Solution 1 (INMO 2020/5)

Let F be the infinite family and d be the distance between two closest parallel lines. Let the parallel diagonals PQ and RS have lengths $2R\sin\frac{2\pi}{n}$ and $2R\sin\frac{4\pi}{n}$. If lines $MP, NR \in F$ such that $QM \perp PM$ and $SN \perp RN$ then $\triangle PQM \sim \triangle RSN$. As QM and SN are integral multiples of d, $\frac{RS}{PQ} = \frac{SN}{QM}$ is rational.

$$\frac{RS}{PQ} = \frac{2R\sin\frac{4\pi}{n}}{2R\sin\frac{2\pi}{n}} = 2\cos\frac{2\pi}{n}$$

So $\cos\frac{2\pi}{n}$ is rational. By Niven's theorem, $\cos\frac{2\pi}{n} \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ and this gives $n = 3, 4, 6$. It can be easily checked that all three solutions work.

Solution 2 (BAMO 2018/5)

At a vertex, $\frac{\pi(n-2)}{n} = \theta_1 + \cdots + \theta_k$ where $cos\theta_1$, $sin\theta_1$ $\ldots \in \mathbb{Q}$. So $\cos\frac{2\pi}{n}, \sin\frac{2\pi}{n} \in \mathbb{Q}$. By Niven's theorem, $n = 4$. It is possible for this case by dividing the square into 3×4 rectangles.

Solution 3 (Black MOP 2010)

Assign complex numbers $1, \omega^1, \omega^2, \ldots, \omega^{n-1}$ to the lamps where 1 is the lamp which is initially on and $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. If the lamps on a regular polygon are toggled, the sum of their weights is zero. So if we add half of the weights of each lamp when toggled, the sum is zero. 1 is toggled an odd number of times, say $2k+1$ times. $\omega^1, \omega^2, \ldots, \omega^{n-1}$ each are toggled an even number of times, $2r_1, 2r_2, \ldots, 2r_{n-1}$. So $0 = \frac{2k+1}{2} + r_1\omega^1 + r_2\omega^2 + \cdots + r_{n-1}\omega^{n-1}$
or $\frac{-1}{2} = k + r_1\omega^1 + r_2\omega^2 + \cdots + r_{n-1}\omega^{n-1}$. The left hand side is not an algebraic integer, but the right hand side is an algebraic integer, a contradiction. So it is not possible for any n.

Solution 5 ($\infty\bullet$MO 2018/5)

The characteristic polynomial of this sequence is

$$X^k = c_1X^{k-1} + c_2X^{k-2} + \cdots + c_k$$

with roots $r_1, \ldots, r_k$. So

$$a_n = d_1r_1^n + d_2r_2^n + \cdots + d_kr_k^n$$

We take $a_1, \ldots, a_k$ such that $d_1 = \ldots = d_k = 1$.
$r_1^n + \cdots + r_k^n$ is rational by the theorem of symmetric polynomials, and it is an algebraic integer, so it is a rational integer.
$\frac{(r_1+\cdots+r_k)^p - (r_1^p+\cdots+r_k^p)}{p}$ is rational, and expanding by binomial theorem (by induction), it is an algebraic integer. So

$$a_p = r_1^p + \cdots + r_k^p \equiv (r_1 + \cdots + r_k)^p = a_1^p \bmod p$$

Solution 6 (APMO 2018/5)

Assume $P(0) = 0$. Let p be a large prime. If $P(x) - p = f(x)g(x)$ such that $g(0) = 1$ then α is a root of $g(x)$ such that $|\alpha| \leq 1$. Then $\left|P(\alpha) - p\right| \geq p - \left|P(\alpha)\right| > 0$. Therefore $P(x) - p$ is irreducible over the integers. Choose real k such that $P(k) - p = 0$. Then $P(x) - p \mid P(mx) - P(mk)$. As their degree is same, we compare coefficients to get $P(x) = cx^n$. If $P(a) = 1$ then $P(a^2) = \frac{1}{c}$ is an integer.
So the answer is $\pm x^n + b$.

Solution 9 (IMO 2006/5)

If $Q(t) = t$, then $P(t) = t$ or we get a sequence of distinct integers $x_1, x_2, \ldots x_m$ where $t = x_1$ and $m \leq k$.
$(x_1 - x_2) \mid P(x_1) - P(x_2) = (x_2 - x_3)$ and similar equations give $|x_1 - x_2| = |x_2 - x_3| = \ldots = |x_k - x_1|$
As they are distinct, $x_1 = x_3 = x_5 = \ldots$ and $x_2 = x_4 = \ldots$ so $P(P(t)) = t$.
Let $P(a) = b$ and $P(b) = a$ where $a \neq b$. If such a and b do not exist then $Q(t) = t \implies P(t) = t$ which has at most n roots.
Now if $P(c) = d$ and $P(d) = c$ where these are not equal to any of a or b, then $(b - c) \mid P(b) - P(c) = (a - d)$ and similar divisibility relations give $|a - d| = |b - c|$ and $|a - c| = |b - d|$. Denote these as x-coordinates of points A, B, C, D on the x-axis and M is the midpoint of AB. If we fix A, B, C then there are two positions for D satisfying $AD = BC$. But for only one of these, $AC = BD$ and M is the midpoint of CD in this case. So $a + b = c + d$. This is a constant and also holds for $c = d$.
So $Q(t) = t \implies H(t) = a + b - P(t) - t = 0$. But $H(x)$ has degree n, so has at most n roots.

Solution 10 (TSTST 2016/1)

Perform the division algorithm on A and B for the variable x, so $A = B \cdot Q + R$ where R is a polynomial in x with coefficients as rational functions of y. The degree of R is less than the degree of B with respect to x. Now, it is given that for infinitely many values of y, $R = 0$, i.e. each coefficient of R is zero. So R is the zero polynomial. Taking the rational functions of y in the coefficients of y into the numerator and denominator by adding all terms, we can write Q as $Q = \frac{Q_1(x,y)}{F(y)}$. Doing the same by interchanging x and y, we write in lowest terms $\frac{A(x,y)}{B(x,y)} = \frac{Q_1(x,y)}{F(y)} = \frac{Q_2(x,y)}{G(x)}$. So $F(y) \mid Q_1(x,y)G(x)$, so $F(y)$ and $G(x)$ are constant polynomials.

Solution 12 (HMIC 2014/4)

Let $\zeta_j = e^{\frac{2\pi i}{k_j}}$. Define

$$B = \{a_1\zeta_1^{r_1} + a_2\zeta_2^{r_2} + \cdots + a_n\zeta_n^{r_n} \ \mid \ \gcd(r_j, k_j) = 1,\ 1 \leq j \leq n\}$$

$$P(X) = \prod_{\beta \in B} (X - \beta)$$

By the theorem of symmetric polynomials, $P(X)$ has integer coefficients. As $P(\alpha) = 0$, the minimal polynomial of α divides $P(X)$. So the Galois conjugates of α belong to B. Now we get a condition from $\alpha\overline{\alpha} = 1$. The Galois conjugates of α satisfy this, so they lie on the unit circle. By Kronecker's theorem, α is a root of unity.

Solution 13 (OMO 2015 S28)

We factor both sides as

$$\big(P(x)+iQ(x)\big)\big(P(x)-iQ(x)\big) = (x-1)^2(x+1)^2\prod_{k=1}^{n-1}(x^{2^k}+1)^2 = (x-1)^2(x+1)^2\prod_{k=1}^{n-1}(x^{2^{k-1}}-i)^2(x^{2^{k-1}}+i)^2$$

Now we can distribute $(x^{2^{k-1}}-i)^2(x^{2^{k-1}}+i)^2$ as $\big((x^{2^{k-1}}-i)^2,\ (x^{2^{k-1}}+i)^2\big)$, $\big((x^{2^{k-1}}+i)^2,\ (x^{2^{k-1}}-i)^2\big)$ or $\big((x^{2^{k-1}}-i)(x^{2^{k-1}}+i),\ (x^{2^{k-1}}-i)(x^{2^{k-1}}+i)\big)$ as $P(x)+iQ(x)$ and $P(x)-iQ(x)$ are conjugates. Multiplying by $1,-1,i,-i$ we get 4 ordered pairs after the distribution. So the answer is $4\cdot 3^{n-1}$.

Solution 14 (Taiwan TST 2016/3J/1)

Let $\lambda=\beta^3$, the other two roots of the equation $x^3=x^2+1$ are complex conjugates z and $\overline{z}$. $\beta^{900}+z^{900}+\overline{z}^{900}$ is an integer, so we can look at 2 times the real part of z^{900}, which is less than 4^{-100}.

Solution 16 (USA TST 2017/6)

If $1+\omega+\omega^2=0$, $g(x)=p(x-\omega)^2(x-\omega^2)^2$ and $f(x)=(x+1)^p-x^p-1$, then $f(\omega)=f(\omega^2)=f'(\omega)=f'(\omega^2)=0$ only when $3\mid(p-1)$.
So $p(x^2+xy+y^2)\mid(x+y)^p-x^p-y^p$. For every such prime, we choose x and y such that $(xy^{-1})^3\equiv 1 \bmod p^2$ and $p^2\mid g(x)$.

Solution 20 (Shortlist 2003 N7)

Let $2a_k=b_k+\frac{1}{b_k}$. Then $b_{k+1}=b_k^{\pm 2}$. As $b_0=2\pm\sqrt{3}$,

$$a_n=\frac{(2+\sqrt{3})^{2^n}+(2-\sqrt{3})^{2^n}}{2}$$

We work in $\mathbb{F}_{p^2}$. If $p\mid a_n$, then $\alpha=(2+\sqrt{3})=\frac{(1+\sqrt{3})^2}{2}=m^2$ as $8\mid p^2-1$ and $\alpha^{2^{n+1}}=m^{2^{n+2}}=-1$. So 2^{n+3} is the order of m and divides p^2-1.

Mini Survey

(a)

It took around 11 hours.

(b)

Problems 2, 3 and 14 were especially nice.