MAT224: Linear Algebra II

Arnav Patil

University of Toronto

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0 Preface and Acknowledgements

This notes package is based on *A Course in Linear Algebra* by Damiano and Little, which is the primary text for MAT224: Linear Algebra II at the University of Toronto. I self-learned this course over summer 2024 because I wanted to sharpen my linear algebra before heading into second-year studies in ECE, especially since I struggled with MAT188 in Fall 2023. This text also is more proof-based, and discusses a few other topics not covered in MAT188. Nonetheless, my notes for MAT188 can be found here [click me!], if you're interested.

To stay consistent with the text, I will be including the various math statements that are present in the book.

Definition:
An explanation of the mathematical meaning of the word.

Theorem:
A statement that has been proven to be true.

Proposition:
A less important but nonetheless interesting true statement.

Lemma:
A true statement used in proving other true statements.

Corollary:
A true statement that is a simple deduction from a theorem or proposition.

1 Vector Spaces

1.1 Vector Spaces

For the purposes of these notes, I will assume that the reader is familiar with the concept of a 'vector' as well as vector addition and scalar multiplication. First, we will start with a formal definition for a real vector space.

Definition:
A real vector space is a set of vector V together with:

1. An operation called *vector addition*, and
2. An operation called *scalar multiplication*.

There are also eight **axioms** of math that these vectors and functions must satisfy:

1. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$,
2. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$,
3. There exists a vector $\vec{0} \in V$ with the property $\vec{x} + \vec{0} = \vec{x}$,
4. For every vector $\vec{x}$, there exists $\vec{x} + -\vec{x} = \vec{0}$,
5. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$,
6. $(c + d)\vec{x} = c\vec{x} + d\vec{x}$,
7. $(cd)\vec{x} = c(d\vec{x})$, and
8. $1\vec{x} = \vec{x}$.

We see that in $\mathbb{R}^n$there is only one additive identity, which is the zero vector $\vec{0}$. There is also only one additive inverse for each vector in a vector space. This leads us to the following:

Proposition:
Let V be a vector space. Then, we see:

1. The zero vector $\vec{0}$ is unique,
2. For all $\vec{x} \in V, 0\vec{x} = \vec{0}$,
3. For each $\vec{x} \in \vec{V}$, the additive inverse $-\vec{x}$ is unique, and
4. For all $\vec{x} \in \vec{V}$ and all $c \in \mathbb{R}^n$, $(-c)\vec{x} = -(c\vec{x})$.

Now we will go and prove each of the above statements.

1. Suppose $\vec{0}$ and $\vec{0}'$ both satisfy axiom 3 above. Then $\vec{0} + \vec{0}' = \vec{0}$, since $\vec{0}'$ is an additive identity. Conversely, $\vec{0} + \vec{0}' = \vec{0}' + \vec{0} = \vec{0}'$, which gives us $\vec{0} = \vec{0}'$, or there is only one additive identity in V.
   (a) Assuming we have two examples of a given object, then proving those objects are the same, is one common way of proving that something is unique.
2. We have $0\vec{x} = (0 + 0)\vec{x} = 0\vec{x} + 0\vec{x}$ by axiom 6. If we add the inverse of $0\vec{x}$ to both sides, we obtain $\vec{0} = 0\vec{x}$.

3. Using the same logic as part a), we use axioms 1, 3, and 4 to give $\vec{x}+-\vec{x}+(-\vec{x})' = (\vec{x}+-\vec{x})+(-\vec{x})' = \vec{0}+(-\vec{x})' = (-\vec{x})'$. On the other hand, axiom 2 gives us $\vec{x}+-\vec{x}+(-\vec{x})' = \vec{x}+(-\vec{x})'+-\vec{x} = (\vec{x}+(-\vec{x})')+-\vec{x} = \vec{0}+-\vec{x} = -\vec{x}$. Therefore, we have $-\vec{x} = (-\vec{x})'$ and the additive inverse of $\vec{x}$ is unique.

4. $c\vec{x}+(-c)\vec{x} = (c+-c)\vec{x} = 0\vec{x} = \vec{0}$ by axiom 6. Hence $(-c)\vec{x}$ is the additive inverse of $c\vec{x}$ by part c). Therefore, we can prove $(-c)\vec{x} = -(c\vec{x})$. ■

1.2 Subspaces

We define vector sum and scalar multiplication of $C(\mathbb{R})$ as usual for a function. If $f, g \in C(\mathbb{R})$ and $c \in \mathbb{R}$, then both $(f+g)(x)$ and $cf(x)$ are functions defined for all $\vec{x} \in \mathbb{R}$. We claim this set, $C(\mathbb{R})$ is a vector space. Because of this, we may summarize the two following facts:

Lemma:
Let $f, g \in C(\mathbb{R})$, and let $c \in \mathbb{R}$. Then,

1. $f, g \in C(\mathbb{R})$, and
2. $cf \in C(\mathbb{R})$.

Proof:

1. By the limit sum rule from calculus, for all $a \in \mathbb{R}$ we have:

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

   Since f and g are continuous, this last expression equals $f(a)+g(a) = (f+g)(a)$. Hence $f+g$ is continuous.

2. By the limit product rule, we have:

$$\lim_{x\to a}(cf)(x) = \lim_{x\to a} cf(x) = (\lim_{x\to a} c)\cdot(\lim_{x\to a} f(x)) = cf(a) = (cf)(a)$$

   therefore cf is also continuous. ■

Proving the eight axioms proves unnecessary as we see that their properties are already established in the process of proving that the sets in the above lemma are vector spaces. This provides us with the following definition:

Definition:
Let V be a vector space and let $W \subseteq V$ be a subset. Then W is a subspace of V if W is a vector space itself under the operations of vector addition and scalar multiplication.

We can alternatively state the following requirements for a space W to be a vector subspace V:

1. The set W must be closed under vector addition,
2. The set W must be closed under scalar multiplication, and
3. The zero vector $\vec{0}$ must be contained within the set.

Examples [in the book] imply the following, which we state as a theorem:

Theorem:
Let V be a vector space. Then the intersection of any collection of subspaces of V is a subspace of V.

Proof: Consider a collection of subspaces of V. At the very minimum, the intersection of all these sets is nonempty since each subspace must contain the zero vector $\vec{0}$.
Another important application of this theorem is to show that the set of solutions to any systems of linear equations is a subspace of $\mathbb{R}^n$.

1.3 Linear Combinations

We will begin this section with some terminology.

Definition:
Let S be a subset of a vector space V.

1. A *linear combination* of vectors in S is any sum $a_1\vec{x}_1 + ... + a_n\vec{x}_n$,
2. If $S \neq \varnothing$, the set of all linear combinations of vectors in S is called the **span** of S, denoted $\text{Span}(S)$. If $S = \varnothing$ we define $\text{Span}(S) = \{\vec{0}\}$, and
3. If $W = \text{Span}(S)$, we sat S *spans or generates* W.

Simply put, the span of S is the set of all vectors that can be built with **all** linear combinations of the vectors.

We can generally say that the span of a set of vectors is a subspace of the vector space from where those vectors was chosen. This is summarized in the theorem below:

Theorem:
Let V be a vector space and let S be any subset of V. Then $\text{Span}(S)$ is a subspace of V.

Proof: $\text{Span}(S)$ is nonempty by definition (the zero vector). We can write $\vec{x} = a_1\vec{x}_1 + ... + a_n\vec{x}_n$ and $\vec{y} = b_1\vec{y}_1 + ... + b_m\vec{y}_m$. Then for any scalar c, we have:

$$\begin{aligned} c\vec{x} + \vec{y} &= c(a_1\vec{x}_1 + ... + a_n\vec{x}_n) + (b_1\vec{x}'_1 + ... + b_m\vec{x}'_m) \\ &= ca_1\vec{x}_1 + ... + ca_n\vec{x}_n + b_1\vec{x}'_1 + ... + b_m\vec{x}'_m \end{aligned}$$

Since these are all linear combinations of the vectors in the set S, we have $c\vec{x} + \vec{y} \in \text{Span}(S)$, and therefore, $\text{Span}(S)$ is a subspace of V. ■

Proving this theorem is important because every subspace of a given vector space can be constructed this way. It gives us the following definition.

Definition:
Let W_1 and W_2 be subspaces of a given vector space V. The sum of W_1 and W_2 is:

$$W_1 + W_2 = \{\vec{x} \in V | \vec{x} = \vec{x}_1 + \vec{x}_2, \text{ for some } \vec{x}_1 \in W_1 \text{ and } \vec{x}_2 \in W_2\}$$