Solution 1 (Canada 1987)

It is possible when n is even, achieved when pairs of MOPpers are kept close to each other, far from the other pairs. If $n = 3$, the MOPpers closest to each other shoot each other, and the other MOPper isn't drenched. So it isn't possible in this case. Assume it isn't possible for $n = 2k - 1$, where k is a positive integer. If $n = 2k + 1$, we consider all the distances between the MOPpers. The smallest distance is between two MOPpers who shoot each other. Now the others don't shoot these MOPpers, and they can't all get drenched independently, by the induction hypothesis. So it isn't possible for all odd n.

Solution 3 (USAMO 2007/2)

Define a void to be the largest disc that can be placed in an empty space, without overlapping the other discs. The smallest such void is possible when it is surrounded by three discs of radius 5, tangent to each other. So the diameter of the void is at least $2 \cdot 5 \cdot \left(\frac{2}{\sqrt{3}} - 1\right) = 1.55 > \sqrt{2}$. As it is larger than the diagonal of a unit square, at least one lattice point lies inside a void. So such a covering is not possible.

Solution 8 (Caro-Wei)

Consider a random procedure to find such a set S. If vertex i has degree d_i, then the probability that it belongs to S is $\frac{1}{d_i+1}$. The expected value of $|S|$ is $\sum\limits_{i=1}^{n} \frac{1}{d_i+1}$. This is at least $\frac{n^2}{n+\sum d_i} = \frac{n}{d+1}$ by Titu's inequality. By pigeonhole principle, There is some set S which has at least the expected number of vertices.

Solution 9 (TSTST 2019/4)

Let the dominations of the coins be $a_1 \leq a_2 \leq \ldots \leq a_{100}$, where $0 < a_i \leq 1$. $a_{52} + \cdots + a_{100} \leq 49$ gives $1 \leq a_1 + \cdots + a_{51} \leq 51a_{51}$. Similarly $51a_{50} \leq a_{50} + \cdots + a_{100} \leq 50$. Now let

$$D = (a_1 + a_3 + a_5 + \cdots + a_{49}) + (a_{52} + a_{54} + \cdots + a_{100}) - (a_2 + a_4 + a_6 + \cdots + a_{50}) - (a_{51} + a_{53} + \cdots + a_{99})$$

Then

$$D = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{49} - a_{50}) + a_{51} + (a_{52} - a_{53}) + \cdots + (a_{98} - a_{99}) + a_{100} \leq -\frac{1}{51} + 1$$

$$-D = -a_1 + (a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{48} - a_{49}) + a_{50} + (a_{51} - a_{52}) + \cdots + (a_{99} - a_{100}) \leq \frac{50}{51}$$

So $|D| \leq \frac{50}{51}$ if we split them this way. But if $a_1 = a_2 = \ldots = a_{51} = 51$ and $a_{52} = \ldots = a_{100} = 1$, then $\frac{50}{51}$ is the least possible absolute difference. So we can take $C = \frac{50}{51}$.

Solution 10 (USAMO 2008/3)

We use induction on n. For $n = 1$, at least 1 path is needed.
Let $a_1 = (n-1, 0), a_2 = (n-2, 0), a_3 = (n-2, 1), a_4 = (n-3, 1), \ldots, a_{2n-1} = (0, n-1)$. $b_1, \ldots, b_{2n-1}$ are the reflections of these points in the line $y = -\frac{1}{2}$. If a_1 and b_1 are not joined, then join them. This does not increase the number of paths. If a_2 is not joined to a_1, join them and remove other connections with a_2. Continue this way till a_{2n-1}. Do the same with $b_1, \ldots$ till b_{2n-1}. These algorithms do not increase the number of paths. Now we have a path from a_{2n-1} to b_{2n-1} and the remaining figure has at least $n - 1$ paths by the induction hypothesis. So there are at least $(n - 1) + 1 = n$ paths.

Solution 11 (China TST 2015)

Let the colours denote vertices and the kids are the edges in a multigraph (without loops). Each vertex has degree at most $2n$. There are at most k kids with different coloured candies. So we can select at most k disjoint pairs of vertices connected with an edge.

Take a maximal set S of such pairs. This consists of k edges. Consider one of these edges uv. As S is maximal, there isn't any edge between vertices not in S. Also, u and v can't be connected to two different vertices not in S. So they are connected to the same vertex, or one of them is not connected to any such vertex. Both cases show that there are at most $2n$ such vertices for each edge in S. There are a total of at most $2nk$ such edges.

Points in S belong to at most n edges in S. This also follows from S being maximal. So there are at most $3nk$ edges in total.
Equality holds when there are k isolated triangles, each with n edges between two vertices.

Solution 13

Let E be the set of edges, and D_j is the set of neighbours of vertex v_j (with complex number $x_j = a_j + ib_j$).

$$\sum_{v_r \in D_j} (x_r - x_j) = \sum_{v_r \in D_j} (a_r - a_j) + \sum_{v_r \in D_j} i(b_r - b_j) = 0$$

so $\sum\limits_{v_r \in D_j} (a_r - a_j) = 0$ and $\sum\limits_{v_r \in D_j} (b_r - b_j) = 0$. Thus it suffices to solve the problem for real numbers.
Adding a sufficiently large positive real number to $a_1, a_2, \ldots$, we can also assume that they are positive real numbers.

$$2|E| = \sum_j |D_j| = \sum_j \frac{\sum\limits_{v_r \in D_j} a_r}{a_j} = \sum_{jr \in E} \left(\frac{a_j}{a_r} + \frac{a_r}{a_j} \right)$$

so $\sum\limits_{jr \in E} \frac{(a_j - a_r)^2}{a_j a_r} = 0$. This gives $a_1 = a_2 = \ldots$ and similarly $b_1 = b_2 = \ldots$ so $x_1 = x_2 = \ldots$.

Mini Survey

0.1 (a)

The problem set took 5 hours.

0.2 (b)

Problem 11 was especially nice.