Solution 1 (JMO 2016, Zuming Feng)

We can find complex numbers x, y, z, t such that $a = x^2$, $b = y^2$, $c = z^2$, $p = t^2$, and the midpoints of arcs $\widehat{BC}, \widehat{CA}, \widehat{AB}$ opposite A, B, C are $-yz, -zx, -xy$. Taking $-yt$ as the midpoint of arc $\widehat{BP}$ not containing A, we can compute $I_B = -yt - xt - xy$ and $I_C = zt + xt - xz$. We prove that $M = -yz$ is the desired fixed point. We will prove $\frac{p - I_B}{p - I_C} \div \frac{m - I_B}{m - I_C}$ is real. This is equivalent to

$$\frac{(t^2 + yt + xt + xy)(xz - yz - xt - zt)}{(t^2 - zt - xt + xz)(yt + xt + xy - yz)} = \overline{\left(\frac{(t^2 + yt + xt + xy)(xz - yz - xt - zt)}{(t^2 - zt - xt + xz)(yt + xt + xy - yz)}\right)}$$

or

$$y(xz - yz - xt - zt)(xz + yz + zt - xt) = z(yt + xt + xy - yz)(yt - xt - yz - xy)$$

As $AB = AC$, $x^2 = yz$ and the left hand side equals

$$(xyz - y^2z - xyt - yzt)(xz + yz + zt - xt)$$

or

$$(x^3 - x^2y - xyt - x^2t)(xz + x^2 + zt - xt)$$

and the right hand side is

$$(yt + xt + xy - x^2)(x^2t - xzt - x^2z - x^3)$$

Dividing both sides by $-x$, we see that they are equal.

Solution 2 (RMM 2019)

Set Ω as the unit circle. Let F be the midpoint of BD and P' be the intersection of tangents to Ω at D and E. There are 3 degrees of freedom. Let the variables be c, d, e.
$p' = \frac{2de}{d+e}$, $f = \frac{cd}{e}$, $a = 2e - c$. Now it suffices to prove that $\angle AFE = \angle P'AE$. This is equivalent to proving that $\frac{a-f}{e-f} \div \frac{p'-a}{e-a}$ is a real number. But this expression is equal to

$$\frac{2e - c - \frac{cd}{e}}{e - \frac{cd}{e}} \div \frac{\frac{2de}{d+e} - 2e + c}{c - e} = \frac{2e^2 - ce - cd}{e^2 - cd} \div \frac{c - \frac{2e^2}{d+e}}{c - e} = \frac{cd - e^2 + ec - ed}{cd - e^2}$$

But this is equal to its conjugate

$$\overline{\left(1 + \frac{ec - ed}{cd - e^2}\right)} = 1 + \frac{\frac{1}{ec} - \frac{1}{ed}}{\frac{1}{cd} - \frac{1}{e^2}} = 1 + \frac{ed - ec}{e^2 - cd}$$

hence $P = P'$.

Solution 3 (MOP 2006)

Let $(ABCD)$ be the unit circle and rotate the diagram so that P lies on the real axis. $P = p$ is a real number. We compute O_1 using the circumcenter formula by shifting P to 0.

$$o_1 = p + \frac{(a-p)(b-p)(\frac{1}{a} - \frac{1}{b})}{\frac{b}{a} - \frac{a}{b} + (a-b)p + \frac{p}{b} - \frac{p}{a}} = p + \frac{(a-p)(b-p)}{b + a - abp - p} = \frac{ab(1-p^2)}{a + b - abp - p}$$

Applying symmetry,

$$o_2 = \frac{bc(1-p^2)}{b + c - bcp - p},\ o_3 = \frac{cd(1-p^2)}{c + d - cdp - p},\ o_4 = \frac{da(1-p^2)}{d + a - dap - p}$$

Now we evaluate the determinant

$$\begin{vmatrix} \frac{o_1+o_3}{2} & \overline{\left(\frac{o_1+o_3}{2}\right)} & 1 \\ \frac{o_2+o_4}{2} & \overline{\left(\frac{o_2+o_4}{2}\right)} & 1 \\ \frac{0+p}{2} & \overline{\left(\frac{0+p}{2}\right)} & 1 \end{vmatrix}$$

This is a multiple of

$$\begin{vmatrix} ab(c+d-cdp-p)+cd(a+b-abp-p) & (c+d-cdp-p)+(a+b-abp-p) & \frac{1}{1-p^2}(c+d-cdp-p)(a+b-abp-p) \\ bc(d+a-dap-p)+da(b+c-bcp-p) & (d+a-dap-p)+(b+c-bcp-p) & \frac{1}{1-p^2}(b+c-bcp-p)(d+a-dap-p) \\ 1 & 1 & \frac{1}{p} \end{vmatrix}$$

Subtracting the second column from the first, this is a multiple of

$$(1-p^2)\begin{vmatrix} 1 & (a+b+c+d-2p)-abp-cdp & \frac{1}{1-p^2}(c+d-cdp-p)(a+b-abp-p) \\ 1 & (a+b+c+d-2p)-bcp-dap & \frac{1}{1-p^2}(b+c-bcp-p)(d+a-dap-p) \\ 0 & 1 & \frac{1}{p} \end{vmatrix}$$

Now subtract the first column from the second column and expand to get

$$(1-p^2)(ab-bc+cd-ad)-(b+c-bcp-p)(d+a-dap-p)+(c+d-cdp-p)(a+b-abp-p)$$

$$=(1-p^2)(ab-bc+cd-ad)-(b+c)(a+d)-p^2(ad+1)(bc+1)+(a+b)(c+d)+p^2(ab+1)(cd+1)$$

Summing the p^2 terms separately, we see that this is equal to zero. So, the three points are indeed colinear.

Solution 5 (Romania 2003/9.3)

Let (ABC) be the unit circle. The converse is easy to see. We prove the other direction. The triangle is $\triangle ABC$. We are given that $\frac{1}{4}(b+c-\frac{bc}{a}+3a)$, $\frac{1}{4}(c+a-\frac{ca}{b}+3b)$, $\frac{1}{4}(a+b-\frac{ab}{c}+3c)$ are collinear. After scaling and shifting we see that

$$0=\begin{vmatrix} 2a-\frac{bc}{a} & \frac{2}{a}-\frac{a}{bc} & 1 \\ 2b-\frac{ca}{b} & \frac{2}{b}-\frac{b}{ca} & 1 \\ 2c-\frac{ab}{c} & \frac{2}{c}-\frac{c}{ab} & 1 \end{vmatrix}=\begin{vmatrix} \frac{2ab+ac+bc}{ab} & \frac{-2}{ab}-\frac{1}{bc}-\frac{1}{ca} \\ \frac{2bc+ab+ac}{bc} & \frac{-2}{bc}-\frac{1}{ca}-\frac{1}{ab} \end{vmatrix}=\begin{vmatrix} ab+bc+ca & 1 \\ 2abc+a^2(b+c) & 2a+b+c \end{vmatrix}$$

This gives

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=1$$

or $h\overline{h}=1$ where h denotes the orthocenter. So the orthocenter lies on the circumcircle which is possible only when the triangle is right.

Solution 6 (USAMO 2014)

Let (APC) be the unit circle. B is the orthocenter of $\triangle AHC$, so if $K=AB\cap(AHC)$, then $k=-\frac{ch}{a}$. But P is the midpoint of arc $\widehat{CK}$, so $p^2=-c\times\frac{ch}{a}$. $b=a+c+h=a+c-\frac{ap^2}{c^2}$, $y=a+c+p$. Now we compute x by shifting A to zero and using the circumcenter formula.

$$x=a+\frac{(b-a)(p-a)(\bar{b}-\bar{p})}{\overline{(b-a)}(p-a)-(b-a)\overline{(p-a)}}=a+\frac{(c-\frac{ap^2}{c^2})(p-a)(\frac{1}{c}-\frac{c^2}{ap^2}-\frac{1}{p}+\frac{1}{a})}{(p-a)(-\frac{c}{ap^2}(c-\frac{ap^2}{c^2}))+(c-\frac{ap^2}{c^2})(\frac{p-a}{ap})}$$

This simplifies as

$$a+\frac{\frac{1}{c}-\frac{c^2}{ap^2}-\frac{1}{p}+\frac{1}{a}}{\frac{1}{ap}-\frac{c}{ap^2}}=\frac{\frac{1}{c}+\frac{1}{a}-\frac{c^2}{ap^2}-\frac{c}{p^2}}{\frac{1}{ap}-\frac{c}{ap^2}}=\frac{(\frac{1}{c}+\frac{1}{a})(p^2-c^2)a}{(p-c)}=p+c+a+\frac{ap}{c}$$

So $|XY|=|x-y|=\left|\frac{ap}{c}\right|=\frac{|a||p|}{|c|}=1$, which is equal to the circumradius of $\triangle ABC$.

Solution 7 (ELMO 2017)

Let the circle with diameter AH be the unit circle. D and E are the feet of the perpendiculars from B and C. There are 4 degrees of freedom. Let the variables be a, d, e, p. Now, $h = -a$ and $m = \frac{2de}{d+e}$ is the intersection of tangents at D and E. As M lies on chord PQ, $m + \overline{m}pq = p + q$ so

$$q = \frac{p - m}{\overline{m}p - 1} = \frac{p - \frac{2de}{d+e}}{\frac{2p}{d+e} - 1} = \frac{p(d+e) - 2de}{2p - (d+e)}$$

Let J be the orthocenter of $\triangle APQ$.

$$j = a + p + \frac{p(d+e) - 2de}{2p - (d+e)} = a + \frac{2p^2 - 2de}{2p - (d+e)}$$

If T is the reflection of H over M, then AT is the diameter of (ABC). So it suffices to prove that $JT \perp JA$.

$$t = 2m - h = \frac{a(d+e) + 4de}{d+e}$$

$$\frac{j-t}{j-a} = \frac{\frac{2p^2-2de}{2p-(d+e)} - \frac{4de}{d+e}}{\frac{2p^2-2de}{2p-(d+e)}} = \frac{(d+e)(p^2-de) - 2de(2p-(d+e))}{(d+e)(p^2-de)} = \frac{(d+e)p^2 - 4pde + de(d+e)}{(d+e)(p^2-de)}$$

and the conjugate of this expression is

$$\frac{(\frac{1}{d} + \frac{1}{e})\frac{1}{p^2} - \frac{4}{pde} + \frac{1}{de}(\frac{1}{d} + \frac{1}{e})}{(\frac{1}{d} + \frac{1}{e})(\frac{1}{p^2} - \frac{1}{de})} = \frac{(d+e)p^2 - 4pde + de(d+e)}{(d+e)(de - p^2)}$$

So $\frac{j-t}{j-a} + \overline{\left(\frac{j-t}{j-a}\right)} = 0$, and we get $JT \perp JA$.

Solution 8

Let x, y, z be complex numbers such that $a = x^2$, $b = y^2$, $c = z^2$, $I = -(xy+yz+zx)$, and the midpoint of arc $\widehat{BAC}$ $S = yz$. As I lies on chord ST, $yz + t = -(xy+yz+zx) - yzt(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx})$ or $xy + 2yz + zx + t = -\frac{t}{x}(x+y+z)$. This gives $t(2x+y+z) + x^2y + x^2z + 2xyz = 0$. We get

$$t = -\frac{x^2y + x^2z + 2xyz}{2x+y+z}$$

Solution 10 (EGMO 2017/6)

Let (ABC) be the unit circle. $o_1 = b + c,\ o_2 = c + a,\ o_3 = a + b,\ g = \frac{a+b+c}{3},\ g_1 = b + c - bc\overline{g} = \frac{2(b+c)}{3} - \frac{bc}{3a},\ g_2 = \frac{2(c+a)}{3} - \frac{ca}{3b},\ g_3 = \frac{2(a+b)}{3} - \frac{ab}{3c}$.
We prove that all the given circles pass through $k = \frac{ab+bc+ca}{a+b+c}$. First, we prove that O_2O_3AK is cyclic.

$$\frac{o_2 - a}{o_2 - k} \div \frac{o_3 - a}{o_3 - k} = \frac{c}{a + c - \frac{ab+bc+ca}{a+b+c}} \div \frac{b}{a + b - \frac{ab+bc+ca}{a+b+c}} = \frac{c(a^2+ab+b^2)}{(a^2+ac+c^2)b}$$

but this is equal to its conjugate, so it is a real number. By symmetry, it follows that (O_1O_2C) and (O_1O_3B) also pass through K.

$$\frac{g_2 - a}{g_2 - k} \div \frac{g_3 - a}{g_3 - k} = \frac{2bc - ab - ac}{2bc + 2ab - ac - \frac{3b(ab+bc+ca)}{a+b+c}} \div \frac{2bc - ab - ac}{2bc + 2ac - ab - \frac{2c(ab+bc+ca)}{a+b+c}}$$

$$= \frac{(2bc-ab-ac)(a+b+c)}{2b(a^2+c^2) - (a+c)(ac+b^2)} \div \frac{(2bc-ab-ac)(a+b+c)}{2c(a^2+b^2) - (a+b)(ab+c^2)} = \frac{2c(a^2+b^2) - (a+b)(ab+c^2)}{2b(a^2+c^2) - (a+c)(ac+b^2)}$$

The conjugate of this expression is

$$\frac{\frac{2}{c}(\frac{1}{a^2}+\frac{1}{b^2})-(\frac{1}{a}+\frac{1}{b})(\frac{1}{ab}+\frac{1}{c^2})}{\frac{2}{b}(\frac{1}{a^2}+\frac{1}{c^2})-(\frac{1}{a}+\frac{1}{c})(\frac{1}{ac}+\frac{1}{b^2})}=\frac{2c(a^2+b^2)-(a+b)(ab+c^2)}{2b(a^2+c^2)-(a+c)(ac+b^2)}$$

so it is real, which implies K lies on (G_2G_3A). By symmetry, it also lies on (G_1G_2C) and (G_1G_3B). As $\overline{k}=\frac{a+b+c}{ab+bc+ca}=\frac{1}{k}$, it also lies on (ABC).

Solution 11 (Schiffler)

If $AB=AC$, then the Euler lines of $\triangle ABC$ and $\triangle BIC$ coincide, and it is easy to see by symmetry that the other two Euler lines concur on this line.
Now assume $\triangle ABC$ is scalene. Let x,y,z be complex numbers such that $a=x^2$, $b=y^2$, $c=z^2$, $I=-(xy+yz+zx)$. Let the Euler lines of $\triangle ABC$ and $\triangle BIC$ intersect at P. We can write $P=k(a+b+c)$ where k is a real number. Then the centroid G_a of $\triangle BIC$, the midpoint M_a of arc $\widehat{BC}$ not containing A, and P are collinear. $\frac{p-m_a}{g_a-m_a}$ is real, so it is equal to its conjugate.

$$\frac{k(x^2+y^2+z^2)+yz}{y^2+z^2+2yz-xy-xz}=\frac{k(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2})+\frac{1}{yz}}{\frac{1}{y^2}+\frac{1}{z^2}+\frac{2}{yz}-\frac{1}{xy}-\frac{1}{xz}}$$

$$\frac{k(x^2+y^2+z^2)+yz}{(y+z)(y+z-x)}=\frac{k(x^2y^2+y^2z^2+z^2x^2)+x^2yz}{x(y+z)(xz+xy-yz)}$$

$$kx(x^2+y^2+z^2)(xz+xy-yz)+xyz(xz+xy-yz)=k(y+z-x)(x^2y^2+y^2z^2+z^2x^2)+x^2yz(y+z-x)$$

$$k\big((x^2+y^2+z^2)x(x^2-yz)+(y+z-x)(x^4-y^2z^2)\big)+xyz(x^2-yz)=0$$

As $\triangle ABC$ is scalene, $x^2\neq yz$. So,

$$k\big(x^3+(y^2+z^2)x+(y+z)x^2-x^3+yz(y+z)-xyz\big)+xyz=0$$

$$k\big((y^2+z^2)x+(z^2+x^2)y^2+(x^2+y^2)z-xyz\big)+xyz=0$$

k is symmetric in x,y,z, so we will get the same value of k if we repeat the procedure for $\triangle AIB$ and $\triangle CIA$. Thus P lies on all the four Euler lines.

Solution 13 (APMO 2010)

Let BE and CF be the altitudes. $FA\cdot FM=FH\cdot FC=FA\cdot FB$ so F is the midpoint of BM. Let (ABC) be the unit circle. $m=a+c-\frac{ab}{c}$. Similarly $n=a+b-\frac{ac}{b}$. The circumcenter J of $\triangle MNH$ is

$$\frac{(-b-\frac{ab}{c})(-c-\frac{ac}{b})(\frac{(b-c)(a+b+c)}{abc})}{(a+c)(a+b)(\frac{c}{ab^2}-\frac{b}{ac^2})}=\frac{-bc(a+b+c)}{b^2-bc+c^2}$$

As $\frac{j}{h}=\frac{-bc}{b^2-bc+c^2}$ is real, we see that O,H,J are collinear.

Solution 16 (Shortlist 2000 G6)

Let (BCX) be the unit circle. As $\triangle BCX$ and $\triangle ADX$ are oppositely similar, $\frac{x-a}{x-d}=\frac{\bar{x}-\bar{b}}{\bar{x}-\bar{c}}$. So $\frac{b(x-a)}{(x-b)}=\frac{c(x-d)}{(x-c)}$. Denote this value by the complex number k, corresponding to point K.

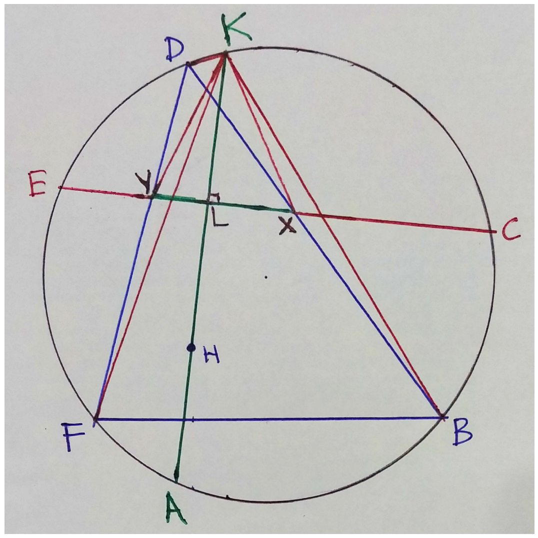
$$k=a+\frac{x(b-a)}{(x-b)}=b+\frac{b(b-a)}{(x-b)}$$

so $|k-a|=|k-b|$ or $KA=KB$.

$$k=d+\frac{x(c-d)}{(x-c)}=c+\frac{c(c-d)}{(x-c)}$$

so $|k-c|=|k-d|$ or $KC=KD$. This shows that $K=Y$. As $\frac{k-b}{k-a}=\frac{b}{x}$, we get $\angle AYB=\angle XOB=2\angle XCB=2\angle ADX$.

Solution 17 (ELMO SL 2018, Michael Ren and Vincent Huang)



Let K be on Ω such that $AK \perp CE$. L is the foot from A to CE. Set Ω as the unit circle. There are 4 degrees of freedom. We use the variables k, b, d, f. If H is the common orthocenter, then $h = b + d + f$ and $l = \frac{k+b+d+f}{2}$. As $LX \perp LK$,

$$0 = \frac{l-x}{l-k} + \frac{\bar{l}-\bar{x}}{\bar{l}-\bar{k}} = \frac{k+b+f+d-2x}{b+f+d-k} + \frac{\left(\frac{1}{k}-\frac{1}{b}-\frac{1}{d}+\frac{1}{f}\right)+\frac{2x}{bd}}{\frac{1}{f}+\frac{1}{b}+\frac{1}{d}-\frac{1}{k}}$$

where we used $\bar{x} = \frac{b+d-x}{bd}$ as X lies on BD. Adding 1 to the second term and -1 to the first term,

$$\frac{k-x}{b+d+f-k} + \frac{bd+fx}{(bd+df+fb)-\frac{bdf}{k}} = 0$$

$$x = \frac{\frac{k}{b+d+f-k} + \frac{bd}{(bd+df+fb)-\frac{bdf}{k}}}{\frac{1}{b+d+f-k} - \frac{f}{(bd+df+fb)-\frac{bdf}{k}}} = \frac{k(bd+df+fb)-bdf+bd(b+d+f-k)}{(bd+df+fb)-\frac{bdf}{k}-f(b+d+f-k)}$$

$$= \frac{(b+d)(bd+kf)}{\left(\frac{k-f}{k}\right)(bd+kf)} = \frac{k(b+d)}{k-f}$$

By symmetry, $y = \frac{k(f+d)}{k-b}$. Now, we claim that $\triangle KXY \sim \triangle KBF$.

$$\begin{vmatrix} k & k & 1 \\ \frac{k(b+d)}{k-f} & b & 1 \\ \frac{k(f+d)}{k-b} & f & 1 \end{vmatrix} = k \begin{vmatrix} 0 & k & 1 \\ \frac{b+d+f-k}{k-f} & b & 1 \\ \frac{b+d+f-k}{k-b} & f & 1 \end{vmatrix} = \frac{k(b+d+f-k)}{(k-b)(k-f)} \begin{vmatrix} 0 & k & 1 \\ k-b & b & 1 \\ k-f & f & 1 \end{vmatrix}$$

Adding the second column and $(-k)$ times the third column to the first column, we see that it is zero. Now $\angle KYX = \angle KFB = \angle KDB = \angle KDX$ (directed angles). Therefore $KXYD$ is cyclic.

Solution 18 (Shortlist 2018 G7)

O_A is the midpoint of O and the intersection of tangents at A and P. So, $o_A = \frac{ap}{a+p}$, $o_B = \frac{bp}{b+p}$, $o_C = \frac{cp}{c+p}$.
The equation for line l_A is $\frac{t-\frac{ap}{a+p}}{(b-c)} = \frac{bc(\bar{t}-\frac{1}{a+p})}{(b-c)}$ or $t = \bar{t}bc + \frac{ap-bc}{a+p}$.
Similarly, the equation for line l_B is $t = \bar{t}ca + \frac{bp-ca}{b+p}$
and the equation for line l_C is $t = \bar{t}ab + \frac{cp-ab}{c+p}$

Let the triangle formed by these lines have vertices x_1, y_1, z_1. We have to prove that $(X_1Y_1Z_1)$ is tangent to line OP. From the above equations, $\overline{z_1} = \frac{p^2+pc+c(a+b)}{c(a+p)(b+p)}$ so

$$x_1 = \frac{abc + bcp + p^2(b+c)}{(b+p)(c+p)},\ y_1 = \frac{abc + cap + p^2(c+a)}{(c+p)(a+p)},\ z_1 = \frac{abc + abp + p^2(a+b)}{(a+p)(b+p)}$$

Shifting these points, we write $x_2 = x_1 - p$, $y_2 = y_1 - p$, $z_2 = z_1 - p$. Line OP is fixed, so we have to prove that $(X_2Y_2Z_2)$ is tangent to line OP. But $x_2 = \frac{a+p}{k}$, $y_2 = \frac{b+p}{k}$, $z_2 = \frac{c+p}{k}$. Here $k = \frac{(a+p)(b+p)(c+p)}{abc-p^3}$. As $k + \overline{k} = 0$, k is purely imaginary. Multiplying by k scales the figure and rotates it by $90°$.
So, if $x_3 = a + p$, $y_3 = b + p$, $z_3 = c + p$, we have to prove that $(X_3Y_3Z_3)$ is tangent to the line through O perpendicular to OP. Now if we subtract p, $(X_3Y_3Z_3)$ shifts to (ABC) and the line through O perpendicular to OP shifts to the tangent at the antipode of P. As these are clearly tangent, we are done.

Solution 19 (IMO 2000)

Set ω as the unit circle. Let x, y, z be the complex numbers for T_1, T_2, T_3.

$$a = \frac{2yz}{y+z},\ b = \frac{2zx}{z+x},\ c = \frac{2xy}{x+y}$$

Now we take the foot from A to T_1T_1. $h_1 = \frac{1}{2}(x + x + \frac{2yz}{y+z} - \frac{2x^2}{y+z})$

$$h_2 = y + \frac{zx}{z+x} - \frac{y^2}{z+x},\ h_3 = z + \frac{xy}{x+y} - \frac{z^2}{x+y}$$

We claim that the vertices of the triangle formed by the images are $P = \frac{yz}{x}$, $Q = \frac{zx}{y}$, $R = \frac{xy}{z}$, which clearly lie on ω.
Let F_1 and F_2 be the reflections of H_1 and H_2 in T_1T_2. $f_1 = x + y - xy\overline{h_1} = \frac{z(x^2+y^2)}{x(y+z)}$ and $f_2 = \frac{z(x^2+y^2)}{y(x+z)}$. Define $r_z = \frac{x^2+y^2}{xy}$. It is equal to its conjugate, so it is real. We can write

$$f_1 = r_z(\frac{yz}{y+z}),\ f_2 = r_z(\frac{xz}{x+z})$$

Multiplying by $\frac{2}{r_z}$, we scale P, F_1, F_2 through the origin. Line F_1F_2 becomes the tangent at T_3. Now, $\frac{\frac{2yz}{r_zx} - z}{z-0} = \frac{y^2-x^2}{y^2+x^2}$ is purely imaginary, so $\frac{2yz}{r_zx}$ lies on the tangent at T_3. Therefore, F_1F_2 passes through P. Similarly the reflection of line H_1H_3 passes through P. So, P is one of the vertices of the triangle formed. By symmetry, we can conclude that the triangle formed is $\triangle PQR$.

Solution 20 (Shortlist 2004 G8)

Let $(ABCD)$ be the unit circle. As $\left|\frac{a-n}{b-n}\right| = \left|\frac{a-m}{b-m}\right|$ and $\angle ANB + \angle BMA = 180°$,

$$\frac{a-n}{b-n} + \frac{a-m}{b-m} = 0$$

$$n = \frac{m(a+b) - 2ab}{2m - (a+b)} = \frac{(c+d)(a+b) - 4ab}{2(c+d) - 2(a+b)}$$

To prove that E, F, N are collinear, we evaluate

$$\begin{vmatrix} \frac{ac(b+d)-bd(a+c)}{ac-bd} & \frac{a+c-b-d}{ac-bd} & 1 \\ \frac{ad(b+c)-bc(a+d)}{ad-bc} & \frac{a+d-b-c}{ad-bc} & 1 \\ \frac{(c+d)(a+b)-4ab}{2(c+d)-2(a+b)} & \frac{(c+d)(a+b)-4cd}{2ab(c+d)-2cd(a+b)} & 1 \end{vmatrix} = k_1 \begin{vmatrix} 2cd & 2 & c+d \\ ad(b+c)-bc(a+d) & a+d-b-c & ad-bc \\ \frac{(c+d)(a+b)-4ab}{2(c+d)-2(a+b)} & \frac{(c+d)(a+b)-4cd}{2ab(c+d)-2cd(a+b)} & 1 \end{vmatrix}$$

$$= k_2 \begin{vmatrix} 0 & 2 & c+d \\ d-c & a+d-b-c & ad-bc \\ \frac{(c+d)((c+d)(a+b)-4cd)+4(cd(a+b)-ab(c+d))}{2(c+d-a-b)} & \frac{(c+d)(a+b)-4cd}{2} & ab(c+d)-cd(a+b) \end{vmatrix}$$

Directly evaluating through the first column, the determinant vanishes.

Solution 22 (IMO 2019)

Let ω be the unit circle. $a = \frac{2ef}{e+f}$, $b = \frac{2fd}{f+d}$, $c = \frac{2de}{d+e}$, $r = -\frac{fe}{d}$, $p = \frac{a-r}{1-r\bar{a}} = \frac{fe(f+e)+2dfe}{2fe+d(f+e)}$. Let K be the intersection of DI and the external A-bisector. If $k = md$ for some real number m, then $0 = \frac{md-a}{a} + \frac{\frac{m}{d}-\bar{a}}{\bar{a}}$ which gives $2 = \frac{m(f+e)(fe+d^2)}{2dfe}$ so $k = \frac{4d^2 fe}{(f+e)(fe+d^2)}$. Let O_1 and O_2 be the circumcenters of $\triangle PFB$ and $\triangle PEC$, respectively. Using $p - f = \frac{f(e-f)(e+d)}{2fe+df+de}$ and $b - f = \frac{f(d-f)}{d+f}$,

$$o_1 - f = \frac{\frac{f^2(e-f)(e+d)(d-f)}{(2fe+df+de)(d+f)}\left(\frac{(f-e)(e+d)}{fe(2d+f+e)} - \frac{(f-d)}{f(d+f)}\right)}{\frac{(f-e)(e+d)(d-f)}{e(2d+f+e)(d+f)} - \frac{(e-f)(e+d)(f-d)}{(d+f)(2fe+df+de)}} = \frac{fe(2d+f+e)(f-d) - f(f-e)(e+d)(d+f)}{(fe+df-de-e^2)(d+f)}$$

so

$$o_1 = \frac{fe(2d+f+e)(f-d)}{(d+e)(d+f)(f-e)}$$

and

$$o_2 = \frac{fe(2d+f+e)(e-d)}{(d+e)(d+f)(e-f)}$$

$$o_1 - o_2 = \frac{fe(2d+f+e)(e+f-2d)}{(d+e)(d+f)(f-e)}$$

Also,

$$p - k = \frac{ef(f+e-2d)((f+e)(fe+d^2)+4dfe)}{(2fe+df+de)(f+e)(fe+d^2)}$$

$\frac{p-k}{o_1-o_2}$ is

$$\frac{((f+e)(fe+d^2)+4dfe)(d+e)(d+f)(f-e)}{(2fe+df+de)(f+e)(fe+d^2)(2d+f+e)}$$

The conjugate of this expression is just (-1) times the expression, so it is purely imaginary. Therefore $PK \perp O_1O_2$ and K lies on the radical axis PQ of these two circles.

Solution 24 (USA TST 2006/6)

$\triangle APC \cong \triangle ABQ$ (SAS) so $\angle APR = \angle ABR$ and $\angle AQR = \angle ACR$. Therefore $APBR$ and $AQCR$ are cyclic. If $\angle PAB = \angle PRB = \beta$, then $\angle BRC = 180^\circ - \beta$ and $\angle BOC = 2\beta$. If L is the midpoint of the arc $\widehat{BRC}$, then $\angle LOC = \beta$.
Let (BCR) be the unit circle and $b = y^2$, $c = z^2$, $l = -yz$. Now, observe that $\left(\frac{p-a}{b-a}\right)c = l$. So, $\frac{p-a}{b-a} = -\frac{z}{y}$ or $z(b-a) + yp - ay = 0$.

$$p = a - yz + \frac{az}{y},\ q = a - yz + \frac{ay}{z}$$

$$\left(\frac{p-q}{a_0}\right) + \overline{\left(\frac{p-q}{a_0}\right)} = \left(\frac{z}{y} - \frac{y}{z}\right) + \left(\frac{y}{z} - \frac{z}{y}\right) = 0$$

So $AO \perp PQ$.

Mini Survey

(a)

I spent around 13 to 16 hours on the problem set.

(b)

Problem 17 seemed instructive. I liked the solution I found for problem 18.
This unit was very helpful in learning to apply complex numbers elegantly. I never knew that better setups make such a difference.