hw4

Original language:

$$
\begin{aligned}
t &::= \lambda x : T.\, t \mid (t\ t) \mid x \\
v &::= \lambda x : T.\, t \\
T &::= T \to T
\end{aligned}
$$

1. New language:

$$
\begin{aligned}
t &::= \lambda x : T.\, t \mid (t\ t) \mid x \mid \text{True} \mid \text{False} \mid \text{if } t_1\ t_2\ t_3 \mid \text{AND } t_1\ t_2 \\
v &::= \lambda x : T.\, t \mid \text{True} \mid \text{False} \\
T &::= T \to T \mid \text{Bool}
\end{aligned}
$$

New rules:
This one just says True and False are of type Bools

$$
\frac{}{\Gamma \vdash \text{True} : \text{Bool}} \quad \frac{}{\Gamma \vdash \text{False} : \text{Bool}}
$$

This is basic typing

$$
\frac{(x \mapsto Bool) \in \Gamma}{\Gamma \vdash x : T} TBool
$$

These are for evaluating (IFApp 1, 2, 3)

$$
\frac{t_1 \to t_1'}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \to \text{if } t_1'\ t_2\ t_3} \quad \frac{t_2 \to t_2'}{\text{if } v_1 \text{ then } t_2 \text{ else } t_3 \to \text{if } v_1\ t_2'\ t_3}
$$

$$
\frac{t_3 \to t_3'}{\text{if } v_1\ v_2\ t_3 \to \text{if } v_1\ v_2\ t_3'}
$$

The IFtrue and IFfalse are trivial, not going to list them

This one is for actually type checking

$$
\frac{\Gamma \vdash v_1 : \text{Bool} \quad \Gamma \vdash v_2 : T \quad \Gamma \vdash v_3 : T}{\Gamma \vdash \text{if } v_1\ v_2\ v_3 : T} TIf
$$

Not going to write down the eval rules for AND, pretend they exist (very similar) with the name ANDApp 1, 2
Type checking:

$$
\frac{\Gamma \vdash v_1 : Bool \quad \Gamma \vdash v_2 : Bool}{\Gamma \vdash AND\ v_1\ v_2 : Bool} TBool
$$

2. Progress
   Case 1:
   if $\Gamma \vdash t : Bool$ then t is a value ELSE IF $\Gamma \vdash t : T$ then t is a value ELSE $t \to t'$ for some t'

Case 2:

$$\Gamma \vdash \text{if } t_1\ t_2\ t_3 : T$$

Assume progress for t_1 and t_2 and t_3
If t_1 or t_2 or t_3 are not values then IFApp 1 2 and 3 will apply (respectively)
If they are all values and t1 is a bool and t2 == t3 then TIf applies, otherwise we are in a broken state

Case 3:

$$\Gamma \vdash AND\ t_1\ t_2 : Bool$$

Assume progress for t_1 and t_2
If t_1 or $t_2$are not values then ANDApp 1 and 2will apply (respectively)
If they are all values then TBool applies

Preservation

Permutation
Case 1: x
$\Gamma \vdash x : Bool \Rightarrow (x \mapsto Bool) \in \Gamma$ by type inversion. By def permutation, $(x \mapsto Bool) \in \Delta$. $\Delta \vdash x : Bool$ by TBool
Case 2: $\Gamma \vdash \text{if } t_1\ t_2\ t_3 : T$
$\Gamma \vdash \text{if } t_1\ t_2\ t_3 : T \Rightarrow (t_1 \mapsto Bool) \in \Gamma, (t_2 \mapsto T) \in \Gamma, (t_3 \mapsto T) \in \Gamma$ by type inversion. Induction hyp says permutation holds and $\Delta \vdash t_1 : Bool$ and $\Delta \vdash t_2 : T$ and $\Delta \vdash t_3 : T$ Thus $\Delta \vdash (\text{if } t_1\ t_2\ t_3) : T$ holds by TIf
Case 3: $\Gamma \vdash AND\ t_1\ t_2$
$\Gamma \vdash AND\ t_1\ t_2 : Bool \Rightarrow (t_1 \mapsto Bool) \in \Gamma, (t_2 \mapsto Bool) \in \Gamma$ by type inversion. Induc hyp says permutation holds and $\Delta \vdash t_1 : Bool$ and $\Delta \vdash t_2 : Bool$ Thus $\Delta \vdash (AND\ t_1\ t_2) : Bool$ Holds by TAnd

Weakening
Case 1: x
$\Gamma \vdash x : Bool \Rightarrow (x \mapsto Bool) \in \Gamma$ by type inversion. With the assumption $x \neq y, (x \mapsto Bool) \in (y \mapsto T), \Gamma$ by def lookup. Thus $(y \mapsto T), \Gamma \vdash x : Bool$ by applying TBool
Case 2: $\Gamma \vdash \text{if } t_1\ t_2\ t_3 : T$
$\Gamma \vdash \text{if } t_1\ t_2\ t_3 : T \Rightarrow (t_1 \mapsto Bool) \in \Gamma, (t_2 \mapsto T) \in \Gamma, (t_3 \mapsto T) \in \Gamma$ by type inversion. Induction hyp says $(y \mapsto S), \Gamma \vdash t_1 : Bool$, $(y \mapsto S), \Gamma \vdash t_2 : T$, and $(y \mapsto S), \Gamma \vdash t_3 : T$ hold thus $(y \mapsto S), \Gamma \vdash (if\ t_1\ t_2\ t_3) : T$ holds by TIf

Case 3: $\Gamma \vdash AND\ t_1\ t_2$
$\Gamma \vdash AND\ t_1\ t_2 : Bool \Rightarrow (t_1 \mapsto Bool) \in \Gamma, (t_2 \mapsto Bool) \in \Gamma$ by type inversion. Induc hyp says $(y \mapsto S), \Gamma \vdash t_1 : Bool$ and $(y \mapsto S), \Gamma \vdash t_2 : Bool$ hold Thus $(y \mapsto S), \Gamma \vdash (AND\ t_1\ t_2) : Bool$ Holds by TAnd

Subst
Case 1: TBool
$t = z \Rightarrow z : Bool$
$(z \mapsto Bool) \in (x \mapsto S), \Gamma$: Type inverting t: Bool and replacing t with z
z = x: $[x \mapsto s]x = s$ Vars are unique. Know Bool = S : s : Bool. If you sub s in for x, you get s. s : Bool so preservation holds
$z \neq x\ [x \mapsto s]z = z$
Case 2: Tif
$t = (if\ t_1\ t_2\ t_3)$
$(x \mapsto X), \Gamma \vdash t : T$
Type inversion
$(x \mapsto S), \Gamma \vdash t_1 : Bool$
$(x \mapsto S), \Gamma \vdash t_2 : T$
$(x \mapsto S), \Gamma \vdash t_3 : T$
Induction hyp
$\Gamma \vdash [x \mapsto s]t_1 : Bool$
$\Gamma \vdash [x \mapsto s]t_2 : T$
$\Gamma \vdash [x \mapsto s]t_3 : T$
By Tif: $\Gamma \vdash [x \mapsto s](if\ t_1\ t_2\ t_3) : T$ so type pres over subst
Case 3: TBool
$t = (AND\ t_1\ t_2)$
$(x \mapsto X), \Gamma \vdash t : Bool$
Type inversion
$(x \mapsto S), \Gamma \vdash t_1 : Bool$
$(x \mapsto S), \Gamma \vdash t_2 : Bool$
Induction hyp
$\Gamma \vdash [x \mapsto s]t_1 : Bool$
$\Gamma \vdash [x \mapsto s]t_2 : Bool$
By TBool: $\Gamma \vdash [x \mapsto s](AND\ t_1\ t_2) : Bool$ so type pres over subst

Proof:
Case 1: x
true and false are vals no t' st t -> t'
Case 2: $\Gamma \vdash \text{if}\ t_1\ t_2\ t_3 : T$
Type inversion
$t_1 : Bool$

$t_2 : T$
$t_3 : T$
5 eval ruls: IFApp 1, 2, 3, and IfTrue/IfFalse
TypeInf cases:
$t_1 : Bool$ (type inv), $t_1 : T_1 \rightarrow t_1' : Bool$ (ind hyp), $t_2 : T$, $t_3 : T (both type inv) \vdash (if\ t_1'\ t_2\ t_3) : T$
Essentially either t1 is bool or it evals to bool in either case if will eval to T
$v_1 : Bool$ (type inv), $t_2 : T \rightarrow t_2' : T$ (ind hyp), $t_2 : T$, $t_3 : T (both type inv) \vdash (if\ v_1\ t_2'\ t_3) : T$
So in this case t1 is now a value v1 and t2 is either T or it evals to T in either case if will eval to T
$v_1 : Bool$ (type inv), $t_3 : T \rightarrow t_3' : T$ (ind hyp), $v_2 : T$, $t_3 : T (both type inv) \vdash (if\ v_1\ v_2\ t_3') : T$
Same thing but with t3 instead
Since we proved subst, subst will hold as well
Preservation holds
Case 3: $\Gamma \vdash AND\ t_1\ t_2$
Type inversion
$t_1 : Bool$
$t_2 : Bool$
4 eval rules (F/F, T/F, T/T, F/T) and 2 to eval both sides
Type inf cases:
$t_1 : Bool$ (type inv), $t_1 : T_1 \rightarrow t_1' : Bool$ (ind hyp), $t_2 : Bool$ (type inv) $\vdash (AND\ t_1'\ t_2) : Bool$
Essentially either t1 is bool or it evals to bool in either case AND will eval to Bool
$v_1 : Bool$ (type inv), $t_2 : T_1 \rightarrow t_2' : Bool$ (ind hyp), $t_2 : Bool$ (type inv) $\vdash (AND\ v_1\ t_2') : Bool$
Essentially either t2 is bool or it evals to bool in either case AND will eval to Bool
Since we proved subst, subst will hold as well
Preservation holds

Done yippee

3. Let

$$\frac{}{\text{Let } x = v_1 \text{ in } t_2 \rightarrow [x \mapsto v_1]t_2} \; \text{ELetV}$$

\quad

\frac{t_1 \rightarrow t_1'}{\text{Let } x = t_1 \text{ in } t_2 \rightarrow \text{Let } x = t_1' \text{ in } t_2} ; \text{ELet}

$$Typerule$$

\frac{\Gamma \vdash t_1 : T_1 \quad (x \mapsto T_1), \Gamma \vdash t_2 : T_2}{\Gamma \vdash \text{Let } x = t_1 \text{ in } t_2 : T_2} ; \text{TLet}

4.$Proof Progress: \$\Gamma \vdash \text{Let } x = t_1 \text{ in } t_2 : T_2\$ Assume progress for t1 and t2 t1 not a val then ELet applies t1$