannot be assumed to exist) of  $\mathbf{x}(t_1), \mathbf{x}(t_2), \dots, \mathbf{x}(t_N)$  for any number  $N$  of time cuts in the interval  $T = [t_0, t_f]$  of interest:

$$f_{\mathbf{x}(t_1), \dots, \mathbf{x}(t_N)}(\xi_1, \dots, \xi_N) = \prod_{j=1}^N f_{\mathbf{x}(t_j)|\mathbf{x}(t_{j-1}), \dots, \mathbf{x}(t_1)}(\xi_j | \xi_{j-1}, \dots, \xi_1) \quad (11-13)$$

by repeated application of Bayes' rule. However, because  $\mathbf{x}(\cdot, \cdot)$  given by (11-12) can be shown to be a *Markov* process, each term in the product in (11-13) is equivalent to  $f_{\mathbf{x}(t_j)|\mathbf{x}(t_{j-1})}(\xi_j | \xi_{j-1})$ , so specification of these *transition probability densities* completely specifies the needed joint density. Thus, for Markov processes, conditional distributions of the form  $F_{\mathbf{x}(t)|\mathbf{x}(t')}(\xi | \xi')$ , or second order joint distributions of the form  $F_{\mathbf{x}(t), \mathbf{x}(t')}(\xi, \xi')$  from which these conditional distributions can be derived via Bayes' rule, are of primary importance because they *totally* describe the process characteristics. When the density functions  $f_{\mathbf{x}(t)|\mathbf{x}(t')}(\xi | \xi')$  and  $f_{\mathbf{x}(t), \mathbf{x}(t')}(\xi, \xi')$  exist, they similarly fulfill this objective.

Moreover, since  $\mathbf{x}(\cdot, \cdot)$  given by (11-12) is *Gauss–Markov*, this complete process specification can also be generated entirely in terms of the mean function and covariance kernel (function): the first two moments of  $f_{\mathbf{x}(t), \mathbf{x}(t')}(\xi, \xi')$ , given by

$$\begin{aligned}\mathbf{m}_x(t) &\triangleq E\{\mathbf{x}(t)\} \\ &= \Phi(t, t_0)\mathbf{m}_x(t_0) + \int_{t_0}^t \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)d\tau\end{aligned}\quad (11-14)$$

$$\mathbf{P}_{xx}(t', t) \triangleq E\{[\mathbf{x}(t') - \mathbf{m}_x(t')][\mathbf{x}(t) - \mathbf{m}_x(t)]^T\}$$

$$= \begin{cases} \Phi(t', t)\mathbf{P}_{xx}(t) & t' \geq t \\ \mathbf{P}_{xx}(t')\Phi^T(t, t') & t' \leq t \end{cases} \quad (11-15a)$$

$$(11-15b)$$

where the covariance matrix  $\mathbf{P}_{xx}(t)$  is given by

$$\begin{aligned}\mathbf{P}_{xx}(t) &\triangleq E\{[\mathbf{x}(t) - \mathbf{m}_x(t)][\mathbf{x}(t) - \mathbf{m}_x(t)]^T\} \\ &= \Phi(t, t_0)\mathbf{P}_0\Phi^T(t, t_0) + \int_{t_0}^t \Phi(t, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^T(\tau)\Phi^T(t, \tau)d\tau\end{aligned}\quad (11-16)$$

Note that the last term in (11-12) contributes zero to the mean in (11-14) and the integral term as given previously in Eq. (11-8) to the covariance matrix in (11-16). Equations (11-14) and (11-16) are the solutions to the ordinary differential equations

$$\dot{\mathbf{m}}_x(t) = \mathbf{F}(t)\mathbf{m}_x(t) + \mathbf{B}(t)\mathbf{u}(t) \quad (11-17)$$

$$\dot{\mathbf{P}}_x(t) = \mathbf{F}(t)\mathbf{P}_x(t) + \mathbf{P}_x(t)\mathbf{F}^T(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t) \quad (11-18)$$

Together, these two equations define the propagation of the Gaussian transition probability density  $f_{\mathbf{x}(t)|\mathbf{x}(t')}(\xi | \xi')$  with time  $t$ , for fixed  $t' \leq t$ , and also of the unconditional density  $f_{\mathbf{x}(t)}(\xi)$ . Even if  $\mathbf{x}(\cdot, \cdot)$  were not Gaussian, these two moments would provide useful information about the central location and spread of possible sample values about that central location for all time, though not a complete depiction.

The preceding results were developed properly on the basis of the stochastic integral and differential defined in (11-7) and (11-9). However, it was also shown that these stochastic differentials obey the formal rules associated with deterministic functions, and similarly that all of the foregoing results could be

obtained formally by white noise interpretations and integrals of the (improper) form  $\int_{t_0}^t \mathbf{A}(\tau) \mathbf{w}(\tau) d\tau$ . Furthermore, the same results can also be generated properly with other meaningful choices of the stochastic integral, as for example

$$\mathbf{l}(t, \cdot) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{i=0}^{N-1} \mathbf{A}_N \left( \frac{t_i + t_{i+1}}{2} \right) [\beta(t_{i+1}, \cdot) - \beta(t_i, \cdot)] \quad (11-19)$$

instead of (11-7). However, as we consider generalizations of the definitions of stochastic integrals, it is important not to expect all definitions to yield identical results and properties, nor to assume that familiar formal rules will be satisfied.

Note that, just as (11-10) replaces the formal differential equation (11-1), the continuous-time measurement process (11-2b) is properly replaced by

$$\mathbf{d}\mathbf{y}(t) = \mathbf{H}(t)\mathbf{x}(t) dt + \mathbf{d}\beta_m(t) \quad (11-20)$$

Here,  $\mathbf{d}\beta_m(t)/dt$  is formally interpreted as  $\mathbf{v}_c(t)$ , and  $\mathbf{d}\mathbf{y}(t)/dt$  as  $\mathbf{z}(t)$ .

We now wish to extend this model to a wider class of problems, but we still wish to do so in a fruitful manner. Although we will be forced to relinquish Gaussianity once nonlinearities are introduced, it will nonetheless be useful (from the standpoints of both providing good descriptions of phenomena and yielding tractable mathematics for estimator and controller development) to so construct this extension as to ensure generation of Markov processes.

One way in which the previous results can be extended is to consider a more generalized driving process on (11-10) than Brownian motion [22, 45, 63]. To preserve the Markov nature of  $\mathbf{x}(\cdot, \cdot)$ , the driving process should have independent increments that are independent of the initial condition (conventionally set as  $\mathbf{x}(t_0) = \mathbf{0}$  with probability one), but the increments need not be Gaussian. *Poisson processes* fit this characterization [23, 25, 56, 63]. A *homogeneous Poisson process* is a counting process  $n_p(\cdot, \cdot)$ , whose components are scalar processes  $n_{pi}(\cdot, \cdot)$

- (1) that have independent increments;
- (2) all of whose sample functions (except possibly for a set of probability zero) start at zero and are step functions with jump value of unity at each discontinuity;
- (3) whose steps occur at Poisson intervals: the probability that the number of jumps between time  $t$  and  $t'$  equals  $k$  is

$$P[n_{pi}(t') - n_{pi}(t) = k] = \frac{\lambda_i^k (|t' - t|)^k}{k!} \exp\{-\lambda_i |t' - t|\} \quad (11-21)$$

where  $\lambda_i > 0$  is the rate parameter for the  $i$ th component; i.e.,

$$P[n_{pi}(t + dt) - n_{pi}(t) = 1] = \lambda_i dt, \quad P[n_{pi}(t + dt) - n_{pi}(t) = 0] = 1 - \lambda_i dt$$

(4) with discontinuities of the first kind; i.e., if  $t_d$  is a point of discontinuity, then

$$n_{Pi}(t_d - \varepsilon, \omega_j) \neq n_{Pi}(t_d, \omega_j) = n_{Pi}(t_d + \varepsilon, \omega_j)$$

for arbitrarily small  $\varepsilon$ .

*Nonhomogeneous Poisson processes* have time-varying rate parameters  $\lambda_i(t)$  associated with each component, and then (11-21) is modified to

$$P[n_{Pi}(t') - n_{Pi}(t) = k] = \frac{[\int_t^{t'} \lambda_i(\tau) d\tau]^k}{k!} \exp\left\{-\int_t^{t'} \lambda_i(\tau) d\tau\right\} \quad (11-22)$$

If the jump magnitudes are not unity and positive but the realizations of random variables, then a *generalized Poisson process* is created. To be specific, consider the  $i$ th component of a simple Poisson process  $n_p(\cdot, \cdot)$ ,  $n_{Pi}(\cdot, \cdot)$  as described previously, and let

$$n_{GPi}(t) = \sum_{j=1}^{n_{Pi}(t)} a_{ij} u_{-1}(t - t_{ij}) \quad (11-23)$$

where the random variables  $a_{ij}$  are independent of  $n_p(\cdot, \cdot)$  and described by a given joint distribution, and  $u_{-1}(t - t_{ij})$  is the unit step function initiating at the  $j$ th arrival time of the Poisson counting process  $n_{Pi}(\cdot, \cdot)$ .

Of particular interest is the scalar case of (11-23) in which the amplitudes  $a_{ij}$  are independent, identically distributed, zero-mean Gaussian random variables with variance  $\sigma_a^2$  and in which  $n_p(\cdot, \cdot)$  is homogeneous. It can be shown [45] that the autocorrelation function of  $n_{GP}(\cdot, \cdot)$  is then

$$E\{n_{GP}(t)n_{GP}(t')\} = \lambda \sigma_a^2 t \quad t \leq t' \quad (11-24)$$

which is identical to the autocorrelation of a Brownian motion. Moreover, the hypothetical derivative of  $n_{GP}(\cdot, \cdot)$ , a process formally composed of Poisson-distributed impulses weighted by Gaussian-distributed coefficients, has an autocorrelation of

$$E\left\{\frac{dn_{GP}(t)}{dt} \frac{dn_{GP}(t')}{dt'}\right\} = \lambda \sigma_a^2 \delta(t - t') \quad (11-25)$$

i.e., it is a zero-mean stationary white noise process with Gaussian amplitudes (though not itself a Gaussian process) with flat power spectral density (thereby able to excite all system modes uniformly). In other words, to second moment properties,  $n_{GP}(\cdot, \cdot)$  is identical to scalar constant-diffusion Brownian motion. Direct extensions to the vector and nonstationary cases are also possible.

A more conclusive statement can be made about the character of an independent increment process  $\mathbf{x}(\cdot, \cdot)$ . First, if  $\mathbf{x}(\cdot, \cdot)$  is continuous in time w.p.1 (for almost all  $\omega$ ), then the increments  $[\mathbf{x}(t + \Delta t, \cdot) - \mathbf{x}(t, \cdot)]$  are Gaussian. Second, if almost all sample paths start from zero and have discontinuities of the first kind that are not fixed (i.e., the limit of  $P[|\mathbf{x}(t) - \mathbf{x}(t')| > 0]$  is zero

as  $t \rightarrow t'$ ) and have jump magnitude of plus one, then the increments are Poisson [45]. Thus, given an arbitrary independent increment process, a decomposition into Gaussian (continuous) and Poisson (discontinuous) parts may be possible [25, 30, 45]. It is then natural to consider a generalization of the stochastic differential equation (11-10) to the form:

$$\mathbf{dx}(t) = \mathbf{F}(t)\mathbf{x}(t)dt + \mathbf{B}(t)\mathbf{u}(t)dt + \mathbf{G}_1(t)d\beta(t) + \mathbf{G}_2(t)d\mathbf{n}_{GP}(t) \quad (11-26)$$

Poisson processes (or conditionally Poisson processes, conditioned on the rate function) have been used as models for the outputs of photon detectors (as, in optical communication systems) and other applications in which discrete-point discontinuities (as, the increasing number of photoconversions in time) is an inherent, important characteristic of the physical phenomenon. In fact, the complete model for optical detectors includes both point processes (detector output due to signal, “dark current,” etc.) and Gaussian noise processes (thermal noise, etc.). Snyder and others have developed rather extensive point process models and estimator and controller results based upon them [15, 26, 35, 45, 58, 59, 61–65]. On the other hand, we will focus upon the Gaussian process case. To date, the mixed process case is essentially intractable as far as estimator and controller development, and most designs are based on either one model type or the other, or they use models adequate to second moments only (based on insights from the Gaussian noise model) to treat the mixed process case. Thus, although (11-26) indicates a useful extension to (11-10), we will not pursue it further in this text.

A second extension is to admit nonlinearities in the system dynamics and measurement equations, while retaining the linear additivity of the driving uncertainties, as modifying (11-10) to the more general form

$$\mathbf{dx}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t]dt + \mathbf{G}(t)d\beta(t) \quad (11-27a)$$

which could be interpreted heuristically as

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] + \mathbf{G}(t)\mathbf{w}(t) \quad (11-27b)$$

again treating white Gaussian noise  $\mathbf{w}(\cdot, \cdot)$  as the hypothetical derivative of Brownian motion  $\beta(\cdot, \cdot)$ . Similarly, discrete-time measurements would be modeled as

$$\mathbf{z}(t_i) = \mathbf{h}[\mathbf{x}(t_i), t_i] + \mathbf{v}(t_i) \quad (11-28)$$

and continuous-time measurements as

$$\mathbf{dy}(t) = \mathbf{h}[\mathbf{x}(t), t]dt + d\beta_m(t) \quad (11-29a)$$

or heuristically as

$$\mathbf{z}(t) = \mathbf{h}[\mathbf{x}(t), t] + \mathbf{v}_c(t) \quad (11-29b)$$

By forcing (11-27) with open-loop (deterministically prespecified) control  $\mathbf{u}(\cdot)$  and Brownian motion  $\beta(\cdot, \cdot)$  with independent increments, the state

process can readily be shown to be Markov (this will be accomplished later as a special case of a more general model form). Later, when feedback control is introduced, we will have to return to this issue. If  $\mathbf{u}$  were a memoryless function of state value, there would be no concern, but perfect knowledge of the entire state vector is not typically available to generate a feedback control.

Note that the preceding model was the assumed form upon which the extended Kalman filter of Chapter 9 was based. It is, in fact, descriptive of many realistic problems of interest. For instance, nonlinear system models driven by additive white Gaussian and time-correlated Gaussian noises, generated as the outputs of linear shaping filters driven by white Gaussian noise, are widely applicable.

Nevertheless, useful extensions can be made beyond this model form. Specifically, a nonlinear stochastic differential equation of the form

$$\mathbf{dx}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), t] dt + \mathbf{G}[\mathbf{x}(t), t] d\beta(t) \quad (11-30)$$

with  $\mathbf{G}[\cdot, \cdot]$  now allowed to be a function of  $\mathbf{x}(\cdot, \cdot)$  as well as time, and  $\beta(\cdot, \cdot)$  being Brownian motion ("Wiener process") as before, is motivated. As previously accomplished in Chapter 4 (Volume 1) for the linear case, to give proper meaning to (11-30) requires a proper definition of a stochastic integral, now of the form

$$\mathbf{I}(t, \cdot) = \int_{t_0}^t \mathbf{A}(\tau, \cdot) d\beta(\tau, \cdot) \quad (11-31)$$

where  $\mathbf{A}(\cdot, \cdot)$  denotes a matrix of stochastic processes such that  $\mathbf{A}(t, \cdot)$  depends at most on the past and present values of  $\beta(\cdot, \cdot)$ , i.e.,  $\{\beta(t', \cdot), t_0 \leq t' \leq t\}$ , but is independent of future values of  $\beta(\cdot, \cdot)$ . Alternative definitions of such a stochastic integral are possible, in analogy with (11-7) and (11-19), yielding results with different characteristics. One in particular, the *Itô stochastic integral* [27, 30], possesses useful properties not shared by other definitions. One primary result is that, if Itô stochastic integrals are used to give meaning to (11-30), then the solution process  $\mathbf{x}(\cdot, \cdot)$  will be *Markov*. Because of this and some other pertinent properties, attention will be concentrated upon the Itô definition of stochastic integrals and stochastic differential equations. Using such a definition, certain formal rules such as the chain rule for differentials will not be obeyed, unlike the case of Wiener stochastic integrals as defined in (11-6). Subtle modeling issues are raised due to this fact, and these will be discussed in the sequel.

Models more general than (11-30) might be proposed, such as the heuristic equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t), t] \quad (11-32)$$

However, such a model would be especially difficult to develop rigorously. Moreover, even if the solution process  $\mathbf{x}(\cdot, \cdot)$  existed and could be characterized,

it generally would *not be Markov*. Because Markov process models are significantly more tractable analytically than more general processes, combined with the fact that they do provide adequate models of physically observed phenomena, we will confine our attention to the generality of (11-30) and (11-31).

### 11.3 MARKOV PROCESS FUNDAMENTALS

In this section, the basic characterization of Markov processes [3, 10, 11, 23, 30, 56] is developed. The natural importance of transition probabilities and densities, and the Chapman–Kolmogorov equation associated with them, is a central thought throughout.

Let  $\mathbf{x}(\cdot, \cdot)$  be a vector stochastic process, and consider

$$F_{\mathbf{x}(t_i)|\mathbf{x}(t_{i-1}), \mathbf{x}(t_{i-2}), \dots, \mathbf{x}(t_j)}(\xi_i | \xi_{i-1}, \xi_{i-2}, \dots, \xi_j)$$

the conditional probability distribution of  $\mathbf{x}(t_i, \cdot)$  as a function of the  $n$ -dimensional vector  $\xi_i$ , given that  $\mathbf{x}(t_{i-1}, \omega_k) = \xi_{i-1}$ ,  $\mathbf{x}(t_{i-2}, \omega_k) = \xi_{i-2}, \dots, \mathbf{x}(t_j, \omega_k) = \xi_j$ . If, for any countable choice of values of  $i$  and  $j$ , it is true that

$$F_{\mathbf{x}(t_i)|\mathbf{x}(t_{i-1}), \mathbf{x}(t_{i-2}), \dots, \mathbf{x}(t_j)}(\xi_i | \xi_{i-1}, \xi_{i-2}, \dots, \xi_j) = F_{\mathbf{x}(t_i)|\mathbf{x}(t_{i-1})}(\xi_i | \xi_{i-1}) \quad (11-33)$$

then  $\mathbf{x}(\cdot, \cdot)$  is a *Markov process*. Thus, the Markov property for stochastic processes is conceptually analogous to the ability to define a system state for deterministic systems: the value of  $\mathbf{x}$  at time  $t_{i-1}$  provides as much information about  $\mathbf{x}(t_i)$  as do the values of  $\mathbf{x}$  at time  $t_{i-1}$  and all previous time instants. Said another way, the time history leading up to  $\mathbf{x}(t_{i-1})$  is of no consequence, and only knowledge of  $\mathbf{x}(t_{i-1})$  itself is required for propagation to the future time  $t_i$ .

There are also Markov-2, Markov-3, ..., Markov- $N$  processes, where for instance a Markov-2 process is such that

$$F_{\mathbf{x}(t_i)|\mathbf{x}(t_{i-1}), \mathbf{x}(t_{i-2}), \dots, \mathbf{x}(t_j)}(\xi_i | \xi_{i-1}, \xi_{i-2}, \dots, \xi_j) = F_{\mathbf{x}(t_i)|\mathbf{x}(t_{i-1}), \mathbf{x}(t_{i-2})}(\xi_i | \xi_{i-1}, \xi_{i-2}) \quad (11-34)$$

Such processes were encountered in conjunction with smoothing problems in Chapter 8. However, if  $\mathbf{x}(\cdot, \cdot)$  is an  $n$ -dimensional Markov-2 process, then the  $2n$ -dimensional process defined at each  $t_i$  by

$$\mathbf{y}(t_i, \cdot) = \begin{bmatrix} \mathbf{x}(t_i, \cdot) \\ \mathbf{x}(t_{i-1}, \cdot) \end{bmatrix} \quad (11-35)$$

is Markov-1. Obvious extensions to Markov- $N$  processes can be made. Thus, we will consider only Markov-1 processes when we speak of a Markov process.

A Markov process  $\mathbf{x}(\cdot, \cdot)$  can evolve as a continuous process in time or as a sequence of random variables at discrete points in time. Also, its probability distribution function can be continuous or have discrete discontinuities. To

start simply, we shall consider the case of discrete probabilities and discrete transition times. Once some fundamental concepts are established for this case, we shall extend to both continuous-state and continuous-time models.

**EXAMPLE 11.1** Consider tossing a coin at discrete time instants  $t_1, t_2, t_3, \dots$ , and let  $x(\cdot, \cdot)$  be the stochastic process such that  $x(t_i, \cdot)$  is a discrete-valued random variable describing the number of heads obtained in  $i$  tosses of the coin. Then  $F_{x(t_i)|x(t_{i-1})}(\xi|\rho)$  is, by definition, the probability (i.e., of the set of  $\omega \in \Omega$ ) that the number of heads to appear in  $i$  tosses is less than or equal to  $\xi$ , given that  $\rho$  heads have occurred in  $i - 1$  tosses. Let  $\rho$  take on an integer value. For  $\xi < \rho$ , this is the probability that less than  $\rho$  heads occur in  $i$  tosses, given that  $\rho$  heads have occurred in  $i - 1$  tosses, and so  $F_{x(t_i)|x(t_{i-1})}(\xi|\rho) = 0$  for this case. For  $\xi$  in the interval  $[\rho, \rho + 1]$ ,  $F_{x(t_i)|x(t_{i-1})}(\xi|\rho) = \frac{1}{2}$ , since it is the probability that less than  $\rho$  heads or  $\rho$  (integer) heads occur on  $i$  tosses, given that  $\rho$  heads occurred on  $i - 1$  tosses, i.e., the probability of a tail being thrown at time  $t_i$ . Finally, for  $\xi \geq \rho + 1$ ,  $F_{x(t_i)|x(t_{i-1})}(\xi|\rho) = 1$ : the probability that  $\rho$  or more heads occur in  $i$  tosses, knowing  $\rho$  heads occurred in  $i - 1$  tosses. Moreover, these evaluations are true regardless of what occurred prior to time  $t_{i-1}$ , so  $x(\cdot, \cdot)$  is Markov. ■

**EXAMPLE 11.2** A graphic example of a process modeled as Markov is given by the description of a frog jumping from one lily pad to another in a pond. There are only a finite number of "state" values in the description of the "system," i.e., the index number of the pad currently occupied by the frog. The frog's leap then constitutes a state transition that occurs at discrete points in time. We can think of this as a discrete-time process by indexing the transitions in time.

To study such a process, the probabilistic nature of the state transition must be specified. Generally, the probability that pad  $j$  is occupied at time  $t_{i+1}$ , given that pad  $k$  is occupied at time  $t_i$ , will be dependent on the value of  $k$ ; e.g., the more distant pad  $j$  is from pad  $k$ , the lower the probability value. If that probability is independent of how the frog got to pad  $k$ , i.e., of the previous state history, then the process is Markov.

One can then describe the probability that any particular state value (lily pad) is assumed at a given point in time. This can be generated after a specified number of transitions from an initial state, which itself might be described only probabilistically. ■

The previous examples introduced the concept of a *transition probability model*. To generalize, suppose that there are  $N$  possible discrete state values which a system of interest can assume at any given time, and let  $x(\cdot, \cdot)$  be a scalar discrete-valued state process that can assume integer values (index numbers,  $j$ ) from 1 to  $N$  at each discrete time  $t_i$ . Associated with each state value  $j$ , one can define a state probability  $p_j(t_i)$ , the probability that the system will be in the state  $j$  at time  $t_i$ . These separate state probabilities can then be arrayed in a vector  $\mathbf{p}(t_i)$  as

$$\mathbf{p}(t_i) = \begin{bmatrix} p_1(t_i) \\ p_2(t_i) \\ \vdots \\ p_N(t_i) \end{bmatrix} = \begin{bmatrix} P(\{\omega: x(t_i, \omega) = 1\}) \\ P(\{\omega: x(t_i, \omega) = 2\}) \\ \vdots \\ P(\{\omega: x(t_i, \omega) = N\}) \end{bmatrix} \quad (11-36)$$

Note that, by its definition, the sum of the  $N$  components of  $\mathbf{p}(t_i)$  equals one for any  $t_i$ :

$$\sum_{j=1}^N p_j(t_i) = 1 \quad (11-37)$$

If the system of interest has the Markov property, then the probability of a transition to state  $j$  by the next discrete time of interest, given that the system now occupies state  $k$ , is a function of  $j$  and  $k$  and not of any history of the system before its arrival at state  $k$ . Therefore, we can define the *state transition probabilities* for a discrete-state Markov process as

$$T_{jk}(t_{i+1}, t_i) = P\{\mathbf{x}(t_{i+1}) = j | \mathbf{x}(t_i) = k\} \quad (11-38)$$

i.e.,  $T_{jk}(t_{i+1}, t_i)$  is the conditional probability that the system will be in state  $j$  at the next time instant  $t_{i+1}$ , given that the state at present time  $t_i$  is  $k$ . Since the  $N$  state values are assumed to be mutually exclusive and collectively exhaustive, it must be true that

$$\sum_{j=1}^N T_{jk}(t_{i+1}, t_i) = 1 \quad k = 1, 2, \dots, N \quad (11-39)$$

which simply states, given that the state is  $k$  at time  $t_i$ , it must assume *some* value at time  $t_{i+1}$ .

Now the state transition probabilities can be used to compose the *state transition probability matrix*  $\mathbf{T}(t_{i+1}, t_i)$ :

$$\mathbf{T}(t_{i+1}, t_i) = \begin{bmatrix} T_{11}(t_{i+1}, t_i) & T_{12}(t_{i+1}, t_i) & \cdots & T_{1N}(t_{i+1}, t_i) \\ T_{21}(t_{i+1}, t_i) & T_{22}(t_{i+1}, t_i) & \cdots & T_{2N}(t_{i+1}, t_i) \\ \vdots & \vdots & & \vdots \\ T_{N1}(t_{i+1}, t_i) & T_{N2}(t_{i+1}, t_i) & \cdots & T_{NN}(t_{i+1}, t_i) \end{bmatrix} \quad (11-40)$$

Note that each column  $k$  depicts the probabilities of transitioning into state 1, state 2, ..., state  $N$  from any given state  $k$ , and in view of (11-39), each of these columns must sum to one. Each row  $j$  relates the probabilities of reaching state  $j$  from state 1, state 2, ..., state  $N$ . Note that the elements of the matrix can be functions of time:  $\mathbf{T}(t_{i+1}, t_i)$  need not be time invariant. With (11-40), the vector of state probabilities at time  $t_{i+1}$ ,  $\mathbf{p}(t_{i+1})$ , can be expressed in terms of the state probabilities at time  $t_i$ ,  $\mathbf{p}(t_i)$ , as

$$\mathbf{p}(t_{i+1}) = \mathbf{T}(t_{i+1}, t_i)\mathbf{p}(t_i) \quad (11-41)$$

The  $j$ th component of this equation is

$$p_j(t_{i+1}) = \sum_{k=1}^N T_{jk}(t_{i+1}, t_i)p_k(t_i) \quad (11-42a)$$

or

$$P\{\mathbf{x}(t_{i+1}) = j\} = \sum_{k=1}^N P\{\mathbf{x}(t_{i+1}) = j | \mathbf{x}(t_i) = k\}P\{\mathbf{x}(t_i) = k\} \quad (11-42b)$$

Thus, it is readily seen that there are  $N$  possible ways  $\mathbf{x}(t_{i+1})$  could have reached the value  $j$ : by means of a state transition to state  $j$  from any one of  $N$  states

at time  $t_i$ . Moreover, (11-41) can be applied recursively to obtain

$$\mathbf{p}(t_i) = \mathbf{T}(t_i, t_{i-1}) \cdots \mathbf{T}(t_2, t_1) \mathbf{T}(t_1, t_0) \mathbf{p}(t_0) \quad (11-43)$$

If the state transition probability matrix is time invariant, so that  $\mathbf{T}(t_{i+1}, t_i) = \mathbf{T} = \text{const}$  for all  $i$ , then this simplifies to

$$\mathbf{p}(t_i) = \mathbf{T}^i \mathbf{p}(t_0) \quad (11-43')$$

Sometimes a graphical model known as a transition probability diagram is employed to aid problem formulation. Figure 11.1 depicts such a diagram for a system composed of three possible states. For instance, to get to state 1 at time  $t_{i+1}$ , one could currently be in state 1 and undergo no state transition, with probability of occurrence  $T_{11}(t_{i+1}, t_i)$ ; or one could be in state 2 at time  $t_i$  and undergo a state transition into state 1, with probability  $T_{21}(t_{i+1}, t_i)$ ; or finally one could be in state 3 and transition into state 1, with probability  $T_{31}(t_{i+1}, t_i)$ . By knowing the nine state transition probabilities depicted in the figure, i.e., knowing the 3-by-3 state transition probability matrix  $\mathbf{T}(t_{i+1}, t_i)$ , and also knowing the probabilities of being in states 1, 2, and 3 at some initial time, one can project the probability of being in any particular state or states at any point in time by propagating the difference equation (11-41).

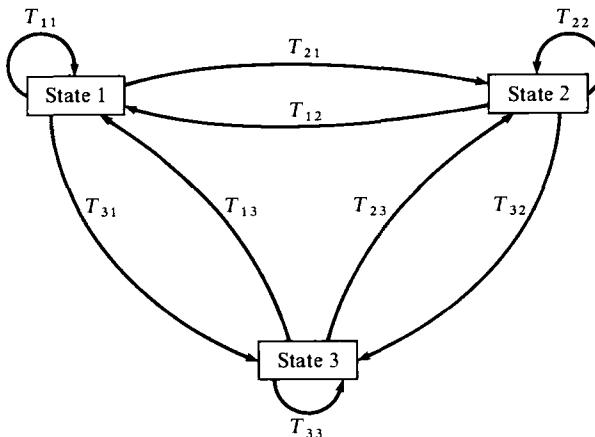


FIG. 11.1 Transition probability diagram.

Now consider an  $M$ -step state transition probability  $T_{jk}(t_{i+M}, t_i)$ , which is the conditional probability that the process will be in state  $j$  at time  $t_{i+M}$  after  $M$  transition time instants, given that the present state of the process at time  $t_i$  is  $k$ :

$$T_{jk}(t_{i+M}, t_i) = P\{\mathbf{x}(t_{i+M}) = j | \mathbf{x}(t_i) = k\} \quad (11-44)$$

Let  $t_{i+I}$  be some intermediate time instant between  $t_i$  and  $t_{i+M}$ , i.e., with  $I < M$ . One can then write

$$T_{jk}(t_{i+M}, t_i) = \sum_{l=1}^N T_{jl}(t_{i+M}, t_{i+I})T_{lk}(t_{i+I}, t_i) \quad (11-45)$$

This states that the conditional probability that the system will be in state  $j$  at time  $t_{i+M}$ , given that it is in state  $k$  at time  $t_i$ , is equal to the summation (over intermediate state value  $l$ ) of the  $N$  possible terms formed as the products of [the probability that the system will transition from state  $k$  at time  $t_i$  to state  $l$  at time  $t_{i+I}$ ] and [the probability that it will transition from that state  $l$  at time  $t_{i+I}$  to state  $j$  at time  $t_{i+M}$ ]. This is the discrete-state version of the *Chapman–Kolmogorov equation* for Markov processes; it and its continuous-state analog will be of significant value subsequently for nonlinear estimator formulations. Note that (11-45) is also the  $jk$ th element of the relation

$$\mathbf{T}(t_{i+M}, t_i) = \mathbf{T}(t_{i+M}, t_{i+I})\mathbf{T}(t_{i+I}, t_i) \quad (11-46)$$

i.e., the state transition probability matrix (like the state transition matrix  $\Phi(t_i, t_j)$  associated with linear deterministic state models of dynamic systems) has the semigroup property: transitioning from  $t_i$  to  $t_{i+M}$  can be achieved as a transition from  $t_i$  to  $t_{i+I}$  and then from there to  $t_{i+M}$ . This property is inherent in the solution form (11-43).

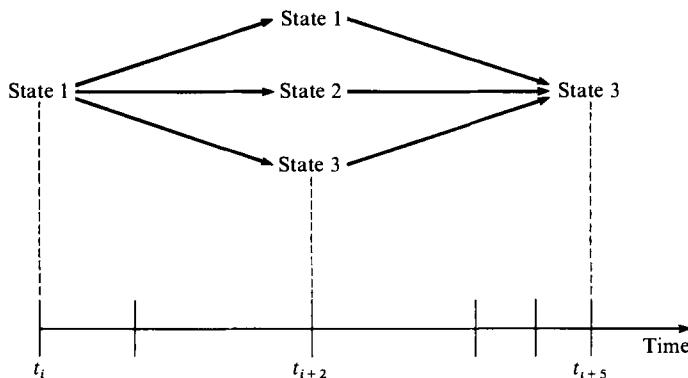
**EXAMPLE 11.3** Recall the 3-state transition probability diagram depicted in Fig. 11.1. Consider the term  $T_{31}(t_{i+5}, t_i)$ , the conditional probability that the process will be in state 3 at time  $t_{i+5}$ , given that the present state at time  $t_i$  is state 1. Then, by (11-45),

$$\begin{aligned} T_{31}(t_{i+5}, t_i) &= \sum_{l=1}^3 T_{3l}(t_{i+5}, t_{i+2})T_{l1}(t_{i+2}, t_i) \\ &= T_{31}(t_{i+5}, t_{i+2})T_{11}(t_{i+2}, t_i) + T_{32}(t_{i+5}, t_{i+2})T_{21}(t_{i+2}, t_i) \\ &\quad + T_{33}(t_{i+5}, t_{i+2})T_{31}(t_{i+2}, t_i) \end{aligned}$$

In other words, there are three possible “routes” of transition through the intermediate time  $t_{i+2}$ , as displayed in Fig. 11.2. The first “route” corresponds to the first term in the preceding summation: the product of [the probability that state 1 is assumed at time  $t_{i+2}$ , given that state 1 is assumed at  $t_i$ ] and [the probability that state 3 is assumed at  $t_{i+5}$ , given that the state at time  $t_{i+2}$  is indeed state 1]. Similarly, the second and third “routes” correspond to the second and third terms in the equation, respectively. ■

**EXAMPLE 11.4** Discrete-state Markov processes provide a useful model for system reliability purposes. Suppose that we have two actuators in parallel, so that the overall actuator system fails only if both actuators fail, and let us investigate the probability of a system failure. For convenience, quantize time into intervals  $\Delta t$  in length.

There are four possible states: state 1 = both actuators are working, state 2 = actuator  $A$  has failed but actuator  $B$  is working, state 3 =  $A$  is operating but  $B$  has failed, and state 4 = both actuators have failed. By using a Markov process model, we are assuming that the state transition probabilities depend only on the current state, not on the previous history leading up to that state. For instance, assume that there is a probability of 0.01 that either actuator will fail in the

FIG. 11.2 Transitions through intermediate time  $t_{i+2}$  for Example 11.3.

next  $\Delta t$  seconds, given that both are operating at the beginning of that time interval. Failures are assumed to be independent of one another. Furthermore, if one actuator has failed at the beginning of an interval, there is a probability of 0.02 that the other will fail in the ensuing  $\Delta t$  sec: since the single actuator takes the entire load, the failure probability is higher. Note that these are *conditional* probabilities, conditioned on the system state at the beginning of the interval, and *not* on any previous history of values.

Figure 11.3 presents the transition probability diagram for this problem. We now evaluate the transition probabilities as depicted there. First, from state 1, it is possible to transition into any of the four states. Since failures are assumed independent of one another, the probability of both *A*

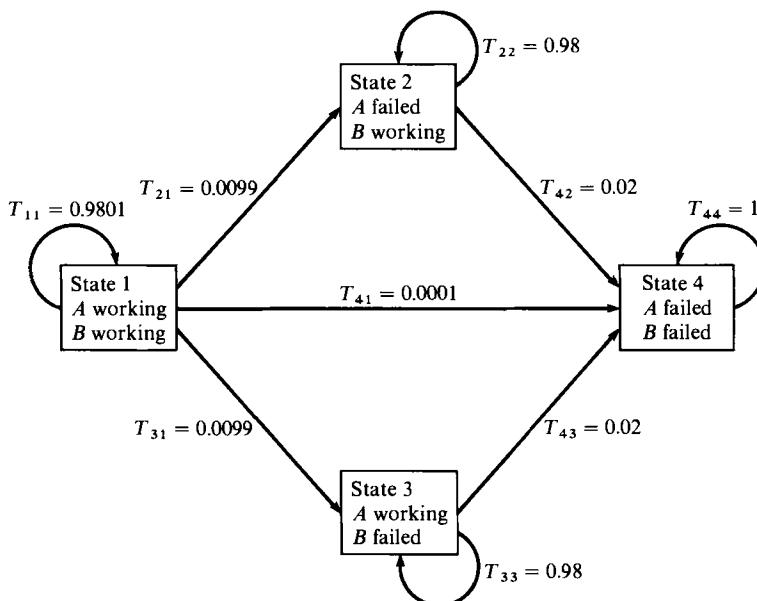


FIG. 11.3 Transition probability diagram for Example 11.4.

and  $B$  failing in the next  $\Delta t$  sec is the product of the separate probabilities of  $A$  failing and  $B$  failing:

$$\begin{aligned} T_{41} &= P\{A \text{ fails and } B \text{ fails} | A \text{ and } B \text{ both working to start}\} \\ &= P\{A \text{ fails} | \text{both working to start}\}P\{B \text{ fails} | \text{both working to start}\} \\ &= (0.01)(0.01) = 0.0001 \end{aligned}$$

To get  $T_{21}$ , we know that  $P\{A \text{ fails} | \text{both working to start}\} = 0.01$ , and that this is the sum

$$P\{A \text{ fails, } B \text{ works} | \text{both working to start}\} + P\{A \text{ fails, } B \text{ fails} | \text{both working to start}\}$$

because these latter events are mutually exclusive and their union is the event  $\{A \text{ fails} | \text{both working to start}\}$ . Thus,

$$0.01 = P\{A \text{ fails, } B \text{ works} | \text{both working to start}\} + 0.0001$$

so that  $T_{21} = 0.01 - 0.0001 = 0.0099$  and similarly for  $T_{31}$ . Finally,  $T_{11}$  is obtained by the fact that the total probability of reaching *some* state out of all possible states is one, so

$$T_{11} = 1 - [T_{21} + T_{31} + T_{41}] = 0.9801$$

From state 2, we can transition only into states 2 or 4: we disallow the possibility of “healing” and recertification of an actuator once it fails (also known as “regeneration”); such a possibility can of course be handled mathematically by nonzero state transition probabilities  $T_{12}$  and  $T_{32}$ . The value of  $T_{42}$  was given as 0.02, and so  $T_{44} = 1 - T_{42} = 0.98$ . Analogously,  $T_{43} = 0.02$  and  $T_{33} = 0.98$ .

From state 4, if we disallow “healing” of failed sensors, we can only transition into state 4, and so  $T_{44} = 1$ . Note this example expresses the transition probabilities as time invariant; time-varying probabilities can also be addressed mathematically.

Thus, the state transition probability matrix is

$$\mathbf{T}(t_{i+1}, t_i) = \begin{bmatrix} 0.9801 & 0 & 0 & 0 \\ 0.0099 & 0.98 & 0 & 0 \\ 0.0099 & 0 & 0.98 & 0 \\ 0.0001 & 0.02 & 0.02 & 1 \end{bmatrix}$$

Note for instance that the first row indicates the probabilities of transitions *into* state 1 from states 1, 2, 3, and 4; the first column depicts probabilities of transitions *from* state 1 into states 1, 2, 3, and 4. Moreover, each column sums to 1: the total probability of reaching *any* of the admissible states from a given state must equal unity.

For reliability purposes, we are interested in the probability of reaching state 4, starting from an initial condition of both actuators working. Thus, we have an initial condition on (11-36) as  $\mathbf{p}(t_0) = [1 \ 0 \ 0 \ 0]^T$ , and (11-41) can be used to propagate the vector of state probabilities in time. We are especially interested in  $p_4(t_i)$ : using the given  $\mathbf{T}(t_{i+1}, t_i)$  and (11-41),  $p_4(t_i)$  is seen to start at zero and to grow, asymptotically approaching 1 as time progresses. Had “healing” of failed sensors been allowed, a nonunity steady state value of  $p_4$  could have been reached.

In this problem, since  $\mathbf{T}(t_{i+1}, t_i) = \mathbf{T}$  is time invariant (a function only of  $\Delta t$ ), the Chapman-Kolmogorov equation (11-46) becomes

$$\mathbf{T}(t_{i+M}, t_i) = \mathbf{T}^M = \mathbf{T}^{(M-1)}\mathbf{T}^I = \mathbf{T}(t_{i+M}, t_{i+1})\mathbf{T}(t_{i+1}, t_i)$$

Thus,  $\mathbf{p}(t_{i+M}) = \mathbf{T}^M \mathbf{p}(t_i)$  can also be evaluated as

$$\mathbf{p}(t_{i+M}) = \mathbf{T}^{(M-1)}\mathbf{T}^I \mathbf{p}(t_i) = \mathbf{T}(t_{i+M}, t_{i+1})\mathbf{p}(t_{i+1})$$

in terms of the probabilities at the intermediate time  $t_{i+1}$ ,  $\mathbf{p}(t_{i+1})$ . ■

Now let us apply the insights gained from the case of discrete states (each with associated discrete probability) to the continuous-state case in which probability distribution functions are continuous. We shall assume henceforth that the conditional density function

$$f_{\mathbf{x}(t)|\mathbf{x}(t')}(\xi|\rho) = \frac{\partial}{\partial\xi_1 \cdots \partial\xi_n} F_{\mathbf{x}(t)|\mathbf{x}(t')}(\xi|\rho) \quad (11-47)$$

exists for all  $t \geq t'$ . Knowledge of this *transition probability density* allows complete specification of the statistical properties of a Markov process  $\mathbf{x}(\cdot, \cdot)$ , as demonstrated earlier. Common notational conventions for this fundamentally important function include

$$f_{\mathbf{x}(t)|\mathbf{x}(t')}(\xi|\rho) \triangleq f_{\mathbf{x}}(\xi, t|\mathbf{x}(t') = \rho) \triangleq f_{\mathbf{x}}(\xi, t|\rho, t') \quad (11-48)$$

and we shall use the last and most common of these for convenience.

In the discrete-state case examined so far, one of a finite number of admissible state values can be assumed at any time, including the intermediate time  $t_{i+1}$ , resulting in the finite sum of transition probability products in the Chapman–Kolmogorov equation, (11-45). When continuous-state problems are considered, we can expect this sum to be replaced by an integration over the range of values that can be assumed. We now demonstrate the validity of this insight. Consider any three times of interest,  $t_1 \leq t_2 \leq t_3$ . By the definition of conditional densities and Bayes' rule, we can write in general

$$f_{\mathbf{x}(t_3), \mathbf{x}(t_2)|\mathbf{x}(t_1)}(\xi, \rho|\eta) = f_{\mathbf{x}(t_3)|\mathbf{x}(t_2), \mathbf{x}(t_1)}(\xi|\rho, \eta) f_{\mathbf{x}(t_2)|\mathbf{x}(t_1)}(\rho|\eta)$$

Since  $\mathbf{x}(\cdot, \cdot)$  is assumed to be a Markov process, this becomes

$$f_{\mathbf{x}(t_3), \mathbf{x}(t_2)|\mathbf{x}(t_1)}(\xi, \rho|\eta) = f_{\mathbf{x}(t_3)|\mathbf{x}(t_2)}(\xi|\rho) f_{\mathbf{x}(t_2)|\mathbf{x}(t_1)}(\rho|\eta)$$

To obtain the conditional marginal density for  $\mathbf{x}(t_3)$ , conditioned on  $\mathbf{x}(t_1) = \eta$ , we can integrate over  $\rho$ , the process value at the intermediate time  $t_2$ :

$$\begin{aligned} f_{\mathbf{x}(t_3)|\mathbf{x}(t_1)}(\xi|\eta) &= \int_{-\infty}^{\infty} f_{\mathbf{x}(t_3), \mathbf{x}(t_2)|\mathbf{x}(t_1)}(\xi, \rho|\eta) d\rho \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{x}(t_3), \mathbf{x}(t_2)|\mathbf{x}(t_1)}(\xi, \rho|\eta) d\rho_1 d\rho_2 \cdots d\rho_n \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{x}(t_3)|\mathbf{x}(t_2)}(\xi|\rho) f_{\mathbf{x}(t_2)|\mathbf{x}(t_1)}(\rho|\eta) d\rho_1 d\rho_2 \cdots d\rho_n \end{aligned} \quad (11-49)$$

Using the notation introduced in (11-48), this becomes the conventional form of the *Chapman–Kolmogorov equation*:

$$f_{\mathbf{x}}(\xi, t_3|\eta, t_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{x}}(\xi, t_3|\rho, t_2) f_{\mathbf{x}}(\rho, t_2|\eta, t_1) d\rho_1 \cdots d\rho_n \quad (11-50)$$

This is directly comparable to (11-45), especially when we note that

$$f_{\mathbf{x}}(\rho, t_2|\eta, t_1) d\rho = P\{\rho \leq \mathbf{x}(t_2) \leq \rho + d\rho | \mathbf{x}(t_1) = \eta\} \quad (11-51)$$

By taking the expectation of the Chapman–Kolmogorov equation, i.e., integrating out the dependence on  $\eta$ , we obtain

$$\begin{aligned} f_{\mathbf{x}(t_3)}(\xi) &= \int_{-\infty}^{\infty} f_{\mathbf{x}(t_3)|\mathbf{x}(t_2)}(\xi|\rho) f_{\mathbf{x}(t_2)}(\rho) d\rho \\ &= \int_{-\infty}^{\infty} f_{\mathbf{x}}(\xi, t_3|\rho, t_2) f_{\mathbf{x}(t_2)}(\rho) d\rho \end{aligned} \quad (11-52)$$

where  $f_{\mathbf{x}(t_2)}(\rho)$  can be considered an initial condition. This is directly analogous to the discrete-state result (11-42).

Further insights about the structural form of propagation relations for transition probability densities can be obtained as well. In the case of discrete states (discrete probabilities, discontinuities in the distribution function) and discrete transition times, the model description for both transition and unconditional probabilities is in the form of a difference equation, i.e., (11-45) and (11-42), respectively. The case of discrete states and continuous time leads to differential equation models for the time evolution of such probabilities. Extending to the case of continuous states and continuous-time Markov processes will result in partial differential equations to describe time evolution of transition probability densities: the forward Kolmogorov equation to be discussed subsequently. An important class of such continuous-time, continuous-state Markov processes is the set of solutions to nonlinear stochastic differential equations defined in the Itô sense. The proper definition of such processes and their characteristics, including time evolution of the associated transition densities, will be developed. To do so, as in Chapter 4 of Volume 1 where linear stochastic differential equations were developed, we first establish the proper definition of Itô stochastic integrals and differentials.

## 11.4 ITÔ STOCHASTIC INTEGRALS AND DIFFERENTIALS

Wiener stochastic integrals of the form of (11-6) have previously been developed, and their properties are directly effected by properties of the Brownian motion or Wiener process  $\beta(\cdot, \cdot)$  used to define the integrals properly. Now we wish to consider the extension to (11-31), initially in the scalar case:

$$I(t, \cdot) = \int_{t_0}^t a(\tau, \cdot) d\beta(\tau, \cdot) \quad (11-53)$$

where  $a(\cdot, \cdot)$  is an *admissible* stochastic process, such that  $a(t, \cdot)$  depends at most on the past and present values of  $\beta(\cdot, \cdot)$ ,  $\{\beta(t'), t_0 \leq t' \leq t\}$ , but is independent of future values of  $\beta(\cdot, \cdot)$  [10, 11, 22, 23, 27, 30–32, 39, 46, 47, 60]. The properties of this stochastic integral will also be a direct consequence of Brownian motion characteristics, so we first consider these [9, 12, 32, 44, 45, 55, 71].

Scalar Brownian motion is defined to be a process with *independent increments* that are *Gaussian*, such that for any  $t_1$  and  $t_2$  in the time set  $T$  of interest,

$$E\{[\beta(t_2) - \beta(t_1)]\} = 0 \quad (11-54a)$$

$$E\{[\beta(t_2) - \beta(t_1)]^2\} = \int_{t_1}^{t_2} q(\tau) d\tau \quad (11-54b)$$

and, by convention, such that  $\beta(t_0) = 0$  with probability 1. From this definition, the following properties were developed in Volume 1: (1)  $\beta(\cdot, \cdot)$  is *Markov*, which is true of any process with independent increments; (2)  $\beta(\cdot, \cdot)$  is *continuous* everywhere with probability one (or, “almost surely,” i.e., all sample functions are continuous except possibly a set of total probability zero) and also in the mean square sense (or, in “quadratic mean”); (3)  $\beta(\cdot, \cdot)$  is *nondifferentiable* everywhere with probability one and in the mean square sense; (4)  $\beta(\cdot, \cdot)$  is *not of bounded variation* with probability one and in the mean square sense [23, 68]; (5)  $E\{\beta(t_i)\beta(t_j)\} = E\{\beta(t_i)^2\}$  for  $t_j \geq t_i$ .

Since  $\beta(\cdot, \cdot)$  is a zero-mean process with independent increments, it is readily shown also to be a *martingale* [10, 14, 39, 50], a stochastic process  $x(\cdot, \cdot)$  for which  $E\{|x(t)|\}$  is finite for all admissible  $t$  and

$$E\{x(t_i) | x(t_{i-1}), x(t_{i-2}), \dots, x(t_0)\} = x(t_{i-1}) \quad (11-55a)$$

for any sequential times  $t_0, t_1, \dots, t_i$ , or if  $x(\cdot, \cdot)$  is defined over some interval  $T$ ,

$$E[x(t) | \{x(\tau), t_0 \leq \tau \leq t' < t\}] = x(t') \quad (11-55b)$$

which can be written more rigorously as

$$E\{x(t) | \mathcal{F}_{t'}\} = x(t') \quad (11-55c)$$

where  $\mathcal{F}_{t'}$  is the minimal  $\sigma$ -algebra generated by  $\{x(\tau), t_0 \leq \tau \leq t' < t\}$ . The martingale property is important in developing the idea of conditional expectations conditioned on knowledge of samples of a continuous-time process (e.g., continuous-time measurements), likelihood ratios for statistical detection theory, and other concepts [10], as well as being pertinent in the discussion of stochastic integrals. Moreover, it can be proven [34, 45] that if  $x(\cdot, \cdot)$  is a martingale that is continuous with probability one, with covariance

$$E\{[x(t_2) - x(t_1)]^2 | \mathcal{F}_{t_1}\} = (t_2 - t_1) \quad (11-56)$$

then  $x(\cdot, \cdot)$  is a Brownian motion with unit diffusion. Throughout our discussion of (11-53), we could restrict our attention to unit-diffusion Brownian motion without loss of generality since

$$\begin{aligned} \int_{t_0}^t a(\tau, \cdot) d\beta(\tau, \cdot) &= \int_{t_0}^t [a(\tau, \cdot) q^{1/2}(\tau)] d\beta'(\tau, \cdot) \\ &\triangleq \int_{t_0}^t a'(\tau, \cdot) d\beta'(\tau, \cdot) \end{aligned} \quad (11-57)$$

if  $E\{[d\beta(t)]^2\} = q(t) dt$  and  $E\{[d\beta'(t)]^2\} = dt$ .

An additional property of Brownian motion has not been discussed previously, since it was not essential to the development of Wiener stochastic integrals. Namely, Brownian motion has the *Levy oscillation property* or *quadratic variation property* [34, 42]: If  $\beta'(\cdot, \cdot)$  is unit-diffusion Brownian motion and  $\{t_0, t_1, \dots, t_N = t_f\}$  is a partition of the interval  $[t_0, t_f]$ , then

$$\lim_{\max|t_{i+1} - t_i| \rightarrow 0} \sum_{i=0}^{N-1} [\beta'(t_{i+1}) - \beta'(t_i)]^2 = (t_f - t_0) \quad (11-58)$$

where the limit exists both in the mean square sense and with probability one. Written another way,

$$[d\beta'(t)]^2 = dt \quad \text{w.p.1; in m.s.} \quad (11-59a)$$

The mean square convergence is readily demonstrated:

$$E \left\{ \sum_{i=0}^{N-1} [\beta'(t_{i+1}) - \beta'(t_i)]^2 \right\} = (t_f - t_0)$$

and the variance of the sum tends to zero in the limit as  $\max|t_{i+1} - t_i| \rightarrow 0$ . Thus, not only is it true that  $E\{[d\beta'(t)]^2\} = dt$ , but  $[d\beta'(t)]^2$  itself equals  $dt$  for all samples except possibly a set of total probability zero. This readily extends to the nonunit-diffusion case:

$$[d\beta(t)]^2 = q(t) dt \quad \text{w.p.1; in m.s.} \quad (11-59b)$$

and also to the vector Brownian motion case:

$$[d\beta(t) d\beta^T(t)] = Q(t) dt \quad \text{w.p.1; in m.s.} \quad (11-59c)$$

This result will have *profound* effect upon the properties of the stochastic integral (11-53). Unlike the Wiener integral case, evaluating it as

$$I(t, \cdot) = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} a(\tau_i, \cdot) [\beta(t_{i+1}, \cdot) - \beta(t_i, \cdot)] \quad (11-60)$$

where  $\tau_i$  is any point in the interval  $[t_i, t_{i+1}]$ , will cause the value and properties of  $I(t, \cdot)$  to be *dependent upon the specific choice of  $\tau_i$*  [19, 32, 34, 67]. Heuristically, (11-59) will also impact the evaluation of truncated Taylor series representations of nonlinear functions of  $\beta(\cdot, \cdot)$ , invalidating the applicability of formal rules for differentials and the like.

Let  $a(\cdot, \cdot)$  be *admissible* (as defined below Eq. (11-53)) and suppose that  $\int_{t_0}^{t_f} E\{a(\tau, \cdot)^2\} d\tau$  is *finite* with probability one. Under these *sufficient conditions*, Itô [27, 30, 34] showed that the stochastic integral (11-53) can be defined in a manner analogous to that of Chapter 4 for Wiener integrals. First the interval  $[t_0, t]$  is partitioned into  $N$  steps with  $t_0 < t_1 < \dots < t_N = t$ , and the stochastic integral can then be defined for a piecewise constant function  $a_N(\cdot, \cdot)$ , constant over each partition  $[t_i, t_{i+1}]$ , approximating  $a(\cdot, \cdot)$  by letting  $a_N(\tau, \cdot) = a(t_i, \cdot)$

for  $\tau \in [t_i, t_{i+1})$ , as

$$l_N(t, \cdot) = \sum_{i=0}^{N-1} a_N(t_i, \cdot) [\beta(t_{i+1}, \cdot) - \beta(t_i, \cdot)] \triangleq \int_{t_0}^t a_N(\tau, \cdot) d\beta(\tau, \cdot) \quad (11-61)$$

Now we take finer and finer partitions of  $[t_0, t]$ , such that  $\max\{|t_{i+1} - t_i|\} \rightarrow 0$ , and consider the sequence of integrals  $l_N(t, \cdot)$  as  $N \rightarrow \infty$ . If the sequence of  $a_N(\cdot, \cdot)$ 's converges to the random function  $a(\cdot, \cdot)$  in the sense that

$$\int_{t_0}^t E\{[a(\tau, \cdot) - a_N(\tau, \cdot)]^2\} d\tau \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad (11-62)$$

which is assured by the two stated sufficient conditions, then the *Itô stochastic integral* can be defined as the mean square limit:

$$\begin{aligned} l(t, \cdot) &= \text{l.i.m.}_{N \rightarrow \infty} \sum_{i=0}^{N-1} a_N(t_i, \cdot) [\beta(t_{i+1}, \cdot) - \beta(t_i, \cdot)] \\ &\triangleq \int_{t_0}^t a(\tau, \cdot) d\beta(\tau, \cdot) \end{aligned} \quad (11-63)$$

**EXAMPLE 11.5** Consider evaluation of the stochastic integral [10]

$$l(t, \cdot) = \int_0^t \beta(\tau, \cdot) d\beta(\tau, \cdot)$$

in which the admissible  $a(\cdot, \cdot)$  is specifically  $\beta(\cdot, \cdot)$  itself: here  $a(t, \cdot)$  depends only on  $\beta(t, \cdot)$  and not the more generally admissible  $\{\beta(t', \cdot), 0 \leq t' \leq t\}$ . Let  $\beta(\cdot, \cdot)$  have constant diffusion parameter  $q$ . First let us interpret this stochastic integral in the Itô sense and use (11-63) to establish the desired evaluation:

$$\begin{aligned} l(t) &= \text{l.i.m.}_{N \rightarrow \infty} \sum_{i=0}^{N-1} \beta(t_i) [\beta(t_{i+1}) - \beta(t_i)] \\ &= \text{l.i.m.}_{N \rightarrow \infty} \left\{ \frac{1}{2} \sum_{i=0}^{N-1} [\beta^2(t_{i+1}) - \beta^2(t_i)] - \frac{1}{2} \sum_{i=0}^{N-1} [\beta(t_{i+1}) - \beta(t_i)]^2 \right\} \\ &= \frac{1}{2}\beta^2(t) - \frac{1}{2}qt \end{aligned}$$

where the second term follows from the Levy oscillation property (11-59b). Note that this does *not* obey formal integration rules, which would yield simply  $\frac{1}{2}\beta^2(t)$ .

If alternate stochastic integral definitions are considered as in (11-60), with  $\tau_i = t_i + k[t_{i+1} - t_i]$  and  $k \in [0, 1]$ , then [19]

$$\begin{aligned} l(t) &= \text{l.i.m.}_{N \rightarrow \infty} \sum_{i=0}^{N-1} \beta(t_i + k[t_{i+1} - t_i]) [\beta(t_{i+1}) - \beta(t_i)] \\ &= \frac{1}{2}\beta^2(t) + [k - \frac{1}{2}]qt \end{aligned}$$

If  $k$  is chosen to be  $\frac{1}{2}$ , i.e., if we let  $\tau_i$  be the midpoint between  $t_i$  and  $t_{i+1}$ , then the formal integration rule is satisfied. This corresponds to the Stratonovich definition of a stochastic integral [16, 17, 32, 66, 67]. Despite this attractive feature in the scalar case, the Stratonovich integral lacks some properties possessed by the Itô integral that are essential to Markov process descriptions and optimal estimator derivation, so we shall concentrate only on the latter definition. ■

Viewed as a stochastic process,  $I(\cdot, \cdot)$  is itself *admissible*, and it is both *mean square continuous* and *continuous with probability one* [10, 14, 32, 45]. It is also a *martingale of Brownian motion* [19, 20, 34]:

$$E[I(t) | \{\beta(\tau), t_0 \leq \tau \leq t' < t\}] = I(t') \quad (11-64)$$

and has the properties (for admissible  $a(\cdot, \cdot)$  and  $b(\cdot, \cdot)$ )

$$E \left\{ \int_{t_0}^t a(\tau) d\beta(\tau) \right\} = 0 \quad (11-65)$$

$$E \left\{ \left[ \int_{t_0}^t a(\tau) d\beta(\tau) \right] \left[ \int_{t_0}^t b(\tau) d\beta(\tau) \right] \right\} = \int_{t_0}^t E\{a(\tau)b(\tau)\} q(\tau) d\tau \quad (11-66)$$

### Demonstration of (11-65), (11-66), and Mean Square Continuity

These properties follow directly from (11-63). For instance,  $a_N(t_i)$  is independent of  $[\beta(t_{i+1}) - \beta(t_i)]$  for admissible  $a_N$ , so that the expectation of (11-61) yields

$$\begin{aligned} E\{I_N(t)\} &= E \left\{ \sum_{i=0}^{N-1} a_N(t_i) [\beta(t_{i+1}) - \beta(t_i)] \right\} \\ &= \sum_{i=0}^{N-1} E\{a_N(t_i)\} E\{[\beta(t_{i+1}) - \beta(t_i)]\} = 0 \end{aligned}$$

for all  $N$ . Since

$$E \left\{ \text{l.i.m. } x_N \right\} = \lim_{N \rightarrow \infty} E\{x_N\}$$

for any sequence  $\{x_N\}$  with finite second moments (readily demonstrated using the Schwarz inequality; see Problem 11.7), (11-65) is established by

$$E\{I(t)\} = E \left\{ \text{l.i.m. } I_N(t) \right\} = \lim_{N \rightarrow \infty} E\{I_N(t)\} = 0$$

Similarly, (11-66) is valid for step functions  $a_N$  and  $b_N$ :

$$\begin{aligned} E \left\{ \left[ \int_{t_0}^t a_N(\tau) d\beta(\tau) \right] \left[ \int_{t_0}^t b_N(\tau) d\beta(\tau) \right] \right\} \\ = E \left\{ \left[ \sum_{i=0}^{N-1} a_N(t_i) \{ \beta(t_{i+1}) - \beta(t_i) \} \right] \left[ \sum_{j=0}^{N-1} b_N(t_j) \{ \beta(t_{j+1}) - \beta(t_j) \} \right] \right\} \\ = \sum_{i=0}^{N-1} E\{a_N(t_i)b_N(t_i)\} \int_{t_i}^{t_{i+1}} q(\tau) d\tau \\ = \int_{t_0}^t E\{a_N(\tau)b_N(\tau)\} q(\tau) d\tau \end{aligned}$$

where the single summation in the third line is the result of  $a_N(t_i)$ ,  $b_N(t_j)$ , and  $[\beta(t_{i+1}) - \beta(t_i)]$  being uncorrelated with  $[\beta(t_{j+1}) - \beta(t_j)]$  for  $t_j > t_i$  and  $E\{[\beta(t_{j+1}) - \beta(t_j)]\} = 0$  (and similarly for  $t_i > t_j$ ); for  $t_i = t_j$ ,  $a_N(t_i)$  and  $b_N(t_i)$  are uncorrelated with  $[\beta(t_{i+1}) - \beta(t_i)]$ , so the expectation of the product is  $E\{a_N(t_i)b_N(t_i)\}E\{[\beta(t_{i+1}) - \beta(t_i)]^2\}$  as shown. Now assume  $\{a_N(\tau)\}$  and  $\{b_N(\tau)\}$  converge to  $a(\tau)$

and  $b(\tau)$ ,  $t_0 \leq \tau \leq t$ , in the sense of (11-62); then

$$\lim_{N \rightarrow \infty} \int_{t_0}^t E\{\mathbf{a}_N(\tau)b_N(\tau)\}q(\tau)d\tau = \int_{t_0}^t E\{\mathbf{a}(\tau)b(\tau)\}q(\tau)d\tau$$

which can be shown by appropriate application of the Schwarz inequality (see Problem 11.8). Finally, for any sequences  $\{x_N\}$  and  $\{y_N\}$  with finite second moments,

$$E\left\{\left[\underset{N \rightarrow \infty}{\text{l.i.m.}} \mathbf{x}_N\right]\left[\underset{N \rightarrow \infty}{\text{l.i.m.}} \mathbf{y}_N\right]\right\} = \lim_{N \rightarrow \infty} E\{\mathbf{x}_N \mathbf{y}_N\}$$

(see Problem 11.7), so that (11-66) is established by

$$\begin{aligned} & E\left\{\left[\underset{N \rightarrow \infty}{\text{l.i.m.}} \sum_{i=0}^{N-1} \mathbf{a}_N(t_i)\{\beta(t_{i+1}) - \beta(t_i)\}\right]\left[\underset{N \rightarrow \infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \mathbf{b}_N(t_j)\{\beta(t_{j+1}) - \beta(t_j)\}\right]\right\} \\ &= \lim_{N \rightarrow \infty} E\left\{\left[\sum_{i=0}^{N-1} \mathbf{a}_N(t_i)\{\beta(t_{i+1}) - \beta(t_i)\}\right]\left[\sum_{j=0}^{N-1} \mathbf{b}_N(t_j)\{\beta(t_{j+1}) - \beta(t_j)\}\right]\right\} \\ &= \int_{t_0}^t E\{\mathbf{a}(\tau)\mathbf{b}(\tau)\}q(\tau)d\tau \end{aligned}$$

Moreover, since (11-66) is true,  $E\{\left[\int_{t_0}^t \mathbf{a}(\tau)d\beta(\tau)\right]^2\}$  equals  $\int_{t_0}^t E\{\mathbf{a}^2(\tau)\}q(\tau)d\tau$  and is thus finite if  $E\{\mathbf{a}^2(\tau)\} < \infty$ ,  $t_0 \leq \tau \leq t$ , and

$$\lim_{\Delta t \rightarrow 0} E\left\{\left[\int_t^{t+\Delta t} \mathbf{a}(\tau)d\beta(\tau)\right]^2\right\} = \lim_{\Delta t \rightarrow 0} \int_t^{t+\Delta t} E\{\mathbf{a}^2(\tau)\}q(\tau)d\tau = 0$$

since the last Riemann integral is a continuous function of its upper limit. This demonstrates mean square continuity. ■

The properties just described for Itô stochastic integrals were seen to be dependent upon the definition given in (11-63). These properties need *not* be true for alternate definitions of stochastic integrals as given in (11-60).

Other properties of the Itô integral are also natural extensions of properties of Wiener integrals. For instance, a *linear functional* of a random process  $\mathbf{z}(\cdot, \cdot)$  with finite second moments is defined to be a finite sum of the form  $\sum_{i=1}^N k_i \mathbf{z}(t_i)$  or a limit in the mean of such sums, while a *nonlinear functional* of  $\mathbf{z}(\cdot, \cdot)$  is defined as a finite sum  $\sum_{i=1}^N k_i \mathbf{z}(t_i)$ , a finite product  $\prod_{i=1}^M k_i \mathbf{z}(t_i)$ , or a limit in the mean of such sums, products, or combinations thereof. It can be shown [10] that any finite-variance linear functional of Brownian motion  $\{\beta(\tau), 0 \leq \tau \leq t\}$  can be written as a Wiener stochastic integral, as

$$\mathbf{x}(t) = \int_0^t \mathbf{a}(\tau)d\beta(\tau) \quad (11-67a)$$

with  $\int_0^t \mathbf{a}^2(\tau)q(\tau)d\tau$  finite. Analogously, any finite-variance nonlinear functional of Brownian motion can be expressed as an Itô stochastic integral, as

$$\mathbf{x}(t) = \int_0^t \mathbf{a}(\tau)d\beta(\tau) \quad (11-67b)$$

with  $\int_0^t E\{\mathbf{a}^2(\tau)\}q(\tau)d\tau$  finite [4, 5, 19, 20, 28, 34, 39, 72]. In fact, once the Itô integral is established, it is useful to consider the Wiener integral as a special case of Itô integral, with nonrandom function  $\mathbf{a}(t) \equiv \mathbf{a}(t)$  used in its definition.

The concepts presented for the scalar case extend directly to the vector Itô stochastic integral,

$$\mathbf{I}(t, \cdot) = \int_{t_0}^t \mathbf{A}(\tau, \cdot) d\beta(\tau, \cdot) \quad (11-68)$$

with  $\mathbf{A}(\cdot, \cdot)$  a matrix of admissible functions. Then, if the sequence of  $\mathbf{A}_N(\cdot, \cdot)$ 's, achieved by finer and finer partitions, converges to  $\mathbf{A}(\cdot, \cdot)$  in the sense that

$$\int_{t_0}^t E\{[\mathbf{A}(\tau, \cdot) - \mathbf{A}_N(\tau, \cdot)]_{ij}^2\} d\tau \rightarrow 0$$

as  $N \rightarrow \infty$  for all  $i$  and  $j$ , then the stochastic integral is defined as in (11-63):

$$\mathbf{I}(t, \cdot) = \text{l.i.m. } \sum_{N \rightarrow \infty} \sum_{i=0}^{N-1} \mathbf{A}_N(t_i, \cdot) [\beta(t_{i+1}, \cdot) - \beta(t_i, \cdot)] \quad (11-69)$$

with the same properties as already described in the scalar case.

Given the definition of an Itô stochastic integral, the corresponding *stochastic differential* can be established. If

$$\mathbf{I}(t) = \mathbf{I}(t_0) + \int_{t_0}^t \mathbf{A}(\tau) d\beta(\tau) \quad (11-70)$$

then the stochastic differential of  $\mathbf{I}(t)$  is that which would be integrated from time  $t_0$  to time  $t$  to evaluate  $[\mathbf{I}(t) - \mathbf{I}(t_0)]$ :

$$d\mathbf{I}(t) = \mathbf{A}(t) d\beta(t) \quad (11-71)$$

*Heuristically*, it can be interpreted as an infinitesimal difference

$$d\mathbf{I}(t) = \mathbf{I}(t + dt) - \mathbf{I}(t) \quad (11-72)$$

## 11.5 ITÔ STOCHASTIC DIFFERENTIAL EQUATIONS

Consider a dynamical system described by the *nonlinear (Itô) stochastic differential equation* [1, 10, 18, 22, 27–32]

$$d\mathbf{x}(t) = \mathbf{f}[\mathbf{x}(t), t] dt + \mathbf{G}[\mathbf{x}(t), t] d\beta(t) \quad (11-73)$$

where  $\mathbf{x}(\cdot, \cdot)$  is the  $n$ -dimensional state stochastic process,  $\mathbf{f}[\mathbf{x}(t), t]$  is an  $n$ -vector function of  $\mathbf{x}(t)$  and (possibly)  $t$  describing system dynamics,  $\mathbf{G}[\mathbf{x}(t), t]$  is an  $n$ -by- $s$  matrix of functions of (possibly)  $\mathbf{x}(t)$  and  $t$ , and  $\beta(\cdot, \cdot)$  is  $s$ -vector dimensional Brownian motion of mean zero and diffusion  $\mathbf{Q}(t)$ :

$$E\{d\beta(t) d\beta^T(t)\} = \mathbf{Q}(t) dt \quad (11-74a)$$

$$E\{[\beta(t_2) - \beta(t_1)][\beta(t_2) - \beta(t_1)]^T\} = \int_{t_1}^{t_2} \mathbf{Q}(t) dt \quad (11-74b)$$

Equation (11-73) is to be understood in the sense that

$$\mathbf{x}(t) - \mathbf{x}(t_0) = \int_{t_0}^t \mathbf{f}[\mathbf{x}(\tau), \tau] d\tau + \int_{t_0}^t \mathbf{G}[\mathbf{x}(\tau), \tau] d\beta(\tau) \quad (11-75)$$

where the first integral on the right hand side is an ordinary integral for a given sample function of the process. The second integral is a specific form of Itô stochastic integral as in (11-68) and (11-69), but in which we restrict attention to  $\mathbf{G}[\cdot, t]$  being specifically a function of  $\mathbf{x}(t)$  rather than the entire history  $\{\mathbf{x}(\tau), t_0 \leq \tau \leq t\}$ . In fact, (11-73) can be generalized to the case of assuming only that  $\mathbf{f}$  and  $\mathbf{G}$  are admissible, yielding solution processes known as Itô processes. The motivation for the restriction is that solutions to (11-73) will be *Markov*, whereas general Itô processes need not be. Moreover, it is *only* the Itô definition of stochastic integrals that will yield a solution to an equation such as (11-73) that will be Markov.

Itô [29] established the *existence* and *uniqueness* of solutions in the mean square sense to (11-73) *under sufficient conditions* that are very similar to those for ordinary nonlinear differential equations, namely:

(1)  $\mathbf{f}[\cdot, \cdot]$  and  $\mathbf{G}[\cdot, \cdot]$  are real functions that are uniformly Lipschitz in their first argument (a continuity condition): there exists a  $K$ , independent of  $t$ , such that

$$\begin{aligned}\|\mathbf{f}[\mathbf{x} + \Delta\mathbf{x}, t] - \mathbf{f}[\mathbf{x}, t]\| &\leq K\|\Delta\mathbf{x}\| \\ \|\mathbf{G}[\mathbf{x} + \Delta\mathbf{x}, t] - \mathbf{G}[\mathbf{x}, t]\| &\leq K\|\Delta\mathbf{x}\|\end{aligned}$$

for all  $\mathbf{x}$  and  $\Delta\mathbf{x}$  in  $R^n$  and all  $t$  in the interval  $[t_0, t_f]$  of interest, where the appropriate norm definitions for an  $n$ -vector  $\mathbf{v}$  and an  $m$ -by- $n$  matrix  $\mathbf{M}$  are

$$\begin{aligned}\|\mathbf{v}\| &= \left[ \sum_{i=1}^n v_i^2 \right]^{1/2} = [\mathbf{v}^T \mathbf{v}]^{1/2} = [\text{tr}(\mathbf{v}\mathbf{v}^T)]^{1/2} \\ \|\mathbf{M}\| &= \left[ \sum_{i=1}^m \sum_{j=1}^n M_{ij}^2 \right]^{1/2} = [\text{tr}(\mathbf{M}\mathbf{M}^T)]^{1/2}\end{aligned}$$

(2)  $\mathbf{f}[\cdot, \cdot]$  and  $\mathbf{G}[\cdot, \cdot]$  are continuous in their second (time) argument over the interval  $[t_0, t_f]$  of interest;

(3)  $\mathbf{f}[\cdot, \cdot]$  and  $\mathbf{G}[\cdot, \cdot]$  are uniformly bounded according to

$$\|\mathbf{f}[\mathbf{x}, t]\|^2 \leq K(1 + \|\mathbf{x}\|^2), \quad \|\mathbf{G}[\mathbf{x}, t]\|^2 \leq K(1 + \|\mathbf{x}\|^2)$$

(4)  $\mathbf{x}(t_0)$  is any random vector, with finite second moment  $E\{\mathbf{x}(t_0)\mathbf{x}^T(t_0)\}$ , which is independent of the Brownian motion process  $\beta(\cdot, \cdot)$ .

In fact, Itô's proof is analogous to the standard successive approximations proof of existence and uniqueness of solutions for ordinary nonlinear differential equations [8]. The solution is generated constructively by assuming the existence of  $\mathbf{x}_k(\cdot, \cdot)$  with  $\mathbf{x}_k(t_0, \cdot) = \mathbf{x}(t_0, \cdot)$  and then forming  $\mathbf{x}_{k+1}(\cdot, \cdot)$  via

$$\mathbf{x}_{k+1}(t) = \mathbf{x}(t_0) + \int_{t_0}^t \mathbf{f}[\mathbf{x}_k(\tau), \tau] d\tau + \int_{t_0}^t \mathbf{G}[\mathbf{x}_k(\tau), \tau] d\beta(\tau) \quad (11-76)$$

The sequence  $\{\mathbf{x}_k(\cdot, \cdot)\}$  converges (in the mean square sense and with probability one) on any finite interval  $[t_0, t_f]$  to a solution process  $\mathbf{x}(\cdot, \cdot)$  provided that

the four sufficient conditions are satisfied. Once a solution is so generated, its uniqueness in the mean square sense can also be established [9, 29, 32, 45, 73, 74].

The solution process  $\mathbf{x}(\cdot, \cdot)$  has the following important properties [3, 9, 30, 32, 34, 45, 76]:

(1)  $\mathbf{x}(\cdot, \cdot)$  is *mean square continuous*, i.e.,

$$\lim_{t' \rightarrow t} \mathbf{x}(t') = \mathbf{x}(t) \quad (11-77a)$$

or

$$\lim_{t' \rightarrow t} \text{tr} E\{\mathbf{x}(t') - \mathbf{x}(t)\}[\mathbf{x}(t') - \mathbf{x}(t)]^T = 0 \quad (11-77b)$$

and it is also continuous w.p.1.

(2)  $[\mathbf{x}(t) - \mathbf{x}(t_0)]$  and  $\mathbf{x}(t)$  are both independent of the future increments of  $\beta(\cdot, \cdot)$ , i.e.,  $[\beta(t + \tau_2) - \beta(t + \tau_1)]$  for  $\tau_2 > \tau_1 \geq 0$ .

(3)  $\mathbf{x}(\cdot, \cdot)$  is *Markov*. Since, for  $t \geq t'$ ,

$$\mathbf{x}(t) = \mathbf{x}(t') + \int_{t'}^t \mathbf{f}[\mathbf{x}(\tau), \tau] d\tau + \int_{t'}^t \mathbf{G}[\mathbf{x}(\tau), \tau] d\beta(\tau) \quad (11-78)$$

$\mathbf{x}(t)$  depends on  $\mathbf{x}(t')$  and  $\{d\beta(\tau), t' \leq \tau \leq t\}$ , and the latter is independent of  $\mathbf{x}(\sigma)$ ,  $\sigma \leq t'$ . Thus, the conditional probability distribution for  $\mathbf{x}(t)$  given  $\mathbf{x}(t')$  and  $\{\mathbf{x}(\sigma), \sigma < t'\}$  equals the distribution conditioned only on  $\mathbf{x}(t')$ , establishing the Markov nature.

(4) The mean squared value of each component of  $\mathbf{x}(\cdot, \cdot)$  is *bounded* by a finite number,

$$E\{\mathbf{x}_i(t)^2\} < M < \infty \quad (11-79a)$$

for all time, and also

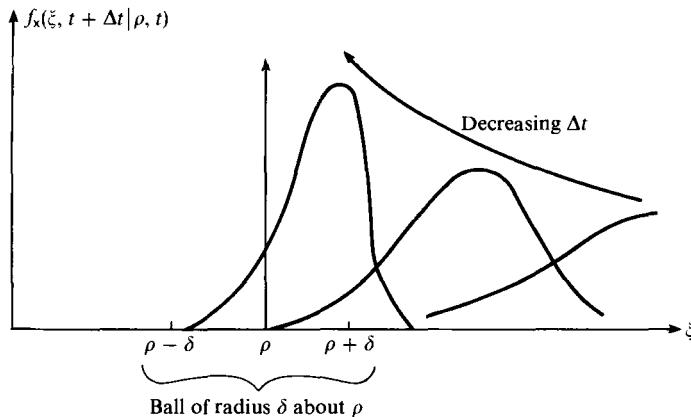
$$\int_{t_0}^{t_f} E\{\mathbf{x}_i(t)^2\} dt < \infty \quad (11-79b)$$

for any  $t_0$  and  $t_f$ .

(5) The probability of a change in  $\mathbf{x}(t)$  in a small interval  $\Delta t$  is of higher order than  $\Delta t$  (a form of continuity property):

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_x(\xi, t + \Delta t | \rho, t) d\xi_1 \cdots d\xi_n = 0, \quad (11-80)$$

where the notation means that the integration over  $\xi_1, \dots, \xi_n$  is to be carried out *outside* the ball of radius  $\delta$  about  $\rho$ . Note that since  $\mathbf{x}(\cdot, \cdot)$  is Markov, the transition probability is an appropriate means of describing fundamental properties. Figure 11.4 displays this property graphically for scalar  $x(\cdot, \cdot)$ : as  $\Delta t \rightarrow 0$ , the transition probability converges to a delta function at  $\Delta t = 0$  and crowds into the region  $\rho \pm \delta$  faster than  $\Delta t \rightarrow 0$ .

FIG. 11.4 Illustrating property (5) of solution process  $\mathbf{x}(\cdot, \cdot)$ .

(6) The *drift* of  $\mathbf{x}(\cdot, \cdot)$  at time  $t$  is  $\mathbf{f}[\mathbf{x}(t), t]$ : recalling (11-73),

$$\begin{aligned} & \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [\xi - \rho] f_x(\xi, t + \Delta t | \rho, t) d\xi_1 \cdots d\xi_n \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} E\{\mathbf{x}(t + \Delta t) - \mathbf{x}(t) | \mathbf{x}(t) = \rho\} \\ &= \mathbf{f}[\rho, t] \end{aligned} \quad (11-81)$$

This says that the mean rate of change in  $\mathbf{x}(\cdot, \cdot)$  going from  $t$  to  $t + \Delta t$  is  $\mathbf{f}[\mathbf{x}(t), t]$  in the limit as  $\Delta t \rightarrow 0$ .

(7) The *diffusion* of  $\mathbf{x}(\cdot, \cdot)$  at time  $t$  is  $\{\mathbf{G}[\mathbf{x}(t), t] \mathbf{Q}(t) \mathbf{G}^T[\mathbf{x}(t), t]\}$ :

$$\begin{aligned} & \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [\xi - \rho][\xi - \rho]^T f_x(\xi, t + \Delta t | \rho, t) d\xi_1 \cdots d\xi_n \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} E\{[\mathbf{x}(t + \Delta t) - \mathbf{x}(t)][\mathbf{x}(t + \Delta t) - \mathbf{x}(t)]^T | \mathbf{x}(t) = \rho\} \\ &= \mathbf{G}[\rho, t] \mathbf{Q}(t) \mathbf{G}^T[\rho, t] \end{aligned} \quad (11-82)$$

This is the covariance of the rate of change of  $\mathbf{x}(\cdot, \cdot)$ , and  $\mathbf{Q}(t)$  is the Brownian motion diffusion defined in (11-74).

(8) The higher order infinitesimals in the progression established by (11-80)–(11-82) are all zero:

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\xi_i - \rho_i)^k f_x(\xi, t + \Delta t | \rho, t) d\xi_1 \cdots d\xi_n = 0 \quad (11-83a)$$

for  $k > 2$ , and a similar relation is true for the general products greater than second degree as well, such as

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\xi_i - \rho_i)(\xi_j - \rho_j)(\xi_k - \rho_k) f_x(\xi, t + \Delta t | \boldsymbol{\rho}, t) d\xi_1 \cdots d\xi_n = 0 \quad (11-83b)$$

This implies that the process does not diffuse “too fast,” not that the transition density stays symmetric or Gaussian-like.

### Formal Calculation of Statistics of $\mathbf{dx}(t)$

It is instructive to calculate the statistics of the differential  $\mathbf{dx}(t)$  in (11-73) to understand properties (5) and (6) and the rate of change of  $\mathbf{x}(\cdot, \cdot)$  as just described. To do this properly would require addressing existence questions, but we present only formal calculations here.

In performing these calculations, we will have need for the fact that, if  $\mathbf{x}$  and  $\mathbf{y}$  are scalar functions of random vectors  $\mathbf{u}$  and  $\mathbf{v}$  respectively, i.e.

$$\mathbf{x} = \psi[\mathbf{u}], \quad \mathbf{y} = \theta[\mathbf{v}]$$

then

$$E\{\mathbf{x}\mathbf{y} | \mathbf{v} = \mathbf{v}(\cdot)\} = E\{\mathbf{x} | \mathbf{v} = \mathbf{v}(\cdot)\} \{\theta[\mathbf{v}(\cdot)]\}$$

This can be established as

$$\begin{aligned} E\{\mathbf{x}\mathbf{y} | \mathbf{v} = \boldsymbol{\eta}\} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi \rho f_{x,y|\mathbf{v}}(\xi, \rho | \boldsymbol{\eta}) d\xi d\rho \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi \rho f_{y|x,v}(\rho | \xi, \boldsymbol{\eta}) f_{x|v}(\xi | \boldsymbol{\eta}) d\xi d\rho \end{aligned}$$

by Bayes' rule. But  $f_{y|x,v}(\rho | \xi, \boldsymbol{\eta}) = \delta(\rho - \theta(\boldsymbol{\eta}))$  so the sifting property of the Dirac delta yields

$$\begin{aligned} E\{\mathbf{x}\mathbf{y} | \mathbf{v} = \boldsymbol{\eta}\} &= \int_{-\infty}^{\infty} \xi \theta(\boldsymbol{\eta}) f_{x|v}(\xi | \boldsymbol{\eta}) d\xi \\ &= \left[ \int_{-\infty}^{\infty} \xi f_{x|v}(\xi | \boldsymbol{\eta}) d\xi \right] [\theta(\boldsymbol{\eta})] \end{aligned}$$

as desired.

Now consider the conditional expectation of  $\mathbf{dx}(t)$ . Recalling (11-73),

$$\begin{aligned} E\{\mathbf{d}\mathbf{x}(t) | \mathbf{x}(t) = \mathbf{x}(t, \cdot)\} &= E\{\mathbf{f}[\mathbf{x}(t), t] dt | \mathbf{x}(t) = \mathbf{x}(t, \cdot)\} + \mathbf{0} \\ &= \mathbf{f}[\mathbf{x}(t), t] dt \end{aligned}$$

where the last equality follows componentwise from the result just established, with  $\mathbf{v} = \mathbf{x}(t)$ ,  $\psi \equiv 1$ , and  $\theta[\mathbf{v}] = f_1[\mathbf{x}(t), t]$ . This establishes property (6).

Now consider the conditional expected value:

$$\begin{aligned} E\{\mathbf{d}\mathbf{x}(t) \mathbf{d}\mathbf{x}^T(t) | \mathbf{x}(t) = \mathbf{x}(t, \cdot)\} &= E\{\mathbf{f}[\mathbf{x}(t), t] \mathbf{f}^T[\mathbf{x}(t), t] dt^2 + \mathbf{f}[\mathbf{x}(t), t] dt \mathbf{d}\beta^T(t) G^T[\mathbf{x}(t), t] \\ &\quad + \mathbf{G}[\mathbf{x}(t), t] \mathbf{d}\beta(t) \mathbf{f}^T[\mathbf{x}(t), t] dt \\ &\quad + \mathbf{G}[\mathbf{x}(t), t] \mathbf{d}\beta(t) \mathbf{d}\beta(t)^T G^T[\mathbf{x}(t), t] | \mathbf{x}(t) = \mathbf{x}(t, \cdot)\} \end{aligned}$$

The first term is second order in  $dt$ ; the second and third terms are zero mean because  $\mathbf{d}\beta(t)$  is independent of  $\mathbf{x}(t)$  and is zero mean. Using the previously established result componentwise yields

the fourth term as

$$\begin{aligned} \mathbf{G}[\mathbf{x}(t), t] E\{\mathbf{d}\beta(t) \mathbf{d}\beta^T(t) | \mathbf{x}(t) = \mathbf{x}(t, \cdot)\} \mathbf{G}^T[\mathbf{x}(t), t] &= \mathbf{G}[\mathbf{x}(t), t] E\{\mathbf{d}\beta(t) \mathbf{d}\beta^T(t)\} \mathbf{G}^T[\mathbf{x}(t), t] \\ &= \mathbf{G}[\mathbf{x}(t), t] \mathbf{Q}(t) \mathbf{G}^T[\mathbf{x}(t), t] dt \end{aligned}$$

again using the independence of  $\mathbf{d}\beta(t)$  and  $\mathbf{x}(t)$ . This establishes property (7). ■

Formal rules of integration and differentiation are not valid for Itô stochastic integrals or differentials based upon them. As a direct consequence of the Itô integral definition and the Levy oscillation property of Brownian motion (11-59), differentials of functions of  $\mathbf{x}(\cdot, \cdot)$  satisfy the *Itô differential rule* [27, 30, 32, 34, 39, 45]. Let  $\mathbf{x}(\cdot, \cdot)$  be the unique solution to (11-73)–(11-75), and let  $\psi[\cdot, \cdot]$  be a scalar real-valued function that has continuous first and second partial derivatives with respect to its first argument  $\mathbf{x}$  and is continuously differentiable in its second argument  $t$ . Then  $\psi$  satisfies the stochastic differential equation

$$d\psi[\mathbf{x}(t), t] = \frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial \mathbf{x}} \mathbf{d}\mathbf{x}(t) + \frac{1}{2} \text{tr}\{\mathbf{G}[\mathbf{x}(t), t] \mathbf{Q}(t) \mathbf{G}^T[\mathbf{x}(t), t]\} \frac{\partial^2 \psi}{\partial \mathbf{x}^2} dt \quad (11-84)$$

where

$$\begin{aligned} \frac{\partial \psi}{\partial t} &\triangleq \left. \frac{\partial \psi[\mathbf{x}, t]}{\partial t} \right|_{\mathbf{x}=\mathbf{x}(t)}, & \frac{\partial \psi}{\partial \mathbf{x}} &\triangleq \left[ \left. \frac{\partial \psi}{\partial x_1} \dots \frac{\partial \psi}{\partial x_n} \right] \right]_{\mathbf{x}=\mathbf{x}(t)} \\ \frac{\partial^2 \psi}{\partial \mathbf{x}^2} &\triangleq \left[ \begin{array}{ccc} \frac{\partial^2 \psi}{\partial x_1^2} & \dots & \frac{\partial^2 \psi}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 \psi}{\partial x_n \partial x_1} & \dots & \frac{\partial^2 \psi}{\partial x_n^2} \end{array} \right] \Big|_{\mathbf{x}=\mathbf{x}(t)} \end{aligned}$$

and tr designates trace. Again, (11-84) is to be understood in the sense that

$$\psi[\mathbf{x}(t), t] - \psi[\mathbf{x}(t_0), t_0] = \int_{t_0}^t d\psi[\mathbf{x}(\tau), \tau] \quad (11-85)$$

It is the last term in (11-84) that causes formal differential rules to be erroneous in this application. The validity of (11-84) can be demonstrated formally by a Taylor series expansion of  $\psi[\mathbf{x} + \mathbf{d}\mathbf{x}, t + dt]$ :

$$\begin{aligned} \psi[\mathbf{x} + \mathbf{d}\mathbf{x}, t + dt] &= \psi[\mathbf{x}, t] + \frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial \mathbf{x}} \mathbf{d}\mathbf{x}(t) + \frac{1}{2} \frac{\partial^2 \psi}{\partial t^2} dt^2 \\ &\quad + \frac{1}{2} \mathbf{d}\mathbf{x}(t)^T \frac{\partial^2 \psi}{\partial \mathbf{x}^2} \mathbf{d}\mathbf{x}(t) + \dots \end{aligned}$$

Now (11-73) is substituted into the last term, and recalling the Levy oscillation property (11-59), we retain only terms up to first order in  $dt$  and second order

in  $d\beta$  to get

$$\begin{aligned}\psi[\mathbf{x} + d\mathbf{x}, t + dt] - \psi[\mathbf{x}, t] &= \frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial \mathbf{x}} d\mathbf{x}(t) + \frac{1}{2} d\beta^T(t) \mathbf{G}^T \frac{\partial^2 \psi}{\partial \mathbf{x}^2} \mathbf{G} d\beta(t) \\ &= \frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial \mathbf{x}} d\mathbf{x}(t) + \frac{1}{2} \text{tr} \left\{ \mathbf{G} d\beta(t) d\beta^T(t) \mathbf{G}^T \frac{\partial^2 \psi}{\partial \mathbf{x}^2} \right\} \\ &= \frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial \mathbf{x}} d\mathbf{x}(t) + \frac{1}{2} \text{tr} \left\{ \mathbf{G} \mathbf{Q}(t) \mathbf{G}^T \frac{\partial^2 \psi}{\partial \mathbf{x}^2} \right\} dt\end{aligned}$$

where the last equality invokes the Levy property directly. Rigorous derivations of (11-84) can also be developed [30, 60]. Equation (11-84) is often combined with (11-73) and written in the form

$$d\psi[\mathbf{x}(t), t] = \frac{\partial \psi}{\partial t} dt + \mathcal{L}\{\psi[\mathbf{x}(t), t]\} dt + \frac{\partial \psi}{\partial \mathbf{x}} \mathbf{G}[\mathbf{x}(t), t] d\beta(t) \quad (11-86a)$$

$$\mathcal{L}\{\psi[\mathbf{x}(t), t]\} = \frac{\partial \psi}{\partial \mathbf{x}} \mathbf{f}[\mathbf{x}(t), t] + \frac{1}{2} \text{tr} \left\{ \mathbf{G}[\mathbf{x}(t), t] \mathbf{Q}(t) \mathbf{G}^T[\mathbf{x}(t), t] \frac{\partial^2 \psi}{\partial \mathbf{x}^2} \right\} \quad (11-86b)$$

where  $\mathcal{L}\{\psi[\mathbf{x}(t), t]\}$  is termed the differential generator of the process.

EXAMPLE 11.6 Consider a simple scalar case of (11-73) with  $f \equiv 0$ ,  $G \equiv 1$ :

$$d\mathbf{x}(t) = d\beta(t)$$

i.e.,  $\mathbf{x}(\cdot, \cdot)$  is scalar Brownian motion itself, and we really have a trivial linear stochastic differential equation, written heuristically as  $\dot{\mathbf{x}}(t) = \mathbf{w}(t)$  with  $\mathbf{w}(\cdot, \cdot)$  white Gaussian noise. Let the diffusion (white noise strength)  $Q$  be constant. Now consider the nonlinear function

$$\psi[\mathbf{x}(t), t] = e^{\mathbf{x}(t)} = e^{\beta(t)}$$

From (11-84), this satisfies the stochastic differential equation

$$d\psi = e^{\mathbf{x}(t)} d\mathbf{x}(t) + \frac{1}{2} Q e^{\mathbf{x}(t)} dt$$

or

$$d[e^{\beta(t)}] = e^{\beta(t)} d\beta(t) + \frac{1}{2} Q e^{\beta(t)} dt$$

Note that, because of the last term, this does not satisfy formal rules for differentials. Letting  $y(t) = e^{\beta(t)}$ , this yields

$$dy(t) = \frac{1}{2} Q y(t) dt + y(t) d\beta(t); \quad y(t_0) = 1 \quad \text{w.p.1}$$

as the appropriate stochastic differential equation in the form of (11-73) or equivalently (11-75) to yield a solution in the form of  $e^{\beta(t)}$ , since  $\beta(t_0) = 0$  w.p.1. Thus, it can be seen that stochastic differential equations do not obey formal rules of integration either. The differential equation that would have been proposed by formal rules,

$$dz(t) = z(t) d\beta(t); \quad z(t_0) = 1 \quad \text{w.p.1}$$

can be shown via (11-84) to have a solution, for  $t_0 = 0$ , of

$$z(t) = e^{[f\beta(t) - (Qt/2)]} \quad \blacksquare$$

**EXAMPLE 11.7** A model that has been used for normal acceleration of an aircraft, i.e., acceleration orthogonal to the plane of its wings, is of the form [36, 37]:

$$\mathbf{a}(t) = a_0 + b e^{c n(t)}$$

where the coefficients  $a_0$ ,  $b$ , and  $c$  are constants associated with a particular class of aircraft, and  $n(\cdot, \cdot)$  is an exponentially time-correlated Gauss–Markov motion, the solution to

$$dn(t) = -(1/T)n(t)dt + d\beta(t)$$

with  $\beta(\cdot, \cdot)$  Brownian motion of constant diffusion  $Q$ . Using (11-84),  $\mathbf{a}(\cdot, \cdot)$  could be modeled directly as the solution to the differential equation

$$\begin{aligned} d\mathbf{a}(t) &= [bce^{cn(t)}] dn(t) + \frac{1}{2}Q[bc^2e^{cn(t)}] dt \\ &= c[\mathbf{a}(t) - a_0][-(1/T)n(t)dt + d\beta(t)] + \frac{1}{2}Qc^2[\mathbf{a}(t) - a_0] dt \\ &= \{[\mathbf{a}(t) - a_0][(Qc^2/2) - (1/T)\ln((\mathbf{a}(t) - a_0)/b)]\} dt + \{c[\mathbf{a}(t) - a_0]\} d\beta(t) \end{aligned}$$

Although this is of the form of (11-73), the original linear dynamics relation with nonlinear output is more desirable computationally for generating  $\mathbf{a}(\cdot, \cdot)$ . ■

**EXAMPLE 11.8** In many physical applications, nonlinear dynamical systems are driven by time-correlated noises that can be modeled as the output of linear shaping filters driven by white Gaussian noise. In such applications, the appropriate Itô stochastic differential equations are readily generated. Let a dynamical system be described heuristically by

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{n}(t), t]$$

where  $\mathbf{f}$  is in fact often linear in  $\mathbf{n}(\cdot, \cdot)$ , and let  $\mathbf{n}$  be the output of a linear shaping filter,

$$\begin{aligned} d\mathbf{x}_f(t) &= \mathbf{F}(t)\mathbf{x}_f(t)dt + \mathbf{G}(t)d\beta(t) \\ \mathbf{n}(t) &= \mathbf{H}(t)\mathbf{x}_f(t) \end{aligned}$$

Then an augmented stochastic differential equation results as

$$\begin{bmatrix} d\mathbf{x}(t) \\ d\mathbf{x}_f(t) \end{bmatrix} = \begin{bmatrix} \mathbf{f}[\mathbf{x}(t), \mathbf{H}(t)\mathbf{x}_f(t), t] \\ \mathbf{F}(t)\mathbf{x}_f(t) \end{bmatrix} dt + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}(t) \end{bmatrix} d\beta(t) \quad ■$$

**EXAMPLE 11.9** Consider the general scalar stochastic differential equation

$$dx(t) = f[x(t), t]dt + G[x(t), t]d\beta(t)$$

with  $x(t_0) = 0$  w.p.l and  $\beta(\cdot, \cdot)$  having diffusion  $Q(t)$ . Now let  $\psi[x(t), t] = x(t)^2$ . Then, by (11-84),  $x(t)^2$  satisfies

$$\begin{aligned} d\psi[x(t), t] &\triangleq d[x^2(t)] = 2x(t)dx(t) + \frac{1}{2}\{2G[x(t), t]^2Q(t)\}dt \\ &= \{2x(t)f[x(t), t] + G[x(t), t]^2Q(t)\}dt + \{2x(t)G[x(t), t]\}d\beta(t) \end{aligned}$$

or, in view of (11-75),

$$x^2(t) = \int_{t_0}^t \{2x(\tau)f[x(\tau), \tau] + G[x(\tau), \tau]^2Q(\tau)\}d\tau + \int_{t_0}^t \{2x(\tau)G[x(\tau), \tau]\}d\beta(\tau)$$

Here, the differential generator of the process  $x^2$  is

$$\mathcal{L}\{\psi[x(t), t]\} = 2x(t)f[x(t), t] + G[x(t), t]^2Q(t) \quad ■$$

In view of Examples 11.5 and 11.6, it might seem appropriate to consider stochastic differential equations based upon the Stratonovich stochastic integral

instead of the Itô definition. The fact is, if we restrict our attention from  $\mathbf{a}(\cdot, \cdot)$  being admissible, i.e.,  $\mathbf{a}(t, \cdot)$  being dependent on at most  $\{\beta(t'), t_0 \leq t' \leq t\}$ , to  $\mathbf{a}(t, \cdot)$  being an explicit function of only  $\beta(t)$ , and if  $a[\beta(t), t]$  has a continuous partial derivative with respect to its first argument and is continuous in its second argument and  $\int_{t_0}^t E\{a[\beta(\tau), \tau]^2\} d\tau$  is finite with probability one, then the mean square limit

$$\text{l.i.m. } \sum_{N \rightarrow \infty} \sum_{i=0}^{N-1} a\left[\frac{1}{2}\beta(t_i) + \frac{1}{2}\beta(t_{i+1}), t_i\right] [\beta(t_{i+1}) - \beta(t_i)]$$

exists and defines the Stratonovich stochastic integral. Moreover, if this is denoted as  $\{\int_{t_0}^t a[\beta(\tau), \tau] d\beta(\tau)\}_S$ , then Stratonovich showed its relation to the Itô integral to be

$$\left\{ \int_{t_0}^t a[\beta(\tau), \tau] d\beta(\tau) \right\}_S = \int_{t_0}^t a[\beta(\tau), \tau] d\beta(\tau) + \frac{1}{2} \int_{t_0}^t Q(\tau) \frac{\partial a[\beta, \tau]}{\partial \beta} dt \quad \text{w.p.1} \quad (11-87)$$

As a result, for the restricted class of functions for which the Stratonovich integral can be defined, there is a one-to-one correspondence between the *scalar Itô equation*

$$d\mathbf{x}(t) = f[\mathbf{x}(t), t] dt + G[\mathbf{x}(t), t] d\beta(t) \quad (11-88a)$$

and the *scalar Stratonovich equation*

$$\begin{aligned} d\mathbf{x}(t) &= \{f[\mathbf{x}(t), t] - \frac{1}{2}Q(t)G[\mathbf{x}(t), t]\partial G[\mathbf{x}(t), t]/\partial \mathbf{x}\} dt \\ &\quad + G[\mathbf{x}(t), t] d\beta(t) \end{aligned} \quad (11-88b)$$

since  $a[\beta(t), t] \triangleq G[\mathbf{x}(t), t]$  and so  $\partial a/\partial \beta = [\partial G/\partial \mathbf{x}][\partial \mathbf{x}/\partial \beta]$ . This can be of use heuristically in generating Itô stochastic differential equations since Stratonovich equations obey formal integration rules.

**EXAMPLE 11.10** Recall Example 11.6 in which an Itô stochastic differential equation was sought so as to yield a solution  $e^{\theta(t)}$ . Using formal rules of integration, the Stratonovich equation (11-88b) would be

$$d\mathbf{x}(t) = \mathbf{x}(t) d\beta(t); \quad \mathbf{x}(t_0) = 1 \quad \text{w.p.1}$$

The associated Itô equation (11-88a) is then found as

$$\begin{aligned} d\mathbf{x}(t) &= \{\frac{1}{2}Q\mathbf{x}(t) \partial \mathbf{x} / \partial \mathbf{x}\} dt + \mathbf{x}(t) d\beta(t) \\ &= \frac{1}{2}Q\mathbf{x}(t) dt + \mathbf{x}(t) d\beta(t) \end{aligned}$$

as found previously by applying the Itô differential rule. ■

The previous discussion and example, as well as Example 11.8, have impact upon numerical *simulation* of stochastic differential equations [2, 24, 38, 51, 52,

54, 66, 74, 75], as for conducting Monte Carlo simulations on a digital computer. In essence, our stochastic differential equation models are meant to provide adequate representations of true physical processes, and we must ask how well solutions, or simulations of solutions, to these equations represent the desired physical process. If a physical process can be generated via

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), t] + \mathbf{G}[\mathbf{x}(t), t]\mathbf{n}(t) \quad (11-89)$$

or even the more general form as seen in Example 11.8, and  $\mathbf{n}(\cdot, \cdot)$  is a time-correlated Gaussian process, the appropriate stochastic differential equation and simulation thereof are readily produced via linear shaping filter design and state augmentation.

**EXAMPLE 11.11** A digital simulation of the augmented system in Example 11.8 can be generated by first developing the equivalent discrete-time model for the shaping filter:

$$\mathbf{x}_f(t_{i+1}) = \Phi(t_{i+1}, t_i)\mathbf{x}_f(t_i) + \mathbf{w}_d(t_i)$$

where  $\Phi(t_{i+1}, t_i)$  is the state transition matrix satisfying

$$\frac{d}{dt}\Phi(t, t_i) = \mathbf{F}(t)\Phi(t, t_i) \quad \text{for all } t \in [t_i, t_{i+1})$$

$$\Phi(t_i, t_i) = \mathbf{I}$$

and  $\mathbf{w}_d(\cdot, \cdot)$  is discrete-time zero-mean white Gaussian noise with  $E\{\mathbf{w}_d(t_i)\mathbf{w}_d^T(t_j)\} = \mathbf{Q}_d(t_i)\delta_{ij}$  and

$$\mathbf{Q}_d(t_i) = \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \tau)\mathbf{G}(\tau)\mathbf{Q}(\tau)\mathbf{G}^T(\tau)\Phi^T(t_{i+1}, \tau) d\tau$$

A full discussion of such simulations is provided by Sections 4.9 and 6.8 and Problem 7.14 of Volume 1.

With  $\mathbf{x}_f(\cdot, \cdot)$  so simulated,  $\mathbf{x}(\cdot, \cdot)$  can be simulated on a sample-by-sample basis using any numerical integration procedure. For instance, Euler integration would yield

$$\mathbf{x}(t_{i+1}, \omega_k) = \mathbf{x}(t_i, \omega_k) + \mathbf{f}[\mathbf{x}(t_i, \omega_k), \mathbf{H}(t_i)\mathbf{x}_f(t_i, \omega_k), t_i]\{t_{i+1} - t_i\} \quad \blacksquare$$

This approach really bypassed the modeling and simulation subtleties of Itô differential equations by accounting for the time correlation (finite bandwidth) of  $\mathbf{n}(\cdot, \cdot)$  in (11-89) and generating an augmented state equation of the form

$$d\mathbf{x}_a(t) = \mathbf{f}_a[\mathbf{x}_a(t), t] + \mathbf{G}_a(t)d\beta(t) \quad (11-90)$$

in which  $\mathbf{G}_a$  is *not* a function of  $\mathbf{x}_a(t)$ .

The question still remains: How would we simulate (11-73), *heuristically* the result of replacing time-correlated  $\mathbf{n}(\cdot, \cdot)$  with white Gaussian  $\mathbf{w}(\cdot, \cdot)$  in (11-89)? To gain some insight, consider the scalar case and a polygonal approximation to scalar Brownian motion,  $\beta_p(\cdot, \cdot)$ , defined over each interval

$[t_i, t_{i+1}]$  via

$$\beta_p(t, \cdot) = \beta(t_i, \cdot) + \frac{\beta(t_{i+1}, \cdot) - \beta(t_i, \cdot)}{t_{i+1} - t_i} \{t - t_i\}, \quad t \in [t_i, t_{i+1}] \quad (11-91)$$

Consider the differential equation

$$dx_p(t) = f[x_p(t), t] dt + G[x_p(t), t] d\beta_p(t) \quad (11-92)$$

Assume  $f[\cdot, \cdot]$  and  $G[\cdot, \cdot]$  satisfy the conditions imposed at the beginning of this section and further that  $\partial G[\cdot, \cdot]/\partial x$  be continuous in its arguments and  $\{Q(t)G[x, t] \partial G[x, t]/\partial x\}$  be uniformly Lipschitz in its first argument  $x$  and continuous in  $t$ , and let  $x_p(t_0) = x(t_0)$ . Then [74]

$$\underset{\max\{t_{i+1} - t_i\} \rightarrow 0}{\text{l.i.m.}} x_p(t, \cdot) = x(t, \cdot) \quad (11-93a)$$

for all  $t$ , where  $x(t, \cdot)$  is the solution to the scalar *Stratonovich* equation

$$dx(t) = f[x(t), t] dt + G[x(t), t] d\beta(t) \quad (11-93b)$$

or, in view of (11-88), the scalar *Itô* equation

$$\begin{aligned} dx(t) &= \{f[x(t), t] + \frac{1}{2}Q(t)G[x(t), t] \partial G[x(t), t]/\partial x\} dt \\ &\quad + G[x(t), t] d\beta(t) \end{aligned} \quad (11-93c)$$

Thus, if (11-91) and (11-92) were the basis of a numerical simulation, then as we let the simulation sample period be decreased, the results would converge in the mean square sense to the solution of (11-93b) or (11-93c).

**EXAMPLE 11.12** A digital simulation could generate a sample of a sequence of independent random variables  $[\beta(t_{i+1}, \cdot) - \beta(t_i, \cdot)]$  for  $i = 0, 1, 2, \dots$ , by independent calls to a realization generator for random variables with mean zero and variance  $[\int_{t_i}^{t_{i+1}} Q(\tau) d\tau]$ . Then, using (11-91),  $\beta_p(\cdot, \omega_k)$  would be a piecewise linear function connecting successive  $\beta(t_i, \omega_k)$  values. Then, on a sample-by-sample basis,

$$dx_p(t, \omega_k) = f[x_p(t, \omega_k), t] dt + G[x_p(t, \omega_k), t] d\beta_p(t, \omega_k)$$

becomes an ordinary nonlinear differential equation to be solved numerically by standard Runge-Kutta or predictor-corrector methods. ■

Extensions to the vector case can be made if  $\beta(\cdot, \cdot)$  has diffusion  $\mathbf{Q}(t) \equiv \mathbf{I}$  for all  $t$ : the appropriate correction term on the  $i$ th component of  $dx(t)$ , corresponding to the scalar term in (11-88b) or (11-93c), is

$$\frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n G_{kj}[x(t), t] \frac{\partial G_{ij}[x(t), t]}{\partial x_k}$$

If  $\beta(\cdot, \cdot)$  has nondiagonal diffusion  $\mathbf{Q}(t)$ , such results are not valid, but this is not a severe restriction in view of a vector extension of (11-57):

$$\begin{aligned} \int_{t_0}^t \mathbf{G}[\mathbf{x}(\tau), \tau] d\beta(\tau) &= \int_{t_0}^t \{\mathbf{G}[\mathbf{x}(\tau), \tau] \mathbf{Q}^{1/2}(\tau)\} d\beta'(\tau) \\ &\triangleq \int_{t_0}^t \mathbf{G}'[\mathbf{x}(\tau), \tau] d\beta'(\tau) \end{aligned} \quad (11-94)$$

with  $\mathbf{Q}^{1/2}(t)$  defined as in Chapter 7 of Volume 1, and evaluated for example by Cholesky decompositions described there.

Thus, it might appear that, for modeling and simulation purposes, Stratonovich stochastic integrals and differential equations are more natural than the Itô type. However, the Stratonovich definition is not as useful as the Itô form for many reasons [10, 19, 20, 24, 32–34, 40, 41, 48, 51, 52, 77]. Properties (11-64)–(11-66) are no longer valid, and Stratonovich stochastic differential equations do *not model Markov processes*, making estimation problems substantially more complicated. In fact, since the Stratonovich integral is defined only for the restricted class of  $\alpha$  functions (functions of  $\beta(t)$  only, rather than  $\{\beta(\tau), t_0 \leq \tau \leq t\}$ ), it is not applicable to continuous-measurement estimation problems in which (recalling (11-29a))  $E\{\mathbf{h}[\mathbf{x}(t), t]\}|\{\mathbf{y}(\tau), t_0 \leq \tau < t\}$ , i.e., functionals of  $\{\beta(\tau), t_0 \leq \tau < t\}$ , appear in integrands. Furthermore, the motivation to use Stratonovich results, the preservation of formal calculus rules, is not fulfilled in the general vector case [6, 7] with  $\beta(\cdot, \cdot)$  of nondiagonal diffusion  $\mathbf{Q}(t)$  or in extensions to  $\beta(\cdot, \cdot)$  being replaced by a non-Gaussian independent-increment process.

On the other hand, it is fruitful to pursue Itô stochastic integrals, which can be extended to non-Gaussian zero-mean independent-increment processes replacing  $\beta(\cdot, \cdot)$ . Properties (11-64)–(11-66) do remain valid, and generalized stochastic differential equations do still yield Markov solutions. The Itô differential rule as stated in (11-84) is not valid, but it can be modified [30, 45, 63] to handle the generalized case as well.

## 11.6 FORWARD KOLMOGOROV EQUATION

The Itô stochastic differential equations of the previous section provide a means of describing a large and important class of Markov processes known as diffusions. Being Markov, these processes are characterized by their transition probability densities, and it is natural to ask how these transition densities propagate in time. It is shown that the *forward Kolmogorov equation*, or *Fokker–Planck equation*, is the partial differential equation that these transition probability densities must satisfy in their propagation forward in time [9, 12, 13, 21, 40, 49, 53, 55, 57, 69, 70]. As might be suspected from the name, there are also backward Kolmogorov equations, which will be important and fully dis-

cussed in conjunction with stochastic optimal control problems in Chapter 13 of Volume 3.

In the special case of linear stochastic differential equations driven by Brownian motion (or, heuristically, white Gaussian noise), the transition densities remain Gaussian. Thus, although the forward Kolmogorov equation is valid for this special case, we previously did not exploit it. Instead, we generated expressions for propagating means, covariances, and covariance kernels, which totally defined the time propagation of Gaussian densities. Here, these moments do not define the transition density completely, and in fact we will not even be able to generate proper differential equations for these moments themselves without knowledge of the entire density. Thus, the forward Kolmogorov equations will be of primary importance both for stochastic process characterization in this chapter and for filter time propagation relationships to be derived in the next chapter.

Given the Markov process  $\mathbf{x}(\cdot, \cdot)$  generated as the solution to the Itô stochastic differential equation

$$\mathbf{dx}(t) = \mathbf{f}[\mathbf{x}(t), t] dt + \mathbf{G}[\mathbf{x}(t), t] d\beta(t) \quad (11-95)$$

with  $\beta(\cdot, \cdot)$  being Brownian motion of diffusion  $\mathbf{Q}(t)$  for all  $t \in [t_0, t_f]$ , the transition probability density for  $\mathbf{x}(\cdot, \cdot)$ ,  $f_{\mathbf{x}}(\xi, t | \rho, t')$  as defined in (11-48), satisfies the *forward Kolmogorov equation*:

$$\begin{aligned} \frac{\partial f_{\mathbf{x}}(\xi, t | \rho, t')}{\partial t} &= - \sum_{i=1}^n \frac{\partial}{\partial \xi_i} \{ f_{\mathbf{x}}(\xi, t | \rho, t') f_i[\xi, t] \} \\ &\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial \xi_i \partial \xi_j} [f_{\mathbf{x}}(\xi, t | \rho, t') \{ \mathbf{G}[\xi, t] \mathbf{Q}(t) \mathbf{G}^T[\xi, t] \}_{ij}] \end{aligned} \quad (11-96)$$

assuming the existence of the continuous partial derivatives as indicated. In (11-96),  $f_i[\xi, t]$  is the  $i$ th component of  $\mathbf{f}[\xi, t]$  as given in (11-95), not to be confused with the transition density  $f_{\mathbf{x}}(\xi, t | \rho, t')$ . Similarly,  $\{ \mathbf{G} \mathbf{Q} \mathbf{G}^T \}_{ij}$  is the  $i-j$  element of the  $n$ -by- $n$  matrix  $\mathbf{G} \mathbf{Q} \mathbf{G}^T$ .

*Formal proof* Given the Itô stochastic differential equation (11-95),  $\mathbf{x}(\cdot, \cdot)$  can be described via

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^t \mathbf{f}[\mathbf{x}(\tau), \tau] d\tau + \int_{t_0}^t \mathbf{G}[\mathbf{x}(\tau), \tau] d\beta(\tau)$$

We want to show that  $f_{\mathbf{x}}(\xi, t | \rho, t')$ , i.e., the conditional density of  $f_{\mathbf{x}(t)|\mathbf{x}(t')}(\xi | \rho)$ , for the process  $\mathbf{x}(\cdot, \cdot)$  does in fact satisfy the forward Kolmogorov equation (11-96). To do so [9, 32], we first assume that the derivatives

$$\partial f_{\mathbf{x}} / \partial t, \quad \partial [f_{\mathbf{x}} f_i] / \partial \xi_i, \quad \partial^2 [f_{\mathbf{x}} \{ \mathbf{G} \mathbf{Q} \mathbf{G}^T \}_{ij}] / \partial \xi_i \partial \xi_j$$

exist and are continuous for all  $i$  and  $j$ . This proof will be formal in that existence of appropriate limits is assumed rather than proven; rigorous proofs include such existence [29].

This proof will entail integration by parts of certain functions over  $R^n$ , i.e., with infinite upper and lower limits. As a general approach to this kind of problem, we first evaluate results for integrands that are nonzero only over a finite region of  $R^n$  and that smoothly go to zero at the boundary

of this region, allowing integration by parts to be performed simply. Then, the region over which the integrand can take on nonzero values is expanded to fill as much of  $R^n$  as required. To this end, we define a hypercube in  $R^n$ :

$$\{\xi : a_1 \leq \xi_1 \leq b_1, a_2 \leq \xi_2 \leq b_2, \dots, a_n \leq \xi_n \leq b_n\}$$

with an *interior* defined as the set  $S$

$$S = \{\xi : a_1 < \xi_1 < b_1, a_2 < \xi_2 < b_2, \dots, a_n < \xi_n < b_n\}$$

and a *boundary* denoted as  $B_S$ . Figure 11.5 depicts this for the two-dimensional case. Further define a nonnegative scalar function  $r(\xi)$  such that

- (1)  $r(\xi) > 0$  if  $\xi \in S$ ;
- (2)  $r(\xi) = 0$  if  $\xi \notin S$ ;
- (3)  $r(\xi)$  has continuous first and second derivatives;
- (4)  $\partial r(\xi)/\partial \xi_i = 0$  and  $\partial^2 r(\xi)/\partial \xi_i \partial \xi_j = 0$  if  $\xi \in B_S$ , for all  $i$  and  $j$ .

In other words,  $r(\xi)$  can assume arbitrary positive values in the interior  $S$ , is zero on the boundary and outside the hypercube, and goes to zero smoothly at the boundary.

Writing the partial of  $f_{\mathbf{x}}$  with respect to  $t$  as the appropriate limit, the properties of  $r(\xi)$  allow us to write

$$\int_S \frac{\partial f_{\mathbf{x}}(\xi, t | \rho, t')}{\partial t} r(\xi) d\xi = \lim_{\Delta t \rightarrow 0} \int_{-\infty}^{\infty} \frac{f_{\mathbf{x}}(\xi, t + \Delta t | \rho, t') - f_{\mathbf{x}}(\xi, t | \rho, t')}{\Delta t} r(\xi) d\xi$$

where we note the integrations are over  $n$ -dimensional hypervolumes. Since  $\mathbf{x}(\cdot, \cdot)$  is Markov (property (3) in Section 11.5), the Chapman–Kolmogorov equation (11-50) is valid and yields

$$f_{\mathbf{x}}(\xi, t + \Delta t | \rho, t') = \int_{-\infty}^{\infty} f_{\mathbf{x}}(\xi, t + \Delta t | \zeta, t) f_{\mathbf{x}}(\zeta, t | \rho, t') d\zeta$$

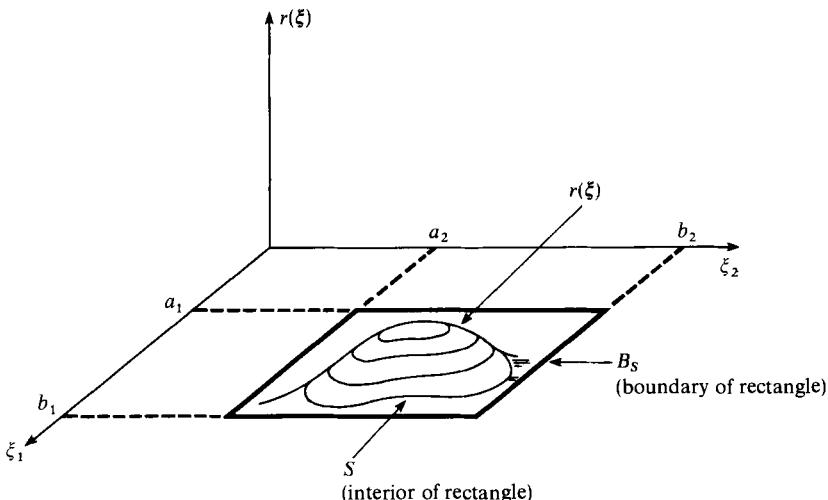


FIG. 11.5 Hypercube interior, boundary, and  $r(\xi)$  for two-dimensional case.

Substituting this into the preceding equation yields

$$\begin{aligned} \int_S \frac{\partial f_s(\xi, t | \rho, t')}{\partial t} r(\xi) d\xi &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\{ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f_s(\xi, t + \Delta t | \zeta, t) f_s(\zeta, t | \rho, t') d\zeta \right] r(\xi) d\xi \right. \\ &\quad \left. - \int_{-\infty}^{\infty} f_s(\xi, t | \rho, t') r(\xi) d\xi \right\} \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\{ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f_s(\xi, t + \Delta t | \zeta, t) r(\zeta) d\zeta \right] f_s(\zeta, t | \rho, t') d\zeta \right. \\ &\quad \left. - \int_{-\infty}^{\infty} f_s(\xi, t | \rho, t') r(\xi) d\xi \right\} \end{aligned}$$

after changing the order of integration. Now reverse the roles of the dummy variables  $\xi$  and  $\zeta$  in the first term to obtain

$$\int_S \frac{\partial f_s(\xi, t | \rho, t')}{\partial t} r(\xi) d\xi = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} f_s(\xi, t | \rho, t') \left\{ \int_{-\infty}^{\infty} f_s(\zeta, t + \Delta t | \xi, t) r(\zeta) d\zeta - r(\xi) \right\} d\xi$$

Now  $r(\zeta)$  can be expanded in a Taylor series about the point  $\xi$  to obtain

$$\begin{aligned} r(\zeta) &= r(\xi) + \frac{\partial r(\xi)}{\partial \zeta} [\zeta - \xi] + \frac{1}{2} [\zeta - \xi]^T \frac{\partial^2 r(\xi)}{\partial \zeta^2} [\zeta - \xi] + o\{||\zeta - \xi||^3\} \\ &= r(\xi) + \sum_{i=1}^n \frac{\partial r}{\partial \xi_i} [\zeta_i - \xi_i] + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 r}{\partial \xi_i \partial \xi_j} [\zeta_i - \xi_i][\zeta_j - \xi_j] + o\{||\zeta - \xi||^3\} \end{aligned}$$

Substituting this into the previous expression provides

$$\begin{aligned} \int_S \frac{\partial f_s(\xi, t | \rho, t')}{\partial t} r(\xi) d\xi &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} f_s(\xi, t | \rho, t') \left\{ \left[ \int_{-\infty}^{\infty} f_s(\zeta, t + \Delta t | \xi, t) d\zeta \right] r(\xi) - r(\xi) \right. \\ &\quad \left. + \sum_{i=1}^n \frac{\partial r(\xi)}{\partial \xi_i} \int_{-\infty}^{\infty} f_s(\zeta, t + \Delta t | \xi, t) [\zeta_i - \xi_i] d\zeta \right. \\ &\quad \left. + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 r(\xi)}{\partial \xi_i \partial \xi_j} \int_{-\infty}^{\infty} f_s(\zeta, t + \Delta t | \xi, t) [\zeta_i - \xi_i][\zeta_j - \xi_j] d\zeta \right. \\ &\quad \left. + \int_{-\infty}^{\infty} o\{||\zeta - \xi||^3\} f_s(\zeta, t + \Delta t | \xi, t) d\zeta \right\} d\xi \end{aligned}$$

But, the first two terms (on the right hand side) sum to zero, since  $\int_{-\infty}^{\infty} f_s(\zeta, t + \Delta t | \xi, t) d\zeta = 1$ . Now we take the limit as  $\Delta t \rightarrow 0$ , using properties (6)–(8) of Itô differential equation solutions in Section 11.5, to obtain

$$\begin{aligned} \int_S \frac{\partial f_s(\xi, t | \rho, t')}{\partial t} r(\xi) d\xi &= \int_{-\infty}^{\infty} f_s(\xi, t | \rho, t') \left\{ \sum_{i=1}^n \frac{\partial r(\xi)}{\partial \xi_i} f_i[\xi, t] \right. \\ &\quad \left. + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 r(\xi)}{\partial \xi_i \partial \xi_j} [\mathbf{G}(\xi, t) \mathbf{Q}(t) \mathbf{G}^T(\xi, t)]_{ij} \right\} d\xi \end{aligned}$$

Since the first and second partial derivatives of  $r$  with respect to  $\xi$  are zero outside the interior set  $S$ , the integration over all  $\xi$  on the right can be replaced by an integration only over  $S$ . The expressions

on the right hand side are now integrated by parts, using the fact that the partials of  $r$  are zero on the boundary  $B_S$ . For instance, the first term in the first sum is evaluated as

$$\int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} \underbrace{\left[ \int_{a_1}^{b_1} f_{\mathbf{x}} f_1 (\partial r / \partial \xi_1) d\xi_1 \right]}_u \underbrace{d\xi_2 \cdots d\xi_n}_v$$

Integration by parts on the bracketed term, i.e.,

$$\int_{a_1}^{b_1} u dv = uv \Big|_{a_1}^{b_1} - \int_{a_1}^{b_1} v du$$

yields

$$\int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} \left[ 0 - \int_{a_1}^{b_1} r \left\{ \frac{\partial(f_{\mathbf{x}} f_1)}{\partial \xi_1} \right\} d\xi_1 \right] d\xi_2 \cdots d\xi_n$$

The double summation terms require two integrations by parts. Performing the integrations and collecting terms yields

$$\begin{aligned} \int_S r(\xi) & \left[ \frac{\partial f_{\mathbf{x}}(\xi, t | \rho, t')}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial \xi_i} \{ f_{\mathbf{x}}(\xi, t | \rho, t') f_i[\xi, t] \} \right. \\ & \left. - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial \xi_i \partial \xi_j} [f_{\mathbf{x}}(\xi, t | \rho, t') \{ \mathbf{G}[\xi, t] \mathbf{Q}(t) \mathbf{G}^T[\xi, t] \}_{ij}] \right] d\xi = 0 \end{aligned}$$

Since  $r(\xi)$  is an arbitrary positive function in  $S$ , this implies that the term within the outer brackets must be zero, which is in fact the forward Kolmogorov equation. Note that  $S$  can be expanded as large as necessary without altering the result. ■

An alternative derivation of the forward Kolmogorov equation using characteristic functions is developed in Problem 11.14 [34, 45, 53].

**EXAMPLE 11.13** For scalar Brownian motion, the stochastic differential equation is

$$dx(t) = q^{1/2}(t) dB(t)$$

where the diffusion of  $B(\cdot, \cdot)$  is assumed to be unity and the diffusion of  $x(\cdot, \cdot)$  is then  $q(t)$ ; this is a degenerate case of (11-95) with  $f[x(t), t] \equiv 0$  and  $G[x(t), t] = G(t) = q^{1/2}(t)$ . The forward Kolmogorov equation for the Brownian motion  $x(\cdot, \cdot)$  is then

$$\frac{\partial f_x(\xi, t | \rho, t')}{\partial t} = \frac{1}{2} q(t) \frac{\partial^2 f_x(\xi, t | \rho, t')}{\partial \xi^2}$$

Note that this is also the equation for the diffusion of heat in an infinite length rod, and hence the terminology “diffusion” for  $q(t)$ . ■

The appropriate initial condition for (11-96) is simply  $f_x(\xi, t' | \rho, t') = \delta(\xi - \rho)$ . But we can view  $\mathbf{x}(t')$  itself as a random variable with associated dummy variable  $\rho$ , instead of considering  $\rho$  merely as a given realization of  $\mathbf{x}(t')$ . Thus,  $f_x(\xi, t | \rho, t')$  itself can be a random variable, a function of  $\mathbf{x}(t')$ . Its expected value is (recalling (11-52), the expectation of the Chapman–Kolmogorov equation):

$$f_{x(t)}(\xi) = \int_{-\infty}^{\infty} f_x(\xi, t | \rho, t') f_{\mathbf{x}(t')}(\rho) d\rho \quad (11-97)$$

where now  $f_{\mathbf{x}(t')}(p)$  is considered as an initial condition. Now take (11-96), multiply through by  $f_{\mathbf{x}(t')}(p) dp$ , and integrate out the dependence on  $p$ . By so doing, one can conclude that  $f_{\mathbf{x}(t)}(\xi)$  also satisfies the forward Kolmogorov equation:

$$\begin{aligned} \frac{\partial f_{\mathbf{x}(t)}(\xi)}{\partial t} &= - \sum_{i=1}^n \frac{\partial}{\partial \xi_i} \{ f_{\mathbf{x}(t)}(\xi) f_i[\xi, t] \} \\ &\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial \xi_i \partial \xi_j} [f_{\mathbf{x}(t)}(\xi) \{ \mathbf{G}[\xi, t] \mathbf{Q}(t) \mathbf{G}^T[\xi, t] \}_{ij}] \end{aligned} \quad (11-98)$$

Solutions to the forward Kolmogorov equation are not generally simple to obtain analytically [3]. However, as the next example shows, solutions can be generated for applications involving linear system models.

**EXAMPLE 11.14** Consider the scalar linear system model

$$dx(t) = a(t)x(t) dt + g(t)d\beta(t)$$

where  $\beta(\cdot, \cdot)$  is scalar Brownian motion of diffusion  $q(t)$ . Thus, we can identify

$$f[\xi, t] = a(t)\xi, \quad G[\xi, t]Q(t)G^T[\xi, t] = g^2(t)q(t)$$

The forward Kolmogorov equation becomes

$$\frac{\partial f_x}{\partial t} = - \left[ a(t)\xi \frac{\partial f_x}{\partial \xi} + a(t)f_x \right] + \frac{1}{2} g^2(t)q(t) \frac{\partial^2 f_x}{\partial \xi^2}$$

This can be used either to propagate  $f_x(\xi, t | \rho, t')$  forward from  $f_x(\xi, t' | \rho, t') = \delta(\xi - \rho)$ , or to propagate  $f_{\mathbf{x}(t)}(\xi)$  from an initial condition of  $f_{\mathbf{x}(t')}(p)$ . To describe either density's time propagation, it is easiest to employ the characteristic function; for instance,

$$\phi_x(\mu, t) = \int_{-\infty}^{\infty} e^{j\mu\xi} f_{\mathbf{x}(t)}(\xi) d\xi$$

Multiplying both sides of (11-98) by  $e^{j\mu\xi}$  and integrating yields

$$\frac{\partial \phi_x(\mu, t)}{\partial t} = -a(t)\phi_x(\mu, t) + \int_{-\infty}^{\infty} \left\{ -a(t)\xi \frac{\partial \phi_x}{\partial \xi} + \frac{1}{2} g^2(t)q(t) \frac{\partial^2 \phi_x}{\partial \xi^2} \right\} e^{j\mu\xi} d\xi$$

The left hand side is obtained by writing the original partial as the appropriate limit,

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \{ f_{\mathbf{x}(t+\Delta t)}(\xi) - f_{\mathbf{x}(t)}(\xi) \}$$

multiplying each term by  $e^{j\mu\xi}$  and integrating, and then taking the indicated limit. Now perform integration by parts, assuming that  $f_{\mathbf{x}(t)}(\xi) \rightarrow 0$  as  $\xi \rightarrow \pm\infty$ , i.e., the boundary condition at the edge of the space is that the density has asymptotically approached zero height. First consider the term

$$\int_{-\infty}^{\infty} -a(t)\xi e^{j\mu\xi} \frac{\partial f_x}{\partial \xi} d\xi = -a(t) \underbrace{\int_{-\infty}^{\infty} \xi e^{j\mu\xi} \frac{\partial f_x}{\partial \xi} d\xi}_u \underbrace{dv}_v$$

Since  $\int_{-\infty}^{\infty} u dv = uv \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v du$ , this is equal to

$$-a(t) \left[ 0 - \int_{-\infty}^{\infty} (e^{j\mu\xi} + j\mu\xi e^{j\mu\xi}) f_x d\xi \right] = a(t) \left[ \phi_x(\mu, t) + \mu \frac{\partial \phi_x(\mu, t)}{\partial \mu} \right]$$

Similarly, by two integrations by parts,

$$\frac{1}{2} g^2(t) q(t) \int_{-\infty}^{\infty} \frac{\partial^2 f_x}{\partial \xi^2} e^{j\mu\xi} d\xi = -\frac{1}{2} g^2(t) q(t) \mu^2 \phi_x(\mu, t)$$

Substituting these results into the expression for  $\partial \phi_x / \partial t$  yields

$$\partial \phi_x / \partial t = a(t) \mu \partial \phi_x / \partial \mu - \frac{1}{2} g^2(t) q(t) \mu^2 \phi_x$$

From our previous work, we know that  $x(\cdot, \cdot)$  is in fact a Gaussian process, so we will assume that  $\phi_x(\mu, t)$  is the characteristic function for a scalar Gaussian  $x(\cdot, \cdot)$ , namely

$$\phi_x(\mu, t) = e^{j\mu m_x(t) - (1/2)\mu^2 P_x(t)}$$

so that

$$\begin{aligned} \partial \phi_x / \partial t &= [j\mu m_x(t) - \frac{1}{2}\mu^2 \dot{P}_x(t)] \phi_x(\mu, t) \\ \partial \phi_x / \partial \mu &= [jm_x(t) - \mu P_x(t)] \phi_x(\mu, t) \end{aligned}$$

Substituting these expressions into the equation above yields

$$[j\mu m_x(t) - \frac{1}{2}\mu^2 \dot{P}_x(t)] \phi_x(\mu, t) = a(t) \mu [jm_x(t) - \mu P_x(t)] \phi_x(\mu, t) - \frac{1}{2} g^2(t) q(t) \mu^2 \phi_x(\mu, t)$$

Now divide through by  $\phi_x(\mu, t)$ , and equate the real and imaginary parts of the equation separately (since these must be true individually for the total equation to be true), generating

$$\dot{m}_x(t) = a(t) m_x(t), \quad \dot{P}_x(t) = 2a(t) P_x(t) + g^2(t) q(t)$$

These can be recognized as the scalar form of the equations for propagating the mean and covariance describing  $f_{x(t)}(\xi)$ , as derived previously, i.e., (11-17) and (11-18). ■

**EXAMPLE 11.15** The previous example generalizes readily to the case of vector  $\mathbf{x}(\cdot, \cdot)$ :

$$d\mathbf{x}(t) = \mathbf{F}(t)\mathbf{x}(t) dt + \mathbf{G}(t) d\beta(t)$$

where  $\beta(\cdot, \cdot)$  is Brownian motion of diffusion  $\mathbf{Q}(t)$ . The forward Kolmogorov equation becomes

$$\frac{\partial f_{\mathbf{x}}}{\partial t} = - \left[ \frac{\partial f_{\mathbf{x}}}{\partial \xi} \mathbf{F}(t) \xi + f_{\mathbf{x}} \text{tr}\{\mathbf{F}(t)\} \right] + \frac{1}{2} \text{tr} \left\{ \mathbf{G}(t) \mathbf{Q}(t) \mathbf{G}^T(t) \frac{\partial^2 f_{\mathbf{x}}}{\partial \xi^2} \right\}$$

and the corresponding time propagation for the characteristic function is

$$\frac{\partial \phi_{\mathbf{x}}}{\partial t} = \boldsymbol{\mu}^T \mathbf{F}(t) \frac{\partial \phi_{\mathbf{x}}^T}{\partial \boldsymbol{\mu}} - \frac{1}{2} \boldsymbol{\mu}^T \mathbf{G}(t) \mathbf{Q}(t) \mathbf{G}^T(t) \boldsymbol{\mu} \phi_{\mathbf{x}} \quad ■$$

**EXAMPLE 11.16** Consider the nonlinear system described by the  $n$ -dimensional vector stochastic differential equation

$$d\mathbf{x}(t) = \mathbf{f}[\mathbf{x}(t), t] dt + \mathbf{G}(t) d\beta(t)$$

This is the special case of (11-95) upon which the extended Kalman filter was formulated in Chapter 9:  $\mathbf{G}$  is not a function of  $\mathbf{x}(\cdot, \cdot)$ , but merely a deterministic function of time. In this case, the

forward Kolmogorov equation is

$$\begin{aligned}\frac{\partial f_{\mathbf{x}}}{\partial t} &= - \sum_{i=1}^n \frac{\partial}{\partial \xi_i} [f_{\mathbf{x}} f_i] + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left\{ \frac{\partial^2}{\partial \xi_i \partial \xi_j} [f_{\mathbf{x}}] \right\} \{ \mathbf{G}(t) \mathbf{Q}(t) \mathbf{G}^T(t) \}_{ij} \\ &= - \sum_{i=1}^n \frac{\partial}{\partial \xi_i} [f_{\mathbf{x}} f_i] + \frac{1}{2} \text{tr} \left\{ \left[ \frac{\partial^2 f_{\mathbf{x}}}{\partial \xi_i \partial \xi_j} \right] [\mathbf{G}(t) \mathbf{Q}(t) \mathbf{G}^T(t)] \right\} \quad \blacksquare\end{aligned}$$

Rather than attempting to propagate the entire density  $f_{\mathbf{x}}(\xi, t | \rho, t')$  or  $f_{\mathbf{x}(t)}(\xi)$ , we might try to achieve a *partial* description of the density by propagating a finite number of its moments [40, 41, 76]. Unlike the case involving linear dynamics driven by white Gaussian noise, we can only expect a partial description since  $f_{\mathbf{x}(t)}(\xi)$  will not generally be of Gaussian or other form described completely by the first two, or any finite number, of moments. However, even this partial description cannot be obtained precisely: propagation of the mean and covariance of  $\mathbf{x}(t)$  for all  $t$  will *not be feasible without knowledge of the entire density  $f_{\mathbf{x}(t)}(\xi)$  for all  $t$* ! In fact, if  $\mathbf{x}(\cdot, \cdot)$  is the Markov solution to (11-95), then its mean  $\mathbf{m}_x(t)$  and covariance  $\mathbf{P}_x(t)$  propagate according to

$$\dot{\mathbf{m}}_x(t) = E\{\mathbf{f}[\mathbf{x}(t), t]\} \quad (11-99)$$

$$\begin{aligned}\dot{\mathbf{P}}_x(t) &= [E\{\mathbf{f}[\mathbf{x}(t), t] \mathbf{x}^T(t)\} - E\{\mathbf{f}[\mathbf{x}(t), t]\} \mathbf{m}_x^T(t)] \\ &\quad + [E\{\mathbf{x}(t) \mathbf{f}^T[\mathbf{x}(t), t]\} - \mathbf{m}_x(t) E\{\mathbf{f}^T[\mathbf{x}(t), t]\}] \\ &\quad + E\{\mathbf{G}[\mathbf{x}(t), t] \mathbf{Q}(t) \mathbf{G}^T[\mathbf{x}(t), t]\} \quad (11-100)\end{aligned}$$

as will now be shown.

### Mean and Covariance Propagation Equations

Let  $\mathbf{x}(\cdot, \cdot)$  be the solution to (11-95) and define the mean and covariance of  $\mathbf{x}(t, \cdot)$  for all  $t$  as

$$\begin{aligned}\mathbf{m}_x(t) &= \int_{-\infty}^{\infty} \xi f_{\mathbf{x}(t)}(\xi) d\xi \\ \mathbf{P}_x(t) &= \int_{-\infty}^{\infty} \xi \xi^T f_{\mathbf{x}(t)}(\xi) d\xi - \mathbf{m}_x(t) \mathbf{m}_x^T(t)\end{aligned}$$

Differentiating these expressions with respect to time yields

$$\begin{aligned}\dot{\mathbf{m}}_x(t) &= \int_{-\infty}^{\infty} \xi \frac{\partial f_{\mathbf{x}(t)}(\xi)}{\partial t} d\xi \\ \dot{\mathbf{P}}_x(t) &= \int_{-\infty}^{\infty} \xi \xi^T \frac{\partial f_{\mathbf{x}(t)}(\xi)}{\partial t} d\xi - \dot{\mathbf{m}}_x(t) \mathbf{m}_x^T(t) - \mathbf{m}_x(t) \dot{\mathbf{m}}_x^T(t)\end{aligned}$$

Since  $f_{\mathbf{x}(t)}(\xi)$  satisfies the forward Kolmogorov equation, we can evaluate  $\dot{\mathbf{m}}_x(t)$  as

$$\begin{aligned}\dot{\mathbf{m}}_x(t) &= - \int_{-\infty}^{\infty} \sum_{i=1}^n \frac{\partial}{\partial \xi_i} \{ f_{\mathbf{x}(t)}(\xi) f_i[\xi, t] \} \xi d\xi \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial \xi_i \partial \xi_j} [f_{\mathbf{x}(t)}(\xi) \{ \mathbf{G}[\xi, t] \mathbf{Q}(t) \mathbf{G}^T[\xi, t] \}_{ij}] \xi d\xi\end{aligned}$$

Writing out the first term on the right hand side yields

$$-\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial \xi_1} [f_x f_1] + \frac{\partial}{\partial \xi_2} [f_x f_2] + \cdots + \frac{\partial}{\partial \xi_n} [f_x f_n] \right\} \xi d\xi_1 d\xi_2 \cdots d\xi_n$$

Let us evaluate the first of these  $n$  terms by parts to get

$$\begin{aligned} & -\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial}{\partial \xi_1} [f_x f_1] \xi d\xi_1 d\xi_2 \cdots d\xi_n \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[ -\int_{-\infty}^{\infty} \underbrace{\xi \frac{\partial}{\partial \xi_1} [f_x f_1]}_{u} d\xi_1 \right] d\xi_2 \cdots d\xi_n \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[ -f_x f_1 \xi \Big|_{\xi_1=-\infty}^{\xi_1=\infty} + \int_{-\infty}^{\infty} f_x f_1 \frac{\partial \xi}{\partial \xi_1} d\xi_1 \right] d\xi_2 \cdots d\xi_n \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f_x f_1 \frac{\partial \xi}{\partial \xi_1} d\xi_1 \right] d\xi_2 \cdots d\xi_n \end{aligned}$$

Treating the other  $(n-1)$  terms similarly yields the first right hand side term of the  $\dot{\mathbf{m}}_x(t)$  equation as

$$+ \int_{-\infty}^{\infty} f_{x(t)}(\xi) \sum_{i=1}^n \left\{ f_i[\xi, t] \frac{\partial \xi}{\partial \xi_i} \right\} d\xi$$

But note that  $\partial \xi / \partial \xi_1 = [1 \ 0 \ 0 \ \cdots \ 0]^T$ ,  $\partial \xi / \partial \xi_2 = [0 \ 1 \ 0 \ \cdots \ 0]^T$ , etc., so that the summation term above is just  $\mathbf{f}[\xi, t]$ , yielding

$$+ \int_{-\infty}^{\infty} f_{x(t)}(\xi) \mathbf{f}[\xi, t] d\xi = + E\{\mathbf{f}[\mathbf{x}(t), t]\}$$

Integrating the second term in the  $\dot{\mathbf{m}}_x(t)$  equation by parts twice results in

$$\frac{1}{2} \int_{-\infty}^{\infty} f_{x(t)}(\xi) \sum_{i=1}^n \sum_{j=1}^n \left( \frac{\partial^2 \xi}{\partial \xi_i \partial \xi_j} \{ \mathbf{G}[\xi, t] \mathbf{Q}(t) \mathbf{G}^T[\xi, t] \}_{ij} \right) d\xi = \mathbf{0}$$

where equality to zero follows from the fact that

$$\frac{\partial^2 \xi}{\partial \xi_i \partial \xi_j} = \frac{\partial}{\partial \xi_i} \left[ \frac{\partial \xi}{\partial \xi_j} \right] = \frac{\partial}{\partial \xi_i} [\mathbf{e}_j] = \mathbf{0}$$

since  $\mathbf{e}_j$  is an  $n$ -vector of zeros except for a unit  $j$ th component. Thus we have obtained (11-99).

Similarly, the  $\dot{\mathbf{P}}_x(t)$  equation becomes

$$\begin{aligned} \dot{\mathbf{P}}_x(t) &= - \int_{-\infty}^{\infty} \sum_{i=1}^n \frac{\partial}{\partial \xi_i} \{ f_{x(t)}(\xi) f_i[\xi, t] \} \xi \xi^T d\xi \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial \xi_i \partial \xi_j} [f_{x(t)}(\xi) \{ \mathbf{G}[\xi, t] \mathbf{Q}(t) \mathbf{G}^T[\xi, t] \}_{ij}] \xi \xi^T d\xi \\ &- \dot{\mathbf{m}}_x(t) \mathbf{m}_x^T(t) - \mathbf{m}_x(t) \dot{\mathbf{m}}_x^T(t) \end{aligned}$$

By integrating the first term by parts, we obtain

$$- \int_{-\infty}^{\infty} \sum_{i=1}^n \frac{\partial}{\partial \xi_i} \{ f_{x(t)}(\xi) f_i[\xi, t] \} \xi \xi^T d\xi = + \int_{-\infty}^{\infty} f_{x(t)}(\xi) \sum_{i=1}^n \left( f_i[\xi, t] \frac{\partial (\xi \xi^T)}{\partial \xi_i} \right) d\xi$$

But

$$\begin{aligned}\partial(\xi\xi^T)/\partial\xi_1 &= [1 \ 0 \ 0 \ \cdots \ 0]^T\xi^T + \xi[1 \ 0 \ 0 \ \cdots \ 0] \\ \partial(\xi\xi^T)/\partial\xi_2 &= [0 \ 1 \ 0 \ \cdots \ 0]^T\xi^T + \xi[0 \ 1 \ 0 \ \cdots \ 0], \text{ etc.}\end{aligned}$$

}

So this first term becomes

$$\int_{-\infty}^{\infty} f_{\mathbf{x}(t)}(\xi) \{ \mathbf{f}[\xi, t] \xi^T + \xi \mathbf{f}^T[\xi, t] \} d\xi = E\{\mathbf{f}[\mathbf{x}(t), t] \mathbf{x}^T(t)\} + E\{\mathbf{x}(t) \mathbf{f}^T[\mathbf{x}(t), t]\}$$

Performing a double integration by parts on the second term in the  $\dot{\mathbf{P}}_x(t)$  relation generates

$$\begin{aligned}&\frac{1}{2} \int_{-\infty}^{\infty} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial\xi_i \partial\xi_j} [f_{\mathbf{x}(t)}(\xi) \{ \mathbf{G}[\xi, t] \mathbf{Q}(t) \mathbf{G}^T[\xi, t] \}_{ij}] \xi \xi^T d\xi \\ &= \frac{1}{2} \int_{-\infty}^{\infty} f_{\mathbf{x}(t)}(\xi) \sum_{i=1}^n \sum_{j=1}^n \left( \{ \mathbf{G}[\xi, t] \mathbf{Q}(t) \mathbf{G}^T[\xi, t] \}_{ij} \frac{\partial^2(\xi\xi^T)}{\partial\xi_i \partial\xi_j} \right) d\xi\end{aligned}$$

But

$$\begin{aligned}\frac{\partial^2(\xi\xi^T)}{\partial\xi_1^2} &= \frac{\partial}{\partial\xi_1} \{ [1 \ 0 \ 0 \ \cdots \ 0]^T \xi^T + \xi[1 \ 0 \ 0 \ \cdots \ 0] \} \\ &= \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\end{aligned}$$

and similarly for the other such terms, so that this integral becomes

$$\frac{1}{2} \left( 2 \int_{-\infty}^{\infty} f_{\mathbf{x}(t)}(\xi) \{ \mathbf{G}[\xi, t] \mathbf{Q}(t) \mathbf{G}^T[\xi, t] \} \right) = E\{\mathbf{G}[\mathbf{x}(t), t] \mathbf{Q}(t) \mathbf{G}^T[\mathbf{x}(t), t]\}$$

Substituting these results into the  $\dot{\mathbf{P}}_x(t)$  equation generates (11-100). ■

Note that (11-99) and (11-100) are analogous to the results

$$\dot{\mathbf{m}}_x(t) = \mathbf{F}(t) \mathbf{m}_x(t) \quad (11-101)$$

$$\dot{\mathbf{P}}_x(t) = \mathbf{F}(t) \mathbf{P}_x(t) + \mathbf{P}_x(t) \mathbf{F}^T(t) + \mathbf{G}(t) \mathbf{Q}(t) \mathbf{G}^T(t) \quad (11-102)$$

corresponding to the linear equation

$$d\mathbf{x}(t) = \mathbf{F}(t) \mathbf{x}(t) dt + \mathbf{G}(t) d\beta(t) \quad (11-103)$$

Moreover, (11-99) and (11-100) reduce to these results for the case of  $\mathbf{f}[\mathbf{x}(t), t] = \mathbf{F}(t) \mathbf{x}(t)$ ,  $\mathbf{G}[\mathbf{x}(t), t] = \mathbf{G}(t)$ . However, unlike this special case, to calculate  $\dot{\mathbf{m}}_x(t)$  and  $\dot{\mathbf{P}}_x(t)$  for propagation purposes, we generally require the *entire density*  $f_{\mathbf{x}(t)}(\xi)$  in order to evaluate the expectations in (11-99) and (11-100).

Looking ahead to the optimal filtering problem based upon nonlinear dynamics models, we can gain an appreciation for the unique complexity of the results as compared to linear filtering. An obvious extension of the preceding discussion reveals that the *entire density*  $f_{\mathbf{x}(t)|\mathbf{z}(t_i)}(\xi | \mathbf{Z}_i)$  would be required merely

to compute the propagations of the conditional mean and conditional covariance of  $\mathbf{x}(t)$  from sample time  $t_i$  to sample time  $t_{i+1}$ . The measurement updates will also be more complex, as will estimators for the continuous-measurement case. These topics will be the subject of the next chapter.

## 11.7 SUMMARY

In general, a stochastic process  $\mathbf{x}(\cdot, \cdot)$  is characterized by a joint distribution  $F_{\mathbf{x}(t_1), \dots, \mathbf{x}(t_N)}(\xi_1, \dots, \xi_N)$  for all times  $t_i$  in a time set  $T$  of interest, an uncountably infinite number of times for continuous-time processes. If we restrict our attention to *Markov* processes, this complete description is also provided by the transition probability distribution  $F_{\mathbf{x}(t)|\mathbf{x}(t')}(\xi|\rho)$  or its derivative (assumed here to exist), the transition probability density. Besides being descriptive of a large class of physically observed phenomena, Markov processes thus will furnish substantial benefit in mathematical tractability of process characterization and subsequent estimator and stochastic controller derivations.

Itô nonlinear stochastic differential equations of the form of (11-73)–(11-75) generate solution processes which are in fact Markov. Proper development of these equations is based upon the definition of the Itô stochastic integral as in (11-63) and (11-69), and the associated Itô stochastic differential as in (11-70) and (11-71). The properties of these integrals, differentials, and solutions to differential equations were seen to be affected particularly by the Levy oscillation property of Brownian motion, (11-59). Formal rules of calculus were seen to be invalid, and instead the Itô differential rule, (11-84), is a fundamental result for exploiting Itô stochastic differential equations for nonlinear dynamics models.

Since the solutions to Itô differential equations are Markov, the time propagation of the transition probability density  $f_{\mathbf{x}}(\xi, t|\rho, t')$  is of primary importance for characterizing the process itself. The forward Kolmogorov equation (11-96) was shown to be the partial differential equation  $f_{\mathbf{x}}(\xi, t|\rho, t')$  satisfies through such time propagation.

Nonlinear state dynamics models in the form of Itô stochastic differential equations can be combined with either sampled-data measurement models of the form in (11-28), or continuous-time measurements as modeled by (11-29), to generate overall nonlinear system models. Those will be the basis not only of describing physical phenomena mathematically, but also of estimator and stochastic controller formulations in subsequent chapters.

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## PROBLEMS

**11.1** A linear stochastic system satisfies the differential equation

$$d^2x(t)/dt^2 + x(t) = n(t)$$

where  $n(\cdot, \cdot)$  is exponentially time-correlated noise, of mean zero and autocorrelation

$$E\{n(t)n(t+\tau)\} = \sigma_n^2 e^{-\alpha|\tau|}$$

Is  $\mathbf{x}(\cdot, \cdot)$  a Markov process? Explain your answer fully, indicating the Markov nature of any and all processes associated with this problem description.

**11.2** Show that the solution process (11-12) to the linear stochastic differential equation (11-1) or (11-10) is in fact Markov, as claimed in the text below (11-13).

**11.3** Show that if  $\mathbf{x}(\cdot, \cdot)$  is Markov-2, then  $\mathbf{y}(\cdot, \cdot)$  defined in (11-35) is Markov-1 as claimed.

**11.4** The electronic calculator market is highly competitive. Consider the simplified model of success probability as follows. There are two states: (1) your calculator is in public favor, and (2) your calculator is not in public favor. Suppose that if you are in state 1 at the beginning of a 3-month period, there is a 50% chance of going out of favor by the end of the quarter. If you are in state 2 at the beginning of a quarter, there is only a 40% chance of returning to state 1 by the end of the quarter. What is the long-term (steady state) probability that your calculator will be in the public favor?

Suppose you had an initial advantage in the market, so that your calculator is totally successful: the probability of being in state 1 is 100%. Suppose further that you require the probability of being in public favor to be at least 45% to remain profitable. After how many months will you have to introduce a new model to maintain a competitive edge?

**11.5** Consider the reliability problem in Example 11.4.

(a) Plot the time history of  $p_4(t_i)$ , the probability of system failure versus time.

(b) Consider the same problem, but modified by the fact that if actuator  $A$  has failed, the probability of actuator  $B$  failing in the next  $\Delta t$  sec is 0.03 instead of 0.02.  $B$  is no more likely to fail than  $A$  if both are working to start, but assume  $B$  is less able to operate reliably than  $A$  when it is subjected to the full load. Carry through all evaluations to the time history of  $p_4(t_i)$ .

(c) Now assume that  $B$  is generally a less reliable actuator than  $A$ . Given that both are working, the probability that  $A$  will fail in the next  $\Delta t$  sec is still 0.01, but the probability that  $B$  will fail is 0.02. Failures are still independent of one another. Assume that failure probabilities conditioned on one actuator already failed are as in part (b). Again plot  $p_4(t_i)$  versus time.

(d) Return to the original problem as in part (a). However, now assume that online repair (or healing) and recertification of the actuator as operational is possible. Mathematically, let  $T_{12} = T_{13} = 0.25$  instead of 0, and rework the problem.

(e) Repeat part (d) but with  $T_{12} = T_{13} = 0.5$ , and compare results.

**11.6** Explicitly demonstrate the mean square convergence results of (11-59), corresponding to the Levy oscillation property, by the means described in the text.

**11.7** (a) Show that the expectation and limit-in-the-mean (l.i.m.) operations are interchangeable; i.e., that for a sequence of scalar random variables  $x_N$  for  $N = 1, 2, \dots$ ,

$$E\left\{\text{l.i.m. } x_N\right\} = \lim_{N \rightarrow \infty} E\{x_N\}$$

as claimed in the demonstration of (11-65).

To show this, consider the Schwarz inequality for elements  $x_1$  and  $x_2$  of a linear vector space  $X$  with inner product  $\langle \cdot, \cdot \rangle$  defined on it (to establish a measure of "length" of elements):

$$\langle x_1, x_2 \rangle^2 \leq \langle x_1, x_1 \rangle \langle x_2, x_2 \rangle$$

which, for the space of scalar random variables with finite second moments, becomes

$$E\{|x_1 x_2|\}^2 \leq E\{|x_1|^2\} E\{|x_2|^2\}$$

(for real random variables, the magnitude notation  $||$  can be removed). Let  $x_1 = \{x_N - x\}$  and  $x_2 \equiv 1$  in the preceding expression, where  $x = \text{l.i.m. } x_N$ .

- (b) Similarly show that for sequences  $\{x_N\}$  and  $\{y_N\}$  with finite second moments,

$$E \left\{ \left[ \lim_{N \rightarrow \infty} x_N \right] \left[ \lim_{N \rightarrow \infty} y_N \right] \right\} = \lim_{N \rightarrow \infty} E\{x_N y_N\}$$

used in the demonstration of (11-66). (Note the important special case of  $x_N \equiv y_N$ .) To do this, define  $x = \lim_{N \rightarrow \infty} x_N$  and  $y = \lim_{N \rightarrow \infty} y_N$ , and write

$$|E\{xy\} - E\{x_N y_N\}| = |E\{y - y_N\}x + E\{(x - x_N)y\} + E\{(x - x_N)(y - y_N)\}|$$

and relate this to the sum of three individual magnitude terms, and then apply the Schwarz inequality to these three terms.

### 11.8 Show that

$$\lim_{N \rightarrow \infty} \int_{t_0}^t E\{\mathbf{a}_N(\tau)\mathbf{b}_N(\tau)\}q(\tau) d\tau = \int_{t_0}^t E\{\mathbf{a}(\tau)\mathbf{b}(\tau)\}q(\tau) d\tau$$

as claimed in the demonstration of (11-66). To do so, first show explicitly what it means for “ $\{\mathbf{a}_N(\tau)\}$  to converge to  $\mathbf{a}(\tau)$ ,  $t_0 \leq \tau \leq t$ , in the sense of (11-62).” Then write

$$\begin{aligned} & \left| \int_{t_0}^t E\{\mathbf{a}_N(\tau)\mathbf{b}_N(\tau) - \mathbf{a}(\tau)\mathbf{b}(\tau)\}q(\tau) d\tau \right| \\ &= \left| \int_{t_0}^t E\{\mathbf{a}_N(\tau)[\mathbf{b}_N(\tau) - \mathbf{b}(\tau)] + [\mathbf{a}_N(\tau) - \mathbf{a}(\tau)]\mathbf{b}(\tau)\}q(\tau) d\tau \right| \\ &\leq \left| \int_{t_0}^t E\{\mathbf{a}_N(\tau)[\mathbf{b}_N(\tau) - \mathbf{b}(\tau)]\}q(\tau) d\tau \right| + \left| \int_{t_0}^t E\{[\mathbf{a}_N(\tau) - \mathbf{a}(\tau)]\mathbf{b}(\tau)\}q(\tau) d\tau \right| \end{aligned}$$

and show that this converges to zero as  $N \rightarrow \infty$  by applying the Schwarz inequality.

**11.9** Evaluate the results of applying the Itô differential rule in Example 11.9 for the specific choice of  $f \equiv 0$  and  $G \equiv 1$ , and relate this to the evaluation of the Itô stochastic integral given in Example 11.5.

**11.10** Let  $\mathbf{x}(\cdot, \cdot)$  satisfy the scalar version of the general stochastic differential equation described by (11-73) and (11-74). Use the Itô differential rule to show that

$$d \ln[\mathbf{x}(t)] = \frac{dx(t)}{x(t)} - \frac{1}{2} \frac{G^2[x(t), t]Q(t)}{x^2(t)}$$

**11.11 (a)** Use the Itô differential rule to derive the “fundamental theorem of Itô stochastic calculus [32]”: if  $\psi[\beta(t)]$  is a twice continuously differentiable real scalar function of scalar Brownian motion of diffusion  $Q(t)$  for all  $t$ , then for all  $t > t'$ ,

$$\int_{t'}^t \frac{\partial \psi[\beta(\tau)]}{\partial \beta(\tau)} d\beta(\tau) = \psi[\beta(t)] - \psi[\beta(t')] - \frac{1}{2} \int_{t'}^t \frac{\partial^2 \psi[\beta(\tau)]}{\partial \beta(\tau)^2} Q(\tau) d\tau$$

- (b) Use this result to evaluate  $\int_{t'}^t \beta(\tau) d\beta(\tau)$ .
- (c) Similarly evaluate  $\int_{t'}^t \beta^2(\tau) d\beta(\tau)$ .
- (d) By induction, evaluate  $\int_{t'}^t \beta^N(\tau) d\beta(\tau)$ .

**11.12 (a)** Consider the general coupled scalar stochastic differential equations

$$dx_1(t) = f_1[x_1(t), x_2(t), t] dt + G_1[x_1(t), x_2(t), t] d\beta(t)$$

$$dx_2(t) = f_2[x_1(t), x_2(t), t] dt + G_2[x_1(t), x_2(t), t] d\beta(t)$$

Show that the product  $[x_1(t)x_2(t)]$  obeys

$$\begin{aligned} d[x_1(t)x_2(t)] &= x_1(t)dx_2(t) + x_2(t)dx_1(t) \\ &\quad + G_1[x_1(t), x_2(t), t]G_2[x_1(t), x_2(t), t]Q(t)dt \end{aligned}$$

by use of the Itô differential rule. Show that this reduces to the result of Example 11.9 if  $x_1 = x_2 = x$ .

(b) Now apply this product rule to derive the covariance propagation equations for the case of linear stochastic differential equations

$$dx(t) = F(t)x(t)dt + G(t)d\beta(t)$$

each component of which can be written as

$$dx_i(t) = \sum_{k=1}^n F_{ik}(t)x_k(t)dt + \sum_{k=1}^s G_{ik}(t)d\beta_k(t)$$

or, equivalently,

$$d[x_i(t) - E\{x_i(t)\}] = \sum_{k=1}^n F_{ik}(t)[x_k(t) - E\{x_k(t)\}] + \sum_{k=1}^s G_{ik}(t)d\beta_k(t)$$

First consider the simple case in which the diffusion of  $\beta(\cdot, \cdot)$  is  $Q(t) \equiv I$  for all time  $t$ . Generate an expression for  $d[x_i(t) - E\{x_i(t)\}][x_j(t) - E\{x_j(t)\}]$ . Take the expectation of this to yield

$$dP_{ij}(t) = \sum_{k=1}^n \{F_{ik}(t)P_{kj}(t) + F_{jk}(t)P_{ki}(t)\} + \sum_{k=1}^s G_{ik}(t)G_{jk}(t)$$

or

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)G^T(t)$$

Show how this generalizes to the usual equation, with the last term above becoming  $G(t)Q(t)G^T(t)$ , for the case of general diffusion  $Q(t)$  instead of  $Q(t) \equiv I$ .

**11.13** Let  $x_1(\cdot, \cdot)$  and  $x_2(\cdot, \cdot)$  be two  $n$ -dimensional solutions to separate Itô stochastic differential equations driven by the same Brownian motion of diffusion  $Q$ :

$$\begin{aligned} dx_1(t) &= f_1[x_1(t), t]dt + G_1[x_1(t), t]d\beta(t) \\ dx_2(t) &= f_2[x_2(t), t]dt + G_2[x_2(t), t]d\beta(t) \end{aligned}$$

and consider scalar functions  $\psi_1[x_1(t), t]$  and  $\psi_2[x_2(t), t]$ , each having continuous second partial derivatives with respect to their first argument and each being continuously differentiable in their second arguments. Use the Itô differential rule (as applied to the augmented process  $[x_1^T : x_2^T]^T$ ) to show

$$(a) \quad d[\psi_1\psi_2] = \psi_1d\psi_2 + \psi_2d\psi_1 + \text{tr}\{G_1QG_2^T\} \frac{\partial\psi_1}{\partial x_1} \frac{\partial\psi_2^T}{\partial x_2} dt$$

$$\begin{aligned} (b) \quad d[\psi_1\psi_2^{-1}] &= \psi_2^{-1}d\psi_1 - \psi_1\psi_2^{-2}d\psi_2 - \psi_2^{-1} \text{tr}\{G_1QG_2^T\} \frac{\partial\psi_1}{\partial x_1} \frac{\partial\psi_2^T}{\partial x_2} dt \\ &\quad + \psi_1\psi_2^{-3} \text{tr}\{G_2QG_2^T\} \frac{\partial\psi_2}{\partial x_2} \frac{\partial\psi_2^T}{\partial x_2} dt \end{aligned}$$

**11.14** Let  $x(\cdot, \cdot)$  satisfy the general scalar stochastic differential equation

$$dx(t) = f[x(t), t]dt + G[x(t), t]d\beta(t)$$

with  $x(t_0) = 0$  with probability one and with  $\beta(\cdot, \cdot)$  as Brownian motion having diffusion  $Q(t)$  for all  $t$ .

(a) Let  $\psi[x(t), t] = e^{ax(t)}$  and use the Itô differential rule to derive  $d[e^{ax(t)}]$ . Integrate this from  $t_0$  to  $t$  to obtain a useful implicit formula for  $e^{ax(t)}$  as

$$\begin{aligned} e^{ax(t)} &= 1 + \int_{t_0}^t e^{ax(\tau)} \{ a f[x(\tau), \tau] + \frac{1}{2} a^2 G^2[x(\tau), \tau] Q(\tau) \} d\tau \\ &\quad + \int_{t_0}^t a e^{ax(\tau)} G[x(\tau), \tau] d\beta(\tau) \end{aligned}$$

(b) Consider the function

$$\alpha(t) = \exp\{j\mu[x(t) - x(t')]\}$$

and by applying the Itô differential rule and taking the conditional expectation, conditioned on knowledge that  $x(t') = \rho$ , derive an expression for the conditional characteristic function for  $x(t)$  conditioned on  $x(t') = \rho$  (i.e., the Fourier transform of the transition probability density  $f_{x(t)|x(t')}(\xi | \rho)$ ) as

$$\begin{aligned} \phi_{x(t)|x(t')}(\mu | \rho) &\triangleq E\{\alpha(t) | x(t') = \rho\} \\ &= 1 + \int_{t'}^t E\left[\alpha(\tau) \left\{ j\mu f[x(\tau), \tau] - \frac{1}{2} \mu^2 G^2[x(\tau), \tau] Q(\tau) \right\} | x(t') = \rho\right] d\tau \end{aligned}$$

Show that taking the partial of this with respect to time  $t$  and then taking the inverse Fourier transform yields the forward Kolmogorov equation, the scalar version of (11-96).

(c) As another application of  $\psi[x(t), t] = \exp\{ax(t)\}$ , consider the special case in which the stochastic differential equation has the form given by  $f[x(t), t] = -\frac{1}{2}k^2(t)$  and  $G[x(t), t] = k(t)$ , where  $k(\cdot)$  is a square integrable deterministic function, and where  $a \equiv 1$ , to yield

$$L(t) = \exp\left\{\int_{t_0}^t k(\tau) d\beta(\tau) - \frac{1}{2} \int_{t_0}^t k^2(\tau) d\tau\right\}$$

As it turns out, this is the likelihood ratio associated with detecting a signal  $k(\cdot)$  in additive white Gaussian noise [33, 34]. By applying the Itô differential rule, show that  $L(t)$  can be expressed as the Itô integral

$$L(t) = 1 + \int_{t_0}^t k(\tau) L(\tau) d\beta(\tau)$$

Note that this demonstrates that  $L(\cdot, \cdot)$  is a martingale with  $E\{L(t)\} = 1$ . Also, show by induction that the likelihood ratio can be expressed by the series representation

$$L(t) = 1 + \sum_{n=1}^{\infty} \int_{t_0}^t \int_{t_0}^{t_n} \cdots \int_{t_0}^{t_2} k(\tau_1) \cdots k(\tau_n) d\beta(\tau_1) \cdots d\beta(\tau_n)$$

Moreover, from the Itô integral expression, show that

$$E\{L^2(t)\} = 1 + \int_{t_0}^t k^2(\tau) E\{L^2(\tau)\} Q(\tau) d\tau$$

which implies that

$$E\{L^2(t)\} = \exp\left\{\int_{t_0}^t k^2(\tau) Q(\tau) d\tau\right\}$$

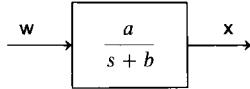
since both expressions obey the same initial conditions and differential equations.

**11.15** A scalar Brownian motion (Gaussian) process  $\mathbf{x}(\cdot, \cdot)$  has the following statistics:

$$\begin{aligned} E\{\mathbf{x}(t)\} &= 0 \\ E\{[\mathbf{x}(t) - \mathbf{x}(t')]^2\} &= t - t' \quad \text{for } t \geq t' \end{aligned}$$

What is the transition probability  $f_x(\xi, t; 0, 0)$ ? Show that this transition probability satisfies the appropriate forward Kolmogorov equation.

**11.16** Consider the scalar linear system with white dynamic driving noise,  $\mathbf{w}(\cdot, \cdot)$ :



Write the transition probability for  $\mathbf{x}(t)$  and show that it satisfies the forward Kolmogorov equation.

**11.17** A scalar Brownian motion process  $\beta(\cdot, \cdot)$  has the statistics

$$E\{\beta(t)\} = 0, \quad E\{[\beta(t_2) - \beta(t_1)]^2\} = 5|t_2 - t_1|, \quad \beta(0) = 0 \quad \text{w.p.1}$$

The process  $\mathbf{y}(\cdot, \cdot)$  is defined as

$$\mathbf{y}(t) = |\beta(t)|$$

Obtain the forward Kolmogorov equation for the  $\beta(\cdot, \cdot)$  process. Using this result, obtain an expression for the derivative of the mean of  $\mathbf{y}(t)$ . Solve for the mean of  $\mathbf{y}(t)$ .

**11.18** Consider the simulation issues raised at the end of Section 11.5, specifically pertaining to the simulation of the simple case in which  $\mathbf{G}$  is a function only of time and not of  $\mathbf{x}(\cdot, \cdot)$ :

$$d\mathbf{x}(t) = \mathbf{f}[\mathbf{x}(t), t] dt + \mathbf{G}(t) d\beta(t)$$

This is specifically the class of problems for which extended Kalman filters were developed in Chapter 9, and we wish to explore the truth model simulation for a Monte Carlo performance evaluation tool [54].

(a) First recall the simulation of “equivalent discrete-time models” for *linear* stochastic differential equations, as discussed in Example 11.11 and Sections 4.9 and 6.8 and Problem 7.14 of Volume 1. Depict such a simulation in detail, specifying all appropriate transformations to be performed on independent scalar unit-variance zero-mean white Gaussian sequences as produced from pseudorandom noise generators.

(b) For sample intervals  $\Delta t$  short compared to system characteristic times, first order approximations

$$\begin{aligned} \Phi(t_{i+1}, t_i) &\cong \mathbf{I} + \mathbf{F}(t_i) \Delta t \\ \int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \tau) \mathbf{G}(\tau) \mathbf{Q}(\tau) \mathbf{G}^T(\tau) \Phi^T(t_{i+1}, \tau) d\tau &\cong [\mathbf{G}(t_i) \mathbf{Q}(t_i) \mathbf{G}^T(t_i)] \Delta t \end{aligned}$$

often suffice. Show that these yield approximate simulation equations as

$$\mathbf{x}(t_{i+1}) = [\mathbf{I} + \mathbf{F}(t_i) \Delta t] \mathbf{x}(t_i) + \mathbf{w}_d(t_i)$$

where  $\mathbf{w}_d(\cdot, \cdot)$  is zero-mean white Gaussian discrete-time noise with

$$E\{\mathbf{w}_d(t_i) \mathbf{w}_d^T(t_i)\} = [\mathbf{G}(t_i) \mathbf{Q}(t_i) \mathbf{G}^T(t_i)] \Delta t$$

and discuss how to implement such a simulation.

(c) Show that this is equivalent to using Euler integration (first order numerical integration) to write

$$\mathbf{x}(t_{i+1}) = \mathbf{x}(t_i) + [\dot{\mathbf{x}}(t_i)] \Delta t$$

where specifically  $\hat{\mathbf{x}}(t_i)$  is simulated according to

$$\hat{\mathbf{x}}(t_i) = \mathbf{F}(t_i)\mathbf{x}(t_i) + \mathbf{G}(t_i)\mathbf{w}_d(t_i)$$

where  $\mathbf{w}_d(\cdot, \cdot)$  is zero-mean white Gaussian discrete-time noise with

$$E\{\mathbf{w}_d(t_i)\mathbf{w}_d^T(t_i)\} = \mathbf{Q}(t_i)/(\Delta t)$$

(d) Show that the approximate method of (c) is directly extendable to simulating the non-linear differential equation at the beginning of this problem. Discuss implementation of this simulation, including Cholesky square roots or  $U$ - $D$  covariance factorizations. In some available software tools [54], the stochastic driving term effects are simulated by such an Euler integration approximation, while the simulation of the effects of the  $\mathbf{f}[\mathbf{x}(t), t]$  term are in fact handled through higher order integration techniques, such as fifth order Runge-Kutta integration.

(e) Compare these simple results to the more complicated issues raised in Examples 11.11 and 11.12 for the more general case of  $\mathbf{G}$  being a function of both  $\mathbf{x}(t)$  and  $t$ .

**11.19** A scalar linear system is described by the stochastic differential equation

$$dx(t) = -2x(t)dt + d\beta(t)$$

where  $\beta(\cdot, \cdot)$  is scalar Brownian motion with statistics

$$E\{\beta(t)\} = 0, \quad E\{[\beta(t) - \beta(t')]^2\} = 5|t - t'|$$

and the initial value of  $x(t)$  is known exactly as

$$x(0) = 3$$

- (a) Derive an expression for  $f_{x(t)|x(0)}(\xi|\rho) = f_x(\xi, t|\rho, 0)$ .
- (b) At discrete time points  $t_i$ , *perfect* measurements of the square of  $x(t_i)$  are available:

$$z(t_i) = x(t_i)^2$$

Develop recursion formulas for calculating the mean of  $x(t_i)$ , conditioned on the history of measurements up to time  $t_i$ .

(c) Given measurements  $z(1) = 4$ ,  $z(2) = 1$ , calculate the mean of  $x(2)$  conditioned on the measurements.

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# CHAPTER 12

## Nonlinear estimation

### 12.1 INTRODUCTION

This chapter applies the nonlinear stochastic system models of the previous chapter to the problem of estimating the system state, given real-time data available from the actual system and the products of our modeling efforts. Unlike the case of estimation based upon linear system models driven by white Gaussian noises, the full-scale estimator turns out to be generally intractable without imposing simplifying assumptions: propagating and updating either an entire density function for the state conditioned on observed measurements or an infinite number of describing parameters for that density is not implementable, as seen in Section 12.2.

Section 12.3 discusses means of generating approximate conditional moment estimators. The appropriateness of approximations to be made is problem dependent. Taylor series representations for dynamics and measurement nonlinearities, as expanded about the current state estimate, are combined with assumptions about the conditional state density to yield the *truncated* and *Gaussian second order filters*. Computational and performance considerations lead to both *modified* forms of these and *first order filters with bias correction terms*. *Assumed density filters* and *higher order moment filters* which do not depend on such Taylor series representations are also developed.

Alternatives to minimum mean square error (MMSE), conditional moment, estimators are discussed in subsequent sections. *Conditional quasi-moment*, *conditional mode*, and *statistically linearized estimators* are considered.

Finally, Section 12.7 presents the estimation problem and solutions for the case of continuous, instead of sampled-data, measurements being available. Analogies to the earlier sections are made in order to gain better insights into the problem and corresponding filter design.

## 12.2 NONLINEAR FILTERING WITH DISCRETE-TIME MEASUREMENTS: CONCEPTUALLY

Consider a system described by the Itô stochastic differential state equation

$$\mathbf{dx}(t) = \mathbf{f}[\mathbf{x}(t), t] dt + \mathbf{G}[\mathbf{x}(t), t] d\beta(t) \quad (12-1)$$

subject to an initial condition specifying the density function for  $\mathbf{x}(t_0)$ , with  $\beta(\cdot, \cdot)$  being  $s$ -vector dimensional Brownian motion of diffusion  $\mathbf{Q}(t)$  for all  $t$  of interest (see Eq. (11-74)). Note that (12-1) admits systems driven by known deterministic inputs  $\mathbf{u}(t)$  for all time  $t$  as well as by  $\beta(\cdot, \cdot)$  simply by writing  $\mathbf{f}[\mathbf{x}(t), t] = \mathbf{f}'[\mathbf{x}(t), \mathbf{u}(t), t]$ ; feedback control in which  $\mathbf{u}$  is a function of perfectly known  $\mathbf{x}(t)$  and  $t$  is similarly admitted; feedback control as a function of current noise-corrupted measurements, the time history of such measurements, or a state estimate based on that time history, will be discussed in subsequent chapters. Let discrete-time measurements be available from the system, in the form of

$$\mathbf{z}(t_i) = \mathbf{h}[\mathbf{x}(t_i), t_i] + \mathbf{v}(t_i) \quad (12-2)$$

with  $\mathbf{v}(\cdot, \cdot)$   $m$ -dimensional discrete-time white Gaussian noise independent of  $\beta(\cdot, \cdot)$ , with mean zero and covariance  $\mathbf{R}(t_i)$  for all sample times  $t_i$ :  $E\{\mathbf{v}(t_i)\mathbf{v}^T(t_j)\} = \mathbf{R}(t_i)\delta_{ij}$ . We desire to establish, if possible, the conditional density for  $\mathbf{x}(t_i)$ , conditioned on the measurements up to and including time  $t_i$ :  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_i)}(\xi|\mathcal{L}_i)$ . If we could accomplish this objective, then we could define various estimators that are optimal with respect to some specified criterion, such as the conditional mean (minimum mean square error, or MMSE, estimate) or the conditional mode (maximum a posteriori, or MAP, estimate).

Conceptually, this can be accomplished in the following manner. First consider the time propagation of  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1})$  from sample time  $t_{i-1}$  to time  $t_i$ . As shown in the previous chapter, the solution to (12-1) is a Markov process, completely characterized by the transition probability density  $f_{\mathbf{x}}(\xi, t|\rho, t')$ . If we know  $f_{\mathbf{x}}(\xi, t|\rho, t_{i-1})$  for all  $t \in [t_{i-1}, t_i]$ , then the conditional density  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1})$  can be established for all  $t$  in that interval, and thus be propagated through that sample period, via

$$f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1}) = \int_{-\infty}^{\infty} f_{\mathbf{x}}(\xi, t|\rho, t_{i-1}) f_{\mathbf{x}(t_{i-1})|\mathbf{z}(t_{i-1})}(\rho|\mathcal{L}_{i-1}) d\rho \quad (12-3)$$

using the *Chapman–Kolmogorov equation* valid for Markov  $\mathbf{x}(\cdot, \cdot)$  (see (11-50)–(11-52)) and  $f_{\mathbf{x}(t_{i-1})|\mathbf{z}(t_{i-1})}(\rho|\mathcal{L}_{i-1})$  from the previous measurement update. Thus, we could obtain  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1})$  from  $f_{\mathbf{x}(t_{i-1})|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1})$  by using the *forward Kolmogorov equation* (11-96) to evaluate  $f_{\mathbf{x}}(\xi, t_i|\rho, t_{i-1})$  and then employing (12-3), or by attempting to use these same concepts to derive an expression (and/or suitable approximation) to propagate  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1})$  directly. In fact, it can be shown that  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1})$  itself satisfies the

forward Kolmogorov equation in the interval  $[t_{i-1}, t_i]$ , starting from the density  $f_{\mathbf{x}(t_{i-1})|\mathbf{z}(t_{i-1})}(\xi | \mathcal{L}_{i-1})$  obtained from the previous measurement update.

Now we wish to incorporate the measurement  $\mathbf{z}(t_i, \omega_k) = \zeta_i$  that becomes available at time  $t_i$ . As for the case of linear estimation, *Bayes' rule* can be applied repeatedly to establish

$$f_{\mathbf{x}(t_i)|\mathbf{z}(t_i)}(\xi | \mathcal{L}_i) = \frac{f_{\mathbf{z}(t_i)|\mathbf{x}(t_i), \mathbf{z}(t_{i-1})}(\zeta_i | \xi, \mathcal{L}_{i-1}) f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi | \mathcal{L}_{i-1})}{f_{\mathbf{z}(t_i)|\mathbf{z}(t_{i-1})}(\zeta_i | \mathcal{L}_{i-1})} \quad (12-4)$$

The second numerator term was just discussed, and is provided by the previous propagation. For the first numerator term, if  $\mathbf{v}(\cdot, \cdot)$  is assumed to be a white Gaussian discrete-time noise of mean zero and covariance  $\mathbf{R}(t_i)$ , then it can be readily shown that

$$\begin{aligned} f_{\mathbf{z}(t_i)|\mathbf{x}(t_i), \mathbf{z}(t_{i-1})}(\zeta_i | \xi, \mathcal{L}_{i-1}) &= f_{\mathbf{z}(t_i)|\mathbf{x}(t_i)}(\zeta_i | \xi) \\ &= \frac{1}{(2\pi)^{m/2} |\mathbf{R}(t_i)|^{1/2}} \\ &\times \exp\left\{-\frac{1}{2} [\zeta_i - \mathbf{h}(\xi, t_i)]^T \mathbf{R}^{-1}(t_i) [\zeta_i - \mathbf{h}(\xi, t_i)]\right\} \end{aligned} \quad (12-5)$$

For the denominator in (12-4), the concept of marginal densities and Bayes' rule can be combined to yield

$$\begin{aligned} f_{\mathbf{z}(t_i)|\mathbf{z}(t_{i-1})}(\zeta_i | \mathcal{L}_{i-1}) &= \int_{-\infty}^{\infty} f_{\mathbf{z}(t_i), \mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\zeta_i, \xi | \mathcal{L}_{i-1}) d\xi \\ &= \int_{-\infty}^{\infty} f_{\mathbf{z}(t_i)|\mathbf{x}(t_i)}(\zeta_i | \xi) f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi | \mathcal{L}_{i-1}) d\xi \end{aligned} \quad (12-6)$$

Thus, knowledge of  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi | \mathcal{L}_{i-1})$  from propagation and  $f_{\mathbf{z}(t_i)|\mathbf{x}(t_i)}(\zeta_i | \xi)$  from (12-5) provide sufficient information to evaluate  $f_{\mathbf{z}(t_i)|\mathbf{z}(t_{i-1})}(\zeta_i | \mathcal{L}_{i-1})$  via (12-4), but the result entails an integration as indicated in Eq. (12-6).

Conceptually, we have all we need, but actual evaluations are not possible. The computation of the entire density function  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_i)}(\xi | \mathcal{L}_i)$  is generally intractable because time propagations involve partial integro-differential equations (derived by means of the forward Kolmogorov equation) and measurement updates involve functional integral difference equations (derived by means of Bayes' rule). One might attempt to generate a *partial or approximate depiction* of the conditional density as accurately as possible with a *small* number of *appropriate parameters*. The evolution of these parameters would then constitute the nonlinear filter for a given problem. However, as was foretold in Section 6 of the preceding chapter, even the partial description of the conditional density by these parameters will not be feasible without knowledge of the entire density

itself to evaluate certain conditional expectations. Said another way, the optimal nonlinear estimator will be infinite dimensional. To obtain practically feasible algorithms, *expansions truncated to some low order or other approximations* are required both in the time propagation and measurement update of the nonlinear filter. The suitability of approximations to be made is problem dependent, giving rise to many estimator forms, but some general trends can and will be evaluated in this chapter.

There are a number of possible means of parameterizing the conditional density function. Expressing it in terms of a *complete orthogonal series* and then *truncating* the series at a specified order is a valid approach, and series of Hermite polynomials will be particularly motivated and approximate filters in terms of the series coefficients, known as *quasi-moments*, will be developed. *Conditional mode* filters will also be discussed. However, attention will be concentrated on parameterization via *moments*, and filters of this form are developed in the next section.

### 12.3 CONDITIONAL MOMENT ESTIMATORS

Consider the time propagation of the conditional mean and covariance,

$$\hat{\mathbf{x}}(t/t_{i-1}) \triangleq E\{\mathbf{x}(t) | \mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \quad (12-7)$$

$$\hat{\mathbf{P}}(t/t_{i-1}) \triangleq E\{[\mathbf{x}(t) - \hat{\mathbf{x}}(t/t_{i-1})][\mathbf{x}(t) - \hat{\mathbf{x}}(t/t_{i-1})]^T | \mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \quad (12-8)$$

as generated for a particular measurement history realization  $\mathbf{Z}(t_{i-1}, \omega_k) = \mathbf{Z}_{i-1}$ . (Note that a parallel development can be made by conditioning on  $\mathbf{Z}(t_{i-1}, \cdot)$  instead of  $\mathbf{Z}(t_{i-1}, \omega_k)$ , viewing the conditional expectation as a random variable mapping itself and thereby defining  $\hat{\mathbf{x}}(t/t_{i-1})$  and  $\hat{\mathbf{P}}(t/t_{i-1})$ .) Between sample times  $t_{i-1}$  and  $t_i$ , i.e., for all  $t \in [t_{i-1}, t_i]$ , the densities  $f_{\mathbf{x}(t)}(\xi)$  and  $f_{\mathbf{x}(t)|\mathbf{Z}(t_{i-1})}(\xi | \mathbf{Z}_{i-1})$  both satisfy the forward Kolmogorov equation, so the propagation of  $\hat{\mathbf{x}}(t/t_{i-1})$  and  $\hat{\mathbf{P}}(t/t_{i-1})$  can be obtained from (11-99) and (11-100), replacing the unconditional expectations with expectations conditioned on  $\mathbf{Z}(t_{i-1}, \omega_k) = \mathbf{Z}_{i-1}$ :

$$\hat{\mathbf{x}}(t/t_{i-1}) = \widehat{\mathbf{f}[\mathbf{x}(t), t]} \quad (12-9)$$

$$\begin{aligned} \hat{\mathbf{P}}(t/t_{i-1}) &= \{\widehat{\mathbf{f}[\mathbf{x}(t), t]}\mathbf{x}^T(t) - \widehat{\mathbf{f}[\mathbf{x}(t), t]}\widehat{\mathbf{x}}^T(t/t_{i-1})\} \\ &\quad + \{\widehat{\mathbf{x}(t)\mathbf{f}^T[\mathbf{x}(t), t]} - \widehat{\mathbf{x}}(t/t_{i-1})\widehat{\mathbf{f}^T[\mathbf{x}(t), t]}\} \\ &\quad + \widehat{\mathbf{G}[\mathbf{x}(t), t]\mathbf{Q}(t)\mathbf{G}^T[\mathbf{x}(t), t]} \end{aligned} \quad (12-10)$$

where

$$\widehat{(\cdot)} \triangleq E\{(\cdot) | \mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \quad (12-11)$$

and this notation is to be adopted throughout this section. These are propagated to sample time  $t_i$  to yield  $\hat{\mathbf{x}}(t_i^-) \triangleq \hat{\mathbf{x}}(t_i/t_{i-1})$  and  $\mathbf{P}(t_i^-) \triangleq \mathbf{P}(t_i/t_{i-1})$ . To incorporate the next measurement at time  $t_i$ , the conditional mean can be calculated as

$$\begin{aligned}\hat{\mathbf{x}}(t_i^+) &\triangleq \hat{\mathbf{x}}(t_i/t_i) = E\{\mathbf{x}(t_i)|\mathbf{Z}(t_i) = \mathbf{Z}_i\} \\ &= \int_{-\infty}^{\infty} \xi f_{\mathbf{x}(t_i)|\mathbf{z}(t_i)}(\xi|\mathbf{Z}_i) d\xi\end{aligned}\quad (12-12)$$

where  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_i)}(\xi|\mathbf{Z}_i)$  is obtained from (12-4)–(12-6), expressed equivalently as

$$\begin{aligned}\hat{\mathbf{x}}(t_i^+) &= \frac{\int_{-\infty}^{\infty} \xi f_{\mathbf{z}(t_i)|\mathbf{x}(t_i)}(\mathbf{z}_i|\xi) f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathbf{Z}_{i-1}) d\xi}{\int_{-\infty}^{\infty} f_{\mathbf{z}(t_i)|\mathbf{x}(t_i)}(\mathbf{z}_i|\xi) f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathbf{Z}_{i-1}) d\xi} \\ &= \overbrace{\mathbf{x}(t_i) f_{\mathbf{z}(t_i)|\mathbf{x}(t_i)}(\mathbf{z}_i|\mathbf{x}(t_i))} / \overbrace{f_{\mathbf{z}(t_i)|\mathbf{x}(t_i)}(\mathbf{z}_i|\mathbf{x}(t_i))}\end{aligned}\quad (12-13)$$

using the same notation as in Eq. (12-11). Similarly,

$$\begin{aligned}\mathbf{P}(t_i^+) &= \int_{-\infty}^{\infty} \xi \xi^T f_{\mathbf{x}(t_i)|\mathbf{z}(t_i)}(\xi|\mathbf{Z}_i) d\xi - \hat{\mathbf{x}}(t_i^+) \hat{\mathbf{x}}^T(t_i^+) \\ &= \overbrace{\mathbf{x}(t_i) \mathbf{x}^T(t_i) f_{\mathbf{z}(t_i)|\mathbf{x}(t_i)}} / \overbrace{f_{\mathbf{z}(t_i)|\mathbf{x}(t_i)}} - \hat{\mathbf{x}}(t_i^+) \hat{\mathbf{x}}^T(t_i^+)\end{aligned}\quad (12-14)$$

The conditional expectations in Eqs. (12-9)–(12-14) require knowledge of the entire conditional density, i.e., knowledge of all higher order moments as well as the first two. By making certain assumptions about the density and/or higher order moments, approximations to these full-scale estimator equations can be generated that do not require evaluation of an infinite number of moments, yielding computationally feasible algorithms. One might assume that the conditional density is nearly symmetric so that third and higher order odd central moments are essentially zero, and also that it is concentrated sufficiently closely about the mean that the fourth and higher order even central moments are small enough to be neglected: this set of assumptions gives rise to the *truncated second order filter*. (Scaling of variables may be required to ensure that higher order even moments decrease in relative magnitude instead of increase.) Or, one might assume that the conditional density is nearly Gaussian, so that third and higher order odd central moments are again essentially zero, and the fourth and higher order even central moments can be expressed in terms of the covariance: this yields the *Gaussian second order filter*. Typically, sixth and higher order even moments are also assumed small enough to be neglected, and fourth order central moments are expressed in terms of second moments. These filters are based upon a further approximation of representing the functions  $\mathbf{f}$  and  $\mathbf{h}$  in (12-1) and (12-2) via a truncated Taylor series expanded about the state estimate, but there are also *assumed density filters* that do not require such series approximations. Finally, there are also *higher order moment filters* that maintain estimates of moments beyond second order, based upon different types of approximations such as cumulant truncation.

Let us first consider the *truncated second order filter* [7, 52, 54]. To aid understanding of basic concepts without the overburdening notation inherent in the vector case, we restrict our immediate attention to *scalar* systems with scalar measurements and driving noise ( $n = m = s = 1$ ). First consider the approximate evaluation of the conditional expectation of  $f[\mathbf{x}(t), t]$  in Eq. (12-9). Perform a Taylor series expansion of  $f[\mathbf{x}(t), t]$  about the conditional mean, i.e., current estimate,  $\hat{\mathbf{x}}(t/t_{i-1})$ :

$$\begin{aligned} f[\mathbf{x}(t), t] &= f[\hat{\mathbf{x}}(t/t_{i-1}), t] + \frac{\partial f[\hat{\mathbf{x}}(t/t_{i-1}), t]}{\partial \mathbf{x}} \{ \mathbf{x}(t) - \hat{\mathbf{x}}(t/t_{i-1}) \} \\ &\quad + \frac{1}{2} \frac{\partial^2 f[\hat{\mathbf{x}}(t/t_{i-1}), t]}{\partial \mathbf{x}^2} \{ \mathbf{x}(t) - \hat{\mathbf{x}}(t/t_{i-1}) \}^2 + \dots \end{aligned} \quad (12-15)$$

where  $\partial f[\hat{\mathbf{x}}(t/t_{i-1}), t]/\partial \mathbf{x}$  denotes  $\partial f[\mathbf{x}, t]/\partial \mathbf{x}$  evaluated with  $\mathbf{x} = \hat{\mathbf{x}}(t/t_{i-1})$ , and so on. Taking the conditional expectation of this series carried out to second order yields

$$\widehat{f[\mathbf{x}(t), t]} \cong f[\hat{\mathbf{x}}(t/t_{i-1}), t] + \frac{1}{2} \frac{\partial^2 f[\hat{\mathbf{x}}(t/t_{i-1}), t]}{\partial \mathbf{x}^2} P(t/t_{i-1}) \quad (12-16)$$

When the conditional expectation is taken, recall that  $\hat{\mathbf{x}}(t/t_{i-1})$  is a *function* of  $\mathbf{Z}(t_{i-1})$ , so that, for example,

$$\widehat{f[\hat{\mathbf{x}}(t/t_{i-1})]} \triangleq E\{f[\hat{\mathbf{x}}(t/t_{i-1})] | \mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} = f[\hat{\mathbf{x}}(t/t_{i-1})]$$

where  $\hat{\mathbf{x}}(t/t_{i-1})$  is a particular realization of  $\hat{\mathbf{x}}(t/t_{i-1})$ . Similarly,

$$\begin{aligned} E\left\{ \frac{\partial^2 f[\hat{\mathbf{x}}(t/t_{i-1}), t]}{\partial \mathbf{x}^2} [\mathbf{x}(t) - \hat{\mathbf{x}}(t/t_{i-1})]^2 | \mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1} \right\} \\ = \frac{\partial^2 f[\hat{\mathbf{x}}(t/t_{i-1}), t]}{\partial \mathbf{x}^2} E\{[\mathbf{x}(t) - \hat{\mathbf{x}}(t/t_{i-1})]^2 | \mathbf{Z}(t_{i-1}) = \mathbf{Z}_{i-1}\} \\ = \frac{\partial^2 f[\hat{\mathbf{x}}(t/t_{i-1}), t]}{\partial \mathbf{x}^2} P(t/t_{i-1}) \end{aligned}$$

In a similar manner, the expectations in the scalar  $\dot{P}(t/t_{i-1})$  equation, (12-10), can be shown to equal, to second order,

$$[\hat{\mathbf{x}} - \hat{\mathbf{x}}^*] = [\hat{\mathbf{x}}^* - \hat{\mathbf{x}}^*] \cong \frac{\partial f[\hat{\mathbf{x}}(t/t_{i-1}), t]}{\partial \mathbf{x}} P(t/t_{i-1}) \quad (12-17a)$$

$$\begin{aligned} \widehat{G[\mathbf{x}(t), t]^2 Q(t)} &\cong G^2[\hat{\mathbf{x}}(t/t_{i-1}), t] Q(t) + \frac{\partial G[\hat{\mathbf{x}}(t/t_{i-1}), t]^2}{\partial \mathbf{x}} P(t/t_{i-1}) Q(t) \\ &\quad + G[\hat{\mathbf{x}}(t/t_{i-1}), t] \frac{\partial^2 G[\hat{\mathbf{x}}(t/t_{i-1}), t]}{\partial \mathbf{x}^2} P(t/t_{i-1}) Q(t) \end{aligned} \quad (12-17b)$$

Therefore, the *time propagation of the state estimate* between  $t_{i-1}$  and  $t_i$  is accomplished by solving

$$\dot{\hat{x}}(t/t_{i-1}) = f[\hat{x}(t/t_{i-1}), t] + \frac{1}{2} \frac{\partial^2 f[\hat{x}(t/t_{i-1}), t]}{\partial x^2} P(t/t_{i-1}) \quad (12-18)$$

$$\dot{P}(t/t_{i-1}) = 2 \frac{\partial f[\hat{x}(t/t_{i-1}), t]}{\partial x} P(t/t_{i-1}) + \widehat{G[x(t), t]^2 Q(t)} \quad (12-19)$$

where  $\widehat{G[x(t), t]^2 Q(t)}$  is given in (12-17b).

Note the additional term in (12-18) compared to the extended Kalman filter estimate propagation, Eq. (9-65). When nonlinearities are significant, the second order filter yields performance that is superior to that of a first order filter on the average, due primarily to this “bias correction term” [4] and a similar term in residual generation for measurement updating. Ignoring this term does generally result in a more biased estimate, and this will be discussed more fully later. The other difference from the extended Kalman filter is the covariance differential equation driving term, given by (12-17b) rather than simply  $G^2(t)Q(t)$ . In developing the extended Kalman filter,  $G$  was assumed to be a function only of  $t$ , and not of  $x(t)$  and  $t$ . Under this assumption, (12-17b) reduces to  $G^2(t)Q(t)$ . Without this assumption, a truncated first order filter would introduce the second of three terms in (12-17b) beyond the computations of the extended Kalman filter, whereas the current second order filter introduces two additional terms.

For measurement updating at sample time  $t_i$ , one could attempt to approximate the expectation integrations inherent in (12-13) and (12-14). This requires a series expansion of  $f_{z(t_i)|x(t_i)}(\zeta | \xi)$ , and it turns out that the filter is very sensitive to the particular approximation employed [52, 54, 131]. Kramer [67] and Jazwinski [52, 54] have suggested an alternate, better behaved approximation technique. We assume that the conditional mean and covariance after measurement incorporation can be expressed as a power series in the residual (innovations),

$$\{z(t_i) - \hat{z}(t_i^-)\} \triangleq \{z(t_i) - E[z(t_i) | \mathbf{Z}(t_{i-1}) = \mathbf{Z}(t_{i-1}, \cdot)]\} \quad (12-20)$$

For computational tractability, this power series is truncated at first order terms:

$$\hat{x}(t_i^+) = \mathbf{a}_0 + \mathbf{a}_1 \{z(t_i) - \hat{z}(t_i^-)\} \quad (12-21a)$$

$$P(t_i^+) = \mathbf{b}_0 + \mathbf{b}_1 \{z(t_i) - \hat{z}(t_i^-)\} \quad (12-21b)$$

The structural form of (12-21a) is motivated in part by the first order filter update,  $\hat{x}(t_i^+) = \hat{x}(t_i^-) + K(t_i)[z(t_i) - \hat{z}(t_i^-)]$ , and  $\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1$ , and  $\hat{z}(t_i^-)$  are all random variables that are viewed as explicit functions of  $\mathbf{Z}(t_{i-1})$ . Note that (12-21b) is a variance equation driven by a random residual forcing function,

thereby admitting negative computed values of  $P(t_i^+)$ : this unacceptable attribute will be discussed in more detail subsequently, and will motivate an alternate assumed form with  $b_1$  set identically to zero.

Now the coefficients in (12-21) are determined by manipulation of Bayes' rule into an appropriate form. To do so, consider two scalar functions of random variables  $x(t_i)$  and  $z(t_i)$ , respectively,  $\psi[\cdot]$  and  $\theta[\cdot]$ . Then the desired relation is

$$\begin{aligned} E\{\psi[x(t_i)]\theta[z(t_i)]|\mathbf{Z}(t_{i-1}) = \mathcal{L}_{i-1}\} \\ = E\{E\{\psi[x(t_i)]|\mathbf{Z}(t_i) = \mathbf{Z}(t_i, \cdot)\}\theta[z(t_i)]|\mathbf{Z}(t_{i-1}) = \mathcal{L}_{i-1}\} \end{aligned} \quad (12-22)$$

where the inner conditional expectation is itself a random variable since it is conditioned on  $\mathbf{Z}(t_i, \cdot)$ . The validity of (12-22) can be demonstrated by taking Eq. (12-4), which is equivalently written as

$$f_{x(t_i)|\mathbf{Z}(t_i)}(\xi|\mathcal{L}_i) = \frac{f_{x(t_i), z(t_i)|\mathbf{Z}(t_{i-1})}(\xi, \zeta_i|\mathcal{L}_{i-1})}{f_{z(t_i)|\mathbf{Z}(t_{i-1})}(\zeta_i|\mathcal{L}_{i-1})} \quad (12-23a)$$

or

$$f_{x(t_i), z(t_i)|\mathbf{Z}(t_{i-1})}(\xi, \zeta_i|\mathcal{L}_{i-1}) = f_{x(t_i)|\mathbf{Z}(t_i)}(\xi|\mathcal{L}_i)f_{x(t_i)|\mathbf{Z}(t_{i-1})}(\zeta_i|\mathcal{L}_{i-1}) \quad (12-23b)$$

and using this result to generate

$$\begin{aligned} E\{\psi[x(t_i)]\theta[z(t_i)]|\mathbf{Z}(t_{i-1}) = \mathcal{L}_{i-1}\} \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(\xi)\theta(\zeta_i)f_{x(t_i), z(t_i)|\mathbf{Z}(t_{i-1})}(\xi, \zeta_i|\mathcal{L}_{i-1})d\xi d\zeta_i \\ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \psi(\xi)f_{x(t_i)|\mathbf{Z}(t_i)}(\xi|\mathcal{L}_i)d\xi \right] \theta(\zeta_i)f_{z(t_i)|\mathbf{Z}(t_{i-1})}(\zeta_i|\mathcal{L}_{i-1})d\zeta_i \\ = E\{E\{\psi[x(t_i)]|\mathbf{Z}(t_i) = \mathbf{Z}(t_i, \cdot)\}\theta[z(t_i)]|\mathbf{Z}(t_{i-1}) = \mathcal{L}_{i-1}\} \end{aligned}$$

as desired. Evaluation of the coefficient  $a_0$  is now obtained by letting  $\psi[x(t_i)] = x(t_i)$  and  $\theta[z(t_i)] = 1$  in (12-22):

$$E\{x(t_i)|\mathbf{Z}(t_{i-1}) = \mathcal{L}_{i-1}\} = E\{\hat{x}(t_i^+)|\mathbf{Z}(t_{i-1}) = \mathcal{L}_{i-1}\} \quad (12-24)$$

Recognizing the left hand side as the realization  $\hat{x}(t_i^-)$  and substituting the assumed form (12-21a) into the right hand side yields

$$\begin{aligned} \hat{x}(t_i^-) &= E\{a_0 + a_1[z(t_i) - \hat{z}(t_i^-)]|\mathbf{Z}(t_{i-1}) = \mathcal{L}_{i-1}\} \\ &= a_0 + a_1[\hat{z}(t_i^-) - \hat{z}(t_i^-)] = a_0 \end{aligned} \quad (12-25)$$

Note that conditioning on  $\mathbf{Z}(t_{i-1}, \cdot)$  instead of  $\mathcal{L}_{i-1}$  would yield  $\hat{x}(t_i^-) = a_0$ . Similarly,  $a_1$  is obtained by letting  $\psi[x(t_i)] = x(t_i)$  and  $\theta[z(t_i)] = \{z(t_i) - \hat{z}(t_i^-)\}$  in (12-22),  $b_0$  by letting  $\psi[x(t_i)] = \{x(t_i) - \hat{x}(t_i^+)\}^2$  and  $\theta[z(t_i)] = 1$ , and  $b_1$  by letting  $\psi[x(t_i)] = \{x(t_i) - \hat{x}(t_i^+)\}^2$  and  $\theta[z(t_i)] = \{z(t_i) - \hat{z}(t_i^-)\}$ . In this manner,

the coefficients in (12-21) are evaluated as

$$a_0 = \hat{x} \triangleq E\{\mathbf{x}(t_i) | \mathbf{Z}(t_{i-1}) = \mathcal{Z}_{i-1}\} = \hat{x}(t_i^-) \quad (12-26a)$$

$$a_1 = [\widehat{\mathbf{x}\mathbf{h}} - \widehat{\mathbf{x}\mathbf{h}}][\widehat{(\mathbf{h} - \hat{\mathbf{h}})^2} + R]^{-1} \quad (12-26b)$$

$$b_0 = P(t_i^-) - a_1[\widehat{\mathbf{x}\mathbf{h}} - \widehat{\mathbf{x}\mathbf{h}}] \quad (12-26c)$$

$$b_1 = \left\{ \widehat{[(\mathbf{x} - \hat{\mathbf{x}})^2 \mathbf{h}]} - 2a_1 \widehat{[(\mathbf{h} - \hat{\mathbf{h}})(\mathbf{x} - \hat{\mathbf{x}})\mathbf{h}]} + a_1^2 \widehat{[(\mathbf{h} - \hat{\mathbf{h}})^2 \mathbf{h}]} + R\hat{\mathbf{h}} \right\} \\ \times \{\widehat{(\mathbf{h} - \hat{\mathbf{h}})^2} + R\}^{-1} \quad (12-26d)$$

using the notation of (12-11) and dropping time arguments for compactness. Now the various expectations in Eq. (12-26) are approximated by generating Taylor series representations, truncating to second order, and taking conditional expectations of the resulting terms individually, to yield

$$\widehat{z}(t_i^-) = \widehat{h[\mathbf{x}(t_i), t_i]} \cong h[\widehat{\mathbf{x}}(t_i^-), t_i] + \frac{1}{2} \frac{\partial^2 h[\widehat{\mathbf{x}}(t_i^-), t_i]}{\partial x^2} P(t_i^-) \quad (12-27a)$$

$$[\widehat{\mathbf{x}\mathbf{h}} - \widehat{\mathbf{x}\mathbf{h}}] \cong \frac{\partial h[\widehat{\mathbf{x}}(t_i^-), t_i]}{\partial x} P(t_i^-) \quad (12-27b)$$

$$\widehat{(\mathbf{h} - \hat{\mathbf{h}})^2} \cong \frac{\partial h[\widehat{\mathbf{x}}(t_i^-), t_i]^2}{\partial x} P(t_i^-) - \frac{1}{4} \frac{\partial^2 h[\widehat{\mathbf{x}}(t_i^-), t_i]^2}{\partial x^2} P^2(t_i^-) \quad (12-27c)$$

$$b_1 \cong -\frac{1}{2} P^2(t_i^-) \frac{\partial^2 h}{\partial x^2} [\widehat{(\mathbf{h} - \hat{\mathbf{h}})^2} + R]^{-1} \left\{ 1 - 3P(t_i^-) \frac{\partial h^2}{\partial x} [\widehat{(\mathbf{h} - \hat{\mathbf{h}})^2} + R]^{-1} \right. \\ \left. - P(t_i^-) \frac{\partial h^2}{\partial x} [\widehat{(\mathbf{h} - \hat{\mathbf{h}})^2} + R]^{-2} \left[ \frac{1}{4} P^2(t_i^-) \frac{\partial^2 h^2}{\partial x^2} + R \right] \right. \\ \left. + 2P^2(t_i^-) \frac{\partial h^4}{\partial x} [\widehat{(\mathbf{h} - \hat{\mathbf{h}})^2} + R]^{-2} \right\} \quad (12-27d)$$

where  $\widehat{\mathbf{x}}(t_i^-) = \widehat{\mathbf{x}}(t_i/t_{i-1})$ ,  $P(t_i^-) = P(t_i/t_{i-1})$ , and in (12-27d) the expectation is evaluated via (12-27c) and  $h$ ,  $\partial h/\partial x$ , and  $\partial^2 h/\partial x^2$  are all evaluated at  $[\widehat{\mathbf{x}}(t_i^-), t_i]$ .

Combining these results in the assumed form (12-21) yields the *update equations for the truncated second order filter* as

$$A_{TS}(t_i) = \frac{\partial h[\widehat{\mathbf{x}}(t_i^-), t_i]^2}{\partial x} P(t_i^-) - \frac{1}{4} \frac{\partial^2 h[\widehat{\mathbf{x}}(t_i^-), t_i]^2}{\partial x^2} P^2(t_i^-) + R(t_i) \quad (12-28)$$

$$K_{TS}(t_i) = P(t_i^-) \frac{\partial h[\widehat{\mathbf{x}}(t_i^-), t_i]}{\partial x} A_{TS}^{-1}(t_i) \quad (12-29)$$

$$\widehat{\mathbf{x}}(t_i^+) = \widehat{\mathbf{x}}(t_i^-) + K_{TS}(t_i) \left\{ z_i - h[\widehat{\mathbf{x}}(t_i^-), t_i] - \frac{1}{2} \frac{\partial^2 h[\widehat{\mathbf{x}}(t_i^-), t_i]}{\partial x^2} P(t_i^-) \right\} \quad (12-30)$$

$$P(t_i^+) = P(t_i^-) - K_{TS}(t_i) \frac{\partial h[\hat{x}(t_i^-), t_i]}{\partial x} P(t_i^-) \\ + b_1 \left\{ z_i - h[\hat{x}(t_i^-), t_i] - \frac{1}{2} \frac{\partial^2 h[\hat{x}(t_i^-), t_i]}{\partial x^2} P(t_i^-) \right\} \quad (12-31)$$

where  $b_1$  is defined by (12-27c,d). The same relations but with  $b_1 \equiv 0$  in (12-31) yield what is called the *modified truncated second order filter update equations* [54]. Compare these results to the first order extended Kalman filter update relations, (9-61)–(9-64). First of all, the truncated second order filter gain  $K_{TS}(t_i)$  has an additional term in the matrix being inverted,  $A_{TS}(t_i)$  versus  $[H(t_i)P(t_i^-)H^T(t_i) + R(t_i)]$ . Second, there is an additional term in the residual generation in (12-30), and this is the second “bias correction term” mentioned earlier. Finally, one is motivated to use the *modified* second order filter from a computational standpoint as well as from a desire to preclude negative values for computed  $P(t_i^+)$ : the computational burden of the second order filter above that of the extended Kalman filter is severe (especially in the vector case), whereas the additional computations for the modified second order filter are considerably more manageable. Had the power series (12-21b) been proposed as only zero-order in the residuals, this modified second order filter would have been derived directly.

Now consider the *Gaussian second order filter* for scalar systems [38, 52, 54, 102]. Whereas the truncated second order filter ignores all central moments of  $\mathbf{x}$  above second order, the Gaussian second order filter accounts for the fourth central moments as well, by approximating them with the values they would assume if the density were in fact Gaussian:

$$\overline{(\mathbf{x}_i - \hat{\mathbf{x}}_i)(\mathbf{x}_j - \hat{\mathbf{x}}_j)(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_l - \hat{\mathbf{x}}_l)} = P_{ij}P_{kl} + P_{ik}P_{jl} + P_{il}P_{jk} \quad (12-32a)$$

where  $\mathbf{P}$  is the conditional covariance matrix, or, in the scalar case,

$$\widehat{(\mathbf{x} - \hat{\mathbf{x}})^4} = 3P^2 \quad (12-32b)$$

Following the same reasoning as for the truncated second order filter, the *time propagation of the estimate* between sample times  $t_{i-1}$  and  $t_i$  is accomplished by means of integrating the relations

$$\hat{x}(t/t_{i-1}) = f[\hat{x}(t/t_{i-1}), t] + \frac{1}{2} \frac{\partial^2 f[\hat{x}(t/t_{i-1}), t]}{\partial x^2} P(t/t_{i-1}) \quad (12-33)$$

$$\dot{P}(t/t_{i-1}) = 2 \frac{\partial f[\hat{x}(t/t_{i-1}), t]}{\partial x} P(t/t_{i-1}) + \widehat{G[x(t), t]^2 Q(t)} \quad (12-34)$$

forward from  $\hat{x}(t_{i-1}/t_{i-1}) = \hat{x}(t_{i-1}^+)$ ,  $P(t_{i-1}/t_{i-1}) = P(t_{i-1}^+)$ , using the results of the measurement update at time  $t_{i-1}$ . These are identical in form to (12-18)

and (12-19), except that  $\widehat{G^2Q}$  is now evaluated as

$$\begin{aligned} \widehat{G[x(t), t]^2 Q(t)} &\cong G^2[\hat{x}(t/t_{i-1}), t]Q(t) + \frac{\partial G[\hat{x}(t/t_{i-1}), t]}{\partial x} P(t/t_{i-1})Q(t) \\ &+ G[\hat{x}(t/t_{i-1}), t] \frac{\partial^2 G[\hat{x}(t/t_{i-1}), t]}{\partial x^2} P(t/t_{i-1})Q(t) \\ &+ \frac{3}{4} \frac{\partial^2 G[\hat{x}(t/t_{i-1}), t]}{\partial x^2} P^2(t/t_{i-1})Q(t) \end{aligned} \quad (12-35)$$

where the additional fourth term is due to the difference in assumptions of the two filter formulations. As before, the approximation technique (12-21) suggested by Kramer and Jazwinski is applied to obtain the update equations for measurement incorporation. The expectations involved in (12-26) are again approximated, yielding (12-27a) and (12-27b) as evaluated previously, but (12-27c) and (12-27d) are replaced by

$$\widehat{(h - \hat{h})^2} \cong \frac{\partial h[\hat{x}(t_i^-), t_i]}{\partial x} P(t_i^-) + \frac{1}{2} \frac{\partial^2 h[\hat{x}(t_i^-), t_i]}{\partial x^2} P^2(t_i^-) \quad (12-27c')$$

$$\begin{aligned} b_1 &\cong P^2(t_i^-) \frac{\partial^2 h}{\partial x^2} [\widehat{(h - \hat{h})^2} + R]^{-1} \left\{ 1 - \frac{9}{2} P(t_i^-) \frac{\partial h^2}{\partial x} [\widehat{(h - \hat{h})^2} + R]^{-1} \right. \\ &\quad \left. + \frac{1}{2} P(t_i^-) \frac{\partial h^2}{\partial x} [\widehat{(h - \hat{h})^2} + R]^{-2} \left[ 7P(t_i^-) \frac{\partial h^2}{\partial x} + R \right] \right\} \end{aligned} \quad (12-27d')$$

Note particularly that both the sign and magnitude of the second term in (12-27c') differ from those of (12-27c). Thus, the *measurement update* equations for the Gaussian second order filter are:

$$A_{GS}(t_i) = \frac{\partial h[\hat{x}(t_i^-), t_i]}{\partial x} P(t_i^-) + \frac{1}{2} \frac{\partial^2 h[\hat{x}(t_i^-), t_i]}{\partial x^2} P^2(t_i^-) + R(t_i) \quad (12-36)$$

$$K_{GS}(t_i) = P(t_i^-) \frac{\partial h[\hat{x}(t_i^-), t_i]}{\partial x} A_{GS}^{-1}(t_i) \quad (12-37)$$

$$\hat{x}(t_i^+) = \hat{x}(t_i^-) + K_{GS}(t_i) \left\{ z_i - h[\hat{x}(t_i^-), t_i] - \frac{1}{2} \frac{\partial^2 h[\hat{x}(t_i^-), t_i]}{\partial x^2} P(t_i^-) \right\} \quad (12-38)$$

$$\begin{aligned} P(t_i^+) &= P(t_i^-) - K_{GS}(t_i) \frac{\partial h[\hat{x}(t_i^-), t_i]}{\partial x} P(t_i^-) \\ &+ b_1 \left\{ z_i - h[\hat{x}(t_i^-), t_i] - \frac{1}{2} \frac{\partial^2 h[\hat{x}(t_i^-), t_i]}{\partial x^2} P(t_i^-) \right\} \end{aligned} \quad (12-39)$$

with  $b_1$  defined by (12-27c', d'). Note that the basic form of the update is the same as before, but with  $A_{GS}(t_i)$  and  $b_1$  evaluated differently. Equations (12-33)–

(12-39) but with  $b_1 \equiv 0$  in (12-39) yield the *modified Gaussian second order filter*, with no random forcing function on the variance relations [4, 54, 72]. Since these are only approximate equations, such a random forcing function could yield negative computed variances, especially if the dynamic driving noise strength  $Q(t)$  is small and operating time is long. For this reason, the modified filters are often the preferred form. Moreover, since its inherent approximations are usually more applicable, the modified Gaussian is often the better of the two possible modified second order filters. Note that any of the four second order filters reduce to the extended Kalman filter if  $G$  is a function only of time and if  $\partial^2 f / \partial x^2$  and  $\partial^2 h / \partial x^2$  are neglected. Similarly, any of the four reduce to the Kalman filter if dynamics and measurement models are linear.

**EXAMPLE 12.1** Consider estimation of the scalar  $x$  described by

$$dx(t) = -[x(t) + ax^3(t)] dt + [1 + bx^2(t)] d\beta(t)$$

based on sampled data measurements of the form

$$z(t_i) = \sin x(t_i) + v(t_i)$$

with  $\beta(\cdot, \cdot)$  Brownian motion of diffusion  $Q$ , and  $v(\cdot, \cdot)$  white Gaussian zero-mean discrete-time noise of variance  $R$ , independent of  $\beta(\cdot, \cdot)$ , and let  $x(t_0)$  be described with mean  $\bar{x}_0$  and variance  $P_0$ .

The first order filter for this problem would be described by time propagation from  $t_{i-1}$  to  $t_i$  as

$$\begin{aligned}\dot{\hat{x}}(t/t_{i-1}) &= -[\hat{x}(t/t_{i-1}) + a\hat{x}^3(t/t_{i-1})] \\ \dot{P}(t/t_{i-1}) &= -2[1 + 3a\hat{x}^2(t/t_{i-1})]P(t/t_{i-1}) + [1 + b\hat{x}^2(t/t_{i-1})]^2 Q \\ &\quad + [2b\hat{x}(t/t_{i-1})]^2 P(t/t_{i-1})Q\end{aligned}$$

Note that the last term in the  $\dot{P}(t/t_{i-1})$  equation is  $[\partial G / \partial x]^2 PQ$ , a term that does not appear in extended Kalman filtering. The measurement update at time  $t_i$  would be

$$\begin{aligned}K(t_i) &= \frac{P(t_i^-)[\cos \hat{x}(t_i^-)]}{P(t_i^-)[\cos^2 \hat{x}(t_i^-)] + R} \\ \hat{x}(t_i^+) &= \hat{x}(t_i^-) + K(t_i)\{z_i - \sin \hat{x}(t_i^-)\} \\ P(t_i^+) &= P(t_i^-) - K(t_i)[\cos \hat{x}(t_i^-)]P(t_i^-)\end{aligned}$$

The *modified truncated second order filter* for this problem has a time propagation of (12-18)–(12-20):

$$\begin{aligned}\dot{\hat{x}}(t/t_{i-1}) &= -[\hat{x}(t/t_{i-1}) + a\hat{x}^3(t/t_{i-1})] - 3a\hat{x}(t/t_{i-1})P(t/t_{i-1}) \\ \dot{P}(t/t_{i-1}) &= -2[1 + 3a\hat{x}^2(t/t_{i-1})]P(t/t_{i-1}) + [1 + b\hat{x}^2(t/t_{i-1})]^2 Q \\ &\quad + [2b\hat{x}(t/t_{i-1})]^2 P(t/t_{i-1})Q + 2b[1 + b\hat{x}^2(t/t_{i-1})]P(t/t_{i-1})Q\end{aligned}$$

and a measurement update of (12-28)–(12-31) with  $b_1 \equiv 0$ :

$$\begin{aligned}K_{TS}(t_i) &= \frac{P(t_i^-)[\cos \hat{x}(t_i^-)]}{[\cos^2 \hat{x}(t_i^-)]P(t_i^-) - \frac{1}{4}[\sin^2 \hat{x}(t_i^-)]P^2(t_i^-) + R} \\ \hat{x}(t_i^+) &= \hat{x}(t_i^-) + K_{TS}(t_i)\{z_i - \sin \hat{x}(t_i^-) + \frac{1}{2}[\sin \hat{x}(t_i^-)]P(t_i^-)\} \\ P(t_i^+) &= P(t_i^-) - K_{TS}(t_i)[\cos \hat{x}(t_i^-)]P(t_i^-)\end{aligned}$$

The time propagation for the *modified Gaussian second order filter* would be the same except for an additional term,

$$+ 3b^2 P^2(t/t_{i-1}) Q$$

added to the  $\dot{P}(t/t_{i-1})$  equation. The update would differ from the preceding result only in the evaluation of the gain denominator:

$$K_{GS}(t_i) = \frac{P(t_i^-)[\cos \hat{x}(t_i^-)]}{[\cos^2 \hat{x}(t_i^-)]P(t_i^-) + \frac{1}{2}[\sin^2 \hat{x}(t_i^-)]P^2(t_i^-) + R} \quad \blacksquare$$

Second order filters provide performance generally superior to that of first order techniques, such as the extended Kalman filter, especially when nonlinearities are significant and noise strengths  $Q(t)$  and  $R(t_i)$  associated with the system model, (12-1) and (12-2), are small [4, 22, 31, 54, 91, 112, 114, 130, 145]. Usually, the primary benefit is due to the *bias correction terms* [4] in state estimate propagation and residual generation:

$$\dot{\hat{x}}(t/t_{i-1}) = f[\hat{x}(t/t_{i-1}), t] + \hat{b}_p(t/t_{i-1}) \quad (12-40a)$$

$$\hat{z}(t_i^-) = h[\hat{x}(t_i^-), t_i] + \hat{b}_m(t_i^-) \quad (12-40b)$$

where the propagation bias term  $\hat{b}_p(t/t_{i-1})$  and measurement bias term  $\hat{b}_m(t_i^-)$  are given by

$$\hat{b}_p(t/t_{i-1}) = \frac{1}{2} \frac{\partial^2 f[\hat{x}(t/t_{i-1}), t]}{\partial x^2} P(t/t_{i-1}) \quad (12-41a)$$

$$\hat{b}_m(t_i^-) = \frac{1}{2} \frac{\partial^2 h[\hat{x}(t_i^-), t_i]}{\partial x^2} P(t_i^-) \quad (12-41b)$$

The performance enhancement to be gained is related to the size of the second partial derivatives and error variances in these expressions. Intuitively, the greater the magnitude or harshness of the nonlinearity, the greater the benefit of accounting for the higher order terms. Moreover, the improvement is in fact often exhibited in the form of estimate bias reduction. This is as expected since, for example, the residuals  $\{z(t_i) - \hat{z}(t_i^-)\}$  using (12-40b) are zero mean to second order, whereas  $\{z(t_i) - h[\hat{x}(t_i^-), t_i]\}$  are zero mean only to first order. The importance of the bias correction term  $\hat{b}_p(t/t_{i-1})$  is reduced if the dynamics equations are driven by a high strength noise, and similarly the significance of  $\hat{b}_m(t_i^-)$  is lowered if there is considerable measurement corruption noise: biases due to neglecting higher order effects are masked by the substantial spreading of the density functions. Looking at (12-28) or (12-36), it can be seen that measurement nonlinearities will be significant if  $\{[\partial^2 h/\partial x^2]^2 P^2(t_i^-)\}$  is large compared to  $R(t_i)$ , and empirical results [22, 31, 54, 114, 131] have corroborated that nonlinear effects are most important when noise inputs are small while estimation error variance is relatively large.

In fact, this is exploited in the tuning of filters by purposely adding “pseudonoise” to account for model inadequacies such as neglected higher order

terms, as a frequently viable alternative to accounting explicitly for these terms with substantially more complicated filter computations [88]. Often a first order filter, as an extended Kalman filter, is designed and tuned for a given application. If performance (as indicated by a Monte Carlo analysis [96, 100]) is not acceptable, *first order filters based on better models* are considered as well as higher order filters. When biases due to neglected nonlinearities make it essential to design higher order filters, the modified Gaussian second order filter is often developed first. However, especially in the vector case to follow, the full-scale second order filter may not actually be implemented. Rather, a *first order filter with bias correction terms*, incorporating (12-40) and (12-41) but without altering the covariance or gain expressions of the first order filter, may be generated to obtain the essential benefit of second order filtering without the computational penalty of additional, time-consuming (matrix) second moment calculations. ■

**EXAMPLE 12.2** The *first order filter with bias correction terms* for the problem defined in Example 12.1 would have the same  $\hat{P}(t/t_{i-1})$ ,  $K(t_i)$ , and  $P(t_i^+)$  relations as the first order filter results depicted earlier, but with  $\hat{x}(t/t_{i-1})$  and  $\hat{x}(t_i^+)$  equations as for either second order filter.

First order filters with bias correction terms can also be established with covariance and gain calculations as given by the extended Kalman filter, or even as given by the linearized Kalman filter. It has also been suggested [3, 79] that (12-40) be incorporated into a first order filter design, with the biases treated as additional states to be estimated. Many of these ideas will be brought out further in the examples to follow the development of the second order filters for the vector case.

The *modified truncated second order filter* [54, 130] for the vector case updates the state estimate  $\hat{x}(t_i^-)$  with the measurement  $\mathbf{z}(t_i, \omega_j) = \mathbf{z}_i$  at sample time  $t_i$  via

$$\dot{\mathbf{A}}_{TS}(t_i) = \mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)] \mathbf{P}(t_i^-) \mathbf{H}^T[t_i; \hat{\mathbf{x}}(t_i^-)] - \hat{\mathbf{b}}_m(t_i^-) \hat{\mathbf{b}}_m^T(t_i^-) + \mathbf{R}(t_i) \quad (12-42)$$

$$\mathbf{K}_{TS}(t_i) = \mathbf{P}(t_i^-) \mathbf{H}^T[t_i; \hat{\mathbf{x}}(t_i^-)] \mathbf{A}_{TS}^{-1}(t_i) \quad (12-43)$$

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}_{TS}(t_i) \{ \mathbf{z}_i - \mathbf{h}[\hat{\mathbf{x}}(t_i^-), t_i] - \hat{\mathbf{b}}_m(t_i^-) \} \quad (12-44)$$

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{K}_{TS}(t_i) \mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)] \mathbf{P}(t_i^-) \quad (12-45)$$

where  $\mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)]$  is defined as the  $m$ -by- $n$  partial derivative matrix:

$$\mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)] \triangleq \left. \frac{\partial \mathbf{h}[\mathbf{x}, t_i]}{\partial \mathbf{x}} \right|_{\mathbf{x}=\hat{\mathbf{x}}(t_i^-)} \quad (12-46)$$

and the bias correction term  $\hat{\mathbf{b}}_m(t_i^-)$  is the  $m$ -vector with  $k$ th component given as

$$\hat{b}_{mk}(t_i^-) \triangleq \frac{1}{2} \operatorname{tr} \left\{ \frac{\partial^2 h_k[\hat{\mathbf{x}}(t_i^-), t_i]}{\partial \mathbf{x}^2} \mathbf{P}(t_i^-) \right\} \quad (12-47)$$

These relations are direct extensions of (12-28)–(12-31), (12-40b), and (12-41b). The estimate is *propagated* forward to the next sample time  $t_{i+1}$  by using the

initial conditions provided by (12-44) and (12-45):

$$\hat{\mathbf{x}}(t_i/t_i) = \hat{\mathbf{x}}(t_i^+) \quad (12-48a)$$

$$\mathbf{P}(t_i/t_i) = \mathbf{P}(t_i^+) \quad (12-48b)$$

and integrating

$$\dot{\hat{\mathbf{x}}}(t/t_i) = \mathbf{f}[\hat{\mathbf{x}}(t/t_i), t] + \hat{\mathbf{b}}_p(t/t_i) \quad (12-49)$$

$$\dot{\mathbf{P}}(t/t_i) = \mathbf{F}[t; \hat{\mathbf{x}}(t/t_i)]\mathbf{P}(t/t_i) + \mathbf{P}(t/t_i)\mathbf{F}^T[t; \hat{\mathbf{x}}(t/t_i)] + \widehat{\mathbf{G}[\mathbf{x}(t), t]\mathbf{Q}(t)\mathbf{G}^T[\mathbf{x}(t), t]} \quad (12-50)$$

where the bias correction term  $\hat{\mathbf{b}}_p(t/t_i)$  is the  $n$ -vector with  $k$ th component

$$\hat{b}_{pk}(t/t_i) \triangleq \frac{1}{2} \text{tr} \left\{ \frac{\partial^2 f_k[\hat{\mathbf{x}}(t/t_i), t]}{\partial \mathbf{x}^2} \mathbf{P}(t/t_i) \right\} \quad (12-51)$$

$\mathbf{F}[t; \hat{\mathbf{x}}(t/t_i)]$  is the  $n$ -by- $n$  partial derivative matrix

$$\mathbf{F}[t; \hat{\mathbf{x}}(t/t_i)] \triangleq \left. \frac{\partial \mathbf{f}[\mathbf{x}, t]}{\partial \mathbf{x}} \right|_{\mathbf{x}=\hat{\mathbf{x}}(t/t_i)} \quad (12-52)$$

and the last term in (12-50) is an  $n$ -by- $n$  matrix with  $ij$  element as (dropping time and  $\hat{\mathbf{x}}(t/t_i)$  arguments for convenience):

$$\begin{aligned} \widehat{[\mathbf{G}\mathbf{Q}\mathbf{G}^T]}_{ij} &= \sum_{k=1}^s \sum_{l=1}^s \left[ G_{ik} Q_{kl} G_{lj}^T + \text{tr} \left\{ \left( \frac{\partial G_{ik}}{\partial \mathbf{x}} \mathbf{Q}_{kl} \frac{\partial G_{lj}^T}{\partial \mathbf{x}} \right) \mathbf{P} \right\} \right. \\ &\quad \left. + \frac{1}{2} G_{ik} Q_{kl} \text{tr} \left\{ \frac{\partial^2 G_{lj}^T}{\partial \mathbf{x}^2} \mathbf{P} \right\} + \frac{1}{2} \text{tr} \left\{ \mathbf{P} \frac{\partial^2 G_{ik}}{\partial \mathbf{x}^2} \right\} Q_{kl} G_{lj}^T \right] \end{aligned} \quad (12-53)$$

where  $s$  is the dimension of the dynamic driving noise. Upon integrating these to the next sample time,  $\hat{\mathbf{x}}(t_{i+1}^-)$  and  $\mathbf{P}(t_{i+1}^-)$  are defined as

$$\hat{\mathbf{x}}(t_{i+1}^-) = \hat{\mathbf{x}}(t_{i+1}/t_i) \quad (12-54a)$$

$$\mathbf{P}(t_{i+1}^-) = \mathbf{P}(t_{i+1}/t_i) \quad (12-54b)$$

for use in the next measurement update. These equations are similarly direct extensions of (12-17)–(12-19), (12-40a), and (12-41a). Note the direct comparison between this filter and the extended Kalman filter given by (9-61)–(9-69).

Now consider the *modified Gaussian second order filter* for the vector case [4, 54, 72, 130]. The *measurement update* at time  $t_i$  is given by

$$\mathbf{A}_{GS}(t_i) = \mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)]\mathbf{P}(t_i^-)\mathbf{H}^T[t_i; \hat{\mathbf{x}}(t_i^-)] + \hat{\mathbf{B}}_m(t_i^-) + \mathbf{R}(t_i) \quad (12-55)$$

$$\mathbf{K}_{GS}(t_i) = \mathbf{P}(t_i^-)\mathbf{H}^T[t_i; \hat{\mathbf{x}}(t_i^-)]\mathbf{A}_{GS}^{-1}(t_i) \quad (12-56)$$

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}_{GS}(t_i)\{\mathbf{z}_i - \mathbf{h}[\hat{\mathbf{x}}(t_i^-), t_i] - \hat{\mathbf{b}}_m(t_i^-)\} \quad (12-57)$$

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{K}_{GS}(t_i)\mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)]\mathbf{P}(t_i^-) \quad (12-58)$$

where  $\mathbf{H}[t_i; \hat{\mathbf{x}}(t_i^-)]$  and  $\hat{\mathbf{b}}_m(t_i^-)$  are as defined in (12-46) and (12-47), and  $\hat{\mathbf{B}}_m(t_i^-)$

is an  $m$ -by- $m$  matrix with the  $kl$ th element as

$$\hat{B}_{mkl}(t_i^-) \triangleq \frac{1}{2} \text{tr} \left\{ \frac{\partial^2 h_k[\hat{\mathbf{x}}(t_i^-), t_i]}{\partial \mathbf{x}^2} \mathbf{P}(t_i^-) \frac{\partial^2 h_l[\hat{\mathbf{x}}(t_i^-), t_i]}{\partial \mathbf{x}^2} \mathbf{P}(t_i^-) \right\} \quad (12-59)$$

Note the difference in form between this and the corresponding term in (12-42); both reduce to a scalar times  $[\partial^2 h / \partial x^2]^2 P^2(t_i^-)$  in the scalar case. These results are the vector case of (12-36)–(12-39). The *propagation* relations corresponding to the scalar case of (12-33)–(12-35) are identical to (12-48)–(12-52) and (12-54), but an additional term appears in  $\widehat{\mathbf{GQG}^\top}$  as compared to the previous result (12-53). To express the result conveniently, define

$$\mathbf{G}'[\mathbf{x}(t), t] \triangleq \mathbf{G}[\mathbf{x}(t), t] \mathbf{Q}^{1/2}(t) \quad (12-60)$$

as in (11-94), and then  $\widehat{\mathbf{GQG}^\top} = \widehat{\mathbf{G'G'}^\top}$  is an  $n$ -by- $n$  matrix with  $kj$  element as [54; 85, Appendix E] (see Problem 10.9)

$$\begin{aligned} [\widehat{\mathbf{GQG}^\top}]_{ij} &= \sum_{k=1}^s \left[ G'_{ik} G'_{kj}^\top + \text{tr} \left\{ \left( \frac{\partial G'_{ik}^\top}{\partial \mathbf{x}} \frac{\partial G'_{kj}}{\partial \mathbf{x}} \right) \mathbf{P} \right\} \right. \\ &\quad + \frac{1}{2} G'_{ik} \text{tr} \left\{ \frac{\partial^2 G'_{kj}^\top}{\partial \mathbf{x}^2} \mathbf{P} \right\} + \frac{1}{2} \text{tr} \left\{ \mathbf{P} \frac{\partial^2 G'_{ik}}{\partial \mathbf{x}^2} \right\} G'_{kj}^\top \\ &\quad \left. + \frac{1}{4} \text{tr} \left\{ \frac{\partial^2 G'_{ik}}{\partial \mathbf{x}^2} \mathbf{P} \right\} \text{tr} \left\{ \frac{\partial^2 G'_{kj}^\top}{\partial \mathbf{x}^2} \mathbf{P} \right\} + \frac{1}{2} \text{tr} \left\{ \frac{\partial^2 G'_{ik}}{\partial \mathbf{x}^2} \mathbf{P} \frac{\partial^2 G'_{kj}^\top}{\partial \mathbf{x}^2} \mathbf{P} \right\} \right] \end{aligned} \quad (12-61)$$

Note that the first four terms replicate (12-53), and the last two terms correspond to the last term of (12-35) for the scalar case. These filter relations are also directly comparable to the extended Kalman filter, (9-61)–(9-69).

**EXAMPLE 12.3** To demonstrate the performance difference between a first order (extended Kalman) and the modified Gaussian second order filters, a problem [4] that has significant nonlinearities in both the state dynamics and measurement equations is explored. Consider the estimation of the altitude, velocity, and constant ballistic coefficient of a vertically falling body, as shown in Fig. 12.1. Sampled-data measurements of range (only) are provided by a radar, corrupted by wideband noise assumed to be well modelled as white. Note in the figure that the body is assumed

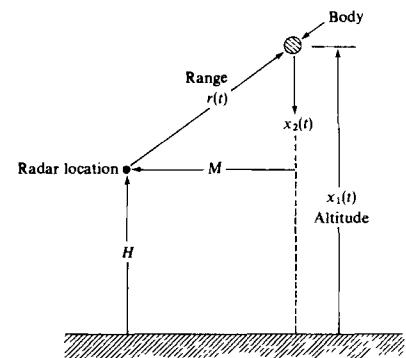


FIG. 12.1 Geometry of estimation problem.  
From [4], © 1968 IEEE.

to be falling vertically to simplify this example, the horizontal miss distance  $M$  between the radar location and the trajectory of the body is assumed to be known (100,000 ft), the altitude  $H$  of the radar is to be *fixed* at one of three values corresponding to three different scenarios, and that vertical velocity  $x_2$  is defined as positive downward while altitude  $x_1$  is positive upward.

Assuming the effect of gravity to be negligible compared to atmospheric deceleration as a further simplification, the two basic state equations become

$$\begin{aligned}\dot{x}_1(t) &= -x_2(t) \\ \dot{x}_2(t) &= -\left[\frac{C_D A \rho(t)}{2m}\right] x_2^2(t) + w(t)\end{aligned}$$

where  $C_D$  is the constant drag coefficient of the body and  $A$  is its reference area used for drag evaluation,  $m$  is its constant mass and  $\rho(t)$  is the mass density of the atmosphere. The air density is approximated by the exponential function

$$\rho(t) = \rho_0 \exp\{-\gamma x_1(t)\}$$

where  $\rho_0$  is sea level density ( $1.22 \text{ kg/m}^3 = 0.076 \text{ lb/ft}^3$ ) and  $\gamma = 5 \times 10^{-5} \text{ ft}^{-1}$ . Thus, if we define a third state variable  $x_3 \triangleq [C_D A \rho_0]/[2m]$  as a constant whose value is not known precisely a priori, inversely related to the ballistic coefficient  $\beta \triangleq mg/C_D A$  as  $x_3 = g\rho_0/2\beta$ , the state equations can be written as

$$\begin{aligned}\dot{x}_1(t) &= -x_2(t) &\triangleq f_1 \\ \dot{x}_2(t) &= -x_2^2(t)x_3(t) \exp\{-\gamma x_1(t)\} + w(t) &\triangleq f_2 + w \\ \dot{x}_3(t) &= 0 + w'(t) &\triangleq f_3 + w'\end{aligned}$$

$w(\cdot, \cdot)$  and  $w'(\cdot, \cdot)$  are independent zero-mean white Gaussian noises of strengths  $Q$  and  $Q'$ , respectively, and can be considered “pseudonoises” for tuning purposes ( $Q = Q' = 0$  for the “truth model” in simulations to follow). These state equations can be put into the form

$$\mathbf{dx}(t) = \mathbf{f}[\mathbf{x}(t)] dt + \mathbf{G} d\beta(t)$$

i.e., time invariant and with  $\mathbf{G}$  not a function of  $\mathbf{x}(\cdot, \cdot)$ .

Radar measurements, assumed to be available once a second, are described in terms of variables of Fig. 12.1 as

$$z(t_i) = \{M^2 + [x_1(t_i) - H]^2\}^{1/2} + v(t_i)$$

with  $v(\cdot, \cdot)$  white Gaussian discrete-time noise of mean zero and

$$E\{v(t_i)^2\} = R = \text{const} = 10,000 \text{ ft}^2$$

The purpose of allowing  $H$  to be set at 0, 100K ft, or 200K ft is to alter the effect of the measurement nonlinearity relative to the dynamics nonlinearities. As seen in Fig. 12.1, when the target body passes through the radar location altitude (i.e.,  $x_1 = H$ ), the range data is least useful in updating the state estimates; this is an observability problem in that variations in  $x_1$  are imperceptible from range at  $x_1 = H$ , and little information is provided about  $x_1$  from range when  $x_1$  is nearly equal to  $H$ . Figure 12.2 plots the altitude and velocity time histories of a body with true  $x_3$  value set at  $3 \times 10^{-3} \text{ ft}^{-1}$ , and thus these time periods of least useful data were at about 5 sec for  $H = 200\text{K}$  ft, 10 sec for  $H = 100\text{K}$  ft, and at the end of simulation time for  $H = 0$ .

True initial conditions were  $x_1(0) = 300\text{K}$  ft,  $x_2(0) = 20\text{K}$  ft/sec, and  $x_3(0) = 3 \times 10^{-3} \text{ ft}^{-1}$ , while the filter assumed initial conditions modeled as independent Gaussian random variables, with means and variances

$$\begin{aligned}\hat{x}_1(0) &= 300\text{K} \text{ ft} & P_{11}(0) &= 10^6 \text{ ft}^2 \\ \hat{x}_2(0) &= 20\text{K} \text{ ft/sec} & P_{22}(0) &= 4 \times 10^6 \text{ ft}^2/\text{sec}^2 \\ \hat{x}_3(0) &= 3 \times 10^{-5} \text{ ft}^{-1} & P_{33}(0) &= 10^{-4} \text{ ft}^{-2}\end{aligned}$$

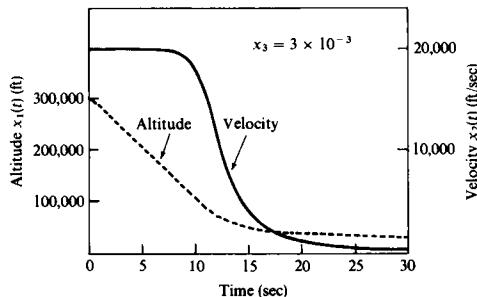
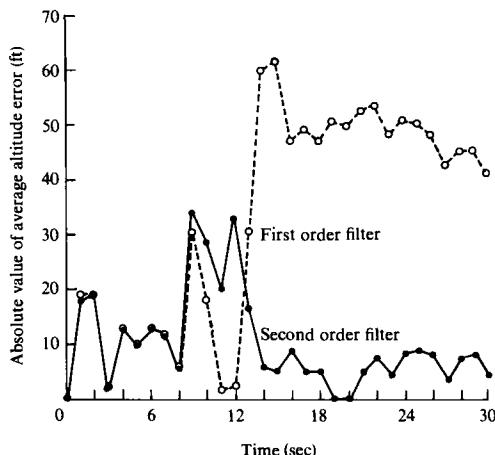


FIG. 12.2 Altitude and velocity time histories for true target. From [4], © 1968 IEEE.

Thus, the initial state estimates of altitude and velocity were exact, while the estimate of  $x_3$  was initially very poor. Simulation results revealed that large initial errors in  $x_1$  and  $x_2$  were reduced substantially after only one or two measurements (respectively), so good initial values were used here to allow reasonable resolution of plotted error statistics. The error in the  $x_3$  estimate corresponded to a light true target ( $\beta \cong 16.6$ ) while the filter assumed it to be heavy ( $\beta \cong 1660$ ). Note from Fig. 12.2 that correct estimation of  $x_3$  cannot take place until the body descends into the denser atmosphere and deceleration characteristics allow the body's density to become apparent: the most notable dynamics nonlinearities started taking effect at about 9 sec, with maximum deceleration occurring between 9 and 16 sec in the simulation. Until this time, one could expect little difference in performance between estimators of different order.

A Monte Carlo analysis involving 50 simulation runs (each having different measurement noise samples) was conducted to evaluate an extended Kalman filter versus a modified Gaussian second order filter (using the same 50 noise history samples for each). To accentuate the differences,  $Q$  and  $Q'$  were assumed zero in the filters as well as in the true simulations. For each run, the error vector  $[x(t_i) - \hat{x}(t_i^+)]$  was generated for all  $t_i$ , and true average and rms values computed as sample statistics. These actual rms errors can be compared to the filter-computed  $P(t_i^+)$  value as well.

Figures 12.3–12.5 plot the absolute value of the average errors in  $x_1$  (altitude),  $x_2$  (velocity), and  $x_3$ , respectively, for the case of the radar located on the ground ( $H = 0$ ). Here the measurement

FIG. 12.3 Average altitude estimation errors of the two filters ( $H = 0$ ). From [4], © 1968 IEEE.

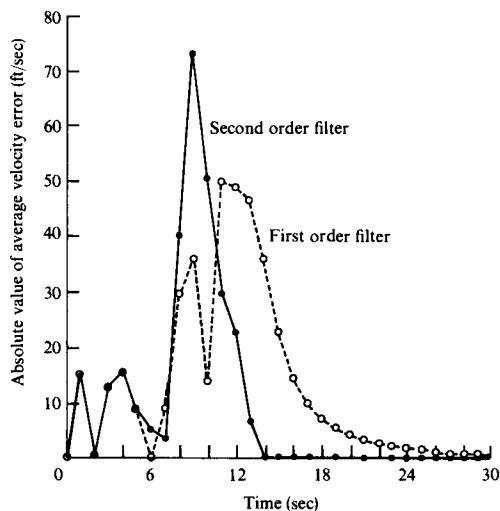


FIG. 12.4 Average velocity estimation errors of the two filters ( $H = 0$ ). From [4], © 1968 IEEE.

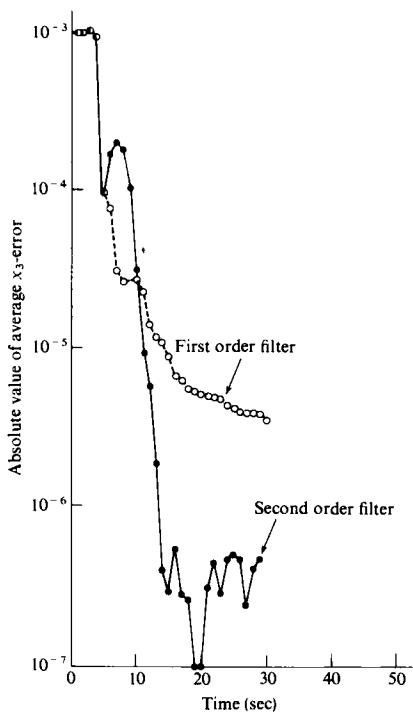


FIG. 12.5 Average parameter  $x_3$  estimation errors of the two filters ( $H = 0$ ). From [4], © 1968 IEEE.

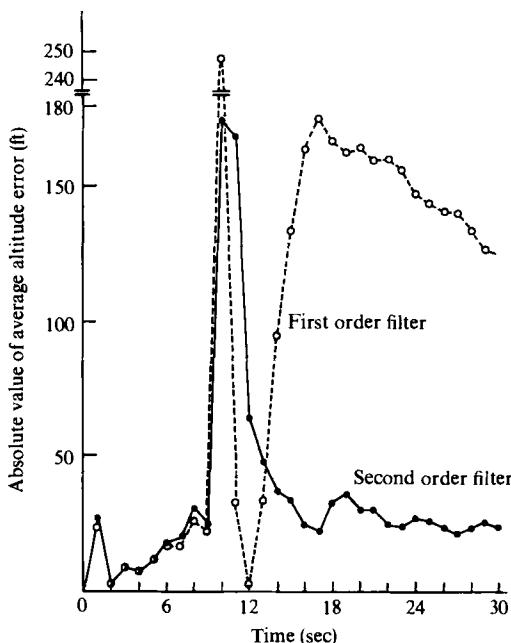


FIG. 12.6 Average altitude estimation errors of the two filters ( $H = 100$ K ft). From [4], © 1968 IEEE.

nonlinearity was more pronounced towards the end of the trajectory, after the maximum deceleration region (period of most significant effect of dynamics nonlinearities). As expected, the performances were essentially indistinguishable until about 9 sec, after which the second order filter achieved an order of magnitude better estimate of  $x_3$  and better precision in the other estimates as well.

Figure 12.6 is directly comparable to Fig. 12.3, but for the case of  $H = 100$ K ft. Here the measurement nonlinearity and observability problem were most severe at about 10 sec, causing estimate degradation. Upon recovery, the second order filter again outperformed the first order significantly; an order of magnitude improvement in  $x_3$  estimation was achieved as in Fig. 12.3.

Figure 12.7 plots the same results for  $H = 200$ K ft, for which the measurement information was least useful at 5 sec, resulting in increased estimation errors at this time. Again, an order of magnitude enhancement was realized in  $x_3$  estimation by using the second order filter. Figure 12.8 depicts the corresponding actual rms altitude errors and the average second order calculation of  $\sqrt{P_{11}}$ ; the first order filter calculations were within one percent of these computed values. Thus, the marked improvement in performance of the second order filter did not appear to be due to the more complex covariance and gain calculations. To corroborate this, an extended Kalman filter with bias correction terms, as given by (12-44), (12-47), (12-49) and (12-51), was evaluated, and the results were virtually the same as those of the second order filter.

In Fig. 12.8 it can be seen that the filter-computed error covariance was a better representation of *true* error covariance in the second order filter than in the extended Kalman filter, but both filters underestimated their own errors. This was due to setting  $Q$  and  $Q'$  to zero in the filters. Because of its better internal model (second order terms included), the second order filter would require less pseudonoise addition for tuning. Nevertheless, both filters would yield improved precision, and

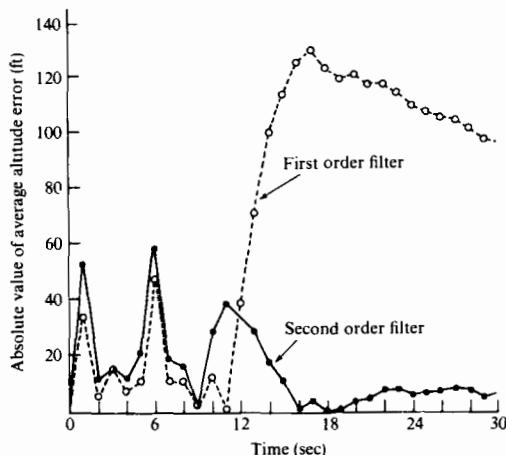


FIG. 12.7 Average altitude estimation errors of the two filters ( $H = 200\text{K ft}$ ). From [4]. © 1968 IEEE.

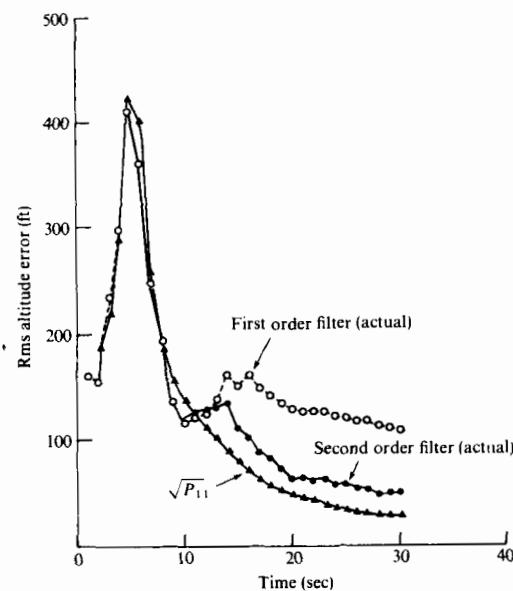


FIG. 12.8 Actual RMS altitude errors and filter-computed values ( $H = 200\text{K ft}$ ). From [4]. © 1968 IEEE.

their differences would be reduced, by adding enough pseudonoise to each to accomplish proper tuning (i.e., yielding smallest rms errors; matching computed variances to true rms errors is more conservative than matching to true error variances, but still does not guarantee elimination of biases and serious performance degradation in the lower order filter). ■

**EXAMPLE 12.4** Residual monitoring can be used in an approximate fashion for reasonableness checking, sensor failure detection, and adaptation in nonlinear filtering, as discussed in Volume I for linear filtering. By monitoring the residuals in real-time operation, one can determine if they are well approximated as white, zero-mean, with covariance given by  $\mathbf{A}_{TS}(t_i)$  or  $\mathbf{A}_{GS}(t_i)$ .

Consider the outputs of an aircraft's flight control system rate gyros and Euler angle orientation information provided by either the inertial navigation system (INS) or attitude and heading reference system (AHRS). Inherent functional redundancy among the signals from these different sensors can be exploited rather than resorting solely to hardware duplication to achieve fault detection and high reliability in the overall data system [12, 29, 86–88, 92, 93].

If  $\omega_x$ ,  $\omega_y$ , and  $\omega_z$  are roll rate, pitch rate, and yaw rate, respectively, as would be measured by the rate gyros, and if  $\psi$ ,  $\theta$ , and  $\phi$  are Euler angles of yaw, pitch, and roll relative to a reference coordinate frame, then the functional relationships among the variables are

$$\begin{aligned}\dot{\theta} &= \omega_y \cos \phi - \omega_z \sin \phi \\ \dot{\phi} &= \omega_x + \omega_y \tan \theta \sin \phi + \omega_z \tan \theta \cos \phi \\ \dot{\psi} &= [\omega_y \sin \phi + \omega_z \cos \phi] / \cos \theta\end{aligned}$$

These provide the basis of a dynamics model by letting  $\mathbf{x} = [\theta, \phi, \psi]^T$  and letting the rate gyro outputs be denoted as

$$\mathbf{u} \triangleq \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \triangleq \mathbf{w}_{\text{true}} + \mathbf{w}$$

where  $\mathbf{w}(\cdot, \cdot)$  is zero-mean white Gaussian noise with  $E[\mathbf{w}(t)\mathbf{w}^T(t+\tau)] = Q\delta(\tau)$ . Then  $\omega_x$  can be replaced by  $[u_1 - w_1]$ , and so forth, in the previous differential equations, and a realization of  $\mathbf{u}(\cdot, \cdot)$  would be available as real-time rate gyro data. Thus is generated

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} (u_2 - w_2) \cos x_2 - (u_3 - w_3) \sin x_2 \\ (u_1 - w_1) + (u_2 - w_2) \tan x_1 \sin x_2 + (u_3 - w_3) \tan x_1 \cos x_2 \\ [(u_2 - w_2) \sin x_2 + (u_3 - w_3) \cos x_2] / \cos x_1 \end{bmatrix}$$

or

$$d\mathbf{x}(t) = f[\mathbf{x}(t), \mathbf{u}(t)] dt + G[\mathbf{x}(t)] d\beta(t)$$

The discrete-time attitude measurements from the INS can be modeled as true values plus white Gaussian noise:

$$\mathbf{z}(t_i) \triangleq \begin{bmatrix} z_1(t_i) \\ z_2(t_i) \\ z_3(t_i) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta(t_i) \\ \phi(t_i) \\ \psi(t_i) \end{bmatrix} + \begin{bmatrix} v_1(t_i) \\ v_2(t_i) \\ v_3(t_i) \end{bmatrix} \triangleq H\mathbf{x}(t_i) + \mathbf{v}(t_i)$$

Thus a three-state nonlinear filter can be generated to combine rate gyro and INS outputs. A separate filter of identical structure (but different  $R$ ) can combine rate gyro and AHRS data. If abnormal residual growth is confined to the first filter, the failure can be isolated as occurring in the INS; if only in the second filter, the failure can be declared in the AHRS; if residuals in both filters are exhibiting abnormal behavior, the fault most likely occurred in the rate gyros, since their outputs feed both filters (simultaneous failures in the INS and AHRS are of much lower probability).

As discussed in Section 5.4 of Volume 1, a failure can be indicated by a likelihood function, defined in terms of a conditional joint density for the  $N$  most recent values of a particular residual component, passing a predetermined threshold. For the  $k$ th residual component, this becomes

$$\begin{aligned}L_{N_k}(t_i) &= \sum_{j=i-N+1}^i \ln f_{r_k(t_j)|r_k(t_{j-1}), \dots, r_k(t_1)}(\rho_j | \rho_{j-1}, \dots, \rho_1) \\ &\cong c_k(t_i) - \frac{1}{2} \sum_{j=i-N+1}^i \frac{r_k^2(t_j)}{\sigma_k^2(t_j)}\end{aligned}$$

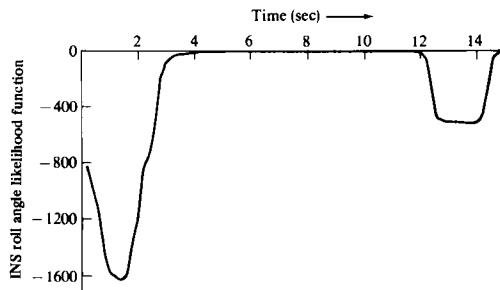


FIG. 12.9 Roll state likelihood function during turn: no failures. From [87], © 1976 IEEE.

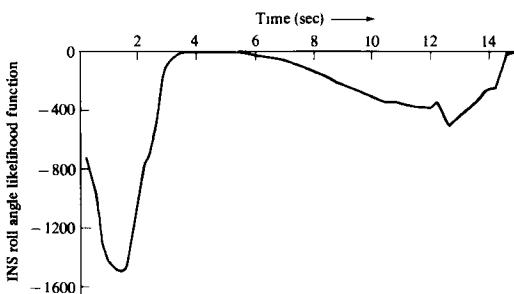


FIG. 12.10 Roll state likelihood function during turn: INS gyro float leak. From [87], © 1976 IEEE.

using the *approximating* description of the residuals as zero-mean, white and Gaussian, with  $\sigma_k^2$  as computed through  $A_{TS}(t_i)$  or  $A_{GS}(t_i)$ . By appropriately choosing  $N$  and the threshold for each  $L_{N_k}$ , false and missed alarms can be minimized for “hard failures,” those which cause abrupt, large changes in  $L_{N_k}$ .

“Soft failures” require more sophisticated detection logic. Figure 12.9 depicts the likelihood function corresponding to the second residual (on the roll state  $\phi$ ) of the INS/rate gyro filter during a turn, with no failures, in a simulated flight (actually for a first order filter, but representative of second order filter characteristics as well). The transients seen in the figure correspond to times of rapid roll rates (snap rolls) at the beginning and end of a coordinated turn. At these times, the true residual magnitudes far surpass the statistical characterization ( $\sigma_k^2(t_i)$  terms), particularly because of the very simple dynamics model and first order integration method employed in the filter for computational simplicity. Essentially, the filter’s internal model does *not* provide an adequate representation of the dynamics at these times, and the likelihood function undergoes a transient magnitude growth. Figure 12.10 portrays the same trajectory simulation, but with an INS gyro float leak “soft” failure. Although the failure characteristic is evident, the growth due to this true failure never exceeds the magnitudes encountered in normal operation.

One solution to this difficulty is “time-to-failure-declaration” parameters, specifying a time interval length over which a threshold must be surpassed consistently before declaring a failure. This allows tighter thresholds, and thereby a higher sensitivity to “soft” or hard-to-discriminate failures, without causing false alarms due to transitory threshold surpassing. Moreover, a second threshold, of magnitude greater than that attained under any normal circumstances, can be associated with each likelihood function to allow *immediate* declaration of “hard” failures.

More sophisticated methods, as using generalized likelihood ratios [30, 32, 34, 143, 144], have also been applied to this problem. ■

*Higher order filters* can be generated by including higher order terms from the Taylor series expansions of  $\mathbf{f}$  and  $\mathbf{h}$ . However, the severe computational disadvantage makes such filters unattractive, and first or second order filters based on better models are preferable.

In some cases, it is beneficial to utilize Eqs. (12-9)–(12-14), and the vector case equivalent of (12-21) and (12-26), without such series representations for  $\mathbf{f}$  and  $\mathbf{h}$ , instead evaluating the needed conditional expectations based on an assumed conditional density function form [44, 68–70]. Typically,  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1})$  is assumed to be Gaussian with mean  $\hat{\mathbf{x}}(t/t_{i-1})$  and covariance  $\mathbf{P}(t/t_{i-1})$  as computed within the algorithm itself. This is especially convenient when nonlinearities simply involve powers of  $\mathbf{x}$  (or products of components  $x_j$ ,  $x_k$ , etc., in the vector case). Moreover, since no Taylor series for  $\mathbf{f}$  and  $\mathbf{h}$  are required, these functions are not required to be differentiable with respect to  $\mathbf{x}$ . Thus, this form of *assumed density filter* is applicable to problems involving such discontinuous nonlinearities as relays and saturations, for which the previous filters are not suitable.

**EXAMPLE 12.5** Consider the problem of Examples 12.1 and 12.2. For propagations, (12-9) and (12-10) yield

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \hat{\mathbf{f}} & = -[\hat{\mathbf{x}} + a\hat{\mathbf{x}}^3] \\ \dot{\hat{\mathbf{P}}} &= 2\hat{\mathbf{f}}\hat{\mathbf{x}} - 2\hat{\mathbf{f}}\hat{\mathbf{x}} + \widehat{\mathbf{G}^2Q} & = -2[\hat{\mathbf{x}}^2 + a\hat{\mathbf{x}}^4] + 2[\hat{\mathbf{x}} + a\hat{\mathbf{x}}^3]\hat{\mathbf{x}} + [1 + 2b\hat{\mathbf{x}}^2 + b^2\hat{\mathbf{x}}^4]Q\end{aligned}$$

Now expressing  $\widehat{\mathbf{x}}^3$  and  $\widehat{\mathbf{x}}^4$  under the assumption that  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1})$  is Gaussian, i.e.,

$$\begin{aligned}\widehat{\mathbf{x}}^3 &= \int_{-\infty}^{\infty} \xi^3 f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1}) d\xi = 3\hat{\mathbf{x}}P + \hat{\mathbf{x}}^3 \\ \widehat{\mathbf{x}}^4 &= \int_{-\infty}^{\infty} \xi^4 f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1}) d\xi = 3P^2 + 6\hat{\mathbf{x}}^2P + \hat{\mathbf{x}}^4\end{aligned}$$

results in an estimator given by

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= -\hat{\mathbf{x}} - a\hat{\mathbf{x}}^3 - 3a\hat{\mathbf{x}}P \\ \dot{\hat{\mathbf{P}}} &= -2[1 + 3a\hat{\mathbf{x}}^2 + 3\hat{\mathbf{x}}P]P + [(1 + b\hat{\mathbf{x}}^2)^2 + 2bP + 3b^2P^2 + 6b^2\hat{\mathbf{x}}^2P]Q\end{aligned}$$

This agrees with the Gaussian second order filter of Example 12.1 except for the appearance of the  $3\hat{\mathbf{x}}P$  in the first bracketed term of  $\dot{\hat{\mathbf{P}}}$ , the difference due to retaining versus neglecting the last term of

$$\widehat{\mathbf{x}}^2 = (\hat{\mathbf{x}}^2 + P)^2 = \hat{\mathbf{x}}^4 + 2\hat{\mathbf{x}}^2P + P^2 \cong \hat{\mathbf{x}}^4 + 2\hat{\mathbf{x}}^2P$$

and yielding a term that is dominated by the other terms in those brackets.

Measurement updating can be accomplished by using (12-13) and (12-14), and evaluating the conditional expectations under the assumption that  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi|\mathcal{L}_{i-1})$  is Gaussian. However, it is usually better to invoke the residual power series assumption (12-21), with  $\mathbf{b}_1 \equiv 0$ , producing (12-26), in which the conditional expectations can be evaluated under the Gaussian assumption. If  $h[\mathbf{x}(t_i), t_i]$  had been a simple power of  $\mathbf{x}(t_i)$  instead of  $\sin \mathbf{x}(t_i)$ , these results could be written down without

further approximation, but here such terms as  $\hat{h}$  must be left in the form

$$\hat{h} = \int_{-\infty}^{\infty} (\sin \xi) f_{\mathbf{x}(t_i) | \mathbf{z}(t_{i-1})}(\xi | \mathbf{Z}_{i-1}) d\xi$$

to be integrated numerically or approximated in some fashion. The results are, using  $\hat{x}$  and  $P$  as propagated from the previous relations,

$$\begin{aligned}\hat{x}(t_i)^+ &= \hat{x} + [\widehat{xh} - \hat{x}\hat{h}] \widehat{[(h - \hat{h})^2 + R]}^{-1} \{z(t_i) - \hat{h}\} \\ P(t_i)^+ &= P - [\widehat{xh} - \hat{x}\hat{h}] \widehat{[(h - \hat{h})^2 + R]}^{-1} [\widehat{xh} - \hat{x}\hat{h}]\end{aligned}\blacksquare$$

*Higher order moment filters* can be generated by writing explicit moment equations as (12-9)–(12-14) and higher order moments as well, again using the forward Kolmogorov equation (11-96) or Itô differential rule (11-84) to create propagation relations and Bayes' rule to establish conditional density update relationships. For instance, by taking the conditional expectation of (11-84) and using the notation of (12-11), we get

$$\widehat{\frac{d\psi[\mathbf{x}(t), t]}{dt}} = \widehat{\frac{\partial \psi}{\partial t}} + \widehat{\frac{\partial \psi}{\partial \mathbf{x}}} \mathbf{f}[\mathbf{x}(t), t] + \frac{1}{2} \text{tr} \left\{ \mathbf{G}[\mathbf{x}(t), t] \mathbf{Q}(t) \mathbf{G}^T[\mathbf{x}(t), t] \frac{\partial^2 \psi}{\partial \mathbf{x}^2} \right\} \quad (12-62)$$

and by letting the scalar  $\psi[\mathbf{x}(t), t]$  be successively set to  $x_j(t)$ ,  $[x_j(t)x_k(t)]$ ,  $[x_j(t)x_k(t)x_l(t)]$ , etc., individual scalar components of all order moments can be generated. Then an assumption about the densities or moments would again be involved to allow approximate estimator algorithms to be developed in implementable form. One useful technique is *cumulants truncation* [97, 110, 111, 120], and in fact the Gaussian second order filter can be viewed as a special case of truncated cumulants filter.

Consider the scalar case. In both second order filters discussed previously, we wrote expressions for various terms assuming that higher order central moments associated with  $f_{\mathbf{x}(t) | \mathbf{z}(t_i)}(\xi | \mathcal{Z}_i)$  could be ignored. These central moments are the coefficients of a Taylor series for the conditional characteristic function for  $\mathbf{x}(t)$  minus its conditional mean  $\hat{x}(t)$ ,

$$E\{e^{j\mu[x(t) - \hat{x}(t)]} | \mathbf{Z}(t_i) = \mathcal{Z}_i\}$$

while noncentral moments are the coefficients of a Taylor series for the conditional characteristic function

$$E\{e^{j\mu x(t)} | \mathbf{Z}(t_i) = \mathcal{Z}_i\}$$

*Cumulants* are the coefficients of a Taylor series for

$$\ln E\{e^{j\mu x(t)} | \mathbf{Z}(t_i) = \mathcal{Z}_i\}$$

It is often a better approximation to set higher order cumulants to zero than to ignore higher order moments (especially if  $[x(t) - \hat{x}(t)]$  realizations are generally larger than one, so that successive higher order moments are larger rather than decreasing toward zero). If  $\mathbf{x}(t)$  is Gaussian, then the first two

cumulants are the conditional mean and variance, and all higher order cumulants are identically zero (not just “negligible”): for  $x(t)$  Gaussian, the characteristic function is  $\exp\{j\mu\hat{x}(t/t_i) - \frac{1}{2}P(t/t_i)\mu^2\}$  so that

$$\begin{aligned}\ln E\{e^{j\mu x(t)} | \mathbf{Z}(t_i) = \mathcal{L}_i\} &= j\mu\hat{x}(t/t_i) - \frac{1}{2}\mu^2 P(t/t_i) \\ &= \sum_{k=1}^{\infty} c_k \frac{(j\mu)^k}{k!}\end{aligned}\quad (12-63)$$

with  $c_1 = \hat{x}(t/t_i)$ ,  $c_2 = P(t/t_i)$ ,  $c_3 = c_4 = \dots \equiv 0$ . Thus, by assuming  $c_k = 0$  for all  $k > 2$ , we are in fact making the same Gaussian density assumption used previously. If we truncate at a level higher than  $k = 2$ , we obtain higher order non-Gaussian corrections that are appropriate in the sense that cumulants do go to zero with increasing order (though not necessarily monotonically). One is not guaranteed of uniformly improving performance by going to higher order, and performance evaluations and comparisons must be accomplished via Monte Carlo simulations in general.

In the general scalar case, the cumulants  $c_k$  and noncentral moments  $m_k$  can be directly related by exponentiating the series for  $[\ln E\{e^{j\mu x(t)} | \mathbf{Z}(t_i) = \mathcal{L}_i\}]$  and equating coefficients to those of like powers in the series for  $[E\{e^{j\mu x(t)} | \mathbf{Z}(t_i) = \mathcal{L}_i\}]$ , yielding [26]

$$\begin{aligned}c_1 &= m_1 &= m_1 \\ c_2 &= m_2 - m_1^2 &= m_2^c \\ c_3 &= m_3 - 3m_1m_2 + 2m_1^3 &= m_3^c \\ c_4 &= m_4 - 3m_2^2 - 4m_1m_3 + 12m_1^2m_2 - 6m_1^4 = m_4^c - 3m_2^c &= m_4^c - 3m_2^c \\ c_5 &= &= m_5^c - 10m_2^c m_3^c \\ &\vdots &\end{aligned}\quad (12-64)$$

as expressed also in terms of central moments  $m_k^c$ . Note that setting  $c_3$  and  $c_4$  to zero equates the third central moment to zero and invokes (12-32b), respectively.

**EXAMPLE 12.6** Reconsider the problem introduced in Example 12.1. Letting the  $k$ th conditional moments be denoted as  $m_k(t)$ , (12-62) yields successively, for  $\psi = x, x^2, x^3$ :

$$\begin{aligned}\dot{m}_1 &= \hat{f} &= -[m_1 + am_3] \\ \dot{m}_2 &= 2\hat{x}f + \widehat{G^2Q} &= -2[m_2 + am_4] + [1 + 2bm_2 + b^2m_4]Q \\ \dot{m}_3 &= 3\hat{x}^2f + 3\widehat{G^2Qx} &= -3[m_3 + am_5] + 3[m_1 + 2bm_3 + b^2m_5]Q\end{aligned}$$

Expressions for  $m_4$  and  $m_5$  can be generated for a third order truncated cumulant filter by setting  $c_4$  and  $c_5$  to zero in (12-64), i.e.,

$$\begin{aligned}m_4 &= 3m_2^2 + 4m_1m_3 - 12m_1^2m_2 + 6m_1^4 \\ m_5 &= 10m_2m_3 + 5m_1m_4 - 30m_1m_2^2 - 20m_1^2m_3 + 60m_1^3m_2 - 24m_1^5\end{aligned}$$

Substituting these into the differential equations above define the propagation relations for the filter. Measurement updates can be generated, for example, by performing power series expansions as in (12-21), using Bayes' Rule as in (12-26), and employing cumulant truncations to express the resulting moments.

Note that a second order cumulant truncation filter time propagation would be based on the  $\dot{m}_1$  and  $\dot{m}_2$  equations above, with  $m_4$  expressed as already given but  $m_3$  provided by setting  $c_3 = 0$  in (12-64) as well:

$$m_3 = 3m_1m_2 - 2m_1^3$$

This yields

$$\begin{aligned}\dot{m}_1 &= -m_1 - 3am_1m_2 + 2am_1^3 \\ \dot{m}_2 &= -2[m_2 + 3am_2^2 - 2am_1^4] + [1 + 2bm_2 + 3b^2m_2^2 - 2b^2m_1^4]Q\end{aligned}$$

or, using  $P \triangleq m_2^c = m_2 - m_1^2$ ,

$$\begin{aligned}\dot{m}_1 &= -m_1 - am_1^3 - 3am_1P \\ \dot{P} &= -2[1 + 3am_1^2 + 3aP]P + [(1 + bm_1^2)^2 + 2bP + 3b^2P^2 + 6b^2m_1^2P]Q\end{aligned}$$

which agrees with the result of Example 12.5. ■

Finally, the results of this section can also be applied to the case in which the dynamics are described by a discrete-time (or discretized) model [2, 40, 50], instead of (12-1),

$$\mathbf{x}(t_i) = \boldsymbol{\phi}[\mathbf{x}(t_{i-1}), \mathbf{u}(t_{i-1}), t_{i-1}] + \mathbf{G}_d[\mathbf{x}(t_{i-1}), t_{i-1}]\mathbf{w}_d(t_{i-1}) \quad (12-65)$$

with  $\mathbf{w}_d(\cdot, \cdot)$  discrete-time, zero-mean, white Gaussian noise with covariance

$$E\{\mathbf{w}_d(t_i)\mathbf{w}_d^T(t_j)\} = \mathbf{Q}_d(t_i)\delta_{ij} \quad (12-66)$$

The true conditional mean and covariance would propagate as (using the notation of (12-11)):

$$\hat{\mathbf{x}}(t_i) = \hat{\boldsymbol{\phi}} \quad (12-67a)$$

$$\mathbf{P}(t_i) = \widehat{\boldsymbol{\phi}\boldsymbol{\phi}^T} + \widehat{\mathbf{G}_d\mathbf{Q}_d\mathbf{G}_d^T} - \widehat{\boldsymbol{\phi}}\widehat{\boldsymbol{\phi}}^T \quad (12-67b)$$

where

$$\boldsymbol{\phi} \triangleq \boldsymbol{\phi}[\mathbf{x}(t_{i-1}), \mathbf{u}(t_{i-1}), t_{i-1}] \quad (12-67c)$$

$$\mathbf{G}_d \triangleq \mathbf{G}_d[\mathbf{x}(t_{i-1}), t_{i-1}] \quad (12-67d)$$

These would replace (12-9) and (12-10), again giving only a partial description of the conditional density, and again requiring approximation for development of implementable estimation algorithms. Appropriately truncated Taylor series for  $\boldsymbol{\phi}$  and  $\mathbf{G}_d$  can be developed, or assumed density filters evaluating the required conditional expectations in (12-67) as though the conditional density were Gaussian and of mean  $\hat{\mathbf{x}}(t_{i-1}^+)$  and covariance  $\mathbf{P}(t_{i-1}^+)$  can be generated.

## 12.4 CONDITIONAL QUASI-MOMENTS AND HERMITE POLYNOMIAL SERIES

The previous section discussed parameterizing the conditional density  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_i)}(\xi | \mathcal{L}_i)$  with moments to attempt to achieve useful approximations for estimation, but there are other possible parameterizations as well. One could generate a complete orthogonal series representation and then truncate the series at a specified order and perhaps make other approximations as required to obtain an implementable algorithm. It is particularly appropriate to consider *Hermite polynomial series* and the associated parameterization via *quasi-moments*, as will now be presented [20, 26–28, 38, 47, 59, 65, 66, 76–78, 116, 131, 132, 135].

To motivate Hermite polynomials, consider the scalar process  $\exp\{\beta(t) - \frac{1}{2}t\}$ , as discussed in Examples 11.5 and 11.6, where  $\beta(\cdot, \cdot)$  is scalar Brownian motion with unit diffusion. For Itô integrals and stochastic differential equations, this exponential plays the same role as  $\exp\{\beta(t)\}$  would if  $\beta(\cdot)$  were a deterministic continuous function of bounded variation. By applying the Itô differential rule, it can be shown that

$$\exp\{\beta(t) - \frac{1}{2}t\} = \sum_{i=0}^{\infty} \frac{H_i[t, \beta(t)]}{i!} \quad (12-68)$$

where  $H_i[t, \beta(t)]$  is the Hermite polynomial defined for  $i \geq 0$  as

$$H_i[t, \beta(t)] = (-t)^i \exp\{-\beta^2/2t\} \partial^i [\exp\{-\beta^2/2t\}] / \partial \beta^i \quad (12-69)$$

This can be compared to the case involving deterministic  $\beta(\cdot)$ :

$$\exp\{\beta(t)\} = \sum_{i=0}^{\infty} \frac{\beta(t)^i}{i!} \quad (12-70)$$

It can be seen that in Itô theory, the Hermite polynomials are the counterparts of the ordinary powers for deterministic functions [59]. Thus, when dealing with Itô nonlinear stochastic differential equations, instead of using moments

$$m_i(t) \triangleq \int_{-\infty}^{\infty} \xi^i f_{\mathbf{x}(t)}(\xi) d\xi \quad (12-71a)$$

or central moments

$$m_i^c(t) \triangleq \int_{-\infty}^{\infty} (\xi - m_1)^i f_{\mathbf{x}(t)}(\xi) d\xi \quad (12-71b)$$

to parameterize the density  $f_{\mathbf{x}(t)}(\xi)$ , it might be *more natural* to use “quasi-moments” defined as

$$q_i(t) \triangleq \int_{-\infty}^{\infty} H_i(\xi) f_{\mathbf{x}(t)}(\xi) d\xi \quad (12-72)$$

where

$$H_i(\xi) = H_i[t, \beta] \Big|_{t=1, \beta=\xi} \quad (12-73a)$$

or defined recursively as

$$\begin{aligned} H_0(\xi) &= 1 \\ H_1(\xi) &= \xi \\ &\vdots \\ H_{i+1}(\xi) &= \xi H_i(\xi) - i H_{i-1}(\xi) \end{aligned} \quad (12-73b)$$

The extension to conditional quasi-moments would be obvious.

Now consider expanding some density of interest,  $f_{\mathbf{x}(t)}(\xi)$ , in a series representation, and let  $f_{\mathbf{x}(t)}^0(\xi)$  be some *reference* density. An arbitrary (measurable, square integrable, real) density function  $f_{\mathbf{x}(t)}(\xi)$  can be expressed in terms of a *complete orthonormal set* of basis functions  $[\phi_i(\xi)p_0(\xi)]$  via a generalized Gram–Charlier series

$$\begin{aligned} f_{\mathbf{x}(t)}(\xi) &= \sum_{i=0}^{\infty} k_i \phi_i(\xi) p_0^2(\xi); \quad p_0(\xi) = \sqrt{f_{\mathbf{x}(t)}^0(\xi)} \\ &= \sum_{i=0}^{\infty} k_i \phi_i(\xi) f_{\mathbf{x}(t)}^0(\xi) \end{aligned} \quad (12-74)$$

where the coefficients  $k_i$  are “general quasi-moments” of  $f_{\mathbf{x}(t)}(\xi)$ :

$$k_i = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi_i(\xi) f_{\mathbf{x}(t)}(\xi) d\xi \quad (12-75)$$

For scalar  $\mathbf{x}(t)$  and  $f_{\mathbf{x}(t)}^0(\xi)$  chosen to be a Gaussian density with zero mean and unit variance,

$$f_{\mathbf{x}(t)}^0(\xi) = (1/\sqrt{2\pi}) \exp\{-\xi^2/2\} \quad (12-76)$$

the  $\phi_i(\xi)$  functions in (12-74) are the scaled Hermite polynomials

$$\phi_i(\xi) = \frac{1}{\sqrt{i!}} H_i(\xi) \quad (12-77)$$

and the coefficients  $k_i$  are the *conventional quasi-moments*  $q_i$  defined in (12-72). Moreover, the functions  $[\phi_i(\xi)p_0(\xi)]$  are indeed orthogonal:

$$\begin{aligned} \int_{-\infty}^{\infty} [\phi_i(\xi)p_0(\xi)][\phi_j(\xi)p_0(\xi)] d\xi &= \int_{-\infty}^{\infty} \phi_i(\xi)\phi_j(\xi)p_0^2(\xi) d\xi \\ &= \int_{-\infty}^{\infty} \phi_i(\xi)\phi_j(\xi)f_{\mathbf{x}(t)}^0(\xi) d\xi \\ &= \frac{1}{\sqrt{i!}\sqrt{j!}} \int_{-\infty}^{\infty} H_i(\xi)H_j(\xi)f_{\mathbf{x}(t)}^0(\xi) d\xi = 0, \quad i \neq j \end{aligned} \quad (12-78)$$

Said another way,  $H_i(\xi)$  and  $H_j(\xi)$  are orthogonal relative to the “standardized” Gaussian density (with mean 0 and variance 1).

If we did not truncate the series, we could represent *any* density *perfectly* via (12-74) and (12-75). However, as a further motivation for Hermite polynomials to be considered in particular, we seek *good approximations* to *nearly Gaussian* density functions with as *few* parameters required as possible. Quasi-moments are the coefficients in the expansion of the ratio:

$$f_{\mathbf{x}(t)}(\xi)/f_{\mathbf{x}(t)}^G(\xi)$$

in a series of multidimensional Hermite polynomials, where  $f_{\mathbf{x}(t)}^G(\xi)$  is a Gaussian zero-mean density of specified covariance. If  $f_{\mathbf{x}(t)}(\xi)$  were in fact Gaussian with zero mean and the specified covariance, the ratio would be equal to one and *all quasi-moments would equal zero*. Since the multidimensional Hermite polynomials do form a complete orthogonal set of eigenfunctions relative to the Gaussian density, *any* ratio  $[f_{\mathbf{x}(t)}(\xi)/f_{\mathbf{x}(t)}^G(\xi)]$  can be expanded in a series of multidimensional Hermite polynomials, provided  $\int_{-\infty}^{\infty} [f_{\mathbf{x}(t)}(\xi)/f_{\mathbf{x}(t)}^G(\xi)]^2 d\xi$  is finite (i.e., assuming that you are working in a Hilbert space). Additionally, approximation theory then yields that any such ratio can be approximated to *any* desired precision (in the integrated square error sense) by an appropriate *finite* number of terms in the series.

Thus, we can approximate  $f_{\mathbf{x}(t)|\mathbf{z}(t_i)}(\xi|\mathcal{Z}_i)$  with a truncated series representation in terms of quasi-moments instead of moments. However, convergence of such expansions is limited, a finite number of given quasi-moments would erroneously produce a unique reconstructed density (an infinite number of densities actually satisfies a finite number of moment or quasi-moment specifications), and density approximations based on finite segments of the series may yield a density function approximation which can assume negative values [47, 72]. Some of these objections can be removed by expanding the square root of the density instead of the density itself [47], but two facts remain. Conventional quasi-moments are directly related to moments; for example, in the scalar case, (12-71)–(12-73) yield  $q_0 = m_0 = 1$ ,  $q_1 = m_1$ ,  $q_2 = m_2 - m_0$ ,  $q_3 = m_3 - 3m_1$ , etc. Moreover, for estimation purposes, only the lowest order quasi-moments or moments are approximated (as especially the first two), and there is little benefit from considering quasi-moments instead of moments as in the previous section. Algorithmic details are available in the references at the beginning of this section.

## 12.5 CONDITIONAL MODE ESTIMATORS

Rather than developing approximate relations for the conditional mean, i.e., the minimum mean square error (MMSE) estimate, one can seek an approximate algorithm for the *conditional mode*, the maximum a posteriori (MAP) estimate

[54]. The mode during time propagations from  $t_{i-1}$  to  $t_i$  is defined by

$$\partial f_{\mathbf{x}(t)|\mathbf{z}(t_{i-1})}(\xi | \mathcal{Z}_{i-1}) / \partial \xi^T \Big|_{\xi = \hat{\mathbf{x}}_{MAP}(t/t_{i-1})} \equiv \mathbf{0} \quad (12-79)$$

provided the second derivative is positive definite to assure that the mode is well defined and unique. The time derivative of (12-79) must also be identically zero:

$$\begin{aligned} \mathbf{0} &= \frac{d}{dt} [\partial f_{\mathbf{x}|\mathbf{z}} / \partial \xi^T] \\ &= \frac{\partial}{\partial t} [(\partial f_{\mathbf{x}|\mathbf{z}} / \partial \xi)^T] + [\partial^2 f_{\mathbf{x}|\mathbf{z}} / \partial \xi^2] \dot{\hat{\mathbf{x}}}_{MAP}(t/t_{i-1}) \\ &= \left[ \frac{\partial}{\partial \xi} (\partial f_{\mathbf{x}|\mathbf{z}} / \partial t) \right]^T + [\partial^2 f_{\mathbf{x}|\mathbf{z}} / \partial \xi^2] \dot{\hat{\mathbf{x}}}_{MAP}(t/t_{i-1}) \end{aligned}$$

or

$$\dot{\hat{\mathbf{x}}}_{MAP}(t/t_{i-1}) = -[\partial^2 f_{\mathbf{x}|\mathbf{z}} / \partial \xi^2]^{-1} [\partial (\partial f_{\mathbf{x}|\mathbf{z}} / \partial t) / \partial \xi]^T \quad (12-80)$$

as the desired conditional mode equation, where  $(\partial f_{\mathbf{x}|\mathbf{z}} / \partial t)$  is given by the forward Kolmogorov equation and  $f_{\mathbf{x}|\mathbf{z}}$  and its partials are evaluated at  $\hat{\mathbf{x}}_{MAP}(t/t_{i-1})$ . The right hand side of (12-80) involves various order partials of the density that must be evaluated, and time propagation relations for them involve still higher order partials, again yielding an infinite dimensional estimator: as before, approximations are required to generate an implementable finite dimensional filter. One of the terms that appears in the right hand side of (12-80) is the  $n$ -by- $n$  matrix

$$\Sigma(t/t_{i-1}) \triangleq - (f_{\mathbf{x}|\mathbf{z}}) [\partial^2 f_{\mathbf{x}|\mathbf{z}} / \partial \xi^2]^{-1} \quad (12-81)$$

For the conditions under which  $f_{\mathbf{x}(t)|\mathbf{z}(t_{i-1})}$  is Gaussian and the MAP estimator becomes the Kalman filter,  $\Sigma(t/t_{i-1})$  is in fact the conditional covariance, which is also computed in that filter structure. To propagate  $\Sigma(t/t_{i-1})$  here, differentiate (12-81) to yield

$$\begin{aligned} d\Sigma(t/t_{i-1})/dt &= -[\partial^2 f_{\mathbf{x}|\mathbf{z}} / \partial \xi^2]^{-1} (df_{\mathbf{x}|\mathbf{z}} / dt) \\ &\quad + f_{\mathbf{x}|\mathbf{z}} [\partial^2 f_{\mathbf{x}|\mathbf{z}} / \partial \xi^2]^{-1} [d(\partial^2 f_{\mathbf{x}|\mathbf{z}} / \partial \xi^2) / dt] [\partial^2 f_{\mathbf{x}|\mathbf{z}} / \partial \xi^2]^{-1} \end{aligned} \quad (12-82)$$

where

$$df_{\mathbf{x}|\mathbf{z}} / dt = \partial f_{\mathbf{x}|\mathbf{z}} / \partial t + [\partial f_{\mathbf{x}|\mathbf{z}} / \partial \xi] \dot{\hat{\mathbf{x}}}_{MAP} = \partial f_{\mathbf{x}|\mathbf{z}} / \partial t \quad (12-83)$$

in view of (12-79), but no such simplification occurs for  $[d(\partial^2 f_{\mathbf{x}|\mathbf{z}} / \partial \xi^2) / dt]$ ; again these are evaluated using the forward Kolmogorov equation.

For the measurement update, the partial of (12-4) with respect to  $\xi$  is again set to  $\mathbf{0}^T$ . The denominator of (12-4) is not a function of  $\xi$ , so this is equivalent

to solving

$$\mathbf{0}^T = [\partial f_{z|x}/\partial \xi] f_{x|z} + f_{z|x} [\partial f_{x|z}/\partial \xi] \quad (12-84)$$

or

$$[\partial h/\partial x] R^{-1} [z_i - h] f_{x|z} + [\partial f_{x|z}/\partial \xi]^T = \mathbf{0} \quad (12-85)$$

for  $\hat{x}_{MAP}(t_i^+)$ . By differentiating (12-84) again, the update relation for  $\Sigma$  becomes

$$\begin{aligned} \Sigma^{-1}(t_i^+) = & -[(\partial^2 f_{z|x}/\partial \xi^2) f_{x|z} + (\partial f_{z|x}/\partial \xi)^T (\partial f_{x|z}/\partial \xi)] \\ & + (\partial f_{x|z}/\partial \xi)^T (\partial f_{z|x}/\partial \xi) + (\partial^2 f_{x|z}/\partial \xi^2) f_{z|x}] \div [f_{z|x} f_{x|z}] \end{aligned} \quad (12-86)$$

The mode estimator is then defined by appropriate approximations to (12-80), (12-82), (12-85), and (12-86). Especially when discrete dynamics are proposed, this formulation has led to useful filter forms, exploiting the simplifying fact that the denominator of (12-4) is not involved in the estimate generation as it was in the previous sections [13, 24, 25, 95, 98, 99].

## 12.6 STATISTICALLY LINEARIZED FILTER

If  $\mathbf{G}$  in (12-1) is a function of only time instead of  $\mathbf{x}(t)$  and  $t$ , then *statistical linearization* can be used to develop nonlinear estimators which, like the latter estimators of Section 12.3, do *not* employ series representations of  $\mathbf{f}$  and  $\mathbf{h}$  [44, 82, 138]. For this reason, it is a method that does not require  $\mathbf{f}$  and  $\mathbf{h}$  to be differentiable, thereby admitting such important nonlinearities as saturation. But this advantage is gained at the expense of requiring evaluation of conditional expectations, i.e., of knowing the entire density. Typically, the density is approximated as Gaussian, and the resulting implementable algorithm often has *better* characteristics than those based on truncated series expansions of  $\mathbf{f}$  and  $\mathbf{h}$  about the conditional mean approximate estimate.

Consider approximating  $\mathbf{f}[\mathbf{x}(t), t]$  by a linear approximation of the form

$$\mathbf{f}[\mathbf{x}(t), t] \cong \mathbf{f}_0(t) + \mathcal{F}(t)\mathbf{x}(t) \quad (12-87)$$

that is “best” in the sense that the error in the representation,

$$\mathbf{e}(t) = \mathbf{f}[\mathbf{x}(t), t] - \mathbf{f}_0(t) - \mathcal{F}(t)\mathbf{x}(t) \quad (12-88)$$

has minimum (generalized) mean square error

$$J = E\{\mathbf{e}(t)^T \mathbf{W} \mathbf{e}(t) | \mathbf{Z}(t_{i-1}) = \mathcal{L}_{i-1}\} \quad (12-89)$$

for all  $t \in [t_{i-1}, t_i]$ , where  $\mathbf{W}$  is a general  $n$ -by- $n$  weighting matrix. Note that  $f_{x(t)|z(t_{i-1})}(\xi | \mathcal{L}_{i-1})$  would be required to evaluate this expectation. Setting the partials of (12-89) with respect to  $\mathbf{f}_0(t)$  and  $\mathcal{F}(t)$  simultaneously to zero yields,

using the notation of (12-11),

$$\mathbf{f}_0(t) = \widehat{\mathbf{f}[\mathbf{x}(t), t]} - \mathcal{F}(t)\hat{\mathbf{x}}(t) \quad (12-90a)$$

$$\mathcal{F}(t) = \{\widehat{\mathbf{f}[\mathbf{x}(t), t]}\mathbf{x}^T(t) - \widehat{\mathbf{f}[\mathbf{x}(t), t]}\hat{\mathbf{x}}^T(t)\}\mathbf{P}^{-1}(t) \quad (12-90b)$$

with  $\mathbf{P}(t)$  being the conditional covariance of  $\mathbf{x}(t)$ , conditioned on  $\mathbf{z}(t_{i-1}) = \mathbf{z}_{i-1}$ .  $\mathcal{F}(t)$  is seen to be intimately related to *describing functions*: in the scalar zero-mean case, (12-90b) becomes  $(\mathbf{f}\mathbf{x}/\mathbf{x}^2)$ , the describing function gain for an odd-function nonlinearity (such as a symmetric saturation) driven by a zero-mean input [44, 45]. Similarly,  $\mathbf{h}[\mathbf{x}(t_i), t_i]$  can be approximated by the first two terms in a power series

$$\mathbf{h}[\mathbf{x}(t_i), t_i] \cong \mathbf{h}_0(t_i) + \mathcal{H}(t_i)\mathbf{x}(t_i) \quad (12-91)$$

with the coefficients  $\mathbf{h}_0(t_i)$  and  $\mathcal{H}(t_i)$  statistically optimized to produce

$$\mathbf{h}_0(t_i) = \widehat{\mathbf{h}[\mathbf{x}(t_i), t_i]} - \mathcal{H}(t_i)\hat{\mathbf{x}}(t_i^-) \quad (12-92a)$$

$$\mathcal{H}(t_i) = \{\widehat{\mathbf{h}[\mathbf{x}(t_i), t_i]}\mathbf{x}^T(t_i) - \widehat{\mathbf{h}[\mathbf{x}(t_i), t_i]}\hat{\mathbf{x}}^T(t_i^-)\}\mathbf{P}^{-1}(t_i^-) \quad (12-92b)$$

Once the statistically optimized approximations (12-87) and (12-91) are introduced into the problem description, (12-1) with  $\mathbf{G} = \mathbf{G}(t)$  and (12-2), and its solution, (12-9)–(12-14), the *statistically linearized filter* can be expressed as a time propagation

$$\dot{\hat{\mathbf{x}}}(t) = \widehat{\mathbf{f}[\mathbf{x}(t), t]} \quad (12-93)$$

$$\dot{\mathbf{P}}(t) = \mathcal{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathcal{F}^T(t) + \mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t) \quad (12-94)$$

with  $\mathcal{F}(t)$  given as in (12-90b), and  $\widehat{\mathbf{f}\mathbf{x}^T}$ ,  $\widehat{\mathbf{f}}$ , and  $\widehat{\mathbf{x}}$  calculated assuming  $\mathbf{x}(t)$  to be Gaussian with mean  $\widehat{\mathbf{x}}(t)$  and covariance  $\mathbf{P}(t)$ . The measurement update at time  $t_i$  is given by

$$\mathbf{K}_{SL}(t_i) = \mathbf{P}(t_i^-)\mathcal{H}^T(t_i)[\mathcal{H}(t_i)\mathbf{P}(t_i^-)\mathcal{H}^T(t_i) + \mathbf{R}(t_i)]^{-1} \quad (12-95)$$

$$\hat{\mathbf{x}}(t_i^+) = \hat{\mathbf{x}}(t_i^-) + \mathbf{K}_{SL}(t_i)\{\mathbf{z}_i - \widehat{\mathbf{h}[\mathbf{x}(t_i), t_i]}\} \quad (12-96)$$

$$\mathbf{P}(t_i^+) = \mathbf{P}(t_i^-) - \mathbf{K}_{SL}(t_i)\mathcal{H}(t_i)\mathbf{P}(t_i^-) \quad (12-97)$$

with  $\mathcal{H}(t_i)$  given by (12-92b), and  $\widehat{\mathbf{h}\mathbf{x}^T}$ ,  $\widehat{\mathbf{h}}$ , and  $\widehat{\mathbf{x}}$  computed as though  $\mathbf{x}(t_i)$  were Gaussian with mean  $\widehat{\mathbf{x}}(t_i^-)$  and covariance  $\mathbf{P}(t_i^-)$ , as generated by the previous propagation. Note that these relations are intimately related to the assumed density filters of Section 12.3 (assuming  $\mathbf{G}$  to be a function only of  $t$ ) that directly implement conditional moment relations (12-9)–(12-14), (12-21), and (12-26) and evaluate conditional expectations by assuming a Gaussian conditional density. This is to be expected since the conditional mean does provide the minimum mean square error estimate, as derived fundamentally under similar assumptions in this section.

Structurally, the covariance and gain equations, (12-94), (12-95), and (12-97), are the same as those for an extended Kalman filter, but with  $\mathcal{F}(t)$  replacing  $\partial\mathbf{f}[\hat{\mathbf{x}}(t/t_i), t]/\partial\mathbf{x}$  and  $\mathcal{H}(t_i)$  replacing  $\partial\mathbf{h}[\hat{\mathbf{x}}(t_i^-), t_i]/\partial\mathbf{x}$ . Because of its relation to describing functions,  $\mathcal{F}(t)$  accounts for the probability that  $[\mathbf{x}(t) - \hat{\mathbf{x}}(t)]$ , approximated as Gaussian with zero mean and covariance  $\mathbf{P}(t)$ , will take on values large enough such that

$$\mathbf{f}[\hat{\mathbf{x}}(t), t] + \{\partial\mathbf{f}[\hat{\mathbf{x}}(t), t]/\partial\mathbf{x}\}\{\mathbf{x}(t) - \hat{\mathbf{x}}(t)\}$$

is no longer an adequate approximation to  $\mathbf{f}[\mathbf{x}(t), t]$ , and similarly for  $\mathcal{H}(t)$ . As a result, gains computed via (12-95) tend to be more conservative, and performance better, than achieved with an extended Kalman filter, especially for cases involving large error covariance magnitudes.

**EXAMPLE 12.7** Consider Example 12.1, but let  $b \equiv 0$  so that  $\mathbf{G}$  is not a function of  $\mathbf{x}(t)$ . Then the time propagation is identical to that of Example 12.5 and the second order cumulant truncation filter of Example 12.6, with  $b \equiv 0$ .

For the update,  $\hat{\mathbf{h}}$  and  $\widehat{\mathbf{h}\mathbf{x}}$  can be evaluated as

$$\begin{aligned}\hat{\mathbf{h}} &= \int_{-\infty}^{\infty} (\sin \xi) f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi | \mathcal{Z}_{i-1}) d\xi \\ \widehat{\mathbf{h}\mathbf{x}} &= \int_{-\infty}^{\infty} \xi (\sin \xi) f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi | \mathcal{Z}_{i-1}) d\xi\end{aligned}$$

assuming  $f_{\mathbf{x}(t_i)|\mathbf{z}(t_{i-1})}(\xi | \mathcal{Z}_{i-1})$  to be Gaussian with mean  $\hat{\mathbf{x}}(t_i^-)$  and variance  $P(t_i^-)$ ; the integrals could be integrated numerically and perhaps curve-fitted with appropriate extrapolation functions of  $\hat{\mathbf{x}}(t_i^-)$  and  $P(t_i^-)$ . Once computed, these expectations can be used to evaluate (12-92b) and (12-95)–(12-97). ■

*Statistical approximations* based on power series of higher order than in (12-87) and (12-91) can be sought, as can higher order moment estimators based on non-Gaussian cumulant truncation forms. However, the algorithm specified by (12-93)–(12-97) is particularly more attractive from a computational standpoint.

## 12.7 NONLINEAR FILTERING WITH CONTINUOUS-TIME MEASUREMENTS

Again let a continuous-time system be described by the Markov solution  $\mathbf{x}(\cdot, \cdot)$  to the Itô stochastic differential equation

$$d\mathbf{x}(t) = \mathbf{f}[\mathbf{x}(t), t] dt + \mathbf{G}[\mathbf{x}(t), t] d\beta(t) \quad (12-98)$$

with  $\beta(\cdot, \cdot)$  Brownian motion of diffusion  $\mathbf{Q}(t)$ :

$$E\{d\beta(t) d\beta^T(t)\} = \mathbf{Q}(t) dt \quad (12-99a)$$

$$E\{[\beta(t_2) - \beta(t_1)][\beta(t_2) - \beta(t_1)]^T\} = \int_{t_1}^{t_2} \mathbf{Q}(t) dt \quad (12-99b)$$

However, now consider an  $m$ -vector *continuous-time* measurement process described by

$$\mathbf{dy}(t) = \mathbf{h}[\mathbf{x}(t), t] dt + \mathbf{d}\beta_m(t) \quad (12-100)$$

where  $\beta_m(\cdot, \cdot)$  is a Brownian motion independent of  $\beta(\cdot, \cdot)$ , and of diffusion  $\mathbf{R}_c(t)$  for all  $t \in [t_0, t_f]$ :

$$E\{\mathbf{d}\beta_m(t) \mathbf{d}\beta_m^T(t)\} = \mathbf{R}_c(t) dt \quad (12-101)$$

$$E\{\mathbf{d}\beta_m(t) \mathbf{d}\beta^T(t)\} = \mathbf{0} \quad (12-102)$$

Heuristically, this corresponds to

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), t] + \mathbf{G}[\mathbf{x}(t), t]\mathbf{w}(t) \quad (12-98')$$

$$\dot{\mathbf{y}}(t) \triangleq \mathbf{z}(t) = \mathbf{h}[\mathbf{x}(t), t] + \mathbf{v}_c(t) \quad (12-100')$$

with  $\mathbf{w}(\cdot, \cdot)$  and  $\mathbf{v}_c(\cdot, \cdot)$  independent zero-mean white Gaussian noises with

$$E\{\mathbf{w}(t)\mathbf{w}^T(t + \tau)\} = \mathbf{Q}(t) \delta(\tau) \quad (12-99')$$

$$E\{\mathbf{v}_c(t)\mathbf{v}_c^T(t + \tau)\} = \mathbf{R}_c(t) \delta(\tau) \quad (12-101')$$

$$E\{\mathbf{v}_c(t)\mathbf{w}^T(t + \tau)\} = \mathbf{0} \quad (12-102')$$

The subscript  $c$  on  $\mathbf{R}_c(t)$  and  $\mathbf{v}_c(\cdot, \cdot)$  denotes continuous-time, to distinguish them from  $\mathbf{R}(t_i)$  and  $\mathbf{v}(\cdot, \cdot)$  associated with the discrete-time measurement (12-2). As in the previous sections, we wish to establish the time history of the conditional density of the state  $\mathbf{x}(t, \cdot)$ , conditioned on the entire history of measurements observed up to time  $t$ , which in this case becomes  $f_{\mathbf{x}(t)}(\xi | \{\mathbf{y}(\tau), t_0 \leq \tau \leq t\})$  or  $f_{\mathbf{x}(t)}(\xi | \{\mathbf{z}(\tau), t_0 \leq \tau \leq t\})$ . In view of the discussion of conditional densities and expectations in Section 3.7 of Volume 1, this is more rigorously described as  $f_{\mathbf{x}(t)}(\xi | \mathcal{F}_y[t_0, t])$ , where  $\mathcal{F}_y[t_0, t]$  is the minimal  $\sigma$ -algebra generated by the measurement process  $\{\mathbf{y}(\tau, \cdot), t_0 \leq \tau \leq t\}$ , but we will adopt the less rigorous notation. Once such a conditional density time propagation were established, it could conceptually be used for real-time estimation to generate  $f_{\mathbf{x}(t)}(\xi | \{\mathbf{y}(\tau) = \mathbf{y}(\tau), t_0 \leq \tau \leq t\})$  or  $f_{\mathbf{x}(t)}(\xi | \{\mathbf{z}(\tau) = \mathbf{z}(\tau), t_0 \leq \tau \leq t\})$ , given the observed sample of the measurement process,  $\mathbf{y}(\tau)$  or  $\mathbf{z}(\tau)$ , for all  $\tau \in [t_0, t]$ .

It can be shown that the conditional density satisfies the *Kushner equation* [14, 38, 54, 68–71, 89, 102, 134, 136, 137, 141, 147], sometimes denoted as the Kushner–Stratonovich equation,

$$\begin{aligned} \frac{\partial f_{\mathbf{x}}}{\partial t} = & - \sum_{i=1}^n \frac{\partial}{\partial \xi_i} \{f_{\mathbf{x}} f_i\} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial \xi_i \partial \xi_j} \{f_{\mathbf{x}} [\mathbf{G} \mathbf{Q} \mathbf{G}^T]_{ij}\} \\ & + \{\mathbf{h}[\xi, t] - \widehat{\mathbf{h}[\mathbf{x}(t), t]}\}^T \mathbf{R}_c^{-1}(t) \{\mathbf{z}(t) - \mathbf{h}[\mathbf{x}(t), t]\} f_{\mathbf{x}} \end{aligned} \quad (12-103)$$

where

$$\widehat{\mathbf{h}[\mathbf{x}(t), t]} \triangleq \int_{-\infty}^{\infty} \mathbf{h}[\xi, t] f_{\mathbf{x}(t)}(\xi | \{\mathbf{y}(\tau), t_0 \leq \tau \leq t\}) d\xi \quad (12-104)$$

provided the indicated derivatives exist. The first two terms of (12-103) correspond directly to the forward Kolmogorov equation (11-96), which can be used to propagate the conditional density for discrete-time measurements,  $f_{\mathbf{x}(t)|\mathbf{z}_{(t,-1)}}(\xi|\mathcal{L}_{i-1})$ , from one sample time to the next. The last term in (12-103) is due to the continuous-time incorporation of measurement information, in contradistinction to the discrete-time updates seen previously. Without continuous-time measurement information, i.e., with  $\mathbf{R}_c^{-1} \equiv \mathbf{0}$ , the Kushner equation reduces to the forward Kolmogorov equation. As written, (12-103) is a stochastic partial differential equation, yielding a sample solution for each sample of the measurement process,  $\mathbf{y}(\tau, \omega_k) = \mathbf{y}(\tau)$  for all  $\tau \in [t_0, t]$  and corresponding  $\mathbf{z}(\tau) = \mathbf{d}\mathbf{y}(\tau)/d\tau$ . Arguments on the existence and uniqueness of solutions to such equations are available [36, 89, 108].

This relation can be established from the sampled-data results of the previous sections in the limit as the sample period goes to zero. Basically, (12-4) is written, replacing  $\mathbf{Z}(t_{i-1})$  by  $\{\mathbf{y}(\tau), t_0 \leq \tau \leq t\}$  and  $\mathbf{z}(t_i)$  by  $\mathbf{d}\mathbf{y}(t)$  to write

$$f_{\mathbf{x}(t)}(\xi | \{\mathbf{y}(\tau), t_0 \leq \tau \leq t\}, \mathbf{d}\mathbf{y}(t))$$

$$= \left[ \frac{f_{\mathbf{d}\mathbf{y}(t)}(\eta | \mathbf{x}(t) = \xi, \{\mathbf{y}(\tau), t_0 \leq \tau \leq t\})}{f_{\mathbf{d}\mathbf{y}(t)}(\eta | \{\mathbf{y}(\tau), t_0 \leq \tau \leq t\})} \right] f_{\mathbf{x}(t)}(\xi | \{\mathbf{y}(\tau), t_0 \leq \tau \leq t\}) \quad (12-105)$$

and expanding the bracketed term to order  $dt$ , recalling that, by the Levy oscillation property (see (11-59)),

$$[\mathbf{d}\mathbf{B}_m(t) \mathbf{d}\mathbf{B}_m^T(t)] = \mathbf{R}_c(t) dt \quad \text{w.p.1; in m.s.} \quad (12-106)$$

After some algebra [68, 69, 89], the bracketed term in (12-105) can be shown equal to

$$[\cdot] = 1 + \{\mathbf{h}[\xi, t] - \widehat{\mathbf{h}[\mathbf{x}(t), t]}\}^T \mathbf{R}_c^{-1}(t) \{\mathbf{d}\mathbf{y}(t) - \widehat{\mathbf{h}[\mathbf{x}(t), t]} dt\} + \mathcal{O}(dt) \quad (12-107)$$

where  $\lim_{\Delta t \rightarrow 0} \mathcal{O}(\Delta t)/\Delta t = 0$  and  $\widehat{\mathbf{h}[\mathbf{x}(t), t]}$  is given by (12-104). Rigorous equality in (12-107) is established by noting that  $\mathbf{x}(\cdot, \cdot)$  is a martingale relative to  $\{\mathbf{y}(\tau), t_0 \leq \tau \leq t\}$ , (see (11-55) and (11-64)) and then invoking martingale convergence results [35]. Then the partial derivative

$$\begin{aligned} & \frac{\partial f_{\mathbf{x}(t)}(\xi | \{\mathbf{y}(\tau), t_0 \leq \tau \leq t\})}{\partial t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{f_{\mathbf{x}(t+\Delta t)}(\xi | \{\mathbf{y}(\tau), t_0 \leq \tau \leq t + \Delta t\}) - f_{\mathbf{x}(t)}(\xi | \{\mathbf{y}(\tau), t_0 \leq \tau \leq t\})}{\Delta t} \end{aligned}$$

or the partial derivative for the corresponding characteristic function can be evaluated, using the Chapman–Kolmogorov equation (as in the derivation of (11-96)) and Eq. (12-102), to express the first numerator term conveniently [68, 69, 89]. Carrying this through yields the result (12-103) as claimed. Extensions to the case of independent-increment point-process descriptions of

dynamics driving noise and measurement corruptions, instead of Brownian motions as considered here, are also available [39, 89, 104, 106, 107, 121–125].

Once the fundamentally important Kushner equation (12-103) is available to describe the evolution of the conditional density  $f_{\mathbf{x}(t)}(\xi | \mathbf{y}(\tau), t_0 \leq \tau \leq t)$ , the conditional mean and covariance propagations can be evaluated analogously to (12-9) and (12-10) [70, 115]. They satisfy the stochastic differential equations

$$\begin{aligned} d\hat{\mathbf{x}}(t) &= \widehat{\mathbf{f}[\mathbf{x}(t), t]} dt + \{\widehat{\mathbf{x}(t)\mathbf{h}^T[\mathbf{x}(t), t]} \\ &\quad - \widehat{\mathbf{x}(t)\mathbf{h}^T[\mathbf{x}(t), t]}\} \mathbf{R}_c^{-1}(t) \{\mathbf{dy}(t) - \widehat{\mathbf{h}[\mathbf{x}(t), t]} dt\} \end{aligned} \quad (12-108)$$

$$\begin{aligned} d\mathbf{P}(t) &= \{\widehat{[\mathbf{x} - \hat{\mathbf{x}}]\mathbf{f}^T} + \widehat{\mathbf{f}[\mathbf{x} - \hat{\mathbf{x}}]^T} + \widehat{\mathbf{GQG}^T} - \widehat{[\mathbf{x} - \hat{\mathbf{x}}]\mathbf{h}^T\mathbf{R}_c^{-1}(t)\mathbf{h}[\mathbf{x} - \hat{\mathbf{x}}]^T}\} dt \\ &\quad + \widehat{[\mathbf{x} - \hat{\mathbf{x}}][\mathbf{x} - \hat{\mathbf{x}}]^T} \{[\widehat{\mathbf{h} - \hat{\mathbf{h}}}]^T \mathbf{R}_c^{-1}(t) [\mathbf{dy}(t) - \widehat{\mathbf{h}} dt]\} \end{aligned} \quad (12-109a)$$

or, on an element-by-element basis,

$$\begin{aligned} \mathbf{dP}_{ij}(t) &= \{\widehat{[\mathbf{x}_i f_j - \hat{\mathbf{x}}_i \hat{f}_j]} + \widehat{[f_i x_j - \hat{f}_i \hat{x}_j]} + \widehat{[\mathbf{GQG}^T]}_{ij} \\ &\quad - \widehat{[\mathbf{x}_i \mathbf{h} - \hat{\mathbf{x}}_i \hat{\mathbf{h}}]^T \mathbf{R}_c^{-1}(t) [\mathbf{h} \mathbf{x}_j - \hat{\mathbf{h}} \hat{\mathbf{x}}_j]}\} dt \\ &\quad + \{\widehat{\mathbf{x}_i \mathbf{x}_j \mathbf{h}} - \widehat{\mathbf{x}_i \mathbf{x}_j \hat{\mathbf{h}}} - \widehat{\hat{\mathbf{x}}_i \mathbf{x}_j \mathbf{h}} - \widehat{\hat{\mathbf{x}}_i \mathbf{x}_j \hat{\mathbf{h}}} + 2\widehat{\mathbf{x}_i \hat{\mathbf{x}}_j \hat{\mathbf{h}}}\}^T \mathbf{R}_c^{-1}(t) \{\mathbf{dy}(t) - \widehat{\mathbf{h}} dt\} \end{aligned} \quad (12-109b)$$

for  $i, j = 1, 2, \dots, n$ , where  $\mathbf{x}_i$  is the  $i$ th component of  $\mathbf{x}$ ,  $f_j$  is the  $j$ th component of  $\mathbf{f}[\mathbf{x}(t), t]$ ,  $\mathbf{h}$  represents  $\mathbf{h}[\mathbf{x}(t), t]$ , and all terms written with the caret symbol are conditional expectations of the corresponding functions as in (12-104). Note that (12-108) and (12-109) can be divided through by  $dt$  heuristically, identifying  $\mathbf{dy}(t)/dt$  as  $\mathbf{z}(t)$ . Also note the structure of the filter gain, the term premultiplying the residual in (12-108). The indicated conditional expectations require knowledge of the entire density function, i.e., of all higher moments as well as the first two, as was the case in Section 12.3 [108].

Various assumptions about the conditional density, higher order moments, or higher order cumulants allow development of *approximations* to these full-scale equations in the form of finite-dimensional, implementable filter algorithms [72, 74, 113]. In the special case of *linear* dynamics and measurement models, they reduce *without approximation* to the continuous-time, continuous-measurement *Kalman filter* of Chapter 5 (Volume 1). Only in this special case do the conditional covariance equations truly decouple from the conditional mean and measurement residual driving function, thereby becoming precomputable. If  $\mathbf{G}$  is a function only of time and not of  $\mathbf{x}(t)$ , Taylor series representations for  $\mathbf{f}$  and  $\mathbf{h}$  expanded about  $\hat{\mathbf{x}}(t)$  and truncated at first order terms, yield the *extended Kalman filter* given by (9-80)–(9-82). As in Section 12.3, Taylor series carried to second order can generate either truncated or Gaussian second order filters, depending on the assumptions made about the conditional density

for the state. The *truncated second order filter* [7, 52, 54, 91, 114] is given by

$$\begin{aligned}\mathbf{d}\hat{\mathbf{x}}(t) &= \{\mathbf{f}[\hat{\mathbf{x}}(t), t] + \hat{\mathbf{b}}_p(t)\} dt \\ &+ \mathbf{P}(t)\mathbf{H}^T[t; \hat{\mathbf{x}}(t)]\mathbf{R}_c^{-1}(t)\{\mathbf{dy}(t) - \mathbf{h}[\hat{\mathbf{x}}(t), t]\} dt - \hat{\mathbf{b}}_m(t) dt\end{aligned}\quad (12-110)$$

$$\begin{aligned}\mathbf{d}\mathbf{P}(t) &= \{\mathbf{F}[t; \hat{\mathbf{x}}(t)]\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T[t; \hat{\mathbf{x}}(t)] + \widehat{\mathbf{G}\mathbf{Q}\mathbf{G}^T} \\ &- \mathbf{P}(t)\mathbf{H}^T[t; \hat{\mathbf{x}}(t)]\mathbf{R}_c^{-1}(t)\mathbf{H}[t; \hat{\mathbf{x}}(t)]\mathbf{P}(t)\} dt \\ &- \mathbf{P}(t)[\hat{\mathbf{b}}_m^T(t)\mathbf{R}_c^{-1}(t)\{\mathbf{dy}(t) - \mathbf{h}[\hat{\mathbf{x}}(t), t]\} dt - \hat{\mathbf{b}}_m(t) dt]\end{aligned}\quad (12-111)$$

where

$$\mathbf{F}[t; \hat{\mathbf{x}}(t)] \triangleq \left. \frac{\partial \mathbf{f}[\mathbf{x}, t]}{\partial \mathbf{x}} \right|_{\mathbf{x}=\hat{\mathbf{x}}(t)} \quad (12-112)$$

$$\mathbf{H}[t; \hat{\mathbf{x}}(t)] \triangleq \left. \frac{\partial \mathbf{h}[\mathbf{x}, t]}{\partial \mathbf{x}} \right|_{\mathbf{x}=\hat{\mathbf{x}}(t)} \quad (12-113)$$

and  $\hat{\mathbf{b}}_p(t)$  and  $\hat{\mathbf{b}}_m(t)$  are defined componentwise by

$$\hat{\mathbf{b}}_{pk}(t) \triangleq \frac{1}{2} \operatorname{tr} \left\{ \frac{\partial^2 f_k[\hat{\mathbf{x}}(t), t]}{\partial \mathbf{x}^2} \mathbf{P}(t) \right\} \quad (12-114)$$

$$\hat{\mathbf{b}}_{mk}(t) \triangleq \frac{1}{2} \operatorname{tr} \left\{ \frac{\partial^2 h_k[\hat{\mathbf{x}}(t), t]}{\partial \mathbf{x}^2} \mathbf{P}(t) \right\} \quad (12-115)$$

and  $\widehat{\mathbf{G}\mathbf{Q}\mathbf{G}^T}$  is as given in (12-53). Compare these relations to the discrete-time measurement filter of the corresponding form, (12-42)–(12-54), and notice that the filter gain is now  $\mathbf{P}(t)\mathbf{H}^T[t; \hat{\mathbf{x}}(t)]\mathbf{R}_c^{-1}(t)$ . The *Gaussian second order filter* [23, 38, 54, 72, 91, 113, 114] is described by the same relations, except for (12-111) being replaced by

$$\begin{aligned}\mathbf{d}\mathbf{P}(t) &= \{\mathbf{F}(t; \hat{\mathbf{x}}(t))\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T[t; \hat{\mathbf{x}}(t)] + \widehat{\mathbf{G}\mathbf{Q}\mathbf{G}^T} \\ &- \mathbf{P}(t)\mathbf{H}^T[t; \hat{\mathbf{x}}(t)]\mathbf{R}_c^{-1}(t)\mathbf{H}[t; \hat{\mathbf{x}}(t)]\mathbf{P}(t)\} dt \\ &+ \sum_{k=1}^m \left[ \mathbf{P}(t) \frac{\partial^2 h_k[\hat{\mathbf{x}}(t), t]}{\partial \mathbf{x}^2} \mathbf{P}(t) \mathbf{e}_k^T \mathbf{R}_c^{-1}(t) \{\mathbf{dy}(t) - \mathbf{h}[\hat{\mathbf{x}}(t), t]\} dt - \hat{\mathbf{b}}_m(t) dt \right]\end{aligned}\quad (12-116)$$

where  $\widehat{\mathbf{G}\mathbf{Q}\mathbf{G}^T}$  is now given as in (12-61) and  $\mathbf{e}_k$  is an  $m$ -dimensional vector with a one as its  $k$ th component and zeros elsewhere. This is comparable to the discrete-time measurement version, (12-48)–(12-52) and (12-54)–(12-61). As discussed for the discrete-time measurement case, the stochastic driving terms on (12-111) and (12-116) can cause significant difficulties, since  $\mathbf{P}(t)$  is not necessarily positive definite, and *modified second order filters* (the same filter relations but with these driving terms removed) and *first order filters with bias correction terms* are therefore motivated as viable alternatives.

**EXAMPLE 12.8** Recall the scalar example introduced in Example 12.1, but now assume continuous measurements are available as

$$\mathbf{y}(t) = \sin \mathbf{x}(t) dt + \mathbf{d}\beta_m(t) \quad \text{or} \quad z(t) = \sin \mathbf{x}(t) + v_c(t)$$

with  $\beta_m(\cdot, \cdot)$  scalar Brownian motion of diffusion  $R_c$ , or  $v_c(\cdot, \cdot)$  zero-mean white Gaussian noise of strength  $R_c$ :

$$E\{\mathbf{d}\beta^2(t)\} = R_c dt, \quad E\{v_c(t)v_c(t+\tau)\} = R_c \delta(\tau)$$

The truncated second order filter for this application is given by

$$\begin{aligned} \dot{\hat{x}}(t) &= -[\hat{x}(t) + a\hat{x}^3(t)] - 3a\hat{x}(t)P(t) + P(t)[\cos \hat{x}(t)]\{z(t) - \sin \hat{x}(t) + \frac{1}{2}[\sin \hat{x}(t)]P(t)\}/R_c \\ \dot{P}(t) &= -2[1 + 3a\hat{x}^2(t)]P(t) + [1 + b\hat{x}^2(t)]^2Q + [2b\hat{x}(t)]^2P(t)Q \\ &\quad + 2b[1 + b\hat{x}^2(t)]P(t)Q - P^2(t)[\cos \hat{x}(t)]^2/R_c \\ &\quad - \frac{1}{2}P^2(t)[\sin \hat{x}(t)]\{z(t) - \sin \hat{x}(t) + \frac{1}{2}[\sin \hat{x}(t)]P(t)\}/R_c \end{aligned}$$

The Gaussian second order filter shares the same  $\dot{\hat{x}}(t)$  equation, but the  $\dot{P}(t)$  equation has an additional term of  $[+3b^2P^2(t)Q]$  due to the different  $\widehat{\mathbf{G}\mathbf{Q}\mathbf{G}^\top}$  evaluation, and the last term above is doubled in magnitude and of opposite sign. Modified filters would have this last term on the  $\dot{P}(t)$  equation removed, and a first order filter with bias correction would differ only in  $\widehat{\mathbf{G}\mathbf{Q}\mathbf{G}^\top}$ . ■

In actual implementation for cases involving time-invariant systems and stationary noises (or described by slowly varying parameters that are approximated quasi-statically as constant), one often considers the *steady state constant gain* first order filter with bias correction terms because of the reduced computational loading. This entails an assumed nominal for evaluation of partials in  $\mathbf{P}(t)$  and gain calculations, but full accounting for nonlinearities and new evaluation point in the state dynamics and residual generation expressions of (12-110).

Continuous-measurement *assumed density filters* can also be derived without series representations for  $\mathbf{f}$  and  $\mathbf{h}$ , evaluating required conditional expectations in (12-108) and (12-109) on the basis of *assumed* conditional density function forms, typically Gaussian with mean  $\hat{\mathbf{x}}(t)$  and covariance  $\mathbf{P}(t)$ , as in Section 12.3. Moreover, *higher order moment filters* can be derived by writing explicit moment equations using *Kushner's equation* in a form analogous to (12-62): for scalar  $\psi[\mathbf{x}(t), t]$  that is twice differentiable with respect to  $\mathbf{x}(t)$ ,

$$\begin{aligned} \widehat{\frac{d\psi[\mathbf{x}(t), t]}{dt}} &= \widehat{\frac{\partial \psi}{\partial t}} + \widehat{\frac{\partial \psi}{\partial \mathbf{x}}} \mathbf{f}[\mathbf{x}(t), t] \\ &\quad + \frac{1}{2} \text{tr} \left\{ \widehat{\mathbf{G}[\mathbf{x}(t), t] \mathbf{Q}(t) \mathbf{G}^\top [\mathbf{x}(t), t]} \widehat{\frac{\partial^2 \psi}{\partial \mathbf{x}^2}} \right\} \\ &\quad + \{ \widehat{\psi \mathbf{h}[\mathbf{x}(t), t]} - \widehat{\psi} \widehat{\mathbf{h}[\mathbf{x}(t), t]} \}^T \mathbf{R}_c^{-1}(t) \{ \mathbf{z}(t) - \widehat{\mathbf{h}[\mathbf{x}(t), t]} \} \end{aligned} \quad (12-117)$$

Letting  $\psi[\mathbf{x}(t), t]$  be successively higher order products of  $\mathbf{x}(t)$  components yields (12-108), (12-109), and higher order scalar moments. As in the discrete-measurement case, *cumulants truncation* can be applied to achieve the final

equations for implementation [97, 110, 111, 120]. As in the discrete-time measurement case, conditional mode estimators can also be developed [33, 54, 71].

Another means of deriving continuous-measurement filters rigorously is by application of integral techniques in the form of a *representation theorem* originally due to Bucy [14–19, 21, 35, 43, 62–64, 89, 94]. For the problem described in the beginning of this section, it can be shown that the conditional density  $f_{\mathbf{x}(t)}(\xi | \{\mathbf{y}(\tau); t_0 \leq \tau \leq t\})$  can be expressed by the representation

$$f_{\mathbf{x}(t)}(\xi | \{\mathbf{y}(\tau); t_0 \leq \tau \leq t\}) = \frac{E[\exp(H) | \mathbf{x}(t) = \xi, \{\mathbf{y}(\tau); t_0 \leq \tau \leq t\}]}{E[\exp(H) | \{\mathbf{y}(\tau); t_0 \leq \tau \leq t\}]} f_{\mathbf{x}(t)}(\xi) \quad (12-118)$$

$$H = \int_{t_0}^t \mathbf{h}^T[\mathbf{x}(\tau), \tau] \mathbf{R}_c^{-1}(\tau) d\mathbf{y}(\tau) - \frac{1}{2} \int_{t_0}^t \mathbf{h}^T[\mathbf{x}(\tau), \tau] \mathbf{R}_c^{-1}(\tau) \mathbf{h}[\mathbf{x}(\tau), \tau] d\tau \quad (12-119)$$

where the expectations are over  $\{\mathbf{x}(\tau); t_0 \leq \tau \leq t\}$  given the minimal  $\sigma$ -algebra generated by  $\{\mathbf{y}(\tau); t_0 \leq \tau \leq t\}$ . Function space integration and other concepts from functional analysis, as well as martingale properties, are required for rigorous proof [41, 63, 64, 94, 148], though more heuristic approaches [14, 54, 89] provide considerable insight. By explicitly representing the conditional density in this manner, the theorem provides an alternate means of deriving the propagation relations for both the theoretically optimal and the approximate, finite-dimensional filters for the nonlinear estimation problem. An analogous result also exists for problems with discrete-time measurements [14, 89].

As mentioned in Section 12.2, an equation of the form of (12-98) can represent a system driven by deterministic control inputs  $\mathbf{u}(t)$ , or feedback controls as a function of the current perfectly known state  $\mathbf{u}[\mathbf{x}(t), t]$  by writing

$$\mathbf{f}[\mathbf{x}(t), t] = \mathbf{f}'[\mathbf{x}(t), \mathbf{u}(t), t] \quad (12-120a)$$

or

$$\mathbf{f}[\mathbf{x}(t), t] = \mathbf{f}''[\mathbf{x}(t), \mathbf{u}[\mathbf{x}(t), t], t] \quad (12-120b)$$

The same remarks also pertain to  $\mathbf{h}$  in (12-100). However, a number of “feedback estimation problems” are well modelled by replacing (12-100) with a stochastic functional differential equation,

$$d\mathbf{y}(t) = \mathbf{h}[\mathbf{x}(t), \{\mathbf{y}(\tau); t_0 \leq \tau \leq t\}, t] dt + d\mathbf{p}_m(t) \quad (12-121)$$

For instance, in many signal tracking applications, such as correlation tracking, the measurement is specifically a function of the difference between some components of the current system state (representable as  $\mathbf{T}_s \mathbf{x}(t)$ ) and some components of a filter state estimate ( $\mathbf{T}_f \hat{\mathbf{x}}(t)$ ):

$$\mathbf{h}[\mathbf{x}(t), \{\mathbf{y}(\tau); t_0 \leq \tau \leq t\}, t] = \mathbf{h}'[\mathbf{T}_s \mathbf{x}(t) - \mathbf{T}_f \hat{\mathbf{x}}(t), t] \quad (12-122)$$

It can be shown [11, 23, 43] that a function  $\psi[\cdot, \cdot]$  twice differentiable with respect to its first argument satisfies the Kushner moment equation, (12-117), when  $\mathbf{h}$  is generalized as in (12-121), and that (12-118) and (12-119) also apply under this generalization. The general stochastic control problem in which  $\mathbf{f}$  is also allowed to be a function of  $\{\mathbf{y}(\tau); t_0 \leq \tau \leq t\}$ , as through a feedback of estimated states,  $\mathbf{u}[\hat{\mathbf{x}}(t), t]$ , will be discussed in subsequent chapters.

**EXAMPLE 12.9** Consider the tracking of a pseudorandom code having an autocorrelation function as given in Fig. 12.11, by a receiver able to generate the code sequence in order to lock onto an incoming signal of this form [11]. A feedback loop as shown in Fig. 12.12 keeps the receiver's locally generated code in near coincidence with the incoming code. Many current code tracking loops develop a detector output as the difference between an "early" correlation (the result of multiplying the incoming code by a local code advanced by one time unit known as a "chip" relative to the expected arrival time of the input code) and a "late" correlation (using a one-chip-delayed local code). Let  $\mathbf{s}(t)$  denote the pseudorandom code as generated locally, and model the incoming code  $\mathbf{i}(t)$  as

$$\mathbf{i}(t) = \mathbf{s}(t) + \mathbf{n}(t)$$

where  $\mathbf{n}(\cdot, \cdot)$  is zero-mean white Gaussian noise with strength  $N(t)$ :  $E\{\mathbf{n}(t)\mathbf{n}(t+\tau)\} = N(t)\delta(\tau)$ . Define the detector error characteristic  $D_t(e)$  as the expected value of the detector output for a

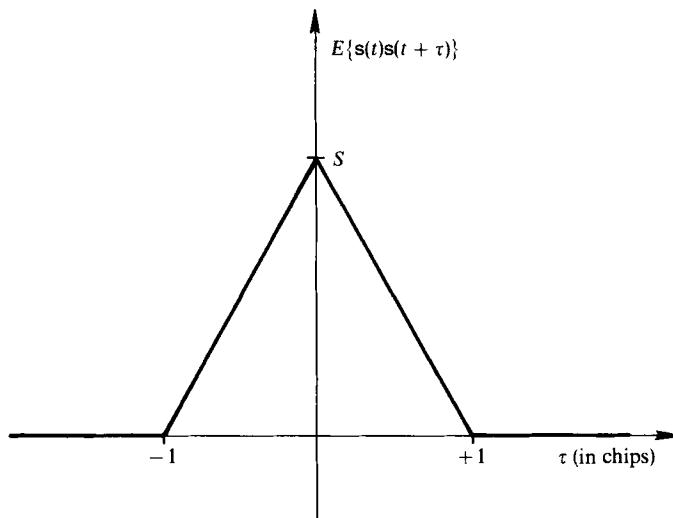


FIG. 12.11 Autocorrelation of pseudorandom code.

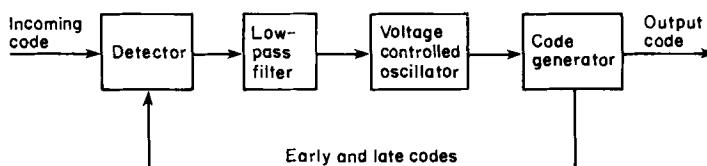


FIG. 12.12 Code tracking loop.

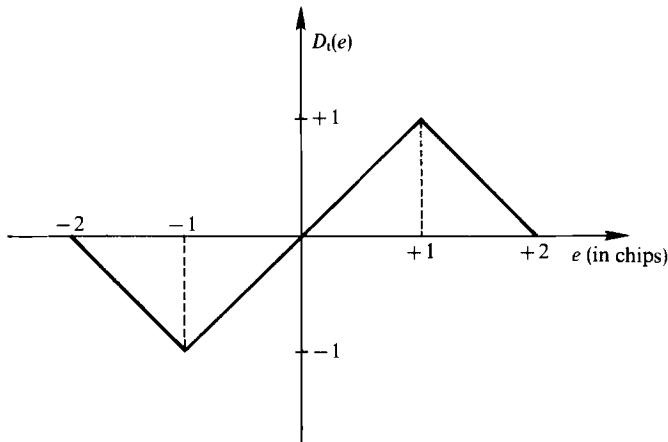


FIG. 12.13 Detector error characteristic.

phase tracking error of  $e$  chips normalized by the signal power  $S$ , assuming  $N \equiv 0$ , i.e.,  $i(t) = s(t)$ ,

$$D_t(e) \triangleq \left. \frac{E\{i(t)[s(t+e-1) - s(t+e+1)]\}}{S} \right|_{i(t) = s(t)}$$

This is plotted in Fig. 12.13. Letting  $N(t)$  be nonzero does not affect the mean value of the detector output, but contributes only to its covariance:

$$\begin{aligned} E\{i(t)[s(t+e-1) - s(t+e+1)]/S\} &= D_t(e) \\ E\{(i(t)[s(t+e-1) - s(t+e+1)]S^{-1} - D_t(e))(i(\tau)[s(\tau+e-1) - s(\tau+e+1)]S^{-1} - D_t(e))\} \\ &= [2N(t)/S]\delta(t-\tau) \end{aligned}$$

Now the incoming signal phase  $\theta(t)$  in chips is assumed to be described adequately by the dynamics model

$$d\theta(t) = d\beta(t)$$

where  $\beta(\cdot, \cdot)$  is scalar Brownian motion with diffusion  $Q(t)$ . The detector output measurement is given by

$$dy(t) = D_t[\theta(t) - \hat{\theta}(t)]dt + d\beta_m(t)$$

where  $\beta_m(\cdot, \cdot)$  is Brownian motion independent of  $\beta(\cdot, \cdot)$  and of diffusion  $2N(t)/S$ , or

$$z(t) \triangleq dy(t)/dt = D_t[\theta(t) - \hat{\theta}(t)] + v_c(t)$$

where  $v_c(\cdot, \cdot)$  is zero-mean white Gaussian noise of strength  $R_c(t) = 2N(t)/S$ . Then (12-108) and (12-109) yield, for  $x \triangleq \theta$ ,

$$\begin{aligned} d\hat{\theta}(t) &= [\widehat{\theta D_t} - \widehat{\theta \hat{D}_t}][2N(t)/S]^{-1}\{dy(t) - \widehat{D}_t dt\} \\ dP(t) &= \{Q(t) - [\widehat{\theta D_t} - \widehat{\theta \hat{D}_t}]^2[2N(t)/S]^{-1}\}dt + \overbrace{[\theta - \hat{\theta}]^2[D_t - \widehat{D}_t]}[2N(t)/S]^{-1}\{dy(t) - \widehat{D}_t dt\} \end{aligned}$$

Noting that  $D_t(\cdot)$  is an antisymmetric function, if one assumes the conditional density to be symmetric about its mean, then

$$\widehat{D}_t \triangleq \overbrace{D_t[\theta(t) - \hat{\theta}(t)]} = 0$$

and the coefficient premultiplying the residual in  $dP(t)$  is also zero. If one further assumes the conditional density is Gaussian with mean  $\hat{\theta}(t)$  and variance  $P(t)$ ,  $\widehat{\theta}D_t$  can be evaluated as

$$\widehat{\theta}D_t = \frac{1}{\sqrt{2\pi P(t)}} \int_{-\infty}^{\infty} \xi D_t(\xi) \exp\left\{\frac{-\xi^2}{2P(t)}\right\} d\xi = P(t)H[D_t, P(t)]$$

where  $H[D_t, P(t)]$  is the *describing function gain* [45] for the nonlinearity  $D_t(\cdot)$ . Thus, the filter equations become

$$\begin{aligned} d\hat{\theta}(t) &= P(t)H[D_t, P(t)]R_c^{-1}(t)dy(t) \\ \dot{P}(t) &= Q(t) - P^2(t)H^2[D_t, P(t)]R_c^{-1}(t) \end{aligned}$$

with  $R_c(t) = 2N(t)/S$ , which has the structure of a Kalman filter linearized with the describing function gains. Such an interpretation is typical for the Gaussian approximation, except that the stochastic driving term on  $P(t)$  is generally present.

An extended Kalman filter for this problem would be described by the same form of equations but with  $H[D_t, P(t)]$  replaced by

$$\left. \frac{\partial D_t[\theta - \hat{\theta}]}{\partial \theta} \right|_{\theta=\hat{\theta}} = \left. \frac{\partial D_t[0]}{\partial \theta} \right|_{\theta=0} = 1$$

The partial is evaluated at  $\theta = \hat{\theta}$ , and thus there is no accounting for the shape of  $D_t$  other than its slope at  $e = 0$ ; this is an inherent weakness of methods based on Taylor series representation for  $f$  and/or  $h$ . Because this is a correlation tracking problem with an assumed linear dynamics model, the variance and gain expressions of the extended Kalman filter are not dependent on the state trajectory or measurement history, as they generally would be. Therefore, a steady state, constant gain extended Kalman filter can be generated if  $Q$  and  $R_c$  are assumed constant. Empirical tests [11] have shown the filter-computed variance to be a good representation of actual error statistics up to a standard deviation  $\sqrt{P}$  of about  $\frac{1}{2}$  chip, above which the filter underestimates its own errors significantly and performance degrades unacceptably.

In contrast, the describing function  $H[D_t, P(t)]$  in the filter just derived takes into account the probability that  $[\theta - \hat{\theta}]$ , assumed Gaussian with mean zero and variance  $P(t)$ , lies outside the linear region of  $D_t$ . For small  $P(t)$ , this filter approaches the extended Kalman filter form, but as  $P(t)$  increases, it tends to have more conservative gains and to provide adequate performance over a larger domain.

**EXAMPLE 12.10** As seen in Fig. 12.13, the detector in the previous example provides no information for signal tracking purposes if the magnitude of the phase tracking error exceeds 2 chips. This range can be extended by correlating the incoming code with more advanced and delayed bits, and using a weighted sum of such correlation products to drive the loop filter. Figures 12.14a and 12.14b portray the detector error characteristics of two such methods using  $K$  advanced and  $K$  delayed bits instead of one of each as in Example 12.9; the associated linear and flat error characteristics are given by

$$\begin{aligned} D_t^L(e) &\triangleq \left. \frac{E\{i(t) \sum_{k=-K}^K [-ks(t+e-k)]\}}{S} \right|_{i(t)=s(t)} \\ D_t^F(e) &\triangleq \left. \frac{E\{i(t) \sum_{k=-K}^K [-\text{sgn}(k)s(t+e-k)]\}}{S} \right|_{i(t)=s(t)} \end{aligned}$$

with detector output means and covariance kernels of

$$\{D_t^L(e), [(2K+1)(K+1)KN(t)/3S]\delta(t-\tau)\}, \quad \{D_t^F(e), [2KN(t)/S]\delta(t-\tau)\}$$

respectively, replacing  $\{D_t(e), [2N(t)/S]\delta(t-\tau)\}$  of Example 12.9. The post-detection noise is seen to increase with increasing detector range, more so in the linear extension case than the flat case.

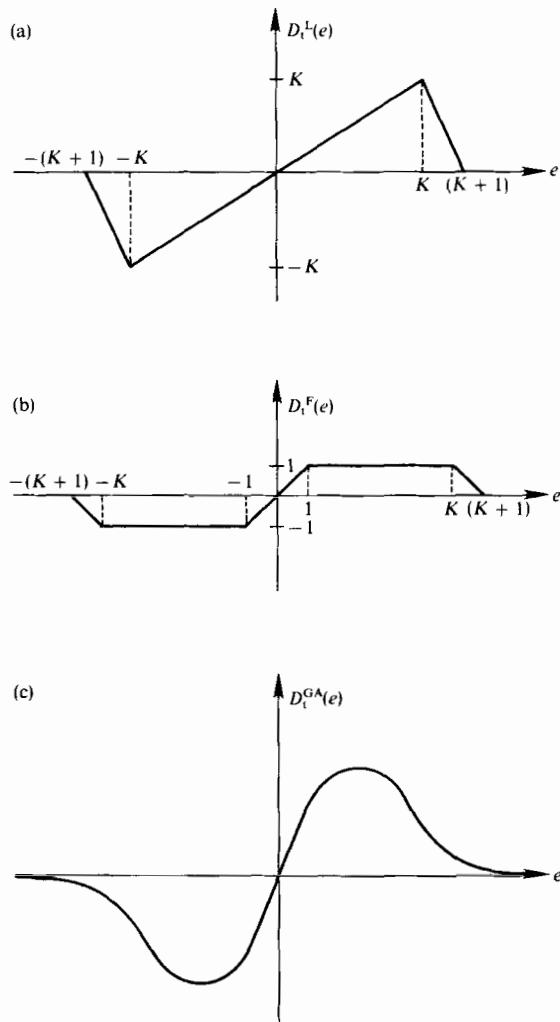


FIG. 12.14 Extended range detector error characteristics. (a) Linear detector error characteristic  $D_t^L$ . (b) Flat detector error characteristic  $D_t^F$ . (c) General antisymmetric error characteristic  $D_t^{GA}$ .

Therefore, one may seek a best  $K$  within these detector characteristics, or even a best characteristic functional form as in Fig. 12.14c, to yield desired tracking performance for a specified noise and code phase dynamics environment. As long as the detector characteristic is antisymmetric, the filter equations of the previous example are valid, with  $D_t^L$ ,  $D_t^F$ , or  $D_t^{GA}$  replacing  $D_t$ . Note that the replacement of  $H[D_t^{\cdot}, P(t)]$  by 1 in the extended Kalman filter for all cases (assuming  $dD_t^{GA}(0)/de = 1$ ) is particularly inappropriate since one envisions using the detectors in the regions over which their characteristics differ markedly.

In actual implementation,  $Q(t)$  can be established adaptively to account for varying code phase dynamics, as due to transmitting and/or receiving vehicle dynamics, and  $R_c(t)$  can be adapted to environmental noise, as due to jamming. Moreover, better dynamics models can be incorporated to reflect knowledge of phase dynamics characteristics. ■

Another method that has been used to generate estimator results is the *innovations process approach* [42, 46, 55, 56, 58–61]. Basically one seeks to transform the measurement history  $\{\mathbf{y}(\tau); t_0 \leq \tau \leq t\}$  to a Brownian motion  $\{\mathbf{v}(\tau); t_0 \leq \tau \leq t\}$  by a causal (nonanticipative) and causally invertible transformation;  $\mathbf{v}(\cdot, \cdot)$  is then the *innovations* process. Estimation of the solution  $\mathbf{x}(t)$  to (12-98) and (12-99) given the innovations history is simpler than estimation based upon the original measurement history described by (12-100)–(12-102). Given (12-98) and (12-99), and a *unit-diffusion* Brownian motion (or Wiener process)  $\mathbf{v}(\cdot, \cdot)$  such that

$$E\{\mathbf{d}\mathbf{v}(t)\mathbf{d}\beta^T(t + \tau)\} \equiv 0, \quad \tau \geq 0 \quad (12-123)$$

the conditional mean (MMSE estimate) can be shown to satisfy

$$\widehat{\mathbf{dx}}(t) = \widehat{\mathbf{f}[\mathbf{x}(t), t]} dt + \widehat{\mathbf{x}(t)\mathbf{v}^T(t)}\mathbf{d}\mathbf{v}(t) \quad (12-124)$$

The process  $\mathbf{v}(\cdot, \cdot)$  described by

$$\mathbf{d}\mathbf{v}(t) = \sqrt{R_c(t)^{-1}}\{\mathbf{dy}(t) - \widehat{\mathbf{h}[\mathbf{x}(t), t]} dt\} \quad (12-125)$$

is in fact such a Wiener process, and substitution of (12-125) and (12-100) into (12-124) yields the previous result of (12-108). This also demonstrates that the nonlinear filter residual  $\{\mathbf{z}(t) - \widehat{\mathbf{h}[\mathbf{x}(t), t]}\}$  is zero-mean white Gaussian noise of the same covariance as the measurement noise [42, 59]:

$$\cdot E\{(\mathbf{z}(t) - \widehat{\mathbf{h}[\mathbf{x}(t), t]})(\mathbf{z}(\tau) - \widehat{\mathbf{h}[\mathbf{x}(\tau), \tau]})^T\} = R_c(t)\delta(t - \tau) \quad (12-126)$$

If  $\widehat{\mathbf{h}[\mathbf{x}(t), t]}$  could be evaluated without approximation. Note that the Gaussianity of the nonlinear filter residual is valid only in the continuous-measurement case. Finding a means of generating innovations processes in a general and/or practical computational manner is then the crux of estimator design by this method, though one of difficult tractability.

Continuous-time state estimation problems based on both continuous and discrete measurements can be solved by combining the results of this section and Section 12.3. To be more specific, the discrete-time measurement updates given by (12-12)–(12-14) or the vector equivalent of (12-21) and (12-26), or the various approximations thereto, can be combined with the appropriate equations for incorporating the continuous measurements between the discrete-time sample points, namely (12-108) and (12-109), or approximations to these.

In the development of appropriate nonlinear filter structures, the insights gained from simpler linear estimation problems have often been exploited [126–129, 142]. Martingales [35, 51, 119], use of orthogonality in a more

general Hilbert space setting [5, 6, 37, 39, 81, 84], reproducing kernel Hilbert space methods [57], invariant embedding [8, 33, 101], least squares [49], and algebraic structure [83] approaches have also been used as alternative means of generating estimator results. Estimators have also been developed for the case of uncertainties described as unknown but bounded or as members of a given set [9, 103, 117, 118], for distributed parameter models of systems [73, 90, 140], and for the cases of smoothing [10, 80] and finite memory filtering [53, 54]. Furthermore, stochastic approximations [1, 44, 48, 75, 105, 109, 139, 146] have been used to produce estimators which are not necessarily optimum in any specific statistical sense, but which have certain desirable convergence characteristics.

## 12.8 SUMMARY

The nonlinear filtering problem for continuous-time systems with Markov state descriptions (12-1), given discrete-time noise-corrupted measurements (12-2), was described conceptually in Section 12.2. Basically, one attempts to depict the conditional density for  $\mathbf{x}(t)$ , conditioned on measurements available up to that time. A partial description of this density is provided by Eqs. (12-9) and (12-10) for propagating the *first two moments* between measurement sample times (as derived from the forward Kolmogorov equation), and update equations given by either (12-12)–(12-14) or the vector equivalents of (12-21) and (12-26), based upon Bayes' rule. The latter, better computational, update relations are dependent upon the assumption that the moments after update can be expressed by a truncated power series in the residual. However, even this partial description is an infinite-dimensional filter in general, since the conditional expectations in these relations require knowledge of the entire conditional density, i.e., of moments of all orders.

To develop implementable algorithms, various approximations can be made. One class of nonlinear filter exploits a Taylor series representation for the dynamics  $\mathbf{f}$  function and the measurement  $\mathbf{h}$  function. Within this class, the *truncated second order filter* is described for scalar problems via (12-17)–(12-19) and (12-27)–(12-31); the *modified* form removes the stochastic driving term from the covariance relation by setting  $b_1 \equiv 0$  in (12-31), and is described for the vector case by (12-42)–(12-54). Alternately, one can make generally better modeling assumptions to yield the *Gaussian second order filter*, (12-33)–(12-39), or the *modified* form thereof by setting  $b_1 \equiv 0$  in (12-39). The modified Gaussian second order filter for the vector case problem, (12-48)–(12-52) and (12-54)–(12-61), is perhaps the most applicable of this form of filter. Based on a tradeoff of computational loading and performance, a *first order filter with bias correction terms*, based on first order covariance and gain computations, but with a residual as given by (12-44) and (12-47) and a state propagation as by (12-49) and (12-51), is often a viable alternative to full-scale second order filters.

*Assumed density filters* do not inherently involve Taylor series for  $\mathbf{f}$  and  $\mathbf{h}$ , and therefore are more generally applicable and often outperform filters based on such representations. Fundamentally, these filters implement the full-scale moment equations (12-9)–(12-14) and the vector equivalent of (12-21) and (12-26), evaluating required conditional expectations as though the conditional state density were of an assumed form, as especially Gaussian with mean  $\hat{\mathbf{x}}(t/t_{i-1})$  and covariance  $\mathbf{P}(t/t_{i-1})$  as computed by the algorithm for all  $t \in [t_{i-1}, t_i]$ .

*Higher order moment filters* can be generated by writing higher order moment equations explicitly, as via the moment equation (12-62) derived from the forward Kolmogorov equation and expansions of the form of (12-21) and (12-26). *Cumulants truncation* as described in (12-64) can be used in the approximate evaluation of required conditional expectations.

The corresponding results for a discrete-time dynamics model, (12-65) and (12-66), were delineated by (12-67) and approximations thereto.

Section 12.4 described *conditional quasi-moments* as an alternative parameterization to moments of the conditional state density. Hermite polynomial series were motivated as a natural replacement for power series involving moments, and the resulting filter does provide good approximations for nearly Gaussian densities with few parameters required. However, for most problems of interest, there is little benefit to this form over conditional moment estimators, since they are both typically truncated in practice at reasonably low order, under which condition they differ only slightly.

Instead of generating an MMSE estimate as just described, the MAP estimate can be given by the *conditional mode*. Equations (12-80), (12-82), (12-85), and (12-86), and suitable approximations to these, yield this estimator.

For the special case of  $\mathbf{G}$  being a function only of time  $t$  and not of  $\mathbf{x}(t)$ , a *statistically linearized filter* as given by (12-90b) and (12-92b)–(12-97) can be employed. Like assumed density filters, it does not depend upon series representations for  $\mathbf{f}$  and  $\mathbf{h}$ . Structurally the algorithm has the form of an extended Kalman filter, but with partials of  $\mathbf{f}$  and  $\mathbf{h}$  evaluated at the current estimate being replaced by statistically optimized matrices that are directly related to describing function gains for the associated nonlinearities. Its performance often surpasses that of an extended Kalman filter by a considerable margin.

Finally, Section 12.7 considered the case of continuous-time state estimation based upon continuous-time measurements being available, as described by (12-98)–(12-102). The *Kushner equation* (12-103) describes the propagation of the conditional state density, and it forms a fundamental basis for estimation in the same manner as the forward Kolmogorov equation did for propagations between sampled-data measurements. The *moment equations* (12-108), (12-109), and the general (12-117) derived from the Kushner equation provide an infinite-dimensional filter analogous to the discrete-time measurement results (12-9)–(12-14), (12-21), (12-26), and (12-62). As in the discrete-time measurement case,

approximations yield the *truncated second order filter* ((12-110)–(12-115)), *Gaussian second order filter* ((12-110), (12-112)–(12-116)), *modified second order filters* (removing the stochastic driving term from (12-111)), *first order filter with bias correction terms* (additionally removing the second order derivative terms from  $\mathbf{GQG}^T$ ), *assumed density filter* ((12-108) and (12-109) with Gaussian density assumption for expectation evaluation particularly), and *higher order moment filter* ((12-117) with cumulant truncation). Other fundamental tools for filter generation are the *representation theorem*, (12-118) and (12-119), and the *innovations process approach*, (12-123)–(12-126).

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## PROBLEMS

**12.1** (a) Show that  $f_{x(t)}(\xi)$  satisfies the forward Kolmogorov equation.

(b) Show that  $f_{x(t)|Z(t_{i-1})}(\xi | \mathcal{Z}_{i-1})$  satisfies the forward Kolmogorov equation in the interval  $[t_{i-1}, t_i]$ , starting from the density  $f_{x(t_{i-1})|Z(t_{i-1})}(\xi | \mathcal{Z}_{i-1})$  obtained from the measurement update at sample time  $t_{i-1}$  (as claimed below (12-3)).

**12.2** (a) Show that, for any scalar random variables  $y$  and  $z$ , and scalar functions  $g(\cdot)$  and  $h(\cdot)$ ,

$$E\{g(z)h(y)|z = \zeta\} = g(\zeta)E\{h(y)|z = \zeta\}$$

and

$$E\{g(z)h(y)|z = z(\cdot)\} = g(z)E\{h(y)|z = z(\cdot)\}$$

(b) Demonstrate how this extends to the vector case, as needed for (12-16) and many other results of this chapter.

**12.3** Obtain the vector equivalent of Eqs. (12-21) and (12-26).

**12.4** (a) In the scalar case, show that the truncated second order, modified truncated second order, Gaussian second order, and modified Gaussian second order filters all reduce to the extended Kalman filter if  $G$  is a function only of  $t$  (and not of  $x(t)$  as well) and if  $\partial^2 f / \partial x^2$  and  $\partial^2 h / \partial x^2$  are neglected.

(b) Show that both modified second order filters in the vector case similarly reduce to the extended Kalman filter under these assumptions.

(c) Show that all four second order filters of part (a) reduce to the standard Kalman filter if dynamics and measurement models are linear.

(d) Show that both modified second order filters of part (b) reduce to the standard Kalman filter if dynamics and measurement models are linear.

**12.5** (a) Derive the expression (12-53) for  $\widehat{\mathbf{GQG}}^T$  in the modified truncated second order filter, and show that it reduces to (12-17b) in the scalar case.

(b) Derive the expression (12-61) for  $\widehat{\mathbf{GQG}}^T$  in the modified Gaussian second order filter by writing the appropriate terms in summation notation and invoking the form developed in Problem 10.9. Show that this reduces to (12-35) in the scalar case.

**12.6** Consider a system that can be modeled by means of the scalar equation

$$\dot{x}(t) = t^{-1}u(t)\cos x(t) + [u(t)w(t) - t]x^3(t) + 4e^{-x(t)}[u(t) + w(t)] + 2w(t)$$

where  $u(\cdot)$  is a prespecified open-loop control function of time, and  $w(\cdot, \cdot)$  is a white Gaussian noise of mean zero and strength  $Q(t)$ .

A scalar measurement is available at discrete time instants  $t_i$ , of the form

$$z(t_i) = \frac{2\sin t_i}{1+x(t_i)} + v(t_i)$$

where  $v(\cdot, \cdot)$  is white Gaussian discrete-time noise of mean zero and variance  $R(t_i)$ .

Write out the explicit equations for the modified truncated and modified Gaussian second order filters for this problem. Explain the assumptions inherent in each of these filters. Can you also generate an extended Kalman filter for this problem?

**12.7** Show that the relationships of (12-64) are correct between cumulants  $c_i$  and noncentral moments  $m_i$ . Show the relationships to central moments  $m_i^c$  are also correct.

**12.8** Consider state estimation based on a scalar model of the form

$$dx(t) = -[a_1x(t) + a_2x^2(t) + a_3x^3(t)]dt + [b_0 + b_1x(t) + b_2x^2(t)]d\beta(t)$$

based on sampled-data measurements described by

$$z(t_i) = [c_1x(t_i) + c_2x^2(t_i) + c_3x^3(t_i)] + v(t_i)$$

where  $\beta(\cdot, \cdot)$  is Brownian motion of constant diffusion  $Q$ ,  $x(t_0)$  is described by mean  $\bar{x}_0$  and variance  $P_0$ , and  $v(\cdot, \cdot)$  is a zero-mean white Gaussian discrete-time noise of variance  $R$ . Let  $x(t_0)$ ,  $\beta(\cdot, \cdot)$ , and  $v(\cdot, \cdot)$  be independent of each other. For this problem, generate and compare

(a) the linearized Kalman filter.

(b) the first order nonlinear filter. Under what conditions on model coefficients is this the extended Kalman filter?

(c) the first order filter with precomputed gains; the first order filter with constant gains. (What is the difference between these two, and again when does this become the extended Kalman filter with precomputed or constant gains?)

(d) the first order filter with bias correction terms.

(e) the precomputed-gain or constant-gain first order filter with bias correction terms.

(f) the truncated second order filter.

(g) the modified truncated second order filter.

(h) the Gaussian second order filter. Under what conditions on model coefficients does this result have the same time propagation relations as in part (f)? Under what conditions are the measurement update equations the same as in (f)?

(i) the modified Gaussian second order filter.

(j) the moment propagation equations given by (12-9), (12-10), and (12-62) for the full-scale (infinite-dimensional) filter, and measurement updates generated by (12-13) and (12-14).

- (k) the second order Gaussian assumed density filter (not involving Taylor series approximations as in (a)–(i)) based on the first two moment propagation relations from (j), and a measurement update based upon (12-21) with  $b_1 \equiv 0$  and (12-26) instead of (12-13) and (12-14).
- (l) the second order assumed density filter based on cumulants truncation; express in terms of central moments as well as noncentral moments. When is this equivalent to the results in (k)?
- (m) the third order assumed density filter based on cumulants truncation; express in terms of central moments as well as noncentral.
- (n) the full-scale conditional mode estimator equations; compare these to the results in (j).
- (o) the statistically linearized filter; compare these results to those in (a)–(e), and (k) and (l). Specifically compare the describing function gains to the corresponding terms in the first order filters.

By explicitly generating each of these filters, both similarities and differences of the various approximate solutions should become more evident. Under what conditions on the model coefficients do the distinguishing characteristics disappear for this particular scalar problem?

**12.9** (a) Write out the explicit equations for the extended Kalman filter and modified Gaussian second order filter for Example 12.3. The purpose of this is to gain an appreciation for the computational differences between the filters. Note that  $\mathbf{f}$  and  $\mathbf{h}$  are both nonlinear here, but that  $\mathbf{G}$  is a function only of time  $t$ .

(b) Write out the extended Kalman filter with bias correction terms and compare to the results in part (a).

(c) Evaluate the linearized Kalman filter for this problem.

(d) Evaluate the precomputed-gain and constant-gain extended Kalman filters for this problem, and compare to the previous results.

(e) Consider the filters of part (d) with bias correction terms.

(f) Generate the equations to define the modified truncated second order filter.

(g) Evaluate second order and third order assumed density filters for this problem.

(h) Evaluate the statistically linearized filter for this problem, and compare to the previous results. Especially compare the describing function gains here to the corresponding gains of the extended Kalman filter.

**12.10** Repeat parts (a)–(o) of Problem 12.8, but for the following vector-case problems (in parts (f)–(i), consider only the modified filters):

(a) the satellite orbit determination problem described by Example 9.8 and Problems 9.7 and 9.13. Here  $\mathbf{f}$  is nonlinear,  $\mathbf{h}$  is linear, and  $\mathbf{G}$  is a function only of time  $t$ .

(b) the uncertain parameter estimation problem of spacecraft thrust-vector controlling, as discussed in Example 9.9 and Problem 9.9. With the uncertain parameter treated as an additional state,  $\mathbf{f}$ ,  $\mathbf{h}$ , and  $\mathbf{G}$  have the same structural form as in part (a).

(c) the pointing and tracking problem portrayed in Example 9.10, with  $\mathbf{f}$  linear,  $\mathbf{h}$  nonlinear and  $\mathbf{G}$  a function only of  $t$ . Compare these filter propagation equations to those for the higher dimensional *linear* dynamics model in Problem 9.12c, and the *nonlinear* dynamics model of Problem 9.12d.

(d) the residual monitoring/sensor failure detection problem in Example 12.4. Note specifically that here, unlike the previous cases,  $\mathbf{G}$  is a function of  $\mathbf{x}(t)$  as well as  $t$ .

**12.11** In air-to-air tracking with radar measurements, one can choose to express state variables in an inertial coordinate system, yielding linear dynamics and nonlinear measurement models. Or, one can use tracker line-of-sight coordinates and achieve another description with nonlinear dynamics and linear measurement models. If tracking performance were comparable for filters based on these two different models, which would be preferable from a computational standpoint?

**12.12** Assume that a *discrete-time* dynamics model, as (12-65) and (12-66), and discrete-time measurement model (12-2), adequately describe a system of interest. Develop the filters of parts (a)–(o) of Problem 12.8 for this problem formulation.

**12.13** Apply the results of the previous problem to the scalar problem defined by models

$$\begin{aligned} \mathbf{x}(t_{i+1}) &= [a_1 \mathbf{x}(t_i) + a_2 \mathbf{x}^2(t_i) + a_3 \mathbf{x}^3(t_i)] + [b_0 + b_1 \mathbf{x}(t_i) + b_2 \mathbf{x}^2(t_i)] w_d(t_i) \\ \mathbf{z}(t_i) &= [c_1 \mathbf{x}(t_i) + c_2 \mathbf{x}^2(t_i) + c_3 \mathbf{x}^3(t_i)] + v(t_i) \end{aligned}$$

where  $w_d(\cdot, \cdot)$  and  $v(\cdot, \cdot)$  are independent zero-mean white Gaussian noises of variances  $Q_d$  and  $R$ , respectively, and both are independent of the initial  $\mathbf{x}(t_0)$  of mean  $\bar{\mathbf{x}}_0$  and variance  $P_0$ .

**12.14** (a) A scalar linear system is described by the stochastic differential equation

$$dx(t) = -2x(t)dt + d\beta(t)$$

where  $\beta(\cdot, \cdot)$  is scalar Brownian motion with statistics

$$E\{\beta(t)\} = 0, \quad E\{[\beta(t) - \beta(t')]^2\} = S|t - t'|$$

The initial value of the state is known exactly:  $x(0) = 1$ . Derive an expression for the transition probability density for  $x(t)$ . Why is this function of primary interest in estimation?

(b) At time  $t = 1$ , a measurement of the state is available. The measurement is corrupted by an error which is independent of the past history. The measurement is

$$z(1) = x(1) + v$$

The error  $v$  can only assume the discrete values  $+1, 0$ , and  $-1$  with the following probabilities:

$$\text{Prob}\{v = -1\} = \frac{1}{4}, \quad \text{Prob}\{v = 0\} = \frac{1}{2}, \quad \text{Prob}\{v = 1\} = \frac{1}{4}$$

Develop an expression for the mean of  $x(1)$ , conditioned on the measurement  $z(1) = \zeta$ . Calculate this conditional mean of  $x(1)$  if the measured value is

$$z(1) = \zeta = 2$$

(c) How would your approach and/or results to this estimation problem change if (consider the 5 parts independently)

- (1) the Brownian motion were replaced by a non-Gaussian driving process?
  - (2) the system description were nonlinear?
  - (3)  $v$  could assume a value from a continuous range of values, with the probability of its magnitude being defined by a Gaussian density function;  $v$  independent of  $x(1)$ ?
  - (4) same as 3 but  $v$  correlated with  $x(1)$  as  $E\{x(1)v\} = \sigma_{xv}$ ?
  - (5) same as 3 but with a non-Gaussian density function describing  $v$ ?
- (d) Other estimates of  $x(1)$  exist besides the conditional mean. What other logical definitions of an “optimal” estimate of  $x(1)$  could be made? Under what conditions do some of these alternatives yield the same value for the “optimal” estimate?

**12.15** Consider the scalar system model

$$dx(t) = ax^2(t)dt + bx(t)d\beta(t)$$

$$dy(t) = x(t)dt + d\beta_m(t)$$

where  $\beta(\cdot, \cdot)$  and  $\beta_m(\cdot, \cdot)$  are independent unit-diffusion Brownian motions, or, in terms of white noise notation,

$$\dot{x}(t) = ax^2(t) + bx(t)w(t), \quad z(t) = x(t) + v_c(t)$$

where  $w(\cdot, \cdot)$  and  $v_c(\cdot, \cdot)$  are independent zero-mean unit-strength white Gaussian noises.

(a) Show that the conditional  $k$ th order noncentral moments,  $m_k(t)$ , for  $k = 1, 2, 3, \dots$ , based on Kushner's equation can be written with use of (12-117) as

$$\begin{aligned}\dot{m}_1(t) &= am_1(t) + [m_2(t) - m_1^2(t)]\{z(t) - m_1(t)\} \\ \dot{m}_2(t) &= 2am_3(t) + b^2m_2(t) + [m_3(t) - m_1(t)m_2(t)]\{z(t) - m_1(t)\} \\ \dot{m}_3(t) &= 3am_4(t) + 3b^2m_3(t) + [m_4(t) - m_1(t)m_3(t)]\{z(t) - m_1(t)\} \\ &\vdots\end{aligned}$$

Express this also in terms of conditional central moments.

- (b) Generate the second order assumed density filter based on a Gaussian assumed density.
- (c) Using the fundamental relationships given by (12-64), generate the second order filter that results from cumulant truncation: letting  $c_k = 0$  for all  $k \geq 3$ . Show that this is the same as the result obtained in (b).
- (d) Similarly generate the third order filter based on cumulant truncation above third order: letting  $c_k = 0$  for all  $k \geq 4$ .
- (e) Express the results of (b)–(d) in terms of conditional central moments as well as noncentral moments.

(f) Compare these results to those of filters based upon Taylor series approximations:

- (1) truncated second order,
- (2) Gaussian second order,
- (3) first order with bias correction terms,
- (4) first order without bias correction terms,
- (5) linearized Kalman filter.

(g) Why are modified truncated second order filters and modified Gaussian second order filters not included in the list in part (f)?

**12.16** Consider the basic signal phase tracking problem described as follows. A sinusoidal signal of carrier frequency  $\omega_c$ , as corrupted by additive zero-mean white Gaussian noise  $v_c(\cdot, \cdot)$  of strength  $R_c$  is available from a receiver:

$$z(t) = \cos[\omega_c t + \theta(t)] + v_c(t), \quad E\{v_c(t)v_c(t + \tau)\} = R_c \delta(\tau)$$

where the uncertain phase  $\theta(\cdot, \cdot)$  to be tracked is modeled as a first order Gauss–Markov process:

$$\dot{\theta}(t) = -b\theta(t) + w(t), \quad E\{w(t)w(t + \tau)\} = [2b\sigma^2] \delta(\tau)$$

where  $\sigma$  and  $b$  are the rms value and bandwidth, respectively, of  $\theta(\cdot, \cdot)$ , i.e.,  $[1/b]$  is the correlation time of the phase process.

- (a) Show that the extended Kalman filter for this problem has the form

$$\begin{aligned}\dot{\hat{\theta}}(t) &= -b\hat{\theta}(t) - \{P(t)/R_c\} \sin[\omega_c t + \hat{\theta}(t)] \{z(t) - \cos[\omega_c t + \hat{\theta}(t)]\} \\ \dot{P}(t) &= -2bP(t) + Q(t) - P^2(t) \sin^2[\omega_c t + \hat{\theta}(t)]/R_c\end{aligned}$$

This can be implemented as a phase-locked loop: the current  $\hat{\theta}(t)$  is used to generate two local oscillator outputs,  $\cos[\omega_c t + \hat{\theta}(t)]$  and  $\sin[\omega_c t + \hat{\theta}(t)]$ . The first of these is subtracted from the current measurement, and this residual multiplied by the  $\sin[\omega_c t + \hat{\theta}(t)]$  (to demodulate  $[\theta(t) - \hat{\theta}(t)]$  and the  $[P(t)/R_c]$  term, and finally passed through a low-pass first order lag filter,  $1/(s + b)$ , to produce the filtered output  $\hat{\theta}(t)$  itself. Show that, in fact, this product does demodulate  $[\theta(t) - \hat{\theta}(t)]$ , that:

$$\begin{aligned}\sin[\omega_c t + \hat{\theta}(t)] \{z(t) - \cos[\omega_c t + \hat{\theta}(t)]\} &\cong \hat{\theta}(t) - \theta(t) + v_c'(t) + \{\sin[\omega_c t + \hat{\theta}(t)] \cos[\omega_c t + \hat{\theta}(t)] \\ &\quad + \text{other high frequency filterable terms}\}\end{aligned}$$

where  $v_c'(t)$  is modulated white noise.

(b) As an alternative to extended Kalman filtering, design the following filters for this application, and compare them to the extended Kalman filter:

- (1) linearized Kalman filter, and steady state version thereof.
- (2) precomputed-gain extended Kalman filter.
- (3) constant-gain extended Kalman filter.
- (4) extended Kalman filter with bias correction terms; consider precomputed-gain and constant-gain approximations.
- (5) truncated second order filter.
- (6) Gaussian second order filter.
- (7) Gaussian second order assumed density filter (without Taylor series approximations for  $f$  and  $h$ ).
- (8) assumed density filter based on cumulants truncation, letting  $c_k = 0$  for all  $k \geq 3$ .
- (9) assumed density filter based on cumulants truncation, letting  $c_k = 0$  for all  $k \geq 4$ .

(10) statistically linearized filter; identify the describing function gains and compare directly to those of the extended Kalman filter and assumed density filters in (7) and (8) above; these gains tend to give better performance than the extended Kalman filter gains, which are based on truncated series representations of  $f$  and  $h$ , and so this filter is less prone to “cycle slipping” that is typical of phase-locked loops based on filters embodying first order series truncation assumptions.

(c) Show the modification to filter structure in (a) when the phase process is modeled as the output of the first order lag  $1/(s + b)$  driven by Brownian motion of diffusion  $Q$ , instead of driven by white Gaussian noise of strength  $Q$  as in (a); note that the structure of the appropriate phase-locked loop tracker is intimately linked to the model for the phase process itself.

- (d) Repeat part (c), but with Brownian motion  $\beta(\cdot, \cdot)$  described by

$$E\{[\beta(t) - \beta(t')]^2\} = \int_t^t (10 + \sin \tau) d\tau$$

- (e) Repeat part (a), but assuming that two scalar measurements are available as

$$\begin{aligned} z_1(t) &= \cos[\omega_c t + \theta(t)] + v_{c1}(t) \\ z_2(t) &= \sin[\omega_c t + \theta(t)] + v_{c2}(t) \end{aligned}$$

where  $v_{c1}(\cdot, \cdot)$  and  $v_{c2}(\cdot, \cdot)$  are assumed *independent*, and each a zero-mean white Gaussian noise of strength  $R_c$ . Develop the extended Kalman filter, and show that, in this very special case, the variance equation is *not* dependent on the state estimate  $\hat{\theta}(t)$ .

- (f) Repeat part (b) for the model in part (c).
- (g) Repeat part (b) for the model in part (d).
- (h) Repeat part (b) for the model in part (e).

**12.17** Consider the sampled-data problem analogous to the previous continuous-measurement problem, in order to generate a digital phase-locked loop tracker. Let the dynamics model be as in Problem 12.16, but let measurements be taken every  $\Delta t$  sec of the form

$$z(t_i) = \cos[\omega_c t_i + \theta(t_i)] + v(t_i)$$

where  $v(\cdot, \cdot)$  is zero-mean white Gaussian discrete-time noise of variance  $R$ . Generate the “equivalent discrete-time system model” for this problem.

- (a) Repeat (a) of the previous problem, including a demodulation interpretation.
- (b) Repeat (b) of the previous problem, adding modified truncated and modified Gaussian second order filters to the list of designs to consider.
- (c) Repeat (c) of the previous problem, with sampled-data measurements.
- (d) Repeat (d) of the previous problem, with sampled-data measurements.

- (e) Show how the results of (e) of the previous problem are altered for sampled-data measurements

$$\begin{aligned} z_1(t_i) &= \cos[\omega_c t_i + \theta(t_i)] + v_1(t_i) \\ z_2(t_i) &= \sin[\omega_c t_i + \theta(t_i)] + v_2(t_i) \end{aligned}$$

where  $v_1(\cdot, \cdot)$  and  $v_2(\cdot, \cdot)$  are independent of each other, and each a zero-mean white Gaussian discrete-time noise of variance  $R$ .

- (f) Repeat part (b) for the model in part (c).
- (g) Repeat part (b) for the model in part (d).
- (h) Repeat part (b) for the model in part (e).

**12.18** Using the moment equations (12-108) and (12-109) based upon the Kushner equations, derive (a) the Kalman filter, (b) the extended Kalman filter, (c) the truncated second order filter, (d) the Gaussian second order filter, and (e) the Gaussian assumed density filter, based on suitable assumptions.

**12.19** Using (12-117), derive the third order assumed density filter based upon cumulants truncation for the scalar case.

**12.20** (a) Show that the mean and variance of  $D_t(e)$  as given in Example 12.9 are correct.  
 (b) Demonstrate the validity of Fig. 12.13 for the detector error characteristic for this example.  
 (c) Show that the expressions for  $D_t^L(e)$  and  $D_t^F(e)$  of Example 12.10 do have the means and variance kernels as claimed.

**12.21** (a) Use the representation theorem of (12-118) and (12-119) to derive the continuous-time measurement Kalman filter for the linear models

$$\begin{aligned} d\mathbf{x}(t) &= \mathbf{F}(t)\mathbf{x}(t)dt + \mathbf{G}(t)d\beta(t) \\ d\mathbf{y}(t) &= \mathbf{H}(t)\mathbf{x}(t)dt + d\beta_m(t) \end{aligned}$$

where  $\beta(\cdot, \cdot)$  and  $\beta_m(\cdot, \cdot)$  are independent Brownian motions of diffusions  $\mathbf{Q}(t)$  and  $\mathbf{R}_c(t)$ , for all  $t$ , respectively.

- (b) Use the innovations approach of (12-123)–(12-126) to derive this same result.

**12.22** Derive the estimator results for the case of both continuous-time and discrete-time measurements being available. Specifically, let the state evolve according to (12-1), and let both certain discrete-time measurements  $\mathbf{z}_d(t_i)$  as described by (12-2) and certain other continuous-time measurements  $\mathbf{z}_c(t)$  as described by (12-100) or (12-100') be available from the system. Develop estimators of the forms depicted in parts (a)–(o) of Problem 12.8 for this formulation.

# Index

## A

Accuracy, maximally achievable, 72, *see also* Performance analysis  
Adaptive filtering, 69, 120, 129, 136, 141, 232, 256  
Adaptive noise estimation, 69, 120, 129, 136, 141, 156  
Additive, *see* Linearly additive noise  
Adequate model, 24, 25, 160, 190, 224, 225, *see also* Models  
Adjoint, 10, 15, 21  
Admissible, 175, 179, 181  
Age-weighting of data, 28, 63, *see also* Fading memory filter  
Algebra,  $\sigma$ -algebra, 176  
Algebraic operations, *see* Constant gain  
Algebraic Riccati equation, *see* Constant gain  
Algebraic structure, 257  
Algorithms, *see* Filter; Performance analysis; Software programs for design  
Aliasing, 75  
Almost everywhere, *see* Convergence  
Almost surely, *see* Convergence  
Ambiguity function, 97, 99, 141  
A posteriori probability, 213, 241, *see also* MAP estimate; Measurement, update  
Approximate nonlinear filter, 214, *see also* Filter  
Approximations, 39, 81, 83, 89, 101, 109, 123, 214, 218, 234, 241, 242  
A priori probability, 35, 73, 76, 130, *see also* Initial conditions; Initial covariance matrix; Mean  
Artificial lower bounding of covariance, 27, 135, *see also* Compensation

Assumed density filter, 216, 235, 244, 250, 254  
Asymptotic, 71, 73, 96  
Asymptotic behavior, 71, 73, 96, 102, 133, 136  
Asymptotic efficiency, 71, 96  
Asymptotic normality, 71, 96, 136, 139  
Asymptotic unbiasedness, 71, 96, 136, 138  
Augmentation, *see* State, augmented  
Autocorrelation, *see* Correlation  
Average, *see* Expectation; Time average ensemble, 81  
Averaging, 103, 141

## B

Backward filter, 2, 5, 9  
Backward Kolmogorov equation, 192  
Basis functions, 240  
Batch processing, 74  
Bayes' rule, 35, 76, 77, 80, 130, 162, 214, 219, 236  
Bayesian estimation, 73, 129, 214, 236  
Benchmark of performance, 27, 82  
Best, *see* Optimal criteria  
Bias, 52, 72, 109, 138, 224, *see also* Random bias  
Bias correction terms, 218, 221, 224–226, 231, 249, 250  
Block diagram of multiple model filtering algorithm, 132  
Bounded, 182, 183  
Bounded matrix, 27  
Bounded variation, 176  
Brownian motion (Wiener process), 39, 51, 160, 175–176, 179–181

## C

- Canonical form, 114, 115  
 Causal, 256  
 Central moments, 215, 236, 237, *see also* Covariance  
 Chapman–Kolmogorov equation, 171, 174, 194, 213, 247  
 Characteristic function, 196, 209, 236  
     conditional 209, 236, 247  
 Characteristic polynomial, *see* Eigenvalue  
 Characterization of stochastic process, 161, 174, 202  
 Cholesky square root, 21, 192, 210–211, *see also* Factorization  
 Coefficient, correlation, 28  
 Combining states, *see* Order reduction  
 Compensation for model inadequacies, 23, 24, 27, 224–225, *see also* Tuning  
 Complete, 239  
 Completely observable, 114  
 Computational aspects, *see* Computer  
 Computed covariance, 23, 27, 52, 231, *see also* Conditional covariance matrix; Tuning  
 Computer  
     loading, 10, 17, 35, 37–38, 73, 74, 81, 88, 101–103, 110, 114–116, 216, 221, 225, 234, 245  
     memory, 17, 37–38, 74, 103, 111, 116  
     programs for design, *see* Software programs for design  
 Condition number, 102  
 Conditional characteristic function, 209, 236, 247  
 Conditional covariance matrix  
     bounds on, 25, 27, 72, 97, 99, 101, 102, *see also* Benchmark of performance  
     for continuous-time states with continuous measurements, 58, 248, 249  
     with discrete measurements, 9–10, 15–17, 30, 32, 44–45, 131, 133, 215–216, 225–227, 238, 244  
     for discrete-time state with discrete measurements, 238  
 Conditional density function  
     for Gauss–Markov process, 162  
     for Markov process, 162, 213–214, 248, 251  
 Conditional distribution function, 162  
 Conditional expectation, 1, 176, 215, 220, *see also* Conditional moments; Filter  
 Conditional Gaussian density, *see* Assumed density filter; Gaussian second order filter; Kalman filter; Statistically linearized filter  
 Conditional information matrix, 81, 85–88, 120  
 Conditional mean, 213, 215–216, 248, 256  
     for continuous-time state with continuous measurements, 58, 248–249  
     with discrete measurements, 213, 215  
     for discrete-time state with discrete measurements, 238  
 Conditional mode, 72, 213, 241, 251  
     for continuous-time state with continuous measurements, 251  
     with discrete measurements, 241  
     for discrete-time state with discrete measurements, 243  
 Conditional moments  
     central, 215, *see also* Conditional covariance matrix  
     for continuous-time state with continuous measurements, 58, 248–249  
     with discrete measurements, 215  
     for discrete-time state with discrete measurements, 238  
     noncentral, 236, *see also* Correlation  
 Conditional probability  
     density, 162, 213–214, 248, 251  
     distribution, 162  
 Conditional quasi-moments, 239  
 Conditional variance, *see* Conditional covariance matrix  
 Confidence, 139, *see also* Information, Uncertainties  
 Conservative filter, *see* Robustness, Tuning  
 Consistent estimator, 71, 96, 136, 138  
 Constant-gain  
     approximation, 57, 58, 250, 266, 267, 270  
     extended Kalman filter, 57, 58, 266, 267, 270  
     steady state filter, 57, 58, 137, 250, 266, 267, 270  
 Continuity  
     mean square, 161, 176, 179, 183  
     with probability one, 161, 176, 179, 183  
 Continuous, 39, 161, 176, 179, 182, 183, 191  
 Continuous time  
     control, 39, 160, 161, 213, 251  
     estimation, 20–21, 57, 245  
     filtering, 57, 245  
     linear dynamic system model, 39–41, 160–161, 165

- measurement, 57, 160, 165, 246, 251  
nonlinear dynamic system model, 39, 165,  
166, 181, 213  
Control, 2, 10, 37, 39, 86, 160, 161, 213, 251  
Control input estimation in smoothing, 10  
Controllability, stochastic, 24  
Convergence, 58, 82, 133, 241, 247, 257  
in the mean (mean square), 161, 176, 177,  
182, 183  
in probability, 71  
with probability one (almost sure), 161, 176,  
182, 183  
rate, 82  
Coordinates, shifted, *see* Perturbation; Taylor series  
Correlated noise  
spatial, 54  
temporal, *see* Shaping filter  
Correlation, *see also* Covariance  
autocorrelation, 137, 252  
coefficient, 28  
cross correlation, 28, 136, 185, 256  
estimate of, 120, 123, 137  
function, *see* Correlation kernel  
kernel, 39, 70, 137, 160, 162, 164, 256  
matrix, 40, 70, 120, 137, 160, 162, 176, 181,  
185  
methods for self tuning, 136  
spatial, 54  
temporal, *see* Shaping filter  
time, *see* Shaping filter  
Corruption, measurement, *see* Measurement noise  
Covariance  
analysis, 48  
bounds, 25, 27, 72, 97, 99, 101, 102, *see also*  
Benchmark of performance  
conditional, *see* Conditional covariance matrix  
cross covariance, 28, 136, 256  
error, 137, 224, 231, *see also* Conditional covariance matrix  
estimate of, 120, 123, 129, 136, 141, *see also*  
Monte Carlo  
factorization, 1, 9, 21, 27, 116, 192, 210, 211  
filter-computed, 23, 27, 52, 231, *see also*  
Conditional covariance matrix; Tuning function, *see* Covariance kernel  
Kalman filter, 9, 44–45, 58, 84–87, 131, 248  
kernel, 39, 40, 70, 160, 162, 164  
matching for self tuning, 141  
matrix, 39, 70, 120, 160, 162, 176, 181  
matrix for continuous linear system model,  
162, 201  
matrix for continuous nonlinear system model, 199–201  
matrix for discrete linear system model, 238  
notation, xv  
true error, *see* Performance analysis; Truth model  
Cramér–Rao lower bound, 72, 97, 99, 101,  
102  
Criteria, optimality, 1, 213, *see also* Estimate  
Cross correlation, 28, 136, 185, 256  
Cross covariance, 28, 136, 256  
Cumulant, 236, 237  
Cumulant truncation, 236, 237, 250  
Curve fitting, 111
- D**
- Data  
processing, *see* Measurement update  
spurious, 232  
Delta function, 185  
Density function  
conditional, *see* Conditional density function; Estimate; Filter  
for continuous linear system, 162, 201  
for continuous nonlinear system, 192, 197,  
199–201  
for discrete linear system, 84  
for discrete nonlinear system, 238  
Gaussian, 129, 132  
joint, 161, 162, 202, 233  
Density, power spectral, 54  
Describing function, 244, 254  
Design  
filter, *see* Adequate model; Performance analysis; Tuning  
software, *see* Monte Carlo; Performance analysis  
Design model, *see* Models  
Deterministic control input, 2, 10, 37, 39, 86,  
160, 161, 213, 251  
Difference equations, *see also* Propagation stochastic, 70, 238  
Differentiable, 161, 176, 191, 235, 243  
Differential, 161, 181  
Differential equations, *see also* Propagation stochastic, 160, 165, 166, 181  
Differential generator, 187, 188  
Differential rule, Itô, 186, 236

- Diffusion, 161, 181, 184, 187, 196, *see also Kolmogorov's equations*  
 Digital computation, *see Computer*  
 Dirac delta, 185  
 Discontinuity, 164  
 Discrete-time  
     control, 2, 10, 37, 86  
     dynamics, 2, 70, 238, 243  
     equivalent, 2, 70, 210  
     estimation, *see Filter; Smoothing*  
     filtering, *see Filter*  
     Markov process, 167  
     measurement, 2, 40, 70, 160, 213  
     model, 70, 238  
     model, equivalent, 2, 70, 210  
     process, 70, 167, 238  
     system, *see Discrete-time dynamics and measurement*  
 Discrete-valued random vector, 71, 130, 167, 171  
 Discretization, 2, 70, 130, 136  
 Distance measure, 178, 181  
 Distributed computation, 144  
 Distributed parameter system, 256  
 Distribution, 161, 162, *see also Density function*  
 Divergence, 23, 24, 38, *see also Compensation for model inadequacies; Tuning*  
 Drift, 184  
 Dummy variable, xiii  
 Dynamic programming, 18  
 Dynamic system model, *see System model (linear); System model (nonlinear)*  
     continuous-time linear, 41, 160–162, 165  
     continuous-time nonlinear, 39, 165, 166, 181, 199, 213  
     discrete-time linear, 2, 70  
     discrete-time nonlinear, 238  
 Dynamics noise, 2, 39, 70, 120, 160, 213
- E**
- Edgeworth series, 239  
 Efficient estimate, 71  
 Eigenvalue, 58, 75, 115  
     assignment or placement, 58  
 Eigenvector, 58, 75, 115  
     assignment or placement, 58  
 Ensemble average, 81  
 Epsilon technique of Schmidt, 31
- Equilibrium state, 57  
 Equivalent discrete-time model, 2, 70, 210, 211  
 Ergodic, 123  
 Erroneous model, *see Adequate model; Compensation for model inadequacies*  
 Error  
     analysis, *see Performance analysis*  
     bounds, 72, 82, 97, 99, 101, 102  
     compensation, *see Compensation for model inadequacies*  
     covariance, 137, 244, 231, *see also Conditional covariance*  
     covariance, computed, 23, 27, 52, 231, *see also Conditional covariance; Tuning*  
 model, *see Models*  
 sensitivity, *see Sensitivity; Robustness*  
 states, *see Perturbation*  
 tracking, 54–56, 109  
 true, *see Performance analysis*  
 Estimate, *see also Filter*  
     asymptotically efficient, 71, 96  
     asymptotically unbiased, 71, 96, 136  
 Bayes, 73, 129, 213–215, 236  
     best linear, *see Kalman filter*  
     bias, 52, 72, 109, 138, 224  
     computed, 23, 27, 52, 231, *see also Performance analysis; Reduced-order; Tuning*  
 conditional mean, *see Bayesian estimation; Conditional mean*  
 consistent, 71, 96, 136, 138  
 continuous-time, continuous-measurement, 20–21, 58, 245  
 continuous-time, discrete-measurement, *see Filter*  
 covariance, 120, 123, 128, 129, 136, 141, *see also Conditional covariance matrix; Monte Carlo*  
 discrete-time, 9, 84–88, 120–123, 238, 243  
 error, *see Error*  
 least squares, 18, 72, 156–158, 257  
 linear, *see Kalman filter*  
 for Markov process, 162, 163–165, 213  
 maximum a posteriori, (MAP), 213, 241, 251  
 maximum likelihood, 18, 35, 71, 74, 80, 96, 101, 120  
 of mean, *see Conditional mean; Monte Carlo*  
 minimum error variance, unbiased, 72, 213, 243, 256, *see also Conditional mean*

- minimum mean squared error, *see* Conditional mean  
 minimum variance reduced order, 25  
 of moments, 123, 128, *see also* Conditional moments; Monte Carlo MVRO, 25  
 notation, xvi  
 parameter, 48, 68, 84, 120, 129, 141  
 predicted, *see* Propagation recursive, 37, 131, 213  
 smoothed, 1, 58, 257  
 of statistical parameters, *see* Monte Carlo unbiased linear, 72  
 unknown but bounded, 71, 72, 257  
 of variance, *see* Estimate of covariance weighted least squares, 18, 72, 156–158, 257  
**E**  
 Estimator, *see* Estimate Euclidean norm, 22  
 Euclidean space, xiii, 22  
 Existence, 178, 181, 182, 247, 251  
 Expansion point, *see* Equilibrium state; Linearization; Nominal; Relinearizations; Taylor series  
 Expectation, 199  
 conditional, 1, 176, 215, 220, *see also* Conditional expectation  
 Expected value, *see* Expectation  
 Exponential age weighting of data, 28, 63, *see also* Fading memory filter  
 Exponentially time-correlated process, 53, 139, 188, 205  
 Extended Kalman filter, 44–59, 198, 218, 227, 231, 245, 254  
 constant-gain steady state, 57, 58, 254  
 continuous-time measurements, 57  
 iterated, 58  
 precomputed gain, 57, 58  
**F**  
 Factorization, 1, 9, 21, 27, 116, 192, 210, 211, 227  
 Fading memory filter, 28, 31, 63, 156  
 Failure detection, 232  
 Feedback control, 86, 213, 251  
 Feedback gain, 86  
 Filter,\* *see also* Estimate  
 adaptive, 69, 120, 129, 136, 141, 232, 256  
 \*Numbers in parentheses are equation numbers.
- age-weighting, 28, 31, 63, 156, (9-10)–(9-11)  
 approximate nonlinear, 214  
 assumed density, 216, 235, 244, 250, 254, (12-9)–(12-14), (12-21), (12-26), (12-108)–(12-109)  
 bias correction terms included, 218, 221, 224–226, 231, 249, 250, (12-44), (12-47), (12-49), (12-51)  
 conditional mode, 213, 241, 251, (12-80)–(12-86)  
 conditional moment, 215, 225–227, 238, 248–249  
 conditional quasi-moment, 239  
 constant-gain, 57, 58, 137, 250, 266, 267, 270  
 continuous-time measurements, 58, 245, 248, 249  
 control inputs included, 9, 37, 44, 58, 213, 233, 238, 251  
 cumulant truncation, 236, 250, (12-64)  
 discrete-time dynamics, 9, 238, 243, (8-31)–(8-33), (12-67)  
 extended Kalman, 44–59, 198, 218, 227, 231, 245, 254, (9-61)–(9-69), (9-80)–(9-82)  
 fading memory, 28, 31, 63, 156, (9-10)–(9-11)  
 finite memory, 33, 74, 80, 257, (9-25)–(9-29)  
 first order, 42, 218, 225, 249, *see also* Extended Kalman filter  
 first order filter with bias correction terms, 218, 224, 225, 249, (12-40)–(12-41)  
 Gaussian second order, 216, 221, 226, 249, (12-33)–(12-39), (12-110), (12-112)–(12-116)  
 higher order moments, 216, 235, 236, 250, (12-21), (12-26), (12-62), (12-117)  
 infinite dimensional, 213, 215, 238, 242, 248, (12-9), (12-10), (12-12)–(12-14), (12-21), (12-26), (12-62), (12-117)  
 inverse covariance, 3, 6, 9, 35, 54, *see also* Information  
 Kalman, *see* Kalman filter  
 Kalman–Bucy, 248  
 limited memory, 33, 74, 80, 257, (9-25)–(9-29)  
 linearized Kalman, 42  
 mode, 213, 241, 251  
 modified Gaussian second order, 223, 224, 226, 227, 249, (12-48)–(12-61)

- Filter (*con't.*)  
modified truncated second order, 221, 223,  
225, 249, (12-42)–(12-54)  
moment, 215, 238, 248  
multiple model adaptive, 129, 131, 142,  
(10-104)–(10-108)  
MVRO, 25  
quasi-moment, 239  
second order: Gaussian, 216, 221, 226, 249,  
(12-33)–(12-39), (12-110), (12-112)–  
(12-116)  
second order: modified Gaussian, 223, 224,  
226, 227, 249, (12-48)–(12-61)  
second order: modified truncated, 221, 223,  
225, 249, (12-42)–(12-54)  
second order: truncated, 217, 218, 220, 225,  
249, (12-17)–(12-19), (12-27)–(12-31)  
statistically linearized, 243, (12-90)–(12-97)  
truncated second order, 217, 218, 220, 225,  
249, (12-17)–(12-19), (12-27)–(12-31)  
Filter-computed covariance, 23, 27, 52, 231,  
*see also* Conditional covariance matrix;  
Tuning  
Filter design, *see* Design  
Filter divergence, 23, 24, 38, *see also* Compensation for model inadequacies; Tuning  
Filter error, *see* Error  
Filter gain, 9, 27, 28, 30, 44, 57, 58, 128, 220,  
222, 225, 226, 244, 249, 266, 267, 270  
compensation of, 23, 24, 27, 28, 30, 32, 69,  
120, 128, 136, 141, *see also* Tuning  
Filter model, *see* System model; Reduced-order; Simplified; design model  
Filter robustness, 143, *see also* Divergence; Tuning  
Filter stability, 24, 34, *see also* Numerical stability  
Filter tuning, *see* Tuning  
Finite dimensional, 214–216, 248  
Finite memory filter, 33, 74, 80, 257  
First moment, *see* Expectation; Mean  
First order approximation, 41, *see also* First order filter; Modified second order filter  
First order density, 162  
First order filter, 42, 218, 225, 249, *see also* Extended Kalman filter  
First order filter with bias correction terms, 218, 224, 225, 249  
First order lag, 49, 53  
First order Markov model, 53, 139, 188, 205  
First variation, *see* Linear perturbation  
Fisher information matrix, 36, 158; *see also* Inverse covariance form filter  
Fixed-gain approximation, *see* Constant gain  
Fixed-interval smoother, 4, 5, 9  
Fixed-lag smoother, 5, 16  
Fixed-length memory filter, 33, 74, 80, 257  
Fixed-point smoother, 5, 15, 17  
Fokker–Planck equation, *see also* Kolmogorov's equations  
Formal rules, 166, 177, 178, 186, 187, 189, 192  
Forward filter, 2, 5, 9  
Forward Kolmogorov equation, 192, 193, 197,  
209, 213, 215, 236, 242  
Fourier transform, 209, *see also* Characteristic function  
Frequency domain, 54, 63, 65, *see also* Stationary; Time-invariant system model  
Function  
characteristic, 196  
density, *see* Density function  
distribution, *see* Distribution  
probability, *see* Probability  
Function of random vector, 185, 218, 219  
Functional analysis, 178, 180, 251  
Functional redundancy, 233  
Fundamental theorem of differential equations, 182  
Fundamental theorem of Itô stochastic calculus, 207
- G**
- Gain matrix, *see* Filter gain  
Gaussian probability density, 77, 132, 241  
conditional, 77, 132, 216, 221, 226, 235, 241,  
243, 244, 249, 250, 254  
Gaussian filter  
assumed density, 216, 235, 244, 250, 254  
second order, 216, 221, 226, 249  
Gaussian process, 161, 164, 176, *see also* Brownian motion; Gauss–Markov process model  
white, *see* White Gaussian noise  
Gaussian random vector, 39, 77, 132, 161,  
164, 176, 241  
Gaussian second order filter, 216, 221, 226,  
249

Gaussian stochastic process, *see* Gaussian process  
 Gaussian white noise, *see* White Gaussian noise  
 Gauss–Markov process model, 53, 139, 160, 162, 188, *see also* System model (linear)  
 Generalized ambiguity function, 97, 141  
     *N*-step, 98  
 Generalized inverse, 31, 123, 138, 142  
 Generalized quadratic cost, 72, 158  
 Gradient, 81, 154  
 Gram–Charlier series, 240

**H**

Half-power frequency, 54  
 Hermite polynomial, 239  
 Hessian, 81  
 Higher order filters, 216, 235, 236, 250  
 Hilbert space, 157, 241, 257  
 History  
     measurement, 2, 83, 160, 165, 213, 215, 246  
     residual, *see* Residual  
 Hypercube, 194  
 Hypothesis conditional probability, 130  
 Hypothesis testing, 73, 130, 139, 232  
 Hypothetical derivative, 164, 165, 246

**I**

Identifiability, 75, 156  
 Identification, *see* System identification; Parameter  
 Implementation, *see* Computer  
 Impulse, 185  
 Impulsive corrections, 26  
 Inaccuracies, model, *see* Adequate model;  
     Numerical precision; System model (linear); System model (nonlinear)  
 Increment, *see* Independent increments  
 Independent, 44, 70  
     in time, *see* White noise  
     and uncorrelated, *see* Correlation  
 Independent increments, 160, 163, 176, 183, 247, *see also* Brownian motion; Poisson process  
 Independent processes, 44, 53, 70

Independent random vectors, 44, 70  
 Infinite dimensional optimal filter, 213, 215, 238, 242, 248  
 Infinitesimal, 184  
 Information, 35, *see also* Information matrix;  
     Inverse covariance form filter  
 Information matrix, 36, 158, *see also* Conditional information matrix  
 Initial conditions, 2, 39, 41, 70, 76, 84, 130, 160, 196, 213  
 Initial covariance matrix, 2, 39, 41, 70, 160, 213  
 Initial transient, 43, 103, 104, 112, 113, 135  
 Inner product, 22, 157, 206  
 Innovations, 18, 256, *see also* Residual Input  
     deterministic control, 2, 10, 37, 39, 86, 160, 161, 213, 251  
     optimal design, 136  
 Integral  
     Itô stochastic, 166, 175, 178, 180, 181, 189  
     Riemann, 160  
     stochastic, *see* Itô stochastic integral; Stratonovich integral; Wiener integral  
 Stratonovich stochastic, 163, 178, 188, 191–193  
     Wiener, 161, 175, 180  
 Integral equation, 181  
 Integration by parts, 196  
 Integration methods, 45, 189, 190–192, 210–211  
 Invariant embedding, 257  
 Inverse, 3, 6, 8–10, 35, 54, 102  
     generalized, 31, 123, 138, 142  
 Inverse covariance form filter, 3, 6, 9, 35, 54, *see also* Information  
 Inversion lemma, matrix, 6  
 Invertibility, 114, 123, 256  
 Iterated extended Kalman filter, 58  
 Iterated linear filter-smoother, 58, 59  
 Iterative scalar measurement updating, 56, 64, 116; *see also* Measurement update  
 Iterative solutions, 80, 81, 102, 154, 155  
 Itô differential, 181  
 Itô differential rule, 186, 207, 236  
 Itô differential equations, 181–183, 191, 213  
     existence and uniqueness of solutions, 182  
     Markov solution, 182, 183  
     properties of solution, 182, 183  
 Itô integral, 166, 175, 178, 180, 181  
     evaluation, 178, 189

- I**
- Itô integral (*con't.*)
    - relation to Stratonovich integral, 189, 191, 192
  - Itô stochastic integral, *see* Itô integral
- J**
- Jacobian, 41, 44, 45, *see also* Partial derivative
  - Joint density and distribution, 161, 162, 202, 233
  - Jointly Gaussian, 162
  - Jordan canonical form, 115
  - Joseph form, 10, 137
- K**
- Kalman filter
    - continuous-time, continuous-measurement, 248
    - discrete-measurement, 2, 9, 15, 16, 17, 24, 42, 44, 78, 85, 87, 131, 223, 265, 266
    - discrete-time, discrete-measurement, 2, 9, 15, 16, 17, 78, 85, 87, 131
    - divergence, 23, 24, 38, *see also* Compensation for model inadequacies; Tuning extended, 44–59, 198, 218, 227, 231, 245, 254
    - gain, *see* Filter gain
    - linearized, 42
    - optimal smoother based on, 2, 5, 9–10, 16, 17
    - perturbation, 42
    - robustness, *see* Robustness; Tuning
    - sensitivity, *see* Sensitivity
    - stability, 24, 34
    - steady state, 57, 58, 137, 250
    - tuning, *see* Tuning
  - Kalman gain, *see* Filter gain
  - approximations, *see* Curve fitting
  - linearized, 42
  - relinearized, 44, *see also* Extended Kalman filter
  - steady state, 57, 58, 137, 250
  - tuning, *see* Tuning
  - Kalman–Bucy filter, 248
  - Kernel
    - correlation, 39, 40
    - covariance, 39, 40
    - cross-correlation, 256
    - cross-covariance, 256
- L**
- Laplace transform, 54
  - Least squares estimation, 18, 72, 156–158, 257
    - weighted, 18, 72, 156–158, 257
  - Least squares fit, 72, 141
  - Levy oscillation property, 177, 186, 247
  - Levy process, *see* Brownian motion
  - Likelihood equation, 75, 78–80, 120, 242
  - Likelihood function, 35, 73, 75–78, 109, 120–122, 140, 233, 242
  - Likelihood ratio, 176, 209, 235
  - Limit in mean, 161, 176, 177, 182, 183, 206
  - Limited memory filter, 33, 74, 80, 257
  - Linear controller, 86
  - Linear filter, *see* Filter; Kalman filter
  - Linear function, 180, *see also* Filter; System model (linear)
  - Linear functional, 180
  - Linear minimum variance filter, *see* Kalman filter
  - Linear models, *see* System model (linear)
  - Linear operations, 22, 157, *see also* Linear function; Linearization
  - Linear perturbation, 40–42, 48, 52, *see also* Perturbation
  - Linear stochastic difference equation, 70
  - Linear stochastic differential equation, 160, 161
  - Linear system model, 70, 160, 161, *see also* System model (linear)
  - Linear transformation, 22, 114, 180, *see also* Linear function
  - Linear unbiased minimum variance estimate, *see* Kalman filter
  - Linearization, 40, 41
  - Linearized Kalman filter, 42
  - Linearly additive noise, 39, 40, 160, 165
  - Lipschitz, 182, 191
  - Local iterations, 58, 81, 84, 138
  - Lower bound
    - artificial bounding of covariance, 27
    - Cramér–Rao, 72, 97, 99, 101, 102
    - performance, *see* Benchmark of performance

- ance; Ambiguity function; Information matrix
- Lower triangular, 21, 192, 210–211, *see also Factorization*
- Low-pass filter, 49, 53
- Luenberger observer, 58
- M**
- MAP estimate, 213, 241, 251
- Mapping, xiii, *see also Function*
- Marginal probability
- density, 130, 162, 214
  - distribution, 162, 202
- Markov process, 136, 159, 162, 163, 166, 167, 176, 183, 192, 213
- Martingale, 176, 179, 209, 247, 251, 256
- Matrix
- block diagonal, *see Cross-correlation*
  - Cholesky square root, 21, 192, 210–211, *see also Factorization*
  - conditional information ( $J$ ), 81, 85–88, 120
  - control input ( $B; B_d$ ), 2, 37, 41, 70, 75, 160
  - controller gain ( $G_c$ ), *see Controller gain*
  - correlation ( $\Psi$ ), *see Correlation*
  - covariance ( $P$ ), *see Covariance*
  - cross-correlation, *see Cross-correlation*
  - cross-covariance, *see Cross-covariance*
  - diagonal, 191, 192
  - dynamics noise strength ( $Q; Q_d$ ), 2, 39, 70, 120, 160, 181, 213
  - factorization, 1, 9, 21, 27, 116, 192, 210, 211, 227
  - filter gain ( $K$ ), *see Filter gain*
  - fundamental ( $F$ ), 41, 160
  - gain, *see Filter gain*
  - Hessian, 81
  - information ( $\mathcal{I}$ ), 36, 37, 158, *see also Conditional information matrix; Inverse covariance form filter*
  - initial covariance ( $P_0$ ), 2, 39, 70, 160, 213
  - inverse, 3, 6, 8–10, 35, 54, 102
  - inversion lemma, 6
  - invertible, 114, 123, 256
  - Jacobian, 41, 44, 45, *see also Partial derivative*
  - Kalman gain, *see Kalman gain*
  - lower triangular, 21, 192, 210–211, *see also Factorization*
  - measurement ( $H$ ), 2, 41, 70, 156, 160
  - measurement noise strength (or covariance) ( $R; R_c$ ), 2, 40, 70, 120, 160, 213, 246
  - noise input ( $G; G_d$ ), 2, 39, 70, 71, 160, 181, 238
  - nonsingular, *see Invertibility*
  - operations, *see Linear transformation*
  - partition, 81
  - positive definite, *see Positive definite*
  - positive semidefinite, *see Positive semidefinite*
  - Riccati, *see Riccati equation*
  - similarity transformation, 114
  - singular, 8, 70, 123, *see also Positive semidefinite*
  - square root, 1, 9, 21, 116, 192, 210–211, 227, *see also Factorization*
  - state transition ( $\Phi$ ), 2, 15, 70, 74, 115, 161, 190
  - state transition probability, 169
  - symmetric, 116
  - system dynamics ( $F$ ), 41, 160
  - trace of, 181
  - transformation, 114
  - unitary, 27, 211
  - upper triangular, 27, 211
  - weighting, 18, 72, 156–158, 257
  - Maximum a posteriori (MAP), 213, 241, 251
  - Maximum likelihood (ML), 18, 35, 71, 74, 80, 96, 101, 120
  - Mean, *see also Expectation*
    - conditional, *see Conditional mean; Bayesian estimation; Filter*  - Mean square
    - continuity, 161, 176, 179, 183
    - convergence, 161, 176, 177, 178, 182
    - derivative, 160, 161, 164, 165, 246
    - limit, 161, 176, 177, 178, 182, 183  - Mean value, 72, *see also Mean*
  - Mean vector, 72, *see also Mean*
    - for continuous linear system models, 162
    - for continuous nonlinear models, 199
    - for discrete-time linear models, 84
    - for discrete-time nonlinear models, 238  - Measurable, 176, 246, 251
  - Measurement
    - continuous-time, 57, 160, 165, 246
    - as controller input, 213, 251
    - discrete-time, 40, 70, 160, 163, 165
    - equation, *see Measurement, discrete-time; Measurement, continuous-time*
    - error, *see Measurement noise*
    - history, 2, 83, 160, 165, 213, 215, 246
    - iterative scalar update, 56, 64, 116, *see also Measurement update*
    - linear continuous-time model, 160, 163

- Measurement (con't.)**
- linear discrete-time model, 2, 70, 160
  - matrix, 2, 41, 70, 156, 160
  - noise, 2, 40, 70, 120, 160, 165, 213, 246
  - nonlinear continuous-time model, 57, 165, 246, 251
  - nonlinear discrete-time model, 40, 165, 213
  - point-process model, 163–165, 247–248
  - Poisson model, 163–165, 247–248
  - residual, *see* Residual
  - sampled-data, 2, 40, 70, 160, 165, 213
  - statistical description, 2, 40, 70, 120, 160, 213, 246
  - spurious, 232
  - time-correlated noise, *see* Shaping filter
  - update, 9, 27, 28, 30, 32, 44, 87, 131, 220, 222, 225, 226, 242, 244, *see also* Filter
  - vector, 2, 40, 57, 70, 160, 163, 165, 213, 246, 251
- Median**, 72
- Memory requirements**, 17, 37–38, 74, 103, 111, 116
- Microprocessor**, 144
- Minimal  $\sigma$ -algebra**, 176, 246, 251
- Minimum mean square error (MMSE)**, 213, 243, 256, *see also* Conditional mean
- Minimum variance estimate**, 72, 213, 243, 256, *see also* Conditional mean
- Minimum variance reduced order (MVRO)**, 25
- Mismodeling**, 69, 133, *see also* Adequate model; Compensation for model inadequacies; Divergence; Robustness; Tuning
- ML estimate**, *see* Maximum likelihood
- MMSE**, *see* Minimum mean square error
- Mode**
- conditional, 72, 213, 241, 251
  - of system response, 75, 115
- Models**, *see also* System model (linear); System model (nonlinear); Measurement dynamics, *see* Dynamic system model
- effect of errors, 23, 24, 38, *see also* Error, Mismodelling
  - equivalent discrete-time linear, 2, 70, 210, 211
  - error models, *see* Error; System model (linear); System model (nonlinear)
  - linear system models, *see* System model (linear)
  - measurement, *see* Measurement
  - nonlinear system models, *see* System model (nonlinear)
- process**, *see* Dynamic system model; Stochastic process; System model (linear); System model (nonlinear)
- reduced-order**, 23, 25, *see also* Adequate model
- simplified**, 109, 121, 128, 216, *see also* Adequate model; Approximations; Reduced-order
- simulation**, 189, 210, *see also* Monte Carlo
- system**, *see* System model (linear); System model (nonlinear)
- time invariant**, *see* Time-invariant system model
- Modified canonical form**, 115
- Modified Gaussian second order filter**, 223, 224, 226, 227, 249
- Modified Jordan canonical form**, 115
- Modified second order filter**, 221–227, 249
- Modified truncated second order filter**, 221, 223, 225, 249
- Moment**
- central, 215, 236, 237, *see also* Covariance
  - conditional, 58, 215, 236, 238, 248–249, *see also* Conditional covariance matrix
  - for continuous linear system models, 162
  - for continuous nonlinear system models, 199
  - for discrete-time linear models, 84
  - for discrete-time nonlinear models, 238
  - estimate of 123, 128, *see also* Conditional moments; Monte Carlo
  - first, *see* Mean
  - noncentral, 236, *see also* Correlation
  - second, *see* Correlation; Covariance
- Moment generating function**, *see* Characteristic function
- Monitoring**, residual, 69, 131, 133, 232
- Monte Carlo analysis**, 48, 51, 54, 93, 105, 190, 210, 225, 229, 231, 237
- Moving window**, 33, 74, 80, 257
- Multiple model filtering algorithm**, 129, 131, 142
- Multiplicative noise**, 166, 181, 213, 238, 245
- Mutually exclusive**, 169
- MVRO estimator**, 25

**N**

- Neglected terms**, 109, 121, 128, 216, *see also* Adequate model; Approximations; Reduced-order
- Newton–Raphson**, 81, 155

- Noise  
 correlated (in time), *see* Shaping filter  
 Gaussian, 161, 164, 176, *see also* Brownian motion, White Gaussian noise  
 generalized Poisson, 164, 165  
 point process, 163–165, 247  
 Poisson, 163–165, 247  
 white, 39, 40, 70, 160, 164–166, 256
- Noise input matrix, 2, 39, 70, 71, 160, 181, 238
- Nominal, 40, *see also* Linearization
- Nonanticipative, *see* Causal
- Noncentral moment, 236, *see also* Correlation
- Nondifferentiability, 161, 164, 165, 176, 235, 243, 246
- Nonlinear effects, 39, 159, 212, *see also* Adequate model; Neglected terms; Nonlinear models
- Nonlinear estimation, 18, 39, 212
- Nonlinear filter, 212, *see also* Filter  
 continuous-time, continuous-measurement, 57, 58, 245  
 continuous-time, discrete-measurement, 39, 213, 215, 239, 241, 243  
 discrete-time, discrete-measurement, 238
- Nonlinear functionals, 180
- Nonlinear models, *see* Dynamic system model; Measurement; System model (nonlinear)
- Nonlinear observers, 58
- Norm, *see* Inner product
- Normal, *see* Gaussian random vector
- Notation, xiii, xvi
- N*-step information matrix, 36, 37, *see also* Conditional information matrix
- Numerical precision, 9, 28, 45, 102, 115, 116, 121, 123, 210, 211, *see also* Factorization; Wordlength
- Numerical stability, 24, 28, 102, 116, 121, *see also* Factorization
- O**
- Observability, 114, 138  
 stochastic, 24
- Observations, *see* Measurements
- Observer, 58
- Off-diagonal, *see* Cross-correlation; Factorization
- Offline calculations, *see* Precomputable
- Off-nominal, *see* Nominal; Perturbation; Robustness; Sensitivity
- Omega ( $\omega$ ), xiii, 63, 65, *see also* Realization of random vector
- Online, 69, 101, *see also* Offline calculations
- Optimal criteria, 1, 213, *see also* Estimate
- Optimal estimate, *see* Estimate; Filter; Smoothing
- Optimal filter, 213–216, 246–248, *see* Filter
- Optimal input design, 136
- Optimal prediction, *see* Filter; Propagation
- Optimal smoothing, *see* Smoothing
- Order reduction, 23, 25, *see also* Adequate model; Approximations; Simplified
- Orthogonal, 239, 256
- Orthogonal projection, 256
- Orthonormal, 240
- Overweighting most recent data, 28, 31, 63, 156
- P**
- Parallel architecture, 144
- Parameter  
 estimation, 48, 68, 84, 120, 129, 136, 141, 143  
 identification, 48, 68, 69, 84, 129, 153  
 sensitivity, 68, 98, 143, *see also* Robustness  
 uncertainties, 48, 68, 143, *see also* Models; Noise
- Parameterization of density function, 214, 215, 239
- Partial derivative, 41, 66, 75, 79, 85, 193, 194, 217, 224, 225, 226, *see also* Expansion point; Taylor series
- Partition, 81
- Penrose pseudoinverse, 31, 123, 138, 142
- Perfect knowledge of entire state, 213, 251
- Performance analysis, 48  
 benchmark, 27, 82  
 covariance, 48  
 error, 48, *see also* Error  
 filter tuning, *see* Tuning
- Monte Carlo, 48, 51, 54, 93, 105, 190, 210, 225, 229, 231, 237
- sensitivity, 48, 68, 71, 98, 99, 143, *see also* Parameter; Robustness
- software, *see* Software programs for design
- truth model, *see* Truth model
- Performance criteria, *see* Criteria, optimality; Estimate
- Perturbation  
 analysis, *see* Robustness; Sensitivity
- Kalman filter, 42

- Perturbation (*con't.*)  
 linear, 40–42, 48, 52  
 measurement, 41  
 model, 41, *see also* Taylor series state, 41
- Phase-locked loop, 269–270
- Phase variable form, 114
- Physically realizable, 256
- Point process, 163–165, 247–248
- Poisson process, 163–165, 247–248  
 generalized, 164, 165  
 homogeneous, 163  
 nonhomogeneous, 164
- Poles of a system, 58, 75, 115  
 placement, 58
- Polygonal approximation, 190
- Positive definite, xiv, 70, *see also* Matrix, measurement noise strength
- Positive semidefinite, xiv, 70, *see also* Matrix, dynamics noise strength; Initial covariance matrix
- Power series in the residual, 218, 222, 235, 238
- Power spectral density, 54, 63, 65, *see also* Correlation; Fourier transform
- Practical aspects, *see* Computer; Design; Implementation; Reduced-Order; Robustness; Tuning
- Precision  
 estimation, 68–69, 71, 72, 97, 101, *see also* Performance analysis  
 knowledge, 68–71, *see also* Covariance  
 numerical, 9, 28, 45, 102, 115, 116, 121, 123, 210, 211, *see also* Factorization; Wordlength
- Precomputable, 41–42, 46, 54, 57, 58, 82, 86, 102, 111–113, 121, 131, 248, 266, 267, 270
- Precomputed-gain extended Kalman filter, 57, 58, 266, 267, 270
- Prediction, *see* Propagation
- Principal axes, *see* Eigenvector
- Prior knowledge, *see* Prior statistics; System model (linear); System model (nonlinear)
- Prior statistics, *see* A priori probability; Initial conditions; Initial covariance matrix; Mean
- Probability  
 a posteriori, 213, 241, *see also* MAP estimate; Measurement update  
 a priori, 35, 70, 73, 76, 130, *see also* Prior statistics  
 conditional, *see* Conditional probability
- density, *see* Density function
- distribution, *see* Distribution
- joint, 161, 162, 202, 233  
 law, *see* Forward Kolmogorov equation  
 marginal, 130, 162, 202, 214  
 model, *see* Dynamic system model; Measurement; Models; System model (linear); System model (nonlinear)
- Procedure, design, *see* Design
- Process, stochastic, *see* Stochastic process
- Process noise, *see* Dynamics noise
- Product space, xiii, 157
- Projection, *see* Orthogonal projection
- Propagation, *see also* Filter  
 covariance, 44, 84, 162, 199–201, 238, 248  
 filter equations, 44, 84, 213, 215, 218, 221, 226, 238, 242, 244, 246, 248  
 mean, 44, 84, 162, 199–200, 238, 248  
 model, 70, 160, 165, 166, 175, 181, 193, 238, *see also* Dynamic system model
- Pseudoinverse, 31, 123, 138, 142
- Pseudonoise, 24, 27, 133, 135
- Q**
- Quadratic  
 cost, 18, 72, 141, 156–158, 257  
 generalized cost, 18, 72, 141, 156–158, 257  
 variation property, 177, 186, 247
- Quantization, 50, *see also* Discretization; Wordlength
- Quasi-moments, 239
- Quasi-static, 71, 73, 74, 102, 122, 123, 250, *see also* Stationary; Steady state; Time-invariant system model
- R**
- Random bias, 25, 51, *see also* Brownian motion
- Random constant, *see* Random bias
- Random process, *see* Stochastic process
- Random sequence, 70, 238, *see also* Discrete-time
- Random variable, *see* Random vector
- Random vector  
 conditional mean of, 213, 215–216, 248, 256  
 continuous, 71, 129, 168, 174, 175  
 correlated, 28, 40, 70, 120, 136, 137, 160, 162, 176, 181, 185, 256  
 covariance of, 28, 39, 70, 120, 136, 160, 162, 176, 181, 256  
 definition of, xiii

- discrete, 71, 130, 167, 171  
 expected value of function of, 185, 219  
 function of, 185, 218, 219  
 Gaussian, 39, 77, 132, 161, 164, 176, 241  
 independent, 44, 70  
 jointly Gaussian, 162  
 mean vector of, 72, 84, 162, 199, 238, *see also* Mean  
 normal, *see* Gaussian random vector  
 orthogonal, 239  
 realization, xiii, 36, 41, 130, 213, 214, 219  
 uncorrelated, 44, 70, *see also* Correlation  
 uniform, 65, 72, 130
- Random walk, 25, 51, *see also* Brownian motion
- Rate, convergence, 82
- Rate parameter, 163
- Realizable, 256, *see also* Causal
- Realization of random vector, xiii, 36, 41, 130, 213, 214, 219
- Reasonableness checking, 232
- Recertification, 173
- Reconstructability, *see also* Observability
- Recursive estimation, 37, 131, 213
- Reduced-order, 23, 25, *see also* Adequate model; Approximation; Simplified
- Reference trajectory, *see* Nominal
- Regularity conditions, 96
- Relinearizations, 42, 44
- Representation theorem, 251
- Reproducing kernel Hilbert space, 257
- Residual, 15, 18, 37, 69, 131, 133, 136, 144, 218, 232, 234, 256  
 monitoring, 69, 131, 133, 232  
 whitening, 136
- Riccati equation  
 algebraic, *see* Constant-gain  
 differential, 58  
 difference, 9  
 steady state solution, *see* Constant-gain
- Riemann integral, 160
- RMS, *see* Root mean square
- Robustness, 143, *see also* Divergence; Tuning
- Root mean square (rms), *see* Correlation; Standard deviation; Variance
- Roundoff errors, *see* Numerical precision
- S**
- Sample, stochastic process, xiii, *see also* Realization of random vector
- Sample frequency, *see* Sample time
- Sample instant, *see* Sample time
- Sample rate, *see* Sample time
- Sample space, xiii
- Sample statistics, 123, 128, *see also* Monte Carlo
- Sample period, *see* Sample time
- Sample time, measurement, 2, 40, 70, 160, 165, 213
- Sample value, *see* Realization
- Sampled data, 2, 40, 70, 160, 165, 213
- Saturation, 235, 243
- Scalar measurement updating, iterative, 56, 64, 116, *see also* Measurement update
- Schmidt epsilon technique, 31
- Schwartz inequality, 179, 180, 206, 207
- Score, 81, 82, 85, 87, 120
- Scoring, 81, 120
- Second order  
 density, 162, *see also* Markov process  
 filter, *see* Filter  
 statistics, *see* Correlation; Covariance
- Self-tuning, 69, 120, 129, 136, 141
- Semi-group property, 171
- Sensitivity, 56, 68, 71, 74, 98, 99, 143, *see also* Error  
 analysis, 48, 68, 71, 98, 99, 143, *see also* Parameter; Robustness  
 system, 85
- Sensor, 232, *see also* Measurement
- Sequence, *see also* Discrete-time control, 2, 10, 37, 86  
 Gaussian, 2, 70, 77, 132  
 Gauss-Markov, 2, 70, 77  
 innovations, *see* Residual  
 Markov, 2, 70, 167, 238  
 measurement, 2, 40, 70, 160, 213  
 noise, 2, 40, 70, 160, 165, 213, 238  
 residual, 15, 37, 69, 132, 136, 144, 218, 232  
 state vector, 2, 70, 77, 167, 238
- Set theoretic, 71, 72, 257
- Shaping filter, 54, 160, 166, 188, 190
- Shifted coordinates, *see* Perturbation; Taylor series
- Sifting property of Dirac delta, 185
- Sigma ( $\sigma$ ), *see* Standard deviation
- $\sigma$ -algebra, 176, 246, 251
- Similarity transformation, 114
- Simple function, 177, 178
- Simplified (controller, filter, model), 109, 121, 128, 216, *see also* Adequate model; Approximations; Model; Neglected terms; Reduced-order
- Simulation, 189, 210, *see also* Monte Carlo

- Singular, 8, 70, 123, *see also* Positive semidefinite  
 Singular value decomposition, 102  
 Smoothability, 11  
 Smoothing, 1, 58, 59, 257  
     continuous measurement, 18, 20  
     fixed-interval, 4, 5, 9  
     fixed-lag, 5, 16  
     fixed-point, 5, 15, 17  
     Fraser form, 5, 16  
     Meditch form, 14, 15  
     nonlinear, 18  
 Snyder's equation, 247, 248  
 Software programs for design, 48, 61 (Ref. 38), 97, 99, 190, 191, 210, *see also* Performance analysis; Tuning  
 Solution  
     deterministic differential equation, 182  
     nominal, 40  
     Riccati equation, *see* Riccati equation  
     stochastic differential equation, 161, 181–186  
 Space  
     Euclidean, 22  
     Hilbert, 157, 241, 257  
     reproducing kernel Hilbert space, 257  
     state, *see* State  
 Spectral analysis, 54, 63, 65, *see also* Eigenvalue  
 Spectral density, 54, 63, 65  
 Spectrum, *see* Eigenvalue; Spectral analysis  
 Spurious data, 232  
 Square root matrix, 1, 9, 21, 116, 192, 210–211, 227, *see also* Factorization  
 Stability  
     filter, 24  
     numerical, 24, 28, 102, 116, 121, *see also* Factorization  
     robustness, 143  
 Standard controllable form, 114  
 Standard deviation, 123, 128, *see also* Monte Carlo; Variance  
 Standard observable form, 114, 115  
 State  
     augmented, 160  
     concept, 167  
     controllable, 24, 114  
     difference equation, *see* Dynamic system model; Discrete-time  
     differential equation, *see* Dynamic system model; Continuous-time; Stochastic differential equations  
     equation, *see* Dynamic system model  
     equilibrium, *see* Equilibrium state; Nominal  
     error, *see* Error; Perturbation  
     estimate, *see* Filter  
     feedback, 213, 251  
     notation, xiii, xvi  
     observable, *see* Observability; Standard observable form  
     propagation, *see* Dynamic system model  
     space, 114, 115, 167  
     steady state, *see* Steady state  
     transition, 2, 15, 70, 74, 115, 161, 169, 190, 238  
     variables, choice, 114, 115  
     vector, *see* Dynamic system model; System model (linear); System model (nonlinear)  
 State space representations, 114, 115  
 State transition matrix, 2, 15, 70, 74, 115, 161, 190  
 State transition probabilities, 169  
 State transition probability matrix, 169  
 Static model, *see* Measurement; Random vector  
 Stationary, 54, 63, 65, 123, 136  
 Statistical approximation, 245  
 Statistical linearization, 243  
 Statistical tests for whiteness, 139, 142  
 Statistically linearized filter, 243  
 Statistics  
     estimates of, 120, 123, 128, 129, 136, 141,  
         *see also* Monte Carlo  
     first order, 162, *see also* First order density; Mean  
     partial description of density, 214, 215, 239  
     second order, 162, *see also* Correlation; Covariance; Monte Carlo  
     sufficient, 72  
 Steady state, *see also* Constant-gain error, *see* Error  
     filter, 14, 52, 57, 58, 112, 137, 250, 266, 267, 270  
     Riccati equation solution, 9, 58  
     gain, 57, 58, 137, 250, 266, 267, 270  
 Stochastic approximations, 58, 73, 257  
 Stochastic controllability, 24  
 Stochastic difference equations, *see also* Discrete-time  
     linear, 2, 70  
     nonlinear, 238  
 Stochastic differential, 39, 161, 175, 181

- Stochastic differential equations, *see also* Stochastic integral  
 linear, 41, 160–162, 165  
 nonlinear, 39, 165, 166, 181, 191, 199, 213
- Stochastic integral  
 Itô, 166, 175, 178, 180, 181, 189  
 Stratonovich, 163, 178, 188, 191–193  
 Wiener, 160, 161, 175, 180
- Stochastic model, *see* Dynamic system model; Measurement; Model; Stochastic process; System model (linear); System model (nonlinear)
- Stochastic observability, 24
- Stochastic process, *see also* Dynamic system model; Measurement; Model  
 admissible, 175, 179, 181  
 bias, 25, 51, 52, 72, 109, 138, 224  
 Brownian motion, 39, 51, 160, 175–176, 179–181  
 characterization, 161, 174, 202  
 continuous-time, *see* Continuous-time correlated, 53, 54, *see also* Correlation correlation kernel of, 39, 70, 137, 160, 162, 164, 256 correlation matrix of, 40, 70, 120, 137, 160, 162, 176, 181, 185 covariance kernel of, 39, 40, 70, 160, 162, 164 covariance matrix of, 39, 70, 84, 120, 160, 162, 176, 181, 199–201, 238 cross-correlation of, 28, 136, 185, 256 cross-covariance of, 28, 136, 185, 256, *see also* Covariance definition, xiii description of, 161, 174, 202 discrete-time, 2, 70, 167, 210, 238 exponentially time-correlated, 53, 139, 188, 205 first order Gauss-Markov, 53, 139, 188, 205, *see also* System model (linear) Gaussian, 161, 164, 176, *see also* Brownian motion; Gauss-Markov process model; White Gaussian noise Gauss–Markov, 53, 139, 160, 162, 188, 205, *see also* System model (linear) independent, 44, 53, 70 independent increment, 160, 163, 176, 183, 247, *see also* Brownian motion; Poisson process Markov, 136, 159, 162, 163, 166, 167, 176, 183, 192, 213 martingale, 176, 179, 209, 247, 251, 256 mean of, *see* Mean vector nonstationary, *see* Stationary normal, *see* Gaussian stochastic process Poisson, 163–165, 247–248 power spectral density, 54, 63, 65, *see also* Correlation; Fourier transform probability laws for, *see* Forward Kolmogorov equation random bias, 25, 51, *see also* Brownian motion random constant, *see* Random bias random walk, 25, 51, *see also* Brownian motion samples, xiii, *see also* Realization simulation, 189, 210, *see also* Monte Carlo stationary, 54, 63, 65, 123, 136 time-correlated, 54, *see also* Shaping filter uncorrelated, *see* White noise; Correlated noise white, *see* White noise white Gaussian, 39, 40, 70, 160, 164–166, 256 Wiener, *see* Brownian motion Storage, 17, 37–38, 74, 103, 111, 116 Stratonovich integral, 163, 178, 188, 191–193 Strength of white noise, 39, 57, 160, 181, 187, 213, 246 Structural model, *see* Dynamic system model; Measurement; Models Subintervals, integration, 45, 190, 210 Suboptimal filter, *see* Approximations; Reduced-order; Filter Sufficient statistic, 72 Superposition, *see* System model (linear) Symmetric, 116, 216 System dynamics matrix, 41, 160 System identification offline, 84, 153, 156 online, 69, 153, 156 simultaneously with state estimation, 48, 68, 74–119, 129–136 System model (linear), *see also* Stochastic model; Dynamic system model; Measurement; Models adequate, *see* Adequate model constant coefficient, *see* Time-invariant system model continuous-time, 39, 41, 160, 161, 165, *see also* Continuous-time measurement discrete-time, 70 dynamics equation, 2, 39, 41, 70, 160, 161  
 165

- System model (linear) (*con't.*)**
- equivalent discrete-time, 2, 70, 210, 211
  - frequency domain, 54, 63, 65, *see also* Stationary; Time-invariant system model
  - matrices, *see* Matrix
  - measurement equation, 2, 70, 160, 163
  - noise, 70, 160, *see also* Dynamics noise; Measurement noise; White Gaussian pseudonoise, *see* Pseudonoise; Tuning state, 70, 160, *see also* State
  - time domain, 70, 160, *see also* State space
  - time-invariant, *see* Time-invariant system model
  - time-varying, *see* Time-varying system
  - transfer function, 54, 63, 65, *see also* Time-invariant system model
  - uncertainty, 70, 160, *see also* Dynamics noise; Measurement noise; Compensation for model inadequacies
- System model (nonlinear), *see also* Stochastic model; Dynamic system model; Measurement; Models**
- adequate, *see* Adequate model
  - constant coefficient, *see* Time-invariant system model
  - continuous-time, 39, 165, 166, 181, 191, 199, 213
  - discrete-time, 238
  - dynamics equation, 39, 165, 166, 181, 191, 199, 213, 238
  - measurement equation, 40, 57, 165, 213, 246, 251
  - noise, 39, 40, 165, 166, 213, 238, 246, 251
  - pseudonoise, *see* Pseudonoise; Tuning state, 39, 165, 166, 181, 191, 199, 213, 238, *see also* State
  - time domain, *see* System model (nonlinear), state
  - time-invariant, *see* Time-invariant system model
  - time-varying, *see* Time-varying system
  - uncertainty, 39, 40, 165, 166, 213, 238, 246, 251, *see also* Dynamics noise; Measurement noise; Compensation for model inadequacies
- Terminal time, 8**
- Time average, 103, 141**
- Time-correlated noise models, *see* Shaping filter**
- Time domain analysis, 160, *see also* State; System model (linear); System model (nonlinear)**
- Time history, 2, 83, 160, 165, 213, 215, 246**
- Time-invariant system model, 54, 57, 58, 71, 115, 136, 170**
- Time propagation, *see* Propagation**
- Time series analysis, 120, 123, 129, 133, 136, 137, 139, 141, *see also* Adaptive; Residual monitoring**
- Time-varying system, *see* Models; System model (linear); System model (nonlinear)**
- Trace, xiv, 181**
- Tracking error, 54–56, 109**
- Tradeoff, *see* Adequate model; Computer; Performance analysis; Tuning**
- Transfer function, 54, 63, 65, *see also* Time-invariant system model**
- Transformation, 22, 114, 180, 256**
- matrix, 114
  - similarity, 114
- Transient, 12, 93, 103, 104, 112, 113, 135**
- Transition density, 162, 174, 183, 192, 213**
- Transition matrix, 2, 15, 70, 74, 115, 161, 169, 190**
- Transition probability, 162, 168, 174, 183, 192, 213**
- Transition probability diagram, 170**
- Transpose, xiv**
- Triangular matrix, 21, 27, 192, 210–211, *see also* Factorization**
- True errors, *see* Monte Carlo; Performance analysis; Truth model**
- Truncated second order filter, 217, 218, 220, 225, 249**
- Truncation errors, 9, 28, 45, 115, 121, 123, *see also* Factorization; Numerical precision; Wordlength**
- Truth model, 25, 48, 98, 228, *see also* Monte Carlo; Performance analysis**
- Tuning, 23, 24, 27, 48, 52, 69, 133–135, 224, 225, 228**

**T**

- Taylor series, 40, 177, 186, 195, 216, 217, 220, 235, 248**
- Terminal condition, 8, 9**

**U**

- U–D covariance factorization, 27, 211, *see also* Factorization**

Unbiased estimate, 72, 96  
 Unbounded variation, 176  
 Uncertain parameters, 48, 68–144, *see also* Parameter  
 Uncertainties, model for, *see* Models; Noise; Random vector; Shaping filter; Stochastic process; System model (linear); System model (nonlinear)  
 Uncontrollable, 24  
 Uncorrelated random vectors, 44, 70, *see also* Correlation  
 Uncorrelated stochastic processes, 44, 70, *see also* Correlation  
 Uniformly distributed, 65, 72, 130  
 Unimodal, 47, 216, 241  
 Uniqueness, 71, 97, 128, 182  
 Unitary matrix, 27, 211  
 Unknown but bounded, 71, 72, 257  
 Unmodelled effects, *see* Neglected terms  
 Unobservable, 24  
 Unreconstructible, *see* Unobservable  
 Unstable, *see* Stability  
 Update, filter, *see* Measurement  
 Upper triangular matrix, 27, 211, *see also* Factorization

**V**

Variable, random, *see* Random variable  
 Variance, *see also* Covariance  
   cross-variance, 28, 136, 256, *see also* Covariance  
   estimator of, 120, 123, 129, 136, 141, *see also* Monte Carlo  
 Variation, first, *see* Linear perturbation  
 Variation, unbounded, 176  
 Vector  
   control input ( $\mathbf{u}$ ), 2, 10, 37, 39, 86, 160, 161, 213, 251

dynamics noise ( $\mathbf{w}$ ), 2, 39, 70, 120, 160, 213  
 Itô differential, 39, 181  
 measurement ( $\mathbf{z}$ ), 2, 40, 57, 70, 160, 163, 165, 213, 246, 251  
 measurement noise ( $\mathbf{v}$ ), 2, 40, 70, 120, 160, 165, 213, 246  
 process, *see* Stochastic process  
 random, *see* Random vector  
 residual ( $\mathbf{r}$ ), *see* Residual  
 state ( $\mathbf{x}$ ), *see* Dynamic system model;  
   State; System model (linear); System model (nonlinear)

**W**

Weight, probability, 72, 109, 131, *see also* Filter gain; Weighted least squares  
 Weighted least squares, 18, 72, 156–158, 257  
 Weighting matrix, *see* Weighted least squares  
 White Gaussian noise, 39, 40, 70, 160, 164–166, 256  
 White noise  
   continuous-time, 39, 160, 164–166, 256  
   discrete-time, 40, 70, 160, 165, 256  
 Whiteness tests, 139, 142  
 Whitening of residuals, 136  
 Wiener integral, 160, 161, 175, 180  
 Wiener process, *see* Brownian motion  
 Wiener stochastic integral, 160, 161, 175, 180  
 With probability one, *see* Convergence, with probability one  
 Wordlength, 1, 9, 27, 65, 115, 116, *see also* Factorization; Numerical precision

**Z**

Zero-mean noise, 70, *see also* Dynamics noise; Measurement noise