## Solution 1 (JMO 2013/2)

Suppose that we fix an arrangement of zeroes on the board. For this arrangement, let $S$ be the set of cells in which exactly one possible number can be filled. Clearly, the zeroes are in $S$. Their nonzero neighbors must be equal to 1, so they are also in $S$. There are no other 1's on the board except these, as 1 must have at least one 0 among its neighbors. Now the neighbors of 1 which do not belong to $\{0, 1\}$ must be 2, so they are in $S$. There are no other 2's on the board. Continuing this way, all cells on the board are in $S$. There is one garden for every arrangement of zeroes. The least number on the board is less than or equal to its neighbors, so it is zero. So there is at least one 0.
The number of gardens is $2^{mn} - 1$.

## Solution 4 (USAMO 2010/2)

Let $s_k$ be the number of switches of a student with height $h_k$ with a smaller student. The number of students between the students with height $h_k$ and $h_{k_1}$ who are smaller than them decreases by $s_k - s_{k-1}$.
But there are at most $k - 2$ such students, so $s_k - s_{k-1} \leq (k - 2)$.
As $s_2 = s_1 = 0$, we add similar inequalities to get

$$s_k \leq (k-2) + (k-1) + \cdots + 1 = \frac{(k-1)(k-2)}{2}$$

So the total number of switches is

$$\sum_{k=1}^{n} s_k \leq \sum_{k=1}^{n} \binom{k-1}{2} = \binom{3}{3} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-1}{2} = \binom{n}{3}$$

## Solution 6 (Shortlist 2001 C7)

For two consecutive columns, the left one is never shorter than the other one. Using this we also get that we never have three consecutive columns with same height, by considering the steps before it to form this.
In the final configuration, The difference between two consecutive columns is 0 or 1. If we take two closest pairs of consecutive columns with same height, then the difference between any two consecutive columns between them is 1. Now we go backwards, using the fact that the left one is never shorter than the right one. Then these two pairs come closer and it is impossible to form them without having three consecutive columns with same height, a contradiction. So there is at most one such pair of consecutive columns with same height in the final configuration. This gives exactly one possible configuration $(62, 61, \ldots, 50, 49, 48, 48, 47, 46, \ldots, 3, 2, 1)$

## Solution 7 (Shortlist 2001 A1)

$f(0, 0, 0) = 0$. Assume $x + y + z > 0$. Let $g(x, y, z) = (x + y + z)f(x, y, z) - 3xyz$. Then the equation gives

$$6g(x,y,z) = g(x-1,y+1,z) + g(x+1,y-1,z) + g(x-1,y,z+1)$$

$$+g(x+1,y,z-1) + g(x,y-1,z+1) + g(x,y+1,z-1)$$

So the average of $g$ for the neighbors of a point in the plane $x+y+z=k$ is the value of $g$ at that point. Take the point on this plane such that $g$ is maximum, then its neighbors should have the same maximum value. Then considering the neighbors of these neighbors and continuing this way till we reach a point with one coordinate zero, all points on this plane must have $g=0$. Therefore the only function that satisfies is $f(x,y,z)=\frac{3xyz}{x+y+z}$ for $x+y+z>0$ and $f(0,0,0)=0$. It is easy to check that this satisfies the conditions.

# Solution 8 (USAMO 2015)

If $\emptyset$ is blue, and exactly $k$ single element sets are blue, then for disjoint sets $T_1$ and $T_2$, $f(T_1 \cup T_2)=f(T_1)f(T_2)$. So we know the color of all the subsets of $S$ using the color of the single element sets. If all sets with size at least 2 are red, the problem condition is satisfied. So there is exactly one coloring for this case.
But we can colour the single element sets blue in $2^n$ ways. So there are $2^n$ colorings if $\emptyset$ is blue.
If $\emptyset$ is red, consider the blue set $M$ which has minimum size. (If there is no blue set then we get a valid coloring).
For any other blue set $T$, $f(T)f(M)=f(T\cup M)f(T\cap M)$ as the left side is positive, $f(T\cap M)>0$ so $T\cap M=M$ or $M\subset T$.
Now looking at $S\setminus M$ suppose $M$ goes to $\emptyset$ in this set (it is blue). Using the previous result, there are $2^{n-|M|}$ colorings.
The total number of colourings is $1+\sum\limits_{m=0}^{n} 2^{n-m}\binom{n}{m}=1+3^n$.

# Solution 9 (Shortlist 2000 C6)

Denote all positive integers by $xp+yq$ where $0\le x<q$. So we get a region in the $xy$-plane. If a lattice point $(x,y)$ for $n\in S$ then the points $(x+1,y)$ and $(x,y+1)$ are in $S$. As the origin is in $S$ we get the subset of the region above the $x$-axis. So the answer is the number of paths from the origin to$(q,-p)$ which do not lie outside the triangle formed by these points and $(q,0)$ as we can take the points on and above that path to belong to $S$. So the answer is $\frac{1}{p+q}\binom{p+q}{p}$.

# Solution 10 (Shortlist 1999 C1)

It is true for $s=1$ and $n=2$. Let the answer be $f(n,s)$.
For $f(n+1,s+1)$ we can add a step $EN$ to the configuration in $f(n,s+1)$ between $E$ and $N$, this can be done in $s+1$ ways. Or we could add a step $EN$ to the configuration in $f(n,s)$ at the beginning or end, or between $EE$, or $NN$, this can be done in $2n+1-s$ ways. So

$$(s+1)f(n+1,s+1)=(s+1)f(n,s+1)+(2n+1-s)f(n,s)$$

as each configuration in $f(n+1,s+1)$ might have come by adding any of the $(s+1)$ steps.
So we can induct on $n$ and assume that it is true for $n$. Then

$$\begin{aligned}(s+1)f(n+1,s+1) &= \binom{n-1}{s}\binom{n}{s}+\binom{n-1}{s-1}\binom{n}{s}+\frac{n}{n-s+1}\binom{n-1}{s-1}\binom{n}{s}\\ &= \binom{n}{s}\binom{n+1}{s}\end{aligned}$$

So the it is true for $f(n+1,s+1)$. As we checked for $s=1$ and $n=2$, the proof is complete.

## Solution 12 (IMO 2013/6)

The number of relatively prime positive integers with sum $k$ is $\phi(k)$. So $N = \phi(2) + \phi(3) + \cdots + \phi(n)$.
If 0 and 1 are consecutive in clockwise order, then $0, 1, t, t+1$ appear in clockwise order for all $1 < t < n$. So $0, 1, 2, \ldots, n$ appear in clockwise order. So there is one labelling for this case.
We ignore the equidistant condition. If 0 and $a$ are consecutive in clockwise order, first we consider the numbers $0, 1, \ldots, a$. Let $x$ be the next number (less than $a$) after $a$ in clockwise order. Taking $0, x, rx, (r+1)x$ or $a, x, rx, (r+1)x$ modulo $a$ for $r > 1$, $(r+1)x$ comes after $rx$ in clockwise order. So the residues of $x, 2x, 3x, \ldots, (a-1)x$ modulo $a$ appear in clockwise order after 0 and $a$. If $d \mid x$ and $d \mid a$ then $d$ divides all numbers less than $a$, which is a contradiction. So $\gcd(a, x) = 1$. Now we have to place $a+1, a+2, \ldots, n$. We found at least $\phi(a)$ labellings for if 0 and $a$ are consecutive in clockwise order, exactly $\phi(n)$ for $a = n$. So $M \geq N + 1$. Now if we add $a + x$ it is between $x$ and $2x$. Using this $a + 2x$ is between $2x$ and $3x$. Continuing like this, we get only one labelling, where $x, a+x, 2a+x, 3a+x, \ldots$ are consecutive, then $2x, a+2x, 2a+2x, \ldots$ are consecutive, and so on. So $M = N + 1$.

## Solution 14 (Shortlist 2013 N7)

Let $f(a,b) = a\lceil b\nu \rceil - b\lfloor a\nu \rfloor = a(\lceil b\nu \rceil - b\nu) + b(a\nu - \lfloor a\nu \rfloor)$. So for fixed $b$ we can make $f(a,b)$ arbitrarily large by increasing $a$, for example $f(a+nb)$.
By evaluation, either

$$f(a+b,b) = f(a,b) \text{ and } f(a,b+a) = f(a,b) + a$$

or

$$f(a,b+a) = f(a,b) \text{ and } f(a+b,b) = b + f(a,b)$$

according as whether $\lfloor a\nu \rfloor + \lceil b\nu \rceil = \lceil a\nu + b\nu \rceil$ or $\lfloor a\nu + b\nu \rfloor$.
Assume that $a$ and $b$ are relatively prime. Now we for fixed $m$, prove by induction that the number of excellent pairs (such that the value of $f$ for the pair is $m$) is equal to the number of ways of expressing $m - f(a,b)$ in the form $ax + by$ where $x, y \geq 0$ and $m - f(a,b) > 0$ for some pair $(a,b)$. But such excellent pairs do not exist when $a, b \geq m$. So there are finitely many values of $a$ and $b$ for which there are excellent pairs. If it is true for $(a+b,b)$ and $(a,b+a)$, then it is true for $(a,b)$.
We take $a = 1$, $b = 1$ to get $m$ excellent pairs.
Now the number of excellent pairs of the form $(da, db)$ where $d \mid m$ is the gcd of $da$ and $db$ is equal to the number of pairs $(a,b)$ such that $f(a,b) = \frac{m}{d}$, which is $\frac{m}{d}$. Summing over all divisors of $m$, the total number of excellent pairs is the sum of divisors of $m$.

## Mini Survey

### (a)

It took around 16 to 18 hours.

### (b)

Problem 14 was instructive. The diagram was very helpful. Problem 1 was very nice.