Solution 1 (Shortlist 2001 A2)

Suppose that it holds for finitely many n. Then there exists M such that $1+a_n \le a_{n-1}\sqrt[n]{2}$ for all $n > M$. Now assume that $a_n > 1$ for all sufficiently large n. Considering an arbitrarily large n, we get the equations

$$2 < 1+a_n \le a_{n-1}\sqrt[n]{2}$$
$$\sqrt[n]{2}(1+a_{n-1}) \le a_{n-2}\sqrt[n]{2}\sqrt[n-1]{2}$$
$$\sqrt[n]{2}\sqrt[n-1]{2}(1+a_{n-2}) \le a_{n-3}\sqrt[n]{2}\sqrt[n-1]{2}\sqrt[n-2]{2}$$
$$\cdots$$
$$\sqrt[n]{2}\sqrt[n-1]{2}\cdots\sqrt[M+2]{2}(1+a_{M+1}) \le a_M\sqrt[n]{2}\sqrt[n-1]{2}\cdots\sqrt[M+1]{2}$$

Adding, we get

$$a_M > \sum_{k=M+1}^{n-1} 2^{-(\frac{1}{M+1}+\frac{1}{M+2}+\cdots+\frac{1}{k})}$$

But

$$\sum 2^{-(1+\frac{1}{2}+\cdots+\frac{1}{k})} > \sum 2^{-1-\log_e n} > \sum 2^{-1-\log_2 n} = \frac{1}{2n}$$

diverges, a contradiction. So we can find large n such that $a_n \le 1$ and $a_{n+1} \le \sqrt[n+1]{2}-1 \le 1$. So $a_n < 1$ for all sufficiently large n. But we also have $a_n \ge \frac{1+a_{n+1}}{\sqrt[n+1]{2}} > \frac{1}{\sqrt[n+1]{2}}$.
Therefore $1+\frac{1}{\sqrt[n+1]{2}} < \sqrt[n]{2}$ for sufficiently large n, a contradiction since $1+\frac{1}{\sqrt[n+1]{2}} > \frac{3}{2}$ and $\sqrt[n]{2} < \frac{3}{2}$ for large n.

Solution 2 (ELMO 2018/5)

$b = m^2$ and $c = m(a_1+a_2+\cdots+a_m)$ works. If $a_1 = \ldots = a_m$ any n satisfies, so assume that not all are equal. Consider

$$f(x) = (\sqrt{n+a_1}+\sqrt{n+a_2}+\cdots+\sqrt{n+a_m}) - m\sqrt{n+\frac{a_1+\cdots+a_m}{m}}$$

where $x > 0$. By Cauchy-Schwartz, $f(x) < 0$.

$$\begin{aligned} 2f'(x) &= \frac{1}{\sqrt{n+a_1}}+\cdots+\frac{1}{\sqrt{n+a_m}} - \frac{m}{\sqrt{n+\frac{a_1+\cdots+a_m}{m}}} \\ &\ge \frac{m^2}{\sqrt{n+a_1}+\sqrt{n+a_2}+\cdots+\sqrt{n+a_m}} - \frac{m}{\sqrt{n+\frac{a_1+\cdots+a_m}{m}}} \\ &\ge 0 \end{aligned}$$

So $f(x)$ increases but is bounded above. So, $f(n)$ converges. As $n \to \infty$, $f(n) \to 0^-$ and there exists N such that $f(n) > -1$ for $n > N$. This is equivalent to the given equation.

Solution 4 (Putnam 2016 B6)

The sum converges absolutely, so we can permute the terms.

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k(2^n k+1)}$$

$$= \sum_{n=0}^{\infty} \int_0^1 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2^n k}}{k} dx$$

$$= \sum_{n=0}^{\infty} \int_0^1 \ln(1 + x^{2^n}) dx$$

$$= \int_0^1 \ln(1 + x + x^2 + x^3 + \cdots) dx$$

$$= \int_0^1 - \ln(1 - x) dx$$

$$= \Big((1 - x)\ln(1 - x) + x\Big)_0^1$$

$$= 1$$

Solution 6 (Shortlist 2002)

Using the inequality, it is clear that $a_i \neq a_j$ for $i \neq j$. Consider $a_1, a_2, \ldots, a_n$ where n is any positive integer. Permute these as $b_1 < b_2 < \ldots < b_n$. Then

$$\begin{aligned}
c &\geq b_n - b_1 \\
&= (b_n - b_{n-1}) + \cdots + (b_2 - b_1) \\
&\geq \frac{1}{b_n + b_{n-1}} + \cdots + \frac{1}{b_2 + b_1} \\
&\geq \frac{(n-1)^2}{n(n+1) - b_1 - b_n} \\
&\geq 1 - \frac{3n - 4}{n^2 + n - 3}
\end{aligned}$$

As it holds for all n, taking $n \to \infty$ we must have $c \geq 1$.

Solution 7 (Pugh)

Let $p \leq a_n \leq q$ for all n. In the sum $\sum_{i=n}^{n+k} a_i b_i$ Let B_n^+ be the sum of the positive b_i's and B_n^- be the sum of the negative b_i's. Then

$$pB_n^+ + qB_n^- \leq \sum_{i=n}^{n+k} a_i b_i \leq qB_n^+ + pB_n^-$$

As B_n^+ and B_n^- converge to zero, $\sum a_n b_n$ is Cauchy and therefore converges.

Solution 9 (IMO 2016)

We have to remove at least 2016 factors, so that no factor is on both sides. This suffices, and we prove that

$$\prod_{r=1}^{n}(x-4r+2)(x-4r+1)=\prod_{r=1}^{n}(x-4r+3)(x-4r)$$

has no real solutions, where $4n=2016$.

$$(x-4r+2)(x-4r+1)=(x-4r+3)(x-4r)+2>(x-4r+3)(x-4r)$$

If $x\notin[1,4n]$ or $x\in(4k,4k+1)$, then we can multiply all these equations to get LHS $>$ RHS. For $x\in\{1,2,\ldots,4n\}$ one side is zero while the other is nonzero. For $x\in(4k+2,4k+3)$, RHS $\div$ LHS is

$$\Big(\frac{x-1}{x-2}\Big)\Big(\frac{4n-x}{4n-1-x}\Big)\Big(\prod_{r=1}^{n}\frac{(x-4r)(x-4r-1)}{(x-4r+1)(x-4r-2)}\Big)$$

All these fractions are greater than 1, so RHS $>$ LHS. For all other cases, the LHS and RHS have opposite sign.

Solution 10 (Putnam 2016 B5)

Let $g(x)=\ln f(e^x)$. This is a function in R^+. If $2\le\frac{x}{y}\le3$ then $2\le\frac{g(x)}{g(y)}\le3$. Multiplying similar equations we get
if $\frac{2^p}{3^q}\le\frac{x}{y}\le\frac{3^p}{2^q}$ then $\frac{2^p}{3^q}\le\frac{g(x)}{g(y)}\le\frac{3^p}{2^q}$.
If $g(x)\le g(y)$ then $\frac{2^p}{3^q}\le\frac{g(2^py)}{g(y)}\cdot\frac{g(x)}{g(3^qx)}\le\frac{g(2^py)}{g(3^qx)}$. If $x>y$ we can take p,q such that $3\le\frac{2^p}{3^q}\le\frac{3x}{y}$. This gives $\frac{g(2^py)}{g(3^qx)}\ge3$ but $\frac{2^py}{3^qx}\le3$, a contradiction.
So if $\frac{2^p}{3^q}\le x\le\frac{3^p}{2^q}$ then $\frac{2^p}{3^q}\le\frac{g(x)}{g(1)}\le\frac{3^p}{2^q}$ and $\frac{g(x)}{g(1)}\le\frac{3^q}{2^pg(1)}g(\frac{2^qx}{3^p})<\frac{3^q}{2^p}$. These give $g(x)\le xg(1)$. Similarly we get $g(x)\ge xg(1)$, so $f(e^x)=e^{xg(1)}$. The answer is $f(x)=x^k$.

Solution 13

Let $H(t)=P(t)-e^t$. As it is differentiable, between two consecutive zeroes of $H(t)$, $H'(t)=0$ at least once, by Rolle's theorem.
But $H'(t)=P'(t)-e^t=P'(t)-P(t)$ is a polynomial of degree n, so it has at most n zeroes. Therefore $H(t)$ has at most $n+1$ zeroes.

Solution 14(USMCA 2019/5)

Let the number on the board after n steps be a_n. We prove that for some constant $c<1$ and for any fixed $a_n\ge1$,

$$f(n)=\frac{E[\sqrt[m]{a_{n+1}}]}{\sqrt[m]{a_n}}\le c$$

for large enough m. It is true for $a_n = 1, 2, 3$. For $a_n > 3$,

$$\begin{aligned} f(n) &= \frac{\sum\limits_{r=0}^{\lceil 2.01a_n \rceil} \sqrt[m]{r}}{(1 + \lceil 2.01a_n \rceil) \sqrt[m]{a_n}} \\ &\leq \frac{\int\limits_{1}^{\lceil 2.01a_n \rceil + 1} \sqrt[m]{r} dr}{(1 + \lceil 2.01a_n \rceil) \sqrt[m]{a_n}} \\ &< \frac{m}{m+1} \sqrt[m]{2.01 + \frac{2}{a_n}} \end{aligned}$$

Now we can use $a_n > 3$ to get $c < 1$ for large m.
If $a_n = 0$ then $a_{n+1} = 0$.
Now, $\frac{E[\sqrt[m]{a_{n+1}}]}{E[\sqrt[m]{a_n}]} \leq c$ and $E[\sqrt[m]{a_n}] \leq k(c^n)$ for some $k > 0$. This proves that the probability that $a_n \neq 0$ is less than 1 percent for some $n > 0$. So the answer is yes.

Solution 15 (Shortlist 2012 A4)

$\left|pf(x) + g(x)\right| = \left|f(x)\right| \left|p + \frac{g(x)}{f(x)}\right| > 0$ for sufficiently large x, so x is bounded in $[-M, M]$. In a sequence of such rational roots corresponding to an increasing sequence of primes, there is a subsequence which converges. Let $p_1 < p_2 < \ldots$ be primes and $x_1, x_2, \ldots$ be the convergent sequence of rational roots corresponding to these primes. It converges to L. As $g(x_n)$ is bounded and $f(x_n) = -\frac{g(x_n)}{p_n}$, we get $f(L) = 0$. Remove all integers from this sequence. If there are infinitely many integers then it is eventually constant and we are done. Now assume $x_n = \frac{a_n}{b_n}$ in lowest terms. Let $f(0) = k$, $g(0) = m$, and $f(x)$ has leading coefficient d.
Then $b_n \mid p_n d$ and $a_n \mid p_n k + m$. $a_n t = p_n k + m$. If $b_n \mid d$ for infinitely many n then we consider that subsequence in which b_n is bounded. So L is rational in this case. Now take the subsequence where it does not divide d. Let $b_n = p_n d_1$.
$x_n = \frac{p_n k + m}{p_n d_1 t} = \frac{k}{d_1 t} + \frac{m}{p_n d_1 t}$. L is the limit as $n \to \infty$ where the numerator is bounded. So it is rational.

Solution 17 (Putnam 1996 B6)

Let

$$f(p, q) = \sum_i e^{a_i p + b_i q}$$

where $p = \ln x$, $q = \ln y$. As the origin is inside the polygon, there are points on both sides of the line $ap + bq = 0$ in the a-b plane. So, $M = \max(a_i p + b_i q)$ is positive, and $\left(\frac{p}{\sqrt{p^2+q^2}}, \frac{q}{\sqrt{p^2+q^2}}\right)$ lies on the unit circle in the p-q plane, for $(p, q) \neq (0, 0)$, so $\frac{M}{\sqrt{p^2+q^2}}$ has a minimum. $\frac{M}{\sqrt{p^2+q^2}} \geq C$ for some $C > 0$.
$f(0, 0) = n$ and $f(p, q) \geq e^M \geq e^{C\sqrt{p^2+q^2}}$.
$f(p, q) > n$ for $\sqrt{p^2 + q^2} > \frac{\ln n}{C}$ so the minimum must lie in the interior of the disc of this radius, which is compact. Both partial derivatives are zero at this point, this gives the equation we need to prove.

Mini Survey

(a)

It took around 12 hours.

(b)

I liked problems 2, 10 and 15.