HOMEWORK 4: QUANTITATIVE MACROECONOMICS

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I. Asimple wealth model.

This part basically introduces us the utility specifications we need to work it, the parametrization, assumptions about income process, etc.

However there is also an small question inside that I want to answer briefly. It asks us about what value of relative risk aversion parameter σ leads the CRRA utility representation to be represented by log-utility. This value is 1. I prove it below:

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1 - \sigma} = \frac{0}{0}$$

which is an indetermination. However as a consequence of applying l'Hopital rule, note that

$$\lim_{\sigma \to 1} \frac{c^{1-\sigma} - 1}{1 - \sigma} = \lim_{\sigma \to 1} \frac{c^{1-\sigma} ln(c)}{1} = \frac{c^0 ln(c)}{1} = ln(c)$$

This proves the statement.

II. Solving the ABHI model.

I will proceed as follows. For each item I will be solving parallelly for both utlity specifications (quadratic and CRRA).

II.1. The recursive formulation. Households problem is:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \sum_{y^r} \beta^t \pi_t(y_t^r) U(c_t(y_t^r)) \tag{1}$$

subject to:

$$a_{t+1}(y_t^r) \geq -y_{min} \sum_{s=0}^{T-(t+1)} (1+r)^{-s} \tag{2}$$

where condition above provides a lower bound for borrowing. Concretely states that assets choice for next period cannot be lower than natural debt limit. Equivalently this means that for any realization of income shock, household must be able to return it's debt. This prevents household to run in Ponzi schemes.

$$c_t(y^r) + a_{t+1}(y_t^r) = y_t(y_t^r) + (1+r)a_t(y_{t-1}^r) \tag{3}$$

If as usual we use budget constraint in 3 to substitute for consumption in 1 we can set the following recursive formulation of the problem where household just explicitly chooses a'.

$$V(a, y) = \max_{a'} \{U(y + (1+r)a - a') + \beta \sum_{y'} \pi_{y'|y} V(a', y')\} \tag{4}$$

s.t.

$$a' \geq -y_{min} \sum_{s=0}^{T-(t+1)} (1+r)^{-s}$$

The foc w.r.t. a' is:

$$U'(c) = \beta \sum_{y'} \pi_{y'|y} V_{a'}(a', y') \tag{5}$$

Applying envelope theorem:

$$V_a(a, y) = U'(c)(1+r) \rightarrow V_{a'}(a', y') = U'(c')(1+r) \tag{6}$$

So that we have the following Euler equation:

$$U'(y + (1+r)a - a') = \beta \pi_{y'|y} U'(y' + (1+r)a' - a'')(1+r) \tag{7}$$

Applying our concrete utility functions (quadratic and CRRA) we've:

- Quadratic utility:

$$\bar{c} - (y + (1+r)a - a') = \beta \pi_{y'|y} [\bar{c} + y' + (1+r)a' - a''](1+r) \tag{8}$$

- CRRA utility:

$$(y + (1+r)a - a')^{-\sigma} = \beta \pi_{y'|y} [y' + (1+r)a' - a'']^{-\sigma}(1+r) \tag{9}$$

II.2. The infinity lived households economy.

You can find the code corresponding to this part in the file:
Codefile1_II.2_and_II.3_quadratic_And_crra.py. Specifically you can find it in the third and fifth sections of that code

The parametrization used is $\rho = 0.06$, $r = 0.4$, $w = 1$, $\gamma = 0.65$, $\sigma_y = 0.28$, $\bar{c} = 100 \cdot y_{max}$ (only for quadratic case), and $\sigma = 2$ (only for CRRA). I only did it with discrete VFI.

An important thing to do is to obtain the cartesian product $ay = Y \cdot A \cdot A'$ which gives the sigma algebra oof all the possible combinations of assets today and tomorrow depending on the realization (r) of the income shock.
Then VFI algorithm can be applied in this environment to obtain Value functions and policies for both income realizations. I present the results below for the two utility specifications that concern us:

Quadratic utility:

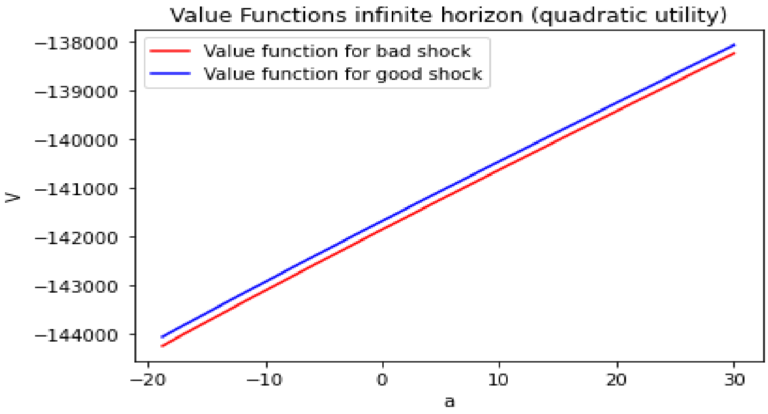


Figure 1:

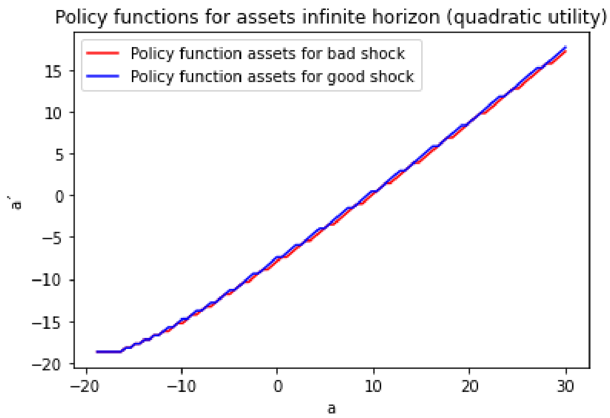


Figure 2:

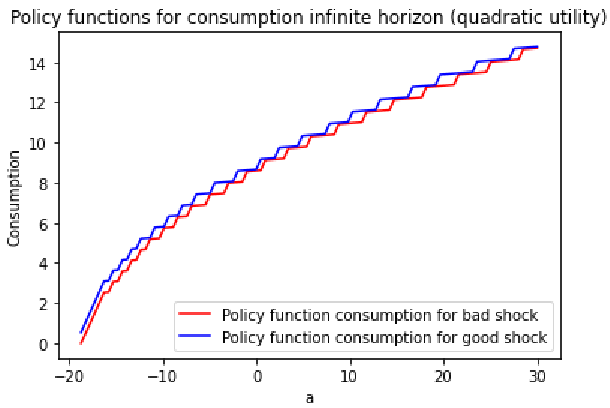


Figure 3:

It can be seen how both value, optimal consumption decision and assets accumulation decisions follow the same pattern, but always lead to slightly higher levels of value, consumption and assets for the positive realization of the shock. This could seem logical, but we will see below that under CRRA utility this will no longer be always the case, in concrete for assets.

CRRA utility:

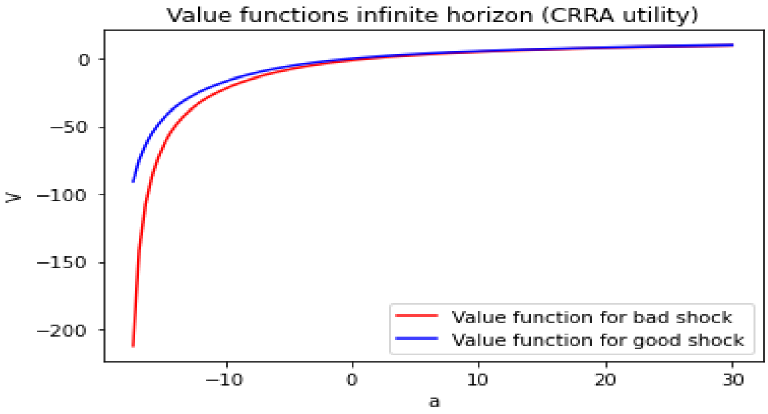


Figure 4:

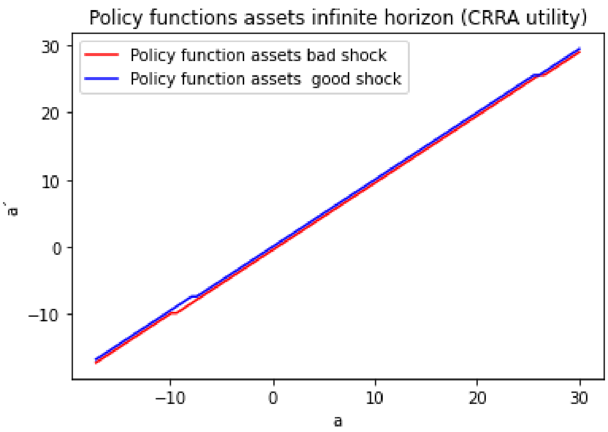


Figure 5:

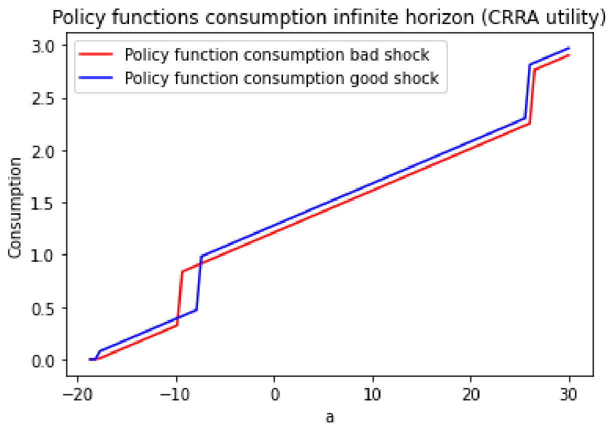


Figure 6:

We can see that with CRRA, in some ranges of values, the optimal asset accumulation for tomorrow (Figure 6) is lower for the positive shock than for the negative. This is due to the role of precautionary savings, which under CRRA utility take an important role (because of the relative risk aversion coefficient σ), whereas this is not the case under quadratic utility. Also the value function is much more hump-shaped with CRRA than under quadratic utility.

II.3. The life-cycle economy.

The code corresponding to this part can also be found in the file:
$Codefile1\_II.2\_and\_II.3\_quadratic\_And\_crra.py$. Specifically you can find it in the second and fourth sections of that code

The parametrization used is the same as in the infinite horizon case, i.e.: $\rho = 0.06$, $r = 0.4$, $w = 1$, $\gamma = 0.65$, $\sigma_y = 0.28$, $\bar{c} = 100 \cdot y_{max}$ (only for quadratic case), and $\sigma = 2$ (only for CRRA).

Here what I do is to solve recursively backward, taking into account the usual terminal condition in finite horizon problems ($V_{T+1} = 0$ for all income state). Below I present my results for both utility specifications that concern us:

<u>Quadratic utility:</u>

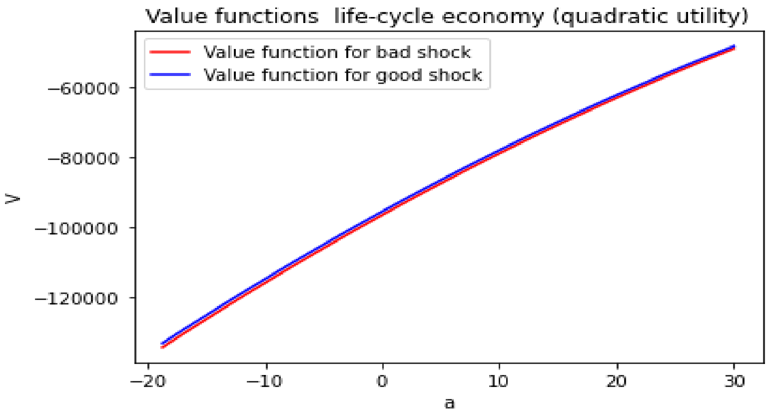


Figure 7:

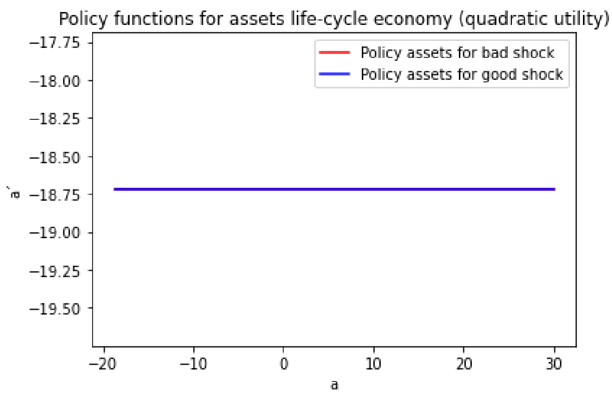


Figure 8:

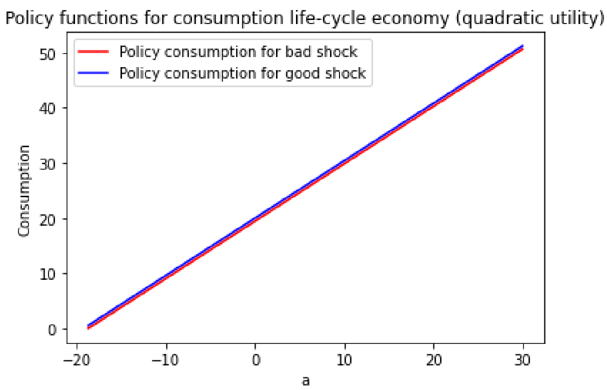


Figure 9:

It is surprising that the optimal choice for assets is inelastic for all y and a. This forecasts a flat asset accumulation profile along the life-cycle. Value and consumption decisions are always slightly higher in the positive shock case.

<u>CRRA utility:</u>

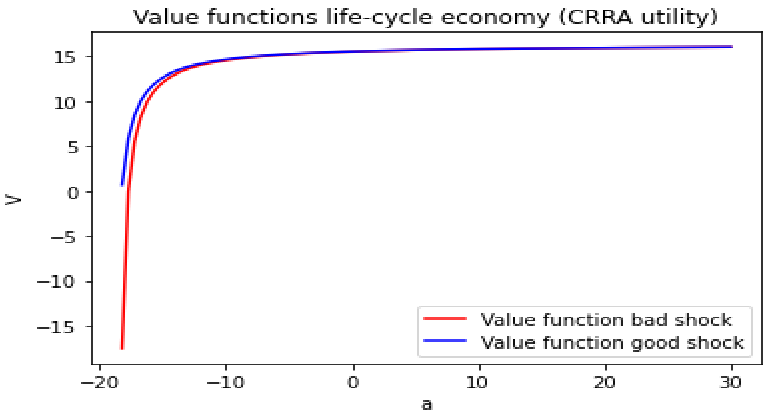


Figure 10:

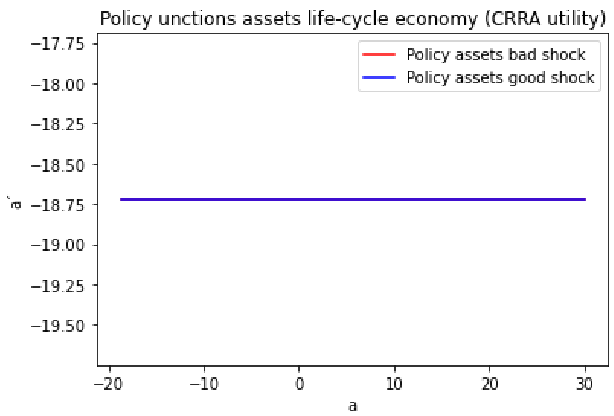


Figure 11:

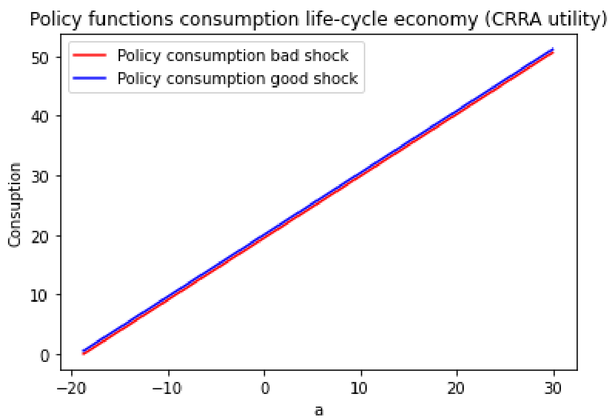


Figure 12:

Although again the value function is more hump-shaped under CRRA utility than under quadratic utility, the shapes of policies do not change too much in this life-cycle economy from quadratic to CRRA preferences.

II.4. Partial Equilibrium.

II.4.1. With certainty.

You can find the code corresponding to this part in the file:
Codefile2_II.4.1_quadratic_and_crra.py.

The parametrization used is $\rho = 0.06$, $r = 0.4$, $w = 1$, $\gamma = 0$ (certainty), $\sigma_y = 0$ (certainty), $\bar{c} = 100$ (only for quadratic case), and $\sigma = 2$ (only for CRRA).

II.4.1.1 Infinity lived households economy.

Here I proceed basically as in part **II.2**, but now since there is no uncertainty, there is, obviously, no difference between the results for the good and bad shock in terms of consumption policy function. After that we are also asked to use the obtained policy functions to draw consumption time profiles. Below I present all this results for both quadratic and CRRA utilities and briefly comment them:

Quadratic utility:

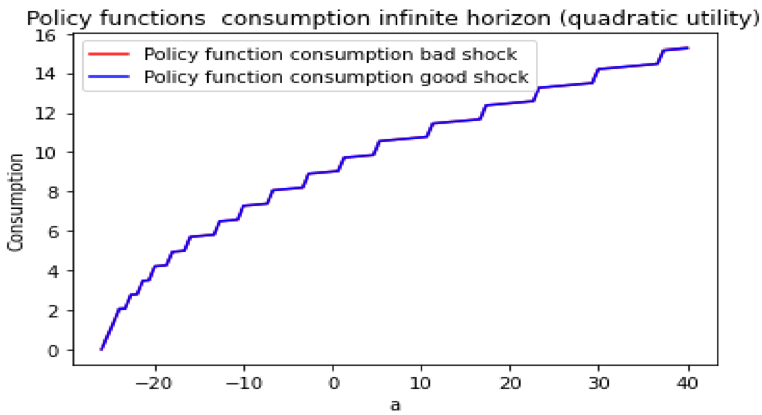


Figure 13:

As commented above, now there is no difference between good and bad shock, basically because under certainty no shocks take place (obviously).

Next page see the simulated paths.

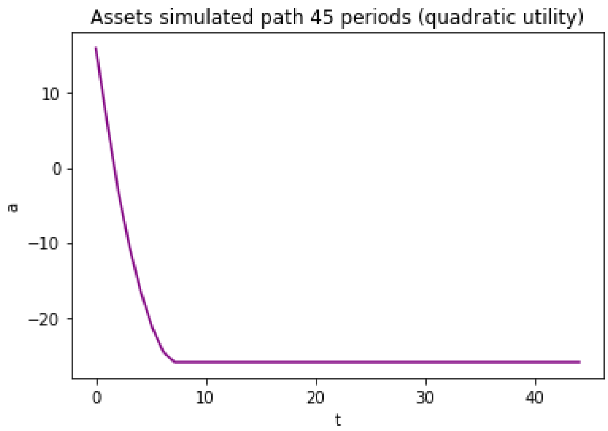


Figure 14:

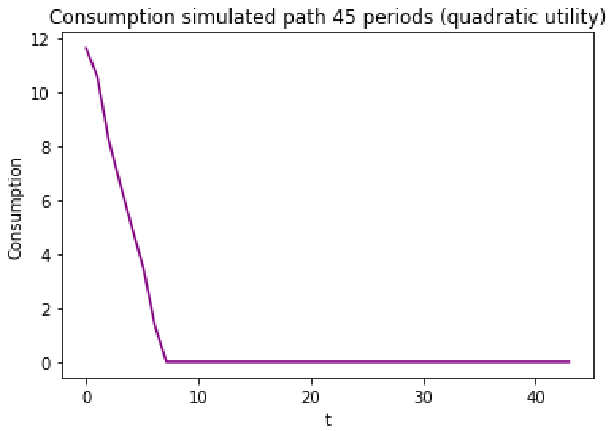


Figure 15:

What is basically observed in this paths is that since the initial assets are high (there is an important initial amount of debt), consumer drastically starts reducing consumption until reaching the 0 lower bound, in order to return it's debt rapidly.

<u>CRRA utility:</u>

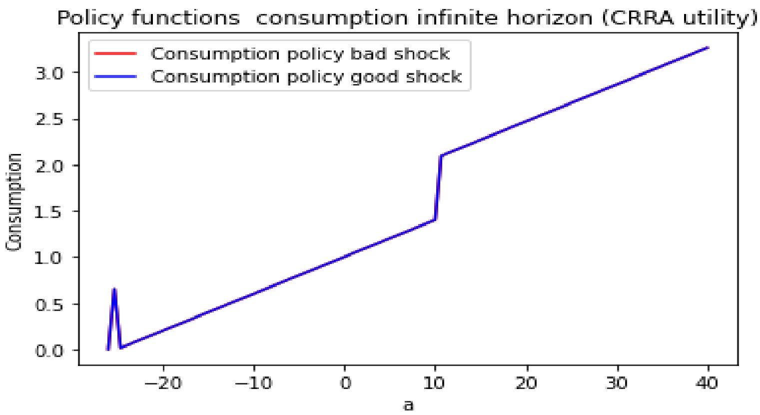


Figure 16:

As commented above, now there is no difference between good and bad shock, basically because under certainty no shocks take place (obviously).

Next page see the simulated paths.

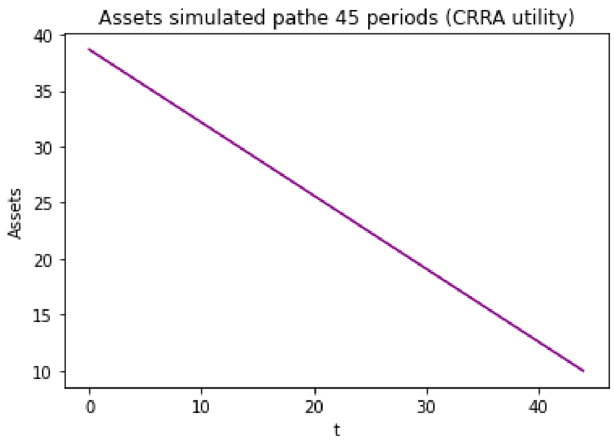


Figure 17:

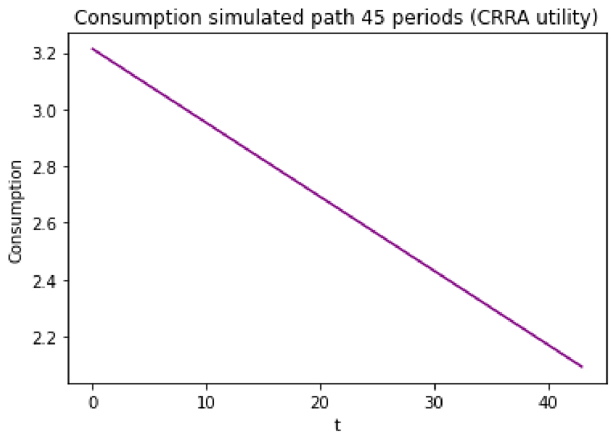


Figure 18:

The simulated profiles are very different from those we found under quadratic preferences. Here households progressively reduce consumption in order to progressively return debt. The path is much more balanced than in the quadratic case.

II.4.1.2 The life-cycle economy.

Notice that now, since we are in finite-horizon, each period has associated a particular value function and policies that already are the optimal ones. This is the case because when we have finite time and can solve backward, since we start the backward iteration with an optimal object ($V_{t+1} = 0$), we are able to solve for an entire backward sequence of **for optimal objects**, i.e. optimal value functions. This is an important difference (and advantatge) with respect the infinite-horizon case, because here (in finite horizon), what we do at each iteration is to find the optimal value function associated with each period, whereas in infinite horizon, since time doesn't have too much meaning (because there is no terminal period T), we use contraction mapping theorem and principle of optimally to approximate a general optimal value function. Remembering this, and solving backward along the 45 periods as in part **II.3.**, but now storing the value function and the policies at each iteration (which recall, are already optimal objects for that period), I can then pick from my stored elements, the policies for consumption at t=5 and at t=40 . Using this procedure for both the quadratic and CRRA utility case gives the results that I show in the next page:

Quadratic utility:

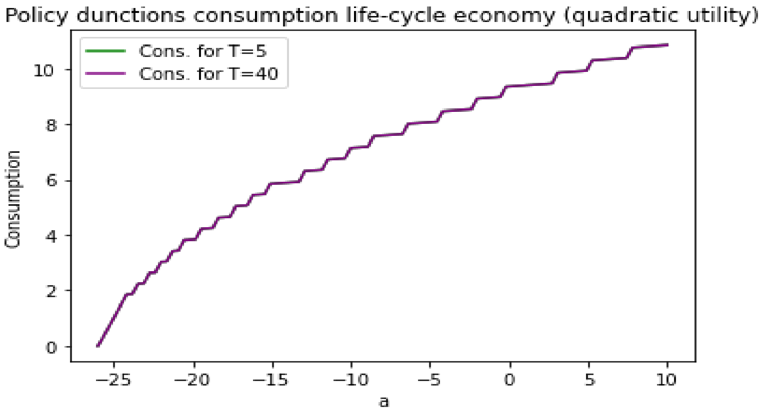


Figure 19:

In the quadratic case the optimal choices remain invariant across both periods (5 and 40)

CRRA utility:

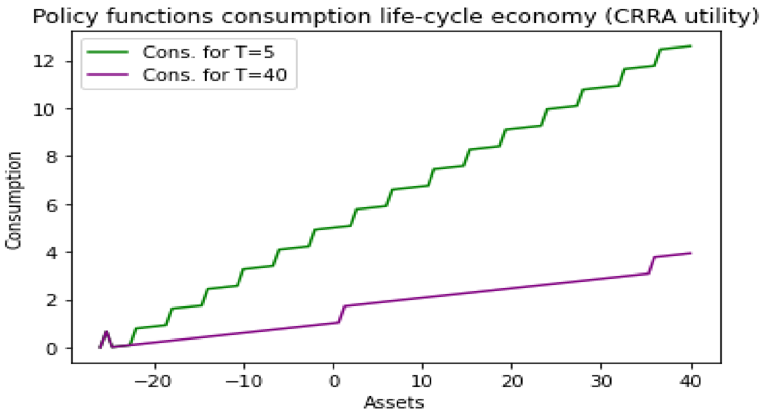


Figure 20:

However, under CRRA, we observe different optimal choice profiles for t=5 and t=40. For the same amount of assets, opstimal consumption is higher at t=5 than at t=40. This differences become higher as assets increase.

II.4.2. With uncertainty.

You can find the code corresponding to this part in the file:
Codefile3_II.4.2_quadratic_and_crra.py.

The parametrization used (initially, then we will change parameters to observe variations) is $\rho = 0.06$, $r = 0.4$, $w = 1$, $\gamma = 0$, $\sigma_y = 0.1$, $\bar{c} = 100$ (only for quadratic case), and $\sigma = 2$ (only for CRRA).

II.4.2.1. Consumption policies

Infinite horizon

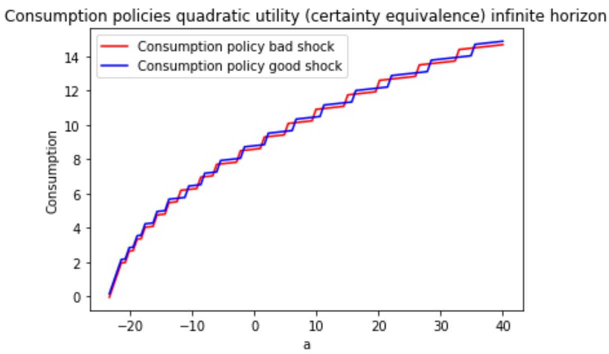


Figure 21:

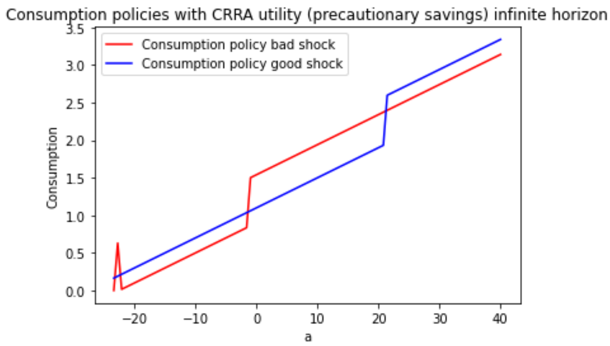


Figure 22:

Finite horizon

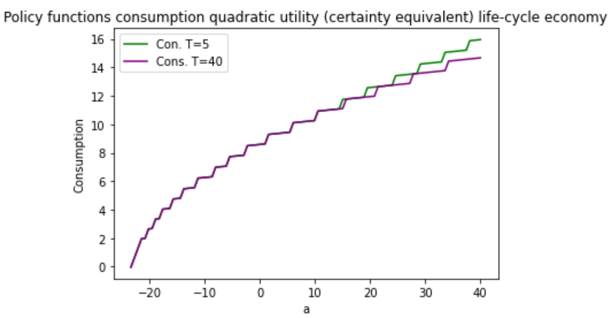


Figure 23:

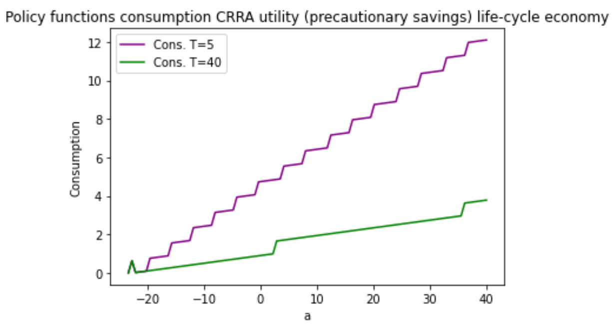


Figure 24:

Regarding the questions in the statement of exercise, the differences between both utility preferences (in ammount of consumption, i.e. see the vertical axe) are higher in the infinite horizon case. This happens because since under CRRA the household is more prudent (i.e. under the lack of insurance markets accumulates precautionary savings (often more than what would be Pareto optimal)), a longer period of prudence (i.e. $T = \infty$) makes more

differences than a shorter period of prudence (T=45).
In infinite horizon, the basic comparison with certainty case is that now we observe differences between the good and bad shock. And in finite horizon, is that for quadratic utility (Figure 23) now we observe different policies depending on period whereas this did not happen with certainty (Figure 19); whereas for CRRA (surprisingly) there are almost no differences in finite horizon.

II.4.2.2. Simulated time paths

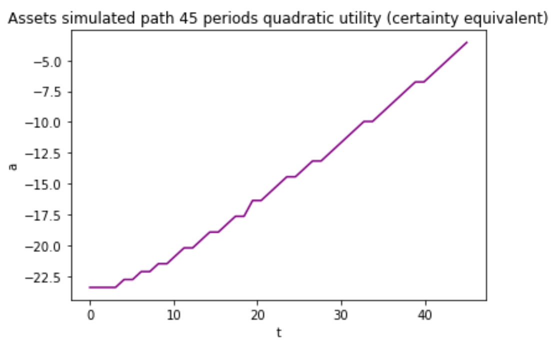


Figure 25:

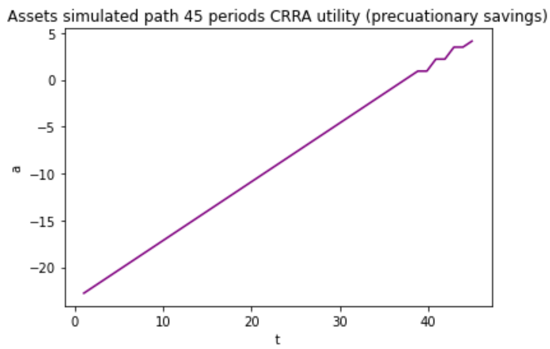


Figure 26:

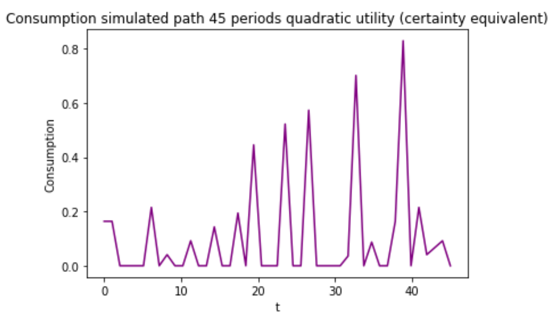


Figure 27:

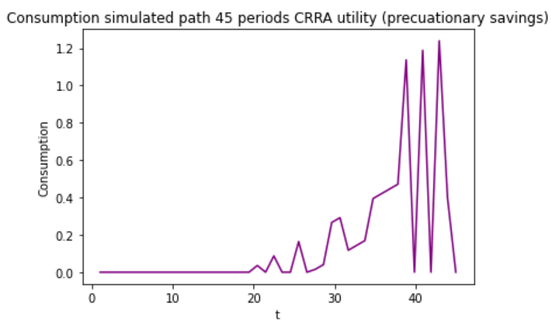


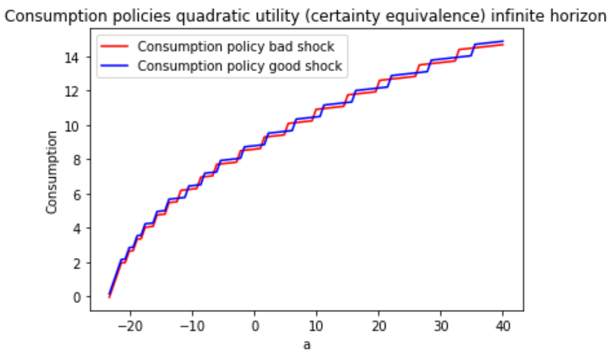
Figure 28:

The main difference is that in the precautionary savings economy, the excessive precautionary savings make consumption to be 0 in the first half of life-cycle, whereas in the second half, as terminal period T=45 approaches, household starts to spend all this excessive savings in consuming. Contrarily, in the certainty equivalence economy we observe a consumption profile more distributed along life. Regarding the assets path, for both cases takes a similar shape.

II.4.2.3. Increase prudence (i.e, increase relative risk aversion coefficient σ)

<u>For $\sigma = 5$:</u>

Here I replicate my figures from **II.4.2.1** and **II.4.2.2** but for $\sigma = 5$



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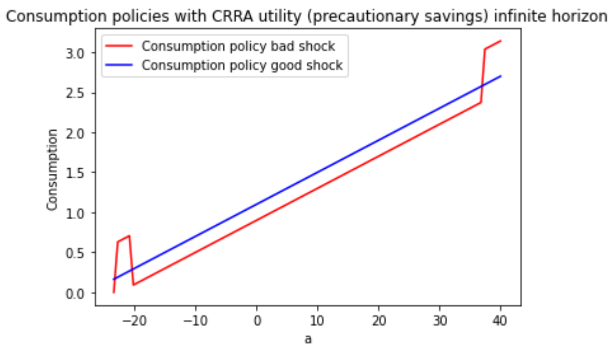

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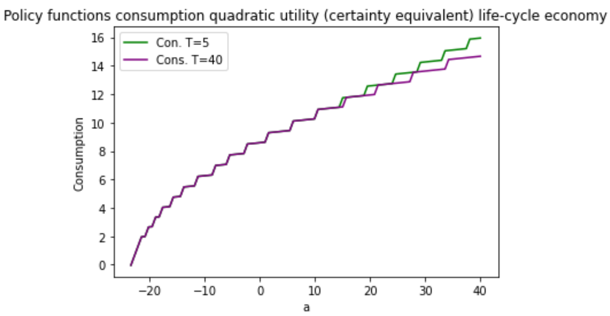

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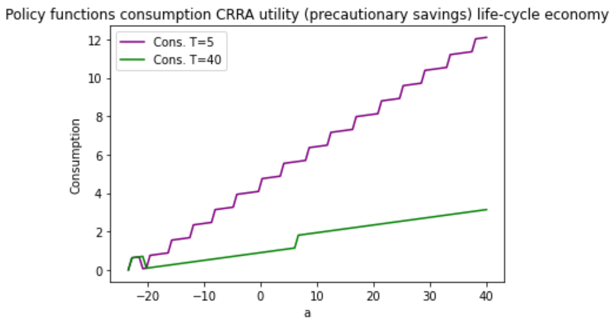


Figure 32:

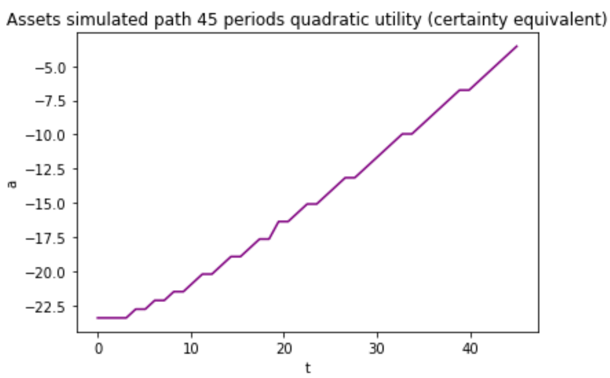
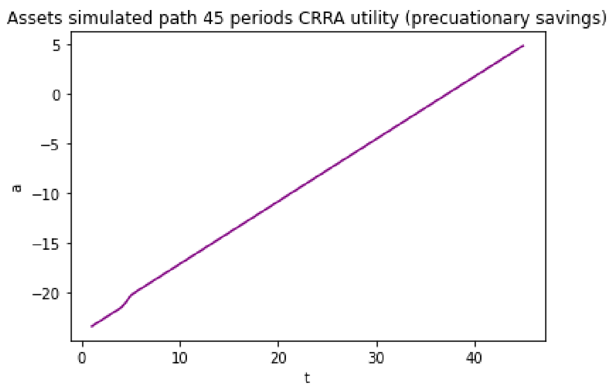


Figure 33:



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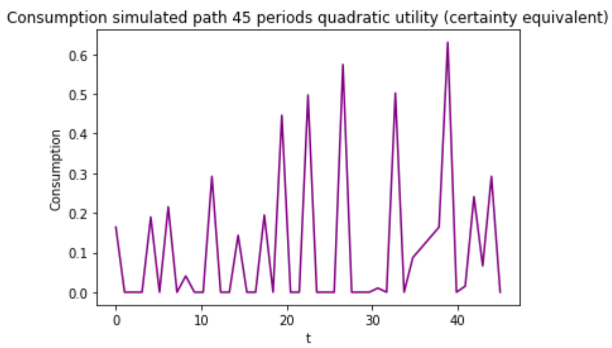


Figure 35:

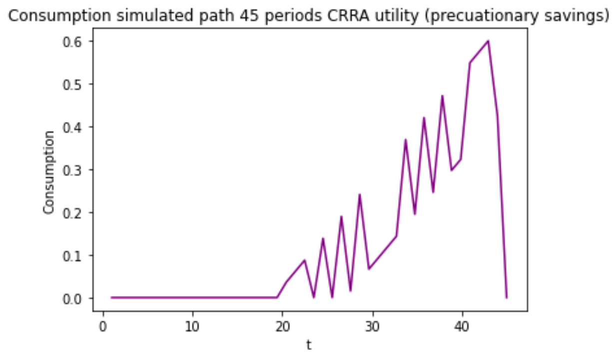


Figure 36:

We can see as an increase in the coefficient of risk aversion σ leads to a higher amount of precautionary savings (due to the lack of insurance markets) that leads also to lower levels of consumption. The shapes of the different curves keep more or less equal but the levels decrease. Probably some public taxation and transfer system such that taxes when good shock takes place and tranfers when bad shock takes place would be pareto improving in the case of CRRA preferences.

<u>For $\sigma = 20$:</u>

Here I replicate my figures from **II.4.2.1** and **II.4.2.2** but for $\sigma = 20$

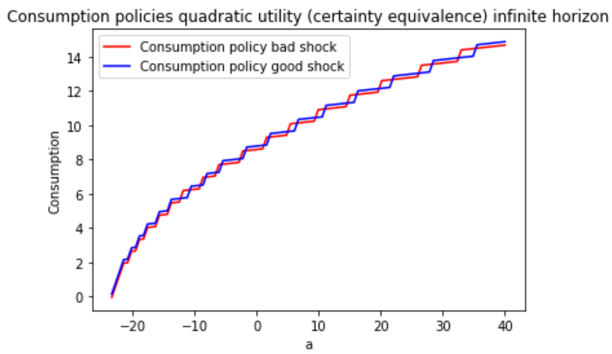


Figure 37:

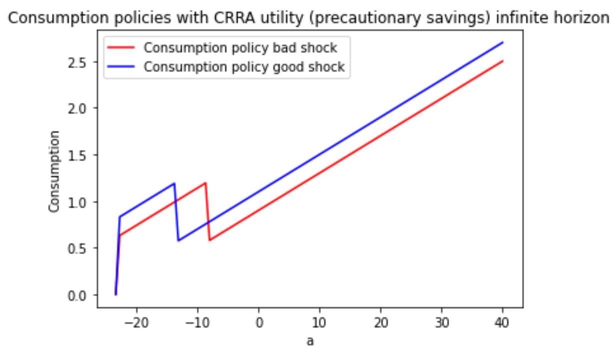


Figure 38:

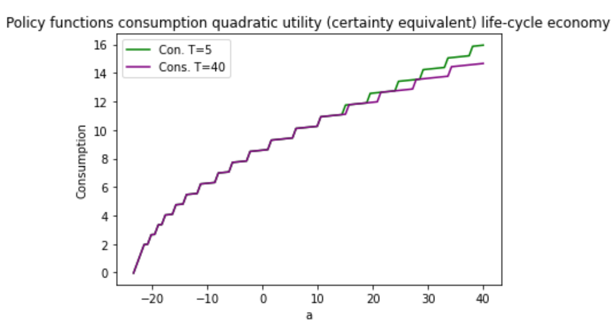


Figure 39:

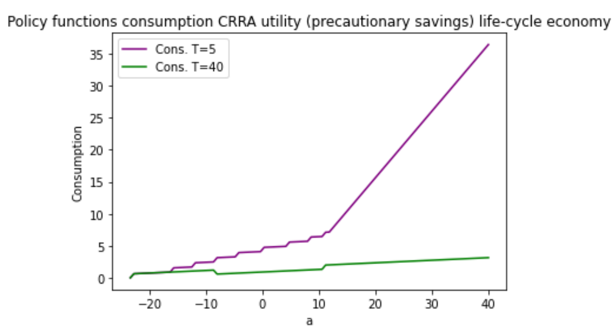
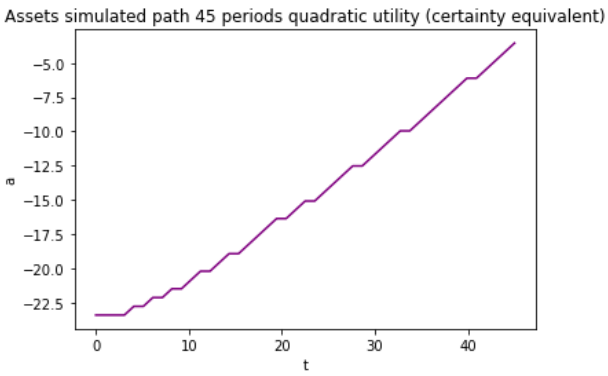


Figure 40:



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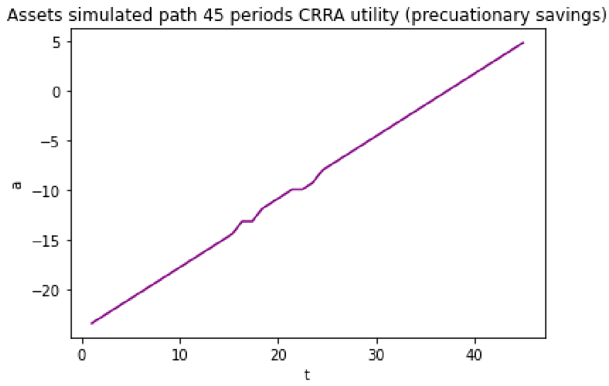
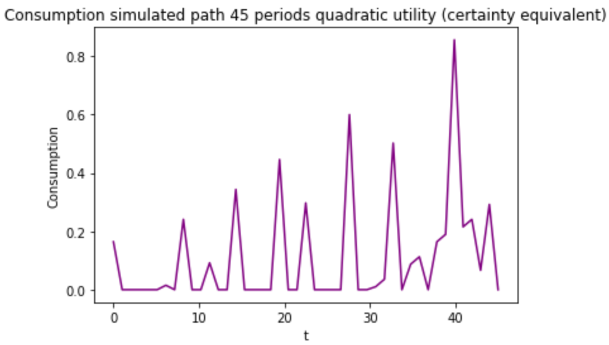


Figure 42:



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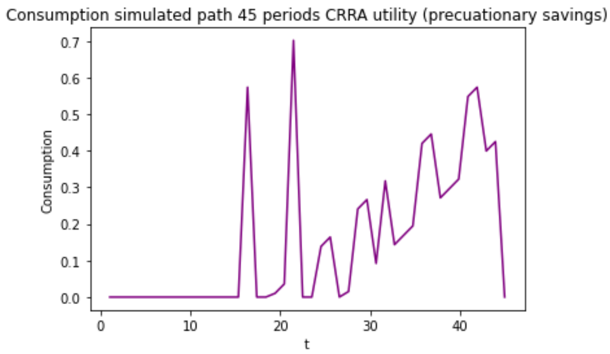

Figure 44:

The changes are not o dramatic as expected for this huge change in σ.

II.4.2.4. Increase income shock variance (σ_y)

Here I replicate my figures from **II.4.2.1** and **II.4.2.2** but for $\sigma_y = 0.5$. Regarding to σ, which we previously changed, I return it to its original value of $\sigma = 2$

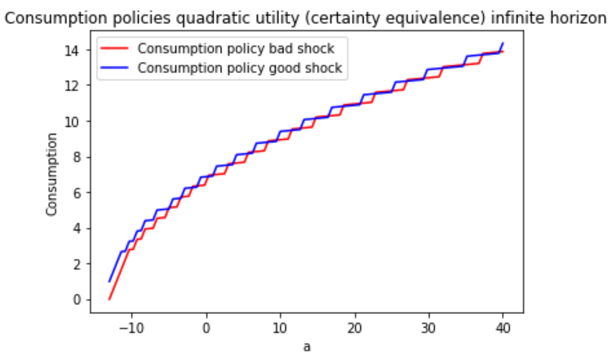

Figure 45:

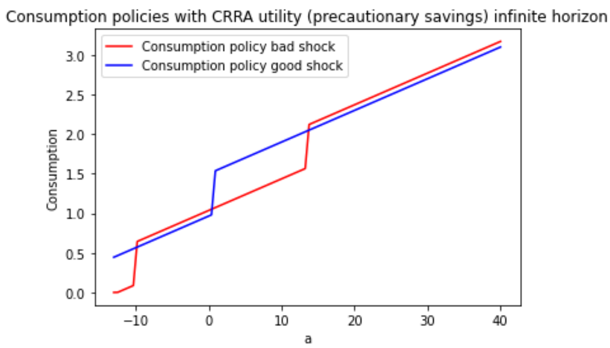

Figure 46:

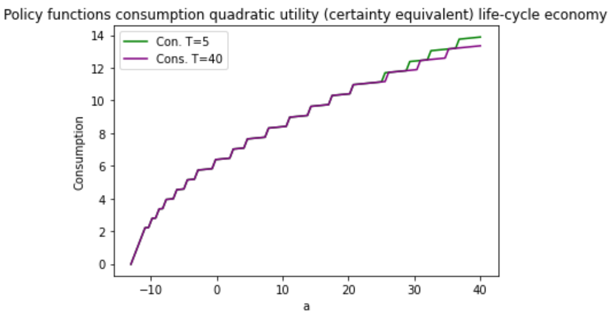

Figure 47:

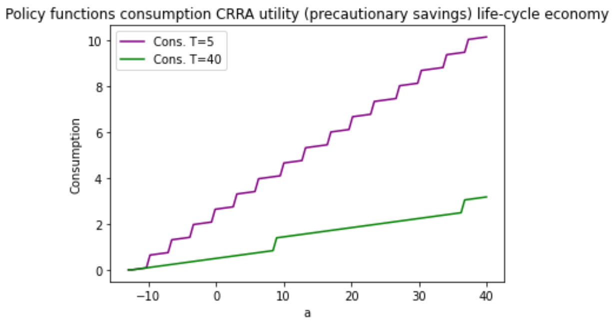

Figure 48:

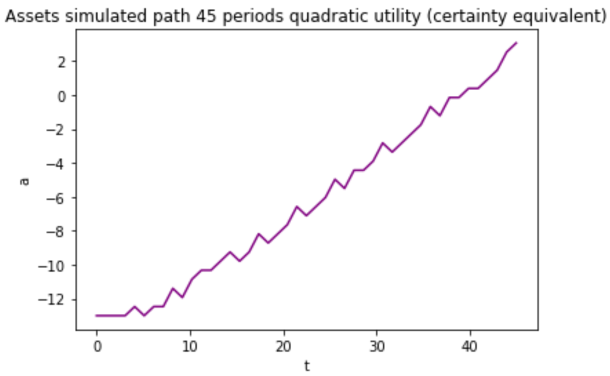

Figure 49:

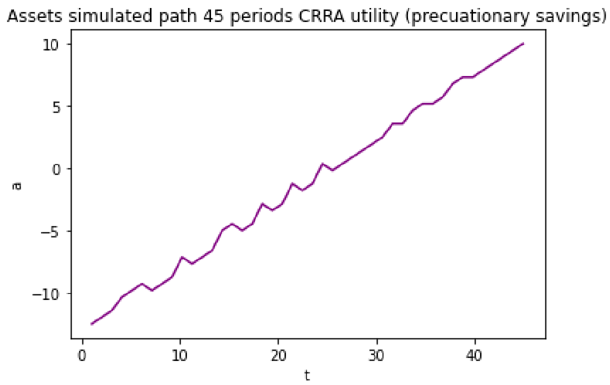

Figure 50:

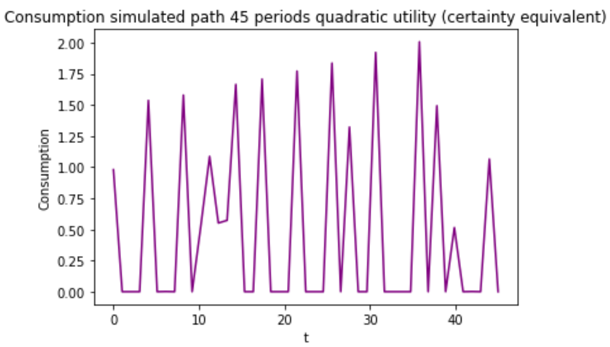

Figure 51:

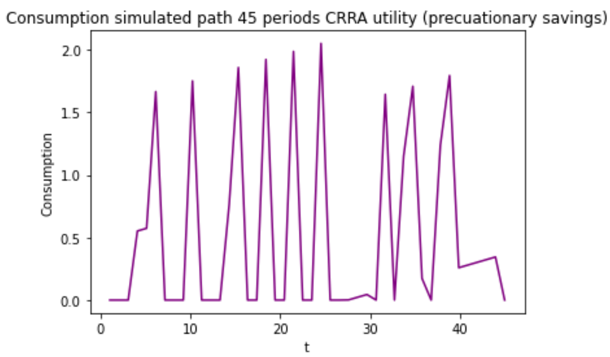

Figure 52:

What happens to the consumption policy is that in Figure 45 consumption optimal choice under positive shock is for all a, higher or equal than in the bad shock, which was not the case in Figures 21, 29 and 37. This is clearly due to the increase of the variance of the shock, which makes the positive shock more positive and the negative shock more negative. Regarding the assets and consumption simulated profiles, we observe much more volatile profiles than in previous cases.

II.4.2.5. Increase persistence of income shocks γ

Here I replicate my figures from **II.4.2.1** and **II.4.2.2** but for $\gamma = 0.95$. As said in the statement I keep $\sigma_y = 0.5$

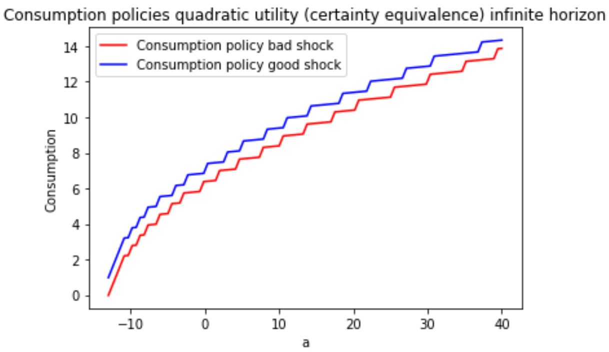

Figure 53:

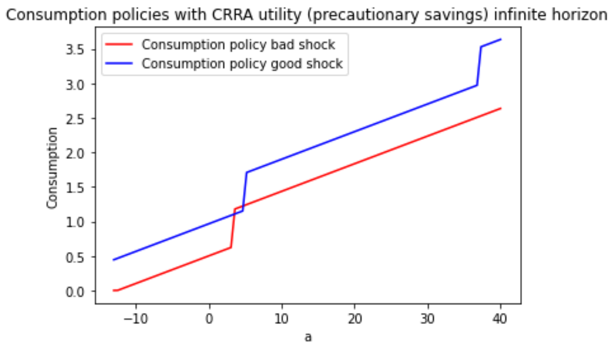

Figure 54:

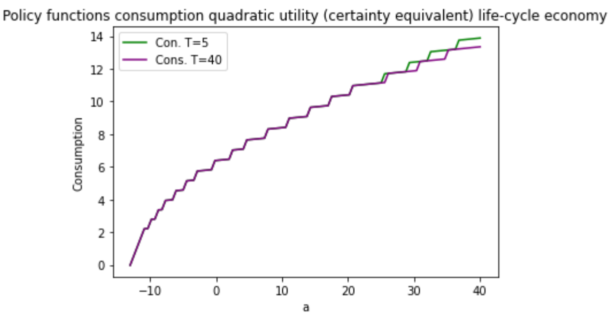

Figure 55:

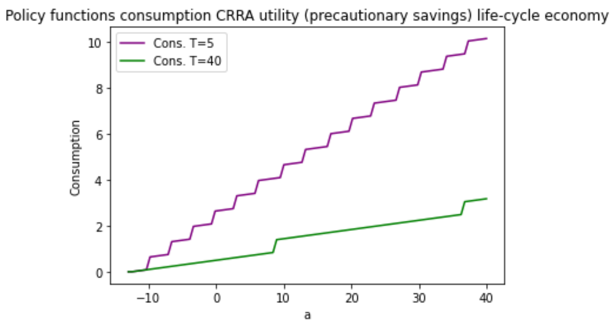

Figure 56:

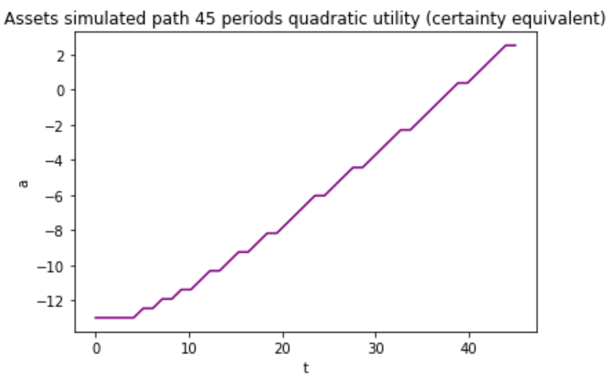

Figure 57:

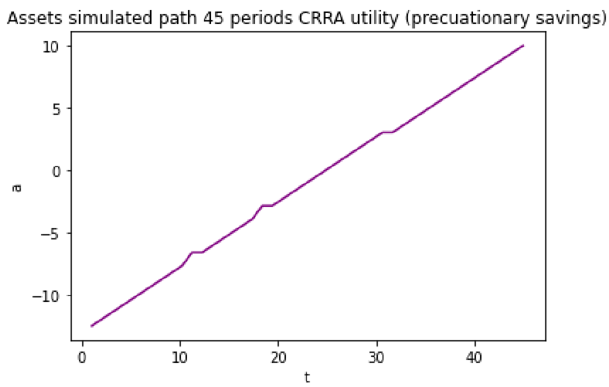

Figure 58:

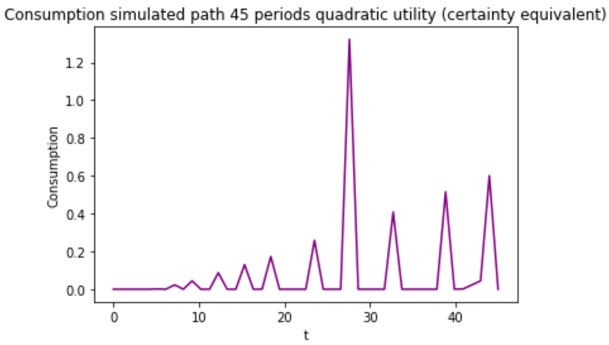

Figure 59:

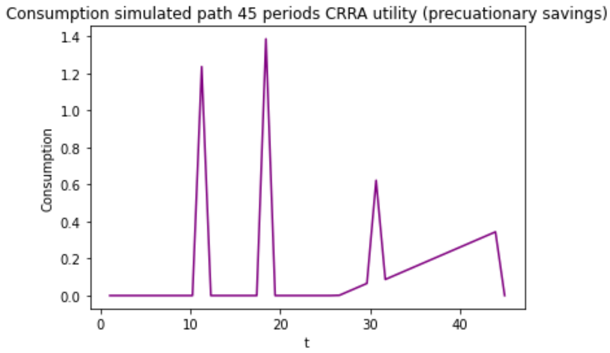

Figure 60:

What basically can be observed is. First: From Figures 53 and 54, we observe that the differences in optimal choices for the same level of a between the good and the bad shock increase (in favour of the good one). In the profiles (Figures 59 and 60) I observe volatility to be more concentrated in time, due to the increased persistency of the negative shock, which notice that also implies a higher persitency of the positive shock.

II.5. General Equilibrium.

II.5.1. The simple ABHI model.

You can find the code for this part in: $Codefile4\_II.5.1\_simple\_ABHI.py$. For this section the code is borrowed from Quant Econ website. It can be found in `https://python.quantecon.org/aiyagari.html`. I just did little modifications. The parametrization is: $\rho = 0.06$, $\sigma_y = 0.5$, $\alpha = 0.33$, $\delta = 0.05$, $\sigma = 2$ and the following transition matrix:

$$\Pi = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

See below the supply and demand curves in the capital market, and the stationary distribution of assets. In the latter, it can be observed that the majority of the population is in a poverty situation in the stationary equilibrium.

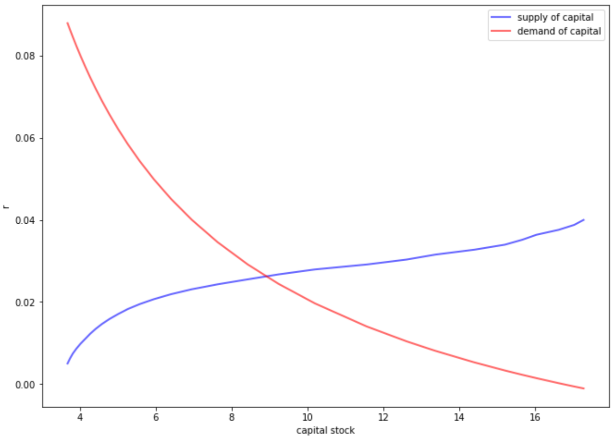

Figure 61:

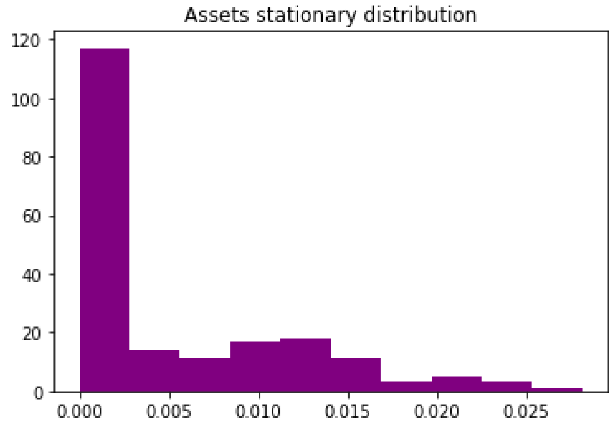

Figure 62:

The Gini coefficient that I obtain is 0.67, whereas the obtained in Krueger, Mitman and Perri (2016) is 0.77.

II.5.1.Solve Ayagari (1994)

You can find the code for this part in: $Codefile5\_II.5.2\_solveAyagari(1994).py$. For this section the code is borrowed from Quant Econ website. It can be found in `https://python.quantecon.org/aiyagari.html`. I just did little modifications. The parametrization is the one in Ayagari (1994): $\rho = 0.04$, $\sigma_y = 0.5$, $\alpha = 0.36$, $\sigma = 2$ $\delta = 0.08$, and the following transition matrix:

$$\Pi = \begin{bmatrix} 0.5 & 0.1 & 0.2 & 0.045 & 0.105 & 0.015 & 0.035 \\ 0.05 & 0.5 & 0.25 & 0.05 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.2 & 0.3 & 0.2 & 0.15 & 0.05 & 0.05 \\ 0.1 & 0.1 & 0.1 & 0.4 & 0.1 & 0.1 & 0.1 \\ 0.05 & 0.05 & 0.05 & 0.05 & 0.6 & 0.1 & 0.1 \\ 0.1 & 0.05 & 0.05 & 0.1 & 0.2 & 0.3 & 0.2 \\ 0.06 & 0.06 & 0.07 & 0.1 & 0.1 & 0.1 & 0.51 \end{bmatrix}$$

With this settings, I get the capital supply and demand curves, and the stationary distribution of assets that you can se below. Comparing Figure 64 with Figure 62, it can be seen that under this Ayagari setting, the economy presents more equality. It can also be seen in the Gini coefficient which now is 0.44, sensibly smaller than in the case above.

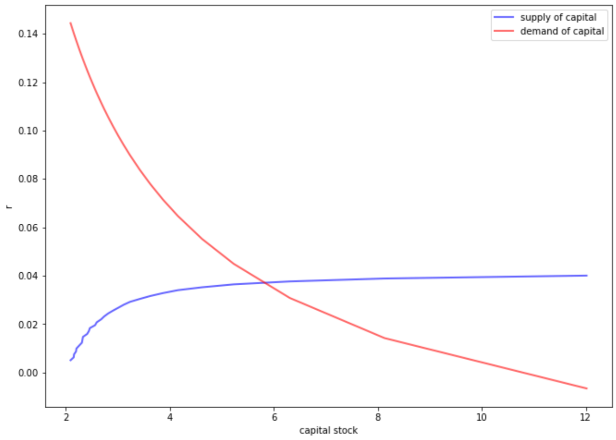

Figure 63:

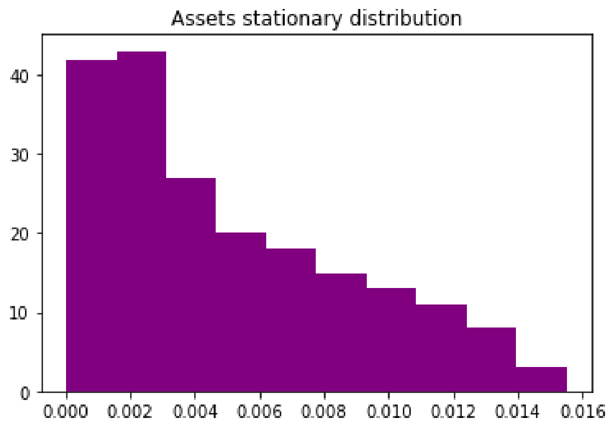

Figure 64: