Solution 1 (China TSTST 2017/3)

Let K be the intersection of l and the radical axis of the corresponding circles for XX_1 and YY_1. Desargues Involution Theorem on the quadrilateral $ABCD$ and line l gives the involutive pairing $(X, X_1), (Y, Y_1), (Z, Z_1)$; the first two imply that it is a negative inversion around K with radius $\sqrt{KX \cdot KX_1}$

Solution 2

Invert around ω and take a homothety with ratio 2 at P. Then $W \to A$, $X \to B$, ... as inversion preserves cross ratios, we are done.

Solution 4 (Shortlist 2006 G5)

We check $CD^2 - C_1D^2 = CA_1^2 - A_1M^2 + DM^2 - JM^2 = CA_1^2 = CJ^2 - C_1J^2$ where M is the midpoint of A_1B_1, so by Carnot's theorem $CC_1 \perp DJ$ and C, E, C_1 are collinear. So, E, A_1, B_1 lie on the circle with diameter CJ. So EC bisects $\angle A_1EB_1$.

$$-1 = (A_1, B_1; D, EC \cap A_1B_1) \stackrel{C}{=} (B, A, D, C_1)$$

so DE bisects $\angle BEA$ and EC is the external bisector. As $CA = CB$, E lies on (ABC). So E is the miquel point of ADA_1C. From this we get that the angles to be found are right angles.

Solution 5 (APMO 2013/5)

Note that $ABCD$ is harmonic. Take a homography that sends $AC \cap BD$ to the center of ω and fixes ω. $\angle DEQ = 135^\circ$ and $\angle QRD = 45^\circ$, $DEQR$ is cyclic and the conclusion follows.

Solution 8 (APMO 2008/3)

Take a projective transformation fixing Γ that takes $BF \cap AC$ to the center of Γ, where $F = AD \cap CE$. Then $LACM$ is a rectangle, so $LAEH$ and $MGDC$ are cyclic. This gives $\angle HKG = 180^\circ - \angle HAG$.

Solution 10 (Iran TST 2008/2)

Using Brianchon's theoerem on the hexagon $BCEYZF$, BY, CZ, EF are concurrent. By, symmetry, AX, BY, CZ concur on l.

Solution 11 (Sharygin 2013/20)

The angle condition shows that C_1 is the Miquel point of $A_1CB_1C_2$. Now by angle chasing C_1C_2 passes through the reflection of C over AB.

Solution 12 (RMM 2013)

Let XY be the chord of ω with midpoint R. Let $K' = PY \cap \omega$, $Z = K'K' \cap YY$, $S = XX \cap YY$, $T = XZ \cap PQ$. P lies on the polar of Z so Z lies on RQ (we used Brokard theorem). Therefore Q is the midpoint of ST, and perspectivity at X for the harmonic quadrilateral formed by $K', X, Y, XZ \cap \omega$ gives K', X, Q collinear, so $K = K'$ and the homothety at K shows that ω and (PKQ) are indeed tangent.

Solution 14 (USAMO 2012/5)

Let $K = B'C' \cap BC$. Using DIT on quadrilateral $ABKB'$, We see that $(PB, PB'), (PC, PC'), (PA, PK)$ are involutive pairs; the first two imply it is reflection over γ, and therefore we get $K = A'$.

Solution 15 (Shortlist 2016 G6)

Using the angle condition, $S = AB \cap CD$ and $T = BC \cap AD$ lie on ω. $\angle DXM = \angle DSB = \angle DTB = \angle DAN$, where $BANC$ is a parallelogram. So X is a HM point in $\triangle BAC$, and X, E, B lie on the apollonius circle of this triangle. Now $\angle XQY = \angle XEA + \angle YFA = 180^\circ - \angle XBP - \angle YDP = \angle XPY$, so Q lies on ω. Now we calculate $\angle EPQ = \angle BXE = 90^\circ + \angle PEC$.

Solution 16 (Serbia 2017/6)

Let Γ, γ, ω denote the circumcircle, A-excircle, and A-mixtilinear incircle. Let the common tangents of Γ, γ meet at S, and E is the point of contact of γ with BC. By DDIT on $AEBC$ with γ and S, after projecting onto BC we get the involutive pairs which are isogonal with respect to the A-bisector. This gives $\angle PAB = \angle CAQ$.

Solution 20 (TSTST 2016, Danielle Wang)

By inversion around the incircle, C_1C_2 and B_1B_2 are medial lines of $\triangle DEF$. Let P, Q, R be the midpoints of EF, FD, DE. Let $X = C_1C_2 \cap AB$, $Y = B_1B_2 \cap AC$, $Z = BY \cap CX$. Then X is the radical center of $(ABC), (DEF), (CC_1C_2)$ and similarly for Y, so Z is the radical center of $(ABC), (BB_1B_2), (CC_1C_2)$, and PZ is the required radical center. Now, triangles BRY and CQX are perspective, so $T = BR \cap CQ$ lies on PZ. Now we can apply barycentric coordinates on $\triangle DEF$ to get that P, T, the midpoint of DK, and $EB \cap CF$ are collinear.

Solution 21 (Shortlist 2000 G3)

Given any three points P, Q, R on an ellipse, and $X = QQ \cap RR$, $Y = RR \cap PP$, $Z = PP \cap QQ$, then PX, QY, RZ are concurrent. To prove this we take a homography preserving the ellipse that sends QR to the major axis and $PQ = PR$.
Now in the problem, reflect H over the sides and join them with O to meet the corresponding sides at D, E, F. Then we take the ellipse passing through D, E, F with foci O, H.

Mini Survey

(a)

It took around 7 hours.

(b)

I enjoyed the lecture notes. Problems 4, 12, 15 stood out.