Solution 1

Suppose there is a solution (a, b, c) other than $(0, 0, 0)$. Then $(a_1, b_1, c_1) = (|a|, |b|, |c|)$ is also a solution. Writing $(a_1^2 - 1)(b_1^2 - 1) = c^2 + 1$, if one of a or b is odd, then c is odd. Checking mod 4, we get a contradiction. So a_1, b_1, c_1 are even. Now $(\frac{a_1}{2}, \frac{b_1}{2}, \frac{c_1}{2})$ satisfies the equation. As these are positive integers, by infinite descent we get a contradiction. So the only solution is $(0, 0, 0)$.

Solution 2 (PUMaC 2016)

If $p \mid 2015$, then $p = 5$ or 403. Now assume p does not divide 2015. As 239 is prime, it is the order of 2016 mod p. So, $239 \mid (p - 1)$. Checking the smallest even multiples of 239, $p = 2 \cdot 239 + 1 = 479$ and $p = 8 \cdot 239 + 1 = 1913$ are prime.
The required sum is $5 + 403 + 479 + 1913 = 2800$.

Solution 3 (Schinzel)

1 satisfies, now assume $n > 1$. Then n is odd. Let p be a prime factor of n. If k is the order of 2 mod p, then $k \mid 2(n - 1)$ and $k \nmid (n - 1)$. So $v_2(k) = 1 + v_2(n - 1)$. As $k \mid (p - 1)$,

$$p \equiv 1 \mod 2^{1+v_2(n-1)}$$

Multiplying similar congruences for other primes dividing n, we get

$$n \equiv 1 \mod 2^{1+v_2(n-1)}$$

so $2^{1+v_2(n-1)} \mid (n - 1)$, a contradiction.

Solution 4

Suppose $p = 8k + 7$ and $p \mid 2^n + 1$.
If n is even, write $2^n \equiv -1 \mod p$, so -1 is a quadratic residue of p

$$1 \equiv (-1)^{\frac{p-1}{2}} \equiv (-1)^{4k+3} \equiv -1 \mod p$$

If n is odd, write $2^{n+1} \equiv -2 \mod p$, so -2 is a quadratic residue of p

$$\begin{aligned} 1 \equiv \left(\frac{-2}{p}\right) \equiv \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right) &\equiv (-1) \cdot (-1)^{(p-1)(\frac{p+1}{8})} \\ &\equiv -(-1)^{2(4k+3)(k+1)} \equiv -1 \mod p \end{aligned}$$

We get a contradiction in both cases.

Solution 7 (Romania TST 2008/3/3)

Assume that $a > 1$ and b is odd. As $3 \nmid 2^a - 1$, a is odd, and we get $2^a - 1 \equiv 7 \mod 12$.
There is a prime p of the form $12k \pm 5$. If 3 is a nonquadratic residue of p then its order mod p is even, but this divides b, a contradiction. So $\left(\frac{3}{p}\right) = 1$.

If $p = 12k+5$ then $\left(\frac{p}{3}\right) = 1$ by the quadratic reciprocity law, but $\left(\frac{p}{3}\right) = \left(\frac{12k+5}{3}\right) = \left(\frac{2}{3}\right) = -1$, a contradiction.
If $p = 12k-5$ then $\left(\frac{p}{3}\right) = -1$ by the quadratic reciprocity law, but $\left(\frac{p}{3}\right) = \left(\frac{12k-5}{3}\right) = \left(\frac{1}{3}\right) = 1$, a contradiction.

Solution 8 (PRIMES 2019 M5)

Checking mod 4, we get $p \equiv 1 \mod 4$.

$$\begin{aligned} \left(\frac{p}{a}\right)s(a) &\equiv \left(\frac{p}{a}\right)a^{\frac{p-1}{2}} \\ &\equiv \left(\frac{p}{a}\right)\left(\frac{a}{p}\right) \\ &\equiv (-1)^{(a-1)(\frac{p-1}{4})} \mod p \end{aligned}$$

If a is odd, $(-1)^{(a-1)(\frac{p-1}{4})} = 1$. If $4 \mid a$, $\frac{p-1}{4}$ is even, and $(-1)^{(a-1)(\frac{p-1}{4})} = 1$. If $4 \mid (a-2)$, $\frac{p-1}{4}$ is odd, and $(-1)^{(a-1)(\frac{p-1}{4})} = -1$. As $p \equiv b^2 \mod a$

$$\left(\frac{p}{a}\right)s(a) \equiv \left(\frac{b^2}{a}\right)s(a) \equiv s(a) \mod p$$

So this function works:

$$s(a) = \begin{cases} -1 & \text{if } 4 \mid (a-2) \\ +1 & \text{if } 4 \nmid (a-2) \end{cases}$$

Solution 9 (TSTST 2013/8)

We prove by induction on k that $f(n+1), \dots, f(n+3^k)$ leave distinct remainders mod 3^k for any n. Suppose it is true for $k-1$.

$$f(n+3^{k-1}) - f(n) = 2^{f(n)} + 2^{f(n+1)} + \dots + 2^{f(n+3^{k-1}-1)} \equiv 2^1 + 2^3 + \dots + 2^{2\cdot 3^{k-1}-1} \mod 3^k$$

by Euler's theorem, as $f(n)$ is odd.

$$v_3(2^1 + 2^3 + \dots + 2^{2\cdot 3^{k-1}-1}) = v_3(2(4^{3^{k-1}} - 1)) - v_3(4-1) = k-1$$

So $f(n+3^{k-1}) - f(n) \equiv f(n+2\cdot 3^{k-1}) - f(n+3^{k-1}) \not\equiv 0 \mod 3^k$. Adding these we get $f(n+2\cdot 3^{k-1}) - f(n) \not\equiv 0 \mod 3^k$. So it is true for k.
Therefore $f(1), f(2), \dots, f(3^{2013})$ leave distinct remainders when divided by 3^{2013}.

Solution 10 (MOP 2007)

We prove that there are infinitely many pairs of positive integers (m, n) such that $\frac{m+1}{n} + \frac{n+1}{m} = 4$. Suppose that (m_1, n) satisfies such that $m_1 \le n$. Write

$$m^2 + (1-4n)m + (n^2+n) = 0$$

The other root is $m_2 = 4n - m_1 - 1 = \frac{n^2+n}{m_1} > n$.
This constructs infinitely many pairs if we go from $(1,1) \to (2,1) \to (2,6) \to (21,6) \to \dots$

Solution 11 (Shortlist 2017 N6)

If $n = 3$ take $(\frac{b+1}{c} + 1, \frac{c+1}{b} + 1, b + c + 1)$ where b and c are positive integers. Their sum is an integer for infinitely many b, c as proved in the previous question. The sum of their reciprocals is always 1.
If $n = 1$ then only (1) works.
If $n = 2$, $b + c = d$ is an integer where b and c are positive rational numbers. Let $b = \frac{p}{q}$ in lowest terms. Then $\frac{1}{b} + \frac{1}{c} = \frac{q}{p} + \frac{1}{d-\frac{p}{q}} = \frac{q^2 d}{p(dq-p)}$ is an integer. As $dq - p \nmid q^2 d$ for $dq - p > 1$ we have $dq = p + 1$. So $\frac{1}{b} + \frac{1}{c} = \frac{q(p+1)}{p} = q + \frac{1}{b}$ is an integer. So $p = 1$ and $b = c = \frac{1}{c}$. $b + c = \frac{2}{c}$ is an integer for $c = 1$, $c = 2$, which give finitely many solutions.
Therefore the answer is $n = 3$.

Solution 13 (EGMO 2016/6)

Let $d = n^2 + k$. It divides n^4, so $n^2 + k \mid k^2$. As $1 \le k \le 2n$,
$k^2 = n^2 + k$ or $k^2 = 2n^2 + 2k$ or $k^2 = 3n^2 + 3k$. If $n \equiv 3, 4 \bmod 7$ then $n^2 \equiv 2 \bmod 7$. The last two possibilities do not hold in this case, which can be checked for all values of $k \bmod 7$. The first possibility occurs only when $n = 1$, as for $n > 1$ we get $(k-1)^2 < n^2 < k^2$. The second possibility can be written as $(k-1)^2 - 2n^2 = 1$. $k = 4, n = 2$ is a solution. This Pell equation has infinitely many solutions for $n \equiv 2, 5, 0 \bmod 7$. The solutions are of the form $7m + 2, 7m + 5, 7m, 7m + 2, 7m + 5, 7m, \ldots$
Similarly the other case is the Pell equation $(2k-3)^2 - 12n^2 = 9$ which has infinitely many solutions for $n \equiv 1, 6 \bmod 7$.

Solution 15 (IMO 2003/2)

If $b = 1$ it is an integer if and only if a is even. Now assume $b > 1$. Let $\frac{a^2}{2ab^2 - b^3 + 1} = k > 0$.
Then $2ab^2 - b^3 + 1 \ge 1 \implies 2a \ge b$.
As $a^2 \ge b^2(2a - b) + 1$ we get $2a = b$ or $a > b$.

$$a^2 - (2b^2 k)a + (b^3 - 1)k = 0$$

So if (a_1, b) satisfies then (a_2, b) satisfies. Suppose that $a_1 \ge a_2 > b$.
$2b^2 k = a_1 + a_2 \le 2a_1$ so

$$b < a_2 = \frac{(b^3 - 1)k}{a_1} \le \frac{(b^3 - 1)k}{b^2 k} = b - \frac{1}{b}$$

which is not true. If $a_1 = \frac{b}{2}$ then $a_2 = \frac{b^4}{2} - \frac{b}{2}$.
The solutions are $(2m, 1), (m, 2m), (8m^4 - m, 2m)$ for some positive integer m.

Solution 17 (Romania TST 2004/2/2)

Let $f(a, b) = k$. $a = b$ gives $k = 3 + \frac{3}{a^2 - 1}$, $a = b = 2 \implies k = 4$. Also, $f(2, 1) = f(1, 2) = 7$.
Now assume $a - b \ge 1$ and $a \ge 3$. Suppose that $a + b$ is minimum.

$$a^2 + (b - kb)a + (b^2 + k) = 0$$

The second root $a_2 = bk - b - a$ is an integer. Using our assumptions

$$(a+b)(b(a-b)-1) > a$$

$$\implies a^2 - b^2 > \frac{a^2+ab+b^2}{ab-1} = k$$

$a_2 = \frac{b^2+k}{a} \leq a$. So $f(a,b) = f(a_2,b)$ with $a_2 + b < a + b$, a contradiction.

Solution 19 (Iran TST 2013)

Let $k = \frac{a^2+b^2+c^2}{2013(ab+bc+ca)}$ be an integer. Dividing by their gcd, we can assume that $gcd(a,b,c) = 1$. We can also assume that 3 does not divide any of a, b, c.

$$(a+b+c)^2 = (2013k+2)(ab+bc+ca)$$

If $p \mid (ab+bc+ca)$ then $p \mid (a+b+c)$. If p divides a then it also divides bc and $b+c$ which is not possible as $gcd(a,b,c) = 1$. So $p \nmid a, b, c$.
$p \mid ((b+c)(a+b+c) - (ab+bc+ca))$ so $p \mid b^2+bc+c^2$. If $p \mid b-c$ then $p = 3$. Otherwise bc^{-1} has order 3 mod p, so $3 \mid p$.
If only one of a, b, c is even, then $ab+bc+ca$ is odd. If two or none of a, b, c is even then $a+b+c$ is odd, so $ab+bc+ca$ must be odd.
Now, there is a prime q such that $q \mid (2013k+2)$, $q \equiv 2 \mod 3$, and q has odd exponent in the prime factorization of $(2013k+2)$. As it has even exponent in $(a+b+c)^2$, it must divide $(ab+bc+ca)$. But this implies $q = 3$ or $3 \mid p$, a contradiction. So $\frac{a^2+b^2+c^2}{2013(ab+bc+ca)}$ is never a positive integer.

Mini Survey

(a)

It took 7 hours.

(b)

The theory was very nice.

Maybe nothing can be done about this, but in example 1.22 walkthrough (a), "Deduce that n is odd" mislead me, to think that this actually had to be proved. I assumed it to be proved and proceeded. But it took me some time to realize that m and n are relatively prime positive integers, given in the question.

My favourite problems on this set were problems 7 and 9.