Solution 1 (Mock BrMO)

Let B be the set of integers which can be written as a sum of elements of a nonempty subset of A. Suppose that in the set A, r integers are positive and k integers are negative.

For the positive ones, $0 < a_1 < \cdots < a_r$ the sequence of $\frac{r(r+1)}{2}$ positive integers

$$a_1 < \cdots < a_r < a_1 + a_r < \cdots < a_{r-1} + a_r < a_1 + a_{r-1} + a_r < \cdots$$

$$\cdots < a_1 + a_3 + \cdots + a_r < a_2 + a_3 + \cdots + a_r < a_1 + a_2 + \cdots + a_r$$

belong to B.

For the negative ones, $0 > b_1 > \cdots > b_k$ the sequence of $\frac{k(k+1)}{2}$ negative integers

$$b_1 > \cdots > b_k > b_1 + b_k > \cdots > b_{k-1} + b_k > b_1 + b_{k-1} + b_k > \cdots$$

$$\cdots > b_1 + b_3 + \cdots + b_k > b_2 + b_3 + \cdots + b_k > b_1 + b_2 + \cdots + b_k$$

belong to B.

So, B has at least $\frac{r(r+1)}{2} + \frac{k(k+1)}{2}$ elements, which is greater than or equal to $\frac{r+k}{2} + \frac{(r+k)^2}{4}$ which is at least 2025 as $r + k = 89$ or 90.

Solution 3 (IMO 2000/4)

There are 12 ways, 3! for keeping numbers congruent to each other mod 3 in the same box, and 3! for keeping 2 to 99 in a box and 1 and 100 in separate boxes. We write n in place of 100 and induct for $n \geq 4$. The base case is easy to check and we suppose that no arrangements other than those described above exist for $4, \ldots, (n-1)$. If some other arrangement exists for n, the trick still works if we delete n. But we should get the above cases after deleting n. Adding n to the any of the boxes now, we can easily get that the trick works only when we keep numbers (including n) congruent to each other mod 3 in the same box, which is among those cases. So we get the result for $n = 100$.

Solution 4 (USAMO 2006/2)

Let the $2k+1$ positive integers be $a_1 < a_2 < \ldots < a_{2k+1}$. We get these equations:

$$a_{k+2} + \cdots + a_{2k+1} \leq \frac{N}{2}$$

$$a_2 + k \leq a_{k+2}$$

$$\cdots\cdots$$

$$a_{k+1} + k \leq a_{2k+1}$$

Using these we get

$$N+1+k^2 \le k^2+a_1+\cdots+a_{2k+1} \le 2(a_{k+2}+\cdots+a_{2k+1})+a_1 \le N+a_1$$

So $a_1 \ge k^2+1$. $(k^2+1, k^2+2, \ldots, k^2+2k+1)$ works. Their sum is greater than or equal to $N+1$, this gives $N \le 2k^3+3k^2+3k$. $a_{k+2}+\cdots+a_{2k+1} \le \frac{N}{2}$ gives $N \ge 2k^3+3k^2+3k$. So $N = 2k^3+3k^2+3k$ is the best possible value.

Solution 5 (Shortlist 2012 C2)

Suppose there are M pairs. As their sums are distinct, the sum of all these sums is at most $(n-M+1)+\cdots+(n-1)+n$. As these pairs consist of $2M$ distinct numbers, the total sum is at least $1+2+\cdots+2M$. So $(n-M)M+\frac{M(M+1)}{2} \ge M(2M+1)$. We get $M \le \lfloor \frac{2n-1}{5} \rfloor$.
Equality holds for this value of M. If $n = 5k+1$ or $5k+2$, consider the $2k$ pairs $(1, 4k+1), (3, 4k), \ldots, (2k-1, 3k+2), (2, 3k), (4, 3k-1), \ldots, (2k, 2k+1)$.
For the other cases there are $2k+1$ pairs $(1, 4k+2), (3, 4k+1), \ldots, (2k-1, 3k+3), (2k+1, 3k+2), (2, 3k+1), (4, 3k), \ldots, (2k, 2k+2)$.

Solution 6 (China 2019/1)

Let $e \ge a, b, c, d$ and $a = p-1, b = q-1, c = r-1, d = s-1, e = t-1$ where p, q, r, s, t are nonnegative and $p+q+r+s+t = 10$.

For the minimum value, we write

$$S = -(2-p-q)(2-q-r)(2-r-s)(8-p-q-r)(8-q-r-s)$$

The last two factors are positive as $t \ge 2$. If the first three factors are positive, we get $S \ge -512$ with equality when $p = q = r = s = 0$. If only one of the first three factors is positive, by AM-GM we get $S \ge -(\frac{18}{5})^5 > -512$. So the minimum is -512.

Similarly for the maximum value, we write

$$S = (p+q-2)(2-q-r)(2-r-s)(8-p-q-r)(8-q-r-s)$$

The last two factors are positive as $t \ge 2$. If the first three factors are positive, we get $S \le 288$ with equality when $q = r = s = 0$, $p = t = 5$. Same if we interchange the sign of the first and third factor. And in another case,

$$S = (2-p-q)(q+r-2)(2-r-s)(8-p-q-r)(8-q-r-s)$$

where all factors are positive, AM-GM on the first three factors gives $S \le 19$. So, the answer is $-512 \le S \le 288$, the extremes attained at (-1, -1, -1, -1, 9) and (-1, -1, -1, 4, 4).

Solution 9 (Taiwan TST Quiz)

The left hand inequality follows by AM-GM, equality holds when $x_1 = x_2 = \ldots = x_n$. If any two consecutive numbers are equal, then all numbers are equal. Now, assume that consecutive numbers are not equal.
If x_i is the largest number, then $x_{i-1} < x_i$ and $x_{i+1} < x_i$. So $x_{i-1} + x_{i+1} = x_i$, as$x_i|(x_{i-1} + x_{i+1})$. So $k_i = 1$ and $k_{i-1} = \frac{x_{i-2}+x_i}{x_{i-1}} = \frac{x_{i-2}+x_{i+1}}{x_{i-1}} + 1$, $k_{i+1} = \frac{x_{i+2}+x_i}{x_{i+1}} = \frac{x_{i+2}+x_{i-1}}{x_{i+1}} + 1$.
Removing x_i decreases $k_1 + \cdots + k_n$ by 3. So, we keep remove the largest number till we get 3 numbers a, b, c such that each divides $a + b + c$. Now if a is the largest then $b + c = 2a$ which implies $a = b = c$, or $b + c = a$ which gives $2b = c$ or $2c$ or $4c$. In all these cases we see that $\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \leq 8$. But $\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} = k_1 + \cdots + k_n - 3(n - 3)$. So $\frac{k_1+\cdots+k_n}{n} \leq \frac{3n-1}{n} < 3$.

Solution 10 (USA TST 2009/7)

(x, y, z) = (2, 4, 6) satisfies. Let x = 2 + a, y = 4 + b, z = 6 + c. The equations reduce to

$$a(a + 3)^2 = -12b$$
$$b(b + 6)^2 = 3c$$
$$c(c + 9)^2 = 27a$$

Multiplying them, we get

$$abc[(a + 3)^2(b + 6)^2(c + 9)^2 + 12 \times 3 \times 27] = 0$$

so one of a, b, c is zero, which implies that all of them are zero. So, (x, y, z) = (2, 4, 6) is the only solution.

Solution 11 (EGMO 2016/5)

The answer is n if $n = k$ or $2k - 1$ and it is $2(n - k + 1)$ if $k < n < 2k - 1$.
If $n = k$, tiles can't be placed perpendicular to each other, so there must be n tiles in the same direction. If $n = k+1$, we can place four tiles at the boundary. Now no tile can be placed inside it. If this square of side $k + 1$ with 4 tiles is present in an optimal configuration for some $n > k + 1$, then there would be a tile in every row and column not intersecting the square with 4 tiles, and we get $2(n - k + 1)$ tiles.
Assume that the answer is less than $2n - 2k + 2$ and not equal to n. Then, either there are at most $n-k$ vertical tiles or there are at most $n-k$ horizontal tiles. Suppose it is the first case. So at least k columns don't have vertical tiles. If two vertical tiles are separated by less than k unit squares, then all columns between them contain vertical tiles. Therefore k consecutive columns don't have vertical tiles, so this whole $k \times n$ rectangle cuts n horizontal tiles. If $n = k$ or $2k - 1$, n is the minimum and for other values, $2(n - k + 1)$ is less than n.

Solution 12 (Shortlist 2013 A4)

$a_1 \leq a_{a_1} \leq n$. $a_1 = n$ gives $a_1 = \cdots = a_n = n$. Now assume $a_1 = k$. Then $a_k \leq n$. Let $a_1 = \cdots = a_{c_k} = k$, $a_{c_k+1} = \cdots = a_{c_k+c_{k+1}} = k+1$, $\ldots$, $a_{c_k+\cdots+c_{n-1}+1} = \cdots = a_{c_k+\cdots+c_{k+(n-k)}} = k + (n-k)$. Here, $c_k + \cdots + c_n = k$.

Now, $a_{k+1} \leq n + c_k$, $a_{k+2} \leq n + c_k + c_{k-1}$, $\ldots$, $a_n \leq n + c_k + \cdots + c_n = n + k$. $a_1 + \cdots + a_n \leq kc_k + (k+1)c_{k+1} + \cdots + nc_n + (n + c_k) + (n + c_k + c_{k-1}) + \cdots + (n + c_k + \cdots + c_n) = nk + (n-k)n = n^2$.

Solution 13 (Shortlist 2013 C6)

Let $2m = 100$. Consider a city C and $a_1, a_2, \ldots, a_t$ be the t cities at a distance 2 to C. Let k_i be the number of cities which are at a distance 2 from a_i and a distance 4 from C. Consider a city adjacent to C and a_i. There are at least $k_i + (t-1)$ cities at a distance of 3 from this city. So $k_i + (t-1) \leq 2m$. This holds for all i. Also, if N is the number of cities which are at a distance of 4 to C, then $N \leq \sum\limits_{i=1}^{t} k_i$.

Now, $N + t(t-1) \leq \sum\limits_{i=1}^{t}(k_i + (t-1)) \leq 2mt$
$2m + 1 \geq \frac{N}{t} + t \geq 2\sqrt{N}$
$N \leq (m + \frac{1}{2})^2 = m^2 + m + \frac{1}{4}$
So N can reach a maximum value of $m^2 + m = 2550$.

Solution 14 (Shortlist 2016 C5)

If n is odd, then the diagonals cannot be perpendicular. Just consider the nth roots of unity on the unit circle. $\omega^{a+x} + \omega^{b+c} = 0$ if $AX \perp BC$, where $\omega = cos\frac{2\pi}{n} + isin\frac{2\pi}{n}$. So $\omega^{a+x-b-c} = -1$ which is not possible if n is odd. So, the answer is (n - 3).
If n is even, if two perpendicular diagonals intersect, other intersecting diagonals must be along these perpendicular directions. Suppose this set of perpendicular diagonals consists of d diagonals. They divide the circumcircle into $2d$ arcs. Now if we draw other diagonals which are not intersecting, their end points can't belong to 2 different arcs. In each arc of length kl, (considering the circumference to be kn) there are $(l-1)$ such diagonals. So there are a total of $d + \sum(l-1) = d + n - 2d = n - d$. This is maximum when $d = 2$, there are two perpendicular diagonals. If there are no perpendicular diagonals, a maximum of $n - 3$ diagonals can be drawn.
So the answer is $n - 3$ if n is odd, and $n - 2$ if n is even.

Solution 17 (Shortlist 2018 C5)

Let $a_1 \leq \ldots \leq a_{2k}$ and $d_1 \leq \ldots \leq d_{2k}$ be the arrival and departure days, not for any particular player. Instead of the number of coins for one player, we calculate $T_i = (d_{2k-i} - a_{i+1} + 1)$.
For $i \leq k$, $(a_{i+1} - 1) \leq \binom{i}{2}$ as there were only i players before day a_{i+1}. Similarly, $\binom{2k}{2} - d_{2k-1} \leq \binom{i}{2}$. So, $T_i \geq \binom{2k}{2} - 2\binom{i}{2}$.
If $i \geq k+1$, we have the stronger inequality $T_i \geq (2k-i)^2$. This follows by induction on $(2k-i)$. $T_{2k-1} \geq 1$ as $d_1 \geq a_{2k}$, because the first player to end should have played with everyone.

Now the total number of coins to be paid is $\sum\limits_{i=0}^{2k-1} T_i = \sum\limits_{i=0}^{k} k(2k-1) + i - i^2 + \sum\limits_{i=k+1}^{2k-1} (2k-i)^2 = (k+1)k(2k-1) + \frac{k(k+1)}{2} - (1^2 + 2^2 + \cdots + k^2) + (1^2 + 2^2 + \cdots + (k-1)^2) = \frac{4k^3+k^2-k}{2}$. Equality holds when we construct a tournament so that $a_i = 1 + \binom{i-1}{2}$ and $d_{2k-1} = \binom{2k}{2} - \binom{i}{2}$ for $i \leq k$, and using the same induction as before, starting from $d_1 = a_{2k}$.

Mini Survey

(a)

It took around 10 to 11 hours.

(b)

Problems 9 and 17 were especially nice. Problem 14 might be a little easy for its placement.