Pricing a Perpetual American Put Option

Arnav Kondagunta

Mentor: Daniela Elizondo
Spring 2025 Directed Reading Program
Rutgers University

May 12, 2025

Perpetual American Put Option

An *American put option* is a contract that grants the holder the right, but not the obligation to sell an underlying asset at a predetermined price, known as the strike price, on or before a specific expiration date.

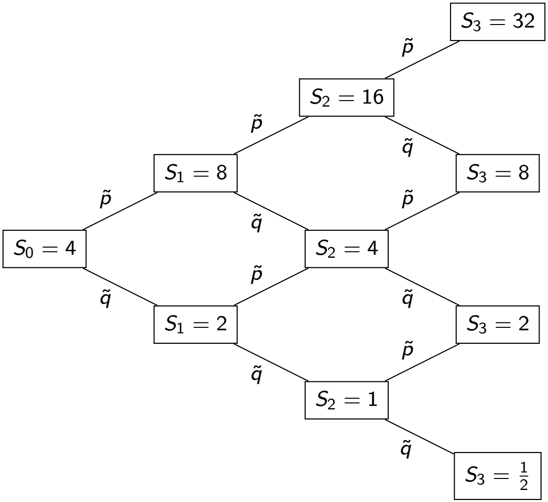
The payoff of a put option for strike price K and at time n is given by

$$(K - S_n)^+ = \max\{K - S_n, 0\}$$

Perpetual refers to the fact that this put has no expiration date.

Given a perpetual American put option with strike price $K = 4$, our goal is to find the value function $v(S)$ and the optimal stopping rule.

Multi-Period Binomial Pricing Model



- Simulates price paths of stock over lifetime of derivative
- $S_{n+1}(H) = 2 \cdot S_n$ and
  $S_{n+1}(T) = \frac{1}{2} \cdot S_n$
- Random walk starting at S_0
- $\tilde{p} = \tilde{q} = \frac{1}{2}$
- Interest rate $r = \frac{1}{4}$

Adapted Stochastic Processes and Martingales

A *stochastic process* is a collection of random variables indexed by some set.

Definition: Martingale

A stochastic process $M = \{M_n\}_n$ is a martingale if it is adapted, integrable, and satisfies

$$M_n = \mathbb{E}_n[M_{n+1}]$$

If $M_n \geq \mathbb{E}_n[M_{n+1}]$, we say this process is a supermartingale.
If $M_n \leq \mathbb{E}_n[M_{n+1}]$, we say this process is a submartingale.

Stopping Times

Definition: Stopping Time

In an N-period binomial model, a stopping time is a random variable τ that takes on values $0, 1, ..., N$ or ∞ and satisfies the condition that if $\tau(\omega_1\omega_2...\omega_n\omega_{n+1}...\omega_N) = n$, then $\tau(\omega_1\omega_2...\omega_n\omega'_{n+1}...\omega'_N) = n$.

A *first passage time* is the time it takes for a stochastic process to reach a state or value for the first time.
Let m be a fixed integer and let τ_m denote the first time a random walk reaches level m.

$$\tau_m = \min\{n : M_n = m\}$$

Optional Sampling Theorem

Theorem: Optional Sampling Theorem

A martingale stopped at a stopping time is a martingale. A supermartingale (or submartingale) stopped at a stopping time is a supermartingale (or submartingale). Furthermore, let τ be a stopping time. Then,

- If M_n is a martingale, then $\mathbb{E}[M_{\tau \wedge n}] = \mathbb{E}[M_n]$
- If M_n is a supermartingale, then $\mathbb{E}[M_{\tau \wedge n}] \geq \mathbb{E}[M_n]$
- If M_n is a submartingale, then $\mathbb{E}[M_{\tau \wedge n}] \leq \mathbb{E}[M_n]$

Testing for a Given Initial Stock Price

Suppose $S_0 = 4$.
Let τ_{-m} be the first time the stock reaches $S_n = 4 \cdot 2^{-m}$. Then,

$$\text{Payoff} = (K - S_{\tau_{-m}})^+ = 4 - 4 \cdot 2^{-m} = 4(1 - 2^{-m})$$

So, the value of the put option at stopping time τ_{-m} is the discounted payoff

$$v(\tau_{-m}) = \tilde{\mathbb{E}}\left[\left(\frac{1}{1+r}\right)^{\tau_{-m}} (K - S_{\tau_{-m}})\right] = 4(1 - 2^{-m}) \cdot \tilde{\mathbb{E}}\left[\left(\frac{4}{5}\right)^{\tau_{-m}}\right]$$

Simplifying, we have

$$v(\tau_{-m}) = 4(1 - 2^{-m}) \cdot \left(\frac{1}{2}\right)^{m}$$

Try Candidate Stopping Times

Try specific values:

- $m = 1 : v = 4(1 - \frac{1}{2}) \cdot \frac{1}{2} = 1$
- $m = 2 : v = 4(1 - \frac{1}{4}) \cdot \frac{1}{4} = 0.75$
- $m = 3 : v = 4(1 - \frac{1}{8}) \cdot \frac{1}{8} = 0.4375$

From these results, we can guess that the optimal stopping policy is at $m = 1$, or exercise when the stock hits 2.

Define the Value Function

Generalizing this observation for $S = 2^j$, we have

$$v(S) = \begin{cases} 4 - S & \text{if } S \leq 2 \text{ (exercise immediately)} \\ 2 \cdot (\frac{1}{2})^{j-1} & \text{if } S = 2^j, j \geq 2 \end{cases}$$

Examples:

- If $S = 1$, then $v(1) = 3$
- If $S = 2$, then $v(2) = 2$
- If $S = 4$, then $v(4) = 1$
- If $S = 8$, then $v(8) = 0.5$

Verifying Optimal Stopping Policy and Value Function

To verify whether our optimal stopping policy and value functions are correct, we must show the following properties:

1. The value of the option must be greater than or equal to its intrinsic value, or $v(S) \geq (4 - S)^+$
2. The discounted value process is a supermartingale, or

$$\mathbb{E}_n\left[\left(\frac{4}{5}\right)^{n+1} v(S_{n+1})\right] \leq \left(\frac{4}{5}\right)^n v(S_n)$$

3. $v(S_n)$ is the smallest process satisfying Properties 1 and 2.

Verifying $v(S) \geq (4 - S)^+$

Given our value function,

$$v(S) = \begin{cases} 4 - S & \text{if } S \leq 2 \text{ (exercise immediately)} \\ 2 \cdot (\frac{1}{2})^{j-1} & \text{if } S = 2^j, j \geq 2 \end{cases}$$

For $j \leq 1$ and $S_n = 2^j$, our function implies that $v(S_n) = 4 - S_n \geq (4 - S_n)^+$. For $j \geq 2$ and $S_n = 2^j$, we have $v(S_n) \geq 0 = (4 - S_n)^+$.

Verifying the Discounted Value Process is a Supermartingale

For $j \leq 0$ and $S_n = 2^j$, we have

$$
\begin{aligned}
\tilde{\mathbb{E}}_n\left[\left(\frac{4}{5}\right)^{n+1} v(S_{n+1})\right] &= \frac{1}{2}\left(\frac{4}{5}\right)^{n+1} v\left(2^{j+1}\right) + \frac{1}{2}\left(\frac{4}{5}\right)^{n+1} v\left(2^{j-1}\right) \\
&= \left(\frac{4}{5}\right)^{n}\left[\frac{2}{5} v\left(2^{j+1}\right) + \frac{2}{5} v\left(2^{j-1}\right)\right] \\
&= \left(\frac{4}{5}\right)^{n}\left[\frac{2}{5}\left(4 - 2^{j+1}\right) + \frac{2}{5}\left(4 - 2^{j-1}\right)\right] \\
&= \left(\frac{4}{5}\right)^{n}\left[\frac{16}{5} - 2^{j}\right] \\
&\leq \left(\frac{4}{5}\right)^{n}\left[4 - 2^{j}\right] = \left(\frac{4}{5}\right)^{n} v(S_n)
\end{aligned}
$$

Verifying the Discounted Value Process is a Supermartingale

For $j \geq 2$ and $S_n = 2^j$, we have

$$
\begin{aligned}
\tilde{\mathbb{E}}_n\left[\left(\frac{4}{5}\right)^{n+1} v(S_{n+1})\right] &= \frac{1}{2}\left(\frac{4}{5}\right)^{n+1} v\left(2^{j+1}\right) + \frac{1}{2}\left(\frac{4}{5}\right)^{n+1} v\left(2^{j-1}\right) \\
&= \left(\frac{4}{5}\right)^{n}\left[\frac{2}{5} v\left(2^{j+1}\right) + \frac{2}{5} v\left(2^{j-1}\right)\right] \\
&= \left(\frac{4}{5}\right)^{n}\left[\frac{2}{5}\left(\frac{4}{2^{j+1}}\right) + \frac{2}{5}\left(\frac{4}{2^{j-1}}\right)\right] \\
&= \left(\frac{4}{5}\right)^{n} \cdot \frac{4}{2^j} = \left(\frac{4}{5}\right)^{n} v(S_n)
\end{aligned}
$$

Verifying the Discounted Value Process is a Supermartingale

For $S_n = 2$, we have

$$\begin{aligned}
\tilde{\mathbb{E}}_n \left[\left(\frac{4}{5}\right)^{n+1} v(S_{n+1}) \right] &= \frac{1}{2}\left(\frac{4}{5}\right)^{n+1} v\left(4\right) + \frac{1}{2}\left(\frac{4}{5}\right)^{n+1} v\left(1\right) \\
&= \left(\frac{4}{5}\right)^{n} \left[\frac{2}{5} v\left(4\right) + \frac{2}{5} v\left(1\right)\right] \\
&= \left(\frac{4}{5}\right)^{n} \left[\frac{2}{5} \cdot 1 + \frac{2}{5} \cdot 3\right] \\
&< \left(\frac{4}{5}\right)^{n} \cdot 2 = \left(\frac{4}{5}\right)^{n} v(S_n)
\end{aligned}$$

Verifying $v(S_n)$ is the smallest process

Suppose $Y_n, n = 1, 2, 3...$ is a process that satisfies

1. $Y_n \geq (4 - S_n)^+, n = 0, 1, 2...$
2. the discounted process $\left(\frac{4}{5}\right)^n Y_n$ is a supermartingale under the risk-neutral probabilities

Now, we show that $v(S_n) \leq Y_n$ for all n:

If $S_n \leq 2$, then $v(S_n) = 4 - S_n \leq Y_n$.
If $S_n = 2^j$ for some $j \geq 2$, let τ denote the first time after time n that the stock price falls to the level 2. Then, we have

$$v(S_n) = \tilde{\mathbb{E}}_n \left[\left(\frac{4}{5}\right)^{\tau - n} (4 - S_\tau) \right] = \tilde{\mathbb{E}}_n \left[\left(\frac{4}{5}\right)^{\tau - n} (4 - S_\tau)^+ \right]$$

Verifying $v(S_n)$ is the smallest process

On the other hand, according to the Optional Sampling Theorem, we know for all $k \geq n$,

$$\begin{aligned}
\left(\frac{4}{5}\right)^n Y_n &= \left(\frac{4}{5}\right)^{\tau \wedge n} Y_{\tau \wedge n} \\
&\geq \tilde{\mathbb{E}}_n \left[\left(\frac{4}{5}\right)^{\tau \wedge k} Y_{\tau \wedge k} \right] \\
&\geq \tilde{\mathbb{E}}_n \left[\left(\frac{4}{5}\right)^{\tau \wedge k} (4 - S_{\tau \wedge k})^+ \right]
\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\begin{aligned}
\left(\frac{4}{5}\right)^n Y_n &\geq \tilde{\mathbb{E}}_n \left[\left(\frac{4}{5}\right)^{\tau} (4 - S_\tau)^+ \right] \\
Y_n &\geq \tilde{\mathbb{E}}_n \left[\left(\frac{4}{5}\right)^{\tau - n} (4 - S_\tau)^+ \right] = v(S_n)
\end{aligned}$$

Final Answer

The value of our Perpetual American Put Option is

$$v(S) = \begin{cases} 4 - S & \text{if } S \leq 2 \text{ (exercise immediately)} \\ 2 \cdot (\frac{1}{2})^{j-1} & \text{if } S = 2^j, j \geq 2 \end{cases}$$

Our optimal stopping policy is to exercise when the stock price falls at or below level 2.

References

Shreve S. E. (2004). *Stochastic calculus for finance i: The binomial asset pricing model* (Vol. 1). Springer Finance.

The End