

# MIE100: Dynamics

Arnav Patil

Department of Electrical and Computer Engineering, University of Toronto

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# 1 Kinematics of Particles

## Rectilinear Motion

$$v = \frac{ds}{dt} = \dot{s} \quad (1)$$

$$a = \frac{dv}{dt} = \ddot{s} \quad (2)$$

- If we eliminate  $dt$ , we get  $v dv = a ds$ .

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a \cdot dt \quad (3)$$

$$\int_{v_1}^{v_2} v \cdot dv = \int_{s_1}^{s_2} a \cdot ds \quad \text{OR} \quad \frac{1}{2}(v_2^2 - v_1^2) \quad (4)$$

## Plane Curvilinear Motion

$$v = \lim_{t \rightarrow 0} \frac{\Delta r}{\Delta t} \rightarrow v = \frac{dr}{dt} = \dot{r} \quad (5)$$

For projectile motion,  $a_x = 0$  and  $a_y = -g$ .

## Normal and Tangential (n-t) Coordinates

$$v = v\hat{e}_t = \rho\dot{\beta}\hat{e}_t \quad (6)$$

where  $\rho$  is the radius of curvature and  $\beta$  is in radians.

For circular motion,

$$a = \frac{v^2}{r}e_n + \dot{v}e_t \quad (7)$$

$$v = r\dot{\theta}, a_n = \frac{v^2}{r} = r\dot{\theta}^2 = v\dot{\theta}, a_t = \dot{v} = r\ddot{\theta} \quad (8)$$

## Polar Coordinates ( $r - \theta$ )

Velocity:

$$v = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \quad (9)$$

Acceleration:

$$a = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta \quad (10)$$

For circular motion with a constant  $r$ , components become:

- $v_r = 0$
- $v_\theta = r\dot{\theta}$
- $a_r = -r\dot{\theta}^2$
- $a_\theta = r\ddot{\theta}$

## Relative Motion (Translating Axis)

$$r_B = r_A + r_{B/A} \rightarrow v_B = v_A + v_{B/A} \rightarrow a_B = a_A + a_{B/A} \quad (11)$$

## 2 Kinetics of Particles

### Newton's Second Law

$$F = ma \quad (12)$$

### Equation of Motion and Solutions of Problems

$$\sum F = ma \quad (13)$$

Remember to determine if motion is constrained or unconstrained.

### Rectilinear Motion

Split up  $\sum F = ma$  into its directional components.

### Curvilinear Motion

$$\sum F_r = ma_r \rightarrow a_r = \ddot{r} - r\dot{\theta}^2 \quad (14)$$

$$\sum F_\theta = ma_\theta \rightarrow a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \quad (15)$$

### Work and Kinetic Energy

$$U = \int F \cdot dr \quad (16)$$

1. Work associated with a constant external force,
2. Work associated with a spring force, and
3. Work associated with weight.

$$U_{1 \rightarrow 2} = \int_1^2 F \cdot dr = \int_{v_1}^{v_2} mv \cdot dv = \frac{1}{2}m(v_2^2 - v_1^2) \quad (17)$$

$$E_K = \frac{1}{2}mv^2 \quad (18)$$

$$U_{1 \rightarrow 2} = T_2 - T_1 = \Delta T \quad (19)$$

Power:  $P = F \cdot v$ .

## Potential Energy

$$V_g = mgh \quad (20)$$

$$\Delta V_e = \frac{1}{2}k(x_2^2 - x_1^2) \quad (21)$$

Work-energy equation:  $T_1 + V_1 + U_{1 \rightarrow 2} = T_2 + V_2$

## Linear Impulse and Linear Momentum

$$\sum F = m\dot{v} \rightarrow \sum F = \dot{G} \quad (22)$$

Impulse:

$$\int_{t_1}^{t_2} \sum F \cdot dt = G_2 - G_1 = \Delta G \rightarrow G_1 + \int_{t_1}^{t_2} \sum F \cdot dt = G_2 \quad (23)$$

## Angular Impulse and Angular Momentum

$$H_O = r \times mv \rightarrow \sum M_O = \dot{H}_O \quad (24)$$

# 3 Kinetics of Systems of Particles

## Generalized Newton's Second Law

For a system of particles:  $mr = \sum m_i r_i$  Kinetic energy expression:  $v_i = v + \dot{\rho}_i$ , where  $\rho_i$  is the velocity of  $m_i$  with regards to a translating reference frame moving with the mass centre  $G$ .

$$T = \frac{1}{2}mv^2 + \sum \frac{1}{2}m_i|\dot{\rho}_i| \quad (25)$$

## Impulse-Momentum

$$G = mv \rightarrow \sum F = \dot{G} \quad (26)$$

Angular momentum:

- About a fixed point O:

$$H_O = \sum (r_i \times m_i v_i) \quad (27)$$

$$\sum M_O = \dot{H}_O \quad (28)$$

- About the mas centre G:

$$H_G = \sum \rho_i \times m_i \dot{r}_i \quad (29)$$

$$\sum M_G = \dot{H}_G \quad (30)$$

- About an arbitrary point P:

$$H_P = H_G + \rho \times mv \quad (31)$$

$$\sum M_P = \dot{H}_G + \rho \times ma \quad (32)$$

## Conservation of Energy and Momentum

$$\Delta T + \Delta V = 0 \rightarrow T_1 + V_1 = T_2 + V_2 \quad (33)$$

$$(H_O)_1 = (H_O)_2 \text{ OR } (H_G)_1 = (H_G)_2 \quad (34)$$

## 4 Plane Kinematics of Rigid Bodies

### Rotations

$$\omega = \omega_0 + \alpha t \quad (35)$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \quad (36)$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \quad (37)$$

Rotation about a fixed axis:

- $v = \omega \times r$
- $a_n = \omega \times \omega \times r = \frac{v^2}{r} = \omega \times v$
- $a_t = \alpha \times r$

### Relative Velocity

**SINCLAIR'S SPAM EQUATION:**  $v_B = v_A + \omega_{AB} \times r_{B/A}$

$$\Delta r_A = \Delta r_B + \Delta r_{A/B} \quad (38)$$

### Instantaneous Centre of Zero Velocity

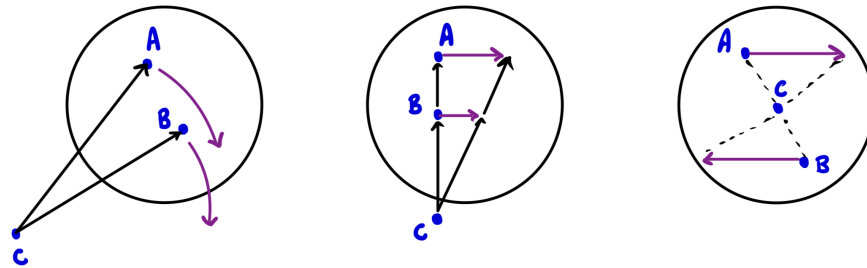


Figure 1: Instantaneous Centre of Zero Velocity

## Relative Acceleration

$$a_A = a_B + a_{A/B} \quad (39)$$

$$(a_{A/B})_n = \frac{v_{A/B}^2}{r} = r\omega^2 \quad (40)$$

$$(a_{A/B})_t = v_{A/B} \dot{\omega} = r\alpha \quad (41)$$

## 5 Plane Kinetics of Rigid Bodies

### General Equations of Motion

$$\sum F = ma \rightarrow \sum M_G = \dot{H}_G \quad (42)$$

$$H_G = I\omega \rightarrow \sum M_G = \dot{H}_G = I\dot{\omega} = I\alpha \quad (43)$$

$$\sum M_P = \dot{H}_G + \rho \times ma \quad (44)$$

This is summed up as:  $M_P = I\alpha + mad$

$$\sum M_O = I_O\alpha \quad (45)$$

Analysis procedure:

1. Kinematics – equations
2. Diagrams – identify knowns and unknowns
3. Equations of motion – use to get extra variables and solvable system

### Translation

For a translating body, our general equations of motion are:

$$\sum F = ma \quad (46)$$

$$\sum M_G = I\alpha = 0 \quad (47)$$

### Fixed-Axis Rotation

Almost same equations are applicable here:

$$\sum F = ma \quad (48)$$

$$\sum M_G = I\alpha \quad (49)$$

$$\sum M_O = I_O\alpha \quad (50)$$

## General Plane Motion

Solving plane motion problems:

1. Choice of coordinate system
2. Choice of moment equations –  $\sum M_P = I\alpha + mad$
3. Choice of constrained vs unconstrained motion
4. Number of unknowns
5. Identify body or system
6. Kinematics equations
7. Consistency in assumptions

## Work-Energy Relations

$$U = \int F \cdot dr \quad \text{OR} \quad U = \int (F \cos \alpha) ds \quad (51)$$

Kinetic energy:

- Translation –  $T = \frac{1}{2}mv^2$
- Fixed axis rotation –  $T = \frac{1}{2}I_O\omega^2$
- General plane motion –  $T = \frac{1}{2}mv^2 + \frac{1}{2}I_O\omega^2$ 
  - Can also be expressed at the IC –  $T = \frac{1}{2}I_C\omega^2$

## 6 Vibration & Time Response

### Free Vibration of Particles

Applying Newton's Second Law in the form  $\sum F_x = m\ddot{x}$ :

- $-kx = m\ddot{x}$  OR  $m\ddot{x} + kx = 0$
- Oscillation of a mass objected to a linear restoring force as described by this equation is called SIMPLE HARMONIC MOTION and is characterized by acceleration which is proportional to the displacement but of the opposite sign.
- $\ddot{x} + \omega_n^2 x = 0$ , which gives us  $\omega_n = \sqrt{k/m}$ .

Solution for undamped free motion:

$$x(t) = A \cos \omega_n t + B \sin \omega_n t \quad (52)$$

$$x(t) = C \sin(\omega_n t + \psi) \quad (53)$$

At  $t = 0$ , we get  $x_0 = A$ , and  $\dot{x}_0 = B\omega_n$ . If  $t = 0$ , then  $x_0 = C \sin \psi$  and  $\dot{x}_0 = C\omega_n \cos \psi$ .

- $\psi = \tan^{-1}\left(\frac{x_0\omega_n}{\dot{x}_0}\right)$

Natural frequency:  $f_n = \frac{\omega_n}{2\pi}$

For damped free vibration:

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (54)$$

If we define  $\zeta = \frac{c}{2m\omega_n}$ , then we can say:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0 \quad (55)$$

For free damped vibration:

$$x = Ae^{rt} \quad (56)$$

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \quad (57)$$

$$x = A_1e^{r_1t} + A_2e^{r_2t} \quad (58)$$

Categories of damped motion:

- $\zeta > 1$  OVERDAMPED
- $\zeta = 1$  CRITICALLY DAMPED
- $\zeta < 1$  UNDERDAMPED

## Forced Vibration of Particles

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = \frac{F_0 \sin \omega t}{m} \quad (59)$$

## Vibration of Rigid Bodies

Use same equations derived throughout the course, just add rotation to the calculations.